Interpolating Values using a Tetrahedral Mesh

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1 Introduction

A computer game featuring localised gravity – the "down" direction of objects changes depending on where you are located in 3D space – is achieved by creating a descrete vector field. Normalised vectors, referred to as "Gravity Vectors", are placed at key locations in the 3D world to define the direction gravity should have at that location.

Ideally, a continuous vector field is desired, as that would remove the need to smooth out sudden changes in gravity.

The gravity vector locations are used to construct a tetrahedral mesh. An arbitrary 3D location, if located inside any one of the tetrahedrons, can be converted into barycentric coordinates and used to calculate a new directional vector based on the interpolated values of the four gravity vectors forming the tetrhahedron.

If the 3D location is located outside the convex hull of the mesh, it is unclear how the vector field should be extrapolated. To address this, we create "infinite tetrahedrons" using the faces of the convex hull. The result is such that a 3D point located outside of the convex hull is projected either onto one of the faces on the convex hull (if the tetrahedron has one vertex located in infinitey), projected onto one of the edges on the convex hull (if the tetrahedron has two vertices located in infinity) or projected onto one of the vertices of the convex hull (if the tetrahedron has three vertices located in infinity). The projected location can be converted to barycentric coordinates as usual and the interpolated vector can be computed.

The result is a continuous vector field, capable of mapping any finite 3D location to a gravitational vector.

2 Theory

Four different cases need to be distinguished: Normal tetrahedrons, a tetrahedron with one vertex located infinitely far away, a tetrahedron with two vertices located infinitely far away, and a tetrahedron with three vertices located infinitely away.

The barycentric coordinates of a 3D point in space need to be computed for all cases. Additionally, a method for checking boundaries needs to exist for all cases. Two 4x4 transformation matrices are proposed to generalise these two requirements. The construction of these matrices is different in every case, and is described in detail in the following sections.

2.1 Transformation Matrices

2.1.1 Transforming from Cartesian to Barycentric

A 3D tetrahedron, a polyhedron having four triangular faces and four vertices, is defined by its four vertices v_1, v_2, v_3 and v_4, where v_n is a 3 dimensional point in Cartesian space $v_n = \begin{bmatrix} x_n & y_n & z_n \end{bmatrix}^T$. The barycentric coordinates are defined so that the first vertex r_1 maps to barycentric coordinates $\lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $r_2 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, etc. and that the sum of barycentric parameters $\sum \lambda_n = 1$.

This is a linear transformation and the problem can be written in matrix form so that $\vec{v} = B^{-1}\vec{\lambda}$ with $B^{-1} = \begin{bmatrix} v_1 | v_2 | v_3 | v_4 \end{bmatrix}$ and $\vec{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T$. The condition $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ can be augmented into the matrix to form the final equation:

$$\begin{bmatrix} v1_x & v2_x & v3_x & v4_x \\ v1_y & v2_y & v3_y & v4_y \\ v1_z & v2_z & v3_z & v4_z \\ 1 & 1 & 1 & 1 \end{bmatrix} \vec{\lambda} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Where x, y, and z define a 3D point in Cartesian space. The barycentric coordinates λ can be obtained by solving this linear equation, thus resulting in the transformation matrix B:

$$B = \begin{bmatrix} v1_x & v2_x & v3_x & v4_x \\ v1_y & v2_y & v3_y & v4_y \\ v1_z & v2_z & v3_z & v4_z \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \tag{1}$$

The projection matrix B will suffice when dealing with a normal tetrahedron (i.e. all four vertices have finite locations).

2.1.2 General Projection

For all other cases where the tetrahedron has infinite vertices, we will need to construct a projection matrix.

Given a subspace $V = \text{span}\left[\vec{e_1}|\vec{e_2}|\ldots\right]$ where $\vec{e_n}$ is the basis for V and $\vec{x}, \vec{e_n} \in \mathbb{R}^N$, the projection of $\vec{x}$ onto V is defined as:

$$\text{proj}_V \vec{x} = A \left(A^\intercal A\right)^{-1} A^\intercal \vec{x} \tag{2}$$

This projection is only valid for $\vec{v_1} = \vec{0}$. If the triangle has an offset in 3D space, then the offset must be subtracted before performing the projection and added back after the projection. A translation matrix is used:

$$T = \begin{bmatrix} 1 & 0 & 0 & -v_{1_x} \\ 0 & 1 & 0 & -v_{1_y} \\ 0 & 0 & 1 & -v_{1_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The final, general projection matrix can be constructed with:

$$\text{proj}_V \vec{x} = TA \left(A^\intercal A\right)^{-1} A^\intercal T^{-1} \vec{x} \tag{4}$$

2.1.3 Barycentric Coordinates of a Face Projection

In the case of a single vertex of the tetrahedron being located infinitely far away, any 3D point located inside its volume will be projected onto the triangle formed by the three finite vertex locations. Thus, a projection matrix must be constructed.

The face of a tetrahedron is defined by the three vertices v_1, v_2 and v_3, where $v_n \in \mathbb{R}^3$. The 3 dimensional Cartesian coordinate $\vec{x} \in \mathbb{R}^3$ is projected onto one of the tetrahedron's triangles using equation 4 where $\vec{a} = \vec{v_2} - \vec{v_1}$ and $\vec{b} = \vec{v_3} - \vec{v_1}$ and the matrix A is defined as $A = \left[a|b\right]$.

The matrices B and proj_V from equations 1 and 4 may be combined to form the final matrix for transforming a 3D coordinate $\vec{x}$ into projected barycentric coordinates $\vec{\lambda}$:

$$\vec{\lambda} = TA \left(A^\intercal A\right)^{-1} A^\intercal T^{-1} B \vec{x} \tag{5}$$

In an effort to be more verbose, equation 5 is broken down and constructed step-by-step using the triangle vertices $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$.

$$\begin{aligned} \vec{a} &= \vec{v_2} - \vec{v_1} \\ \vec{b} &= \vec{v_3} - \vec{v_1} \\ \text{proj}_V &= A \left(A^\intercal A\right)^{-1} A^\intercal \\ \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} &= \begin{bmatrix} a_x & b_x \\ a_y & b_y \\ a_z & b_z \end{bmatrix} \left(\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} a_x & b_x \\ a_y & b_y \\ a_z & b_z \end{bmatrix} \right)^{-1} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \end{aligned}$$

The calculated projection matrix can be sandwiched between the translation matrix T from equation 3 and then be multiplied with the barycentric transformation matrix B from equation 1 to yield the final transformation matrix.

$$\vec{\lambda} = \begin{bmatrix} 1 & 0 & 0 & -v_{1_x} \\ 0 & 1 & 0 & -v_{1_y} \\ 0 & 0 & 1 & -v_{1_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{00} & m_{01} & m_{02} & 0 \\ m_{10} & m_{11} & m_{12} & 0 \\ m_{20} & m_{21} & m_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -v_{1_x} \\ 0 & 1 & 0 & -v_{1_y} \\ 0 & 0 & 1 & -v_{1_z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} v_{1_x} & v_{2_x} & v_{3_x} & v_{4_x} \\ v_{1_y} & v_{2_y} & v_{3_y} & v_{4_y} \\ v_{1_z} & v_{2_z} & v_{3_z} & v_{4_z} \\ 1 & 1 & 1 & 1 \end{bmatrix} \vec{x}$$

2.1.4 Barycentric Coordinates of an Edge Projection

In the case of two vertices of the tetrahedron being located infinitely far away, any 3D point located inside its volume will be projected onto the edge formed by the two finite vertex locations. Thus, a projection matrix must be constructed.

The edge of a tetrahedron is defined by the two vertices v_1 and v_2, where $v_n \in \mathbb{R}^3$. The 3 dimensional Cartesian coordinate $\vec{x} \in \mathbb{R}^3$ is projected onto one of the tetrahedron's edges using equation 4 where the matrix A is defined as $A = \begin{bmatrix} \vec{v_2} - \vec{v_1} \end{bmatrix}$.

The matrices B and proj_V from equations 1 and 4 may be combined to form the final matrix for transforming a 3D coordinate $\vec{x}$ into projected barycentric coordinates $\vec{\lambda}$:

$$\vec{\lambda} = TA \left(A^\intercal A\right)^{-1} A^\intercal T^{-1} B \vec{x} \tag{6}$$

In an effort to be more verbose, equation 6 is broken down and constructed step-by-step using the edge vertices $\vec{v_1}$, $\vec{v_2}$.

$$\begin{aligned} \vec{a} &= \vec{v_2} - \vec{v_1} \\ \text{proj}_V &= A \left(A^\intercal A\right)^{-1} A^\intercal \\ \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \left(\begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \right)^{-1} \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \end{aligned}$$

The calculated projection matrix can be sandwiched between the translation matrix T from equation 3 and then be multiplied with the barycentric transformation matrix B from equation 1 to yield the final transformation matrix.

$$\vec{\lambda} = \begin{bmatrix} 1 & 0 & 0 & -v_{1_x} \\ 0 & 1 & 0 & -v_{1_y} \\ 0 & 0 & 1 & -v_{1_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{00} & m_{01} & m_{02} & 0 \\ m_{10} & m_{11} & m_{12} & 0 \\ m_{20} & m_{21} & m_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -v_{1_x} \\ 0 & 1 & 0 & -v_{1_y} \\ 0 & 0 & 1 & -v_{1_z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} v_{1_x} & v_{2_x} & v_{3_x} & v_{4_x} \\ v_{1_y} & v_{2_y} & v_{3_y} & v_{4_y} \\ v_{1_z} & v_{2_z} & v_{3_z} & v_{4_z} \\ 1 & 1 & 1 & 1 \end{bmatrix} \vec{x}$$

2.1.5 Barycentric Coordinates of a Point Projection

In the case of three vertices of the tetrahedron being located infinitely far away, any 3D point located inside its volume will effectively be projected onto the one finite vertex.

The 3 dimensional Cartesian coordinate $\vec{x} \in \mathbb{R}^3$ is projected onto one of the tetrahedron's vertices $v_1 \in \mathbb{R}^3$. This means one of the barycentric coordinates will always be 1 and the other three barycentric coordinates will have value 0. This can be described with the transformation matrix:

$$P = \begin{bmatrix} 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{7}$$

Where one of the entries in $\vec{\lambda}$ has value 1 and all other entries have value 0.

It should be noted that a projection matrix may be overkill for this particular case. However, it is necessary if one wants to generalise the implementation by using a 4x4 matrix.

2.2 Boundary Check

In this section we discuss how to determine if a point is located inside a tetrahedron or not. This includes cases where vertices of the tetrahedron are located infinitely far away.

The easist case is a normal tetrahedron with no infinite vertices. A 3D point $\vec{x}$ is located inside the tetrahedron if its barycentric coordinates $\vec{\lambda} = \left[\lambda_1, \lambda_2, \lambda_3, \lambda_4\right]^\intercal$ satisfy the condition:

$$0 \leq \lambda_n \leq 1 \tag{8}$$

If one vertex is located in infinity then one of the barycentric parameters λ_1 will always equal 0. Therefore, we need to additionally check which side of the triangle, defined by $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ the point $\vec{x}$ is being projected from. The following condition must be satisfied in addition to the condition described in equation 8

$$(\vec{v_2} - \vec{v_1}) \times (\vec{v_3} - \vec{v_1}) \odot (\vec{x} - \vec{v_1}) \geq 0 \tag{9}$$

If two vertices are located in infinity then two of the barycentric parameters $\lambda_{1,2}$ will always equal 0.

3 Another Approach

Instead of dealing with four different cases and having to build eight different transformation matrices, another approach is to cast a ray from the 3D location $\vec{x}$ to the center of the convex hull of the tetrahedral mesh and determine the location of intercection on the hull's surface.

3.1 Intersection of a Ray and a Triangle

A triangle is defined by its three vertices $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$. A point $\vec{T}(\lambda_1, \lambda_2)$ on a triangle is given by

$$\vec{T}(\lambda_1, \lambda_2) = (1 - \lambda_1 - \lambda_2)\vec{v_1} + u\vec{v_2} + v\vec{v_3} \tag{10}$$

Where (λ_1, λ_2) are the barycentric coordinates of the triangle, which must fulfill the requirement $0 \leq \lambda_n \leq 1$.

A ray $\vec{R}(t)$ with origin $\vec{r_0}$ and normalised direction $\vec{d}$ is defined as

$$\vec{R}(t) = \vec{r_0} + t\vec{d} \tag{11}$$

Computing the intersection between the ray $\vec{R}(t)$ and the triangle $\vec{T}(\lambda_1, \lambda_2)$ is equivalent to $\vec{R}(t) = \vec{T}(\lambda_1, \lambda_2)$, which yields

$$\vec{r_0} + t\vec{d} = (1 - \lambda_1 - \lambda_2)\vec{v_1} + \lambda_1\vec{v_2} + \lambda_2\vec{v_3} \tag{12}$$

Rearranging the terms gives

$$\begin{bmatrix} -d & \vec{v_2} - \vec{v_1} & \vec{v_3} - \vec{v_1} \end{bmatrix} \begin{bmatrix} t \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \vec{r_0} - \vec{v_1} \tag{13}$$

This means the barycentric coordinates (λ_1, λ_2) and the distance t from the ray origin to the intersection point can be found by solving the lienar system of equations above.

Denoting $\vec{a} = \vec{v_2} - \vec{v_1}$, $\vec{b} = \vec{v_3} - \vec{v_1}$ and $\vec{c} = \vec{r_0} - \vec{v_1}$, the solution to equation 13 is obtained by using Cramer's rule:

$$\begin{bmatrix} t \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{\begin{vmatrix} -\vec{d} & \vec{a} & \vec{b} \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} \vec{c} & \vec{a} & \vec{b} \\ -\vec{d} & \vec{c} & \vec{b} \\ -\vec{d} & \vec{a} & \vec{c} \end{vmatrix} \end{bmatrix} \tag{14}$$

From linear algebra, we know that $\begin{vmatrix} \vec{A} & \vec{B} & \vec{C} \end{vmatrix} = -\left(\vec{A} \times \vec{C}\right) \cdot B = -\left(\vec{C} \times \vec{B}\right) \cdot A$. Equation 14 could therefore be rewritten as

$$\begin{bmatrix} t \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{\left(\vec{d} \times \vec{b}\right) \cdot \vec{a}} \begin{bmatrix} (\vec{c} \times \vec{a}) \cdot \vec{b} \\ \left(\vec{d} \times \vec{b}\right) \cdot \vec{c} \\ (\vec{c} \times \vec{a}) \cdot \vec{d} \end{bmatrix} \tag{15}$$

3.2 Finding the circumsphere of a tetrahedron

$$|\vec{x} - \vec{v_1}| = |\vec{x} - \vec{v_2}| \tag{16}$$

$$|\vec{x} - \vec{v_1}| = |\vec{x} - \vec{v_3}| \tag{17}$$

$$|\vec{x} - \vec{v_1}| = |\vec{x} - \vec{v_4}| \tag{18}$$

$$\tag{19}$$