Chapter 4

The Solar System: The Runge–Kutta Methods

4.1 The Solar System

4.1.1 *Newton's Second Law*

We consider the motion of the Earth around the Sun. Let r be the distance and M_s and M_e be the masses of the Sun and the Earth respectively. We neglect the effect of the other planets and the motion of the Sun (i.e. we assume that $M_s >> M_e$). The goal is to calculate the position of the Earth as a function of time. We start from Newton's second law of motion

$$\begin{aligned} M_e \frac{d^2\vec{r}}{dt^2} &= -\frac{GM_eM_s}{r^3}\vec{r} \\ &= -\frac{GM_eM_s}{r^3}(x\vec{i} + y\vec{j}). \end{aligned} \tag{4.1}$$

We get the two equations

$$\frac{d^2x}{dt^2} = -\frac{GM_s}{r^3}x. \tag{4.2}$$

$$\frac{d^2y}{dt^2} = -\frac{GM_s}{r^3}y. \tag{4.3}$$

We replace these two second-order differential equations by the four first-order differential equations

$$\frac{dx}{dt} = v_x. \tag{4.4}$$

$$\frac{dv_x}{dt} = -\frac{GM_s}{r^3}x. \tag{4.5}$$

$$\frac{dy}{dt} = v_y. \tag{4.6}$$

$$\frac{dv_y}{dt} = -\frac{GM_s}{r^3}y. \tag{4.7}$$

We recall

$$r = \sqrt{x^2 + y^2}. \tag{4.8}$$

4.1.2 *Astronomical Units and Initial Conditions*

The distance will be measured in astronomical units (AU) whereas time will be measured in years. One astronomical unit of length (1 AU) is equal to the average distance between the earth and the sun, viz 1 AU $= 1.5 \times 10^{11} m$. The astronomical unit of mass can be found as follows. Assuming a circular orbit we have

$$\frac{M_e v^2}{r} = \frac{G M_s M_e}{r^2}. \tag{4.9}$$

Equivalently

$$G M_s = v^2 r. \tag{4.10}$$

The radius is $r = 1$ AU. The velocity of the earth is $v = 2\pi r/\text{yr} = 2\pi$ AU/yr. Hence

$$G M_s = 4\pi^2 \ \text{AU}^3/\text{yr}^2. \tag{4.11}$$

For the numerical simulations it is important to determine the correct initial conditions. The orbit of Mercury is known to be an ellipse with eccentricity $e = 0.206$ and radius (semimajor axis) $a = 0.39$ AU with the Sun at one of the foci. The distance between the Sun and the center is ea. The first initial condition is $x_0 = r_1$, $y_0 = 0$ where r_1 is the maximum distance from Mercury to the Sun, i.e. $r_1 = (1+e)a = 0.47$ AU. The second initial condition is the velocity $(0, v_1)$ which can be computed using conservation of energy and angular momentum. For example by comparing with the point $(0, b)$ on the orbit where b is the semiminor axis, i.e. $b = a\sqrt{1-e^2}$ the velocity $(v_2, 0)$ there can be obtained in terms of $(0, v_1)$ from conservation of angular momentum as follows

$$r_1 v_1 = b v_2 \Leftrightarrow v_2 = \frac{r_1 v_1}{b}. \tag{4.12}$$

Next conservation of energy yields

$$\frac{1}{2} M_m v_1^2 - \frac{G M_s M_m}{r_1} = \frac{1}{2} M_m v_2^2 - \frac{G M_s M_m}{r_2}. \tag{4.13}$$

In above $r_2 = \sqrt{e^2 a^2 + b^2}$ is the distance between the Sun and Mercury when at the point $(0, b)$. By substituting the value of v_2 we get an equation for v_1. This is given by

$$v_1 = \sqrt{\frac{G M_s}{a} \frac{1-e}{1+e}} = 8.2 \ \text{AU/yr}. \tag{4.14}$$

4.1.3 *Kepler's Laws*

Kepler's laws are given by the following three statements:

- The planets move in elliptical orbits around the sun. The sun resides at one focus.
- The line joining the sun with any planet sweeps out equal areas in equal times.

- Given an orbit with a period T and a semimajor axis a the ratio T^2/a^3 is a constant.

The derivation of these three laws proceeds as follows. We work in polar coordinates. Newton's second law reads

$$M_e \ddot{\vec{r}} = -\frac{GM_s M_e}{r^2}\hat{r}. \tag{4.15}$$

We use $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$ to derive $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ and $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$. Newton's second law decomposes into the two equations

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{4.16}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM_s}{r^2}. \tag{4.17}$$

Let us recall that the angular momentum by unit mass is defined by $\vec{l} = \vec{r} \times \dot{\vec{r}} = r^2\dot{\theta}\hat{r} \times \hat{\theta}$. Thus $l = r^2\dot{\theta}$. Equation (4.16) is precisely the requirement that angular momentum is conserved. Indeed we compute

$$\frac{dl}{dt} = r(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \tag{4.18}$$

Now we remark that the area swept by the vector $\vec{r}$ in a time interval dt is $dA = (r \times rd\theta)/2$ where $d\theta$ is the angle traveled by $\vec{r}$ during dt. Clearly

$$\frac{dA}{dt} = \frac{1}{2}l. \tag{4.19}$$

In other words the planet sweeps equal areas in equal times since l is conserved. This is Kepler's second law.

The second equation (4.17) becomes now

$$\ddot{r} = \frac{l^2}{r^3} - \frac{GM_s}{r^2} \tag{4.20}$$

By multiplying this equation with $\dot{r}$ we obtain

$$\frac{d}{dt}E = 0\ ,\ E = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{GM_s}{r}. \tag{4.21}$$

This is precisely the statement of conservation of energy. E is the energy per unit mass. Solving for dt in terms of dr we obtain

$$dt = \frac{dr}{\sqrt{2\left(E - \frac{l^2}{2r^2} + \frac{GM_s}{r}\right)}} \tag{4.22}$$

However $dt = (r^2 d\theta)/l$. Thus

$$d\theta = \frac{ldr}{r^2\sqrt{2\left(E - \frac{l^2}{2r^2} + \frac{GM_s}{r}\right)}} \tag{4.23}$$

By integrating this equation we obtain (with $u = 1/r$)

$$\theta = \int \frac{ldr}{r^2\sqrt{2\left(E - \frac{l^2}{2r^2} + \frac{GM_s}{r}\right)}}$$

$$= -\int \frac{du}{\sqrt{\frac{2E}{l^2} + \frac{2GM_s}{l^2}u - u^2}}. \quad (4.24)$$

This integral can be done explicitly. We get

$$\theta = -\arccos\left(\frac{u-C}{eC}\right) + \theta' \ , \ e = \sqrt{1 + \frac{2l^2E}{G^2M_s^2}} \ , \ C = \frac{GM_s}{l^2}. \quad (4.25)$$

By inverting this equation we get an equation of ellipse with eccentricity e since $E < 0$, viz

$$\frac{1}{r} = C(1 + e\cos(\theta - \theta')). \quad (4.26)$$

This is Kepler's first law. The angle at which r is maximum is $\theta - \theta' = \pi$. This distance is precisely $(1+e)a$ where a is the semi-major axis of the ellipse since ea is the distance between the Sun which is at one of the two foci and the center of the ellipse. Hence we obtain the relation

$$(1 - e^2)a = \frac{1}{C} = \frac{l^2}{GM_s}. \quad (4.27)$$

From equation (4.19) we can derive Kepler's third law. By integrating both sides of the equation over a single period T and then taking the square we get

$$A^2 = \frac{1}{4}l^2T^2. \quad (4.28)$$

A is the area of the ellipse, i.e. $A = \pi ab$ where the semi-minor axis b is related the semi-major axis a by $b = a\sqrt{1-e^2}$. Hence

$$\pi^2a^4(1 - e^2) = \frac{1}{4}l^2T^2. \quad (4.29)$$

By using equation (4.27) we get the desired formula

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_s}. \quad (4.30)$$

4.1.4 *The Inverse-Square Law and Stability of Orbits*

Any object with mass generates a gravitational field and thus gravitational field lines will emanate from the object and radiate outward to infinity. The number of field lines N is proportional to the mass. The density of field lines crossing a sphere of radius r surrounding this object is given by $N/4\pi r^2$. This is the origin of the inverse-square law. Therefore any other object placed in this gravitational field will experience a gravitational force proportional to the number of field lines which intersect it. If the distance between this second object and the source is increased the force on it will become weaker because the number of field lines which intersect it will decrease as we are further away from the source.

4.2 Euler–Cromer Algorithm

The time discretization is

$$t \equiv t(i) = i\Delta t \ , \ i = 0, ..., N. \tag{4.31}$$

The total time interval is $T = N\Delta t$. We define $x(t) = x(i)$, $v_x(t) = v_x(i)$, $y(t) = y(i)$, $v_y(t) = v_y(i)$. Equations (4.4)–(4.8) become (with $i = 0, ..., N$)

$$v_x(i+1) = v_x(i) - \frac{GM_s}{(r(i))^3} x(i)\Delta t. \tag{4.32}$$

$$x(i+1) = x(i) + v_x(i)\Delta t. \tag{4.33}$$

$$v_y(i+1) = v_y(i) - \frac{GM_s}{(r(i))^3} y(i)\Delta t. \tag{4.34}$$

$$y(i+1) = y(i) + v_y(i)\Delta t. \tag{4.35}$$

$$r(i) = \sqrt{x(i)^2 + y(i)^2}. \tag{4.36}$$

This is Euler algorithm. It can also be rewritten with $\hat{x}(i) = x(i-1)$, $\hat{y}(i) = y(i-1)$, $\hat{v}_x(i) = v_x(i-1)$, $\hat{v}_y(i) = v_y(i-1)$, $\hat{r}(i) = r(i-1)$ and $i = 1, ..., N+1$ as

$$\hat{v}_x(i+1) = \hat{v}_x(i) - \frac{GM_s}{(\hat{r}(i))^3} \hat{x}(i)\Delta t. \tag{4.37}$$

$$\hat{x}(i+1) = \hat{x}(i) + \hat{v}_x(i)\Delta t. \tag{4.38}$$

$$\hat{v}_y(i+1) = \hat{v}_y(i) - \frac{GM_s}{(\hat{r}(i))^3} \hat{y}(i)\Delta t. \tag{4.39}$$

$$\hat{y}(i+1) = \hat{y}(i) + \hat{v}_y(i)\Delta t. \tag{4.40}$$

$$\hat{r}(i) = \sqrt{\hat{x}(i)^2 + \hat{y}(i)^2}. \tag{4.41}$$

In order to maintain energy conservation we employ Euler-Cromer algorithm. We calculate as in the Euler's algorithm the velocity at time step $i+1$ by using the position and velocity at time step i. However we compute the position at time step $i+1$ by using the position at time step i and the velocity at time step $i+1$, viz

$$\hat{v}_x(i+1) = \hat{v}_x(i) - \frac{GM_s}{(\hat{r}(i))^3} \hat{x}(i)\Delta t. \tag{4.42}$$

$$\hat{x}(i+1) = \hat{x}(i) + \hat{v}_x(i+1)\Delta t. \tag{4.43}$$

$$\hat{v}_y(i+1) = \hat{v}_y(i) - \frac{GM_s}{(\hat{r}(i))^3} \hat{y}(i)\Delta t. \tag{4.44}$$

$$\hat{y}(i+1) = \hat{y}(i) + \hat{v}_y(i+1)\Delta t. \tag{4.45}$$

4.3 The Runge–Kutta Algorithm

4.3.1 *The Method*

The problem is still trying to solve the first order differential equation

$$\frac{dy}{dx} = f(x, y). \tag{4.46}$$

In the Euler's method we approximate the function $y = y(x)$ in each interval $[x_n, x_{n+1}]$ by the straight line

$$y_{n+1} = y_n + \Delta x f(x_n, y_n). \tag{4.47}$$

The slope $f(x_n, y_n)$ of this line is exactly given by the slope of the function $y = y(x)$ at the beginning of the interval $[x_n, x_{n+1}]$.

Given the value y_n at x_n we evaluate the value y_{n+1} at x_{n+1} using the method of Runge–Kutta as follows. First the middle of the interval $[x_n, x_{n+1}]$ which is at the value $x_n + \frac{1}{2}\Delta x$ corresponds to the y-value y_{n+1} calculated using the Euler's method, viz $y_{n+1} = y_n + \frac{1}{2}k_1$ where

$$k_1 = \Delta x f(x_n, y_n). \tag{4.48}$$

Second the slope at this middle point $(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1)$ which is given by

$$\frac{k_2}{\Delta x} = f\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1\right) \tag{4.49}$$

is the value of the slope which will be used to estimate the correct value of y_{n+1} at x_{n+1} using again Euler's method, namely

$$y_{n+1} = y_n + k_2. \tag{4.50}$$

In summary the Runge–Kutta algorithm is given by

$$\begin{aligned} k_1 &= \Delta x f(x_n, y_n) \\ k_2 &= \Delta x f\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1\right) \\ y_{n+1} &= y_n + k_2. \end{aligned} \tag{4.51}$$

The error in this method is proportional to Δx^3. This can be shown as follows. We have

$$\begin{aligned} y(x + \Delta x) &= y(x) + \Delta x \frac{dy}{dx} + \frac{1}{2}(\Delta x)^2 \frac{d^2 y}{dx^2} + \ldots \\ &= y(x) + \Delta x f(x, y) + \frac{1}{2}(\Delta x)^2 \frac{d}{dx} f(x, y) + \ldots \\ &= y(x) + \Delta x \left(f(x, y) + \frac{1}{2}\Delta x \frac{\partial f}{\partial x} + \frac{1}{2}\Delta x f(x, y) \frac{\partial f}{\partial y}\right) + \ldots \\ &= y(x) + \Delta x f\left(x + \frac{1}{2}\Delta x, y + \frac{1}{2}\Delta x f(x, y)\right) + O(\Delta x^3) \\ &= y(x) + \Delta x f\left(x + \frac{1}{2}\Delta x, y + \frac{1}{2}k_1\right) + O(\Delta x^3) \\ &= y(x) + k_2 + O(\Delta x^3). \end{aligned} \tag{4.52}$$

Let us finally note that the above Runge–Kutta method is strictly speaking the second-order Runge–Kutta method. The first-order Runge–Kutta method is the Euler algorithm. The higher-order Runge–Kutta methods will not be discussed here.

4.3.2 *Example 1: The Harmonic Oscillator*

Let us apply this method to the problem of the harmonic oscillator. We have the differential equations

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -\frac{g}{l}\theta. \tag{4.53}$$

Euler's equations read

$$\theta_{n+1} = \theta_n + \Delta t \omega_n$$
$$\omega_{n+1} = \omega_n - \frac{g}{l}\theta_n \Delta t. \tag{4.54}$$

First we consider the function $\theta = \theta(t)$. The middle point is $(t_n + \frac{1}{2}\Delta t, \theta_n + \frac{1}{2}k_1)$ where $k_1 = \Delta t \omega_n$. For the function $\omega = \omega(t)$ the middle point is $(t_n + \frac{1}{2}\Delta t, \omega_n + \frac{1}{2}k_3)$ where $k_3 = -\frac{g}{l}\Delta t \theta_n$. Therefore we have

$$k_1 = \Delta t \omega_n$$
$$k_3 = -\frac{g}{l}\Delta t \theta_n. \tag{4.55}$$

The slope of the function $\theta(t)$ at its middle point is

$$\frac{k_2}{\Delta t} = \omega_n + \frac{1}{2}k_3. \tag{4.56}$$

The slope of the function $\omega(t)$ at its middle point is

$$\frac{k_4}{\Delta t} = -\frac{g}{l}\left(\theta_n + \frac{1}{2}k_1\right). \tag{4.57}$$

The Runge–Kutta solution is then given by

$$\theta_{n+1} = \theta_n + k_2$$
$$\omega_{n+1} = \omega_n + k_4. \tag{4.58}$$

4.3.3 *Example 2: The Solar System*

Let us consider the equations

$$\frac{dx}{dt} = v_x. \tag{4.59}$$

$$\frac{dv_x}{dt} = -\frac{GM_s}{r^3}x. \tag{4.60}$$

$$\frac{dy}{dt} = v_y. \tag{4.61}$$

$$\frac{dv_y}{dt} = -\frac{GM_s}{r^3}y. \tag{4.62}$$

First we consider the function $x = x(t)$. The middle point is $(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_1)$ where $k_1 = \Delta t\ v_{xn}$. For the function $v_x = v_x(t)$ the middle point is $(t_n + \frac{1}{2}\Delta t, v_{xn} + \frac{1}{2}k_3)$ where $k_3 = -\frac{GM_s}{r_n}\Delta t\ x_n$. Therefore we have

$$\begin{aligned} k_1 &= \Delta t\ v_{xn} \\ k_3 &= -\frac{GM_s}{r_n^3}\Delta t\ x_n. \end{aligned} \tag{4.63}$$

The slope of the function $x(t)$ at the middle point is

$$\frac{k_2}{\Delta t} = v_{xn} + \frac{1}{2}k_3. \tag{4.64}$$

The slope of the function $v_x(t)$ at the middle point is

$$\frac{k_4}{\Delta t} = -\frac{GM_s}{R_n^3}\left(x_n + \frac{1}{2}k_1\right). \tag{4.65}$$

Next we consider the function $y = y(t)$. The middle point is $(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}k'_1)$ where $k'_1 = \Delta t\ v_{yn}$. For the function $v_y = v_y(t)$ the middle point is $(t_n + \frac{1}{2}\Delta t, v_{yn} + \frac{1}{2}k'_3)$ where $k'_3 = -\frac{GM_s}{r_n}\Delta t\ y_n$. Therefore we have

$$\begin{aligned} k'_1 &= \Delta t\ v_{yn} \\ k'_3 &= -\frac{GM_s}{r_n^3}\Delta t\ y_n. \end{aligned} \tag{4.66}$$

The slope of the function $y(t)$ at the middle point is

$$\frac{k'_2}{\Delta t} = v_{yn} + \frac{1}{2}k'_3. \tag{4.67}$$

The slope of the function $v_y(t)$ at the middle point is

$$\frac{k'_4}{\Delta t} = -\frac{GM_s}{R_n^3}\left(y_n + \frac{1}{2}k'_1\right). \tag{4.68}$$

In the above equations

$$R_n = \sqrt{\left(x_n + \frac{1}{2}k_1\right)^2 + \left(y_n + \frac{1}{2}k'_1\right)^2}. \tag{4.69}$$

The Runge–Kutta solutions are then given by

$$\begin{aligned} x_{n+1} &= x_n + k_2 \\ v_{x(n+1)} &= v_{xn} + k_4 \\ y_{n+1} &= y_n + k'_2 \\ v_{y(n+1)} &= v_{yn} + k'_4. \end{aligned} \tag{4.70}$$

4.4 Precession of the Perihelion of Mercury

The orbit of Mercury is elliptic. The orientation of the axes of the ellipse rotate with time. This is the precession of the perihelion (the point of the orbit nearest to the Sun) of Mercury. Mercury's perihelion makes one revolution every 23000 years. This is approximately 566 arcseconds per century. The gravitational forces of the other planets (in particular Jupiter) lead to a precession of 523 arcseconds per century. The remaining 43 arcseconds per century are accounted for by general relativity.

For objects too close together (like the Sun and Mercury) the force of gravity predicted by general relativity deviates from the inverse-square law. This force is given by

$$F = \frac{GM_s M_m}{r^2}\big(1 + \frac{\alpha}{r^2}\big)\ ,\ \alpha = 1.1 \times 10^{-8}\ \text{AU}^2. \tag{4.71}$$

We discuss here some of the numerical results obtained with the Runge–Kutta method for different values of α. We take the time step and the number of iterations to be $N = 20000$ and $dt = 0.0001$. The angle of the line joining the Sun and Mercury with the horizontal axis when mercury is at the perihelion is found to change linearly with time. We get the following rates of precession

$$\begin{aligned}
\alpha &= 0.0008\ ,\ \frac{d\theta}{dt} = 8.414 \pm 0.019 \\
\alpha &= 0.001\ ,\ \frac{d\theta}{dt} = 10.585 \pm 0.018 \\
\alpha &= 0.002\ ,\ \frac{d\theta}{dt} = 21.658 \pm 0.019 \\
\alpha &= 0.004\ ,\ \frac{d\theta}{dt} = 45.369 \pm 0.017.
\end{aligned} \tag{4.72}$$

Thus

$$\frac{d\theta}{dt} = a\alpha\ ,\ \alpha = 11209.2 \pm 147.2\ \text{degrees}/(\text{yr}.\alpha). \tag{4.73}$$

By extrapolating to the value provided by general relativity, viz $\alpha = 1.1 \times 10^{-8}$ we get

$$\frac{d\theta}{dt} = 44.4 \pm 0.6\ \text{arcsec/century}. \tag{4.74}$$

4.5 Exercises

Exercise 1: Using the Runge–Kutta method solve the following differential equations

$$\frac{d^2r}{dt^2} = \frac{l^2}{r^3} - \frac{GM}{r^2}. \tag{4.75}$$

$$\frac{d^2z}{dt^2} = -g. \tag{4.76}$$

$$\frac{dN}{dt} = aN - bN^2. \tag{4.77}$$

Exercise 2: The Lorenz model is a chaotic system given by three coupled first order differential equations

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = -xz + rx - y$$
$$\frac{dz}{dt} = xy - bz. \tag{4.78}$$

This system is a simplified version of the system of Navier–Stokes equations of fluid mechanics which are relevant for the Rayleigh–Bénard problem. Write down the numerical solution of these equations according to the Runge–Kutta method.

4.6 Simulation 6: Runge–Kutta Algorithm: Solar System

Part I We consider a solar system consisting of a single planet moving around the Sun. We suppose that the Sun is very heavy compared to the planet that we can safely assume that it is not moving at the center of the system. Newton's second law gives the following equations of motion

$$v_x = \frac{dx}{dt}\ ,\ \frac{dv_x}{dt} = -\frac{GM_s}{r^3}x\ ,\ v_y = \frac{dy}{dt}\ ,\ \frac{dv_y}{dt} = -\frac{GM_s}{r^3}y.$$

We will use here the astronomical units defined by $GM_s = 4\pi^2\ \mathrm{AU}^3/\mathrm{yr}^2$.

(1) Write a Fortran code in which we implement the Runge–Kutta algorithm for the problem of solving the equations of motion of the the solar system.
(2) Compute the trajectory, the velocity and the energy as functions of time. What do you observe for the energy.
(3) According to Kepler's first law the orbit of any planet is an ellipse with the Sun at one of the two foci. In the following we will only consider planets which are known to have circular orbits to a great accuracy. These planets are Venus, Earth, Mars, Jupiter and Saturn. The radii in astronomical units are given by

$$a_{\mathrm{venus}} = 0.72\ ,\ a_{\mathrm{earth}} = 1\ ,\ a_{\mathrm{mars}} = 1.52\ ,\ a_{\mathrm{jupiter}} = 5.2\ ,\ a_{\mathrm{saturn}} = 9.54.$$

Verify that Kepler's first law indeed holds for these planets.
In order to answer questions 2 and 3 above we take the initial conditions

$$x(1) = a\ ,\ y(1) = 0\ ,\ v_x(1) = 0\ ,\ v_y(1) = v.$$

The value chosen for the initial velocity is very important to get a correct orbit and must be determined for example by assuming that the orbit is indeed circular and as a consequence the centrifugal force is balanced by the force of gravitational attraction. We get $v = \sqrt{GM_s/a}$.
We take the step and the number of iterations $\Delta t = 0.01$ yr , $N = 10^3 - 10^4$.

Part II

(1) According to Kepler's third law the square of the period of a planet is directly proportional to the cube of the semi-major axis of its orbit. For circular orbits the proportionality factor is equal to 1 exactly. Verify this fact for the planets mentioned above. We can measure the period of a planet by monitoring when the planet returns to its farthest point from the sun.
(2) By changing the initial velocity appropriately we can obtain an elliptical orbit. Check this thing.
(3) The fundamental laws governing the motion of the solar system are Newton's law of universal attraction and Newton's second law of motion. Newton's law of universal attraction states that the force between the Sun and a planet is inversely proportional to the square of the distance between them and it is directed from the planet to the Sun. We will assume in the following that this force is inversely proportional to a different power of the distance. Modify the code accordingly and calculate the new orbits for powers between 1 and 3. What do you observe and what do you conclude.

4.7 Simulation 7: Precession of the perihelion of Mercury

According to Kepler's first law the orbits of all planets are ellipses with the Sun at one of the two foci. This law can be obtained from applying Newton's second law to the system consisting of the Sun and a single planet. The effect of the other planets on the motion will lead to a change of orientation of the orbital ellipse within the orbital plane of the planet. Thus the point of closest approach (the perihelion) will precess, i.e. rotate around the sun. All planets suffer from this effect but because they are all farther from the sun and all have longer periods than Mercury the amount of precession observed for them is smaller than that of Mercury.

However it was established earlier on that the precession of the perihelion of Mercury due to Newtonian effects deviates from the observed precession by the amount 43 arcsecond/century. As it turns out this can only be explained within general relativity. The large mass of the Sun causes space and time around it to be curved which is felt the most by Mercury because of its proximity. This spacetime curvature can be approximated by the force law

$$F=\frac{GM_sM_m}{r^2}(1+\frac{\alpha}{r^2})\ ,\ \alpha=1.1.10^{-8}\ AU^2.$$

(1) Include the above force in the code. The initial position and velocity of Mercury are

$$x_0=(1+e)a\ ,\ y_0=0.$$

$$v_{x0}=0\ ,\ v_{y0}=\sqrt{\frac{GM_s}{a}\frac{1-e}{1+e}}.$$

Thus initially Mercury is at its farthest point from the Sun since a is the semi-major axis of Mercury ($a = 0.39$ AU) and e is its eccentricity ($e = 0.206$) and hence ea is the distance between the Sun and the center of the ellipse. The semi-minor axis is defined by $b = a\sqrt{1-e^2}$. The initial velocity was calculated from applying the principles of conservation of angular momentum and conservation of energy between the above initial point and the point $(0, b)$.

(2) The amount of precession of the perihelion of Mercury is very small because α is very small. In fact it can not be measured directly in any numerical simulation with a limited amount of time. Therefore we will choose a larger value of α for example $\alpha = 0.0008$ AU^2. We also work with $N = 20000$, $dt = 0.0001$. Compute the orbit for these values. Compute the angle θ made between the vector position of Mercury and the horizontal axis as a function of time. Compute also the distance between Mercury and the sun and its derivative with respect to time given by

$$\frac{dr}{dt} = \frac{xv_x + yv_y}{r}.$$

This derivative will vanish each time Mercury reaches its farthest point from the sun or its closest point from the sun (the perihelion). Plot the angle θ_p made between the vector position of Mercury at its farthest point and the horizontal axis as a function of time. What do you observe. Determine the slope $d\theta_p/dt$ which is precisely the amount of precession of the perihelion of Mercury for the above value of α.

(3) Repeat the above question for other values of α say $\alpha = 0.001, 0.002, 0.004$. Each time compute $d\theta_p/dt$. Plot $d\theta_p/dt$ as a function of α. Determine the slope. Deduce the amount of precession of the perihelion of Mercury for the value of $\alpha = 1.1.10^{-8}$ AU^2.