Mahesh Chandra Luintel

Textbook of Mechanical Vibrations

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Mahesh Chandra Luintel
Department of Mechanical and Aerospace
Engineering, Pulcowk Campus
Institute of Engineering
Tribhuvan University
Lalitpur, Nepal

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To Bandana, Surabhi and Aaryab

Preface

The knowledge of dynamics and vibration is essential for engineering researchers, students and professionals of multiple engineering disciplines: mechanical engineering, aerospace engineering, and civil engineering to name a few.

During the initial phase of my academic career, I had to collect several books to prepare lecture notes and assignments for my students to articulate various topics in vibration ranging from discrete systems to continuous systems. After teaching mechanical vibration courses for many undergraduate and postgraduate batches and interacting with students and professionals in the field of vibration, I have tried to present my experience in the form of a book.

This book provides a very concise and clear presentation of classical dynamics and basics of vibration including many examples to provide instant illustration and applications of the results obtained. Each chapter is presented elegantly and requires no special prerequisite knowledge of supporting subjects. Self-explanatory sketches and graphs have been generously used to curtail redundant explanations. Numerous illustrated examples have been included in each chapter to make the fundamentals more clear and to inculcate problem solving approach in the readers. Sufficient review question exercises and problems at the end of each chapter serve as a good source material to practice the application of the basic principles presented in the text. Answers to all the exercise problems have also been provided for the students to evaluate themselves.

I hope this book will be helpful to all researchers, students and professionals. Suggestions for the improvement of the book will be highly appreciated.

April 2023

Mahesh Chandra Luintel
Professor, Department of Mechanical
and Aerospace Engineering
Pulchowk Campus
Institute of Engineering
Tribhuvan University
Lalitpur, Nepal

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About the Author

Dr. Mahesh Chandra Luintel is currently a professor at the Department of Mechanical and Aerospace Engineering, Pulchowk Campus, Institute of Engineering, Tribhuvan University, Nepal. He obtained his B.E. (Mechanical Engineering) from the Pulchowk Campus, Institute of Engineering, Tribhuvan University, Nepal, and M.Tech. (Mechanical Engineering—Solid Mechanics) from the Indian Institute of Technology Kanpur, India, and Ph.D. from the Pulchowk Campus, Institute of Engineering, Tribhuvan University, Nepal. His major areas of research interests include mechanical vibration, finite element method, and rotor dynamics. He has authored three undergraduate text books: *Fundamentals of Thermodynamics and Heat Transfer*, *Engineering Drawing I* and *Engineering Drawing II*. He has published many papers in reputed international journals and conference proceedings. Dr. Luintel was the co-chair of the 17th VETOMAC, an international conference on Vibration Engineering and Technology of Machinery, which was held in Kathmandu, Nepal, in December 2022.

Chapter 1
Basic Concepts of Vibration

1.1 Introduction

Vibration can be defined as a to-and-fro motion of a body or a system about its equilibrium position. Vibration is sometimes also called an oscillation. Any body or system having inertia and elasticity is capable of vibration. Vibration is initiated when a body having inertia is displaced from its equilibrium position due to some external disturbances. A restoring or conservative force is developed due to elasticity of the system which pulls the system back toward its equilibrium, and to-and-fro motion continues if there is not any dissipative force. In the presence of dissipative force, vibration decays and system come to its equilibrium position after some time interval if the external disturbance does not continue after initial disturbance, whereas it may undergo continuous to-and-fro motion as long as the external disturbance continues.

One of the simplest models of a vibrating system is a simple pendulum shown in Fig. 1.1. When the pendulum bob is displaced from its vertical equilibrium position and released, the gravitational force pulls the pendulum bob back toward the equilibrium.

Another common model of a vibrating system is a spring-mass system shown in Fig. 1.2. When the particle of mass m is displaced toward the right from its equilibrium position and released, the restoring force provided by the spring pulls the mass back toward the equilibrium.

For the both systems, at the point of release, potential energy of the system will be maximum and the kinetic energy is zero. When the mass/bob is on the left side, the potential energy is converted to kinetic energy. At the equilibrium position, potential energy of the system will be minimum and the kinetic energy will be maximum which can take the mass/bob toward the right side. Due to this continuous conversion of energy, both the systems oscillate continuously about their equilibrium positions in the absence of non-conservative forces.

In the presence of dissipative force like frictional force, oscillation stops after some interval and the system remains at its equilibrium position.

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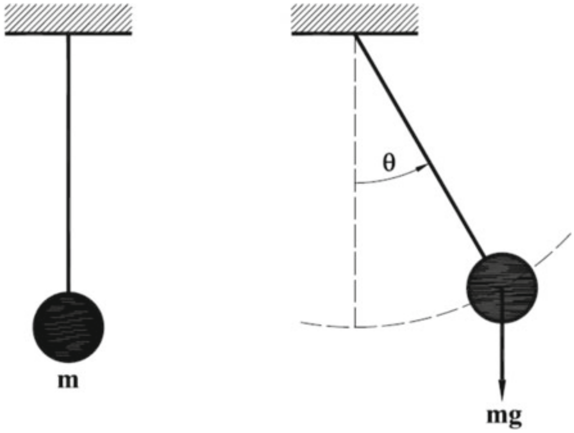


Fig. 1.1 Oscillation of a simple pendulum about its vertical equilibrium position

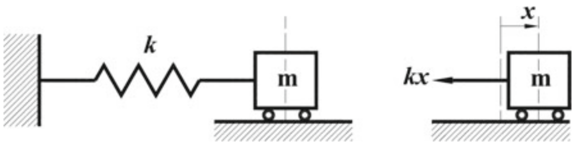


Fig. 1.2 Oscillation of a spring-mass system about its equilibrium position

1.1.1 Causes of Vibration

Every machines or structures undergo vibration to some extent because of different causes. Some of the common causes of vibrations are mentioned below:

(a) **Unbalance**

Unbalance present in the rotating part of any machine is one of the main causes of vibration in many mechanical systems. These unbalances are created due to the presence of key ways on the shaft, uneven distribution of materials due to limitations of fabrication processes, wear and erosion due to operation for certain duration, looseness of the parts, bent shaft, etc.

(b) **Misalignment**

Misalignment produced in the system due to improper assembly or any other reason also leads to increase in vibration of the system.

(c) **Friction**

Dry friction between the two mating surfaces also produces vibration which is also called self-excited vibration.

(d) **External excitation**

External forces imposed to any system can also produce vibrations in the system. These excitations may be periodic, random or of the nature of impact produced external to the vibration system.

(e) **Flow-induced vibration**

Flow-induced vibration is the oscillations of structures immersed in or conveying fluid flow as a result of an interaction between the fluid-dynamic forces and the inertia, damping and elastic forces in the structures.

(f) **Earthquakes**

Ground motion created by earthquake or other effect also produces vibration in the system and may lead to the failure of structures.

(g) **Wind**

Wind blowing on a surface of any structure can produce significant vibration. It may also cause the vibrations of transmission and telephone lines under certain conditions.

1.1.2 Effects of Vibration

Vibration of a system may be undesirable, trivial or essential for functioning of the device for what it is designed for.

Vibrations of machines or structures are usually undesirable. If the vibration of the machine components or structures is not controlled, it will produce excessive loads, excessive stresses, undesirable noise, looseness of parts and which may lead to the partial or complete failure of parts.

In spite of these undesirable effects, vibration phenomena can also be utilized in different systems for some beneficial effects. Suspension systems in automotive vehicles are designed to protect passengers from discomfort when the vehicles run through any terrain. Vibration isolators and vibration absorbers are also used to reduce the vibration of machines. Cushioning is provided to many machines to reduce the effect of impulsive forces during transportation. Many micro-electromechanical or nano-electromechanical systems are triggered by the vibration of tip. Many energy harvester devices are designed to take the energy from the unwanted vibration. Musical instruments produce sound of different frequencies due to vibration.

Vibrations also exist in human body. We hear because of vibration of the eardrum, we walk due to oscillation of legs, and we breathe due to vibration of the lungs.

1.2 Simple Harmonic Motion

Simplest form of the oscillatory or vibratory motion is the periodic motion. Any motion that repeats itself after some time interval is called a periodic motion. Most common example of a periodic motion is a simple harmonic motion.

If the mass m of Fig. 1.2 is displaced by A unit toward right and released, then the subsequent position of the mass in the absence of dissipative force will be a

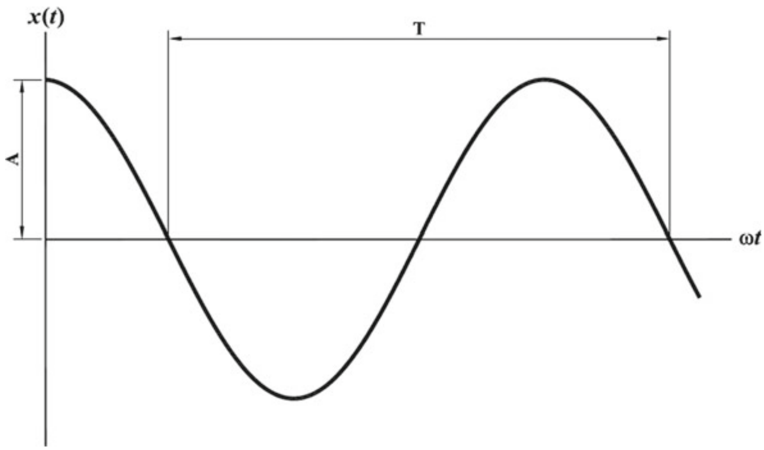


Fig. 1.3 Simple harmonic motion

simple harmonic motion as shown in Fig. 1.3. Instantaneous position of a particle undergoing a simple harmonic motion can be represented by the circular functions, sin or cosine. For this case,

$$x(t) = A\cos\omega t \tag{1.1}$$

where A is amplitude of the cycle which is defined as the maximum displacement from the equilibrium position.

Since the circular function repeats itself in 2π radians and takes a time interval of $T(\omega T = 2\pi)$, then

$$T = \frac{2\pi}{\omega} \tag{1.2}$$

where T is called time period of the cycle measured in s and ω is the circular frequency of the oscillation measured in rad/s.

Similarly, frequency of oscillation f which is the number of cycles measured in Hz can be expressed as.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.3}$$

Velocity and acceleration of the vibrating particle can be determined by the successive differentiation of displacement expression defined by Eq. (1.1) as

$$\dot{x} = -\omega A\sin\omega t \tag{1.4}$$

$$\ddot{x} = -\omega^2 A\cos\omega t \tag{1.5}$$

Comparing Eqs. (1.1) and (1.5), we get

$$\ddot{x} = -\omega^2 x \tag{1.6}$$

With reference to Eq. (1.6), we can recall the definition of simple harmonic motion from mechanics as:

When the acceleration of a particle with rectilinear motion is always proportional to its displacement from a fixed point on the path and is directed toward the fixed point, the particle is said to have simple harmonic motion.

1.3 Vibration Analysis Procedure

Analysis of almost all engineering problems involves three major steps: mathematical modeling, solution of the model and the physical interpretation of the solution.

1.3.1 Mathematical Modeling

First step to study any vibrating system is to develop its mathematical model. For this, components in which the phenomenon of concern is occurring are identified and the interactions between them are represented by drawing a free-body diagram of the system. Then by applying Newton's second law of motion or energy principle, an appropriate mathematical model in the form of equation of motion is derived. The governing equation of motion of any vibrating system will be in the form of an ordinary differential equation or a partial differential equation. If the governing equation is in the form of partial differential equation, the associated boundary conditions should also be determined.

During the development of a mathematical model, some assumptions may be necessary to avoid negligible effects from the analysis and to simplify the problem by retaining appropriate accuracy. Assumptions should also be clearly mentioned to clarify under what conditions the developed model can provide satisfactory result.

1.3.2 Mathematical Solution

The governing equation of the system is then solved by using appropriate mathematical method. Closed form analytical solution is preferred if the governing equation is in simple and standard form. If closed form analytical solution is not possible, then it is solved by using approximate analytical method, and if it also is not possible, the solution is determined by using numerical methods.

The mathematical solution is obtained as an expression for the dependent variable as a function of time if the governing equation is an ordinary differential equation, and it is obtained as an expression for the dependent variable as a function of space (position) and time if the governing equation is a partial differential equation.

1.3.3 Physical Interpretation of Mathematical Solution

The final step of the vibration analysis is to provide the physical interpretation of the mathematical solution obtained. From the mathematical solution information regarding frequency of vibration and amplitude of vibration, mode shapes, etc., should be provided.

1.4 Generalized Coordinates

Position of a particle or a point of a rigid body undergoing vibration continuously changes with time. Therefore, response of any vibrating system is given by an expression in which time is used as an independent variable. To define any instantaneous position of the system, few dependent variables should be used and these variables are called generalized coordinates.

For example, to define the instantaneous position of the bob of the pendulum shown in Fig. 1.4, any of the coordinates θ, x or y can be used. For a given length L of the pendulum, if the value of any one coordinate is given, the other two can be determined mathematically, i.e., if θ is given, then $x = L\sin\theta$ and $y = L(1 - \cos\theta)$.

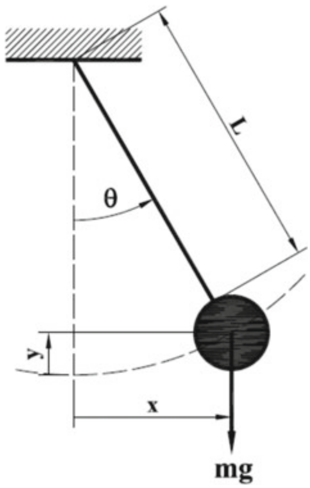


Fig. 1.4 Coordinates (θ, x or y) that can be used to describe the instantaneous position of the pendulum bob

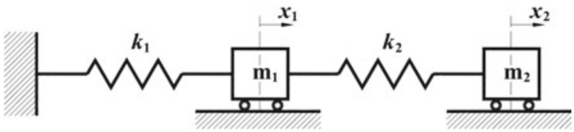


Fig. 1.5 Two masses connected by a spring and a rigid link where x_1 and x_2 are kinematically independent

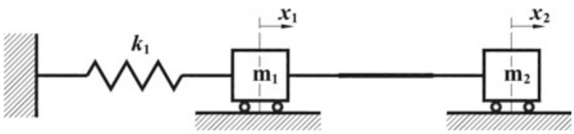


Fig. 1.6 Two masses connected by a spring and a rigid link where x_1 and x_2 are kinematically dependent

For the spring-mass system shown in Fig. 1.2, only one generalized coordinate required to define the instantaneous position of the mass m is x, whereas if two such masses m_1 and m_2 are connected in series by two springs k_1 and k_2 as shown in Fig. 1.5, two generalized coordinates x_1 and x_2 are required.

If the spring k_2 of the system shown in Fig. 1.5 is replaced by a rigid link as shown in Fig. 1.6, then displacements of each mass x_1 and x_2 cannot be independent (for this case $x_1 = x_2$) and these displacements are said to be kinematically dependent. Hence, this system requires only one coordinate to define the instantaneous position of the system.

1.5 Degrees of Freedom

Degree of freedom of any vibratory system is defined as the minimum number kinematically independent coordinates required to define the motion of every particle of the system. A set of kinetically independent coordinates is defined by a set of appropriate generalized coordinates. Degree of freedom of any system is unique, but the set of generalized coordinates may not be unique.

A particle on space can have a maximum of three degrees of freedom, i.e., three displacements u, v and w along the three mutually perpendicular directions x, y and z. If its motion is constrained, then its degree of freedom will be less than three. For example, a simple pendulum (Fig. 1.1) can swing on a plane, and its displacement from the equilibrium position can be conveniently described by the angular displacement θ and hence has one degree of freedom. Similarly, a mass attached to a spring (Fig. 1.2) can move in only one direction on a plane and its displacement from the equilibrium position can be conveniently described by x, and hence it also has one degree of freedom.

Any rigid body on space can have a maximum of six degrees of freedom, i.e., three displacements u, v and w along the three mutually perpendicular directions x, y and z and three rotations θ_x, θ_y and θ_z about the three mutually perpendicular directions x, y and z. Similarly, any rigid body undergoing general plane motion can have a maximum of three degrees of freedom, i.e., two displacements u and v along

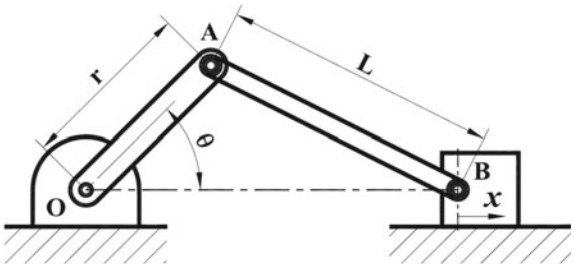


Fig. 1.7 Mechanism of reciprocating engine having one degree of freedom

the two perpendicular directions x and y and one rotation θ_x about the z axis. Again, if these rigid bodies are subjected to some constrains, then there will be reduction in their number of degrees of freedom.

If any system consists of a number of particles or rigid bodies each having some definite degrees of freedom, overall degrees of freedom of the complete system may be equal to or less than the sum of degrees of freedom of each particle or rigid body. This reduction in degrees of freedom is due to kinematic dependency between the particles or rigid bodies.

For example, a system consisting of two particles connected by a spring (Fig. 1.5) has two degrees of freedom, whereas a system consisting of two particles connected by a rigid link (Fig. 1.6) has only one degree of freedom.

For a system consisting of crank OA, connecting rod AB and a reciprocating mass as shown in Fig. 1.7, with the given radius of crank r and length of the connecting rod L, any instantaneous position can be described either by defining the crank rotation θ or the linear displacement of the reciprocating mass x. Hence this system has one degree of freedom.

To describe the vibratory motion of a system consisting of a deformable body such as a cantilever beam as shown in Fig. 1.8, a dependent variable for the transverse deflection of the neutral axis of the beam from the equilibrium position w can be used which is a function of two independent variables x and t. For the same instance, deflection of different points on the beam (such as P_1,P_2, ...) will be different and the given beam consists of infinite number of such points. Hence the system of a cantilever beam is said to have infinite degrees of freedom.

1.6 Discrete and Continuous System

Any vibrating system can be modeled as either a discrete system or a continuous system. If the given system has finite degree of freedom, then it is called a discrete system. If the given discrete system has one degree of freedom, then it is called a single degree of freedom system. Governing equation of the discrete system with single degree of freedom appears in the form of an ordinary differential equation, and the response of this system can be determined by the solution of the ordinary

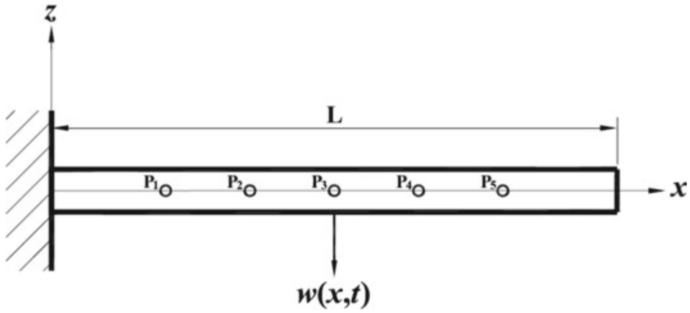


Fig. 1.8 Cantilever beam having infinite degree of freedom

differential equation. Similarly, if the given discrete has a degree of freedom of two or more than two, then it is called a multi-degree freedom system. Governing equation of the multi-degree of freedom system appears in the form of coupled ordinary differential equations (system of ordinary differential equations) and which can also be efficiently expressed in matrix form. Then the response of the multi-degree of freedom system can be determined by solving the eigen-value problem.

If the given system has infinite degrees of freedom, then it is called a continuous system or distributed parameter system. Continuous system can further be classified as one-dimensional continuous system, two-dimensional continuous system and three-dimensional continuous system. Any system in which dependent variable is function of only one spatial coordinate and time is called a one-dimensional system. Longitudinal vibration of a rod, transverse vibration of beam, transverse vibration of a string, etc., are the examples of one-dimensional system. Governing equation of the one-dimensional system appears in the form of a partial differential equation. Similarly, any system in which dependent variable is function of any two spatial coordinate and time is called a two-dimensional system. Transverse vibration of plate is the example of two-dimensional system. Governing equation of the two-dimensional system appears in the form of a coupled partial differential equations. Likewise, any system in which dependent variable is function of all three spatial coordinates and time is called a three-dimensional system. Vibration of any solid mass is the example of three-dimensional system. Governing equation of the three-dimensional system also appears in the form of a coupled partial differential equations.

1.7 Classification of Vibration

Vibrations can be classified by different basis such as according to the existence and nature of external excitation, according to material behavior and according to assumptions used for the development of the model.

Vibration of the system due to its inherent features is called a free vibration. Free vibration in any system is initiated due to initial disturbance in the form of initial displacement or initial velocity or combination of both initial displacement and initial velocity. Vibration of a system due to external force or motion is called a forced vibration.

Vibration of a system in the absence of dissipative forces is called an un-damped vibration, whereas the vibration of a system in the presence of dissipative forces is called a damped vibration. Damped vibration usually occurs due to resistance provided to the vibratory motion by friction, viscosity, material hysteresis, etc. If the effects of such resistances are negligible, we can model the system as an un-damped vibrating system.

If the external excitation to the system can be defined as a function of time, then the resulting vibration is called deterministic vibration, whereas if the external excitation is of random nature (stochastic), then the resulting vibration is called random vibration. Vibration of a system subjected to a periodic external force is the deterministic vibration, whereas vibration of the same system due to earthquake or wind blowing on it is the random vibration.

If the governing equation of motion of the system is linear, then the vibratory response of such system is called linear vibration, whereas if the governing equation of motion of the system is nonlinear, then the vibratory response of such system is called nonlinear vibration. Vibrations occurring in most of the real system are nonlinear, and it can be assumed to be linear if the effects of nonlinearity are negligible.

1.8 Review of Dynamics

Vibration being a subdomain of dynamics, fundamental knowledge of dynamics is essential for the development of mathematical model of any vibrating system. Therefore, a brief review of dynamics is presented which is frequently used throughout this book.

1.8.1 Kinematics

Kinematics for a Particle

Kinematics is the branch of dynamics which deals the study of the geometrical description of motion. The principles of kinematics focus on the relationship between the displacement, velocity, acceleration and time of a body, without reference to the cause of the motion.

The instantaneous position P of a particle on a rigid body with reference to a fixed Cartesian frame $x - y - z$, as shown in Fig. 1.9 is called the position vector of the particle and can be defined as

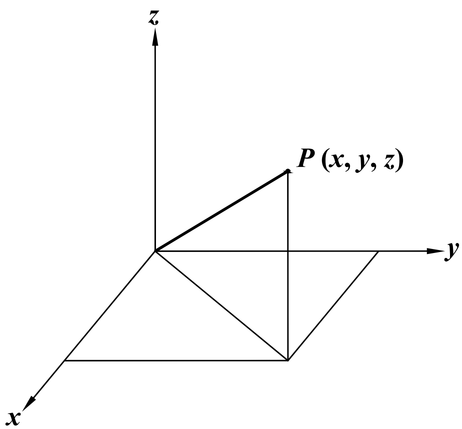


Fig. 1.9 Position vector of a particle ***P***

$$\boldsymbol{r} = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} \tag{1.7}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ be unit vectors parallel to the x, y, and z axes, respectively.

Then the velocity and acceleration of the particle can be determined by the successive differentiation of the position vector as

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{x}(t)\boldsymbol{i} + \dot{y}(t)\boldsymbol{j} + \dot{z}(t)\boldsymbol{k} \tag{1.8}$$

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \ddot{x}(t)\boldsymbol{i} + \ddot{y}(t)\boldsymbol{j} + \ddot{z}(t)\boldsymbol{k} \tag{1.9}$$

Kinematics for a Rigid Body

Now consider two particles A and B on a rigid body shown in Fig. 1.10a undergoing general motion, i.e., the body has a linear motion on the $x - y$ plane, and simultaneously it is rotating about z axis which is perpendicular to the $x - y$ plane. Then the position vector of the particle B is given by

$$\boldsymbol{r}_{\boldsymbol{B}} = \boldsymbol{r}_{\boldsymbol{A}} + \boldsymbol{r}_{\boldsymbol{B}/\boldsymbol{A}} \tag{1.10}$$

where $\boldsymbol{r}_{\boldsymbol{B}/\boldsymbol{A}}$ is the relative position vector of the particle B with respective that of the particle A.

If the point A undergoes a translation $\mathbf{d}\boldsymbol{r}_{A}$ and the point B undergoes a translation $\mathbf{d}\boldsymbol{r}_{\boldsymbol{B}}$ and a rotation of $\mathrm{d}\theta$ about the point A when it moves from B to B' as shown in Fig. 1.10b, the displacements are related as

$$\mathrm{d}\boldsymbol{r}_{\boldsymbol{B}} = \mathrm{d}\boldsymbol{r}_{\boldsymbol{A}} + \mathrm{d}\boldsymbol{r}_{\boldsymbol{B}/\boldsymbol{A}} \tag{1.11}$$

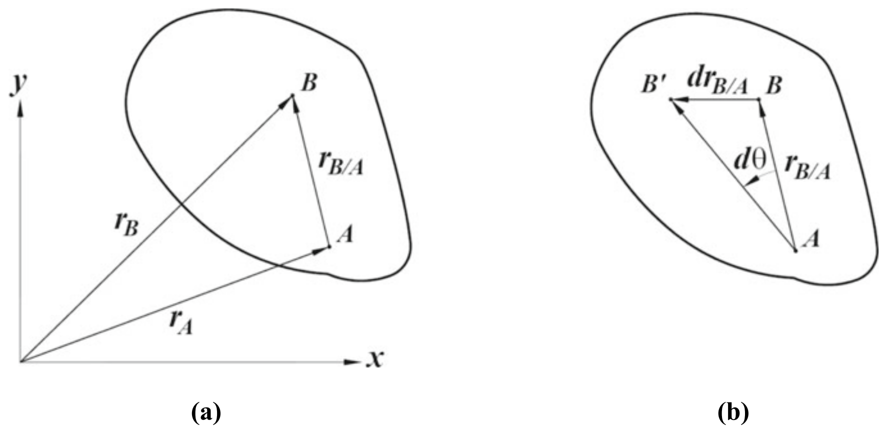


Fig. 1.10 Relative motion of a rigid body

With reference to Fig. 1.10b, the small rotation $\mathrm{d}\theta$ can be related to relative displacement as

$$\mathrm{d}\boldsymbol{\theta} = \frac{\mathrm{d}\boldsymbol{r}_{B/A}}{\left|\boldsymbol{r}_{B/A}\right|} \quad \therefore \quad \mathrm{d}\boldsymbol{r}_{B/A} = \left|\boldsymbol{r}_{B/A}\right| \mathrm{d}\boldsymbol{\theta} \tag{1.12}$$

Differentiating Eq. (1.10) with respect to time, the relationship for the relative velocity can be obtained as

$$\frac{\mathrm{d}\boldsymbol{r}_B}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}_A}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}_{B/A}}{\mathrm{d}t} \tag{1.13}$$

Substituting $\mathbf{d}\boldsymbol{r}_{B/A}$ from Eq. (1.12) into Eq. (1.13),

$$\frac{\mathrm{d}\boldsymbol{r}_B}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}_A}{\mathrm{d}t} + \left|\boldsymbol{r}_{B/A}\right| \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}_A}{\mathrm{d}t} + \left|\boldsymbol{r}_{B/A}\right| \boldsymbol{\omega} \tag{1.14}$$

Equation (1.14) can be expressed as

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A} \tag{1.15}$$

where $\boldsymbol{v}_{B/A}$ is the relative velocity of particle B with respect to that of the particle A and is given by

$$\boldsymbol{v}_{B/A} = \left|\boldsymbol{r}_{B/A}\right| \boldsymbol{\omega} \tag{1.16}$$

Differentiating of Eq. (1.15) with respect to time, the relationship for the relative acceleration can be obtained as

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A} \tag{1.17}$$

1.8.2 Kinetics

Kinetics is the branch of dynamics which deals with the study of the relation between the forces acting on a body, the mass of the body and the motion of the body. Principles of kinetics are applied to predict the motion caused by given forces or to determine the forces required to produce a required motion.

Kinetics for a Particle

The basic law for kinetics of particles is Newton's second law of motion which establishes the relationship between the force and motion as

$$\sum \boldsymbol{F} = m\boldsymbol{a} \tag{1.18}$$

where $\sum \boldsymbol{F}$ is the sum of the forces applied to the particle and $\boldsymbol{a}$ is the acceleration of the particle.

Kinetics for a Rigid Body

A rigid body can be assumed as a collection of particles. Writing an equation similar to Eq. (1.18) for each particle of the rigid body and adding the equations together leads to basic kinetics relationship for a rigid body as

$$\sum \boldsymbol{F} = m\overline{\boldsymbol{a}} \tag{1.19}$$

where $\overline{\boldsymbol{a}}$ is the acceleration of the mass center of the body.

A rigid body can undergo rotational motion also, and a moment equation is required for such problems. The moment equation for a rigid body undergoing planar motion is

$$\sum M_G = \overline{I}\boldsymbol{\alpha} \tag{1.20}$$

where $\boldsymbol{G}$ is the mass center of the rigid body and $\overline{I}$ is the mass moment of inertia about an axis parallel to the z axis that passes through the mass center, $\sum M_G$ is the sum of moments of all forces acting on the rigid body about its mass center and $\boldsymbol{\alpha}$, is the angular acceleration of the rigid body.

Equations (1.19) and (1.20) can be used to solve rigid body problems for planar motion. In general, the force equation of Eq. (1.19) yields two independent equations, and the moment equation of Eq. (1.20) yields one. If the axis of rotation of the rigid body is fixed, Eq. (1.20) may be replaced by

$$\sum M_o = I_o \boldsymbol{\alpha} \tag{1.21}$$

where I_o is the moment of inertia about the axis of rotation.

For example, crank OA of the slider-crank mechanism shown in Fig. 1.7 has a fixed axis of rotation, whereas the connecting rod AB of the system does not have any fixed axis of rotation. Therefore Eq. (1.21) can be applied to study the motion of crank OA but cannot be applied for the connecting rod AB.

Equations of kinetics can be interpreted in another way by using the equivalent system of forces, i.e., a system of forces and moments acting on a rigid body can be replaced by a force equal to the resultant of the force system applied at any point on the body and a moment equal to the resultant moment of the system about the point where the resultant force is applied. It follows that the external forces acting on the rigid body are actually equivalent to the inertial effects produced on the body.

Consider a rigid body subjected to a number of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, ... $\boldsymbol{F}_n$ and number of moments $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, ... $\boldsymbol{M}_n$ as shown in Fig. 1.11a. The given system can be replaced by an equivalent system shown in Fig. 1.11b, which is subjected to a single force equal to $m\bar{\boldsymbol{a}}$ and a single moment equal to $\bar{I}\boldsymbol{\alpha}$. The resultant system defined by $m\bar{\boldsymbol{a}}$ and $\bar{I}\boldsymbol{\alpha}$ is also called the system of effective forces. Hence, using equivalent system of forces Eqs. (1.19) and (1.20) can also be expressed as.

$$\sum \boldsymbol{F}_{\text{ext}} = \sum \boldsymbol{F}_{\text{eff}} \tag{1.22}$$

and

$$\sum \boldsymbol{M}_{\text{ext}} = \sum \boldsymbol{M}_{\text{eff}} \tag{1.23}$$

where moments are taken about any reference point on the rigid body.

Equations (1.22) and (1.23) are modified form of Newton's second law of motion which are also known as of **D' Alembert's principle**.

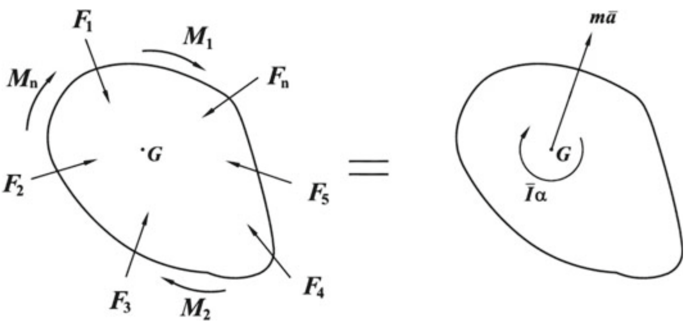


Fig. 1.11 Equivalent system of forces

1.8.3 Principle of Work and Energy

Kinetics problems can also be solved by using work-energy principle. Mathematical form of this principle can be derived by using Newton's second law of motion.

Work-Energy Principle for a Particle

Consider a particle of mass m acted upon by a force F and moving along a rectangular path, the relationship between the force and acceleration is given by

$$F = ma \tag{1.24}$$

Substituting

$$a = V\frac{\mathrm{d}V}{\mathrm{d}s} \tag{1.25}$$

we get

$$F = m\ V\frac{\mathrm{d}V}{\mathrm{d}s}$$

$$\therefore \quad F\ \mathrm{d}s = mV\ \mathrm{d}V \tag{1.26}$$

Integrating equation from an instant t_1 (at which $s = s_1$ and $V = V_1$) to an instant t_2 (at which $s = s_2$ and $V = V_2$),

$$\int_{s_1}^{s_2} F\ \mathrm{d}s = \int_{V_1}^{V_2} mV\ \mathrm{d}V$$

$$\int_{s_1}^{s_2} F\ \mathrm{d}s = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2 \tag{1.27}$$

The left side of Eq. (1.27) represents work done ($U_{1\to2}$) by the force, while the particle is displaced from s_1 to s_2 and the right side represents the change in kinetic energy ($T_2 - T_1$) of the particle. Equation (1.27), then, can also be expressed as

$$U_{1\to2} = T_2 - T_1 \tag{1.28}$$

Hence, principle of work and energy can be state as: *The work done by the force is equal to the change in kinetic energy of the particle.*

The principle of work and energy can be efficiently applied to solve many problems of dynamics. However, in many dynamic systems, the total mechanical energy remains constant, although it may be transformed from one form into another, i.e., from kinetic energy to potential energy or from potential energy to kinetic energy.

Potential energies that should be considered for such problems are gravitational potential energy, elastic potential energy, etc. This modified form of work-energy principle is also known as the principle of conservation of energy.

If the force causing the displacement of the particle is conservative in nature, then work done on the particle can also be expressed as

$$U_{1\to 2} = V_1 - V_2 \tag{1.29}$$

where the function V is called the potential energy, or potential function, of the force $\boldsymbol{F}$.

Substituting $U_{1\to 2}$ from Eq. (1.29) into Eq. (1.28), we get the mathematical form of conservation of energy as

$$T_2 - T_1 = V_1 - V_2$$

$$\therefore \quad T_1 + V_1 = T_2 + V_2 \tag{1.30}$$

Hence, principle of conservation of energy can be state as *the sum of the kinetic energy and of the potential energy of the particle remains constant*.

Since the potential energy in most of the problems includes the gravitation potential energy (V_g) and the elastic potential energy (V_e), Eq. (1.30) can also be expressed as

$$T_1 + \left(V_g\right)_1 + (V_e)_1 = T_2 + \left(V_g\right)_2 + (V_e)_2 \tag{1.31}$$

If the system includes non-conservative forces also, then Eq. (1.31) can be modified as

$$T_1 + \left(V_g\right)_1 + (V_e)_1 + (U_{1\to 2})_{NC} = T_2 + \left(V_g\right)_2 + (V_e)_2 \tag{1.32}$$

where $(U_{1\to 2})_{NC}$ is the work done by the non-conservative forces.

Work-Energy Principle for a Rigid Body

Expressions developed from work-energy principle and conservation of energy can also be applied for the rigid bodies. However, while calculating work done and kinetic energy of the system, the effect of rotational motion should also be considered along with the linear motion considered for the particle.

The work done by a force, $\boldsymbol{F}$, acting on a rigid body during which a particle of the rigid body is displaced from a position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$ is

$$U_{1\to 2} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}.\mathrm{d}\boldsymbol{r} \tag{1.33}$$

where $d\boldsymbol{r}$ is a differential position vector in the direction of motion.

Similarly, the work done by a moment M acting on a rigid body during which angular displacement of a particle about reference point changes from in θ_1 to θ_2 is

$$U_{1\to 2} = \int_{\boldsymbol{\theta}_1}^{\boldsymbol{\theta}_2} M\mathrm{d}\theta \tag{1.34}$$

The total work done on a rigid body undergoing a general plane motion is

$$U_{1\to 2} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} m\overline{\boldsymbol{a}}.\mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{\theta}_1}^{\boldsymbol{\theta}_2} \overline{I}\boldsymbol{\alpha}\mathrm{d}\theta \tag{1.35}$$

The kinetic energy of a rigid body undergoing general plane motion is the sum of the translational kinetic energy and the rotational kinetic energy

$$T = \frac{1}{2}m\overline{V}^2 + \frac{1}{2}\overline{I}\omega^2 \tag{1.36}$$

If the body has a fixed axis of rotation at O, then kinetic energy can be determined as

$$T = \frac{1}{2}I_o\omega^2 \tag{1.37}$$

1.8.4 Principle of Impulse and Momentum

Problems of dynamics can also be solved by applying the principle of impulse and momentum. Although it can be applied for any problems involving time and velocities, it is most suitable for the problems involving impulsive motion.

Principle of Impulse and Momentum for a Particle

Newton's second law of motion can also be expressed as

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{V}) \tag{1.38}$$

where $m\mathbf{V}$ is the linear momentum of the particle.

Multiplying both sides of Eq. (1.38) by $\mathrm{d}t$ and integrating from a time instant t_1 to a time instant t_2, we get

$$\int_{t_1}^{t_2} \boldsymbol{F} \mathrm{d}t = m\boldsymbol{V}_2 - m\boldsymbol{V}_1 \tag{1.39}$$

Equation (1.39) can also be rearranged as

$$m\boldsymbol{V}_1 + \int_{t_1}^{t_2} \boldsymbol{F} \mathrm{d}t = m\boldsymbol{V}_2 \tag{1.40}$$

which is the expression for the principle of linear impulse and momentum.

Principle of Impulse and Momentum for a Rigid Body

The mathematical form of principle of impulse and momentum for a rigid body undergoing general plane motion can be expressed as

$$\overline{I}\omega_1 + \int_{t_1}^{t_2} \sum M_G \mathrm{d}t = \overline{I}\omega_2 \tag{1.41}$$

If the moments are taken about a point O, Eq. (1.41) can also expressed as

$$I_O\omega_1 + \int_{t_1}^{t_2} \sum M_O \mathrm{d}t = I_O\omega_2 \tag{1.42}$$

Solved Examples

Example 1.1

Determine the number of degrees of freedom and recommend a set of appropriate generalized coordinates for each of the system shown in Figure E1.1.

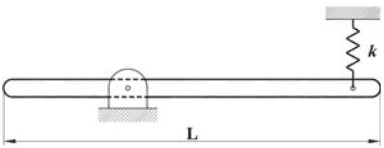


Figure E1.1 (a)

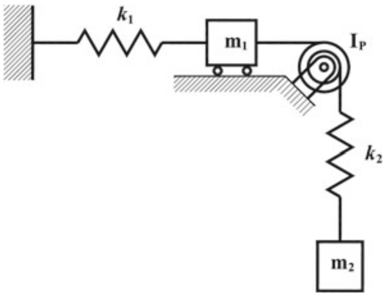


Figure E1.1 (b)

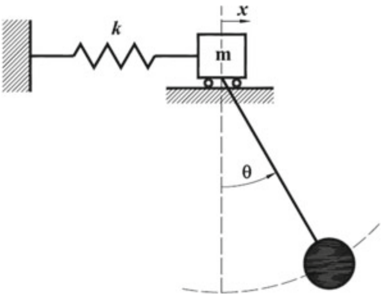


Figure E1.1 (c)

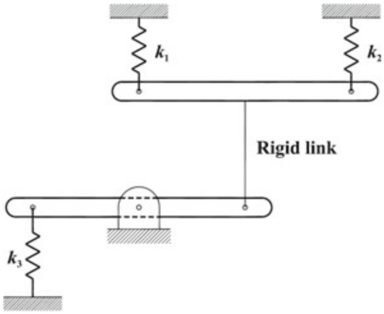


Figure E1.1 (d)

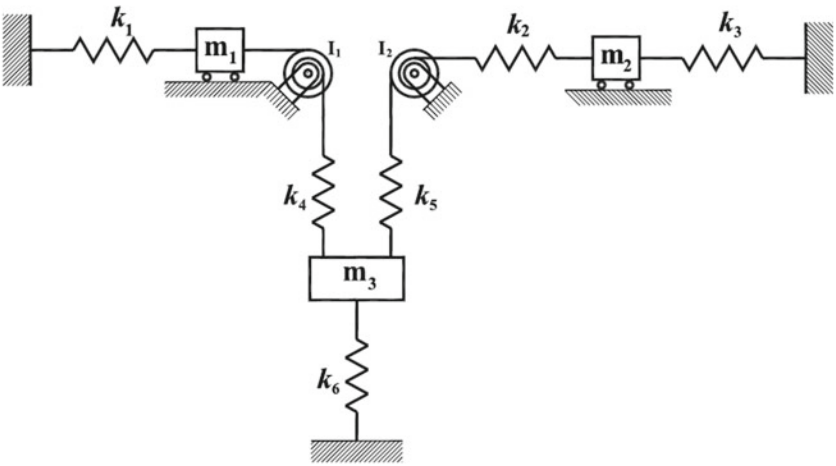


Figure E1.1 (e)

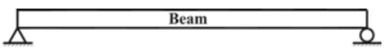


Figure E1.1 (f)

Solution:

(a) Given system consists of only one inertia element (rigid bar). Instantaneous position of the rigid bar can be completely defined by the angular displacement θ of the bar from its equilibrium position, as shown in **Figure ES1.1(a)**. Hence, it has one degree of freedom and θ is an appropriate generalized coordinate.

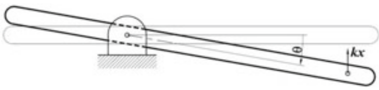


Figure ES1.1(a)

(b) Given system consists of three inertia elements, mass (m_1), pulley (I_P) and mass (m_2). Instantaneous positions of these inertial elements can be defined by linear displacement x_1, angular displacement θ and linear displacement x_2, respectively, as shown in **Figure ES1.1(b)**. Among these displacement variables x_1 and θ are kinematically dependent. Hence, it has only two independent coordinates and has two degrees of freedom and of x_1 and x_2 or θ and x_2 is an appropriate set of generalized coordinates.

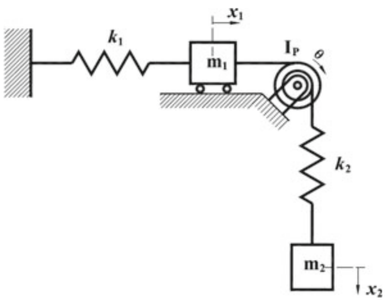


Figure ES1.1(b)

(c) Given system consists of two inertia elements, mass (m) and pendulum bob. Instantaneous positions of these inertial elements can be defined by linear displacement x_1 and angular displacement θ, respectively, as shown in **Figure E1.1(c)**. These displacement variables x and θ are kinematically independent. Hence, it has two degrees of freedom and x and θ is an appropriate set of generalized coordinates.

(d) Given system consists of two inertia elements, both rigid bars. Instantaneous positions of these inertial elements can be defined by angular displacement θ_1 for the lower bar and linear displacement of CG x and angular displacement θ_2 for the upper bar, respectively, as shown in **Figure ES1.1(d)**. All these displacement variables are kinematically independent. Hence, it has three degrees of freedom and θ_1, θ_2 and x is an appropriate set of generalized coordinates.

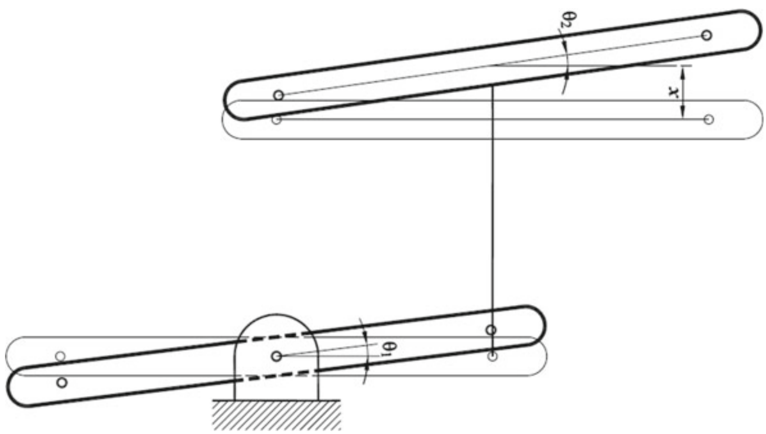


Figure ES1.1(d)

(e) Given system consists of five inertia elements, masses (m_1, m_2 and m_3) and pulleys (I_1 and I_2. Instantaneous positions of these inertial elements can be defined by linear displacements x_1, x_2 and x_3 and angular displacements θ_1 and θ_2, respectively, as shown in **Figure ES1.1(e)**. Among these displacement variables x_1 and θ_1 are kinematically dependent. Hence, it has only four independent coordinates and has four degrees of freedom and x_1, x_2, x_3 and θ_2 or θ_1, x_2, x_3 and θ_2 is an appropriate set of generalized coordinates.

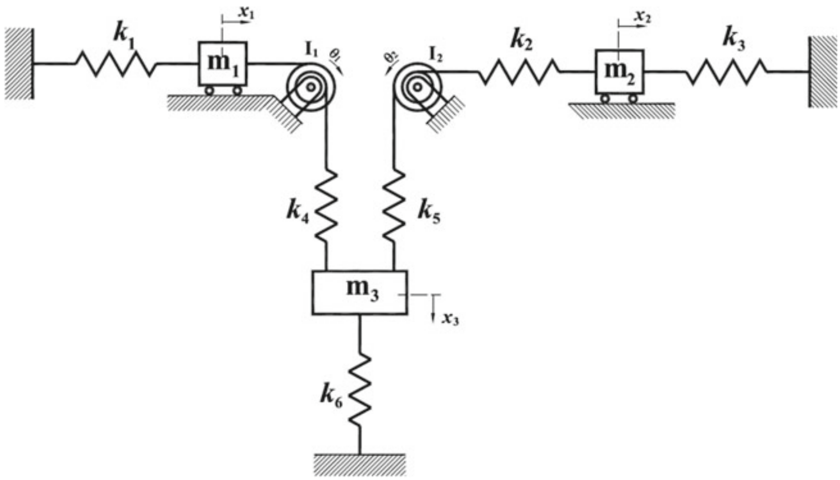


Figure ES1.1(e)

(f) Given system consists of distributed inertia element and its transverse displacement v is a continuous function of x also. Hence the system has an infinite degree of freedom.

Example 1.2

A smooth 3kg collar A, shown in Figure E1.2, is attached to a spring having a stiffness $k = 5$N/m and an un-stretched length of 1m. If the collar is released from rest at A, determine its acceleration and the normal force of the rod on the collar at the instant $y = 1.5$m.

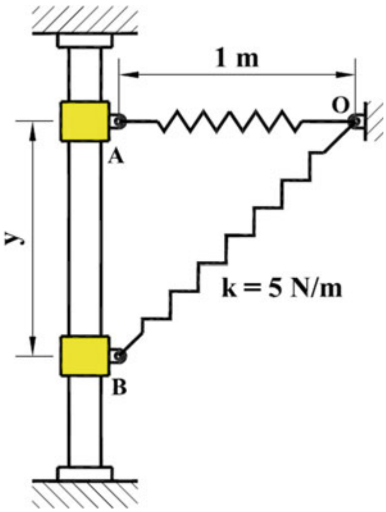


Figure E1.2

Solution

Length of spring in the un-stretched position

$$L_1 = OA = 1\text{m}$$

Length of spring in the stretched position

$$L_2 = OB = \sqrt{(1)^2 + (1.5)^2} = 1.8028\text{m}$$

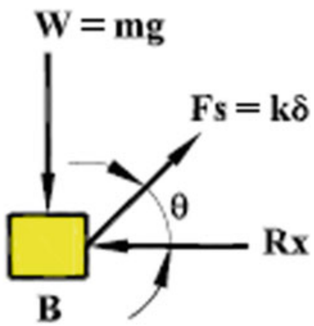


Figure E1.2(a): Free-body diagram of collar

Extension in the spring is given by

$$\delta = L_2 - L_1 = 1.8028 - 1 = 0.8028\text{m}$$

Then free-body diagram of the collar can be drawn when it is at B [**Figure E1.2(a)**]. Weight of the collar (W), spring force (F_s) and inclination of force (θ) can be determined as

$$W = \text{mg} = 3 \times 9.81 = 29.43 \text{ N}$$

$$F_s = k\delta = 5 \times 0.8208 = 4.0139 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{BA}{OA}\right) = \tan^{-1}\left(\frac{1.5}{1}\right) = 0.9828 \text{ rad}$$

Now applying Newton's second law of motion for horizontal direction,

$$\overset{+}{\rightarrow} \sum F_x = ma_x$$

or, $F_s \cos\theta - R_x = 0$

$$\therefore R_x = F_s \cos\theta = 4.0139 \times \cos(0.9828) = 2.2265 \text{ N}$$

Hence the normal force of the rod on the collar is **2.2265 N**.
Similarly, applying Newton's second law of motion for vertical direction,

$$+\downarrow \sum F_y = ma_y$$

or, $W - F_s \sin\theta = ma_y$

$$\therefore \quad a_y = \frac{W - F_s \sin\theta}{m} = \frac{29.43 - 4.0139 \times \sin(0.9828)}{3} = 8.6968 \text{ m/s}^2$$

Therefore, the downward acceleration of the collar when it is at B is **8.698 m/s**^{2}.

Example 1.3
The slider of mass 12kg moves with negligible friction up the inclined guide. The attached spring has a stiffness of 75N/m and is stretched 0.5m in position *A*, where the slider is released from rest. The force *F* is constant and equal to 400N, and the pulley offers negligible resistance to the motion of the cord. Calculate the velocity of the slider as it passes point *B*

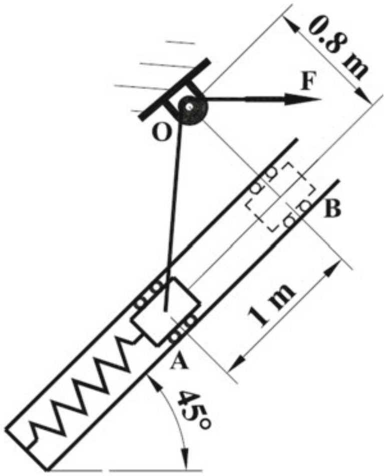


Figure E1.3

Solution

Considering *A* as the datum, gravitational potential energy, elastic potential energy and kinetic energy of the particle for position A can be determined as

$$\left(V_g\right)_1 = 0$$
$$(V_e)_1 = \frac{1}{2}k\,\delta_1^2 = \frac{1}{2} \times 75 \times (0.5)^2 = 9.375\ J$$
$$(T)_1 = 0$$

Similarly, gravitational potential energy, elastic potential energy and kinetic energy of the particle for position B can be determined as

$$\left(V_g\right)_2 = mgh_2 = 12 \times 9.81 \times 1 \times \sin(45) = 83.2406\ J$$
$$(V_e)_2 = \frac{1}{2}k\delta_2^2 = \frac{1}{2} \times 75 \times (0.5 + 1)^2 = 84.375\ J$$
$$(T)_2 = \frac{1}{2}mV_2^2 = \frac{1}{2} \times 12 \times V_2^2 = 6V_2^2$$

Distance traveled by the point of application of force F can be determined as

$$S_{12} = OA - OB = \sqrt{1^2 + (0.8)^2} - 0.8 = 0.4806\text{m}$$

Work done by the non-conservative force is given by

$$(U_{1\to 2})_{NC} = F \times S_{12} = 300 \times 0.48062 = 192.2499J$$

Now, applying energy principle,

$$T_1 + \left(V_g\right)_1 + (V_e)_1 + (U_{1\to 2})_{NC} = T_2 + \left(V_g\right)_2 + (V_e)_2$$

or, $0 + 0 + 9.375 + 192.2499 = 83.2406 + 84.375 + 6{V_2}^2$
or, $34.0093 = 6{V_2}^2$

$$\therefore \quad V_2 = 2.3808 \text{ m/s}$$

Therefore, the velocity of the slider as it passes point B is **2.3808 m/s**.

Example 1.4
Two blocks shown in Figure E1.4 start from rest. Determine the acceleration of each block and the tension in each cord if mass of the block *A* is 20 kg and that of the block *B* is 60 kg. Assume the horizontal plane and the pulley are frictionless and the pulley is of negligible mass.

Solution

Referring to free-body diagram of block A [**Figure E1.4(a)**], Newton's second law of motion can be applied as

$$\overset{+}{\rightarrow} \sum F_x = m_a a_a$$

or, $T_1 = m_a a_a$

$$\therefore \quad T_1 = 20a_a \qquad \text{(a)}$$

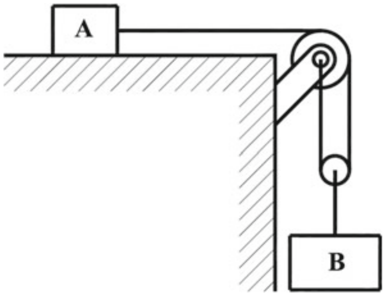


Figure E1.4

Similarly, referring to free-body diagram of block B [**Figure E1.4(b)**], Newton's second law of motion can be applied as

$$
\begin{aligned}
&+\downarrow \sum F_y = m_b a_b \\
&\text{or, } W_b - T_2 = m_b a_b \\
&\text{or, } 60 \times 9.81 - T_2 = 60 a_b \\
&\therefore \quad 588.6 - T_2 = 60 a_b
\end{aligned} \tag{b}
$$

Again, referring to free-body diagram of the pulley [**Figure E1.4(c)**], Newton's second law of motion can be applied as

$$
\begin{aligned}
&+ + \downarrow \sum F_y = m_p a_p \\
&\text{or, } T_2 - 2T_1 = 0 \times a_p \\
&\therefore \quad T_2 = 2T_1
\end{aligned} \tag{c}
$$

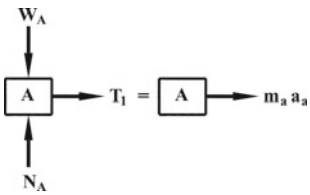


Figure E1.4(a): Free-body diagram of Block *A*

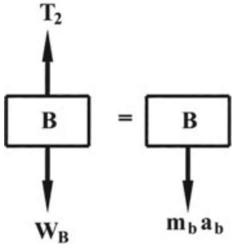


Figure E1.4(b): Free-body diagram of Block *B*

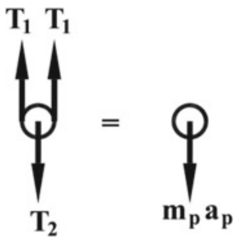


Figure E1.4(c): Free-body diagram of the pulley

Kinematic dependency of motion of block A and block B can be defined in terms of their displacements as

$$x_b = \frac{1}{2}x_a \tag{d}$$

Differentiating twice with respect to time,

$$a_b = \frac{1}{2}a_a \tag{e}$$

Substituting a_B from Eq. (e) and T_2 from Eq. (c) into Eq. (b), we get

$$588.6 - 2T_1 = 30a_a \tag{f}$$

Substituting T_1 from Eq. (a) into Eq. (f), we get

$$588.6 - 40a_a = 30a_a$$

$$\text{or, } 70a_a = 588.6$$

$$\therefore \quad a_a = 8.4086 \text{ m/s}^2$$

Substituting a_a into Eq. (e), we get

$$a_b = \frac{1}{2}a_a = \frac{1}{2} \times 8.4086$$

$$\therefore \quad a_b = 4.2043 \text{ m/s}^2$$

Substituting a_a into Eq. (a), we get

$$T_1 = 20 \times 8.4086$$

$$\therefore \quad T_1 = 168.17 \text{ N}$$

Substituting T_1 into Eq. (c), we get

$$T_2 = 2 \times 168.17$$

$$\therefore \quad T_2 = 336.34 \ N$$

Therefore, the accelerations of block *A* and block *B* are **8.4086 m/s²** and **4.2043 m/s²**, respectively, and the tensions in the cords are **168.17 N** and **336.34 N**, respectively.

Example 1.5
Two rough planes inclined at 45° and 30° to the horizontal and of the same height are placed back to back as shown in Figure E1.5. Two blocks *A* and *B* of masses of 60 and 20kg are placed on the faces and connected by a string over the top of the planes. If the coefficient of friction be 0.25, determine the resulting acceleration.

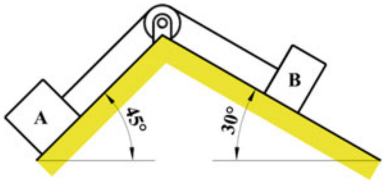


Figure E1.5

Solution

Referring to free-body diagram of block A [**Figure E1.5(a)**], Newton's second law of motion for direction y' can be applied as

$$\overset{+}{\uparrow} \sum F_{y'} = 0$$

$$\text{or, } N_A - W_A \cos 45 = 0$$

$$\text{or, } N_A - 60 \times 9.81 \times \cos 45 = 0$$

$$\therefore \quad N_A = 416.203 \text{ N}$$

Newton's second law of motion for block A for direction x' can be applied as

$$\overset{+}{\leftarrow} \sum F_{x'} = m_a a$$

$$\text{or, } W_A \sin 45 - T - \mu N_A = m_a a$$

$$\text{or, } 20 \times 9.81 \times \sin 45 - T - 0.25 \times 416.203 = m_a a$$

$$\therefore \quad T + 60a = 312.1523 \tag{a}$$

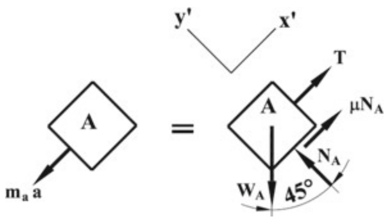


Figure E1.5(a): Free-body diagram of Block A

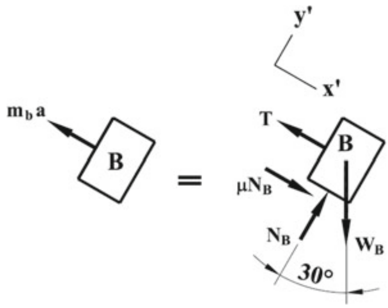


Figure E1.5(a): Free-body diagram of Block B

Referring to free-body diagram of block B [**Figure E1.5(b)**], Newton's second law of motion for direction y' can be applied as

$$\overset{+}{\uparrow} \sum F_{y'} = 0$$

$$\text{or, } N_B - W_B \cos 30 = 0$$

$$\text{or, } N_B - 20 \times 9.81 \times \cos 30 = 0$$

$$\therefore \quad N_B = 169.9142 \text{ N}$$

Newton's second law of motion for block B for direction x'. can be applied as

$$\overset{+}{\leftarrow} \sum F_{x'} = m_b a$$

$$\text{or, } T - W_B \sin 30 - \mu N_B = m_b a$$

$$\text{or, } T - 20 \times 9.81 \times \sin 30 - 0.25 \times 169.9142 = m_b a$$

$$\therefore \quad T - 20a = 140.5785 \tag{b}$$

Solving simultaneous Eqs. (a) and (b) for T and a, we get

$$T = 183.4719 \text{ N} \quad \text{and} \quad a = 2.1447 \text{ m/s}^2$$

Therefore, the acceleration of the system is **2.1447 m/s**^{2}.

Example 1.6
The system shown in Figure E1.6 is released from rest. Block *A* and block *B* have masses of 10 kg and 50 kg, respectively. The spring constant is 250 N/m and is initially un-stretched condition. Determine the magnitude of the velocity of the masses when the block *B* has fallen 1m.

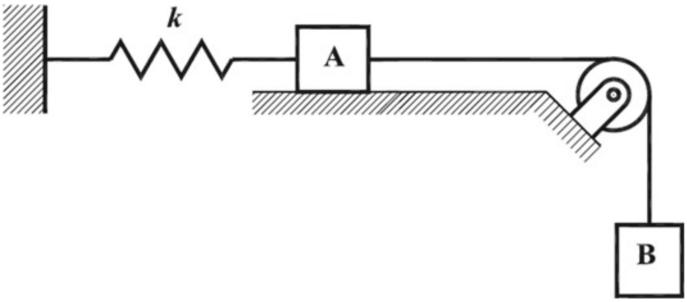


Figure E1.6

Solution

Considering initial position 1 of block B as shown in **Figure E1.6(a)** as the datum, elastic potential energy, gravitational potential energy of block A, gravitational potential energy of block B, kinetic energy of block A and kinetic energy of block B can be determined as

$$(V_e)_1 = \frac{1}{2}k\delta_1^2 = \frac{1}{2} \times 250 \times (0)^2 = 0$$
$$\left(V_g\right)_{A1} = m_a g y = 10 \times 9.81 \times y = 98.1y$$
$$\left(V_g\right)_{B1} = m_b g \times 0 = 0$$
$$(T)_{A1} = \frac{1}{2}m_a(V_{A1})^2 = 0$$
$$(T)_{B1} = \frac{1}{2}m_b(V_{B1})^2 = 0$$

Similarly, elastic potential energy, gravitational potential energy of block A, gravitational potential energy of block B, kinetic energy of block A and kinetic energy of block B for position 2 can be determined as

$$(V_e)_2 = \frac{1}{2}k\ \delta_1^2 = \frac{1}{2} \times 250 \times (1)^2 = 125\ J$$
$$\left(V_g\right)_{A2} = m_a g y = 10 \times 9.81 \times y = 98.1y$$
$$\left(V_g\right)_{B2} = m_b g \times (-1) = 50 \times 9.81 \times (-1) = -490.5\ J$$
$$(T)_{A2} = \frac{1}{2}m_a(V_{A2})^2 = \frac{1}{2} \times 10 \times (V_{A2})^2 = 5(V_{A2})^2$$
$$(T)_{B2} = \frac{1}{2}m_b(V_{B2})^2 = \frac{1}{2} \times 50 \times (V_{B2})^2 = 25(V_{B2})^2$$

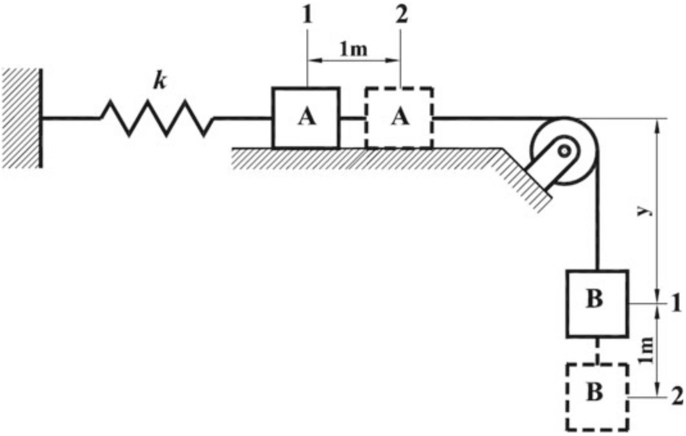


Figure E1.6(a)

Now, applying energy principle,

$$\begin{aligned}
&(V_e)_1 + \left(V_g\right)_{A1} + \left(V_g\right)_{B1} + (T)_{A1} + (T)_{B1} \\
&\quad = (V_e)_2 + \left(V_g\right)_{A1} + \left(V_g\right)_{B1} + (T)_{A1} + (T)_{B1} \\
&\text{or, } 0 + 98.1y + 0 + 0 + 0 = 125 + 98.1y - 490.5 + 5(V_{A2})^2 + 25(V_{B2})^2 \\
&\therefore \quad 5(V_{A2})^2 + 25(V_{B2})^2 = 365.5
\end{aligned}$$

Kinematic dependency shows that velocities of the both blocks should be same, i.e.,

$$V_{A2} = V_{B2} = V_2$$

Then,

$$\begin{aligned}
&5(V_2)^2 + 25(V_2)^2 = 365.5 \\
&\text{or, } 30(V_2)^2 = 365.5 \\
&\text{or, } (V_2)^2 = 12.1833 \\
&\therefore \quad V_2 = 3.4905 \text{ m/s}
\end{aligned}$$

Therefore, the magnitude of the velocity of the masses when the block B has fallen 1m is **3.4905 m/s**.

Example 1.7
A homogeneous bar AB of mass m is pinned at O and held in a horizontal position by a cable at B as shown in Figure E1.7. Determine the angular acceleration of the bar immediately after the cable is cut.

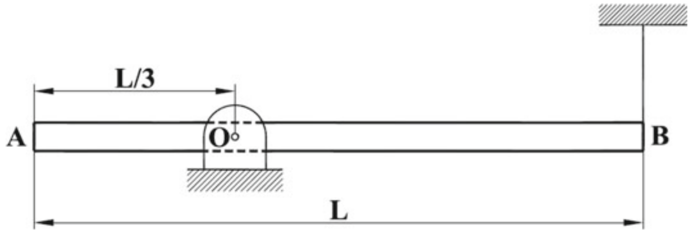


Figure E1.7

Solution

Referring to free-body diagram of the bar as shown in **Figure E1.7(a)**, D' Alembert's principle for angular rotation of the bar about the point O can be applied as

$$\sum(\boldsymbol{M}_{\text{ext}})_O = \sum(\boldsymbol{M}_{\text{eff}})_O$$
$$\text{or, } mg \times \frac{L}{6} = I_O\alpha$$
$$\text{or, } mg \times \frac{L}{6} = \left(\overline{I} + md^2\right)\alpha$$
$$\text{or, } mg \times \frac{L}{6} = \left[\frac{1}{12}mL^2 + m\left(\frac{L}{6}\right)^2\right]\alpha$$
$$\text{or, } \frac{1}{6}mgL = \frac{1}{9}mL^2\alpha$$
$$\therefore \quad \alpha = \frac{3g}{2L}$$

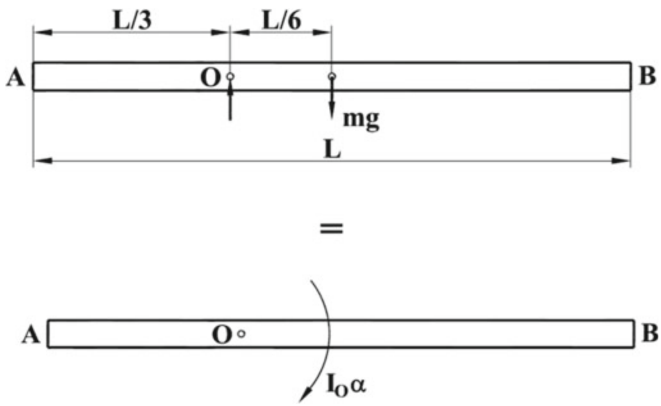


Figure E1.7(a)

Therefore, the angular acceleration of the bar immediately after the cable is cut is $3\mathbf{g}/2\mathbf{L}$.

Example 1.8

A pulley weighing 10**kg and having a radius of gyration of** 25**cm is connected to two blocks *A* and *B* of masses** 30**kg and** 40**kg, respectively, as shown in Figure E1.8. Assuming no axle friction, determine the angular acceleration of the pulley and the acceleration of each block**

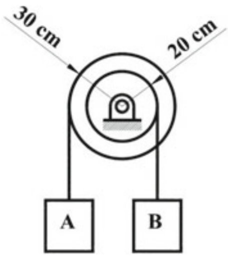


Figure E1.8

Solution

Referring to free-body diagram with external forces and the effective forces for the system shown in **Figure E1.8(a)**, D' Alembert's principle can be applied as

$$\sum (\boldsymbol{M}_{ext})_O = \sum \left(\boldsymbol{M}_{eff}\right)_O$$

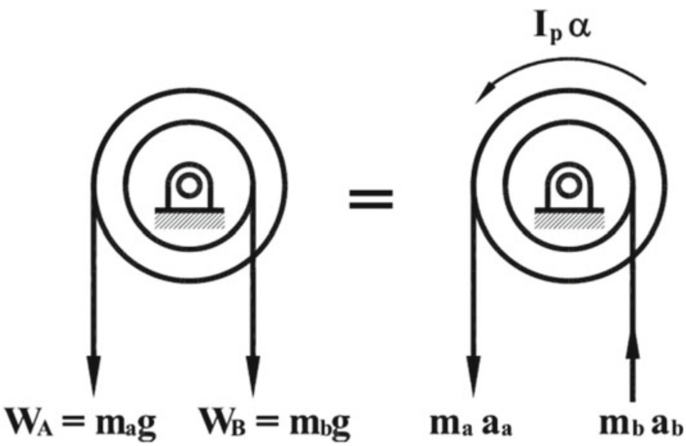


Figure E1.8(a)

$$\text{or, } W_A \times r_A - W_B \times r_B = I_p\alpha + m_a a_a \times r_A + m_b a_b \times r_B$$

Using kinematic dependency of angular rotation of pulley and motion of block A and block B as

$$a_a = r_A \times \alpha = 0.3\alpha$$

and

$$a_b = r_B \times \alpha = 0.2\alpha,$$

we get

$$\begin{aligned}
&W_A \times r_A - W_B \times r_B = I_p\alpha + m_a\alpha \times r_A^2 + m_b\alpha \times r_B^2 \\
&\text{or, } 30 \times 9.81 \times 0.3 - 40 \times 9.81 \times 0.2 = 10 \times 0.25^2 \times \alpha + 30 \times \alpha \times 0.3^2 + 40 \times \alpha \times 0.2^2 \\
&\text{or, } 88.29 - 78.48 = (0.625 + 2.7 + 1.6) \times \alpha \\
&\text{or, } 9.81 = 4.925\alpha \\
&\therefore \quad \alpha = 1.9918 \text{ rad/s}^2
\end{aligned}$$

Then accelerations of each block are given as

$$a_a = 0.3 \times 1.9918 = 0.5976 \text{ m/s}^2$$

and

$$a_b = 0.2 \times 1.9918 = 0.3984 \text{ m/s}^2$$

Therefore, the angular acceleration of the pulley is **1.9918 rad/s²**, acceleration of block A is **0.5976 m/s²** and that of block B is **0.3984 m/s²**.

Example 1.9
The uniform bar AB of mass 20kg and length 5m is released from horizontal position as shown in Figure E1.9. The spring has an un-deformed length of 5m and a stiffness of 20N/m. Determine the angular velocity of the bar when it becomes vertical.

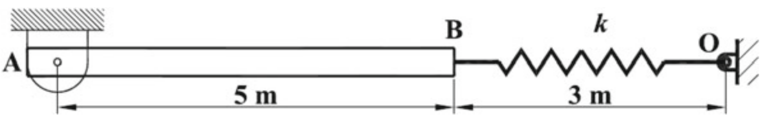


Figure E1.9

Solution

Consider two instantaneous positions of the bar as shown in **Figure E1.9(a)**.

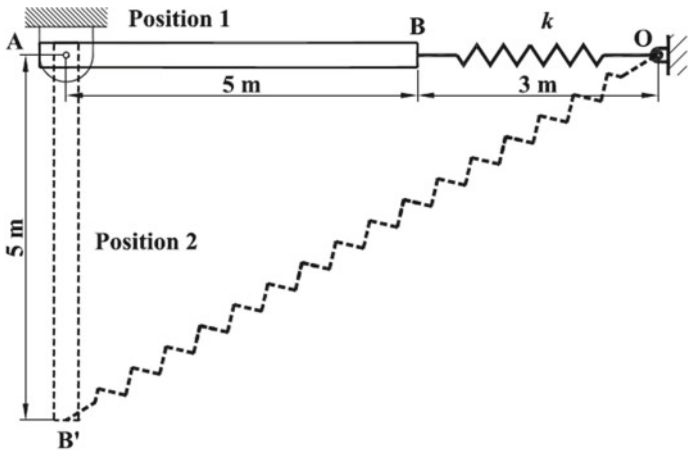


Figure E1.9(a)

Length of the un-deformed spring

$$L_0 = 5 \text{ m}$$

Length of the compressed spring when the bar is in the position 1

$$L_1 = 3 \text{ m}$$

Compression in the spring when the bar is in the position 1

$$\delta_1 = L_1 - L_0 = 2 \text{ m}$$

Length of the elongated spring when the bar is in the position 2

$$L_2 = \sqrt{8^2 + 5^2} = 9.434 \text{ m}$$

Elongation in the spring when the bar is in the position 2

$$\delta_2 = L_2 - L_0 = 4.434 \text{ m}$$

Considering end of the bar B' for position 2 as shown in **Figure E1.9(a)** as the datum, elastic potential energy, gravitational potential energy and kinetic energy of the system for the position 1 can be determined as

$$(V_e)_1 = \frac{1}{2}k\,\delta_1^2 = \frac{1}{2} \times 20 \times (2)^2 = 40\ J$$
$$\left(V_g\right)_1 = mgh_1 = 20 \times 9.81 \times 5 = 981\ J$$

$$(T)_1 = \frac{1}{2}I_A(\omega_1)^2 = 0$$

Similarly, elastic potential energy, gravitational potential energy and kinetic energy of the system for the position 2 can be determined as

$$(V_e)_2 = \frac{1}{2}k\delta_2^2 = \frac{1}{2} \times 20 \times (4.434)^2 = 196.602\ J$$
$$\left(V_g\right)_2 = mgh_2 = 20 \times 9.81 \times 2.5 = 490.5\ J$$
$$(T)_2 = \frac{1}{2}I_A(\omega_2)^2 = \frac{1}{2}\left(\overline{I} + md^2\right)(\omega_2)^2$$
$$= \frac{1}{2}\left(\frac{1}{12} \times 20 \times 5^2 + 20 \times 2.5^2\right)(\omega_2)^2 = 83.333(\omega_2)^2$$

Now, applying energy principle,

$$(V_e)_1 + \left(V_g\right)_1 + T_1 = (V_e)_2 + \left(V_g\right)_2 + T_2$$
$$\text{or, } 40 + 981 + 0 = 196.602 + 490.5 + 83.333(\omega_2)^2$$
$$\text{or, } 83.333(\omega_2)^2 = 333.8981$$
$$\therefore \quad \omega_2 = 2.0017 \text{ rad/s}$$

Therefore, the angular velocity of the bar when it becomes vertical is **2.0017 rad/s.**

Example 1.10

A uniform slender rod AB of mass 10kg and length 1m is initially at rest in a vertical position as shown in Figure E1.10. A 2.5kg sphere moving horizontally to the right with an initial velocity of 4m/s strikes the lower end of the rod. If the coefficient of restitution between the rod and the sphere is 0.80, determine the angular velocity of the rod and the velocity of the sphere immediately after the impact.

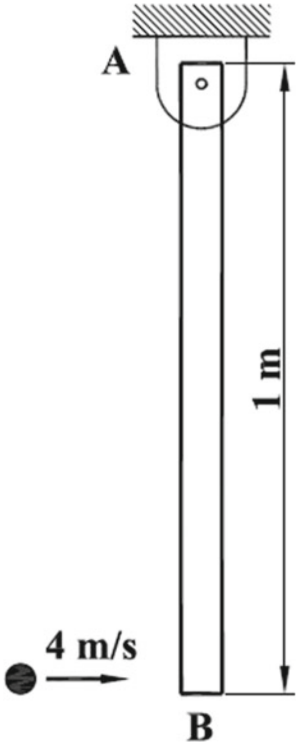


Figure E1.10

Solution

Referring to the impulse-momentum diagram of the system shown in **Figure E1.10(a)** and applying principle of impulse and momentum

$$(\text{system momenta})_1 + (\text{external impulse})_{1\to 2} = (\text{system momenta})_2$$

or, $m_s(V_s)_1 L + m_r\left(\overline{V}_r\right)_1 \frac{L}{2} + \overline{I}_r(\omega_r)_1 + 0 = m_s(V_s)_2 L + m_r\left(\overline{V}_r\right)_2 \frac{L}{2} + \overline{I}_r(\omega_r)_2$

The rod is initially at rest, i.e., $\left(\overline{V}_r\right)_1 = 0$; $(\omega_r)_1 = 0$. Hence the governing equation reduces to

$$m_s(V_s)_1 L = m_s(V_s)_2 L + m_r\left(\overline{V}_r\right)_2 \frac{L}{2} + \overline{I}_r(\omega_r)_2$$

Since the given rod rotates about a fixed point A after the impact, $\left(\overline{V}_r\right)_2 = \frac{L}{2}(\omega_r)_2$. The governing equation further reduces to

$$m_s(V_s)_1 L = m_s(V_s)_2 L + m_r \times \frac{L^2}{4}(\omega_r)_2 + \overline{I}_r(\omega_r)_2$$

Now, substituting remaining system parameters

$$2.5 \times 4 \times 1 = 2.5 \times (V_s)_2 \times 1 + 10 \times 0.25(\omega_r)_2 + \frac{1}{12} \times 10 \times 1^2 \times (\omega_r)_2$$

$$\therefore \quad (V_s)_2 + 1.3333(\omega_r)_2 = 4 \tag{a}$$

Since the coefficient of restitution between the rod and the sphere is 0.80, i.e.,

$$(V_B)_2 - (V_s)_2 = e\big[(V_s)_1 - (V_B)_1\big]$$
$$\text{or, } L(\omega_r)_2 - (V_s)_2 = e\big[(V_s)_1 - L(\omega_r)_1\big]$$
$$\text{or, } 1 \times (\omega_r)_2 - (V_s)_2 = 0.80 \times [4 - 1 \times 0]$$
$$\therefore \quad (\omega_r)_2 - (V_s)_2 = 3.2 \tag{b}$$

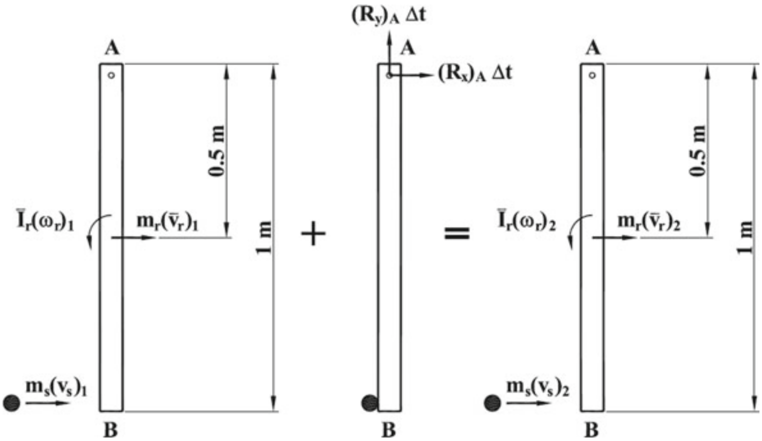


Figure E1.10(a)

Solving simultaneous Eqs. (a) and (b), we get

$$(V_s)_2 = -0.1143 \text{ m/s} \quad \text{and} \quad (\omega_r)_2 = 3.0857 \text{ rad/s}$$

Therefore, the angular velocity of the rod and the velocity of the sphere immediately after the impact are **3.0857 rad/s counterclockwise** and **0.1143 m/s toward left**, respectively.

Review Questions

1. Define vibration. List the common causes of vibration.
2. Explain the undesired and desired effects of vibration.

3. Define simple harmonic motion. Also define amplitude, frequency and time period.
4. Explain the major steps of vibration analysis of any physical system.
5. Define generalized coordinates and degree of freedom of a dynamic system.
6. Differentiate between discrete system and continuous system with examples.
7. Differentiate between: free and forced vibration, un-damped and damped vibration, deterministic and random vibration, linear and nonlinear vibration.
8. Explain common principles or methods that can be used to solve problems of dynamics.

Exercise

1. Determine the number of degrees of freedom and recommend a set of appropriate generalized coordinates for each of the system shown in **Figure P1.1**.

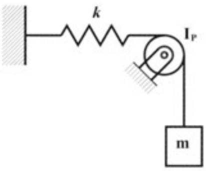


Figure P1.1(a)

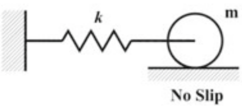


Figure P1.1(b)

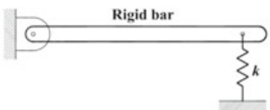


Figure P1.1(c)

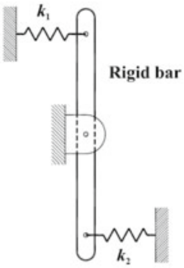


Figure P1.1(d)

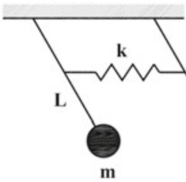


Figure P1.1(e)

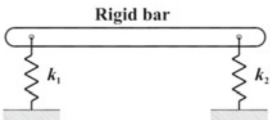


Figure P1.1(f)

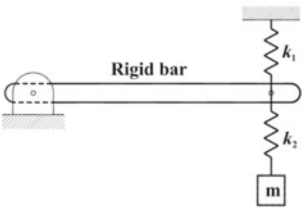


Figure P1.1(g)

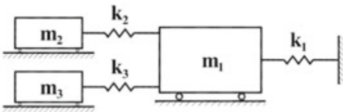


Figure P1.1(h)

Figure E1.10(a)

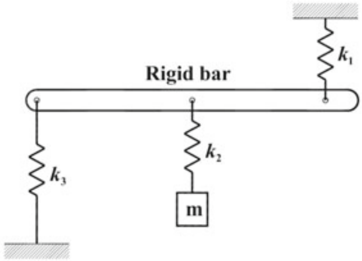


Figure P1.1(i)

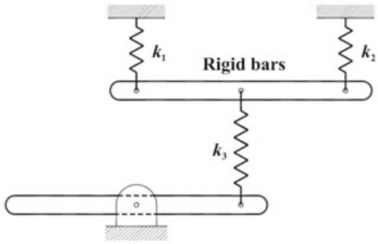


Figure P1.1(j)

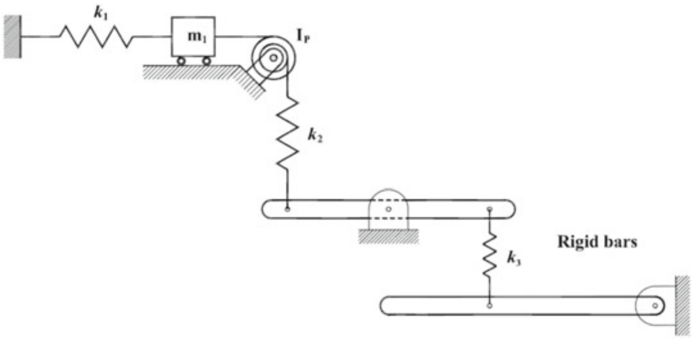


Figure P1.1(k)

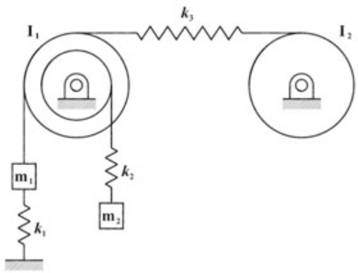


Figure P1.1(l)

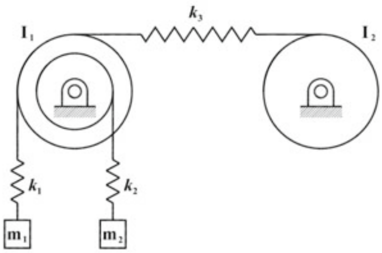


Figure P1.1(m)

Figure E1.10(a)

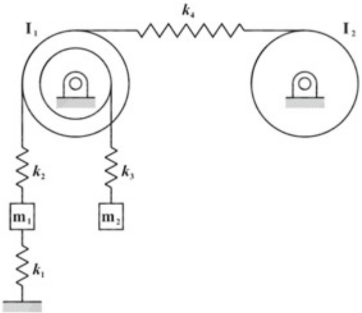


Figure P1.1(n)

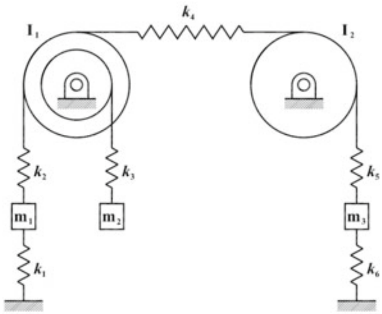


Figure P1.1(o)

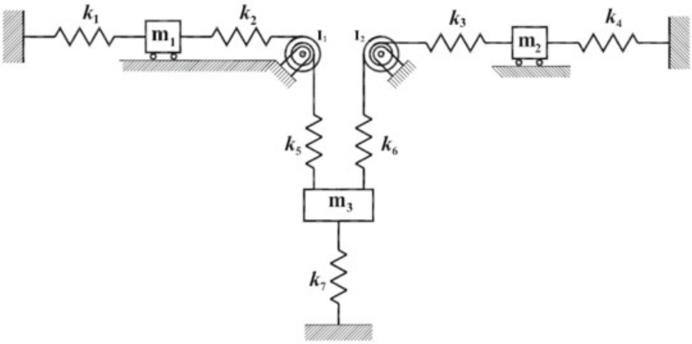


Figure P1.1(p)

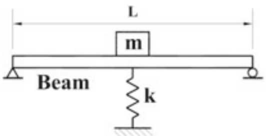


Figure P1.1(q)

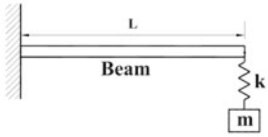


Figure P1.1(r)

Figure E1.10(a)

2. Two blocks shown in **Figure P1.2** start from rest. Determine the acceleration of the system and the tension in the cord, if mass of the block *A* is 20kg and that of the block *B* is 10kg. Assume the horizontal plane and the pulley are frictionless and the pulley is of negligible mass.
3. Two smooth planes inclined at 45^0 and 30^0 to the horizontal and of the same height are placed back to back as shown in **Figure P1.3**. Two blocks *A* and *B* of masses of 60 and 20kg are placed on the faces and connected by a string over the top of the planes. Determine the resulting acceleration and tension in the string.

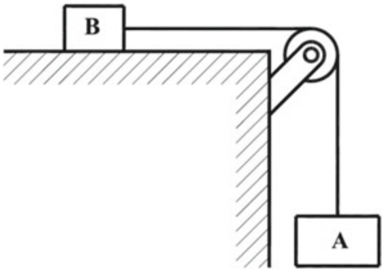


Figure P1.2

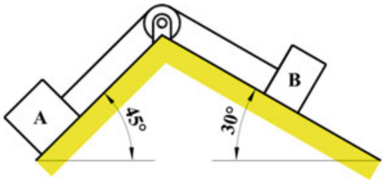


Figure P1.3

4. A system of blocks A and B shown in **Figure P1.4** is released from rest. If mass of the block A is 20kg and that of the block B is 10kg, find the acceleration of each block and the tension in each string. Take coefficient of friction for the contact surface of block A as 0.125.
5. A system of blocks A and B shown in **Figure P1.5** is released from rest. If mass of the block A is 75kg and that of the block B is 50kg, find the acceleration of each block and the tension in the string supporting block B. Take coefficient of friction for the contact surface of block A as 0.2.

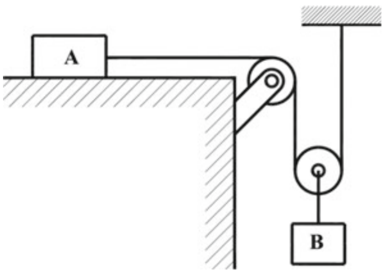


Figure P1.4

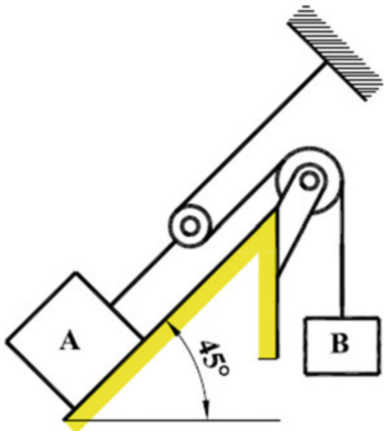


Figure P1.5

6. A system of blocks A, B and C shown in **Figure P1.6** is released from rest. If the mass of blocks A, B and C are 4, 6 and 12kg, determine the acceleration of each and tension in the two strings. Take coefficient of friction for the contact surfaces of bodies A and B as 0.25.
7. Two blocks A and B having masses of 15kg and 10kg, respectively, are connected by a light inextensible cord as shown in **Figure P1.7**. If both the

bodies are released simultaneously, what distance do they move in 1.5sec? Neglect friction between the two bodies and the inclined surfaces.

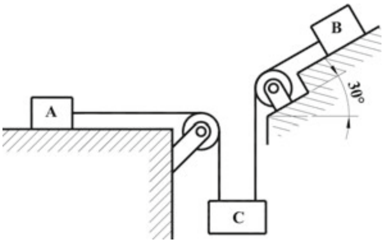


Figure P1.6

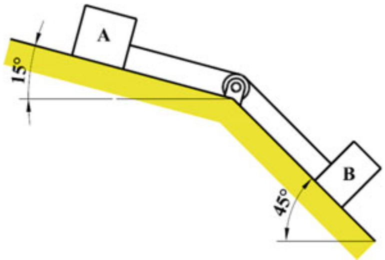


Figure P1.7

8. Determine the resulting motion of the block A assuming the pulleys to be smooth and weightless as shown in **Figure P1.8**. The mass of the block A is 10kg and that of the block B is 20kg. The coefficient of friction between the inclined surface and block A is 0.2. If the system starts from rest, determine the velocity of the block A after 4sec.
9. A block of 8kg mass slides down on a surface inclined at 30^0 as shown in **Figure P1.9**. It is stopped by a spring with a spring constant of 800N/m. If the block slides down 5m before hitting the spring, determine the maximum compression of the spring. The coefficient of friction between the block and the inclined surface is 0.2.

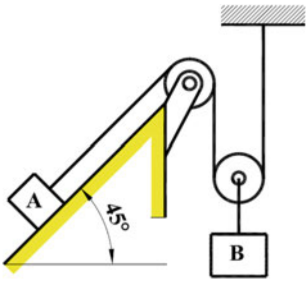


Figure P1.8

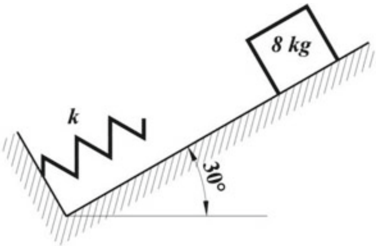


Figure P1.9

10. The 2.5kg collar shown in **Figure P1.10** starts from rest in the position A when a constant force $F = 150$N is applied to the cable, causing the collar to move up the smooth vertical shaft. If the stiffness of the spring is 2500N/m, determine the speed of the collar when the spring is compressed 60mm.
11. The slider of mass 60kg moves under the action of a constant force F of magnitude 400N with negligible friction on a horizontal guide as shown in **Figure P1.11**. The spring has a stiffness $k = 100$N/m and initial extension in the spring

0.25m when the slider is at position A. The block is released from rest at A, determine the velocity of the block as it reaches position B.

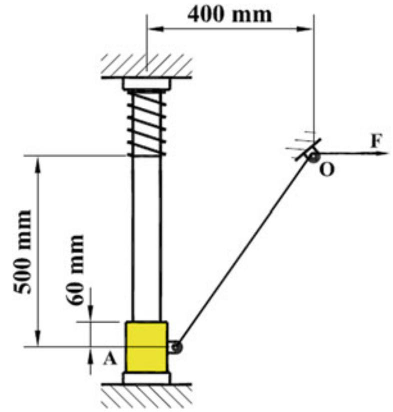


Figure P1.10

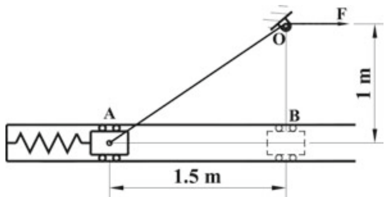


Figure P1.11

12. A 2kg collar shown in **Figure P1.12** slides along the frictionless vertical rod under the actions of gravity and an ideal spring. The spring has a stiffness of 200N/m, and its free length is 1m. The collar is released from rest in position 1. Use the principle of conservation of mechanical energy to determine the speed of the collar in position 2.
13. A smooth 3kg collar A is attached to a spring as shown in **Figure P1.13**. The spring has a stiffness $k = 5\text{N/m}$ and is un-stretched when the collar is in the position A. Determine the speed at which the collar is moving when $y = 1.2\text{m}$,

    (a) if it is released from rest at A.
    (b) it is released at A with an upward velocity of 2.5m/s.

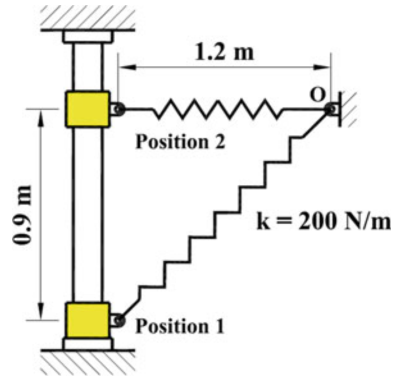


Figure P1.12

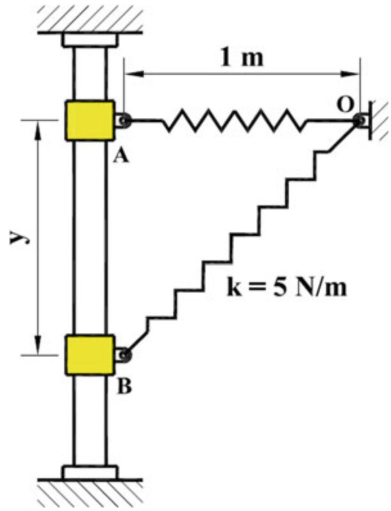


Figure P1.13

14. The spring ($k = 1000\mathrm{N/m}$) is connected to the floor and to the 100kg collar A as shown in **Figure P1.14**. Collar A is at rest, supported by the spring, when the 150kg box B is released from rest in the position shown. Determine the velocities of A and B when B has fallen 0.8m.
15. Two blocks A and B shown in **Figure P1.15**, have masses 40kg and 20kg, respectively, and the coefficient of friction between the block A and the horizontal plane, $\mu = 0.2$. If the system is released, from rest and the block B falls through a vertical distance of 1.5m, what is the velocity acquired by it?

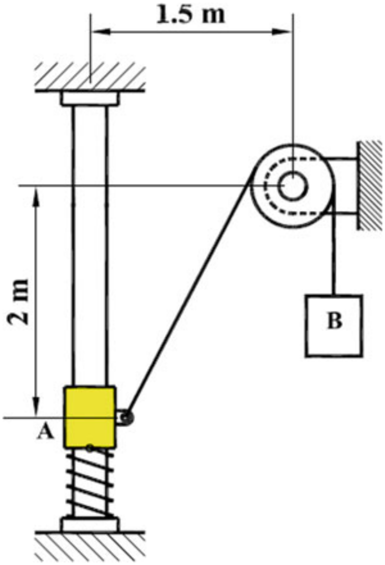


Figure P1.14

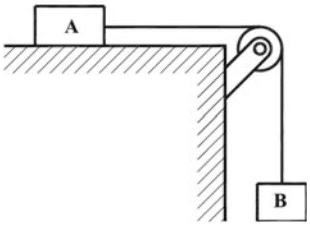


Figure P1.15

16. A homogeneous bar AB of mass 15kg and a length of 1.2m is pinned at O and held in a horizontal position by a cable at B as shown in **Figure P1.16**. Determine the angular acceleration of the bar immediately after the cable is cut.
17. A homogeneous bar AB of mass m is suspended horizontally by a cable at B as shown in **Figure P1.17**. If the cable is cut, determine the velocity of center of mass of the bar when it swings to the vertical position.

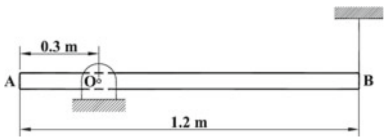


Figure P1.16

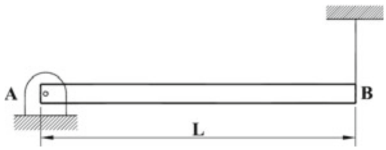


Figure P1.17

18. The cable connected to block A in **Figure P1.18** is wound tightly around a disk of radius 500mm, which is free to rotate about the axle at its mass center O. The masses of block A and disk are 50kg and 20kg, respectively, and the radius of gyration of the disk is 400mm. Determine the angular acceleration of disk and the tension in the cable.
19. The cable connected to block A in **Figure P1.19** is wound tightly around a disk of radius 400mm, which is free to rotate about the axle at its mass center O. An external moment M of magnitude 50Nm is also applied to the disk. The masses of block A and disk are 15kg and 10kg, respectively, and the radius of gyration of the disk is 300mm. Determine the angular acceleration of disk and the tension in the cable.
20. Two blocks are attached to a disk by a cord as shown in **Figure P1.20**. If mass of the block A is 10kg and that of the block B is 15kg, determine the angular acceleration of the disk. The mass moment of inertia of the disk is 0.1kgm^2.
21. The stepped pulley system shown in **Figure P1.21**, when released from rest, determines the acceleration of the blocks, angular acceleration of the pulley and tension in the strings connecting the blocks. The mass of the block A is 10kg and that of the block B is 12kg, mass of the pulley is 30kg, and radius of gyration is 25cm.

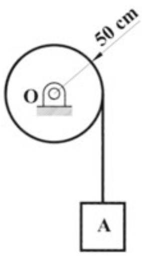


Figure P1.18

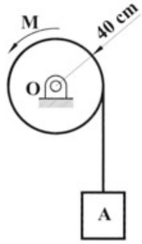


Figure P1.19

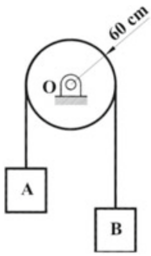


Figure P1.20

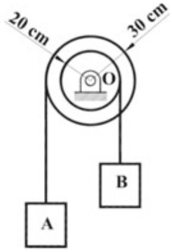


Figure P1.21

Figure E1.10(a)

22. The slender bar of mass 2.5kg shown in **Figure P1.22** is moved to the horizontal position and then released. When the bar is horizontal, the spring is compressed by 25mm. Determine the maximum angle through which the will swing.

23. The center of the thin disk with mass of 2.5kg and radius of 25cm as shown in **Figure P1.23** is displaced 20mm and released. If the stiffness of the spring is 15kN/m, determine the maximum velocity attained by the disk, assuming no slipping between the disk and the surface?

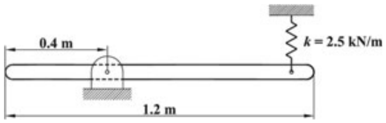


Figure P1.22

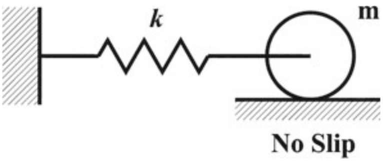


Figure P1.23

24. The uniform 25kg slender bar shown in **Figure P1.24** rotates in a vertical plane about the pin. The spring with spring constant $k = 20\text{N/m}$ and an un-deformed length of 2.5m is attached the one end of the bar. If the bar is released from rest when it is vertical, determine the angular velocity of the bar when it reaches the horizontal position.
25.

The 8kg slender rod AB shown in **Figure P1.25** is pinned at A and is initially at rest. If a 5g bullet is fired into the rod with a velocity of 400m/s, determine the angular velocity of the rod just after the bullet becomes embedded in it.

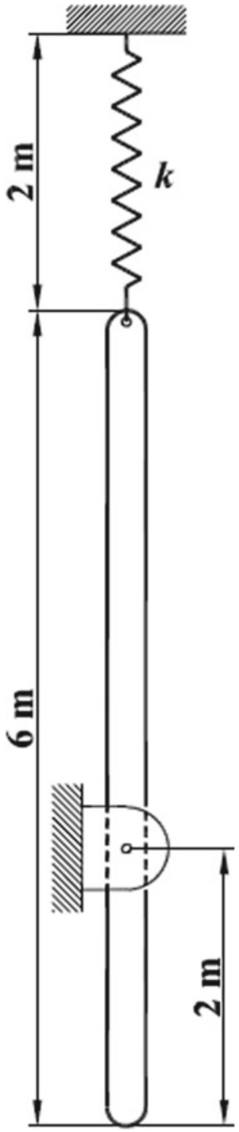


Figure P1.24

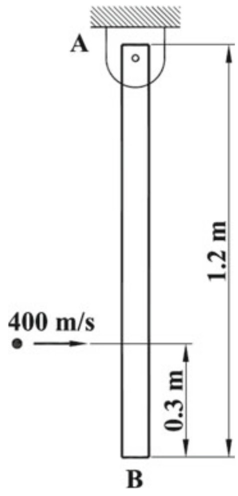


Figure P1.25

Answers

1.
   (a) 1
   (b) 1
   (c) 1
   (d) 1
   (e) 2
   (f) 2
   (g) 2
   (h) 3
   (i) 3
   (j) 3
   (k) 3
   (l) 3
   (m) 4
   (n) 4
   (o) 5
   (p) 5
   (q) $\infty + 1$
   (r) $\infty + 1$
2. 6.54m/s^2; 65.4N
3. 3.98m/s^2; 177.63N
4. 1.09m/s^2, 0.545m/s^2; 46.325N, 92.65N
5. 1.297m/s^2, 2.59m/s^2; 360.38N
6. 5.66m/s^2; 32.46N, 17.29N
7. 4.84m
8. 3.96m/s
9. 0.599m
10. 3.93m/s
11. 2.39m/s
12. 1.83m/s
13. 4.798m/s; 5.41m/s
14. 14.2.62m/s; 1.23m/s
15. 2.43m/s
16. 14.014rad/s^2
17. $\sqrt{3gL/4}$
18. 15.62rad/s^2; 99.97N
19. 2.68rad/s^2; 131.04N
20. 3.23rad/s^2
21. 0.54m/s^2, 0.36m/s^2; 1.81rad/s^2; 92.68N, 122.06N

22. 3.93^0
23. 1.265m/s
24. 0.718rad/s
25. 0.468rad/s

Chapter 2
Modeling of Components of a Vibrating System

2.1 Components of a Vibrating System

The basic components of a vibrating system under idealized conditions are *the mass* (m), *the spring* (k), *the damper* (c) and *the excitation* as shown in Fig. 2.1. The first three components (m, c, and m) describe the physical system and are also called system parameters.

The mass (m) is assumed to be rigid body. It carries energy in the form of kinetic energy in accordance with the velocity of the body.

The spring (k) possesses elasticity and is assumed to be of negligible mass. A spring stores energy in the form of potential energy or strain energy. For basic analysis, a spring is assumed to be linear. A linear spring is one that obeys Hooke's law, which is the spring force is proportional to the spring deformation. The constant of proportionality, measured in force per unit deformation, is called the *stiffness*, or the *spring constant* and its unit is N/m.

The damper (c) has neither mass nor elasticity. Damping force exists only if there is relative motion between the two ends of the damper. The work or energy input to the damper is converted into heat; i.e., it dissipates energy. Hence the damping element is non-conservative. Damping in which damping force is proportional to the velocity is called linear damping. The damping coefficient (c) is measured in force per unit velocity, and its unit is Ns/m.

Energy enters the system through the application of an excitation. Excitation force is applied to the mass of the system.

2.2 Inertia Elements and Kinetic Energy

An inertia element is a body with finite mass. Inertia element possesses kinetic energy when the vibratory motion executes in the system. Kinetic energy is stored when the speed of the body increases and is released when its speed decreases. The

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_2

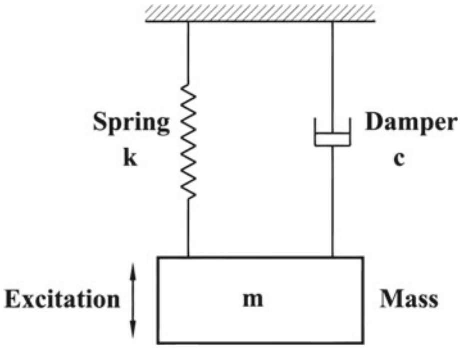


Fig. 2.1 Components of a vibratory system

kinetic energy of a body is a function of the body's inertia properties and its velocity. Mass is taken as inertial property if the system undergoes translational vibratory motion (linear displacement) about its static equilibrium position, and mass moment of inertia is taken as inertial property if the system undergoes rotational vibratory motion (angular displacement) about its static equilibrium position.

2.2.1 Kinetic Energy of a Discrete System Consisting of Particles

If a system consists of a particle of mass m moving with a velocity V, then the kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m V^2 \tag{2.1}$$

If a system consists of n number of particles of mass $m_1, m_2, \ldots, m_n$ moving with velocities $V_1, V_2, \ldots, V_n$ respectively, then the kinetic energy of the system can be expressed as

$$T = \frac{1}{2}\left(m_1 V_1^2 + m_2 V_2^2 + \cdots + m_n V_n^2\right) = \frac{1}{2}\sum_{i=1}^{n} m_i V_i^2 \tag{2.2}$$

2.2.2 Kinetic Energy of a Discrete System Consisting of a Rigid Body

A rigid body can have translational and rotational motion. The total kinetic energy of any rigid body is the sum of its translational kinetic energy and rotational kinetic energy. If a rigid body is in motion in three-dimensional space which has a velocity of center of mass $\overline{V}$ and has the components of angular accelerations along x, y and z axes as ω_x, ω_y and ω_z, respectively, then its total kinetic energy can be expressed as

$$T = \frac{1}{2}m\overline{V}^2 + \frac{1}{2}\left(\overline{I}_x\omega_x^2 + \overline{I}_y\omega_y^2 + \overline{I}_z\omega_z^2 - 2\overline{I}_{xy}\omega_x\omega_y - 2\overline{I}_{yz}\omega_y\omega_z - 2\overline{I}_{zx}\omega_z\omega_x\right) \tag{2.3}$$

where m is the mass of the rigid body and $\overline{I}_x$, $\overline{I}_y$, $\overline{I}_z$, $\overline{I}_{xy}$, $\overline{I}_{yz}$ and $\overline{I}_{zx}$ are the components of inertia tensor $\overline{I}$ about axes through the center of mass.

If the rigid body is undergoing plane motion, then its total kinetic energy can be expressed as

$$T = \frac{1}{2}m\overline{V}^2 + \frac{1}{2}\overline{I}\omega^2 \tag{2.4}$$

where $\overline{I}$ is moment of inertia about an axis through the center of mass, perpendicular to the plane in which the mass center moves, and ω is the angular velocity about the axis.

If the body is rotating about a fixed axis through a point O, Eq. (2.4) can also be expressed as

$$T = \frac{1}{2}I_O\omega^2 \tag{2.5}$$

2.2.3 Kinetic Energy of a Continuous System

In a continuous system the inertial property is spatially distributed within the system. To develop mathematical models of such systems, expressions for kinetic energies of elastic or deformable bodies undergoing different displacements are required. Expressions for kinetic energy for common deformable bodies such as bar, beam and shaft are derived below.

(a) **Kinetic Energy of a Bar Undergoing Longitudinal Deformation**

Consider a bar shown in Fig. 2.2a undergoing longitudinal deformation due to vibration. It has a length of L, density of ρ and modulus of elasticity of E. Longitudinal displacement of any point of the bar at a distance x is defined as $u(x, t)$.

Mass of a small element of the bar shown in Fig. 2.2b is given by

$$\mathrm{d}m = \rho A \mathrm{d}x \tag{2.6}$$

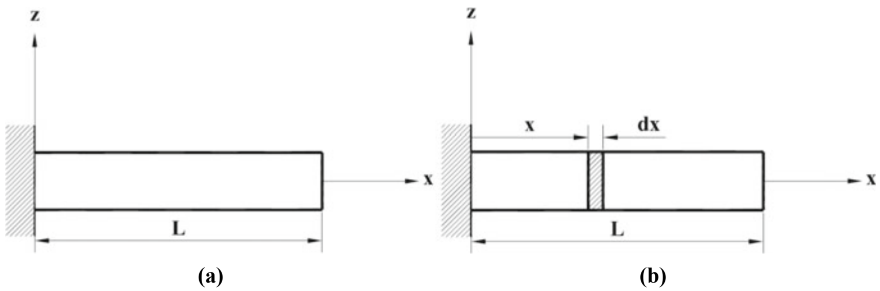


Fig. 2.2 Bar undergoing longitudinal deformation

Kinetic energy of this element is then given by

$$dT = \frac{1}{2}dm\left(\frac{\partial u}{\partial t}\right)^2 = \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{2.7}$$

Then total kinetic energy of the bar can be expressed as

$$T = \int_0^L \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{2.8}$$

If the bar is homogeneous and has a uniform cross-section, Eq. (2.8) can also be expressed as

$$T = \frac{1}{2}\rho A\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \tag{2.9}$$

(b) **Kinetic Energy of a Beam Undergoing Transverse Deflection**

Consider a beam shown in Fig. 2.3 undergoing transverse deformation due to vibration. It has a length of L, density of ρ and modulus of elasticity of E. Transverse displacement of any point of the beam at a distance x due to transverse loading along z-axis creating bending about y-axis is defined as $w(x, t)$.

By following similar procedure explained for the bar, kinetic energy of a beam due to transverse deflections along z-axis can be expressed, respectively, as

$$T = \int_0^L \frac{1}{2}\rho A\left(\frac{\partial w}{\partial t}\right)^2 dx \tag{2.10}$$

Similarly, the kinetic energy of a beam due to transverse deflections along y-axis can be expressed, respectively, as

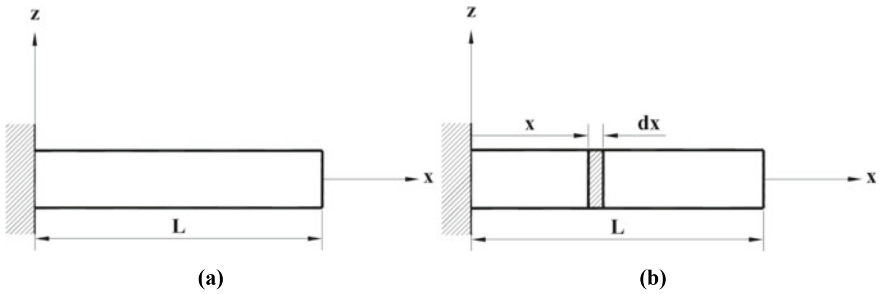


Fig. 2.3 Beam undergoing transverse deformation

$$T = \int_0^L \frac{1}{2}\rho A\left(\frac{\partial v}{\partial t}\right)^2 \mathrm{d}x \tag{2.11}$$

If the beam is homogeneous and has a uniform cross-section, Eq. (2.10) and (2.11) can also be expressed as

$$T = \frac{1}{2}\rho A \int_0^L \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{2.12}$$

$$T = \frac{1}{2}\rho A \int_0^L \left(\frac{\partial v}{\partial t}\right)^2 \mathrm{d}x \tag{2.13}$$

(c) **Kinetic Energy of a Shaft Undergoing Torsional Deformation**

Consider a shaft shown in Fig. 2.4 undergoing angular deformation due to torsional vibration. It has a length of L, density of ρ and shear modulus of elasticity of G. Consider an enlarged cross-section of the shaft shown in Fig. 2.4b, where a point P which is initially at an angular position of ϕ is displaced by θ due to torsional vibration.

Deflection of any point P of the shaft along y-axis due to torsional vibration is given by

$$\begin{aligned} v &= OP' \cos(\phi + \theta) - OP\cos\phi \\ &= OP(\cos\theta\cos\phi - \sin\theta\sin\phi) - OP\cos\phi \end{aligned}$$

Assuming small torsional deformation

$$v = OP\cos\phi - \theta(OP\sin\phi) - OP\cos\phi = -\theta(OP\sin\phi)$$

$$\therefore \quad v = -\theta z \tag{2.14}$$

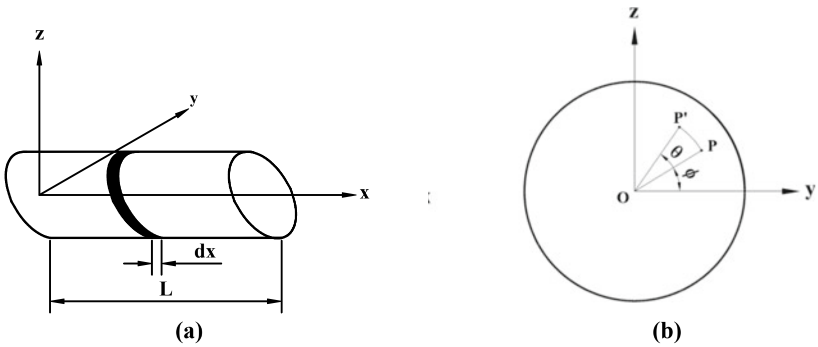


Fig. 2.4 Shaft undergoing torsional deformation

Deflection of any point P of the shaft along z-axis due to torsional vibration is given by

$$\begin{aligned} w &= OP' \sin(\phi + \theta) - OP \sin\phi \\ &= OP(\sin\theta\cos\phi + \cos\theta\sin\phi) - OP\sin\phi \end{aligned}$$

Assuming small torsional deformation

$$\begin{aligned} & w = \theta(OP\cos\phi) + OP\sin\phi - OP\sin\phi = \theta(OP\cos\phi) \\ \therefore \quad & w = \theta y \end{aligned} \tag{2.15}$$

Mass of a small element of the shaft shown in Fig. 2.4a is given by

$$\mathrm{d}m = \rho A \mathrm{d}x \tag{2.16}$$

Kinetic energy of this element is then given by

$$\begin{aligned} \mathrm{d}T &= \frac{1}{2}\mathrm{d}m\left[\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \\ &= \frac{1}{2}\rho A\left[\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x \\ &= \frac{1}{2}\rho A\left(y^2 + z^2\right)\left(\frac{\partial \theta}{\partial t}\right)^2\mathrm{d}x \\ \therefore \quad \mathrm{d}T &= \frac{1}{2}\rho J\left(\frac{\partial \theta}{\partial t}\right)^2\mathrm{d}x \end{aligned} \tag{2.17}$$

where J is the polar moment of area of section of the shaft.

Then total kinetic energy of the shaft can be expressed as

$$T = \int_0^L \frac{1}{2}\rho J \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x \tag{2.18}$$

If the shaft is homogeneous and has a uniform cross-section, Eq. (2.18) can also be expressed as

$$T = \frac{1}{2}\rho J \int_0^L \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x \tag{2.19}$$

2.3 Stiffness Elements and Potential Energy

Elastic behavior of the component of the vibrating system is modeled by a stiffness element and generally represented by a spring with stiffness k. Stiffness of a spring can be defined as the force required to produce unit relative elastic displacement across its two ends. Although spring itself is a distributed system, it can be assumed as massless, if the inertia effect of the spring is negligible in comparison with other inertia elements of the system. Stiffness element stores energy in the form of potential energy or strain energy.

2.3.1 Potential Energy Stored by a Spring

A spring is said to be a linear spring if it follows the force-displacement relationship

$$F = kx \tag{2.20}$$

Potential energy of a spring subject to a linear elastic deformation of x is then given by

$$V = W_{12} = \int_0^x F \mathrm{d}x = \int_0^x kx \, \mathrm{d}x$$

$$\therefore \quad V = \frac{1}{2}kx^2 \tag{2.21}$$

Similarly, potential energy of a torsional spring with torsional stiffness k_t of subject to an elastic deformation of θ is given by

$$\therefore \quad V = \frac{1}{2} k_t \theta^2 \tag{2.22}$$

2.3.2 Potential Energy or Strain Energy Stored by a Continuous System

Expressions for potential energy for common deformable bodies such as bar, beam and shaft are derived below.

(a) **Potential Energy of a Bar Undergoing Longitudinal Deformation**

Strain energy per unit volume of a bar due to longitudinal deformation is given by

$$v_e = \frac{1}{2} \sigma \varepsilon \tag{2.23}$$

If the deformation of the bar is within the elastic range, Hooke's law can be applied as

$$\sigma = E\varepsilon \tag{2.24}$$

Substituting Eq. (2.24) into Eq. (2.23), we get an expression for strain energy per unit volume as

$$v_e = \frac{1}{2} E \varepsilon^2 \tag{2.25}$$

Then the strain energy of the small element of the bar shown in Fig. 2.2b can be determined as

$$\mathrm{d}V = \frac{1}{2} E \varepsilon^2 A \mathrm{d}x \tag{2.26}$$

Linear strain for the undergoing longitudinal deformation is given by

$$\varepsilon = \frac{\partial u}{\partial x} \tag{2.27}$$

Substituting Eq. (2.27) into Eq. (2.26), we get

$$\mathrm{d}V = \frac{1}{2} E \left(\frac{\partial u}{\partial x} \right)^2 A \mathrm{d}x \tag{2.28}$$

Then the total strain energy of the bar shown can be determined as

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \tag{2.29}$$

If the bar is homogeneous and has a uniform cross-section, Eq. (2.29) can also be expressed as

$$V = \frac{1}{2}\mathrm{EA}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \tag{2.30}$$

(b) **Potential Energy of a Beam Undergoing Transverse Deformation**

Consider a beam shown in Fig. 2.3 undergoing transverse vertical deformation $w(x, t)$. If σ_x is the bending stress produced, Hooke's law can be rearranged for strain as

$$\varepsilon = \frac{\sigma_x}{E} \tag{2.31}$$

Substituting Eq. (2.31) into Eq. (2.23), we get an expression for strain energy per unit volume in terms of bending stress as

$$v_e = \frac{\sigma_x^2}{2E} \tag{2.32}$$

For pure bending, bending stress can be expressed in terms of bending moment M as

$$\sigma_x = \frac{\mathrm{Mz}}{I_y} \tag{2.33}$$

Further, bending moment can be related to transverse displacement as

$$M = \mathrm{EI}_y\frac{\partial^2 w}{\partial x^2} \tag{2.34}$$

Substituting Eq. (2.34) into Eq. (2.33), we get an expression for bending stress as

$$\sigma_x = \mathrm{Ez}\frac{\partial^2 w}{\partial x^2} \tag{2.35}$$

Substituting Eq. (2.35) into Eq. (2.32), we get an expression for strain energy per unit volume in terms of transverse displacement $w(x, t)$ as

$$v_e = \frac{1}{2}\mathrm{Ez}^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \tag{2.36}$$

Then the strain energy of the small element of the beam can be determined as

$$\mathrm{d}V = \frac{1}{2}\mathrm{E}z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}A\mathrm{d}x \tag{2.37}$$

Then the total strain energy of the beam can be determined as

$$V = \frac{1}{2}\int_0^L \mathrm{E}z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}A\mathrm{d}x = \frac{1}{2}\int_0^L \mathrm{EI}_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \tag{2.38}$$

Similarly, the total strain energy of the beam due to transverse deflection along y-axis can be determined as

$$V = \frac{1}{2}\int_0^L \mathrm{EI}_z\left(\frac{\partial^2 v}{\partial x^2}\right)^2 \mathrm{d}x \tag{2.39}$$

If the beam is homogeneous and has a uniform cross-section, Eqs. (2.38) and (2.39) can also be expressed as

$$V = \frac{1}{2}\mathrm{EI}_y\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \tag{2.40}$$

$$V = \frac{1}{2}\mathrm{EI}_z\int_0^L \left(\frac{\partial^2 v}{\partial x^2}\right)^2 \mathrm{d}x \tag{2.41}$$

(c) **Potential Energy of a Shaft Undergoing Torsional Deformation**

Referring to Eqs. (2.14) and (2.15), displacement field of any point of a shaft undergoing torsional deformation can be defined as

$$u = 0;\ v = -\theta z;\ w = \theta y \tag{2.42}$$

Then nonzero strain components are determined as

$$\gamma_{xy} = -\frac{1}{2}z\frac{\partial\theta}{\partial x} \tag{2.43}$$

$$\gamma_{xz} = \frac{1}{2}y\frac{\partial\theta}{\partial x} \tag{2.44}$$

Similarly, nonzero stress components are determined as

$$\tau_{xy} = -Gz\frac{\partial\theta}{\partial x} \tag{2.45}$$

$$\tau_{xz} = Gy\frac{\partial\theta}{\partial x} \tag{2.46}$$

Then strain energy per unit volume of a shaft due to torsional deformation is given by

$$v_e = \frac{1}{2}\left(2\tau_{xy}\gamma_{xy} + 2\tau_{xz}\gamma_{xz}\right) = \frac{1}{2}G\left(y^2 + z^2\right)\left(\frac{\partial\theta}{\partial x}\right)^2 \tag{2.47}$$

Then the strain energy of the small element of the shaft can be determined as

$$\mathrm{d}V = \frac{1}{2}G\left(y^2 + z^2\right)\left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}A\mathrm{d}x \tag{2.48}$$

Then the total strain energy of the shaft can be determined as

$$V = \frac{1}{2}\int_0^L G\left(y^2 + z^2\right)\left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}A\mathrm{d}x = \frac{1}{2}\int_0^L \mathrm{GJ}\left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x \tag{2.49}$$

If the shaft is homogeneous and has a uniform cross-section, Eq. (2.49) can also be expressed as

$$V = \frac{1}{2}\mathrm{GJ}\int_0^L \left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x \tag{2.50}$$

2.3.3 Equivalent System and Equivalent Stiffness for Different Combinations of Springs

If a system consists of a number of springs are arranged in different way, then its potential energy can be determined by using equivalent stiffness for the given combination of springs. Methods to determine equivalent stiffness for series and parallel combination of springs are explained below.

(a) **Equivalent Stiffness for a Series Combination of Springs**

Consider n numbers of spring with stiffness values of $k_1, k_2, k_3, \ldots, k_n$ connected in series as in Fig. 2.5a. It can be replaced by an equivalent system shown in Fig. 2.5b consisting of a single spring with stiffness k_{eq} such that total displacement of the system is equal to sum of displacements in each spring, i.e.,

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \cdots + \Delta_n$$

Since the force applied to each spring is same for series combination,

$$\frac{F}{k_{\mathrm{eq}}} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3} + \cdots + \frac{F}{k_n}$$
$$\therefore \quad \frac{1}{k_{\mathrm{eq}}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots + \frac{1}{k_n} \tag{2.51}$$

(b) **Equivalent Stiffness for a Parallel Combination of Springs**

Consider n numbers of spring with stiffness values of k_1, k_2, k_3, …, k_n connected in parallel as in Fig. 2.6a. It can be replaced by an equivalent system shown in Fig. 2.6b consisting of a single spring with stiffness k_{eq} such that total force applied to the system is equal to sum of force exerted in each spring, i.e.,

$$F = F_1 + F_2 + F_3 + \cdots + F_n$$

Since the deformation in each spring is same for parallel combination,

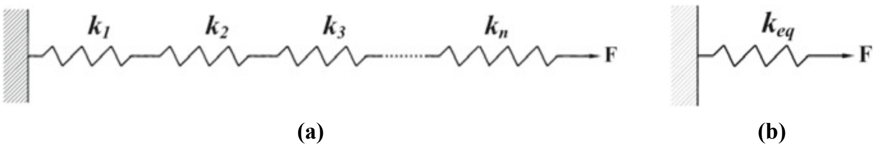


Fig. 2.5 Series combination of springs and equivalent system

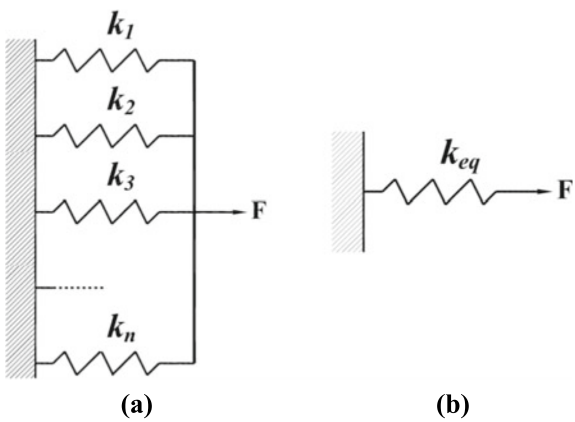


Fig. 2.6 Parallel combination of springs and equivalent system

$$k_{\text{eq}}\Delta = (k_1 + k_2 + k_3 + \cdots + k_n)\Delta$$
$$\therefore \quad k_{\text{eq}} = k_1 + k_2 + k_3 + \cdots + k_n \tag{2.52}$$

2.3.4 Equivalent System and Equivalent Stiffness for Continuous System with Negligible Weight

To carry vibration analysis, sometimes it becomes convenient to determine the equivalent stiffness of a system such that it can be converted into an equivalent single degree of freedom system. To determine an equivalent stiffness of the system, displacement of the system at a point where the concentrated mass or rigid body is attached should be determined. This method will give satisfactory result when the weight of the continuous system such as bar, beam or shaft is negligible in comparison with that of the rigid body attached to it.

(a) **Equivalent system for a Bar with a Concentrated Mass**

Consider a bar which is fixed at the upper end and a concentrated mass M attached at its lower end as shown in Fig. 2.7a. The cross-sectional area of the bar is A, length is L, and the modulus of elasticity of the material of the bar is E. This system can be analyzed by considering its equivalent system with system parameters k_{eq} and M shown in Fig. 2.7b.

To determine the equivalent stiffness of the system, displacement of the lower end of bar where the mass M is attached should be determined. Displacement at free end of a uniform bar subjected to an axial load F is given by

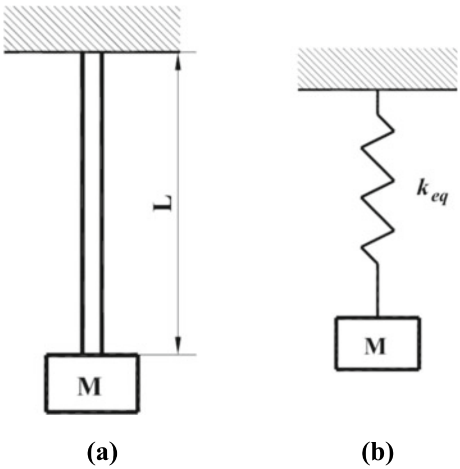


Fig. 2.7 Equivalent system for a concentrated mass attached to a bar

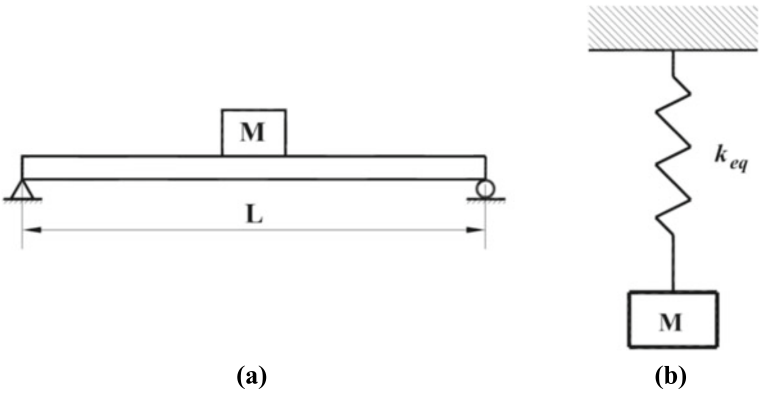


Fig. 2.8 Equivalent system for a simply supported beam with a concentrated mass at its mid-span

$$\delta = \frac{\mathrm{FL}}{\mathrm{AE}} \tag{2.53}$$

Then equivalent stiffness of the system is given by

$$k_{\mathrm{eq}} = \frac{F}{\delta} = \frac{\mathrm{AE}}{L} \tag{2.54}$$

(b) **Equivalent system for a Beam with a Concentrated Mass**

Consider a simply supported beam with a concentrated mass M attached at its mid-span as shown in Fig. 2.8a. The moment of area of the cross-section of the beam is I, length is L, and the modulus of elasticity of the material of the bar is E. This system can be analyzed by considering its equivalent system with system parameters k_{eq} and M shown in Fig. 2.7b.

To determine the equivalent stiffness of the system, displacement at the mid-span of the beam where the mass M is attached should be determined. Displacement at the mid-span of a simply supported beam subjected to a concentrated load F is given by

$$\delta = \frac{\mathrm{FL}^3}{48\mathrm{EI}} \tag{2.55}$$

Then equivalent stiffness of the system is given by

$$k_{\mathrm{eq}} = \frac{F}{\delta} = \frac{48\mathrm{EI}}{L^3} \tag{2.56}$$

By following similar procedure, equivalent stiffness for a beam fixed at both ends with a concentrated mass M at its mid-span as shown in Fig. 2.9 can be determined as

$$k_{\mathrm{eq}} = \frac{192\mathrm{EI}}{L^3} \tag{2.57}$$

Similarly, equivalent stiffness for a cantilever beam with a concentrated mass M at its free end as shown in Fig. 2.10 can be determined as

$$k_{\mathrm{eq}} = \frac{3\mathrm{EI}}{L^3} \tag{2.58}$$

(c) Equivalent system for a Shaft with a Rigid Disk at its Free End

Consider a shaft which is fixed at the left end and a rigid disk of mass moment of inertia I attached at its free end as shown in Fig. 2.11a. The polar moment of inertia of its section is J, length of the shaft is L, and the shear modulus of elasticity of the material of the bar is G. This system can be analyzed by considering its equivalent system with system parameters k_{eq} and I shown in Fig. 2.11b.

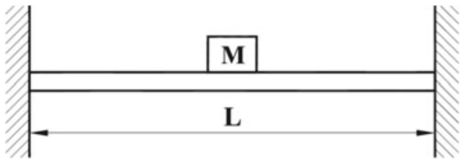


Fig. 2.9 Fixed beam with a concentrated mass at its mid-span

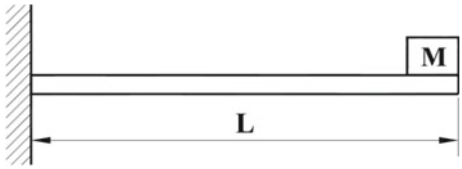


Fig. 2.10 Cantilever beam with a concentrated mass at its free end

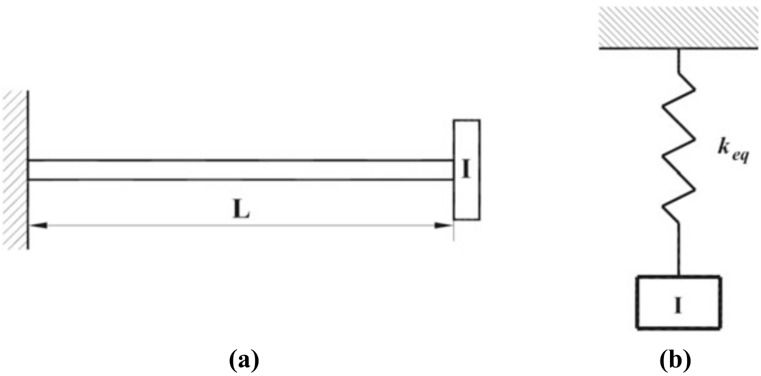


Fig. 2.11 Equivalent system for a shaft with a rigid disk at its free end

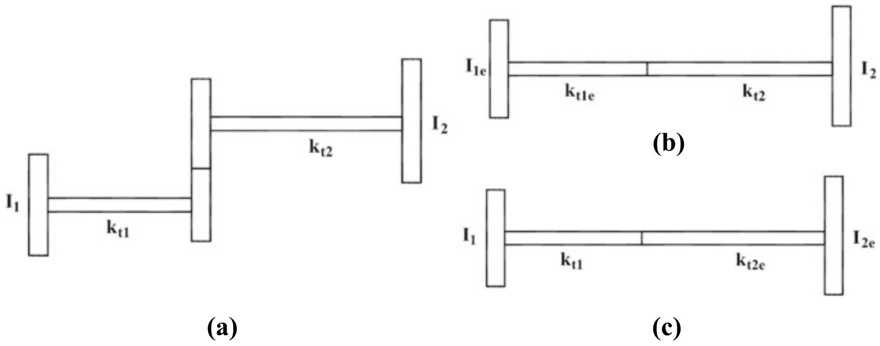


Fig. 2.12 Geared system and its equivalent models

To determine the equivalent stiffness of the system, angular displacement of the freer end of the shaft where the disk is attached should be determined. Angular displacement at free end of a shaft subjected to a torque T is given by

$$\theta = \frac{\mathrm{TL}}{\mathrm{GJ}} \tag{2.59}$$

Then equivalent stiffness of the system is given by

$$k_{\mathrm{eq}} = \frac{T}{\theta} = \frac{\mathrm{GJ}}{L} \tag{2.60}$$

(d) **Equivalent system for a Geared System**

Consider a power transmission system shown in Fig. 2.12a, in which a disk-shaft system $(I_1 - k_{t1})$ is driving another disk-shaft system $(I_2 - k_{t2})$ through a geared system having negligible inertia effect with a gear ratio of $n\left(= \dot{\theta}_2/\dot{\theta}_1\right)$.

Analysis of such system can be carried out by using an equivalent system shown in Fig. 2.12b, if the angular vibration amplitude of driven disk θ_2 as a generalized coordinate. Equivalent parameters ($I_{1e}$and k_{t1e}) for this can be determined as follows:

$$T_1 = \frac{1}{2} I_1 \dot{\theta}_1^2 = \frac{1}{2}\left(\frac{I_1}{n^2}\right)\dot{\theta}_2^2 \quad \therefore I_{1e} = \frac{I_1}{n^2} \tag{2.61}$$

$$V_1 = \frac{1}{2} k_{t1} \theta_1^2 = \frac{1}{2}\left(\frac{k_{t1}}{n^2}\right)\theta_2^2 \quad \therefore k_{1e} = \frac{k_{t1}}{n^2} \tag{2.62}$$

Analysis of the same system can also be carried out by using an equivalent system shown in Fig. 2.12c, if the angular vibration amplitude of driving disk θ_1 as a generalized coordinate. Equivalent parameters (I_{2e}andk_{t2e}) for this can be determined as follows:

$$T_2 = \frac{1}{2} I_2 \dot{\theta}_2^2 = \frac{1}{2}\left(n^2 I_2\right)\dot{\theta}_1^2 \quad \therefore I_{2e} = n^2 I_2 \tag{2.63}$$

$$V_2 = \frac{1}{2} k_{t2} \theta_2^2 = \frac{1}{2}\left(n^2 k_{t1}\right)\theta_1^2 \quad \therefore k_{2e} = n^2 k_{t1} \tag{2.64}$$

2.4 Damper and Energy Dissipation

In the absence of external force, vibration of any real system usually dies out after some interval. This is due to the resistive effect imposed upon the system. Such resistive effect through which energy of the system is dissipated such that it comes to equilibrium position after some interval is called damping.

2.4.1 Types of Damping

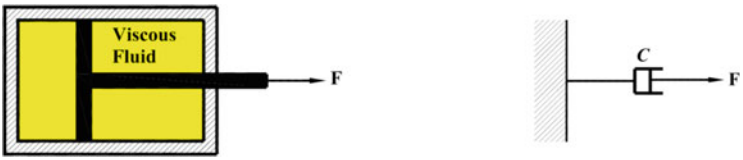
Damping providing resistance to most of the vibrating system can be classified into three types: viscous damping, Coulomb damping and structural damping.

(a) **Viscous Damping**

In viscous damper, resistance to vibration motion is provided by the viscosity of the fluid. It is a most commonly used model for the damping effect. Systems such as hydraulic dashpots and shocks absorbers can be effectively modeled by this type of damping. Simple mechanism of a viscous damper is shown in Fig. 2.13a, and its schematic representation is shown in Fig. 2.13b.

Due to damping, there exists a velocity difference between two ends of the damper. A damper is said to be a linear damper, if the damping force is proportional to the velocity of the particle to which it is attached. The force developed in the linear viscous damper is given by

$$F = c\dot{x} \tag{2.65}$$



(a) Mechanism of a Viscous Damper **(b) Schematic Representation of a Damper**

Fig. 2.13 Viscous damper

(b) **Coulomb Damping**

Resistance provided to the vibratory motion due dry friction between two surfaces is called Coulomb damping. If the coefficient of friction between the block and the surface shown in Fig. 2.14a is μ, then the magnitude of frictional force resisting the motion is given by

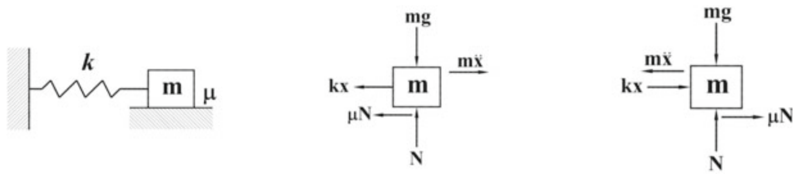
$$F = \mu N \tag{2.66}$$

where N is the normal component of the reaction force.

The frictional force acts toward left when the block moves toward right and acts toward right when the block moves toward left as shown in Fig. 2.15b.

To take the account of direction of the frictional force, Eq. (2.66) can be redefined as

$$F = -\mu N \frac{|\dot{x}|}{\dot{x}} \tag{2.67}$$



(a) Block Sliding on a Rough Surface **(b) Free body Diagrams showing Direction of Frictional Force**

Fig. 2.14 Coulomb damping

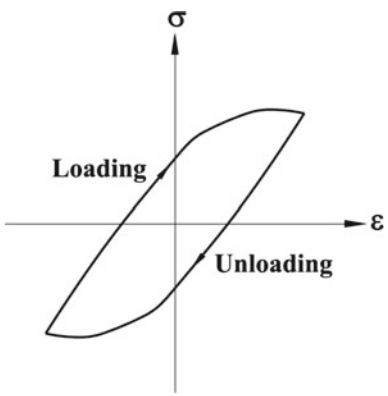


Fig. 2.15 Hysteretic loop formed due to intermolecular friction

(c) **Structural Damping**

Due to to-and-fro motion during vibration, materials are repeatedly subjected to increasing and decreasing stress and strain and energy is dissipated due to intermolecular friction. If stress–strain diagram of such materials is plotted, the path will be different for loading (increasing strain) and unloading (decreasing strain) as shown in Fig. 2.15. The loop formed by these two paths is called hysteresis loop, and area enclosed by the loop is equal to the energy lost during a cycle. This type of energy dissipation is called structural damping, hysteretic damping or solid damping.

For most of the common materials such as steel and aluminum, the energy dissipated is independent of frequency and proportional to the square of the amplitude of vibration and stiffness, i.e.,

$$W_d = \pi k \beta X^2 \tag{2.68}$$

where is k the stiffness, X is the amplitude of vibration, π is any convenient proportionality constant, and β is called the hysteretic damping constant.

2.4.2 *Energy Dissipation Due to Damping*

The damping force opposes the direction of motion. Energy dissipated due to damping is equal to the work done by the damping force and can be determined as

$$W_d = \int_{x_1}^{x_2} F\,\mathrm{d}x = -\int_{x_1}^{x_2} c\dot{x}\,\mathrm{d}x = -\int_{x_1}^{x_2} c\dot{x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = -\int_{t_1}^{t_2} c\dot{x}^2\mathrm{d}t \tag{2.69}$$

2.5 External Excitation

External sources can provide energy to a vibrating system through motion or force/moment as shown in Figs. 2.16 and 2.17.

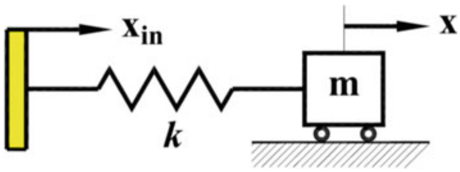


Fig. 2.16 Energy input through motion

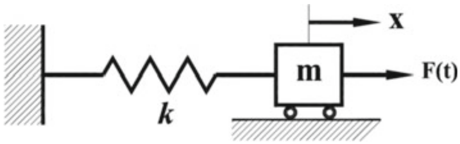


Fig. 2.17 Energy input through force

Energy provided to the system through cam-follower mechanism and disturbance experienced by a machine due to floor vibration are the examples of motion input. The magnitude of energy provided to the system shown in Fig. 2.16 through input motion is given by

$$E_{\text{input}} = \frac{1}{2}\mathrm{k}\mathrm{x}_{\text{in}}^2 \tag{2.70}$$

Energy provided to the system due to unbalance and wind blow on a structure are the examples of force input. Force or moment exerted on a discrete system is dependent upon time and is defined by $F(t)$ as shown in Fig. 2.17. External force or moment exerted on any system may be periodic or aperiodic. A force input $F(t)$ is said to be periodic if $F(t) = F(t + T)$, where T is the period of the force. Most common form of periodic force is a harmonic force defined by

$$F(t) = F_0 \sin \omega t \tag{2.71}$$

where ω is the frequency of the input force which is related to the time period as

$$\omega = \frac{2\pi}{T} \tag{2.72}$$

The input force imposed to a system sometimes may be periodic but not harmonic such as square or triangular pulses of force as shown in Fig. 2.18. To carry out vibration response due to non-harmonic forces, these forces are converted into sum of infinite harmonic components by Fourier series expansion.

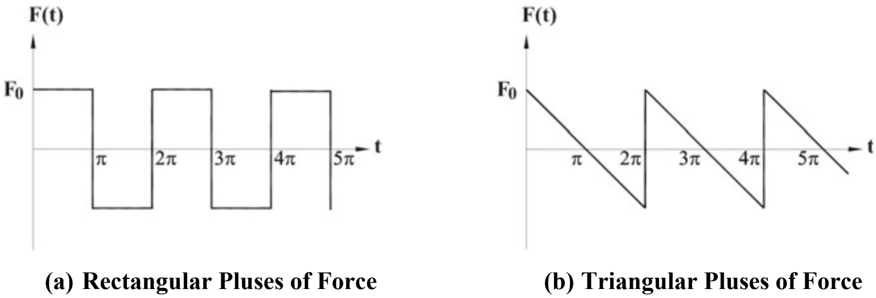
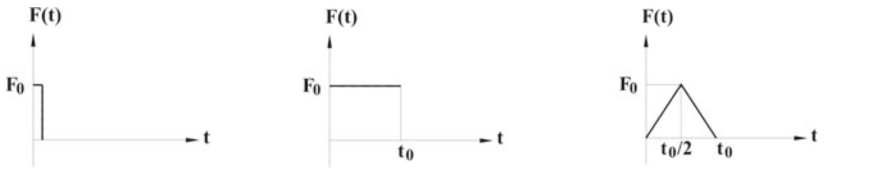


Fig. 2.18 Different forms of periodic forces



(a) Impulsive Force **(b) Truncated Step Force** **(c) Triangular Pulse Force**

Fig. 2.19 Transient forces

Input force which remains active for very short interval is called transient forces. Common examples of transient forces are impulse, truncated step function and triangular pulse as shown in Fig. 2.19a–c, respectively. These transient forces can be expressed as time-dependent force as:

(a) Impulsive force (Fig. 2.19a)

$$F(t) = F_o \delta(t) \tag{2.73}$$

(b) Truncated step function (Fig. 2.19b)

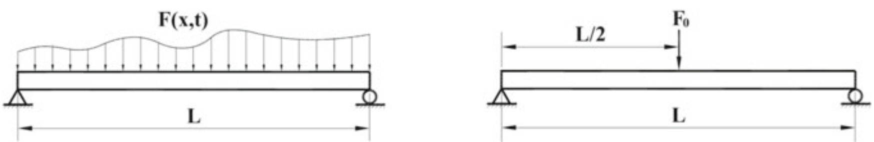
$$F(t) = F_o - F_o u(t - t_0) \tag{2.74}$$

(c) Triangular pulse (Fig. 2.19c)

$$\begin{aligned} F(t) = {} & \frac{2F_o}{t_0} t - \frac{2F_o}{t_0} u\left(t - \frac{t_0}{2}\right) + 2F_o\left(1 - \frac{t}{t_0}\right) u\left(t - \frac{t_0}{2}\right) \\ & - 2F_o\left(1 - \frac{t}{t_0}\right) u(t - t_0) \end{aligned} \tag{2.75}$$

Force or moment exerted on a continuous system is dependent upon both space and time, and for a one-dimensional continuous system it can be defined by $F(x, t)$. Few examples of such forces are distributed force shown in Fig. 2.20a and a concentrated load applied at certain specific point of the system shown in Fig. 2.20b.

A concentrated force shown in Fig. 2.20b can also be defined as a function of x and t as



(a) Distributed Force **(b) A Concentrated Force**

Fig. 2.20 Different forms of external force applied to a beam

$$F(t) = F_0(t)\delta\left(x - \frac{L}{2}\right) \tag{2.76}$$

Solved Examples

Example 2.1

A particle of mass M is attached to a spring with a stiffness of k and mass m as shown in Figure E2.1. Determine its equivalent single degree of freedom model.

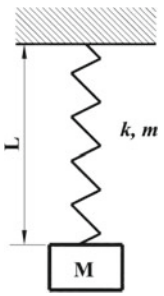


Figure E2.1

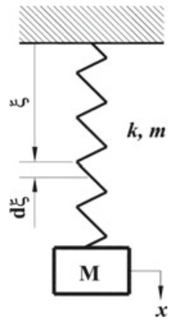


Figure E2.1(a)

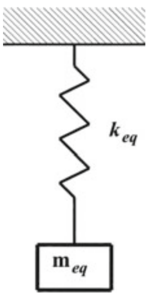


Figure E2.1(b)

Solution

Assume that displacement of the particle M, i.e., the lower end of the spring is x. Referring **Figure E2.1(a)**, displacement of a point in the spring at a distance of ξ from the upper end is given by

$$x_\xi = \frac{x}{L}\xi$$

Similarly, mass of small length $\mathrm{d}\xi$ of the spring is given by

$$\mathrm{d}m = \frac{m}{L}\mathrm{d}\xi$$

Then kinetic energy of small length $\mathrm{d}\xi$ of the spring is given by

$$\mathrm{d}T_s = \frac{1}{2}\mathrm{d}m\left(\dot{x}_\xi\right)^2 = \frac{1}{2}\left(\frac{m}{L}\mathrm{d}\xi\right)\left(\frac{\dot{x}}{L}\xi\right)^2 = \frac{1}{2}\frac{m}{L^3}\dot{x}^2\xi^2\mathrm{d}\xi$$

Total kinetic energy of the spring is then given by

$$T_s = \frac{1}{2}\frac{m}{L^3}\dot{x}^2 \int_0^L \xi^2 \mathrm{d}\xi = \frac{1}{2}\left(\frac{m}{3}\right)\dot{x}^2$$

Then total kinetic energy of the system including inertia effect of both the spring and the particle is given by

$$T = \frac{1}{2}\left(\frac{m}{3}\right)\dot{x}^2 + \frac{1}{2}(M)\dot{x}^2 = \frac{1}{2}\left(M + \frac{m}{3}\right)\dot{x}^2$$

Therefore, equivalent of mass of the spring-mass system with inertia effect of spring is

$$m_{\mathrm{eq}} = M + \frac{m}{3}$$

Since, the inertia effect of spring does not affect its stiffness, equivalent stiffness of the system is same as the stiffness of the spring, i.e.,

$$k_{\mathrm{eq}} = k$$

Hence the equivalent single degree of freedom model of the given system is shown in **Fig. E2.1 (b)**, where $\boldsymbol{k_{eq}} = \boldsymbol{k}$ and $\boldsymbol{m_{eq}} = \boldsymbol{M} + \boldsymbol{m}/3$.

Example 2.2

Determine $\boldsymbol{m_{eq}}$ and $\boldsymbol{k_{eq}}$ for the system of Figure E2.2 Use downward displacement of the block from the static equilibrium position as the generalized coordinate.

Solution

Given system consists of two inertia elements and two stiffness elements. The inertia elements are block with mass m_1 and a pulley with mass moment of inertia I_P, and the stiffness elements are springs with stiffness values of k_1 and k_2. Displacements of two inertia elements are kinematically dependent; therefore it has a one degree of freedom.

When the block is displaced downward by x, clockwise rotation of the pulley will be x/r and compression in the spring with stiffness k_1 will be x and that in the spring with stiffness k_2 will be $2x$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\left(m_1 + \frac{I_P}{r^2}\right)\dot{x}^2$$

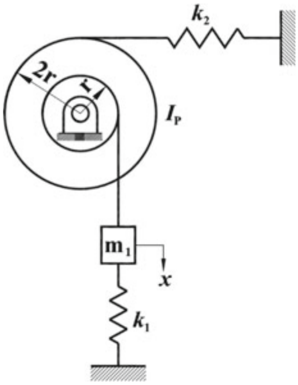


Figure E2.2

Therefore, equivalent mass of the system can be defined as

$$m_{\text{eq}} = m_1 + \frac{I_P}{r^2}$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2(2x)^2 = \frac{1}{2}(k_1 + 4k_2)x^2$$

Therefore, equivalent mass of the system can be defined as

$$k_{\text{eq}} = k_1 + 4k_2$$

Example 2.3

Determine m_{eq} and k_{eq} for the system of Figure E2.3. Use downward displacement of the block of mass $2m$ measured from static equilibrium (x) as the generalized coordinate. Assume the disk is thin and rolls without slip.

Solution

Given system consists of three inertia elements and two stiffness elements. The inertia elements are block with mass $2m$ and a pulley with mass moment of inertia I_P and a thin disk of mass m_d; the stiffness elements are springs with stiffness values of $2k$ and k. Displacements of three inertia elements are kinematically dependent; therefore it has a one degree of freedom.

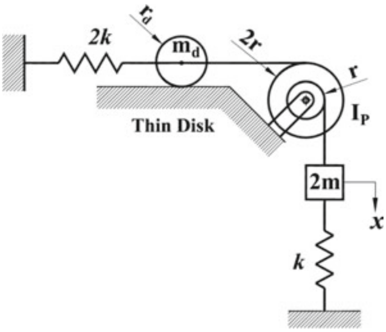


Figure E2.3

When the block is displaced downward by x, clockwise rotation of the pulley will be x/r, linear displacement of the center of the disk is $2x$, compression in the spring with stiffness k will be x, and that in the spring with stiffness $2k$ will be $2x$.

Total kinetic energy of the system can be determined as

$$\begin{aligned} T &= \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m_d(2\dot{x})^2 + \frac{1}{2}I_d\left(\frac{2\dot{x}}{r_d}\right)^2 \\ &= \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m_d(2\dot{x})^2 + \frac{1}{2}\left(\frac{1}{2}m_d r_d^2\right)\left(\frac{2\dot{x}}{r_d}\right)^2 \\ &= \frac{1}{2}\left(2m + 6m_d + \frac{I_P}{r^2}\right)\dot{x}^2 \end{aligned}$$

Therefore, equivalent mass of the system can be defined as

$$m_{\text{eq}} = 2m + 6m_d + \frac{I_P}{r^2}$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}kx^2 + \frac{1}{2}(2k)(2x)^2 = \frac{1}{2}(9k)x^2$$

Therefore, equivalent mass of the system can be defined as

$$k_{\text{teq}} = 9k$$

Example 2.4

Determine the kinetic energy and potential energy of the system shown in Figure E2.4 Mass of the bar is M. Use rotation of the bar (θ) as the generalized coordinate.

(a) **Assume mass of the springs negligible.**
(b) **Assume each spring has a mass of m.**

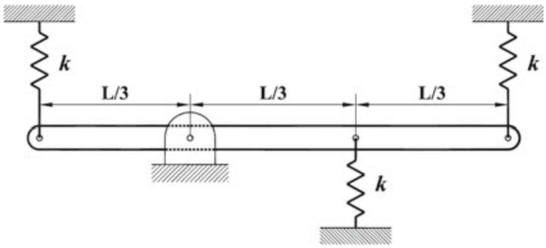


Figure E2.4

Solution

(a)

Given system consists of one inertia elements and three stiffness elements. The inertia element is a bar with mass M, and the stiffness elements are springs each with stiffness values of k. It has a one degree of freedom.

When the bar is rotated in clockwise direction θ, compression in the spring connected to the left end of the bar will be $(L\theta)/3$, compression in the spring connected below the bar will be $(L\theta)/3$, and elongation in the spring connected to the left end of the bar will be $(2L\theta)/3$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}\left[\frac{1}{12}ML^2 + M\left(\frac{L}{6}\right)^2\right]\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}ML^2\right)\dot{\theta}^2$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}k\left(\frac{L}{3}\theta\right)^2 + \frac{1}{2}k\left(\frac{L}{3}\theta\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta\right)^2 = \frac{1}{2}\left(\frac{2}{3}kL^2\right)\theta^2$$

(b)

Equivalent model for the given system considering the inertia effects will be as shown in **Figure E2.4(a)**.

Then, total kinetic energy of the equivalent system can be determined as

$$T = \frac{1}{2}I_0\dot{\theta}^2 + \frac{1}{2}\frac{m}{3}\left(\frac{L}{3}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m}{3}\left(\frac{L}{3}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m}{3}\left(\frac{2L}{3}\dot{\theta}\right)^2$$
$$= \frac{1}{2}\left(\frac{1}{9}ML^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{2}{9}mL^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}ML^2 + \frac{2}{9}mL^2\right)\dot{\theta}^2$$

Since, the inertia effects of springs do not affect potential energy of the system, equivalent stiffness of the system will be same as the stiffness determine for case **(a)**.

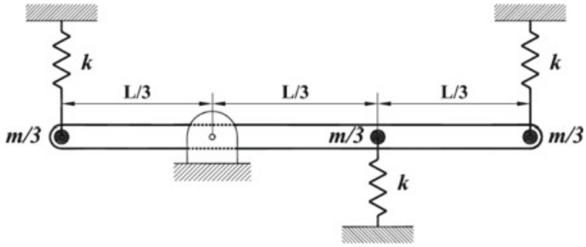


Figure E2.4(a)

Example 2.5

Determine the parameters for an equivalent systems model for the system of Figure E2.5.

(a) **Use the downward displacement of the block from the static equilibrium position x, as the generalized coordinate.**
(b) **Use the counterclockwise angular displacement of the disk measured from the static equilibrium position θ, as the generalized coordinate.**

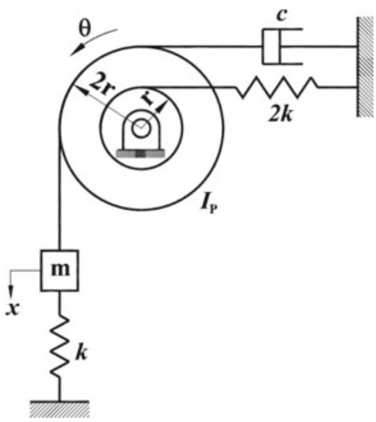


Figure E2.5

Solution

(a)

A given system consists of two inertia elements, two stiffness elements and one damping element. The inertia elements are a particle with mass m and a pulley with mass moment of inertia I_P, and the stiffness elements are springs with stiffness values of $2k$ and k. Displacements of two inertia elements are kinematically dependent; therefore it has a one degree of freedom.

When the particle of mass m is displaced downward by x, rotation of the pulley will be $x/2r$, compression in the spring with stiffness k will be x, elongation in the spring with stiffness $2k$ will be $x/2$, and the velocity of left end of the damper will be $\dot{x}/2r$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}^2$$

Therefore, equivalent mass of the system can be defined as

$$m_{\text{eq}} = m + \frac{I_P}{4r^2}$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}kx^2 + \frac{1}{2}(2k)\left(\frac{x}{2}\right)^2 = \frac{1}{2}\left(\frac{3k}{2}\right)x^2$$

Therefore, equivalent mass of the system can be defined as

$$k_{\text{eq}} = \frac{3k}{2}$$

Work done against the damping is given by

$$W_d = \int -c\dot{x}dx$$

Therefore, equivalent damping of the system can be defined as

$$c_{\text{eq}} = c$$

(b)

When the counterclockwise angular rotation of the pulley is θ, the particle of mass m is displaced downward by $2r\theta$, compression in the spring with stiffness k will be $2r\theta$, elongation in the spring with stiffness $2k$ will be $r\theta$, and the velocity of left end of the damper will be $2r\dot{\theta}$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\left(2r\dot{\theta}\right)^2 + \frac{1}{2}I_P\dot{\theta}^2 = \frac{1}{2}\left(I_P + 4mr^2\right)\dot{\theta}^2$$

Therefore, equivalent inertia of the system can be defined as

$$I_{\text{eq}} = I_P + 4mr^2$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}k(2r\theta)^2 + \frac{1}{2}(2k)(r\theta)^2 = \frac{1}{2}\left(6kr^2\right)\theta^2$$

Therefore, equivalent mass of the system can be determined as

$$k_{teq} = 6kr^2$$

Work done against the damping is given by

$$W_d = \int -c\left(2r\dot{\theta}\right)\mathrm{d}(2r\theta) = \int -\left(4cr^2\right)\dot{\theta}\mathrm{d}\theta$$

Therefore, equivalent damping of the system can be determined as

$$c_{\text{eq}} = 4cr^2$$

Example 2.6

Determine the parameters for an equivalent systems model for the system of Figure E2.6. Mass of the bar is M. Use rotation of the bar (θ) as the generalized coordinate.

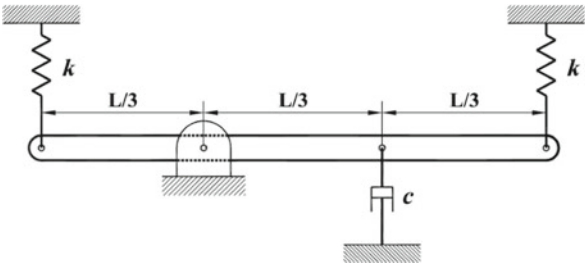


Figure E2.6

Solution

Given system consists of one inertia element and three stiffness elements. The inertia element is a bar with mass M, and the stiffness elements are springs each with stiffness values of k. It has a one degree of freedom.

When the bar is rotated in clockwise direction θ, compression in the spring connected to the left end of the bar will be $(L\theta)/3$, elongation in the spring connected to the right end of the bar will be $(2L\theta)/3$, and the velocity at the upper end of the damper is $\left(L\dot{\theta}\right)/3$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}\left[\frac{1}{12}ML^2 + M\left(\frac{L}{6}\right)^2\right]\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9}ML^2\right)\dot{\theta}^2$$

Therefore, equivalent inertia of the system can be defined as

$$I_{\mathrm{eq}} = \frac{1}{9}ML^2$$

Total potential energy of the system can be determined as

$$V = \frac{1}{2}k\left(\frac{L}{3}\theta\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta\right)^2 = \frac{1}{2}\left(\frac{5}{9}kL^2\right)\theta^2$$

Therefore, equivalent mass of the system can be determined as

$$k_{teq} = \frac{5}{9}kL^2$$

Work done against the damping is given by

$$W_d = \int -c\left(\frac{L}{3}\dot{\theta}\right)d\left(\frac{L}{3}\theta\right) = \int -\left(\frac{1}{9}cL^2\right)\dot{\theta}d\theta$$

Therefore, equivalent damping of the system can be determined as

$$c_{\mathrm{eq}} = \frac{1}{9}cL^2$$

Example 2.7

Determine equivalent inertia and equivalent stiffness of the system shown in Figure E2.7. Mass of the bar is M. Use rotation of the bar (θ) as the generalized coordinate.

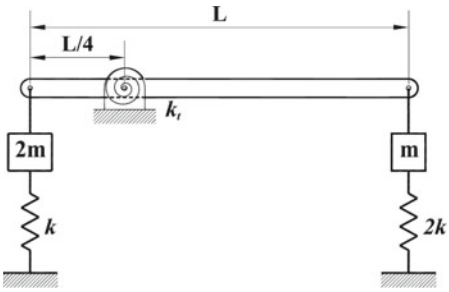


Figure E2.7

Solution

Given system consists of three inertia elements and three stiffness elements. The inertia elements are a bar with mass M and blocs with masses m and $2m$, and the stiffness elements are springs with stiffness values of k and $2k$ and a torsional spring with torsional stiffness k_t. It has a one degree of freedom.

When the bar is rotated in clockwise direction by θ, upward displacement of mass at left end will be $(L\theta)/4$ and downward displacement of mass at right end will be $(3L\theta)/4$.Similarly, compression in the spring connected to the left end of the bar will be $(L\theta)/4$, elongation in the spring connected to the right end of the bar will be $(3L\theta)/4$, and angular deflection of the torsional spring will also be θ.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}I_0\dot{\theta}^2 + \frac{1}{2}(2m)\left(\frac{L}{4}\dot{\theta}\right)^2 + \frac{1}{2}m\left(\frac{3L}{4}\dot{\theta}\right)^2$$
$$= \frac{1}{2}\left[\frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2\right]\dot{\theta}^2 + \frac{1}{2}\left[\frac{1}{8}mL^2 + \frac{9}{16}mL^2\right]\dot{\theta}^2$$
$$= \frac{1}{2}\left(\frac{7}{48}ML^2 + \frac{11}{16}mL^2\right)\dot{\theta}^2$$

Therefore, equivalent inertia of the system can be defined as

$$I_{\text{eq}} = \frac{7}{48}ML^2 + \frac{11}{16}mL^2$$

Total potential energy of the system can be determined as

$$V = \frac{1}{2}k_t\theta^2 + \frac{1}{2}k\left(\frac{L}{4}\theta\right)^2 + \frac{1}{2}(2k)\left(\frac{3L}{4}\theta\right)^2 = \frac{1}{2}\left(k_t + \frac{19}{16}kL^2\right)\theta^2$$

Therefore, equivalent mass of the system can be determined as

$$k_{teq} = k_t + \frac{19}{16}kL^2$$

Example 2.8

Determine the kinetic energy and potential energy of the system shown in Figure E2.8. Mass of the rod is negligible. Use rotation of the bar ($\boldsymbol{\theta}$) as the generalized coordinate.

Solution

Given system consists of one inertia element and one stiffness elements. The inertia element is a concentrated mass m attached at the bottom of weightless rod, and the stiffness element is a spring with stiffness k.

When the angular displacement of the rod is θ, elongation in the spring will be $a\theta$ and linear velocity of the concentrated mass m will be $L\dot{\theta}$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\left(L\dot{\theta}\right)^2 = \frac{1}{2}mL^2\dot{\theta}^2$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}k(a\theta)^2 = \frac{1}{2}ka^2\theta^2$$

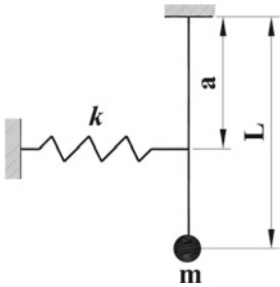


Figure E2.8

Example 2.9

Determine the kinetic energy and potential energy of the system of Figure E2.9. Use x_1 and x_2 as generalized coordinate.

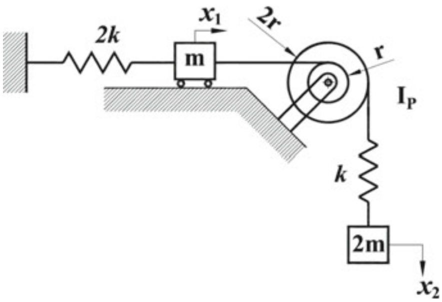


Figure E2.9

Solution

Given system consists of three inertia elements and two stiffness elements. The inertia elements are blocks with masses m and $2m$ and a pulley with mass moment of inertia I_P, and the stiffness elements are springs with stiffness values of $2k$ and k. Displacement of the block with mass m (x_1) and the angular displacement of the pulley (θ) are kinematically dependent ($\theta = x_1/r$), whereas displacement of the block with mass $2m$ (x_2) is kinematically independent with x_1 and θ; therefore, it has two degrees of freedom.

When the block with mass m is displaced by x_1, counterclockwise rotation of the pulley will be x_1/r and the elongation in the spring with stiffness $2k$ will be x_1. Similarly, when the block with mass $2m$ is displaced downward, then the elongation in the spring with stiffness k will be $x_2 - 2x_1$.

Total kinetic energy of the system can be determined as

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}I_p\left(\frac{\dot{x}_1}{r}\right)^2 + \frac{1}{2}(2m)\dot{x}_2^2 \\
&= \frac{1}{2}\left(m + \frac{I_p}{r^2}\right)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2
\end{aligned}
$$

Total potential energy of the system can be determined as

$$
V = \frac{1}{2}(2k)(x_1)^2 + \frac{1}{2}k(x_2 - 2x_1)^2
$$

Example 2.10

Determine the kinetic energy and potential energy of the systems of Figure E2.10 in terms of the specified generalized coordinates $\boldsymbol{x}$, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.

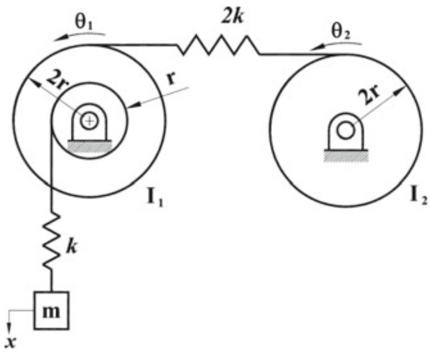


Figure E2.10

Solution

Given system consists of three inertia elements and two stiffness elements. The inertia elements are a block with mass m and two pulleys with mass moment of inertia I_1 and I_2, and the stiffness elements are springs with stiffness values of k and $2k$. Displacement of the block with mass m (x) and the angular displacements of the pulley (θ_1 and θ_2) are kinematically independent; therefore it has three degrees of freedom.

When the block with mass m is displaced by x, counterclockwise rotations of the pulleys are, respectively, θ_1 and θ_2, deformation in the spring with stiffness k will be $x - r\theta_1$ and deformation in the spring with stiffness $2k$ will be $2r(\theta_1 - \theta_2)$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_1\left(\dot{\theta}_1\right)^2 + \frac{1}{2}I_2\left(\dot{\theta}_2\right)^2$$

Total potential energy of the system can be determined as

$$V = \frac{1}{2}k(x - r\theta_1)^2 + \frac{1}{2}(2k)[2r(\theta_1 - \theta_2)]^2$$

Example 2.11

Determine the equivalent stiffness of the system of Figure E2.11.

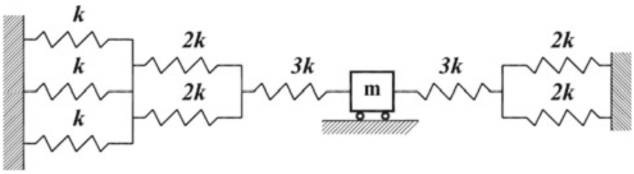


Figure E2.11

Solution

Equivalent stiffness for three parallel springs with stiffness values of k for each spring can be determined as

$$k_{e1} = k + k + k = 3k$$

Similarly, equivalent stiffness for two parallel springs with stiffness values of $2k$ for each spring can be determined as

$$k_{e2} = 2k + 2k = 4k$$

Then the given spring assembly can be reduced to an equivalent model as shown in **Figure E2.11 (a)**.

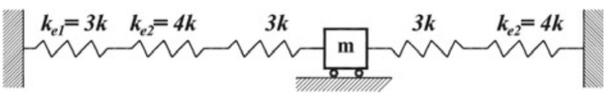


Figure E2.11 (a)

Equivalent stiffness of three series springs connected to the left side of the block is determined as

$$k_{\mathrm{el}} = \frac{k_{e1} \times k_{e2} \times 3k}{k_{e1} \times k_{e2} + k_{e2} \times 3k + k_{e1} \times 3k}$$
$$= \frac{3k \times 4k \times 3k}{3k \times 4k + 4k \times 3k + 3k \times 3k} = \frac{12}{11}k$$

Equivalent stiffness of two series springs connected to the right side of the block is determined as

$$k_{\mathrm{er}} = \frac{3k \times k_{e2}}{3k + k_{e2}} = \frac{3k \times 4k}{3k + 4k} = \frac{12}{7}k$$

Then the given spring assembly shown in **Figure E2.11 (a)** can be reduced to an equivalent model as shown in **Figure E2.11 (b)**.

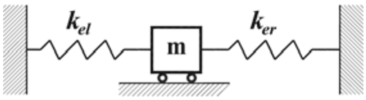


Figure E2.11 (b)

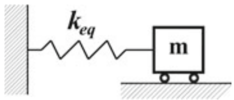


Figure E2.11 (c)

When two springs are connected to opposite sides of a block as shown in **Figure E2.11 (b)**, deflections in both springs will be same; hence this combination is taken as a parallel combination and equivalent stiffness for the equivalent model shown in **Figure E2.11(c)** is given by

$$k_{\mathrm{eq}} = k_{\mathrm{el}} + k_{\mathrm{er}} = \frac{12}{11}k + \frac{12}{7}k = \frac{216}{77}k$$

Example 2.12

A spring has a stiffness of k and an un-stretched length of L. It is divided into three equal parts each having length of $L/3$. What would be the stiffness of each part of the spring?

Solution

Any elastic element becomes more flexible when its length increases and becomes stiffer when its length decreases. Therefore, if a spring is cut into smaller pieces, stiffness of any small piece will be higher than that of the original spring.

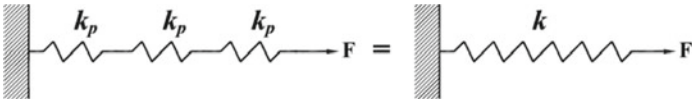


Figure E2.12 (a)

If a given spring having stiffness of k and an un-stretched length of L is divided into three equal parts, each has length of $L/3$ and stiffness of each portion is k_p. When these springs with stiffness values are connected in series as shown in **Figure 2.12 (a)**, their equivalent stiffness should be equal to k, i.e.,

$$\frac{1}{k_p} + \frac{1}{k_p} + \frac{1}{k_p} = \frac{1}{k}$$

$$\text{or, } \frac{3}{k_p} = \frac{1}{k}$$

$$\therefore k_p = 3k$$

Example 2.13

Determine the equivalent stiffness of the beam shown in Figure E2.13 at the location where the concentrated mass is placed. The cross-section of the beam is rectangular with breadth and height of 12mm and 15mm, respectively. Take $E = 210$GPa.

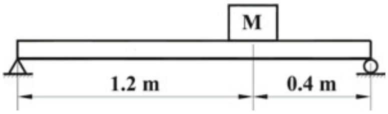


Figure E2.13

Solution

Given : Length of the beam, $L = 1.6$m
Distance of the mass from the left end, $x = 1.2$m
Breadth of the section of the beam, $b = 12$mm
Height of the section of the beam, $h = 15$mm

Then, moment of inertia of the section of the beam is determined as,

$$I = \frac{1}{12}\text{bh}^3 = \frac{1}{12}(0.012)(0.015)^3 = 3.375 \times 10^{-9}\text{m}^4$$

Deflection of a simply supported beam at a distance x from the left end due to a concentrated load F applied at that point as shown in **Figure E2.13 (b)** is given by

$$\delta = \frac{Fx^2(L - x)^2}{3\text{EIL}}$$

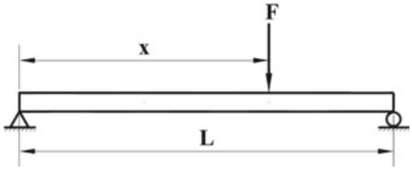


Figure E2.13 (b)

Therefore, equivalent stiffness of the beam at the location where the concentrated mass is placed is given by

$$k_{\text{eq}} = \frac{F}{\delta} = \frac{3\text{EIL}}{x^2(L - x)^2}$$
$$= \frac{3 \times 210 \times 10^9 \times 3.375 \times 10^{-9} \times 1.6}{1.2^2 \times 0.4^2} = 14.766\,\text{kN/m}$$

Example 2.14

Determine the equivalent stiffness of the system shown in Figure E2.14 using the displacement of the block as the generalized coordinate. Take $E = 200\text{GPa}$ and $I = 0.8 \times 10^{-5}\text{m}^4$ for the beam, $k_1 = 0.4\text{MN/m}$, $k_2 = 1\text{MN/m}$ and $k_3 = 0.2\text{MN/m}$.

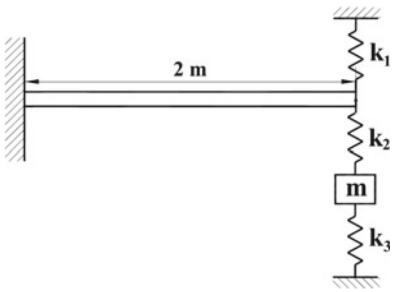


Figure E2.14

Solution

Equivalent stiffness of a cantilever beam when the concentrated mass is located at its free end is given by

$$k_b = \frac{3\mathrm{EI}}{L^3} = \frac{3 \times 200 \times 10^9 \times 0.8 \times 10^{-5}}{2^3} = 0.6\mathrm{MN/m}$$

When the free end of the beam is displaced downward by some amount, the spring with the stiffness k_1 will undergo the same amount of elongation. Hence the equivalent stiffness of the beam k_b acts in parallel with the spring with the stiffness k_1. The equivalent model of the given system will be as shown in **Figure E2.14(a)**.

Equivalent stiffness for two parallel springs with stiffness values of k_b and k_1 can be determined as

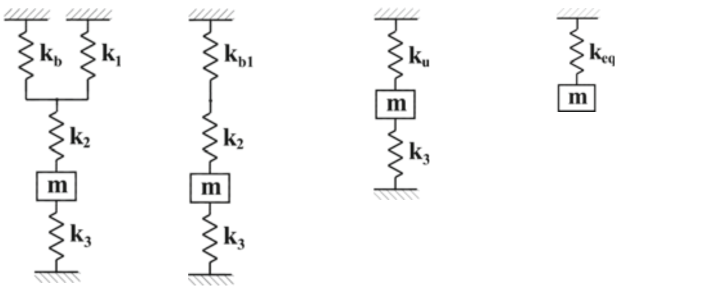


Figure E2.14(a) **Figure E2.14(b)** **Figure E2.14(c)** **Figure E2.14(d)**

$$k_{b1} = k_b + k_1 = 0.6 + 0.4 = 1\mathrm{MN/m}$$

Equivalent stiffness for two series springs with stiffness values of k_{b1} and k_2 shown in **Figure E2.14(b)** can be determined as

$$k_u = \frac{k_{b1} \times k_2}{k_{b1} + k_2} = \frac{1 \times 1}{1 + 1} = 0.5\mathrm{MN/m}$$

Equivalent stiffness of two springs with stiffness values of k_u and k_3 connected on opposite sides of the block shown in **Figure E2.14(c)** can be determined as

$$k_{\mathrm{eq}} = k_u + k_3 = 0.5 + 0.2 = 0.7\,\mathrm{MN/m}$$

Example 2.15

Determine expressions for the kinetic energy and potential of the shaft and the rigid disk attached at its free end as shown in Figure E2.15. Density of the shaft material is ρ, polar moment of inertia of its section is J_{ps}, and the mass moment of inertia of the rigid disk is I_d.

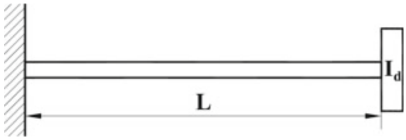


Figure E2.15

Solution

Let $\theta(x, t)$ be the torsional deformation of the continuous shaft. Then kinetic energy of the shaft due to torsional deformation is given by

$$T_s = \frac{1}{2}\int_0^L \rho J_s \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the rigid disk due to torsional deformation is given by

$$T_d = \frac{1}{2} I_d \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L}$$

Then, the kinetic energy of the system is given by

$$T = T_s + T_d = \frac{1}{2}\int_0^L \rho J_s \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2} I_d \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L}$$

Total potential energy of the system is given by

$$V = \frac{1}{2}\int_0^L G J_s \left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x$$

Review Questions

1. What are the common components of a vibrating system? Write down features of each component.
2. Write down the expressions for kinetic and potential energy of the following continuous system **(a)** a bar, **(b)** a beam and **(c)** a shaft.
3. Differentiate between viscous damping, Coulomb damping and hysteretic damping.
4. Explain how equivalent mass, equivalent stiffness and equivalent damping of a system can be determined.

Exercise

1. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.1**. Use
   (a) downward displacement x of the block of mass m as a generalized coordinate.
   (b) counterclockwise rotation θ of the pulley as a generalized coordinate.

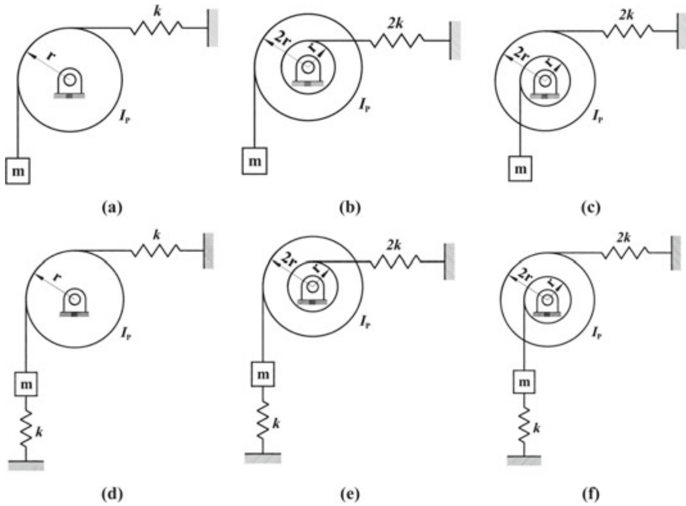


Figure P2.1

2. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.2**. Use
   (a) downward displacement x of the block of mass $2m$ as a generalized coordinate.
   (b) clockwise rotation θ of the pulley as a generalized coordinate.

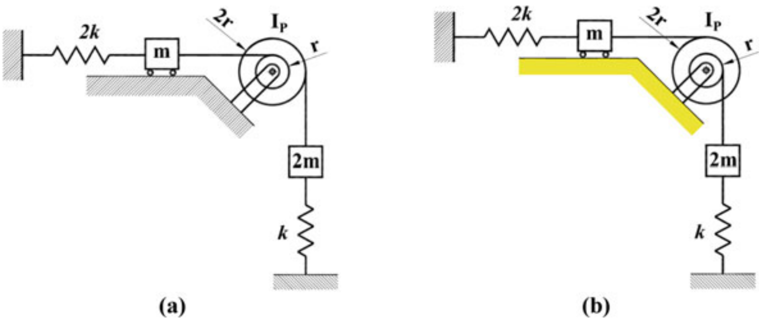


Figure P2.2

3. Determine m_{eq} and k_{eq} for the system of **Figure P2.3**. Use downward displacement x of the block of mass $2m$ from the static equilibrium position as the generalized coordinate.
4. Determine I_{eq} and k_{eq} for the system of **Figure P2.4** Use angular displacement θ of the pulley from the static equilibrium position as the generalized coordinate.

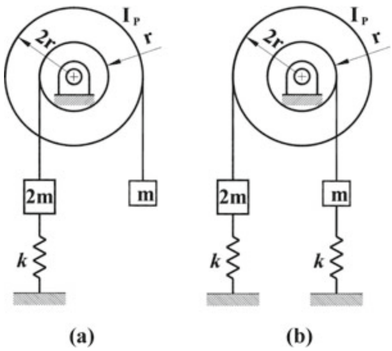


Figure P2.3

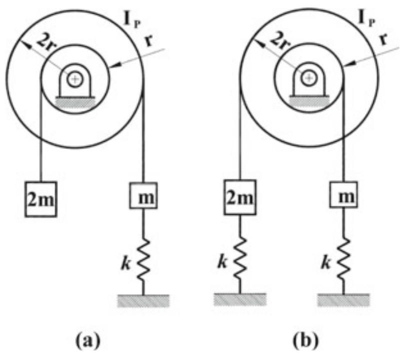


Figure P2.4

5. Determine m_{eq} and k_{eq} for the system of **Figure P2.5** Assume that the disk is thin and rolls without slip. Use
   (a) downward displacement x of the block as a generalized coordinate.
   (b) counterclockwise rotation θ of the pulley as a generalized coordinate.

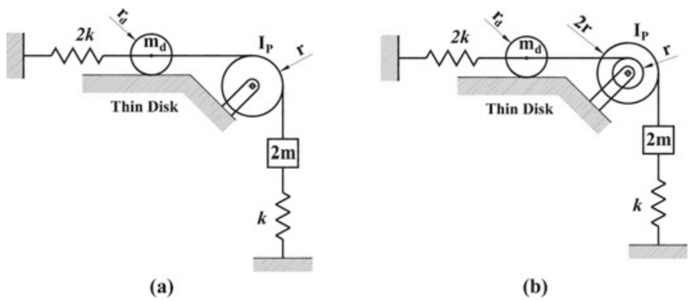


Figure P2.5

6. Determine the parameters for an equivalent system model for the systems of **Figure P2.6** Mass of the bar is M. Use rotation of the bar θ as the generalized coordinate.
7. Determine the parameters for an equivalent system model for the systems of **Figure P2.7**. Mass of the bar is $25kg$, and the spring stiffness is $4kN/m$. Use rotation of the bar θ as the generalized coordinate.

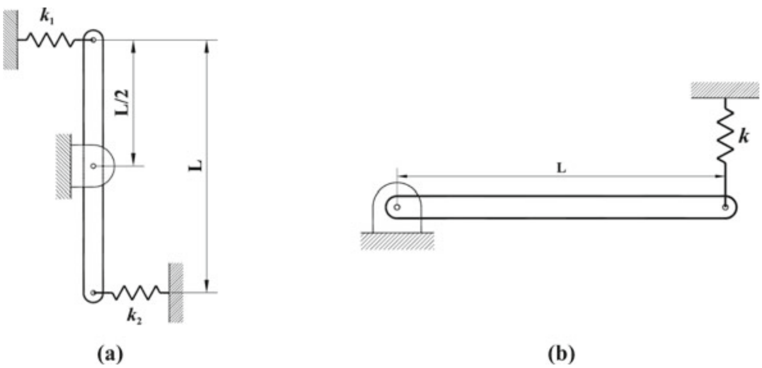


Figure P2.6

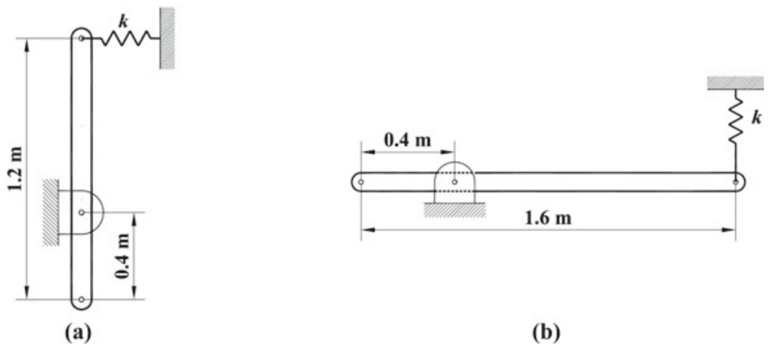


Figure P2.7

8. Determine the parameters for an equivalent systems model for the system of **Figure P2.8** Mass of the bar is M. Use rotation of the bar θ as the generalized coordinate. Each spring has a mass of m.
9. Determine m_{eq}, k_{eq} and c_{eq} for the system of **Figure P2.9**. Use
   (a) downward displacement x of the block of mass as a generalized coordinate.
   (b) counterclockwise rotation θ of the pulley as a generalized coordinate.
10. Determine m_{eq}, k_{eq} and c_{eq} for the system of **Figure P2.10**. Use
   (a) downward displacement x of the block of mass $2m$ as a generalized coordinate.
   (b) counterclockwise rotation θ of the pulley as a generalized coordinate.

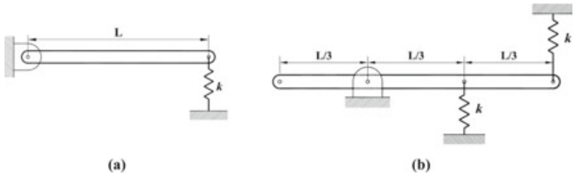


Figure P2.8

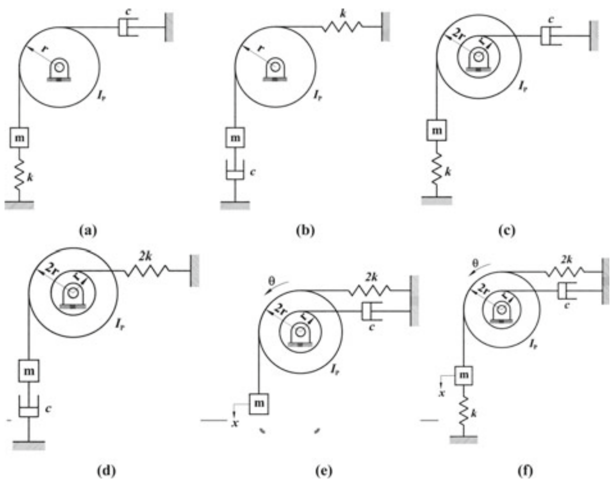


Figure P2.9

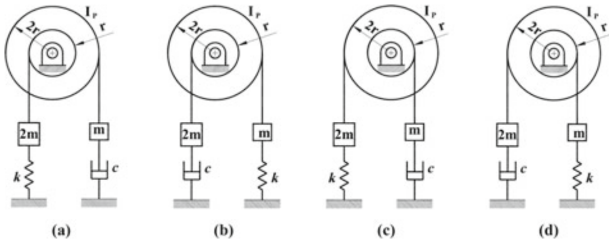


Figure P2.10

11. Determine m_{eq}, k_{eq} and c_{eq} for the system of **Figure P2.11**. Use
    (a) downward displacement x of the block of mass $2m$ as a generalized coordinate.
    (b) clockwise rotation θ of the pulley as a generalized coordinate.

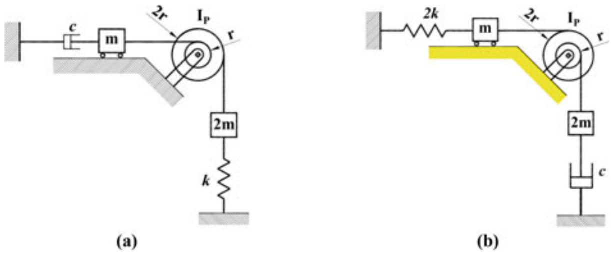


Figure P2.11

12. Determine m_{eq}, k_{eq} and c_{eq} for the system of **Figure P2.12**. Assume that the disk is thin and rolls without slip. Use
    (a) downward displacement x of the block as a generalized coordinate.
    (b) counterclockwise rotation θ of the pulley as a generalized coordinate.
13. Determine the parameters for an equivalent system model for the systems of **Figure P2.13**. Mass of the bar is M. Use rotation of the bar θ as the generalized coordinate.
14. Determine the parameters for an equivalent system model of the system of **Figure P2.14** Use the displacement x of the mass center of the disk measured from the static equilibrium position, as the generalized coordinate. Assume the disk is thin and rolls without slip. Also neglect the inertia effect of the small pulley.
15. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.15** Use vertical displacement x of mass and counterclockwise rotation θ of the pulley as a set of generalized coordinates.

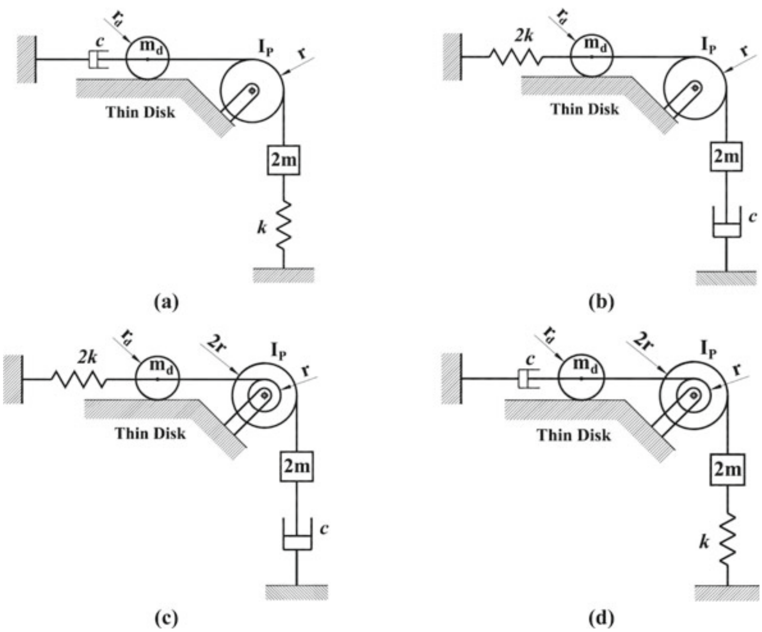


Figure P2.12

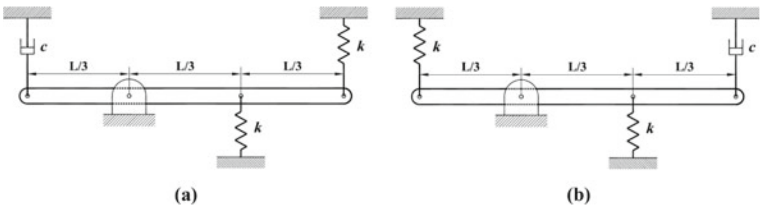


Figure P2.13

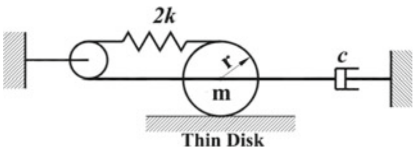


Figure P2.14

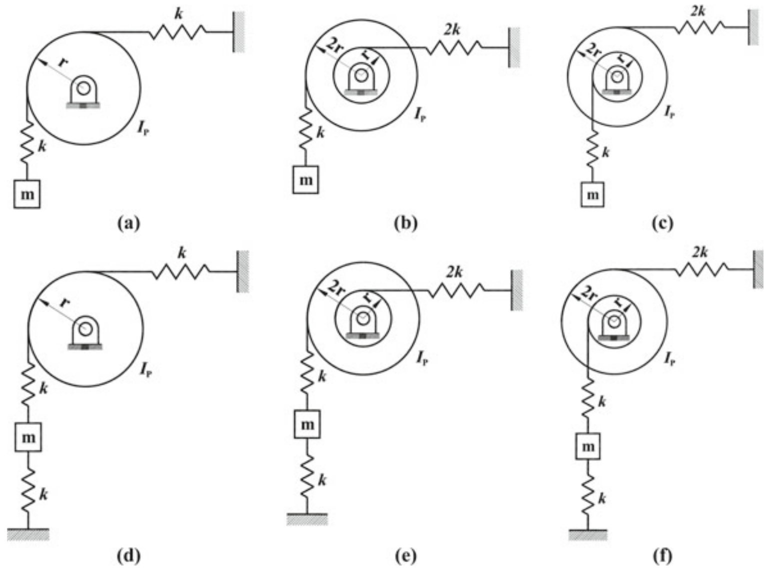


Figure P2.15

16. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.16** Use displacement x_1 of block m and displacement x_2 of block $2m$ as a set of generalized coordinates.

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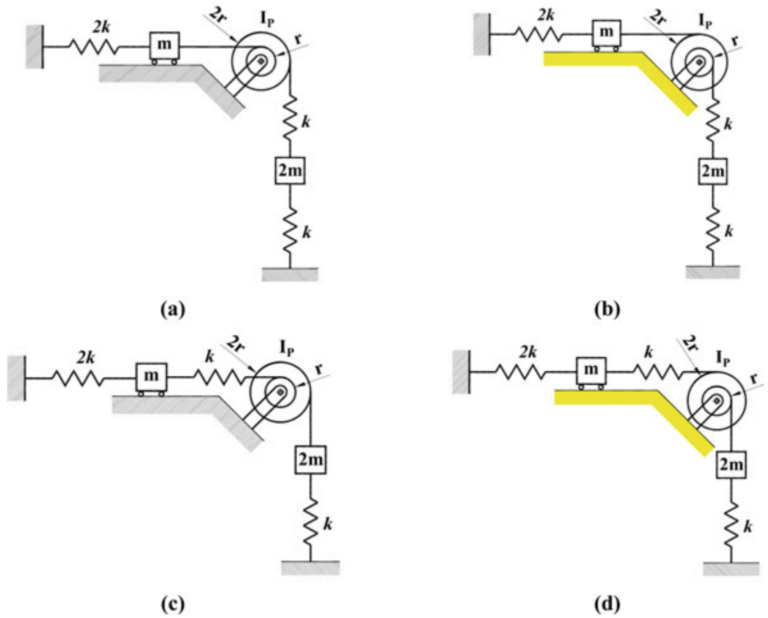


Figure P2.16

17. Determine the kinetic energy and potential energy of the system of **Figure P 2.17**. Assume that the disk is thin and rolls without slip. Use displacement x_1 of block m and displacement x_2 of the center of the disk as a set of generalized coordinates.
18. Determine the kinetic energy and potential energy of the system of **Figure P 2.18** in terms of given generalized coordinates. Assume the disks are thin and roll without slip. Both disks have mass of m and radius r. Use displacements x_1 and x_2 of the centers of the disks as a set of generalized coordinates

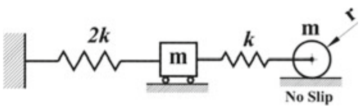


Figure P2.17

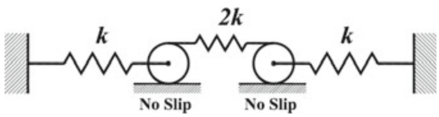


Figure P2.18

19. The rigid body of **Figure P2.19** is a uniform bar of length L and mass M. Determine the kinetic energy and potential energy of the bar. Use
    (a) downward displacement x of the CG of the bar and rotation θ of the bar as generalized coordinates

(b) vertical displacements x_1 and x_2 of the left and right ends of the bar as generalized coordinates.

20. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.20**.

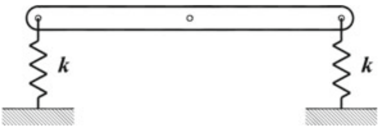


Figure P2.19

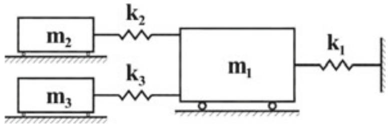


Figure P2.20

21. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.21**. Use displacement x_1 of the block m and displacement x_2 of the block $2m$ and rotation θ of the pulley as a set of generalized coordinates.

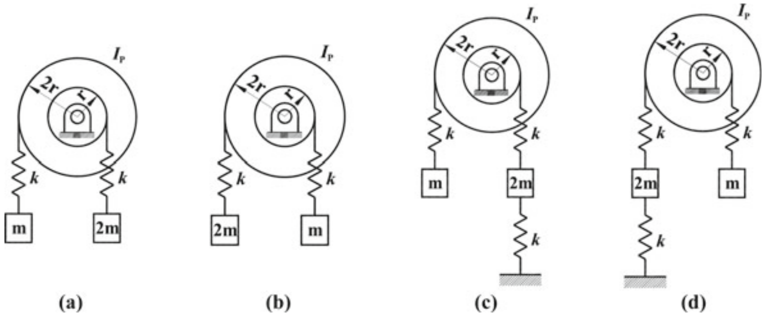


Figure P2.21

22. Determine the kinetic energy and potential energy of the systems shown in **Figure P2.22** The uniform bar has a mass of M. Use displacement x_1 of block m and displacement x_2 of block $2m$ and rotation θ of pulley as a set of generalized coordinates.
23. Determine the equivalent stiffness of the spring assemblies shown in **Figure P2.23**. Each spring has a stiffness of k.

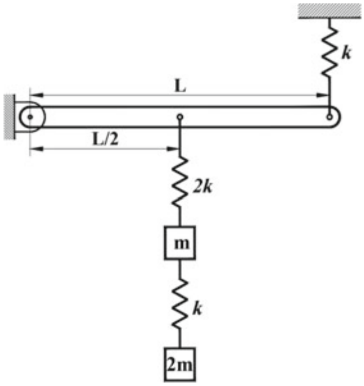


Figure P2.22

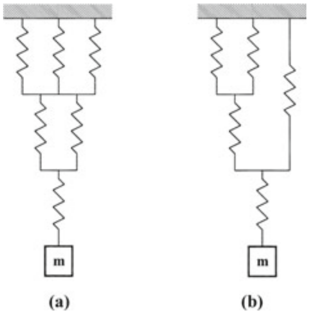


Figure P2.23

24. A spring shown in **Figure P2.24(a)** has a stiffness of k and an un-deformed length of L. The spring is divided into three equal parts and three pieces of the spring, each having stiffness of k_p, are attached to a block as shown in **Figure P2.24(b)**. . Determine the equivalent stiffness of the system.
25. Determine the equivalent stiffness of the system consisting of two bars and spring assembly shown in **Figure P2.25**. Take $A_1 = 2 \times 10^{-6}$ m^2, $E_1 = 210$ GPa for bar 1, $A_2 = 1 \times 10^{-6}$ m^2, $E_2 = 210$ GPa for bar 2 and $k = 100$ kN/m.**Figure P2.26**

Figure P2.24

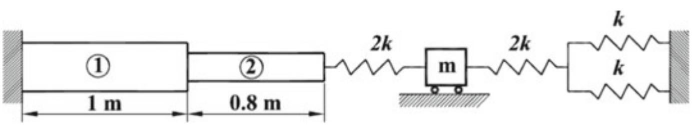


Figure P2.25

26. A machine is attached to a beam as shown in **Figure P2.26**. Mass of the beam is negligible in comparison with that of the machine. Determine equivalent stiffness for the single degree of freedom model of the system when the machine is attached at **(a)** $x = 0.25$ m, **(b)** $x = 0.5$ m and **(c)** $x = 0.75$ m. Take $E = 210$ GPa and $I = 1 \times 10^{-8}$ m^4.

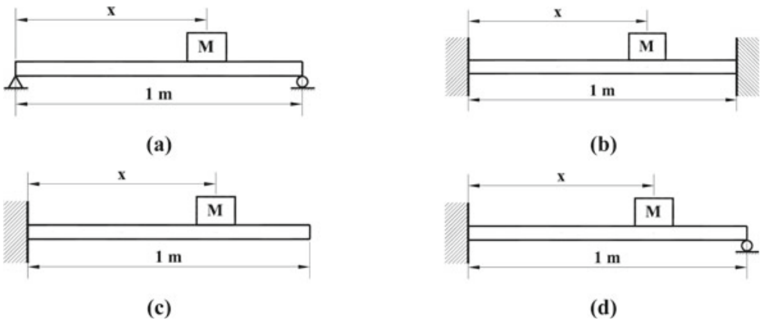


Figure P2.26

27. Determine equivalent stiffness for the single degree of freedom model of the system consisting of a beam and spring and mass assembly shown in **Figure P2.27**. Mass of the beam is negligible in comparison with that of the attached mass. The beam material has a modulus of elasticity of E and moment of inertia of section of I.

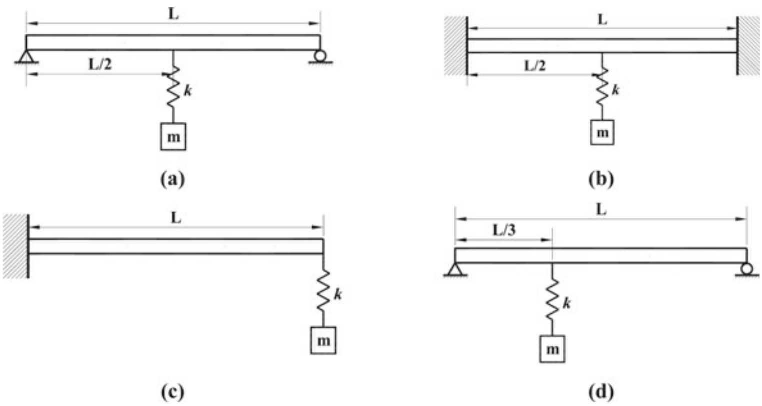


Figure P2.27

28. Determine equivalent stiffness for the single degree of freedom model of the system shown in **Figure P2.28**. Mass of the beam is negligible in comparison with that of the attached mass. Take $E = 210GPa$ and $I = 1 \times 10^{-5}m^4$ for beam; $k_1 = 1MN/m$ and $k_2 = 2MN/m$.

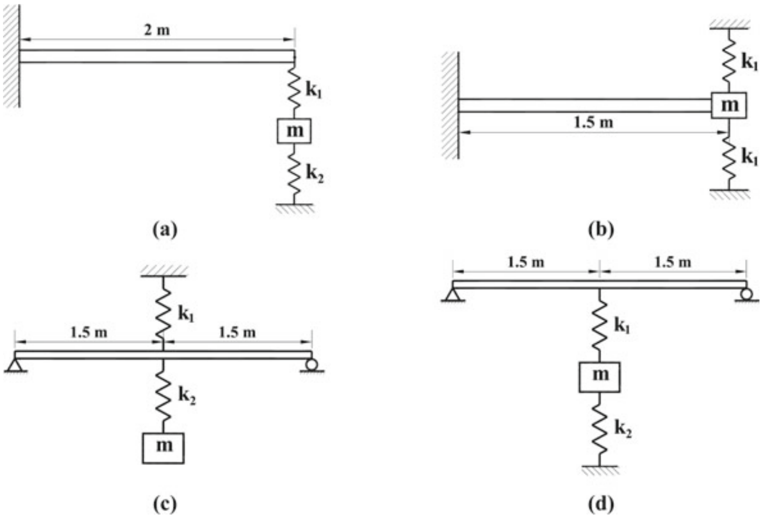


Figure P2.28

29. A concentrated mass M is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P2.29**. The bar material has a density of ρ, and its cross-sectional area is A. Derive expressions for its potential and kinetic energy.
30. A spring of stiffness k is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P2.30**. The bar material has a density of ρ, and its cross-sectional area is A. Derive expressions for its potential and kinetic energy.
31. A beam of length L shown in **Figure 2.31** undergoing traverse vibration is restrained by a spring of stiffness k. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive expressions for its potential and kinetic energy.

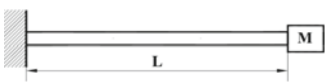


Figure P2.29

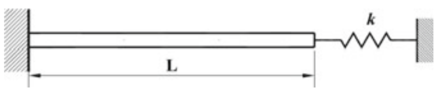


Figure P2.30

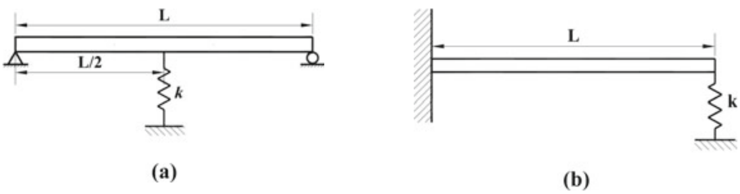


Figure P2.31

32. A rigid disk of mass moment of inertia I_d is attached to shaft of length L undergoing torsional vibration as shown in **Figure 2.32**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J. Derive expressions for its potential and kinetic energy.

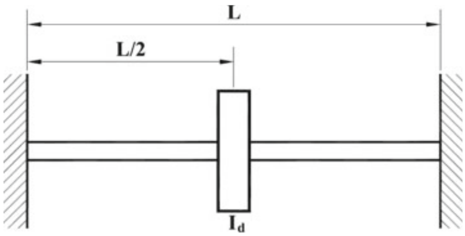


Figure P2.32

Answers

1.

(a) $\frac{1}{2}\left(m+\frac{I_p}{r^2}\right)\dot{x}^2$, $\frac{1}{2}kx^2$; $\frac{1}{2}\left(mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(kr^2\right)\theta^2$

(b) $\frac{1}{2}\left(m+\frac{I_p}{4r^2}\right)\dot{x}^2$, $\frac{1}{2}\left(\frac{k}{2}\right)x^2$; $\frac{1}{2}\left(4mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(2kr^2\right)\theta^2$

(c) $\frac{1}{2}\left(m+\frac{I_p}{r^2}\right)\dot{x}^2$, $\frac{1}{2}(8k)x^2$; $\frac{1}{2}\left(mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(8kr^2\right)\theta^2$

(d) $\frac{1}{2}\left(m+\frac{I_p}{r^2}\right)\dot{x}^2$, $\frac{1}{2}(2k)x^2$; $\frac{1}{2}\left(mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(2kr^2\right)\theta^2$

(e) $\frac{1}{2}\left(m+\frac{I_p}{4r^2}\right)\dot{x}^2$, $\frac{1}{2}\left(\frac{3k}{2}\right)x^2$; $\frac{1}{2}\left(4mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(6kr^2\right)\theta^2$

(f) $\frac{1}{2}\left(m+\frac{I_p}{r^2}\right)\dot{x}^2$, $\frac{1}{2}(9k)x^2$; $\frac{1}{2}\left(mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(9kr^2\right)\theta^2$

2.

(a) $\frac{1}{2}\left(\frac{9m}{4}+\frac{I_p}{4r^2}\right)\dot{x}^2$, $\frac{1}{2}\left(\frac{3k}{2}\right)x^2$; $\frac{1}{2}\left(9mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(6kr^2\right)\theta^2$

(b) $\frac{1}{2}\left(6m+\frac{I_p}{r^2}\right)\dot{x}^2$, $\frac{1}{2}(9k)x^2$; $\frac{1}{2}\left(6mr^2+I_p\right)\dot{\theta}^2$, $\frac{1}{2}\left(9kr^2\right)\theta^2$

3.

(a) $6m+\frac{I_p}{r^2}, k$

(b) $\frac{9m}{4}+\frac{I_p}{4r^2}, \frac{5k}{4}$

4.

(a) $6mr^2+I_p, 4kr^2$

(b) $9mr^2+I_p, 5kr^2$

5.

(a) $2m+\frac{3}{2}m_d+\frac{I_p}{r^2}, 3k; 2mr^2+\frac{3}{2}m_dr^2+I_p, 3kr^2$

(b) $2m + \frac{3}{8}m_d + \frac{I_p}{4r^2}, \frac{3k}{2}; 8mr^2 + \frac{3}{2}m_d r^2 + I_p, 6kr^2$

6.

(a) $\frac{1}{12}ML^2, \frac{L^2}{4}(k_1 + k_2)$
(b) $\frac{1}{3}ML^2, kL^2$

7.

(a) 4kg m^2, 2.56kN.m/rad
(b) 9.3333kg m^2, 5.76kN.m/rad

8.

(a) $\frac{1}{3}(M + m)L^2, kL^2$
(b) $\left(\frac{1}{9}M + \frac{5}{27}m\right)L^2, \frac{5}{9}kL^2$

9.

(a) $m + \frac{I_p}{r^2}, k, c; mr^2 + I_p, kr^2, cr^2$
(b) $m + \frac{I_p}{r^2}, k, c; mr^2 + I_p, kr^2, cr^2$
(c) $m + \frac{I_p}{4r^2}, k, \frac{c}{4}; 4mr^2 + I_p, 4kr^2, cr^2$
(d) $m + \frac{I_p}{4r^2}, \frac{k}{2}, c; 4mr^2 + I_p, 2kr^2, 4cr^2$
(e) $m + \frac{I_p}{4r^2}, 2k, \frac{c}{4}; 4mr^2 + I_p, 8kr^2, cr^2$
(f) $m + \frac{I_p}{4r^2}, 3k, \frac{c}{4}; 4mr^2 + I_p, 12kr^2, cr^2$

10.

(a) $6m + \frac{I_p}{r^2}, k, 4c; 6mr^2 + I_p, kr^2, 4cr^2$
(b) $6m + \frac{I_p}{r^2}, 4k, c; 6mr^2 + I_p, 4kr^2, cr^2$
(c) $\frac{9m}{4} + \frac{I_p}{4r^2}, k, \frac{c}{4}; 9mr^2 + I_p, 4kr^2, cr^2$
(d) $\frac{9m}{4} + \frac{I_p}{4r^2}, \frac{k}{4}, c; 9mr^2 + I_p, kr^2, 4cr^2$

11.

(a) $\frac{9m}{4} + \frac{I_p}{4r^2}, k, \frac{c}{4}; 9mr^2 + I_p, 4kr^2, cr^2$
(b) $6m + \frac{I_p}{r^2}, 8k, c; 6mr^2 + I_p, 8kr^2, cr^2$

12.

(a) $2m + \frac{3}{2}m_d + \frac{I_p}{r^2}, k, c; 2mr^2 + \frac{3}{2}m_d r^2 + I_p, kr^2, cr^2$
(b) $2m + \frac{3}{2}m_d + \frac{I_p}{r^2}, 2k, c; 2mr^2 + \frac{3}{2}m_d r^2 + I_p, 2kr^2, cr^2$
(c) $2m + \frac{3}{8}m_d + \frac{I_p}{4r^2}, \frac{k}{2}, c; 8mr^2 + \frac{3}{2}m_d r^2 + I_p, 2kr^2, 4cr^2$
(d) $2m + \frac{3}{8}m_d + \frac{I_p}{4r^2}, k, \frac{c}{4}; 8mr^2 + \frac{3}{2}m_d r^2 + I_p, 4kr^2, cr^2$

13.

(a) $\frac{1}{9}ML^2, \frac{5}{9}kL^2, \frac{1}{9}cL^2$
(b) $\frac{1}{9}ML^2, \frac{2}{9}kL^2, \frac{4}{9}cL^2$

14. $\frac{3}{2}m, 18k, c$
15.
   (a) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(kr^2)\theta^2$
   (b) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x - 2r\theta)^2 + \frac{1}{2}(2kr^2)\theta^2$
   (c) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(8kr^2)\theta^2$
   (d) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}kx^2 + \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(kr^2)\theta^2$
   (e) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}kx^2 + \frac{1}{2}k(x - 2r\theta)^2 + \frac{1}{2}(2kr^2)\theta^2$
   (f) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}kx^2 + \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(8kr^2)\theta^2$

16.
   (a) $\frac{1}{2}\left(m + \frac{I_P}{r^2}\right)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2, \frac{1}{2}(2k)x_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(x_2 - 2x_1)^2$
   (b) $\frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2, \frac{1}{2}(2k)x_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k\left(x_2 - \frac{x_1}{2}\right)^2$
   (c) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\left(2m + \frac{I_P}{4r^2}\right)\dot{x}_2^2, \frac{1}{2}(2k)x_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k\left(\frac{x_2}{2} - x_1\right)^2$
   (d) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\left(2m + \frac{I_P}{r^2}\right)\dot{x}_2^2, \frac{1}{2}(2k)x_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(2x_2 - x_1)^2$

17. $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}_2^2, \frac{1}{2}(2k)x_1^2 + \frac{1}{2}k(x_2 - x_1)^2$
18. $\frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}_1^2 + \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}_2^2, \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}(2k)(2x_2 - 2x_1)^2$
19.
   (a) $\frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2, \frac{1}{2}k\left(\frac{L}{2}\theta - x\right)^2 + \frac{1}{2}k\left(\frac{L}{2}\theta + x\right)^2$
   (b) $\frac{1}{2}M\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}M\right)(\dot{x}_2 - \dot{x}_1)^2, \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2$

20. $\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2, \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(x_3 - x_1)^2$
21.
   (a) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x_1 - 2r\theta)^2 + \frac{1}{2}k(x_2 - r\theta)^2$
   (b) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x_1 - r\theta)^2 + \frac{1}{2}k(x_2 - 2r\theta)^2$
   (c) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x_1 - 2r\theta)^2 + \frac{1}{2}k(x_2 - r\theta)^2 + \frac{1}{2}kx_2^2$
   (d) $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2, \frac{1}{2}k(x_1 - r\theta)^2 + \frac{1}{2}k(x_2 - 2r\theta)^2 + \frac{1}{2}kx_2^2$

22. $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\dot{\theta}^2, \frac{1}{2}\left(kL^2\right)\theta^2 + \frac{1}{2}(2k)\left(x_1 - \frac{L}{2}\theta\right)^2 + \frac{1}{2}k(x_2 - x_1)^2$
23.
   (a) $\frac{6}{11}k$
   (b) $\frac{5}{8}k$

24. $9k$
25. 189.362 kN/m
26.
   (a) 179.2 kN/m, 100.8 kN/m, 179.2 kN/m
   (b) 955.73 kN/m, 403.2 kN/m, 955.37 kN/m
   (c) 403.2 kN/m, 50.4 kN/m, 14.93 kN/m
   (d) 764.59 kN/m, 230.4 kN/m, 294.07 kN/m

27.

(a) $\frac{48EIk}{kL^3+48EI}$
(b) $\frac{192EIk}{kL^3+192EI}$
(c) $\frac{3EIk}{kL^3+3EI}$
(d) $\frac{243EIk}{4kL^3+243EI}$

28.

(a) 2.4406 MN/m
(b) 4.8667 MN/m
(c) 1.4059 MN/m
(d) 2.7887 MN/m

29. $\frac{1}{2}\int\limits_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\Big|_{x=L}, \frac{1}{2}\int\limits_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$

30. $\frac{1}{2}\int\limits_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x, \frac{1}{2}\int\limits_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}k\{u(x,t)\}^2\Big|_{x=L}$

31.

(a) $\frac{1}{2}\int\limits_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x, \frac{1}{2}\int\limits_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}k\{w(x,t)\}^2\Big|_{x=L/2}$
(b) $\frac{1}{2}\int\limits_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x, \frac{1}{2}\int\limits_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}k\{w(x,t)\}^2\Big|_{x=L}$

32. $\frac{1}{2}\int\limits_0^L \rho J\left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}I_d\left(\frac{\partial \theta}{\partial t}\right)^2\Big|_{x=L/2}, \frac{1}{2}GJ\int\limits_0^L \left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x$

Chapter 3
Derivation of Equation of Motion of a Vibrating System

As explained in Chap. 1, the first step to study any vibrating system is the mathematical modeling. Mathematical modeling is developed to represent the phenomenon occurring in any real system in terms of mathematical expression, which is called governing equation or equation of motion of the system.

Equation of motion can be derived by the direct application of Newton's second law of motion, equivalent system parameters method or conservation of energy principle. These methods can be more efficiently used for discrete system with few degrees of freedom.

Equation of motion for discrete system with relatively higher degrees of freedom and continuous system can also be derived by using variational principle or energy principles. Most common forms of variational formulation of the dynamic system are Hamilton' principle and Lagrange equations which are also modified forms of Newton's second law of motion.

3.1 Classical Methods for Derivation of Equation of Motion

3.1.1 Newton's Second Law of Motion

Most common method to derive an equation of motion of a vibrating system is the application of Newton's second law of motion. Following procedure should be followed to use this method to derive an equation of motion of any system.

(a) Determine degree of freedom of the system and choose any appropriate set of generalized coordinates to describe the instantaneous position of the system.
(b) Draw free-body diagram for the particle or rigid body of the system under static condition and determine static deflection of the system.

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_3

(c) Draw another free-body diagram of the system assume that it is further displaced from the static equilibrium position. Draw representations of all external and reactive forces acting upon the system.
(d) With reference to the free-body diagram, apply Newton's second law of motion or D Alembert's principle as given below.

$$\sum F = m\ddot{x} \quad \text{(For translational motion)} \tag{3.1}$$

$$\sum M = I\ddot{\theta} \quad \text{(For rotational motion)} \tag{3.2}$$

or,

$$\sum F - m\ddot{x} = 0 \quad \text{(For translational motion)} \tag{3.3}$$

$$\sum M - I\ddot{\theta} = 0 \quad \text{(For rotational motion)} \tag{3.4}$$

To demonstrate the method, consider a spring, mass and damper system shown in Fig. 3.1. Instantaneous position of the system can be defined by the vertical displacement of the mass; hence it is a single degree of freedom system and x can be used as a generalized coordinate.

Original or un-stretched position of the spring is shown in Fig. 3.2a. When the mass is attached to the lower end of the spring and the damper, spring undergoes deformation by Δ amount as shown in Fig. 3.2b. The deformation (Δ) of the spring under such static condition is called static displacement. With reference to the free-body diagram of the mass under static condition shown in Fig. 3.2c, equilibrium equation can be written as

$$W = k\Delta \tag{3.5}$$

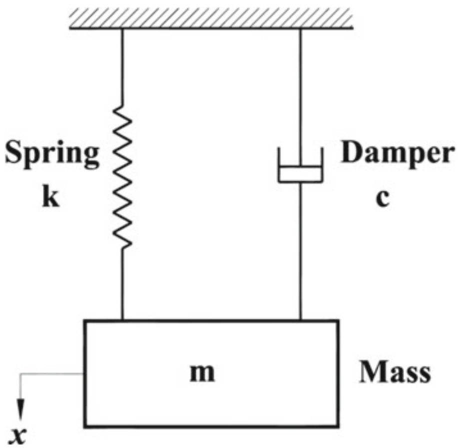


Fig. 3.1 Single degree of freedom system consisting of spring, mass and damper

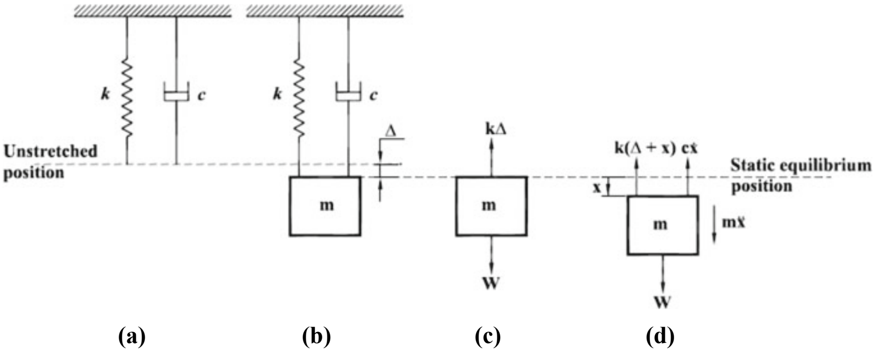


Fig. 3.2 Free-body diagrams for static condition and vibrating conditions

When the mass is further displaced by x amount from its static equilibrium position, and released it undergoes vibratory motion. With reference to free-body diagram of the mass undergoing vibration shown in Fig. 3.2d, Newton's second law of motion can be applied as

$$\sum F = m\ddot{x}$$

$$\text{or,} \quad W - k(\Delta + x) - c\dot{x} = m\ddot{x}$$

$$\text{or,} \quad W - k\Delta - kx - c\dot{x} = m\ddot{x}$$

Substituting $W = k\Delta$, from Eq. (3.5).

$$-kx - c\dot{x} = m\ddot{x}$$

$$\therefore \quad m\ddot{x} + c\dot{x} + kx = 0 \tag{3.6}$$

Equation (3.6) is the equation of motion of the system shown in Fig. 3.1.

3.1.2 Equivalent System Parameters Method

If q is chosen as the general coordinate of the system then the expressions for kinetic energy of the system T, potential energy of the system V and work done against the damping force W_d can be expressed in the form

$$T = \frac{1}{2} m_{eq} \dot{q}^2 \quad \text{or} \quad T = \frac{1}{2} I_{eq} \dot{q}^2 \tag{3.7}$$

$$V = \frac{1}{2} k_{eq} q^2 \tag{3.8}$$

$$W_d = \int -C_{eq} \dot{q} \mathrm{d}q \tag{3.9}$$

where m_{eq}, I_{eq}, k_{eq} and C_{eq} are, respectively, equivalent mass, equivalent inertia, equivalent stiffness and equivalent damping.

Then equation of motion of the system can be written as

$$m_{eq}\ddot{q} + C_{eq}\dot{q} + k_{eq}q = 0 \quad \text{or} \quad I_{eq}\ddot{q} + C_{eq}\dot{q} + k_{eq}q = 0 \tag{3.10}$$

To demonstrate the method, consider a system shown in Fig. 3.3. Instantaneous position of the system can be defined by either the vertical displacement of the mass x or by the angular rotation of the pulley θ.

If x is taken as a generalized coordinate, the total kinetic energy (T), , total potential energy (V) and work done against the damping (W_d) for the system can be expressed as

$$T = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}^2$$
$$V = \frac{1}{2}\left(\frac{3k}{2}\right)x^2$$
$$W_d = \int -c(\dot{x})\mathrm{d}(x)$$

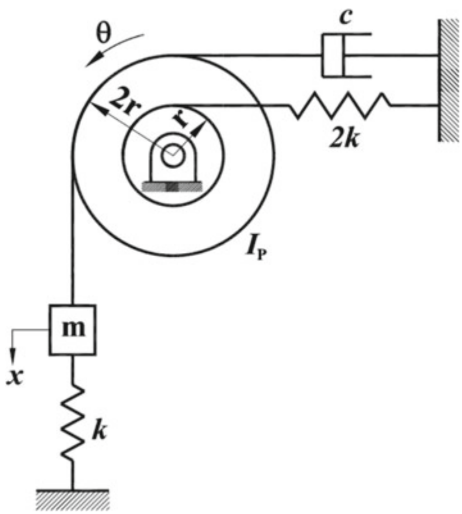


Fig. 3.3 Single degree of freedom system consisting of spring, mass/pulley and damper

Comparing these expressions with Eqs. (3.7), (3.8) and (3.9), we can define equivalent system parameters as

$$m_{eq} = m + \frac{I_P}{4r^2} \quad k_{eq} = \frac{3k}{2} \quad \text{and} \quad c_{eq} = c$$

Then using Eq. (3.10), we can directly write equation of motion of the system as

$$\left(m + \frac{I_P}{4r^2}\right)\ddot{x} + (c)\dot{x} + \left(\frac{3k}{2}\right)x = 0 \tag{3.11}$$

Similarly, if θ is taken as a generalized coordinate, the total kinetic energy (T), total potential energy (V) and work done against the damping (W_d) for the system can be expressed as

$$\begin{aligned} T &= \frac{1}{2}\left(I_P + 4mr^2\right)\dot{\theta}^2 \\ V &= \frac{1}{2}\left(6kr^2\right)\theta^2 \\ W_d &= \int -\left(4cr^2\right)\dot{\theta}\mathrm{d}\theta \end{aligned}$$

Comparing these expressions with Eqs. (3.7), (3.8) and (3.9), we can define equivalent system parameters as

$$I_{eq} = I_P + 4mr^2 \quad k_{eq} = 6kr^2 \quad \text{and} \quad c_{eq} = 4cr^2$$

Then using Eq. (3.10), we can directly write equation of motion of the system as

$$\left(I_P + 4mr^2\right)\ddot{\theta} + \left(4cr^2\right)\dot{\theta} + \left(6kr^2\right)\theta = 0 \tag{3.12}$$

3.1.3 Principle of Conservation of Energy

Principle of conservation of energy can be applied only for a conservative system. A system is said to be conservative if no energy is lost from it or no energy is added to it. If any system does not have any damping effect and external forces acting on it, it can be taken as a conservative system.

Principle of conservation of energy states that total energy of a conservative system remains constant. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy V is stored in the system by virtue of its elastic deformation. Thus, the principle of conservation of energy can be expressed as

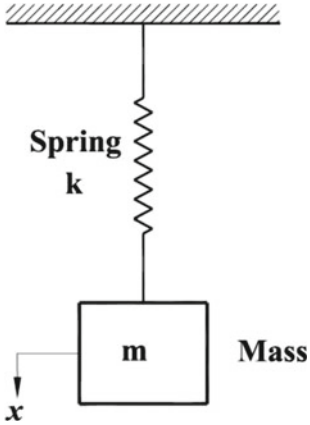


Fig. 3.4 Single degree of freedom system consisting of a spring and a mass

$$(T + V) = \text{constant} \tag{3.13}$$

Equation (3.13) can also be expressed in an alternative form as

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = 0 \tag{3.14}$$

To demonstrate the method, consider a spring and mass system shown in Fig. 3.4. If vertical displacement of the mass (x) is taken as a generalized coordinate, expressions for the total kinetic and potential energy of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad V = \frac{1}{2}kx^2$$

Then substituting T and V into Eq. (3.14), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$\text{or, } m\ddot{x}\dot{x} + kx\dot{x} = 0$$

$$\text{or, } (m\ddot{x} + kx)\dot{x} = 0$$

Since $\dot{x} \neq 0$,

$$m\ddot{x} + kx = 0 \tag{3.15}$$

which is the required equation of motion of the system.

3.2 Variational Formulation of Dynamic System

Variational methods can also be used to derive equation of motion of both discrete and continuous systems. To use this method, principles of variational calculus are used, i.e., an appropriate functional of a dynamic system is derived and the minimization of this functional leads to the equation of motion of the system.

Before going to the procedure of variational method, some common terminologies and principles of variational calculus should be understood.

3.2.1 Independent Variable, Function and Functional

The set of variables which is the minimum number required to determine the state of the system completely is known as the set of independent variables. All other variables describing the system will be dependent on this set, and are called functions. Furthermore, the variables dependent upon the functions are called functionals.

For example, consider a beam shown in Fig. 3.5 undergoing transverse vibration. Instantaneous position of any point of a beam is defined by $w(x, t)$, where x and t are independent variables and w which is dependent upon x and t is a function. Kinetic energy T of the vibrating beam given by

$$T = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{3.16}$$

is dependent upon the function $w(x, t)$, i.e., is a function of $w(x, t)$ and hence is a functional.

Most of the problems of engineering fields have the functional of the form

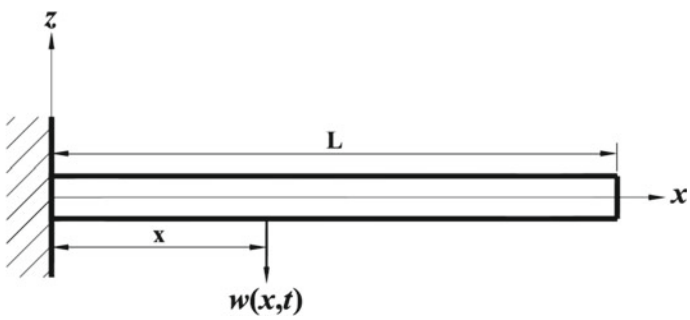


Fig. 3.5 Beam undergoing transverse vibration

$$I = \int_a^b F\left(x, u, u'\right)\mathrm{d}x \tag{3.17}$$

where x is an independent variable, $u(x)$ is a continuously differentiable function on $[a, b]$ that satisfies specified conditions at $x = a$ and $x = b$, $u' = \mathrm{d}u/\mathrm{d}x$ and F is a functional dependent of these variables.

3.2.2 Differentiation and Variation

Consider $u(x)$ shown in Fig. 3.6 as a continuous function of x. Differential of u denoted by $\mathrm{d}u$ is the change in u due to change in x and is defined mathematically as

$$\mathrm{d}u = \lim_{\Delta x \to 0} u(x + \Delta x) - u(x) \tag{3.18}$$

Equation (3.18) can also be expressed as

$$\mathrm{d}u = u(x + \mathrm{d}x) - u(x) \tag{3.19}$$

The operator d used with the function u is called a differential operator.

Consider a functional $F\left(x, u, u'\right)$, which is dependent upon u and u' for an arbitrarily fixed value of independent variable x. Let $u(x)$ has fixed values u_a and u_b, respectively, at points x_a and x_b, as shown in Fig. 3.7. These prescribed values u_a and u_b are called boundary values. Let ε be a small parameter and let $\eta(x)$ be any arbitrary function such that $\eta(x_a) = 0$ and $\eta(x_b) = 0$. A family of functions is then defined by

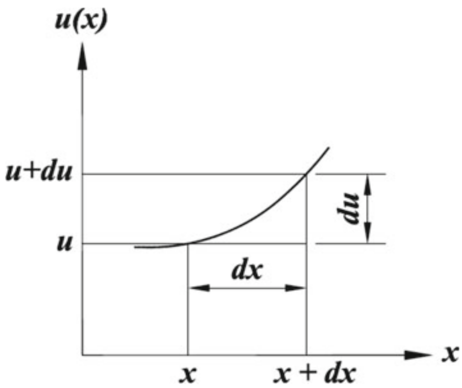


Fig. 3.6 Differentiation of a continuous function $u(x)$

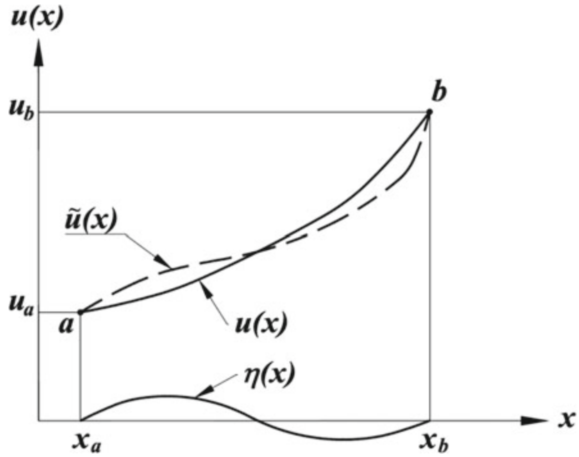


Fig. 3.7 Variation of a continuous function $u(x)$

$$\tilde{u}(x) = u(x) + \varepsilon\eta(x) \tag{3.20}$$

Then the variation in u denoted by δu is the change $\varepsilon\eta(x)$ in u and is defined mathematically as

$$\delta u = \varepsilon\eta(x) \tag{3.21}$$

The operator δ used with the function u is called a variational operator.

The variation δu of a function u represents an admissible change in the function $u(x)$ at a fixed value the independent variable x. This variation δu in the function $u(x)$ causes is a change in the functional F,

$$\Delta F = F\left(x, u + \varepsilon\eta, u' + \varepsilon\eta'\right) - F\left(x, u, u'\right) \tag{3.22}$$

Expanding in powers of ε gives,

$$\begin{aligned}\Delta F &= F\left(x, u, u'\right) + \varepsilon\eta\frac{\partial F}{\partial u} + \varepsilon\eta'\frac{\partial F}{\partial u'} + \frac{(\varepsilon\eta)^2}{2!}\frac{\partial^2 F}{\partial u^2} \\ &\quad + \frac{(\varepsilon\eta)\left(\varepsilon\eta'\right)}{2!}\frac{\partial^2 F}{\partial u \partial u'} + \frac{\left(\varepsilon\eta'\right)^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F\left(x, u, u'\right) \\ &= \varepsilon\eta\frac{\partial F}{\partial u} + \varepsilon\eta'\frac{\partial F}{\partial u'} + \text{Higher order terms of } \varepsilon \end{aligned} \tag{3.23}$$

Then the first variation of F is defined by

$$\begin{aligned}\delta F &= \lim_{\varepsilon \to 0} F\left(x, u + \varepsilon\eta, u' + \varepsilon\eta'\right) - F\left(x, u, u'\right) \\ &= \lim_{\varepsilon \to 0} \Delta F\end{aligned}$$

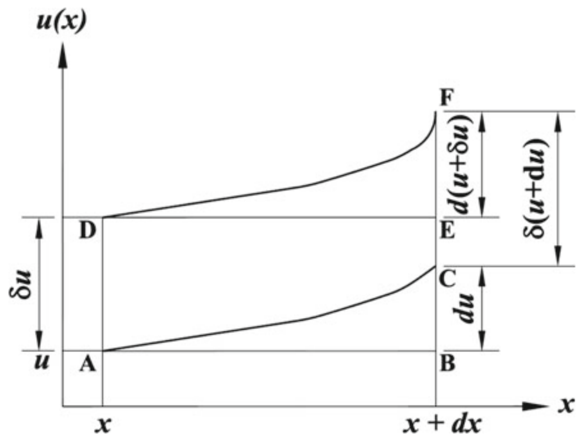


Fig. 3.8 Comparison between differentiation and variation of a continuous function $u(x)$

$$= \varepsilon\eta\frac{\partial F}{\partial u} + \varepsilon\eta'\frac{\partial F}{\partial u'}$$

$$\therefore \delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \tag{3.24}$$

Comparison of differentiation and variation is shown graphically in Fig. 3.8. The function has a value of u at point A. The change in its value du due to change dx in x is its differential and the value of function at point C is $u + \mathrm{d}u$. Whereas the possible change δu in u for given value of independent variable x is its variation and the value of function at point D is $u + \delta u$.

Similarly, the change in functional value of point D due to change dx in x is its differential $\mathrm{d}(u + \delta u)$ and is represented by line EF. Whereas the possible change δ $(u + \mathrm{d}u)$ in $(u + \mathrm{d}u)$ for given value of independent variable $x + \mathrm{d}x$ is its variation and is represented by the line CF. Now, referring to Fig. 3.8, the total distance BF can be expressed as either $BE + EF$ or $BC + CF$, i.e.,

$$\begin{aligned} &\mathrm{BE} + \mathrm{EF} = \mathrm{BC} + \mathrm{CF} \\ &\text{or} \quad \delta u + \mathrm{d}(u + \delta u) = \mathrm{d}u + \delta(u + \mathrm{d}u) \\ &\text{or} \quad \delta u + \mathrm{d}u + \mathrm{d}(\delta u) = \mathrm{d}u + \delta u + \delta(\mathrm{d}u) \\ &\therefore \mathrm{d}(\delta u) = \delta(\mathrm{d}u) \end{aligned} \tag{3.25}$$

Equation (3.25) can also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}(\delta u) = \delta\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \tag{3.26}$$

Equations (3.25) and (3.26) show that differential and variational operators are commutative with each other.

Similarly, integral and variational operators are also commutative with each other, i.e.,

$$\int_a^b (\delta u)\mathrm{d}x = \delta \int_a^b (u)\mathrm{d}x \tag{3.27}$$

Laws of differential calculus for sum, product, quotient and power can also be applied in similar way for the problems of variational calculus. For example, if $F_1(u)$ and $F_2(u)$ are two functionals both dependent upon the function $u(x)$, then these laws can be expressed as

$$\delta(F_1 \pm F_2) = \delta(F_1) \pm \delta(F_2) \tag{3.28}$$

$$\delta(F_1 F_2) = \delta(F_1)F_2 + \delta(F_2)F_1 \tag{3.29}$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2\delta(F_1) - F_1\delta(F_2)}{F_2^2} \tag{3.30}$$

$$\delta(F_1)^n = n(F_1)^{n-1}\delta F_1 \tag{3.31}$$

3.2.3 Fundamental Lemma of Variational Calculus

The fundamental lemma of calculus of variations can be stated as follows:

For any integrable function $f(x)$, if the statement

$$\int_a^b f(x)\eta(x)\mathrm{d}x = 0 \tag{3.32}$$

holds for any arbitrary function $\eta(x)$, for all x in (a, b), then it follows that $f(x) = 0$ in (a, b).

Proof of the fundamental lemma of calculus of variations.

To prove the lemma, assume that $\eta(x)$, which is arbitrary, equal to $f(x)$. Then substituting $\eta(x) = f(x)$ into Eq. (3.32), we get

$$\int_a^b [f(x)]^2\mathrm{d}x = 0$$

Since an integral of a positive function, $[f(x)]^2$, is positive, the above statement implies that $f(x) = 0$ in the domain $\Omega = (a, b)$.

3.3 Euler–Lagrange Equation

As explained earlier, problems of engineering fields are formulated as one of finding the extremum of functionals (i.e., functions of dependent unknowns of the problem).

Consider a problem of finding a function $u = u(x)$ such that

$$u(x_a) = u_a, u(x_b) = u_b \tag{3.33}$$

and

$$I(u) = \int_{x_a}^{x_b} F\big(x, u(x), u'(x)\big)\mathrm{d}x \tag{3.34}$$

is a minimum.

To solve such problems, we list some functions $u_i(x)$ that satisfy the given boundary conditions. The set of all such functions is called a set of admissible functions. Then by applying principles of variational calculus, we can determine the function $u(x)$, which minimizes the functional given in Eq. (3.34).

Condition for minimization of the given functional is given by

$$\delta I(u) = 0$$

or,

$$\int_{x_a}^{x_b} \delta\big\{F\big(x, u(x), u'(x)\big)\big\}\mathrm{d}x = 0$$

or,

$$\int_{x_a}^{x_b} \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right]\mathrm{d}x = 0$$

or,

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial u}\delta u\mathrm{d}x + \int_{x_a}^{x_b} \frac{\partial F}{\partial u'}\delta u'\mathrm{d}x = 0$$

Using integration by parts for the second term,

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial u}\delta u \mathrm{d}x + \frac{\partial F}{\partial u'}\delta u\bigg|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right)\delta u \mathrm{d}x = 0 \tag{3.35}$$

Since $u(x)$ has fixed values of u_a and u_b, respectively, at points $x = x_a$ and $x = x_b$, $\delta(u)|_{x=x_a} = 0$ and $\delta(u)|_{x=x_b} = 0$. Substituting these boundary conditions, Eq. (3.35) reduces to

$$\begin{aligned} &\int_{x_a}^{x_b} \frac{\partial F}{\partial u}\delta u \mathrm{d}x - \int_{x_a}^{x_b} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right)\delta u \mathrm{d}x = 0 \\ &\therefore \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u \mathrm{d}x = 0 \end{aligned} \tag{3.36}$$

Now, applying the fundamental lemma of calculus of variations

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right) = 0 \tag{3.37}$$

Equation (3.37) is called the Euler–Lagrange equation and is used to derive governing equations of problems of dynamics which must be satisfied by a function that extremizes a functional of the form of Eq. (3.34).

Again, consider a functional that depends upon a number of dependent variables defined by

$$I(u) = \int_{x_a}^{x_b} F\left(x, u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'\right)\mathrm{d}x \tag{3.38}$$

Condition for minimization of the given functional is given by

$$\delta I(u) = 0$$

or,

$$\int_{x_a}^{x_b} \delta F \mathrm{d}x = 0$$

or,

$$\int_{x_a}^{x_b}\left[\left\{\frac{\partial F}{\partial u_1}\delta u_1+\frac{\partial F}{\partial u_1'}\delta u_1'\right\}+\left\{\frac{\partial F}{\partial u_2}\delta u_2+\frac{\partial F}{\partial u_2'}\delta u_2'\right\}+\cdots\right.$$
$$\left.+\left\{\frac{\partial F}{\partial u_n}\delta u_n+\frac{\partial F}{\partial u_n'}\delta u_n'\right\}\right]\mathrm{d}x=0$$

or,

$$\int_{x_a}^{x_b}\sum_{i=1}^{n}\left[\left\{\frac{\partial F}{\partial u_i}\delta u_i+\frac{\partial F}{\partial u_i'}\delta u_i'\right\}\right]\mathrm{d}x=0$$

Using integration by parts,

$$\int_{x_a}^{x_b}\sum_{i=1}^{n}\left[\left\{\frac{\partial F}{\partial u_i}\right\}\right]\delta u_i\mathrm{d}x+\sum_{i=1}^{n}\frac{\partial F}{\partial u_i'}\delta u_i\Bigg|_{x_a}^{x_b}-\int_{x_a}^{x_b}\sum_{i=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\partial F}{\partial u_i'}\right\}\right]\delta u_i\mathrm{d}x=0 \quad (3.39)$$

Substituting boundary conditions, Eq. (3.39) reduces to

$$\int_{x_a}^{x_b}\sum_{i=1}^{n}\left[\frac{\partial F}{\partial u_i}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u_i'}\right)\right]\delta u_i\mathrm{d}x=0 \quad (3.40)$$

Since all u_i are independent to each other, each term of the integral should be zero, then the fundamental lemma yields the set of Euler–Lagrange equations as

$$\frac{\partial F}{\partial u_i}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u_i'}\right)=0,\, i=1,2,\ldots,n \quad (3.41)$$

3.4 Hamilton's Principle

Newton's second law of motion or D'Alembert's principle can be used to derive Hamilton's principle, which is the most common tool of variational method that can be used to derive equations of motion of the dynamic systems.

Consider a system of n particles. Newton's second law, for any one particle of the system, can be written as

$$\boldsymbol{F_i}+\sum\boldsymbol{f_i}=\boldsymbol{m_i}\ddot{\boldsymbol{r}}_i \quad (3.42)$$

where $\boldsymbol{F_i}$ is the resultant of all forces external to the system acting on this particle and $\sum \boldsymbol{f_i}$ is the sum of reaction forces acting on this particle from other particles in the system.

Taking dot product of each term of Eq. (3.42) with the variation of position vector, we get

$$\boldsymbol{F_i} \cdot \delta \boldsymbol{r}_i + \sum \boldsymbol{f}_i \cdot \delta \Delta \boldsymbol{r}_i = m_i \ddot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i \tag{3.43}$$

Writing similar equation for each particle of the system and adding all the equations, we get

$$\sum_{i=1}^{n} \boldsymbol{F_i} \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i \tag{3.44}$$

While adding equations, the force acting on particle 1 from particle 2 is equal and opposite to the force acting on particle 2 from particle 1; hence the total virtual work done by the internal forces becomes zero.

Left-hand side of Eq. (3.44) represents the total virtual work done by the external forces and hence it can also be expressed as

$$\delta W = \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_{\mathbf{i}} \cdot \delta \boldsymbol{r}_{\mathbf{i}} \tag{3.45}$$

To simplify right side of Eq. (3.45), consider

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i) = \ddot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i + \dot{\boldsymbol{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\delta \boldsymbol{r}_i) = \ddot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i + \dot{\boldsymbol{r}}_i \cdot (\delta \dot{\boldsymbol{r}}_i) \tag{3.46}$$

Equation (3.46) can be rearranged as

$$\ddot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i) - \dot{\boldsymbol{r}}_i . (\delta \dot{\boldsymbol{r}}_i) \tag{3.47}$$

Again consider

$$\begin{aligned} \delta(\dot{\boldsymbol{r}}_{\boldsymbol{i}} . \dot{\boldsymbol{r}}_{\boldsymbol{i}}) &= \delta \dot{\boldsymbol{r}}_{\boldsymbol{i}} . \dot{\boldsymbol{r}}_{\boldsymbol{i}} + \dot{\boldsymbol{r}}_{\boldsymbol{i}} . \delta \dot{\boldsymbol{r}}_{\boldsymbol{i}} = 2 \dot{\boldsymbol{r}}_{\boldsymbol{i}} . \delta \dot{\boldsymbol{r}}_{\boldsymbol{i}} \\ \therefore \dot{\boldsymbol{r}}_{\boldsymbol{i}} . \delta \dot{\boldsymbol{r}}_{\boldsymbol{i}} &= \frac{1}{2} \delta(\dot{\boldsymbol{r}}_{\boldsymbol{i}} . \dot{\boldsymbol{r}}_{\boldsymbol{i}}) \end{aligned} \tag{3.48}$$

Substituting $\dot{\boldsymbol{r}}_{\boldsymbol{i}} . \delta \dot{\boldsymbol{r}}_{\boldsymbol{i}}$ from Eq. (3.48) into Eq. (3.47), we get

$$\ddot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_i . \delta \boldsymbol{r}_i) - \frac{1}{2} \delta(\dot{\boldsymbol{r}}_i, \dot{\boldsymbol{r}}_i) \tag{3.49}$$

Again, substituting $\dot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i$ from Eq. (3.49) into Eq. (3.44), we get

$$\delta W = \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}}) - \sum_{i=1}^{n} m_i \frac{1}{2}\delta(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\dot{\boldsymbol{r}}_{\boldsymbol{i}}) \tag{3.50}$$

The last term of Eq. (3.50) represents the variation in the total kinetic energy of the system, i.e.,

$$\sum_{i=1}^{n} m_i \frac{1}{2}\delta(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\dot{\boldsymbol{r}}_{\boldsymbol{i}}) = \sum_{i=1}^{n} \delta\left\{m_i \frac{1}{2}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\dot{\boldsymbol{r}}_{\boldsymbol{i}})\right\} = \delta T \tag{3.51}$$

Substituting Eq. (3.51) into Eq. (3.50), we get

$$\begin{aligned} \delta W &= \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}}) - \delta T \\ \therefore \delta W + \delta T &= \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}}) \end{aligned} \tag{3.52}$$

Taking integration of Eq. (3.52) for some finite interval from t_1 to t_2, we get

$$\begin{aligned} &\int_{t_1}^{t_2} (\delta W + \delta T)\mathrm{d}t = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left\{m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}})\right\}\mathrm{d}t = \sum_{i=1}^{n} \int_{t_1}^{t_2} \left\{m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}})\right\}\mathrm{d}t \\ &\therefore \int_{t_1}^{t_2} (\delta W + \delta T)\mathrm{d}t = \sum_{i=1}^{n} m_i (\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\delta \boldsymbol{r}_{\boldsymbol{i}})\big|_{t_1}^{t_2} \end{aligned} \tag{3.53}$$

For given particular states, true and varied paths coincide, i.e., $\delta \boldsymbol{r}_{\boldsymbol{i}}(t = t_1) = 0$ and $\delta \boldsymbol{r}_{\boldsymbol{i}}(t = t_2) = 0$. Hence Eq. (3.53) reduces to

$$\int_{t_1}^{t_2} (\delta W + \delta T)\mathrm{d}t = 0 \tag{3.54}$$

If the system consists of only conservative forces,

$$\delta W = -\delta V \tag{3.55}$$

Substituting δW from Eq. (3.55) into Eq. (3.54), we get

$$\int_{t_1}^{t_2} (\delta T - \delta V)\mathrm{d}t = 0 \tag{3.56}$$

Now defining the Lagrangian functional as

$$L = T - V \tag{3.57}$$

Equation (3.56) reduces to

$$\int_{t_1}^{t_2} \delta L \mathrm{d}t = 0 \tag{3.58}$$

Since integral and variational operators are commutative, Eq. (3.58) can also be expressed as

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = 0 \tag{3.59}$$

Equation (3.59) is known as Hamilton's principle and can be applied for conservative systems.

If the system consists of non-conservative forces also, virtual work is given by

$$\delta W = -\delta V + \delta W_{\mathrm{nc}} \tag{3.60}$$

where δW_{nc} is the virtual work due to all non-conservative forces.

Substituting δW from Eq. (3.60) into Eq. (3.54), we get

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{\mathrm{nc}}) \mathrm{d}t = 0 \tag{3.61}$$

Again, using the Lagrangian functional defined in Eq. (3.57), Eq. (3.61) can also be expressed as

$$\int_{t_1}^{t_2} (\delta L + \delta W_{\mathrm{nc}}) \mathrm{d}t = 0 \tag{3.62}$$

Equation (3.62) is also known as extended Hamilton's principle and can be applied for non-conservative systems.

3.5 Lagrange's Equations for Conservative Discrete Systems

Consider a discrete system having n degrees of freedom. A set of generalized coordinates q_1, q_2, … q_n are used to describe the motion of its inertial elements. As defined in Eq. (3.57), Lagrangian functional is dependent upon the potential energy and kinetic energy of the system which are further functions of time t, displacements q_i and velocities $\dot{q}_i$. Hence the variational formulation to derive the equations of motion can be defined as the minimization of the Lagrangian functional L as

$$I = \int_{t_1}^{t_2} L(t, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)\mathrm{d}t \tag{3.63}$$

Following the procedure of minimization, as explained earlier for derivation of Euler–Lagrange equation, we get

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad i = 1, 2, \ldots, n \tag{3.64}$$

Equation (3.64) is known Lagrange's equations and can be used to derive the governing equations of motion for linear and nonlinear discrete systems.

If the potential energy is the function of displacements (q_i) only and the kinetic energy is the function of velocities ($\dot{q}_i$) only, substituting Eq. (3.57) into Eq. (3.64), we get another form of Lagrange's equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = 0 \quad i = 1, 2, \ldots, n \tag{3.65}$$

3.6 Lagrange's Equations for Non-Conservative Discrete Systems

The work due to the non-conservative forces is dependent upon the displacements (q_i), i.e.,

$$W_{\mathrm{nc}} = W_{\mathrm{nc}}(q_1, q_2, \ldots, q_n) \tag{3.66}$$

Hence the virtual work due to the non-conservative forces can be expressed as

$$\delta W_{\mathrm{nc}} = \sum_{i=1}^{n} \frac{\partial W_{\mathrm{nc}}}{\partial q_i}\delta q_i = \sum_{i=1}^{n} Q_i \delta q_i \tag{3.67}$$

where

$$Q_i = \frac{\partial W_{\mathrm{nc}}}{\partial q_i} \tag{3.68}$$

are called the generalized forces.

Substituting δW_{nc} from Eq. (3.67) into Eq. (3.61), we get

$$\int_{t_1}^{t_2} \left(\delta L + \sum_{i=1}^{n} Q_i \delta q_i \right) \mathrm{d}t = 0 \tag{3.69}$$

Following the procedure explained earlier for derivation of Euler–Lagrange equation, we get variation of the Lagrangian functional as

$$\delta L = \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \tag{3.70}$$

Substituting δL from Eq. (3.70) into Eq. (3.69), we get

$$\sum_{i=1}^{n} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + Q_i \right] \delta q_i \mathrm{d}t = 0 \tag{3.71}$$

Then applying fundamental lemma of calculus of variation, we get

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + Q_i = 0 \quad i = 1, 2, \ldots, n \tag{3.72}$$

Equation (3.72) is the Lagrange's equations for discrete systems consisting of non-conservative effects.

Equation (3.72) can also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \quad i = 1, 2, \ldots, n \tag{3.73}$$

If the potential energy is the function of displacements (q_i) only and the kinetic energy is the function of velocities ($\dot{q}_i$) only, Lagrange's equations for non-conservative discrete systems can also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial V}{\partial q_i} = Q_i \quad i = 1, 2, \ldots, n \tag{3.74}$$

Solved Examples

Example 3.1

Derive equation of motion of the system consisting of a uniform rigid bar of mass *m* and length *L* as shown in Figure E3.1 Use Newton's second law of motion.

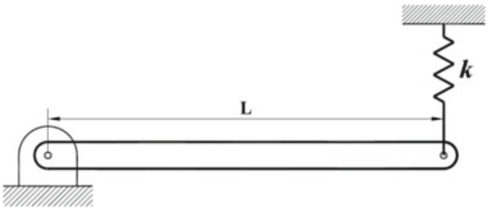


Figure E3.1

Solution

When the bar is attached to the spring, spring undergoes elongation by Δ amount, which is the static deflection of the system. Free-body diagram of the system under this static condition is shown in **Figure E3.1(a)**.

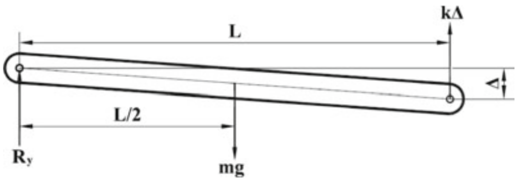


Figure E3.1(a)

Now taking moments of all forces about the left end of the bar, we get

$$mg \times \frac{L}{2} - k\Delta \times L = 0 \qquad \textbf{(a)}$$

If the bar is subjected to further angular displacement of θ from the static equilibrium position and released it undergoes vibratory motion about the static equilibrium position. Free-body diagram for the bar undergoing vibration is shown in **Figure E3.1(b)**

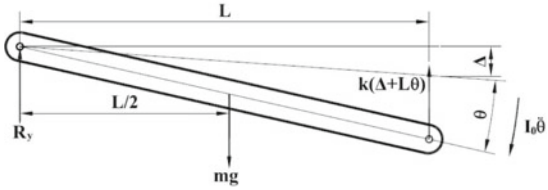


Figure E3.1(b)

Now referring to the free-body diagram shown in **Figure E3.1(b)**, Newton's second law of motion can be applied as

$$\sum M_0 = I_0\ddot{\theta}$$

$$\text{or,}\quad mg \times \frac{L}{2} - k(\Delta + L\theta) \times L = I_0\ddot{\theta}$$

$$\therefore\quad mg \times \frac{L}{2} - k\Delta L - kL^2\theta = I_0\ddot{\theta} \qquad \textbf{(b)}$$

Now using Eq. **(a)** into Eq. **(b)**, we get

$$-kL^2\theta = I_0\theta$$

$$\text{or,}\quad I_0\ddot{\theta} + kL^2\theta = 0$$

$$\text{or,}\quad \left[\frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2\right]\ddot{\theta} + kL^2\theta = 0$$

$$\text{or,}\quad \frac{1}{3}mL^2\ddot{\theta} + kL^2\theta = 0$$

$$\therefore\quad \frac{1}{3}m\ddot{\theta} + k\theta = 0$$

which is the required equation of motion for the given system.

Example 3.2

Derive equation of motion of the system consisting of a uniform bar of mass *m* and length *L* as shown in Figure E3.2.

(a) **Use equivalent system parameters method.**
(b) **Use principle of conservation of energy.**

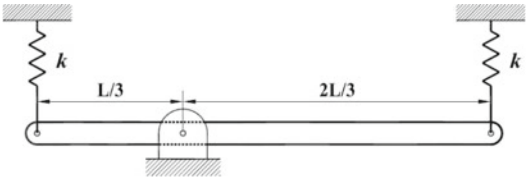


Figure E3.2

Solution

(a) If the angular displacement of the bar (θ) is taken as a generalized coordinate, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2} I_0 \dot{\theta}^2 = \frac{1}{2}\left[\frac{1}{12} mL^2 + m\left(\frac{L}{6}\right)^2\right]\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{9} mL^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2} k\left(\frac{L}{3}\theta\right)^2 + \frac{1}{2} k\left(\frac{2L}{3}\theta\right)^2 = \frac{1}{2}\left(\frac{5}{9} kL^2\right)\theta^2$$

Then equivalent inertia and equivalent stiffness of the system can be determined as

$$I_{eq} = \frac{1}{9} mL^2 \text{ and } k_{eq} = \frac{5}{9} kL^2$$

Then we can write equation of motion directly as,

$$\frac{1}{9} mL^2 \ddot{\theta} + \frac{5}{9} kL^2 \theta = 0$$

$$\therefore \quad m\ddot{\theta} + 5k\theta = 0$$

(b) Using principle of conservation of energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = 0$$

$$\text{or,} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{1}{9} mL^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{5}{9} kL^2\right)\theta^2\right] = 0$$

$$\frac{1}{9} mL^2 \ddot{\theta}\dot{\theta} + \frac{5}{9} kL^2 \theta\dot{\theta} = 0$$

$$\text{or,}\quad \left(\frac{1}{9}mL^2\ddot{\theta} + \frac{5}{9}kL^2\theta\right)\dot{\theta} = 0$$

$$\therefore\quad m\ddot{\theta} + 5k\theta = 0$$

which is the required equation of motion for the given system.

Example 3.3

Derive equation of motion of the system shown in Figure E3.3. Take vertical displacement *x* of the block of mass *m* as a generalized coordinate.

(a) **Use Newton's second law of motion.**
(b) **Use equivalent system parameters method.**

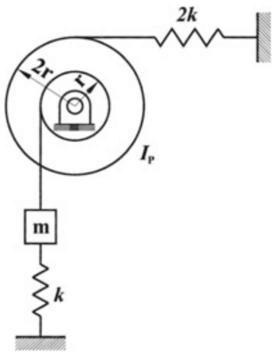


Figure E3.3

Solution

(a) When the block of mass m is attached to the system, the spring with stiffness k will be compressed by Δ, the pulley will rotate in counterclockwise direction by Δ/r and the spring with stiffness $2k$ will be elongated by 2Δ. Free-body diagram of the system under this static condition is shown in **Figure E3.3(a)**.

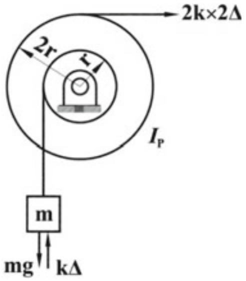


Figure E3.3(a)

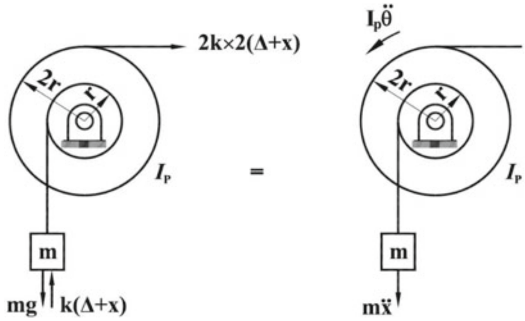


Figure E3.3(b)

Now taking moments of all forces about the center of the pulley, we get

$$mg \times r - k\Delta \times r - 4k\Delta \times 2r = 0$$
$$\therefore \quad mg \times r - 9k\Delta \times r = 0 \qquad (\mathbf{a})$$

If the block is subjected to further displacement of x from the static equilibrium position and released it undergoes vibratory motion about the static equilibrium position. Free-body diagrams for external forces and effective forces acting on the system undergoing vibration are shown in **Figure E3.2(b)**.

Now referring to the free-body diagram, Newton's second law of motion can be applied as

$$\sum M_{\text{external}} = \sum M_{\text{effective}}$$
$$\text{or,} \quad mg \times r - k(\Delta + x) \times r - 2k \times 2(\Delta + x) \times 2r = I_p\ddot{\theta} + m\ddot{x} \times r$$
$$\therefore \quad mg \times r - 9k\Delta \times r - 9kx \times r = I_p\ddot{\theta} + m\ddot{x} \times r \qquad (\mathbf{b})$$

Now using Eq. **(a)** into Eq. **(b)**, we get

$$-9kx \times r = I_p\ddot{\theta} + m\ddot{x} \times r$$
$$\therefore \quad I_p\ddot{\theta} + m\ddot{x} \times r + 9kx \times r = 0$$

Substituting $\theta = x/r$, we get the required equation of motion as

$$I_p\frac{\ddot{x}}{r} + m\ddot{x} \times r + 9kx \times r = 0$$

$$\text{or,} \quad \frac{I_p}{r^2}\ddot{x} + m\ddot{x} + 9kx = 0$$
$$\therefore \quad \left(m + \frac{I_p}{r^2}\right)\ddot{x} + 9kx = 0$$

(b) If the vertical displacement of the block with mass m (x) is taken as a generalized coordinate, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_p\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\left(m + \frac{I_p}{r^2}\right)\dot{x}^2$$

$$V = \frac{1}{2}k(x)^2 + \frac{1}{2}(2k)(2x)^2 = \frac{1}{2}(9k)x^2$$

Then equivalent mass and equivalent stiffness of the system can be determined as

$$m_{eq} = m + \frac{I_p}{r^2} \quad \text{and} \quad k_{eq} = 9k$$

Then we can write equation of motion directly as,

$$\left(m + \frac{I_p}{r^2}\right)\ddot{x} + 9kx = 0$$

Example 3.4

Derive equation of motion of the system shown in Figure E3.4 Use Lagrange's equation. Use vertical displacement of the block of mass $2m$ from the static equilibrium position as the generalized coordinate.

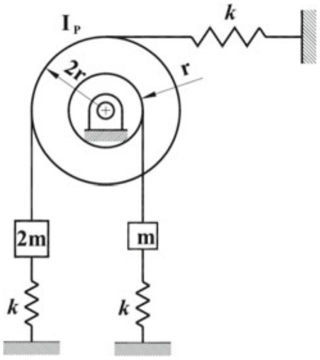


Figure E3.4

Solution

If the vertical displacement of the block with mass $2m$ (x) is taken as a generalized coordinate, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}m\left(\frac{\dot{x}}{2}\right)^2 + \frac{1}{2}I_p\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\dot{x}^2$$

$$V = \frac{1}{2}k(x)^2 + \frac{1}{2}k(x)^2 + \frac{1}{2}k\left(\frac{x}{2}\right)^2 = \frac{1}{2}\left(\frac{9k}{4}\right)x^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\dot{x}^2 - \frac{1}{2}\left(\frac{9k}{4}\right)x^2$$

Now, using Lagrange's equation,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\text{or,} \quad -\frac{9k}{4}x - \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\dot{x}\right] = 0$$

$$\text{or,} \quad -\frac{9k}{4}x - \left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\ddot{x} = 0$$

$$\therefore \quad \left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\ddot{x} + \frac{9k}{4}x = 0$$

which is the required equation of motion for the given system.

Example 3.5

Derive equation of motion of the system shown in Figure E3.5. Take vertical displacement *x* of the block of mass *m* as a generalized coordinate.

(a) **Use Newton's second law of motion.**
(b) **Use equivalent system parameters method.**

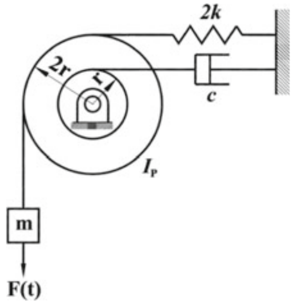


Figure E3.5

Solution

(a) When the block of mass m is attached to the system, the spring with stiffness $2k$ will be elongated by Δ, the pulley will rotate in counterclockwise direction by Δ/r. Free-body diagram of the system under this static condition is shown in **Figure E3.5(a)**.

Now taking moments of all forces about the center of the pulley, we get

$$mg \times 2r - 2k\Delta \times 2r = 0 \qquad (\mathbf{a})$$

If the block is subjected to further displacement of x from the static equilibrium position and released it undergoes vibratory motion about the static equilibrium position. Free-body diagrams for external forces and effective forces acting on the system undergoing vibration is shown in **Figure E3.5(b)**.

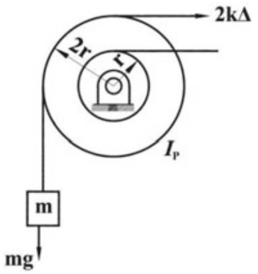


Figure E3.5(a)

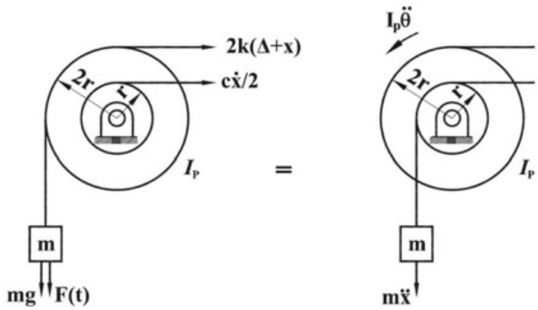


Figure E3.5(b)

Now referring to the free-body diagram, Newton's second law of motion can be applied as

$$\sum M_{\text{external}} = \sum M_{\text{effective}}$$

$$\text{or,} \quad mg \times 2r - 2k(\Delta + x) \times 2r - c \times \frac{\dot{x}}{2} \times r + F(t) \times 2r = I_p\ddot{\theta} + m\ddot{x} \times 2r$$

$$\therefore \quad mg \times r - 2k\Delta \times 2r - 4kx \times r - c \times \frac{\dot{x}}{2} \times r + F(t) \times 2r = I_p\ddot{\theta} + m\ddot{x} \times 2r \qquad (\mathbf{b})$$

Now using Eq. **(a)** into Eq. **(b)**, we get

$$-4kx \times r - c \times \frac{\dot{x}}{2} \times r + F(t) \times 2r = I_p\ddot{\theta} + m\ddot{x} \times 2r$$

$$\therefore \quad I_p\ddot{\theta} + m\ddot{x} \times 2r + c \times \frac{\dot{x}}{2} \times r + 4kx \times r = F(t) \times 2r \qquad (\mathbf{c})$$

Substituting $\theta = x/2r$, we get the required equation of motion as

$$I_p \times \frac{\ddot{x}}{2r} + m\ddot{x} \times 2r + c \times \frac{\dot{x}}{2} \times r + 4kx \times r = F(t) \times 2r$$

$$\text{or,} \quad I_p \times \frac{\ddot{x}}{4r^2} + m\ddot{x} + c \times \frac{\dot{x}}{4} + 2kx = F(t)$$

$$\therefore \quad \left(m + \frac{I_p}{4r^2}\right)\ddot{x} + \frac{c}{4}\dot{x} + 2kx = F(t)$$

(b) If the vertical displacement of the block with mass m (x) is taken as a generalized coordinate, the total kinetic energy (T), potential energy (V) of the system and work done against the damping (W_d) can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_p\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(m + \frac{I_p}{4r^2}\right)\dot{x}^2$$

$$V = \frac{1}{2}(2k)(x)^2 = \frac{1}{2}(2k)x^2$$

$$W_d = -\int c\left(\frac{\dot{x}}{2}\right)\mathrm{d}\left(\frac{x}{2}\right) = -\int\left(\frac{c}{4}\right)\dot{x}\,\mathrm{d}x$$

Then equivalent mass, equivalent stiffness and equivalent damping of the system can be determined as

$$m_{eq} = m + \frac{I_p}{4r^2},\ k_{eq} = 2k \text{ and } c_{eq} = \frac{c}{4}$$

Then we can write equation of motion directly as,

$$\left(m + \frac{I_p}{4r^2}\right)\ddot{x} + \frac{c}{4}\dot{x} + 2kx = F(t)$$

Example 3.6

Derive equation of motion of the system consisting of a uniform bar of mass *m* and length *L* as shown in Figure E3.6. Take angular displacement *θ* of the bar as a generalized coordinate

(a) **Use Newton's second law of motion.**
(b) **Use equivalent system parameters method.**

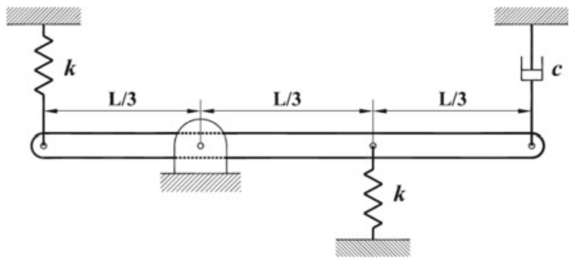


Figure E3.6

Solution

(a) When the bar is attached to the system, it undergoes clockwise rotation by θ_s due its self-weight, then both springs will be compressed by $L\theta_s/3$. Free-body diagram of the system under this static condition is shown in **Figure E3.6(a)**.

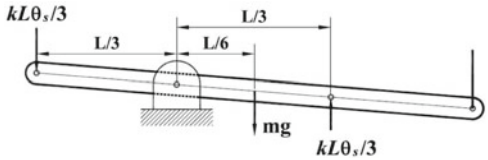


Figure E3.6(a)

Now taking moments of all forces about the pivoted point, we get

$$mg \times \frac{L}{6} - \frac{1}{3}kL\theta_s \times \frac{L}{3} - \frac{1}{3}kL\theta_s \times \frac{L}{3} = 0$$

$$\therefore \quad mg \times \frac{L}{6} - \frac{2}{9}kL^2\theta_s = 0 \qquad \textbf{(a)}$$

If the bar is subjected to further angular displacement of θ from the static equilibrium position and released it undergoes vibratory motion about the static equilibrium position. Free-body diagrams for external forces and effective forces acting on the system undergoing vibration is shown in **Figure E3.6(b)**.

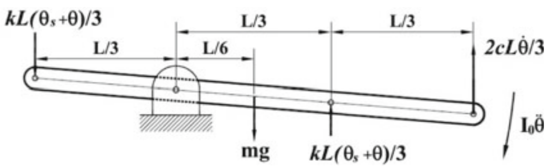


Figure E3.6(b)

Now referring to the free-body diagram shown in **Figure E3.6(b)**, Newton's second law of motion can be applied as

$$\sum M_0 = I_0\ddot{\theta}$$

$$\text{or,}\quad mg\times\frac{L}{6}-\frac{1}{3}kL(\theta_s+\theta)\times\frac{L}{3}-\frac{1}{3}kL(\theta_s+\theta)\times\frac{L}{3}-\frac{2}{3}cL\dot{\theta}\times\frac{2L}{3}=I_0\ddot{\theta}$$

$$\text{or,}\quad mg\times\frac{L}{2}-\frac{2}{9}kL^2\theta_s-\frac{2}{9}kL^2\theta-\frac{4}{9}cL^2\dot{\theta}=I_0\ddot{\theta}\qquad(\mathbf{b})$$

Now using Eq. **(a)** into Eq. **(b)**, we get

$$-\frac{2}{9}kL^2\theta-\frac{4}{9}cL^2\dot{\theta}=I_0\ddot{\theta}$$

$$\text{or,}\quad I_0\ddot{\theta}+\frac{4}{9}cL^2\dot{\theta}+\frac{2}{9}kL^2\theta=0$$

$$\text{or,}\quad \left[\frac{1}{12}mL^2+m\left(\frac{L}{6}\right)^2\right]\ddot{\theta}+\frac{4}{9}cL^2\dot{\theta}+\frac{2}{9}kL^2\theta=0$$

$$\text{or,}\quad \frac{1}{9}mL^2\ddot{\theta}+\frac{4}{9}cL^2\dot{\theta}+\frac{2}{9}kL^2\theta=0$$

$$\therefore\quad m\ddot{\theta}+4c\dot{\theta}+2k\theta=0$$

which is the required equation of motion for the given system.

(b) If the angular displacement of the bar (θ) is taken as a generalized coordinate, the total kinetic energy (T), potential energy (V) of the system and work done against the damping (W_d) can be expressed as

$$T=\frac{1}{2}I_0\dot{\theta}^2=\frac{1}{2}\left[\frac{1}{12}mL^2+m\left(\frac{L}{6}\right)^2\right]\dot{\theta}^2=\frac{1}{2}\left(\frac{1}{9}mL^2\right)\dot{\theta}^2$$

$$V=\frac{1}{2}k\left(\frac{L}{3}\theta\right)^2+\frac{1}{2}k\left(\frac{L}{3}\theta\right)^2=\frac{1}{2}\left(\frac{2}{9}kL^2\right)\theta^2$$

$$W_d=-\int c\left(\frac{2}{3}cL\dot{\theta}\right)\mathrm{d}\left(\frac{2}{3}cL\theta\right)=-\int\left(\frac{4}{9}cL^2\right)\dot{\theta}\,\mathrm{d}\theta$$

Then equivalent inertia, equivalent stiffness and equivalent damping of the system can be determined as

$$m_{eq}=\frac{1}{9}mL^2\quad k_{eq}=\frac{2}{9}kL^2\quad\text{and}\quad c_{eq}=\frac{4}{9}cL^2$$

Then we can write equation of motion directly as,

$$\frac{1}{9}mL^2\ddot{\theta} + \frac{4}{9}cL^2\dot{\theta} + \frac{2}{9}kL^2\theta = 0$$
$$\therefore \quad m\ddot{\theta} + 4c\dot{\theta} + 2k\theta = 0$$

Example 3.7

Derive equation of motion of the system shown in Figure E3.7. Use Lagrange's equation. Use displacement of block m (x_1) and displacement of block $2m$ (x_2) as a set of generalized coordinates

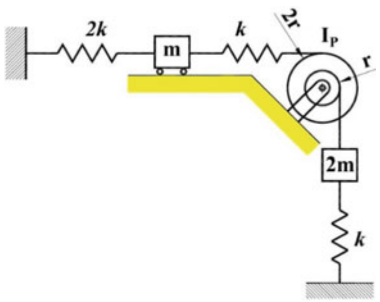


Figure E3.7

Solution

If the displacement of block m (x_1) and displacement of block $2m$ (x_2) as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\left(\frac{\dot{x}_2}{r}\right)^2 = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\left(2m + \frac{I_P}{r^2}\right)\dot{x}_2^2$$

$$V = \frac{1}{2}(2k)(x_1)^2 + \frac{1}{2}k(2x_2 - x_1)^2 + \frac{1}{2}(k)(x_2)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}\left(2m + \frac{I_P}{r^2}\right)\dot{x}_2^2 - \frac{1}{2}(2k)(x_1)^2 - \frac{1}{2}k(2x_2 - x_1)^2 - \frac{1}{2}(k)(x_2)^2$$

Now, using Lagrange' equation for the generalized coordinate x_1,

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$$

$$\text{or,}\quad -2kx_1 + k(2x_2 - x_1) - \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}_1] = 0$$

$$\text{or,}\quad -3kx_1 + 2kx_2 - m\ddot{x}_1 = 0$$

$$\therefore\quad m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0 \qquad (\mathbf{a})$$

Again, using Lagrange' equation for the generalized coordinate x_2,

$$\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0$$

$$\text{or,}\quad -2k(2x_2 - x_1) - kx_2 - \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(2m + \frac{I_P}{r^2}\right)\dot{x}_2\right] = 0$$

$$\text{or,}\quad -5kx_2 + 2kx_1 - \left(2m + \frac{I_P}{r^2}\right)\ddot{x}_2 = 0$$

$$\therefore\quad \left(2m + \frac{I_P}{r^2}\right)\ddot{x}_2 - 2kx_1 + 5kx_2 = 0 \qquad (\mathbf{b})$$

Equations (a) and (b) can be expressed in matrix form for the equation of motion of the system as

$$\begin{bmatrix} m & 0 \\ 0 & 2m + \frac{I_P}{r^2} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 5k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example 3.8

Derive equation of motion of the system shown in Figure E3.8. Use Lagrange's equation. Use vertical displacement of mass (x) and rotation of the pulley (θ) as a set of generalized coordinates

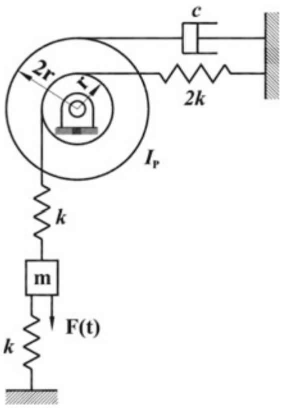


Figure E3.8

Solution

If the displacement of block m (x) and the rotation of the pulley (θ) as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\dot{\theta}\right)^2$$
$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(2k)(r\theta)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\dot{\theta}\right)^2 - \frac{1}{2}kx^2 - \frac{1}{2}k(x - r\theta)^2 - \frac{1}{2}(2k)(r\theta)^2$$

Similarly, work done by the non-conservative forces can be determined as

$$W_{\mathrm{nc}} = Fx - c\left(2r\dot{\theta}\right)(2r\theta) = Fx - 4cr^2\theta\dot{\theta}$$

From which generalized forces Q_x and Q_θ can be determined as

$$Q_x = \frac{\partial W_{\mathrm{nc}}}{\partial x} = F$$
$$Q_\theta = \frac{\partial W_{\mathrm{nc}}}{\partial \theta} = -4cr^2\dot{\theta}$$

Now, using Lagrange' equation for the generalized coordinate x,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + Q_x = 0$$

$$\text{or,}\quad -kx - k(x - r\theta) - \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}] + F = 0$$

$$\text{or,}\quad -2kx + kr\theta - m\ddot{x} + F = 0$$

$$\therefore\quad m\ddot{x} + 2kx - kr\theta = F \qquad (\mathbf{a})$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + Q_\theta = 0$$

$$\text{or,}\quad kr(x - r\theta) - 2kr^2\theta - \frac{d}{dt}\left[I_P\dot{\theta}\right] - 4cr^2\dot{\theta} = 0$$

$$\text{or,}\quad krx - 3kr^2\theta - I_P\ddot{\theta} - 4cr^2\dot{\theta} = 0$$

$$\therefore\quad I_P\ddot{\theta} + 4cr^2\dot{\theta} - krx + 3kr^2\theta = 0 \qquad (\mathbf{b})$$

Equations **(a)** and **(b)** can be expressed in matrix form for the equation of motion of the system as

$$\begin{bmatrix} m & 0 \\ 0 & I_P \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4cr^2 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & -kr \\ -kr & 3kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

Example 3.9

Derive equation of motion of the system shown in Figure E3.9 Use Newton's second law of motion.

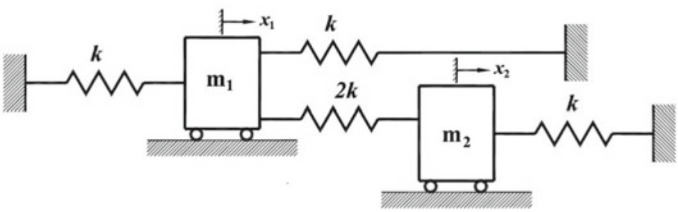


Figure E3.9

Solution

Assuming $x_2 > x_1$, free-body diagrams of block with mass m_1 and m_2 are shown in **Fig. 3.9(a)**.

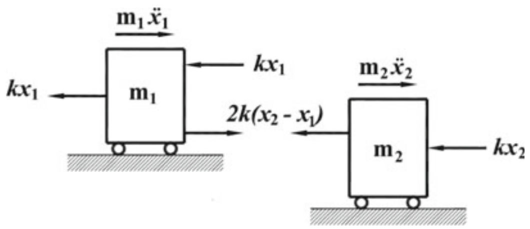


Figure E3.9(a)

Now referring to the free-body diagram of the block with mass m_1 and applying Newton's second law of motion

$$\sum F_1 = m_1\ddot{x}_1$$

$$\text{or,} \quad -kx_1 - kx_1 + 2k(x_2 - x_1) = m_1\ddot{x}_1$$

$$\text{or,} \quad -4kx_1 + 2kx_2 - m_1\ddot{x}_1 = 0$$
$$\therefore \quad m_1\ddot{x}_1 + 4kx_1 - 2kx_2 = 0 \qquad \textbf{(a)}$$

Again, referring to the free-body diagram of the block with mass m_2 and applying Newton's second law of motion

$$\sum F_2 = m_2\ddot{x}_2$$

$$\text{or,} \quad -2k(x_2 - x_1) - kx_2 = m_2\ddot{x}_2$$

$$\text{or,} \quad 2kx_1 - 3kx_2 - m_2\ddot{x}_2 = 0$$
$$\therefore \quad m_2\ddot{x}_2 - 2kx_1 + 3kx_2 = 0 \qquad \textbf{(b)}$$

Equations **(a)** and **(b)** can be expressed in matrix form for the equation of motion of the system as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4k & -2k \\ -2k & 3k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example 3.10

Derive equation of motion of the system shown in Figure E3.10. Use Lagrange's equation. Use displacement of block $m(x_1)$ and displacement of block $2m(x_2)$ and rotation of pulley (θ) as a set of generalized coordinates.

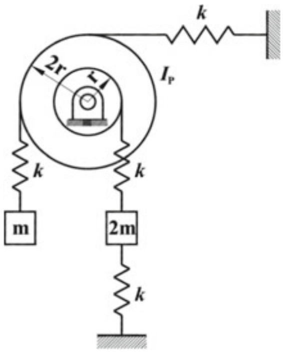


Figure E3.10

Solution

If the displacement of block $m(x_1)$ and displacement of block $2m(x_2)$ and rotation of pulley (θ) are taken as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2$$
$$V = \frac{1}{2}k(x_1 - 2r\theta)^2 + \frac{1}{2}k(2r\theta)^2 + \frac{1}{2}k(x_2 - r\theta)^2 + \frac{1}{2}(k)(x_2)^2$$

Then Lagrangian functional for the system can be determined as

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_P\dot{\theta}^2 - \frac{1}{2}k(x_1 - 2r\theta)^2 \\ &\quad - \frac{1}{2}k(2r\theta)^2 - \frac{1}{2}k(x_2 - r\theta)^2 - \frac{1}{2}(k)(x_2)^2 \end{aligned}$$

Now, using Lagrange' equation for the generalized coordinate x_1,

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$$

$$\text{or,} \quad -k(x_1 - 2r\theta) - \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}_1] = 0$$

$$\text{or,} \quad -kx_1 + 2kr\theta - m\ddot{x}_1 = 0$$
$$\therefore \quad m\ddot{x}_1 + kx_1 - 2kr\theta = 0 \qquad (\mathbf{a})$$

Similarly, using Lagrange' equation for the generalized coordinate x_2,

$$\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0$$

$$\text{or,} \quad -k(x_2 - r\theta) - kx_2 - \frac{\mathrm{d}}{\mathrm{d}t}[2m\dot{x}_2] = 0$$

$$\text{or,} \quad -2kx_2 + kr\theta - 2m\ddot{x}_2 = 0$$

$$\therefore \quad 2m\ddot{x}_2 + 2kx_2 - kr\theta = 0 \qquad (\mathbf{b})$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\text{or,} \quad 2kr(x_1 - 2r\theta) - 4kr^2\theta + kr(x_2 - r\theta) - \frac{\mathrm{d}}{\mathrm{d}t}\left[I_P\dot{\theta}\right] = 0$$

$$\text{or,} \quad 2krx_1 + krx_2 - 9kr^2\theta - I_P\ddot{\theta} = 0$$

$$\therefore \quad I_P\ddot{\theta} - 2krx_1 - krx_2 + 9kr^2\theta = 0 \qquad (\mathbf{c})$$

Equations (**a**), (**b**) and (**c**) can be expressed in matrix form for the equation of motion of the system as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_P \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -2kr \\ 0 & 2k & -kr \\ -2kr & -kr & 9kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example 3.11.

A spring with stiffness k and a concentrated mass M is attached at the left and right ends of a bar of length L respectively as shown in Figure E3.11. The bar material has a density of ρ and its cross-sectional area is A. Derive equation of motion for the longitudinal vibration of the bar using Hamilton's principle.

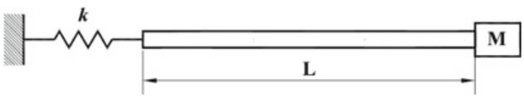


Figure E3.11

Solution

Let $u(x, t)$ be the longitudinal deformation of the continuous bar. Then kinetic energy of the bar due to longitudinal deformation is given by

$$T_b = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the concentrated mass is given by

$$T_M = \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_b + T_M = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\bigg|_{x=L}$$

The potential energy of the bar is given by

$$V_b = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Similarly, the potential energy of the spring is given by

$$V_S = \frac{1}{2}k\,(u)^2\big|_{x=0}$$

Then, the total kinetic energy of the system is given by

$$V = V_b + V_S = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}k\,(u)^2\big|_{x=0}$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\bigg|_{x=L}$$
$$- \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x - \frac{1}{2}k\,(u)^2\big|_{x=0}$$

Now applying Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\mathrm{d}t = 0$$

$$\text{or,}\quad \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{M\left(\frac{\partial u}{\partial t}\right)^2\Big|_{x=L}\right\}\mathrm{d}t$$
$$-\frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x\mathrm{d}t$$
$$-\frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(u)^2\big|_{x=0}\right\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2}\left\{M\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\Big|_{x=L}\right\}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\{k\,u\delta(u)|_{x=0}\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta(u)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\mathrm{d}t + \left\{M\left(\frac{\partial u}{\partial t}\right)\delta(u)\Big|_{x=L}\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\Big|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\{k\,u\delta(u)|_{x=0}\}\mathrm{d}t = 0$$

Since $\delta(u)|_{t_1}^{t_2} = 0$,

$$\text{or,}\quad -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\Big|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,u\delta(u)|_{x=0}\right\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_{0}^{L}\left[\rho A\left(\frac{\partial^2 u}{\partial t^2}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\right]\delta(u)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 u}{\partial t^2}\right)+EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\bigg|_{x=L}\mathrm{d}t-\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)-ku\right\}\delta(u)\bigg|_{x=0}\mathrm{d}t=0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 u}{\partial t^2}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}=0$$

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\rho A\left(\frac{\partial^2 u}{\partial t^2}\right)-EA\left(\frac{\partial^2 u}{\partial x^2}\right)=0$$

The associated boundary conditions are

(a) either $EA\left(\frac{\partial u}{\partial x}\right)-ku=0$ or $\delta(u)=0$ at $x=0$.
(b) either $M\left(\frac{\partial^2 u}{\partial t^2}\right)+EA\left(\frac{\partial u}{\partial x}\right)=0$ or $\delta(u)=0$ at $x=L$.

From the mentioned conditions, the appropriate boundary conditions for the system can be expressed as

$$EA\left(\frac{\partial u}{\partial x}\right)-ku=0\,\text{at}\,x=0\,\text{and}\,M\left(\frac{\partial^2 u}{\partial t^2}\right)+EA\left(\frac{\partial u}{\partial x}\right)=0\,\text{at}\,x=L.$$

Example 3.12

A rigid disk of mass moment of inertia I_d is attached to free end shaft of length L and is subjected to an external torque of magnitude $T(t)$ as shown in Figure E3.12. The shaft material has a density of ρ, shear modulus of elasticity of G and moment of inertia of section of J. Derive equation of motion for the torsional vibration of the shaft using Hamilton's principle.

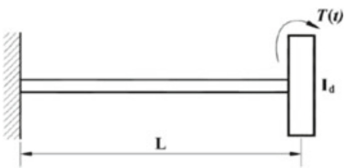


Figure E3.12

Solution

Let $\theta(x, t)$ be the torsional deformation of the continuous shaft. Then kinetic energy of the shaft due to torsional deformation is given by

$$T_s = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial\theta}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the rigid disk is given by

$$T_d = \frac{1}{2}I_d\left(\frac{\partial\theta}{\partial t}\right)^2\bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_b + T_M = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial\theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}I_d\left(\frac{\partial\theta}{\partial t}\right)^2\bigg|_{x=L}$$

The potential energy of the shaft is given by

$$V = \frac{1}{2}\int_0^L GJ\left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial\theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}I_d\left(\frac{\partial\theta}{\partial t}\right)^2\bigg|_{x=L} - \frac{1}{2}\int_0^L GJ\left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x$$

Work done by the external force is given by

$$W_{\mathrm{nc}} = T\theta|_{x=L}$$

where δ_d is the Dirac delta function.

Now applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{\mathrm{nc}})\,\mathrm{d}t = 0$$

$$\text{or,} \quad \frac{1}{2}\delta \int_{t_1}^{t_2} \int_0^L \int_{t_1}^{t_2} (\delta L + \delta W_{nc}) \mathrm{d}t \mathrm{d}t + \frac{1}{2}\delta \int_{t_1}^{t_2} \left\{ I_d \left(\frac{\partial \theta}{\partial t} \right)^2 \bigg|_{x=L} \right\} \mathrm{d}t$$

$$-\frac{1}{2}\delta \int_{t_1}^{t_2} \int_0^L GJ\left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}t + \delta \int_{t_1}^{t_2} T\theta|_{x=L} \mathrm{d}t = 0$$

$$\text{or,} \quad \int_{t_1}^{t_2} \int_0^L \rho J\left(\frac{\partial \theta}{\partial t}\right)\delta\left(\frac{\partial \theta}{\partial t}\right)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} \left\{ V\left(\frac{\partial \theta}{\partial t}\right)\delta\left(\frac{\partial \theta}{\partial t}\right)\bigg|_{x=L} \right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} \int_0^L GJ\left(\frac{\partial \theta}{\partial x}\right)\delta\left(\frac{\partial \theta}{\partial x}\right)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} T\,\delta\theta|_{x=L} \mathrm{d}t = 0$$

$$\text{or,} \quad \int_0^L \rho J\left(\frac{\partial \theta}{\partial t}\right)\delta(\theta)\bigg|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} \int_0^L \rho J\left(\frac{\partial^2 \theta}{\partial t^2}\right)\delta(\theta)\mathrm{d}x\mathrm{d}t + \left\{ I_d\left(\frac{\partial \theta}{\partial t}\right)\delta(\theta)\bigg|_{x=L} \right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} \left\{ I_d\left(\frac{\partial^2 \theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=L} \right\}\mathrm{d}t - \int_{t_1}^{t_2} GJ\left(\frac{\partial \theta}{\partial x}\right)\delta(\theta)\bigg|_{x=0}^{x=L} \mathrm{d}t$$

$$+\int_{t_1}^{t_2} \int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{ GJ\left(\frac{\partial \theta}{\partial x}\right)\right\}\delta(\theta)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} T\,\delta\theta|_{x=L}\mathrm{d}t = 0$$

Since $\delta(\theta)|_{t_1}^{t_2} = 0$

$$\text{or,} \quad -\int_{t_1}^{t_2} \int_0^L \rho J\left(\frac{\partial^2 \theta}{\partial t^2}\right)\delta(\theta)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2} \left\{ I_d\left(\frac{\partial^2 \theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=L} \right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} GJ\left(\frac{\partial \theta}{\partial x}\right)\delta(\theta)\bigg|_{x=0}^{x=L} \mathrm{d}t + \int_{t_1}^{t_2} \int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{ GJ\left(\frac{\partial \theta}{\partial x}\right)\right\}\delta(\theta)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} T\,\delta\theta|_{x=L}\mathrm{d}t = 0$$

$$\text{or,} \quad \int_{t_1}^{t_2} \int_0^L \left[\rho J\left(\frac{\partial^2 \theta}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{ GJ\left(\frac{\partial \theta}{\partial x}\right)\right\}\right]\delta(\theta)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2} \left\{ I_d\left(\frac{\partial^2 \theta}{\partial t^2}\right) + GJ\left(\frac{\partial \theta}{\partial x}\right) - T\right\}\delta(\theta)\bigg|_{x=L} \mathrm{d}t - \int_{t_1}^{t_2} \left\{ GJ\left(\frac{\partial \theta}{\partial x}\right)\right\}\delta(\theta\theta)\bigg|_{x=0} \mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\right\} = 0$$

Since the shaft is uniform and homogeneous, equation of motion can also be expressed as

$$\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) - GJ\left(\frac{\partial^2\theta}{\partial x^2}\right) = 0$$

The associated boundary conditions are

(a) either $GJ\left(\frac{\partial\theta}{\partial x}\right) = 0$ or $\delta(\theta) = 0$ at $x = 0$.
(b) either $I_d\left(\frac{\partial^2\theta}{\partial t^2}\right) + GJ\left(\frac{\partial\theta}{\partial x}\right) - T = 0$ or $\delta(\theta) = 0$ at $x = L$.

From the mentioned conditions, the appropriate boundary conditions for the system can be expressed as

$$\theta = 0 \,\text{at}\, x = 0 \,\text{and}\, I_d\left(\frac{\partial^2\theta}{\partial t^2}\right) + GJ\left(\frac{\partial\theta}{\partial x}\right) - T = 0 \,\text{at}\, x = L.$$

Example 3.13

A beam of length L shown in Figure E3.13 undergoing traverse vibration is restrained by two springs each having stiffness of k. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive equation of motion for the transverse vibration of the beam using Hamilton's principle.

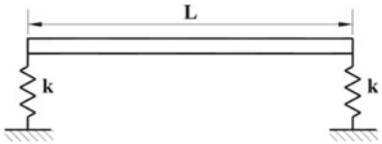


Figure E3.13

Solution

Let $w(x, t)$ be the transverse deformation of the beam due bending about y axis. Then kinetic energy of the beam due to bending is given by

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

The strain energy of the beam is given by

$$V_b = \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Similarly, the potential energy of the springs is given by

$$V_S = \frac{1}{2}k\,(w)^2\Big|_{x=0} + \frac{1}{2}k\,(w)^2\Big|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$V = V_b + V_S = \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}k\,(w)^2\Big|_{x=0} + \frac{1}{2}k\,(w)^2\Big|_{x=L}$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x - \frac{1}{2}k\,(w)^2\Big|_{x=0} - \frac{1}{2}k\,(w)^2\Big|_{x=L}$$

Now applying Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\mathrm{d}t = 0$$

$$\text{or,}\quad \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(w)^2\Big|_{x=0}\right\}\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(w)^2\Big|_{x=L}\right\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=0}\}\mathrm{d}t - \int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(u)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2} EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=0}\}\mathrm{d}t - \int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(u)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2} EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\bigg|_{x=0}^{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\,dt$$

$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=0}\}\mathrm{d}t - \int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

Since $\delta(w)|_{t_1}^{t_2} = 0$,

$$\text{or,}\quad -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}\right\}\mathrm{d}t + \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\bigg|_{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\bigg|_{x=0}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=0}\}\mathrm{d}t - \int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]\delta(w)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}\right\}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} + kw\right]\delta(w)\bigg|_{x=0}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{dx}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - kw\right]\delta(w)\bigg|_{x=L}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0$$

The associated boundary conditions are

(a) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} + kw = 0$ or $\delta(w) = 0$ at $x = 0$.
(b) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - kw = 0$ or $\delta(w) = 0$ at $x = L$.
(c) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = 0$.
(d) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = L$.

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + EI_y\left(\frac{\partial^4 w}{\partial x^4}\right) = 0$$

The corresponding associated boundary conditions are

(a) either $EI_y\left(\frac{\partial^3 w}{\partial x^3}\right) + kw = 0$ or $\delta(w) = 0$ at $x = 0$.
(b) either $EI_y\left(\frac{\partial^3 w}{\partial x^3}\right) - kw = 0$ or $\delta(w) = 0$ at $x = L$.
(c) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = 0$.
(d) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = L$.

From the mentioned conditions, the appropriate boundary conditions for the system can be expressed as

(a) $EI_y\left(\frac{\partial^3 w}{\partial x^3}\right) + kw = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$ at $x = 0$.
(b) $EI_y\left(\frac{\partial^3 w}{\partial x^3}\right) - kw = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$ at $x = L$.

Example 3.14

A simply supported beam of length L shown in Figure E3.14 is subjected to a concentrated load F at its mid-span. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive equation of motion for the transverse vibration of the beam using Hamilton's principle.

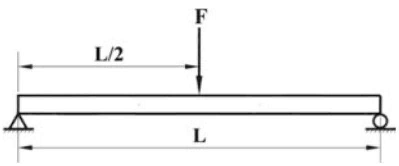


Figure E3.14

Solution

Let $w(x, t)$ be the transverse deformation of the beam due bending about y axis. Then kinetic energy of the beam due to bending is given by

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

The strain energy of the beam is given by

$$V = \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Work done by the external force is given by

$$W_{\mathrm{nc}} = F\, w|_{x=L/2} = \int_0^L F w \delta_d\left(x - \frac{L}{2}\right)\mathrm{d}x$$

where δ_d is the Dirac delta function.
Now applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{\mathrm{nc}})\mathrm{d}t = 0$$

$$\text{or,}\quad \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t$$

$$+\delta\int_{t_1}^{t_2}\int_0^L F w \delta_d\left(x - \frac{L}{2}\right)\mathrm{d}x\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \left\{F\delta_d\left(x - \frac{L}{2}\right)\right\}\delta w\mathrm{d}x\mathrm{d}t = 0$$

$$\text{or,}\quad \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(u)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2} EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \left\{F\delta_d\left(x-\frac{L}{2}\right)\right\}\delta w\mathrm{d}x\mathrm{d}t = 0$$

$$\text{or,}\quad \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\mathrm{d}t$$

$$-\int_{t_1}^{t_2} EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}^{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \left\{F\delta_d\left(x-\frac{L}{2}\right)\right\}\delta w\mathrm{d}x\mathrm{d}t = 0$$

Since $\delta(w)|_{t_1}^{t_2} = 0$,

$$\text{or,}\quad -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L}\right\}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}\right\}\mathrm{d}t + \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L\left\{F\delta_d\left(x-\frac{L}{2}\right)\right\}\delta w\mathrm{d}x\mathrm{d}t = 0$$

$$\text{or,}\quad \int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - F\delta_d\left(x-\frac{L}{2}\right)\right]\delta(w)\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]\delta(w)\Big|_{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]\delta(w)\Big|_{x=L}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = F\delta_d\left(x-\frac{L}{2}\right)$$

The associated boundary conditions are

(a) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0$ or $\delta(w) = 0$ at $x = 0$.
(b) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0$ or $\delta(w) = 0$ at $x = L$.
(c) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = 0$.
(b) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = L$.

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + EI_y\left(\frac{\partial^4 w}{\partial x^4}\right) = F\delta_d\left(x-\frac{L}{2}\right)$$

The corresponding associated boundary conditions are

(a) either $\left(\frac{\partial^3 w}{\partial x^3}\right) = 0$ or $\delta(w) = 0$ at $x = 0$.
(b) either $\left(\frac{\partial^3 w}{\partial x^3}\right) = 0$ or $\delta(w) = 0$ at $x = L$.
(c) either $\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = 0$.
(b) either $\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = L$.

From the mentioned conditions, the appropriate boundary conditions for the system can be expressed as

(a) $w = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$ at $x = 0$.
(b) $w = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$ at $x = L$.

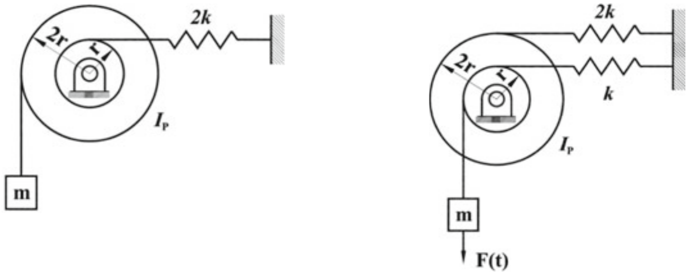
Review Questions

1. List common classical methods used to derive equations of motion of a vibrating system.
2. Define independent variable, function and functional with examples.
3. Write down the difference between differential calculus and variational calculus.
4. State and prove fundamental lemma of calculus of variations.
5. Derive Euler–Lagrange equation for a function defined by $I(u) = \int_{x_a}^{x_b} F\big(x, u(x), u'(x)\big)\mathrm{d}x$.
6. Derive Hamilton's principle and extended Hamilton's principle.
7. Derive Lagrange equations for discrete conservative and non-conservative systems.

Exercise

1. Derive equation of motion of the system shown in **Figure P3.1**. Take vertical displacement x of the block of mass m as a generalized coordinate.

   (a) Use Newton's second law of motion.
   (b) Use equivalent system parameters method.

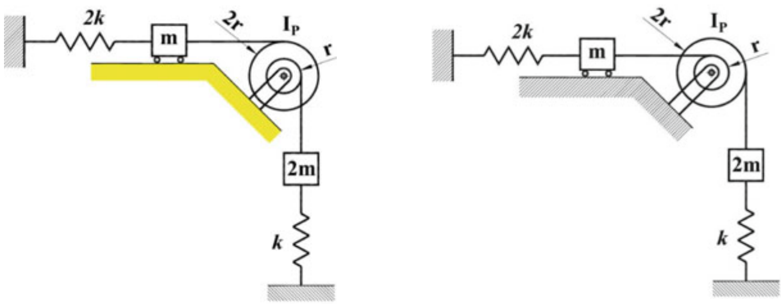


(a) **(b)**

Figure P3.1

2. Derive equation of motion of the system shown in **Figure P3.2**. Take vertical displacement x of the block of mass $2m$ as a generalized coordinate.

   (a) Use equivalent system parameters method.
   (b) Use principle of conservation of energy.

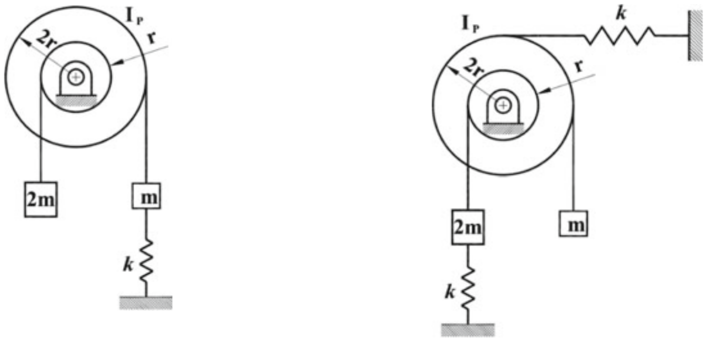


(a) **(b)**

Figure P3.2

3. Derive equation of motion of the system shown in **Figure P3.3**. Take vertical displacement x of the block of mass $2m$ as a generalized coordinate.

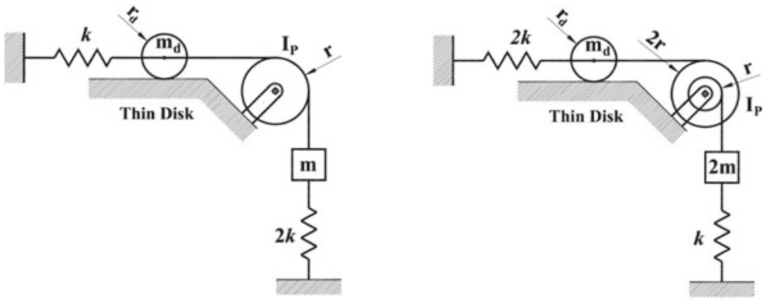
   (a) Use equivalent system parameters method.
   (b) Use principle of conservation of energy.
   (c) Use Lagrange's equation.



(a) **(b)**

Figure P3.3

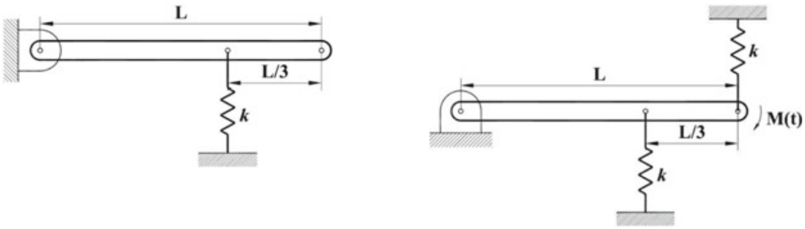
4. Derive equation of motion of the system shown in **Figure P3.4**. Take vertical displacement x of the block as a generalized coordinate. Assume that the disk is thin and rolls without slip.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.



(a) **(b)**

Figure P3.4

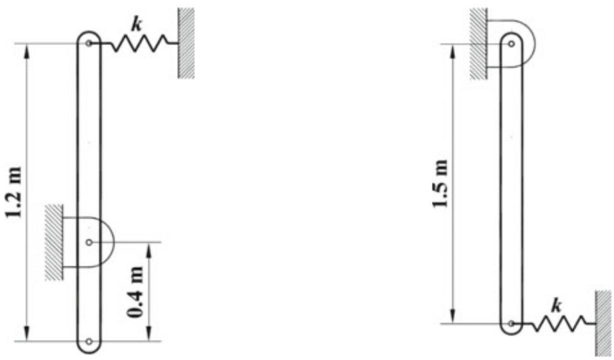
5. Derive equation of motion of the system consisting of a uniform bar of mass m and length L as shown in **Figure P3.5**. Take rotation θ of the bar as the generalized coordinate.
   (a) Use Newton's second law of motion.
   (b) Use equivalent system parameters method.
   (c) Use Lagrange's equation.



(a) **(b)**

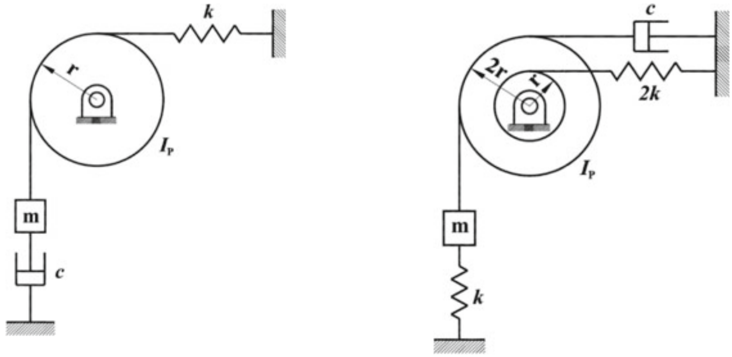
Figure P3.5

6. Derive equation of motion of the system consisting of the systems shown in **Figure P3.6**. Mass of the bar is 24 kg and the spring stiffness is 4 kN/m. Use rotation θ of the bar as the generalized coordinate.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.
7. Derive equation of motion of the system shown in **Figure P3.7**. Take vertical displacement x of the block of mass m as a generalized coordinate.
   (a) Use Newton's second law of motion.
   (b) Use equivalent system parameters method.
8. Derive equation of motion of the system shown in **Figure P3.8** Take vertical displacement x of the block of mass $2m$ as a generalized coordinate.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.



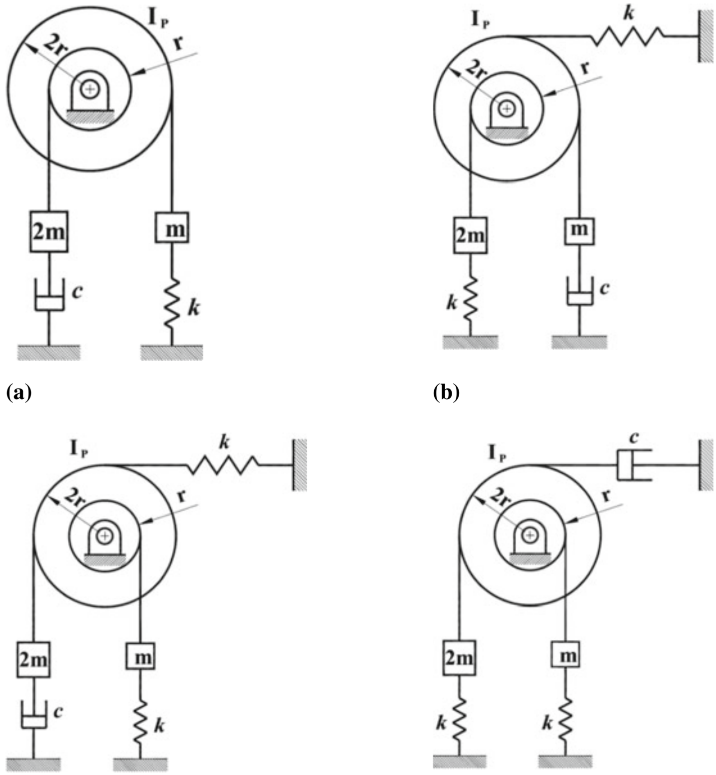
(a) **(b)**

Figure P3.6



(a) (b)

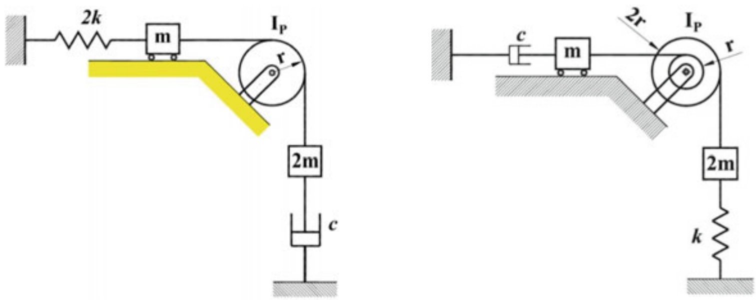
Figure P3.7



(c) (d)

Figure P3.8

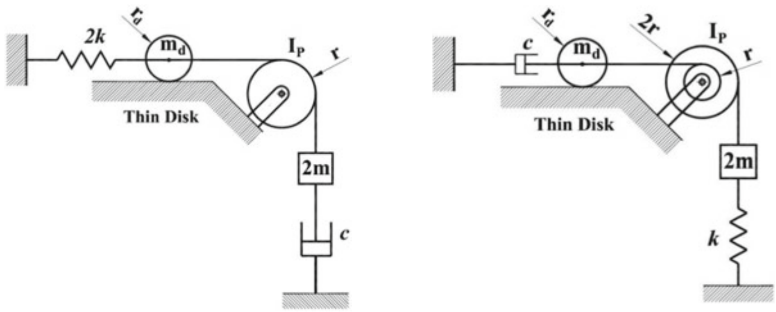
9. Derive equation of motion of the system shown in **Figure P3.9**. Take vertical displacement x of the block of mass $2m$ as a generalized coordinate.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.
10. Derive equation of motion of the system shown in **Figure P3.10**. Take vertical displacement x of the block of mass $2m$ as a generalized coordinate. Assume that the disk is thin and rolls without slip.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.



(a) **(b)**

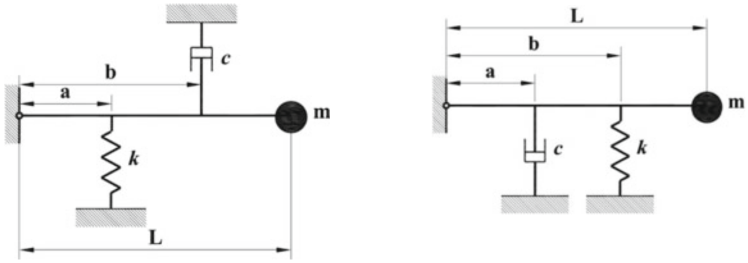
Figure P3.9

11. Derive equation of motion of the system shown in **Figure P3.11** The rod carrying a concentrated mass m has a negligible mass.
   (a) Use equivalent system parameters method.
   (b) Use Lagrange's equation.



(a) **(b)**

Figure P3.10

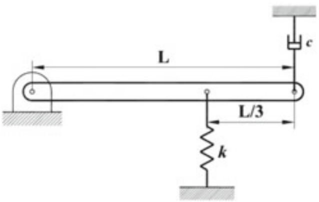


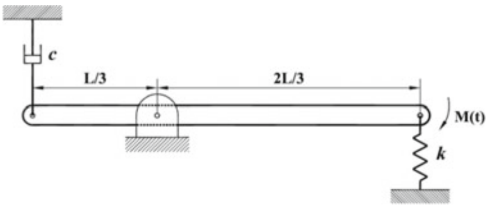
(a) **(b)**

Figure P3.11

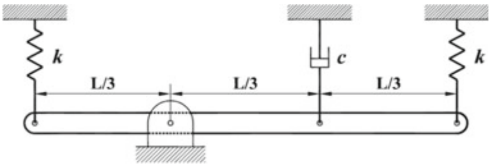
12. Derive equation of motion of the system consisting of a uniform bar of mass m and length L as shown in **Figure P3.12**. Take rotation θ of the bar as the generalized coordinate.
    (a) Use Newton's second law of motion.
    (b) Use Lagrange's equation.
13. Derive equation of motion of the system shown in **Figure P3.13**. Take displacement x of the center of the disk as a generalized coordinate. Assume that the disk is thin and rolls without slip.
    (a) Use equivalent system parameters method.
    (b) Use Lagrange's equation.
14. Derive equation of motion of the system shown in **Figure P3.14**
    (a) Use Newton's second law of motion.
    (b) Use Lagrange's equation.

15. Derive equation of motion of the system shown in **Figure P3.15**. Take vertical displacement x of mass and counterclockwise rotation θ of the pulley as a set of generalized coordinates.
    (a) Use Newton's second law of motion.
    (b) Use Lagrange's equation.



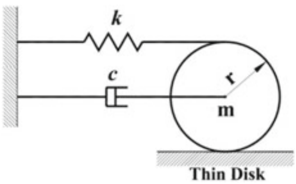


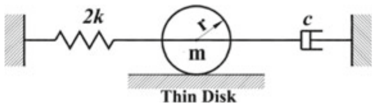
(a) **(b)**



(c)

Figure P3.12





(a) **(b)**

Figure P3.13

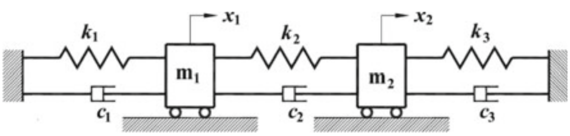


Figure P3.14

16. Derive equation of motion of the system shown in **Figure P3.16**. Use displacement x_1 of the block m and displacement x_2 of the block $2m$ as a set of generalized coordinates. Use Lagrange's equation.
17. Derive equation of motion of a system consisting of a uniform bar of mass m and length L as shown in **Figure P3.17** using Lagrange's equation.

    (a) Use vertical displacements of end points of the bar (x_1 and x_2) as a set of generalized coordinates.
    (b) Use vertical displacement of C. G. of the bar (x) and rotation of bar about C. G (θ) as a set of generalized coordinates.

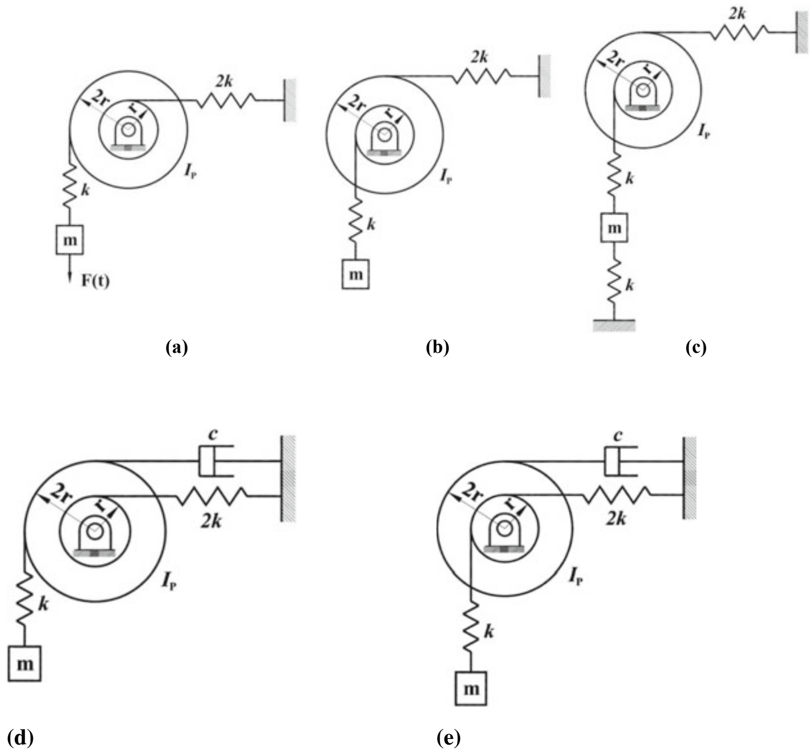
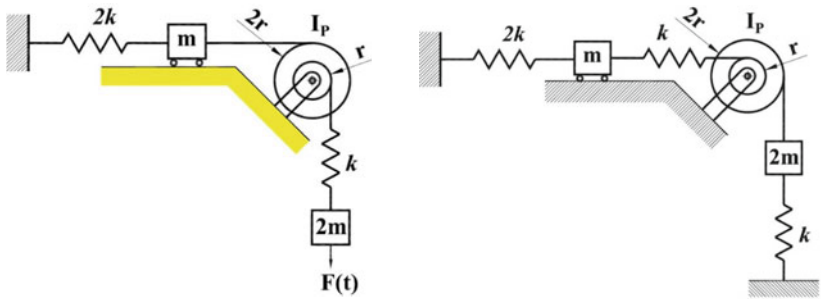


Figure P3.15



(a) **(b)**

Figure P3.16

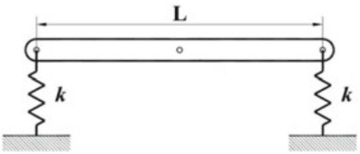
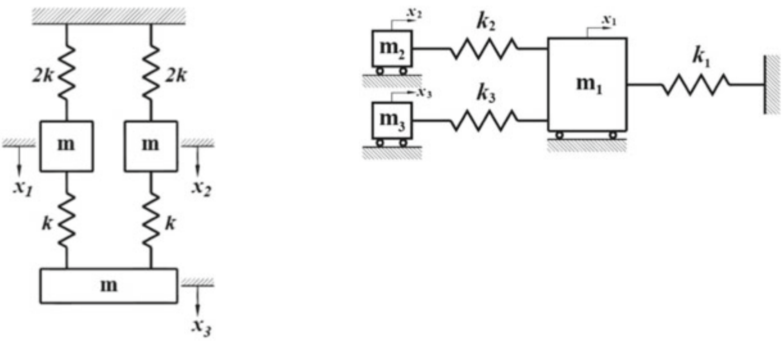


Figure P3.17

18. Derive equation of motion of the system shown in **Figure P3.18**. Use Newton's second law of motion. Use displacement x_1, x_2 and x_3 of the blocks as a set of generalized coordinates.



(a) **(b)**

Figure P3.18

19. Derive equation of motion of the system shown in **Figure P3.19**. Take displacement x_1 of the block m, displacement x_2 of the block $2m$ and rotation θ of the pulley I_2 as a set of generalized coordinates. Use Lagrange's equation.

20. Derive equation of motion of the system shown in **Figure P3.20**. Take displacement x_1 of the block m and displacement x_2 of the block $2m$ and rotation θ of the pulley I_2 as a set of generalized coordinates. Use Lagrange's equation.
21. Derive equation of motion of the system shown in **Figure P3.21**. Take displacement x_1 of the block m_1, displacement x_2 of the block m_2, displacement x_3 of the block m_3 and rotation θ of the pulley I_2 as a set of generalized coordinates. Inner and outer radii of both pulleys are r_1 and r_2, respectively. Use Lagrange's equation.
22. A concentrated mass M is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P3.22**. The bar material has a density of ρ and its cross-sectional area is A. Derive its equation of motion and associated boundary conditions using Hamilton's principle.
23. A spring of stiffness k is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P3.23** The bar material has a density of ρ and its cross-sectional area is A. Derive its equation of motion and associated boundary conditions using Hamilton's principle.

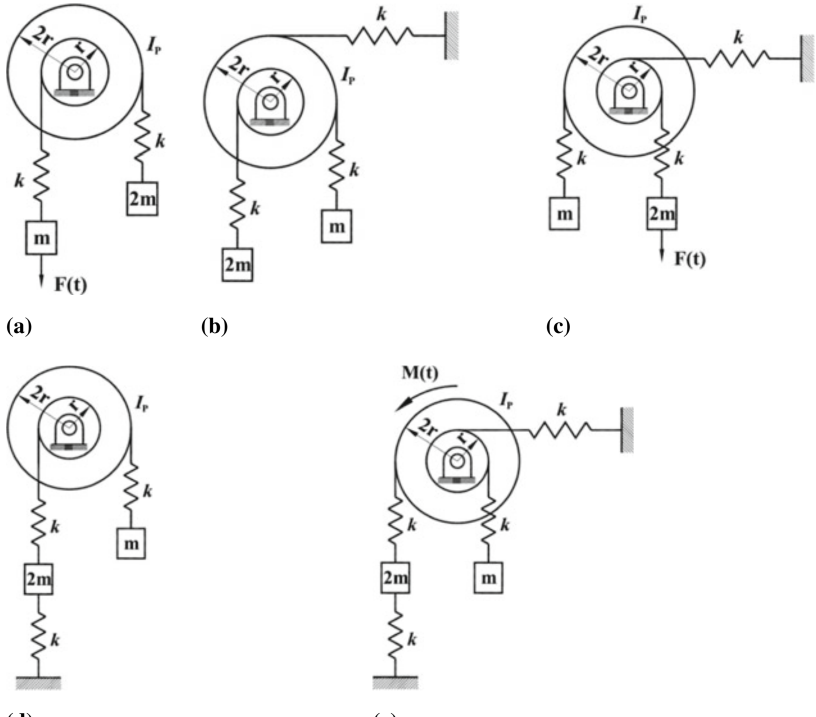
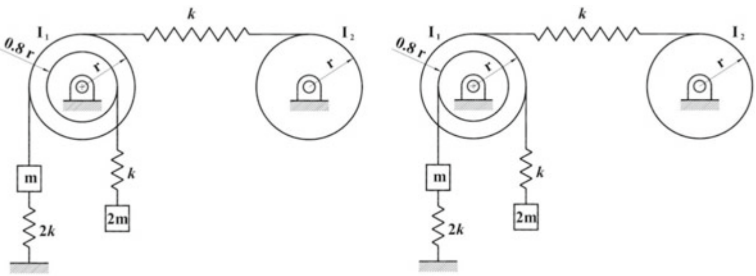


Figure P3.19



(a) (b)

Figure P3.20

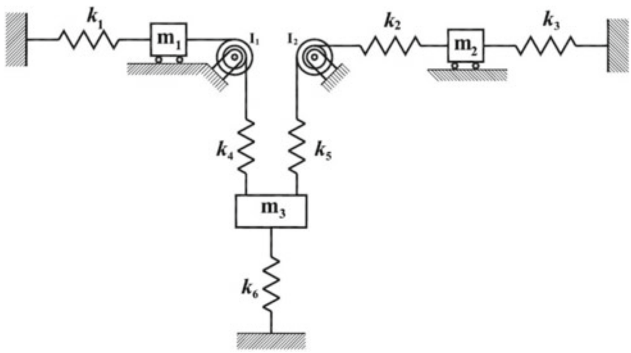


Figure P3.21

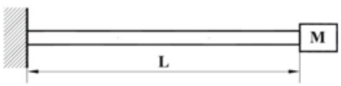


Figure P3.22

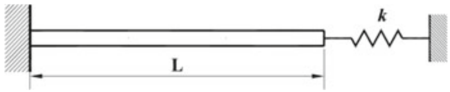


Figure P3.23

24. Two springs of stiffness k are attached to ends of a bar of length L undergoing longitudinal vibration as shown in **Figure P3.24**. The bar material has a density of ρ and its cross-sectional area is A. Derive its equation of motion and associated boundary conditions using Hamilton's principle.

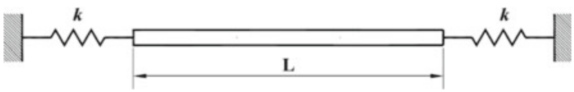


Figure P3.24

25. A concentrated load F is applied at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P3.25**. The bar material has a density of ρ and its cross-sectional area is k. Derive its equation of motion and associated boundary conditions using Hamilton's principle.
26. A spring of stiffness k and a damper with damping constant c are attached at the right end of a bar of length L undergoing longitudinal vibration as shown in **Figure P3.26**. The bar material has a density of ρ and its cross-sectional area is A. Derive its equation of motion and associated boundary conditions using Hamilton's principle.

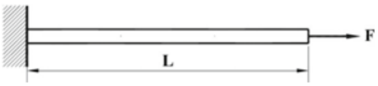


Figure P3.25

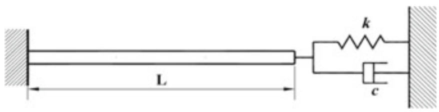
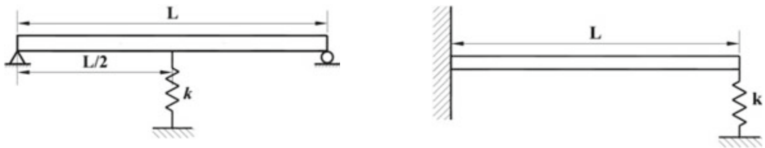


Figure P3.26

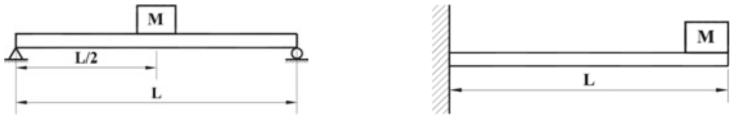
27. A beam of length L shown in **Fig. 2.27** undergoing traverse vibration is restrained by a spring of stiffness k. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive its equation of motion and associated boundary conditions using Hamilton's principle.



(a) **(b)**

Figure P3.27

28. A concentrated mass M is attached to a beam of length L as shown in **Figure P3.28**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive equation of motion for the transverse vibration of the beam and associated boundary conditions using Hamilton's principle.



(a) **(b)**

Figure P3.28

29. A cantilever beam of length L shown in **Figure P3.29** is subjected to a concentrated load F at its mid-span. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive equation of motion for the transverse vibration of the beam and associated boundary conditions using Hamilton's principle.
30. A rigid disk of mass moment of inertia Id is attached to shaft of length L undergoing torsional vibration as shown in **Figure P3.30**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J. Derive equation of motion for the torsional vibration of the shaft and associated boundary conditions using Hamilton's principle.

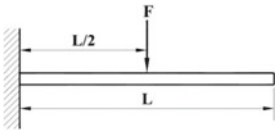


Figure P3.29

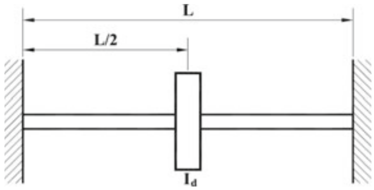


Figure P3.30

31. A rigid disk of mass moment of inertia I_d is attached to free end shaft of length L undergoing torsional vibration as shown in **Figure P3.31**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J. Derive equation of motion for the torsional vibration of the shaft and associated boundary conditions using Hamilton's principle.
32. A torque of $T(t)$ is applied to free end of a shaft of length L undergoing torsional vibration as shown in **Figure P3.32**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J. Derive equation of motion for the torsional vibration of the shaft and associated boundary conditions using Hamilton's principle.

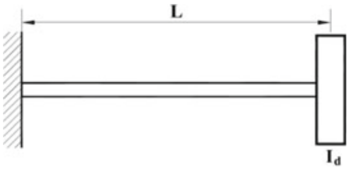


Figure P3.31

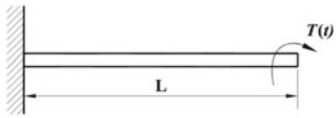


Figure P3.32

Answers

1. (a) $\left(m + \frac{I_p}{4r^2}\right)\ddot{x} + \frac{k}{2}x = 0$ (b) $\left(m + \frac{I_p}{r^2}\right)x + 9kx = F(t)\left(m + \frac{I_p}{r^2}\right)x + 9kx = F(t)$
2. (a) $\left(6m + \frac{I_p}{r^2}\right)\ddot{x} + 9kx = 0$ (b) $\left(\frac{9m}{4} + \frac{I_p}{4r^2}\right)\ddot{x} + +\frac{3}{2}kx = 0$
3. (a) $\left(6m + \frac{I_p}{r^2}\right)\ddot{x} + 4kx = 0$ (b) $\left(6m + \frac{I_p}{r^2}\right)\ddot{x} + 5kx = 0$
4. (a) $\left(m + \frac{3}{2}m_d + \frac{I_p}{r^2}\right)\ddot{x} + 3kx = 0$ (b) $\left(2m + 6m_d + \frac{I_p}{r^2}\right)\ddot{x} + 9kx = 0$
5. (a) $\left(\frac{1}{3}mL^2\right)\ddot{\theta} + \left(\frac{4}{9}kL^2\right)\theta = 0$ (b) $\left(\frac{1}{3}mL^2\right)\ddot{\theta} + \left(\frac{13}{9}kL^2\right)\theta = M(t)$
6. (a) $3.84\ddot{\theta} + 2560\theta = 0$ (b) $18\ddot{\theta} + 9000\theta = 0$
7. (a) $\left(m + \frac{I_p}{r^2}\right)\ddot{x} + c\dot{x} + kx = 0$ (b) $\left(m + \frac{I_p}{4r^2}\right)\ddot{x} + c\dot{x} + \frac{3}{2}kx = 0$

8. (a) $\left(6m+\frac{I_p}{r^2}\right)\ddot{x}+c\dot{x}+4kx=0$ (b) $\left(6m+\frac{I_p}{r^2}\right)x+4c\dot{x}+5kx=0$
   (c) $\left(\frac{9m}{4}+\frac{I_p}{4r^2}\right)\ddot{x}+c\dot{x}+\frac{5}{4}kx=0$ (d) $\left(\frac{9m}{4}+\frac{I_p}{4r^2}\right)x+c\dot{x}+\frac{5}{4}kx=0$
9. (a) $\left(3m+\frac{I_p}{r^2}\right)\ddot{x}+c\dot{x}+2kx=0$ (b) $\left(\frac{9m}{4}+\frac{I_p}{4r^2}\right)\ddot{x}+\frac{c}{4}\dot{x}+kx=0$
10. (a) $\left(2m+\frac{3}{2}m_d+\frac{I_p}{r^2}\right)\ddot{x}+c\dot{x}+2kx=0$ (b) $\left(2m+\frac{3}{8}m_d+\frac{I_p}{4r^2}\right)\ddot{x}+\frac{c}{4}\dot{x}+kx=0$
11. (a) $mL^2\ddot{\theta}+cb^2\dot{\theta}+ka^2\theta=0$ (b) $mL^2\ddot{\theta}+ca^2\dot{\theta}+kb^2\theta=0$
12.

(a) $\left(\frac{1}{3}mL^2\right)\ddot{\theta}+\left(cL^2\right)\dot{\theta}+\left(\frac{4}{9}kL^2\right)\theta=0$
(b) $\left(\frac{1}{9}mL^2\right)\ddot{\theta}+\left(\frac{1}{9}cL^2\right)\dot{\theta}+\left(\frac{4}{9}kL^2\right)\theta=M(t)$
(c) $\left(\frac{1}{9}mL^2\right)\ddot{\theta}+\left(\frac{1}{9}cL^2\right)\ddot{\theta}+\left(\frac{5}{9}kL^2\right)\theta=0$

13. (a) $\frac{3}{2}m\ddot{x}+c\dot{x}+4kx=0$ (b) $\frac{3}{2}m\ddot{x}+c\dot{x}+2kx=0$
14. $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
15.

(a) $\begin{bmatrix} m & 0 \\ 0 & I_p \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k & -2kr \\ -2kr & 6kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$
(b) (b) $\begin{bmatrix} m & 0 \\ 0 & I_p \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k & -kr \\ -kr & 9kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
(c) $\begin{bmatrix} m & 0 \\ 0 & I_p \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} 2k & -kr \\ -kr & 9kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
(d) $\begin{bmatrix} m & 0 \\ 0 & I_p \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 4cr^2 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}+\begin{bmatrix} k & -2kr \\ -2kr & 6kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
(e) $\begin{bmatrix} m & 0 \\ 0 & I_p \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 4cr^2 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}+\begin{bmatrix} k & -kr \\ -kr & 3kr^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

16.

(a) $\begin{bmatrix} m+\frac{I_p}{4r^2} & 0 \\ 0 & 2m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} \frac{9}{4}k & -\frac{1}{2}k \\ -\frac{1}{2}k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ F(t) \end{Bmatrix}$
(b) $\begin{bmatrix} m & 0 \\ 0 & 2m+\frac{I_p}{4r^2} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} 3k & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{5}{4}k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

17.

(a) $\begin{bmatrix} \frac{M}{3} & \frac{M}{6} \\ \frac{M}{6} & \frac{M}{3} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
(b) $\begin{bmatrix} M+m & mL \\ 0 & \frac{1}{12}ML^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{2}kL^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

18.

(a) $$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & 0 & -k \\ 0 & 3k & -k \\ -k & -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(b) $$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2+k_3 & -k_2 & -k_3 \\ -k_2 & k_2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

19.

(a) $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -kr \\ 0 & k & -2kr \\ -kr & -2kr & 5kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \\ 0 \end{Bmatrix}$$

(b) $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -2kr \\ 0 & k & -kr \\ -2kr & -kr & 9kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(c) $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -2kr \\ 0 & k & -kr \\ -2kr & -kr & 6kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(t) \\ 0 \end{Bmatrix}$$

(d) $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -2kr \\ 0 & 2k & -kr \\ -2kr & -kr & 5kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(e) $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -kr \\ 0 & 2k & -2kr \\ -kr & -2kr & 6kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M(t) \end{Bmatrix}$$

20.

(a)

$$\begin{bmatrix} m+\frac{I_1}{r^2} & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 3.64k & -0.8k & -kr \\ -0.8k & k & 0 \\ -kr & 0 & kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(b)

$$\begin{bmatrix} m+1.5625\frac{I_1}{r^2} & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 5.125k & -1.25k & -1.25kr \\ -1.25k & k & 0 \\ -1.25kr & 0 & kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

21.

$$\begin{bmatrix} m_1+\frac{I_1}{r_1^2} & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+\left(\frac{r_2}{r_1}\right)^2 k_4 & 0 & -\frac{r_2}{r_1}k_4 & 0 \\ 0 & k_2+k_3 & 0 & -k_2 r_2 \\ -\frac{r_2}{r_1}k_4 & 0 & k_4+k_5+k_6 & -k_5 r_1 \\ 0 & -k_2 r_2 & -k_5 r_1 & k_2 r_2^2 + k_5 r_1^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

22. $\rho A\ddot{u} - EAu'' = 0; u(0,t) = 0, M\ddot{u}(L,t) + EAu'(L,t) = 0$
23. $\rho A\ddot{u} - EAu'' = 0; u(0,t) = 0, EAu'(L,t) + ku(L,t) = 0$
24. $\rho A\ddot{u} - EAu'' = 0; EAu'(0,t) - ku(0,t) = 0, EAu'(L,t) + ku(L,t) = 0$
25. $\rho A\ddot{u} - EAu'' = 0; u(0,t) = 0, EAu'(L,t) - F = 0$
26. $\rho A\ddot{u} - EAu'' = 0; u(0,t) = 0, EAu'(L,t) + c\dot{u}(L,t) + ku(L,t) = 0$
27.
   (a) $\rho A\ddot{w} + EIw^{iv} + kw\delta_d\left(x - \frac{L}{2}\right) = 0; w(0,t) = 0, w''(0,t) = 0; w(L,t) = 0, w''(L,t) = 0$
   (b) $\rho A\ddot{w} + EIw^{iv} = 0; w(0,t) = 0, w'(0,t) = 0; EIw'''(L,t) - kw(L,t) = 0, w''(L,t) = 0$

28.
   (a) $\rho A\ddot{w} + EIw^{iv} + M\ddot{w}\delta_d\left(x - \frac{L}{2}\right) = 0; w(0,t) = 0, w''(0,t) = 0; w(L,t) = 0, w''(L,t) = 0$
   (b) $\rho A\ddot{w} + EIw^{iv} = 0; w(0,t) = 0, w'(0,t) = 0; M\ddot{w}(L,t) - EIw'''(L,t) = 0, w''(L,t) = 0$

29. $\rho A\ddot{w} + EIw^{iv} + F\delta_d\left(x - \frac{L}{2}\right) = 0; w(0,t) = 0, w'(0,t) = 0; w''(L,t) = 0, w'''(L,t) = 0$
30. $\rho J\theta + I_d\theta\delta_d\left(x - \frac{L}{2}\right) - GJ\theta'' = 0; \theta(0,t) = 0, \theta(L,t) = 0$
31. $\rho J\theta - GJ\theta'' = 0; \theta(0,t) = 0, I_d\theta(L,t) + GJ\theta'(L,t) = 0$
32. $\rho J\theta - GJ\theta'' = 0; \theta(0,t) = 0, GJ\theta'(L,t) - T(t) = 0$

Chapter 4
Response of a Single Degree of Freedom System

Response of any vibrating system can be determined by solving the equation of motion of the system. The simplest model of a vibrating system is single degree of freedom (SDOF) system and basic procedure of vibration analysis can be understood through a SDOF model which can be further extended for a higher degree of freedom system.

4.1 Un-damped Free Response of a SDOF System

Response of a system due to its inherent properties and in the absence of damping is called the un-damped free response of the system. Consider a spring-mass system shown in Fig. 4.1 as a representative model for an un-damped single degree of freedom system.

As described earlier in Chap. 3, equation of motion of the system can be determined as

$$m\ddot{x} + kx = 0 \tag{4.1}$$

Equation (4.1) can also be expressed as

$$\ddot{x} + \frac{k}{m}x = 0 \tag{4.2}$$

Equation (4.2) is a homogeneous second-order linear ordinary differential equation with constant coefficients and which has the general solution of the following form

$$x = Ae^{st} \tag{4.3}$$

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_4

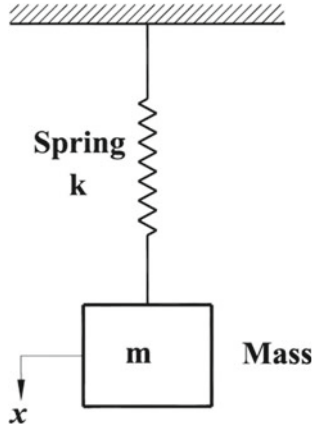


Fig. 4.1 Single degree of freedom system consisting of spring and mass

Differentiating Eq. (4.3) twice, we get

$$\ddot{x} = As^2 e^{st} \tag{4.4}$$

Substituting x and $\ddot{x}$ from Eqs. (4.3) and (4.4), respectively, into Eq. (4.2), we get

$$A\left(s^2 + \frac{k}{m}\right)e^{st} = 0 \tag{4.5}$$

Since $A \neq 0$ and $e^{st} \neq 0$ (for any real time instant), Eq. (4.5) leads to

$$s^2 + \frac{k}{m} = 0 \tag{4.6}$$

which is the characteristic equation of the system.
Substituting

$$\frac{k}{m} = \omega_n^2 \tag{4.7}$$

Equation (4.6) reduces to

$$s^2 + \omega_n^2 = 0 \tag{4.8}$$

Roots of Eq. (4.8) are given as

$$s_{1,2} = \pm i\omega_n \tag{4.9}$$

Since the roots of characteristic equation are complex conjugate pair, the general solution of Eq. (4.2) is given by

$$x(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{4.10}$$

where A and B are the arbitrary constants which can be determined form the initial conditions.

Equation (4.10) is the expression of the free response of an un-damped SDOF system and is the sinusoidal or oscillating function and ω_n represents the natural frequency of oscillation. It can be noted from Eq. (4.7) that natural frequency (ω_n) of the system depends on stiffness (k) and mass (m). Equation (4.7) can also be expressed for the natural frequency in rad/s as

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.11}$$

Natural frequency f_n of the system, in Hz, can be expressed as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{4.12}$$

Similarly, the time period T of free vibration is given by

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{4.13}$$

Solution with Initial Conditions

Free vibration in any system is initiated due to initial disturbance in the form of initial displacement or initial velocity or combination of both initial displacement and initial velocity. Assume that the system is subjected to an initial displacement of x_0 and initial velocity of v_0, respectively, i.e.,

$$\text{at} \quad t = 0 \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{4.14}$$

Substituting $t = 0$ and $x(0) = x_0$ into Eq. (4.10), we get

$$x_0 = B \tag{4.15}$$

Differentiating Eq. (4.10) with respect to t and substituting $t = 0$ and $\dot{x}(0) = v_0$, we get

$$v_0 = A\omega_n$$

$$\therefore \quad A = \frac{v_0}{\omega_n} \tag{4.16}$$

Substituting B and A from Eqs. (4.15) and (4.16), respectively, into Eq. (4.10), we get the expression for the free response of a SDOF system with initial conditions as

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) \tag{4.17}$$

Equation (4.16) can be expressed in terms of sin function only as

$$x(t) = A_1\sin(\omega_n t + \phi_1) \tag{4.18}$$

where

$$A_1 = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + (x_0)^2} \tag{4.19}$$

and

$$\phi_1 = \tan^{-1}\left(\frac{x_0}{v_0/\omega_n}\right) \tag{4.20}$$

.

Equation (4.16) can also be expressed in terms of cos function only as

$$x(t) = A_2\cos(\omega_n t + \phi_2) \tag{4.21}$$

where

$$A_2 = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + (x_0)^2} \tag{4.22}$$

and

$$\phi_2 = \tan^{-1}\left(\frac{v_0/\omega_n}{x_0}\right) \tag{4.23}$$

4.2 Damped Free Response of a SDOF System

Consider a spring-mass-damper system shown in Fig. 4.2 as a representative model for a damped single degree of freedom system.

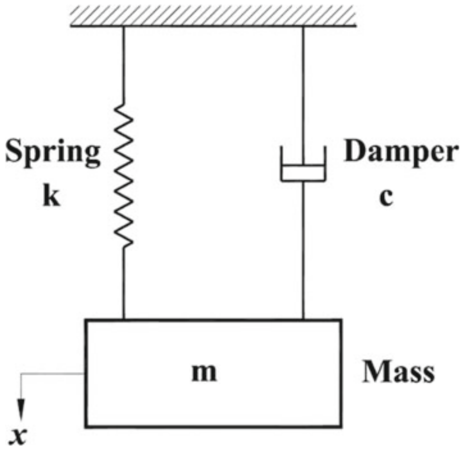


Fig. 4.2 Single degree of freedom system consisting of spring, mass and damper

Equation of motion of the system can be determined as

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.24}$$

Equation (4.24) is a homogeneous second-order linear ordinary differential equation with constant coefficients and which has the general solution of the following form

$$x = Ae^{st} \tag{4.25}$$

Differentiating Eq. (4.25) successively with respect to time, we get

$$\dot{x} = Ase^{st} \tag{4.26}$$

and

$$\ddot{x} = As^2 \mathrm{e}^{st} \tag{4.27}$$

Substituting x, $\dot{x}$ and $\ddot{x}$ from Eqs. (4.25), (4.26) and (4.27), respectively, into Eq. (4.24), we get

$$A\left(ms^2 + cs + k\right)\mathrm{e}^{st} = 0 \tag{4.28}$$

Since $A \neq 0$ and $\mathrm{e}^{st} \neq 0$ (for any real time instant), Eq. (4.28) leads to

$$ms^2 + cs + k = 0 \tag{4.29}$$

which is the characteristic equation of the system.

Roots of Eq. (4.29) are given as

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{4.30}$$

Then general solution of the Eq. (4.24) is given by

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} \tag{4.31}$$

Depending upon the values of system parameters m, c and k, Eq. (4.30) gives three different types of roots. To classify these roots, let us introduce a new parameter called critical damping constant (c_c). Critical damping is the value of damping constant which makes the expression under the square root sign equal to zero, i.e.,

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\therefore \quad \frac{c_c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \tag{4.32}$$

Hence, critical damping constant can be expressed in terms of k and m as

$$c_c = 2\sqrt{km} \tag{4.33}$$

It can also be expressed in terms of m and ω_n as

$$c_c = 2m\omega_n \tag{4.34}$$

Now we can also introduce another new parameter called damping ratio (ξ) which is defined as the ratio of damping constant of the system to the critical damping constant of the system, i.e.,

$$\xi = \frac{c}{c_c} \tag{4.35}$$

Then we can express the roots of the characteristics equation given by Eq. (4.30) in terms of ξ and ω_n as

$$s_{1,2} = -\frac{c}{c_c}\frac{c_c}{2m} \pm \sqrt{\left(\frac{c}{c_c}\frac{c_c}{2m}\right)^2 - \frac{k}{m}} \tag{4.36}$$

Substituting $c_c/2m = \omega_n$ from Eq. (4.32) and $k/m = \omega_n^2$ into Eq. (4.36), we get the expression for roots as

$$s_{1,2} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n \tag{4.37}$$

Depending upon the value of damping ratio ξ, Eq. (4.37) gives following three different types of roots:

(a) When $\xi > 1$ or $c > c_c$, the characteristic equation will have two real and unequal roots and the system is said to be over-damped.
(b) When $\xi = 1$ or $c = c_c$, the characteristic equation will have two real and equal roots and the system is said to be critically damped.
(c) When $\xi < 1$ or $c < c_c$, the characteristic equation will have two complex conjugate pair of roots and the system is said to be under-damped.

4.2.1 Response of an Over-Damped System

If the system is over-damped, then the two real and unequal roots of the characteristics equation are given as

$$\begin{aligned} s_1 &= \left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n \\ s_2 &= \left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n \end{aligned} \tag{4.38}$$

Substituting s_1 and s_2 into Eq. (4.31) the response of over-damped system is given by

$$x(t) = Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t} \tag{4.39}$$

Since both terms of right-hand side of the expression of Eq. (4.39) are exponential functions with negative power, both approach zero when t increases. Therefore response of an over-damped will be aperiodic as shown in Fig. 4.3.

Solution with Initial Conditions

Assume that the system is subjected to an initial displacement of x_0 and initial velocity of v_0, respectively, i.e.,

$$\text{at} \quad t = 0 \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{4.40}$$

Substituting $t = 0$ and $x(0) = x_0$ into Eq. (4.39), we get

$$x_0 = A + B \tag{4.41}$$

Differentiating Eq. (4.39) with respect to t, we get

$$\begin{aligned} \dot{x}(t) = {} & A\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n \mathrm{e}^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} \\ & + B\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n \mathrm{e}^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t} \end{aligned} \tag{4.42}$$

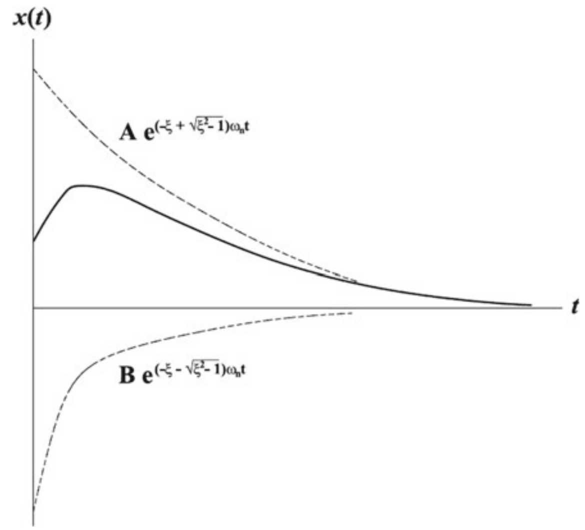


Fig. 4.3 Response of an over-damped system

Substituting $t = 0$ and $\dot{x}(0) = v_0$ into Eq. (4.42), we get

$$v_0 = A\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n + B\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n \tag{4.43}$$

Solving Eqs. (4.42) and (4.43) for A and B, we get

$$A = \frac{v_0 + \left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n x_0}{2\left(\sqrt{\xi^2 - 1}\right)\omega_n} \quad \text{and} \quad B = -\frac{v_0 + \left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n x_0}{2\left(\sqrt{\xi^2 - 1}\right)\omega_n} \tag{4.44}$$

Substituting A and B from Eq. (4.44) into Eq. (4.39), we get

$$\begin{aligned} x(t) = \frac{1}{2\left(\sqrt{\xi^2 - 1}\right)\omega_n}\Bigg[&\left\{v_0 + \left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n x_0\right\}\mathrm{e}^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} \\ &- \left\{v_0 + \left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n x_0\right\}\mathrm{e}^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}\Bigg] \end{aligned} \tag{4.45}$$

4.2.2 Response of a Critically Damped System

If the system is critically damped, then the real and equal roots of the characteristics equation are given as

$$s_1 = s_2 = -\omega_n \tag{4.46}$$

Then, general solution of the second-order linear system with repeated roots is given by

$$x(t) = (A + Bt)\mathrm{e}^{-\omega_n t} \tag{4.47}$$

Since the right-hand side term of the expression of Eq. (4.47) is exponential functions with negative power, $x(t)$ approaches zero when t increases. Therefore response of a critically damped system will also be aperiodic as shown in Fig. 4.4.

Solution with Initial Conditions

Assume that the system is subjected to an initial displacement of x_0 and initial velocity of v_0, respectively, i.e.,

$$\text{at} \quad t = 0 \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{4.48}$$

Substituting $t = 0$ and $x(0) = x_0$ into Eq. (4.47), we get

$$x_0 = A \tag{4.49}$$

Differentiating Eq. (4.47) with respect to t, we get

$$\dot{x}(t) = [B - \omega_n(A + Bt)]\mathrm{e}^{-\omega_n t} \tag{4.50}$$

Substituting $t = 0$ and $\dot{x}(0) = v_0$ into Eq. (4.50), we get

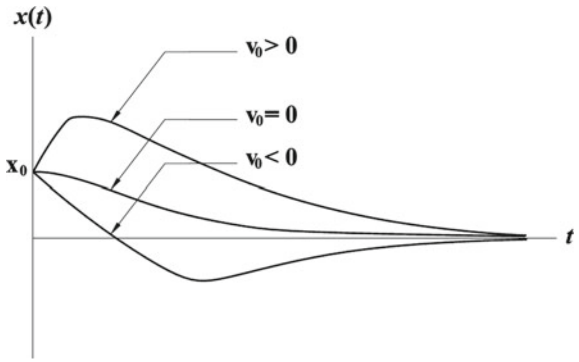


Fig. 4.4 Response of a critically damped system

$$v_0 = B - \omega_n A$$
$$\therefore \quad B = v_0 + \omega_n A \tag{4.51}$$

Substituting A from Eqs. (4.49) into Eq. (4.51), we get

$$B = v_0 + \omega_n x_0 \tag{4.52}$$

Substituting A and B from Eqs. (4.49) and (4.51), respectively, into Eq. (4.47), we get

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]\mathrm{e}^{-\omega_n t} \tag{4.53}$$

It may be noted from Eq. (4.53) and Fig. 4.4 that a critically damped system may cross equilibrium position before reaching finally to rest when the initial displacement and initial velocity have opposite signs. The time instant t_e at which the system crosses the equilibrium position can be determined from Eq. (4.53) as

$$t_e = -\frac{x_0}{v_0 + \omega_n x_0} \tag{4.54}$$

4.2.3 Response of an Under-Damped System

If the system is under-damped, then the two complex conjugate pairs of root of the characteristics equation are given as

$$\begin{aligned} s_1 &= \left(-\xi + i\sqrt{1-\xi^2}\right)\omega_n \\ s_2 &= \left(-\xi - i\sqrt{1-\xi^2}\right)\omega_n \end{aligned} \tag{4.55}$$

Then, general solution of the second-order linear system with complex conjugate pairs of roots is given by

$$x(t) = \mathrm{e}^{-\xi\omega_n t}\left[A\sin\left(\sqrt{1-\xi^2}\right)\omega_n t + B\cos\left(\sqrt{1-\xi^2}\right)\omega_n t\right] \tag{4.56}$$

Equation (4.56) can also be expressed as

$$x(t) = \mathrm{e}^{-\xi\omega_n t}[A\sin(\omega_d t) + B\cos(\omega_d t)] \tag{4.57}$$

where

$$\omega_d = \left(\sqrt{1-\xi^2}\right)\omega_n \tag{4.58}$$

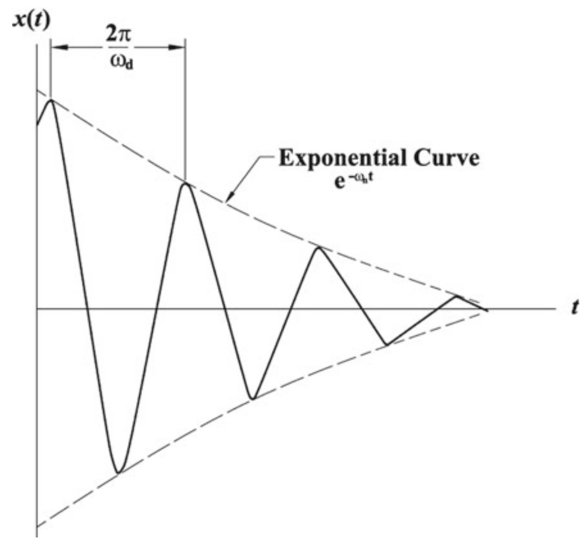


Fig. 4.5 Response of an under-damped system

is called the damped natural frequency of the system.

Equation (4.55) shows that the response of an un-damped system is the product of an exponential function with negative power and a sinusoidal function. Therefore response of an under-damped system will be sinusoidal with decreasing amplitude as shown in Fig. 4.5.

Solution with Initial Conditions

Assume that the system is subjected to an initial displacement of x_0 and initial velocity of v_0, respectively, i.e.,

$$\text{at} \quad t = 0 \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{4.59}$$

Substituting $t = 0$ and $x(0) = x_0$ into Eq. (4.57), we get

$$x_0 = B \tag{4.60}$$

Differentiating Eq. (4.57) with respect to t, we get

$$\dot{x}(t) = \mathrm{e}^{-\xi\omega_n t}[(-\xi\omega_n A - B\omega_d)\sin(\omega_d t) - (\xi\omega_n B - A\omega_d)\cos(\omega_d t)] \tag{4.61}$$

Substituting $t = 0$ and $\dot{x}(0) = v_0$ into Eq. (4.61), we get

$$v_0 = -\xi\omega_n B + A\omega_d$$

$$\therefore \quad A = \frac{v_0 + \xi\omega_n B}{\omega_d} \tag{4.62}$$

Substituting B from Eqs. (4.60) into Eq. (4.62), we get

$$A = \frac{v_0 + \xi\omega_n x_0}{\omega_d} \tag{4.63}$$

Substituting A and B from Eqs. (4.63) and (4.60), respectively, into Eq. (4.57), we get

$$x(t) = \mathrm{e}^{-\xi\omega_n t}\left[\frac{v_0 + \xi\omega_n x_0}{\omega_d}\sin(\omega_d t) + x_0\cos(\omega_d t)\right] \tag{4.64}$$

Logarithmic Decrement

It can be noted from Fig. 4.5 that the amplitude of free vibration of an under-damped system decreases with time. The rate of decay in amplitude will be higher if the damping effect is higher. The rate of decay of any damped system can be determined from logarithmic decrement which is defined as the logarithmic ratio of any two successive amplitudes, i.e.,

$$\delta = \ln\left(\frac{x_1}{x_2}\right) \tag{4.65}$$

Free response of an under-damped system given by Eq. (4.57) can also be expressed in terms of sin function only as

$$x(t) = \mathrm{e}^{-\xi\omega_n t}[A_1\sin(\omega_d t + \phi_1)] \tag{4.66}$$

It may be noted from Fig. 4.6 that the first peak x_1 occurs at the time instant t_1 when $\sin(\omega_d t_1 + \phi_1) = 1$, i.e.,

$$x_1 = A_1\mathrm{e}^{-\xi\omega_n t_1} \tag{4.67}$$

Similarly, the second peak x_2 occurs at the time instant t_2 when $\sin(\omega_d t_2 + \phi_1) = 1$, i.e.,

$$x_2 = A_1\mathrm{e}^{-\xi\omega_n t_2} \tag{4.68}$$

Dividing Eq. (4.67) by Eq. (4.68), we get

$$\frac{x_1}{x_2} = \mathrm{e}^{-\xi\omega_n(t_1-t_2)} = \mathrm{e}^{\xi\omega_n(t_2-t_1)} \tag{4.69}$$

With reference to Fig. 4.6, time period of the damped oscillation can be determined as

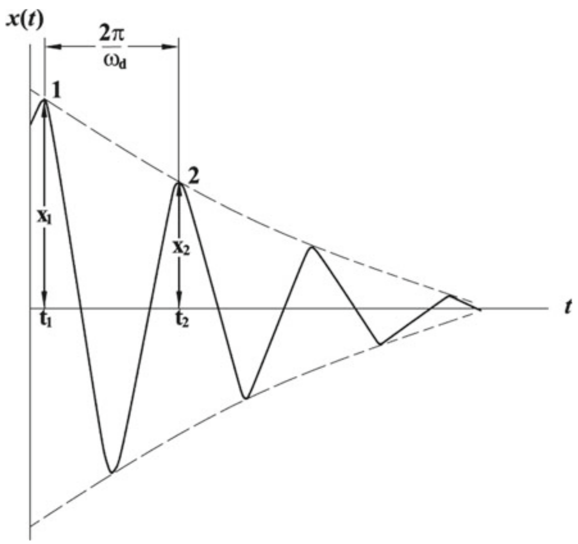


Fig. 4.6 Damped response showing two successive amplitudes

$$T_d = t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\left(\sqrt{1-\xi^2}\right)\omega_n} \tag{4.70}$$

Substituting $(t_2 - t_1)$ from Eq. (4.70) into Eq. (4.69), we get

$$\frac{x_1}{x_2} = \mathrm{e}^{\frac{2\pi\xi}{\left(\sqrt{1-\xi^2}\right)}} \tag{4.71}$$

Taking natural logarithm of both sides, we get

$$\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\therefore \quad \delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{4.72}$$

It can be noted from Eq. (4.71) and Fig. 4.7 that for weak damping, logarithmic decrement is proportional to the damping ratio.

Logarithmic decrement can also be defined in terms of ratio of amplitudes between n cycles. To derive this, Eq. (4.70) can be extended for n successive amplitudes as

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = \frac{x_n}{x_{n+1}} = \mathrm{e}^{\delta} \tag{4.73}$$

where x_{n+1} is the amplitude after n cycles.

Taking products of all ratios of successive amplitudes,

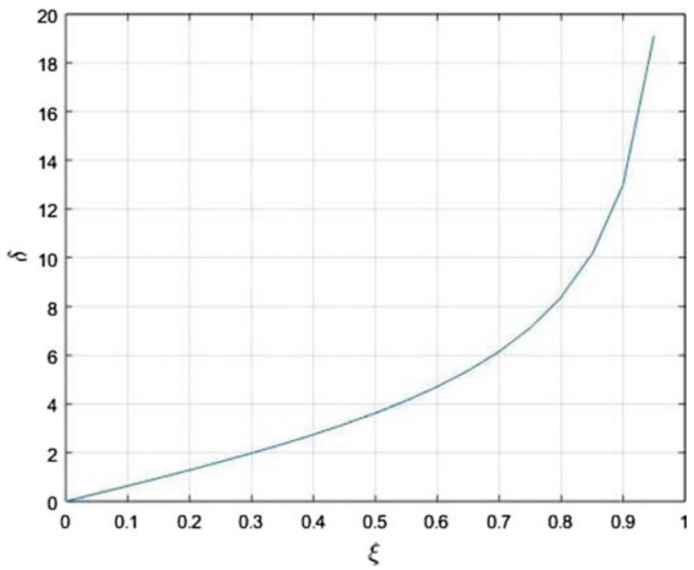


Fig. 4.7 Logarithmic decrement versus damping ratio

$$\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdots \frac{x_n}{x_{n+1}} = \left(\mathrm{e}^{\delta}\right)^n$$
$$\therefore \quad \frac{x_1}{x_{n+1}} = \mathrm{e}^{\delta n} \tag{4.74}$$

Taking natural logarithm of both sides of Eq. (4.74), we get

$$\ln\left(\frac{x_1}{x_{n+1}}\right) = \delta n$$
$$\therefore \quad \delta = \frac{1}{n} \ln\left(\frac{x_1}{x_{n+1}}\right) \tag{4.75}$$

Logarithmic decrement of any system can be determined by observing the amplitudes from the vibration response and from which damping ratio of the system can be determined by using Eq. (4.72).

4.3 Forced Harmonic Response of a SDOF System

Consider a spring-mass-damper system with subject to an external harmonic excitation $F(t) = F_0 \sin \omega t$ as shown in Fig. 4.8. Equation of motion of the system can be determined as

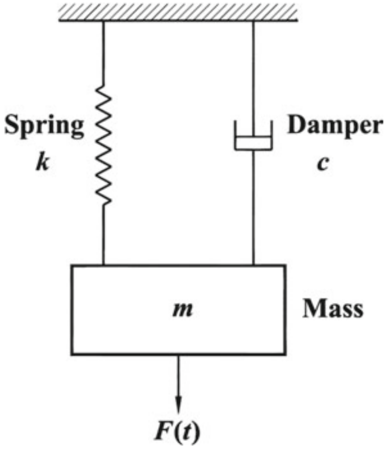


Fig. 4.8 A spring-mass-damper system subjected to a harmonic excitation

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{4.76}$$

Equation (4.73) is a non-homogeneous second-order linear ordinary differential equation with constant coefficients. The general solution of a non-homogeneous linear differential equation is given by the sum of the complementary solution (x_c) and the particular solution $\left(x_p\right)$, i.e.,

$$x(t) = x_c(t) + x_p(t) \tag{4.77}$$

The complementary solution of any differential equation is the solution for the corresponding homogeneous equation. That means it is the response of the system when no external force is applied to the system and is the free response of the system which can be expressed as

$$x_c(t) = \mathrm{e}^{-\xi\omega_n t}[A_1 \sin(\omega_d t + \phi_1)] \tag{4.78}$$

The particular solution of the system is the response due to the imposed external force. For the given harmonic excitation of $F_0 \sin \omega t$, the particular solution can be assumed as

$$x_p(t) = X \sin(\omega t - \phi) \tag{4.79}$$

where X is the amplitude of forced harmonic response and ϕ is phase of the response with respect to the imposed force. Both the amplitude (X) and the phase (ϕ) can be determined substituting the assumed solution (x_p) into the equation of motion of the system.

Differentiating Eq. (4.79) twice with respect to time, we get

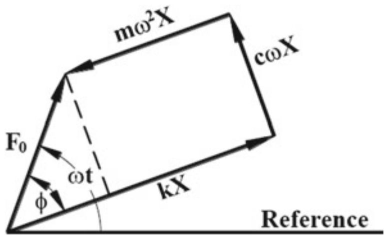


Fig. 4.9 Phasor diagram for forced harmonic response

$$\dot{x}_p(t) = \omega X \cos(\omega t - \phi) = \omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) \tag{4.80}$$

$$\ddot{x}_p(t) = -\omega^2 X \sin(\omega t - \phi) = \omega^2 X \sin(\omega t - \phi + \pi) \tag{4.81}$$

Since $x_p(t)$ is the solution of Eq. (4.76), it should be satisfied in Eq. (4.76). Substituting $x_p(t)$, $\dot{x}_p(t)$ and $\ddot{x}_p(t)$ into Eq. (4.76), we get

$$\begin{aligned} & m\omega^2 X \sin(\omega t - \phi + \pi) + c\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX \sin(\omega t - \phi) \\ & = F_0 \sin \omega t \end{aligned} \tag{4.82}$$

The first term of Eq. (4.82) with magnitude $m\omega^2 X$ represents the inertia force, second term with magnitude $c\omega X$ represents the damping force and the third term with magnitude kX represents the spring force. The resultant of all these forces is the imposed force with magnitude F_0. Similarly, spring force has a phase lag of ϕ with the external force, damping force is perpendicular to the spring force and inertia is further perpendicular to the damping force. Phase relations between all these forces are shown in Fig. 4.9.

With reference to the phasor diagram, relationship between all forces can be established as

$$\begin{aligned} F_0^2 &= \left(kX - m\omega^2 X\right)^2 + (c\omega X)^2 = \left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right] X^2 \\ \therefore \quad X &= \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}} \end{aligned} \tag{4.83}$$

Similarly, phase (ϕ) can be determined as

$$\phi = \tan^{-1}\left\{\frac{c\omega}{k - m\omega^2}\right\} \tag{4.84}$$

Dividing numerator and denominators of both Eqs. (4.83) and (4.84) by k, we get

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} \tag{4.85}$$

and

$$\phi = \tan^{-1}\left\{\frac{\frac{c\omega}{k}}{1 - \frac{m}{k}\omega^2}\right\} \tag{4.86}$$

Simplifying the term $c\omega/k$, we get

$$\frac{c\omega}{k} = \frac{c}{c_c}\frac{c_c\omega}{k} = \xi 2\sqrt{km}\frac{\omega}{k} = 2\xi\sqrt{\frac{m}{k}}\omega = 2\xi\frac{\omega}{\omega_n} \tag{4.87}$$

Substituting $k/m = \omega_n^2$ and $c\omega/k = 2\xi(\omega/\omega_n)$ into Eqs. (4.85) and (4.86), we get

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \tag{4.88}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.89}$$

Substituting X and ϕ from Eqs. (4.88) and (4.89) into Eq. (4.79), we get the expression for the particular solution as

$$x_p(t) = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \sin\left[\omega t - \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\}\right] \tag{4.90}$$

Substituting $x_c(t)$ from Eq. (4.78) and $x_p(t)$ from Eq. (4.90) into Eq. (4.77), we get the expression for the complete general solution as

$$\begin{aligned} x(t) &= \mathrm{e}^{-\xi\omega_n t}[A_1 \sin(\omega_d t + \phi_1)] \\ &\quad + \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \sin\left[\omega t - \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\}\right] \end{aligned} \tag{4.91}$$

The first expression of the complete solution is the complementary solution and in the presence of damping it decays with time and ultimately vanishes. Therefore, this part of the solution is called transient response. The second expression is the particular solution and is a sinusoidal vibration with constant amplitude. This part of the solution exists as long as the external excitation is active and hence called steady state response. It can also be noted that the transient response for a damped system has a frequency equal to damped natural frequency of the system, whereas the steady state response has a frequency equal to that of the external excitation. Hence forced harmonic response generally means the steady state response of the system and the steady state response of any system is expressed in terms of steady state amplitude (X) and its phase (ϕ).

Expressions for steady state response given (4.88) and (4.89) can also be expressed in non-dimensional form as

$$\frac{kX}{F_0} = \frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \tag{4.92}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.93}$$

The left-hand side term of Eq. (4.92), kX/F_0, is called magnification factor or amplitude ratio. Equations (4.92) and (4.93) show that both the amplitude ratio (kX/F_0) and phase (ϕ) are functions of only damping ratio (ξ) and frequency ratio (ω/ω_n).

With the help of Eqs. (4.92) and (4.93), the variation of amplitude ratio (kX/F_0) and phase of the response (ϕ) with the frequency ratio for different values of damping ratio can be plotted as shown in Figs. 4.10 and 4.11, respectively.

Response of the System at Low Frequency Region

When the frequency of the external force (ω) is less than the natural frequency of the system (ω_n) then frequency ratio (ω/ω_n) will be small. It can be noted from Eqs. (4.92) and (4.93) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to 0, \frac{kX}{F_0} \to 1 \quad \text{and} \quad \phi \to 0$$

Hence it can be understood that at low frequency region, amplitude ratio is almost equal to 1 and phase difference between the response and imposed force is almost

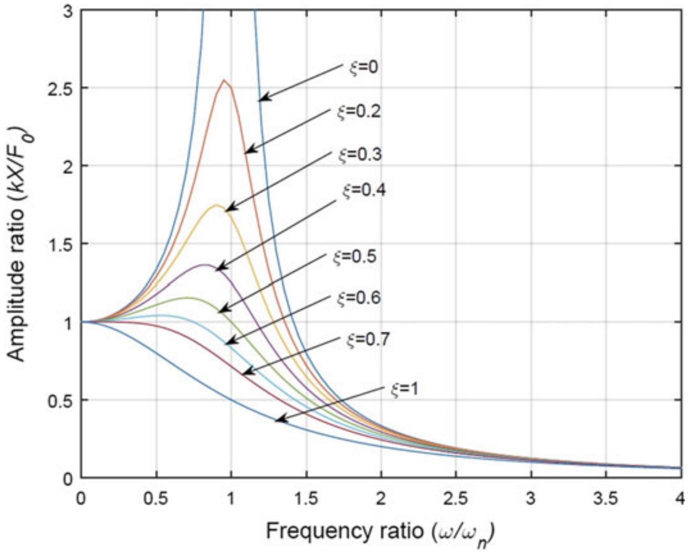


Fig. 4.10 Amplitude response for forced harmonic vibration

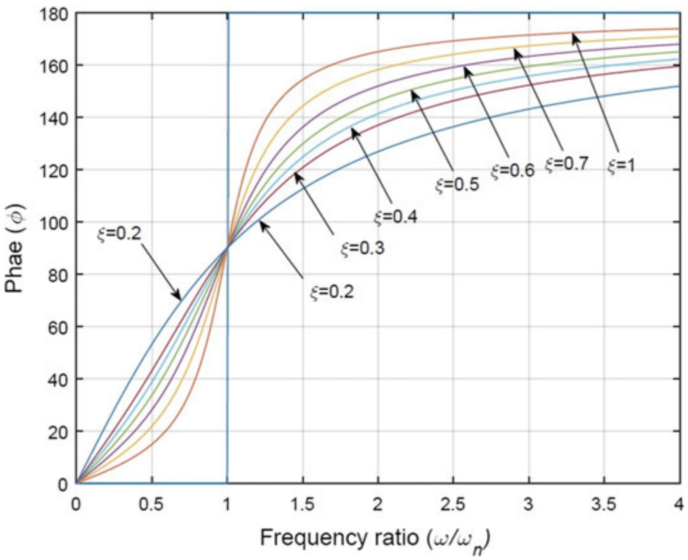


Fig. 4.11 Phase response for forced harmonic vibration

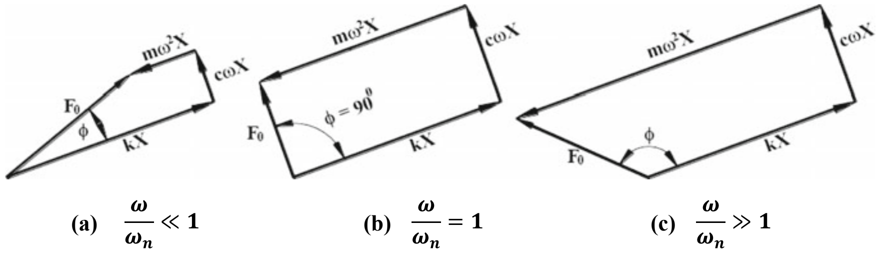


Fig. 4.12 Phasor diagrams for forced harmonic response for different frequency ratios

equal to zero for any values of damping as shown in Figs. 4.10 and 4.11. This behavior can also be understood from the phasor diagram of low frequency ratio shown in Fig. 4.12a. For this region both the inertia and damping force, which involve ω are small, which results in a small phase angle ϕ and the magnitude of the imposed force F_0 is almost equal to the spring force kX.

Response of the System at Resonance Region

When the frequency of the external force (ω) is equal to the natural frequency of the system (ω_n) then frequency ratio ($\omega/\omega n$) will be equal to 1. It can be noted from Eqs. (4.92) and (4.93) that

$$\text{if} \quad \frac{\omega}{\omega_n} = 1, \quad \frac{kX}{F_0} = \frac{1}{2\xi} \quad \text{and} \quad \phi = 90^\circ.$$

Hence it can be understood that in the neighborhood of resonance region, amplitude ratio is proportional to the reciprocal of damping ratio as shown in Fig. 4.10 and phase of the response is almost 90° as shown in Fig. 4.11. These features can also be verified from the phasor diagram shown in Fig. 4.12b. In this region the inertia force is balanced by the spring force, whereas the imposed force overcomes the damping force.

Response of the System at High Frequency Region

When the frequency of the external force (ω) is higher than the natural frequency of the system (ω_n) then frequency ratio (ω/ω_n) will be greater than 1. It can be noted from Eqs. (4.92) and (4.93) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to \infty, \frac{kX}{F_0} \to 0 \quad \text{and} \quad \phi \to 180^\circ.$$

Hence it can be understood that at high frequency region, amplitude ratio is almost equal to zero as shown in Fig. 4.10 and phase of the response is almost 180° as shown in Fig. 4.11. These features can also be verified from the phasor diagram shown in

Fig. 4.12c. In this region the imposed force is expended almost entirely in overcoming the large inertia force.

Frequency Ratio corresponding to Peak Amplitude and Peak Amplitude

It can also be noted from Fig. 4.10 that the maximum amplitude (infinite value) occurs at the resonance frequency if the system is un-damped whereas the peak amplitude occurs not at the resonant frequency but a little toward its left if the system is damped. This shift is found to increase with the increase in damping.

The frequency ratio at which the peak amplitude occurs can be obtained from Eq. (4.92) by differentiating this equation with respect to (ω/ω_n) and equating this differential to zero. For this differentiation of Eq. (4.92) with respect to (ω/ω_n) gives

$$\frac{d\left(\frac{kX}{F_0}\right)}{d\left(\frac{\omega}{\omega_n}\right)} = -\frac{1}{2}\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2\right]^{-\frac{3}{2}} \left[-4\frac{\omega}{\omega_n}\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\} + 8\xi^2\frac{\omega}{\omega_n}\right]$$

If ω_p is the frequency corresponding to the peak amplitude, then $\left\{d\left(\frac{kX}{F_0}\right)/d\left(\frac{\omega}{\omega_n}\right)\right\}\Big|_{\omega=\omega_p} = 0$ gives

$$-\frac{1}{2}\left[\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega_p}{\omega_n}\right)^2\right]^{-\frac{3}{2}} \left[-4\frac{\omega_p}{\omega_n}\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\} + 8\xi^2\frac{\omega_p}{\omega_n}\right] = 0$$

Since $\left[\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega_p}{\omega_n}\right)^2\right] \neq 0$ for a real finite peak amplitude,

$$-4\frac{\omega_p}{\omega_n}\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\} + 8\xi^2\frac{\omega_p}{\omega_n} = 0$$

or,

$$-4\frac{\omega_p}{\omega_n}\left[\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\} - 2\xi^2\right] = 0$$

Again $\omega_p/\omega_n \neq 0$,

$$\left\{1-\left(\frac{\omega_p}{\omega_n}\right)^2\right\} - 2\xi^2 = 0$$

or,

$$\left(\frac{\omega_p}{\omega_n}\right)^2 = 1 - 2\xi^2$$
$$\therefore \quad \frac{\omega_p}{\omega_n} = \sqrt{1 - 2\xi^2} \tag{4.94}$$

Equation (4.94) gives real value of ω_p/ω_n, if $\xi^2 < 0.5$, i.e., $\xi < 0.707$. It may also be observed from Fig. 4.10 that peak amplitude occurs if $\xi < 0.707$ and no peak amplitude occurs for $\xi > 0.707$.

Then the corresponding peak amplitude is obtained by substituting

$$\frac{\omega}{\omega_n} = \frac{\omega_p}{\omega_n} = \sqrt{1 - 2\xi^2}$$

into Eq. (4.92) as

$$X_p = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega_p}{\omega_n}\right)^2}}$$

or,

$$X_p = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(1 - 2\xi^2\right)\right\}^2 + 4\xi^2\left(1 - 2\xi^2\right)}}$$
$$= \frac{\frac{F_0}{k}}{\sqrt{4\xi^4 + 4\xi^2 - 8\xi^4}}$$
$$= \frac{\frac{F_0}{k}}{\sqrt{4\xi^2 - 4\xi^4}}$$
$$\therefore \quad X_p = \frac{F_0}{k}\frac{1}{2\xi\sqrt{1 - \xi^2}} \tag{4.95}$$

4.4 Rotating Unbalance

Rotating unbalance is one of the most common cause of vibration in rotating machineries. Unbalance in rotating machines is a common source of vibration excitation. Unbalance present in a rotating system is usually specified by an unbalanced mass m_u and its eccentricity e.

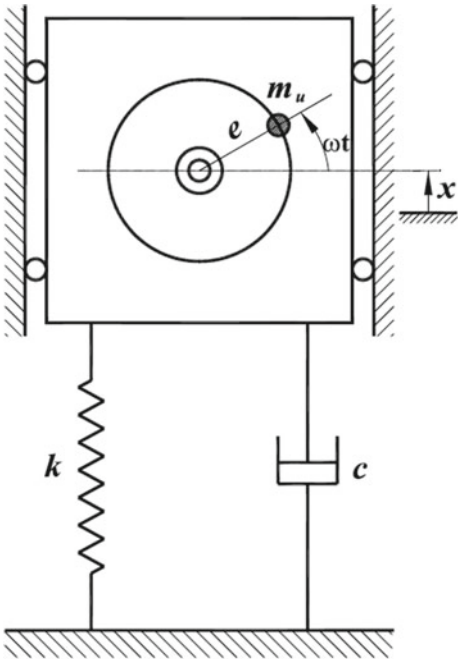


Fig. 4.13 Rotating machine with unbalance

Consider a spring-damper assembly shown in Fig. 4.13 supporting a rotating machine of mass m, which has a rotational speed ω and has a rotating unbalance of m_u and an eccentricity of e. If vertical displacement of the non-rotating mass $(m-m_u)$ from the static equilibrium position is x, then instantaneous vertical displacement of the unbalanced mass from the static equilibrium position will be $(x + e \sin \omega t)$.

Then equation of motion of the system is given by

$$(m - m_u)\ddot{x} + m_u \frac{d^2}{dt^2}(x + e \sin \omega t) = -kx - c\dot{x} \tag{4.96}$$

Simplifying Eq. (4.96), we get

$$m\ddot{x} + c\dot{x} + kx = m_u e \omega^2 \sin \omega t \tag{4.97}$$

Equation (4.97) shows that the rotating unbalance present in the system generates harmonic force whose amplitude is proportional to square of the rotating speed. It can also be noted that Eq. (4.97) that it is identical to Eq. (4.76) with $F_0 = m_u e \omega^2$.

Substituting $F_0 = m_u e \omega^2$ into Eq. (4.88), we can directly express the relationship for steady state amplitude due to rotating unbalance as

$$X = \frac{\frac{m_u e\omega^2}{k}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} = \frac{e\frac{m_u}{m}\frac{m}{k}\omega^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} \tag{4.98}$$

Equation (4.98) can also be expressed in non-dimensional form as

$$\frac{m}{m_u}\frac{X}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} \tag{4.99}$$

Similarly, Eq. (4.93) can also be used to determine the phase of the response due to rotating unbalance as

$$\phi = \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.100}$$

With the help of Eq. (4.99), the variation of non-dimensional amplitude ratio $(mX/m_u e)$ with the frequency ratio for different values of damping ratio can be plotted as shown in Fig. 4.14. Phase response due to rotating unbalance similar to that explained for force harmonic response.

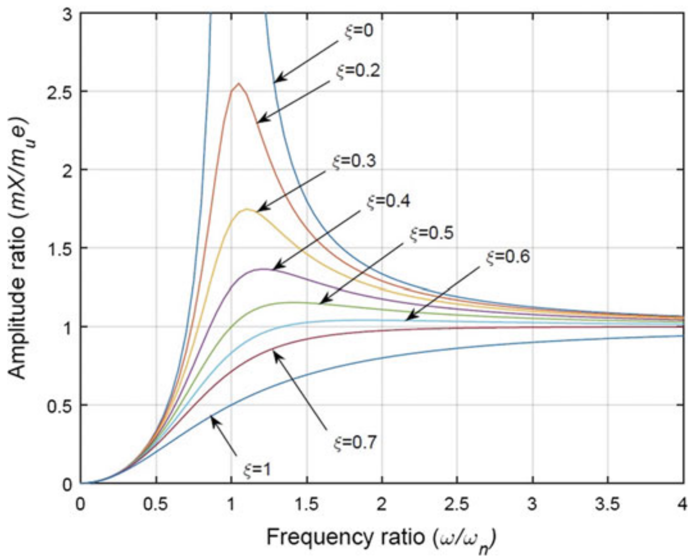


Fig. 4.14 Amplitude response for rotating unbalance

Response of the System at Low Frequency Region

When the frequency of the external force (ω) is less than the natural frequency of the system (ω_n) then frequency ratio (ω/ω_n) will be small. It can be noted from Eq. (4.99) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to 0, \quad \frac{m}{m_u}\frac{X}{e} \to 0.$$

Hence it can be understood that at low frequency region, amplitude ratio is almost equal to zero for any value of damping as shown in Fig. 4.14.

Response of the System at Resonance Region

When the frequency of the external force (ω) is equal to the natural frequency of the system (ω_n) then frequency ratio ($\omega/\omega n$) will be equal to 1. It can be noted from Eq. (4.99) that

$$\text{if} \quad \frac{\omega}{\omega_n} = 1, \quad \frac{m}{m_u}\frac{X}{e} = \frac{1}{2\xi}.$$

Hence it can be understood that in the neighborhood of resonance region, amplitude ratio is proportional to the reciprocal of damping ratio as shown in Fig. 4.14.

Response of the System at High Frequency Region

When the frequency of the external force (ω) is higher than the natural frequency of the system (ω_n) then frequency ratio (ω/ω_n) will be greater than 1. It can be noted from Eq. (4.99) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to \infty, \quad \frac{m}{m_u}\frac{X}{e} \to 1.$$

Hence it can be understood that at high frequency region, amplitude ratio is almost equal to unity for any values of damping as shown in Fig. 4.14.

4.5 Vibration Isolation and Transmissibility

If the machines rotating with high speed are installed directly on the foundation, high magnitude forces generated by vibration are transferred to foundations. This may damage the foundation and also adversely affect the performances of other machines installed on the same foundation. To avoid this usually some devices such as combination of springs and dampers are placed between the machine and the foundation which reduces the force transmitted to the foundations. Such system used to reduce vibration transmitted to the foundation is called a vibration isolator.

Common materials used for vibration isolation are rubber, felt cork, metallic springs, etc. These materials are used for different ranges of operating speeds, such as rubber is used for high frequency vibrations, felt for low frequency ratio, springs for high frequency ratio.

The effectiveness of a vibration isolating system can also be defined in terms of transmissibility ratio (TR) which is defined as the ratio of force transmitted to the foundation (F_T) and the externally imposed force (F_0), i.e.,

$$\mathrm{TR} = \frac{F_T}{F_0} \tag{4.101}$$

It can be understood from Eq. (4.101) that vibration isolation will be effective when the transmissibility ratio is low.

To determine the transmissibility ratio of a system, consider an isolator consisting of a spring of stiffness (k) and a damper of damping constant (c) supporting a machine which is subject to an external harmonic force $F(t)$ as shown in Fig. 4.15.

Phasor diagram of the all involved forces is shown in Fig. 4.16. Force imposed to the system is transmitted to the foundation through the spring and damper, and the magnitude of transmitted force is given by the resultant of the spring force and the damping force, i.e.,

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + \left(\frac{c\omega}{k}\right)^2} \tag{4.102}$$

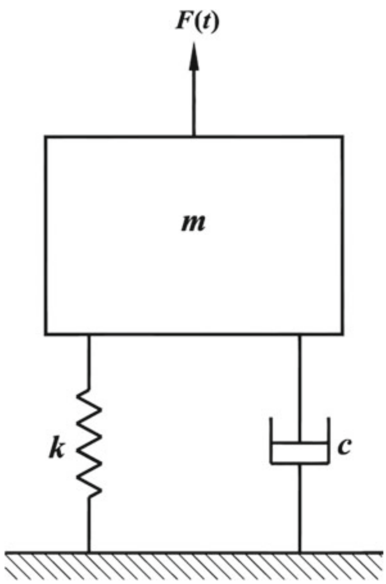


Fig. 4.15 Dynamic force transmitted to the foundation through spring and damper

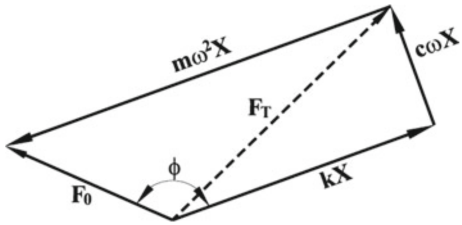


Fig. 4.16 Phasor diagram of forces involved in forced harmonic vibration

Substituting steady state amplitude of vibration (X) due to harmonic excitation $F(t) = F_0 \sin \omega t$ from Eq. (4.85) into Eq. (4.102), we get

$$F_T = F_0 \frac{\sqrt{1 + \left(\frac{c\omega}{k}\right)^2}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} \tag{4.103}$$

Substituting $\frac{c\omega}{k} = 2\xi \frac{\omega}{\omega_n} \frac{m}{k}\omega^2 = \left(\frac{\omega}{\omega_n}\right)^2$, we get

$$F_T = F_0 \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.104}$$

Equation (4.103) can be rearranged to give an expression for the transmissibility ratio as

$$\mathrm{TR} = \frac{F_T}{F_0} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.105}$$

With the help of Eq. (4.104), the variation of transmissibility ratio (TR) with the frequency ratio for different values of damping ratio can be plotted as shown in Fig. 4.17.

Response of the Isolator at Low Frequency Region

For a given operating speed (ω) of the machine, low frequency ratio (ω/ω_n) is obtained if natural frequency of the system (ω_n) is high. High natural frequency of the system is obtained when the stiffness of the system is high. Therefore this region is called spring controlled region.

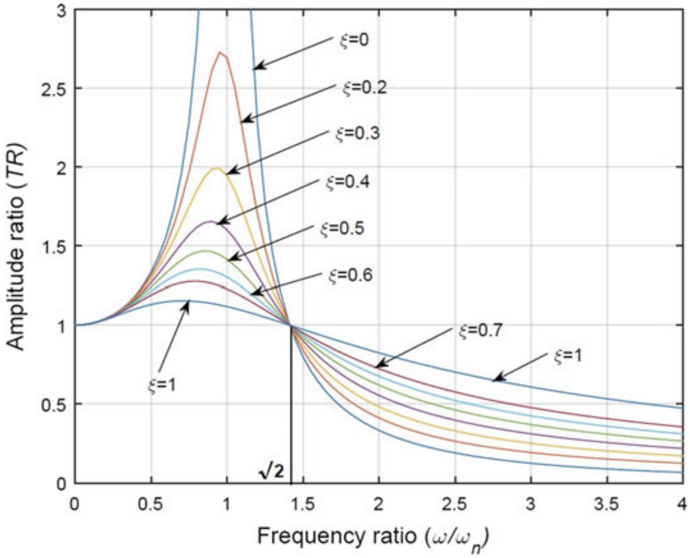


Fig. 4.17 Transmissibility ratio plot for different values of damping ratio

It can be noted from Eq. (4.105) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to 0, \quad \text{TR} \to 1.$$

Hence it can be understood that at low frequency region, transmissibility ratio is almost equal to unity for any values of damping as shown in Fig. 4.17.

Response of the Isolator at Resonance Region

When the frequency of the external force (ω) is equal to the natural frequency of the system (ω_n) then frequency ratio (ω/ω_n) will be equal to 1. It can be noted from Eqs. (4.105) that

$$\text{if} \quad \frac{\omega}{\omega_n} = 1, \quad \text{TR} = \sqrt{1 + \frac{1}{4\xi^2}}.$$

Hence it can be understood that in the neighborhood of resonance region, transmissibility ratio is proportional to the reciprocal of damping ratio as shown in Fig. 4.17. In this region, transmissibility ratio is largely dependent upon the damping; therefore this region is called the damping controlled region.

Response of the Isolator at High Frequency Region

For a given operating speed (ω) of the machine, high frequency ratio (ω/ω_n) is obtained if natural frequency of the system (ω_n) is low. Low natural frequency of the system is obtained when the mass of the system is high. Therefore this region is called mass controlled region.

It can be noted from Eqs. (4.105) that

$$\text{if} \quad \frac{\omega}{\omega_n} \to \infty, \quad \text{TR} \to 0.$$

Hence it can be understood that at high frequency region, transmissibility ratio is almost equal to zero for any values of damping as shown in Fig. 4.17.

Response of the Isolator at Frequency Ratio of $\sqrt{2}$

It can be noted from Eqs. (4.105) that

$$\text{if} \quad \frac{\omega}{\omega_n} = \sqrt{2}, \quad \text{TR} = 1.$$

It can also be noted from Fig. 4.17 that all the transmissibility curves start from unity value, pass through the unit transmissibility again at $\omega/\omega_n = \sqrt{2}$ and after that they tend to zero as ω/ω_n tends to ∞. Hence vibration isolation will be effective in the region where $\omega/\omega_n > \sqrt{2}$.

4.6 Response of a System to an External Motion

External motion input is also one of the causes of vibration. Energy provided to the system through cam-follower mechanism, disturbance experienced by a machine due to floor vibration are the examples of motion input. To study the response of a system due to motion, consider an assembly of a spring, mass and damper subjected to a base motion $y(t)$ as shown in Fig. 4.18.

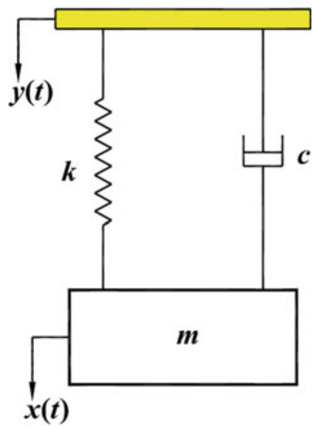


Fig. 4.18 SDOF system subjected to a base motion

Then equation of motion of the system is given by

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \tag{4.106}$$

Substituting $x(t) - y(t) = z(t)$ into Eq. (4.106), we get

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4.107}$$

If the base motion is harmonic, i.e., $y(t) = Y \sin \omega t$, Eq. (4.107) reduces to

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t \tag{4.108}$$

Comparing Eq. (4.107) with Eq. (4.76), we can directly express the steady state response for dependent variable $z(t)$ as

$$z = Z \sin(\omega t - \phi) \tag{4.109}$$

where

$$Z = \frac{\frac{m\omega^2 Y}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.110}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.111}$$

Then steady state response for dependent variable $x(t)$ can be expressed as

$$x = Z \sin(\omega t - \phi) + Y \sin \omega t \tag{4.112}$$

Substituting Z and ϕ, respectively, form Eqs. (4.110) and (4.111) into Eq. (4.112) and simplifying, it can be expressed in the form

$$x = X \sin(\omega t - \psi) \tag{4.113}$$

where

$$X = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2} Y}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.114}$$

and

$$\psi = \tan^{-1}\left\{\frac{2\xi\left(\frac{\omega}{\omega_n}\right)^3}{\left[1 + (4\xi^2 - 1)\left(\frac{\omega}{\omega_n}\right)^2\right]}\right\} \tag{4.115}$$

Equation (4.114) can be expressed in non-dimensional form as

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.116}$$

With the help of Eqs. (4.116) and (4.115), the variation of amplitude ratio (X/Y) and phase of the response (ψ) with the frequency ratio for different values of damping ratio can be plotted as shown in Figs. 4.19 and 4.20, respectively.

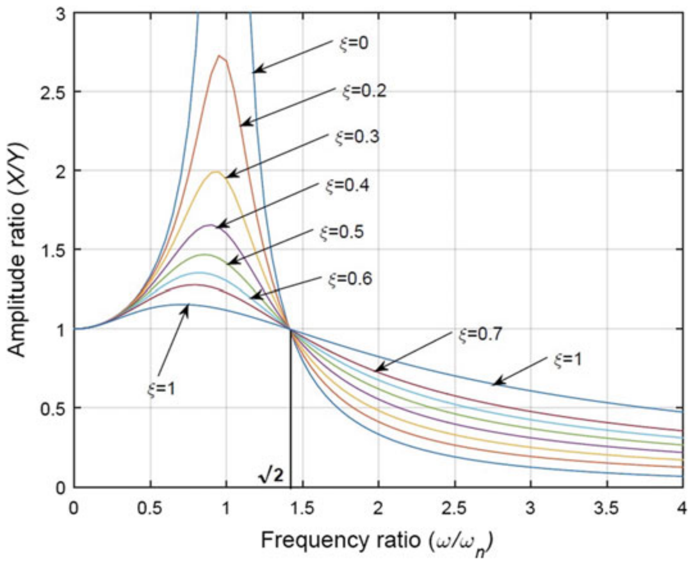


Fig. 4.19 Amplitude response for harmonic vibration due to external motion

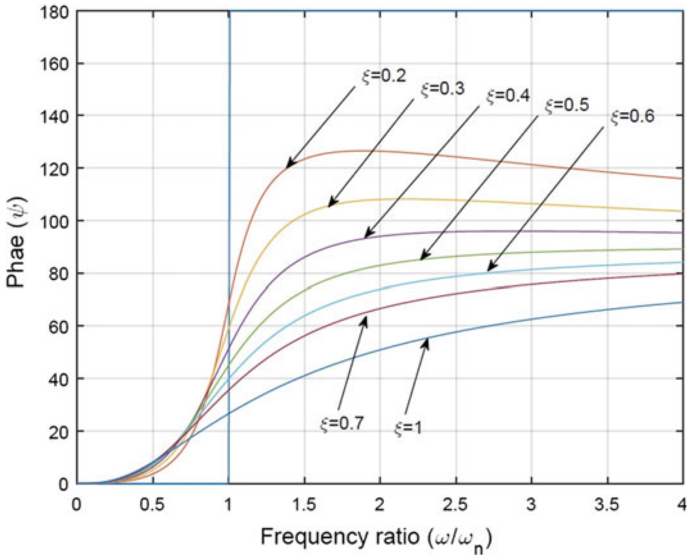


Fig. 4.20 Phase response for harmonic vibration due to external motion

It can be noted from Fig. 4.19 that external motion is amplified if the frequency ratio (ω/ω_n) is less than $\sqrt{2}$ and is attenuated if the frequency ratio (ω/ω_n) is greater than $\sqrt{2}$.

4.7 Vibration Measuring Instruments

Response of a system due to external motion can also be used for the vibration measurement. Such response can be calibrated to determine the displacement, velocity or acceleration of any vibrating system.

The instruments which are used to measure the displacement, velocity or acceleration of a vibrating body are called vibration measuring instrument. Figure 4.21 shows the essential elements of a vibration measuring instrument. It consists of a seismic mass m supported by springs with an equivalent stiffness k and a damper with a damping constant c inside a case which is to be fastened to the vibrating body. The motion is to be measured is y and the relative motion $z(= x - y)$ between the mass m and the supporting case is sensed.

From the derivation of previous section, we can directly express the steady state response for dependent variable $z(t)$ as

$$z = Z \sin(\omega t - \phi) \tag{4.117}$$

where

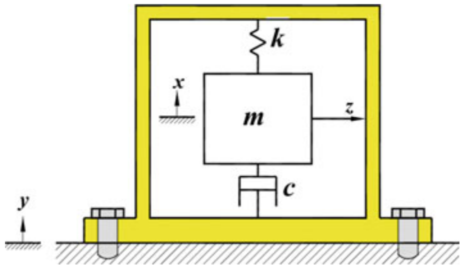


Fig. 4.21 Components a vibration measuring instrument

$$Z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.118}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.119}$$

Equation (4.118) can be expressed in non-dimensional form as

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \tag{4.120}$$

With the help of Eqs. (4.120) and (4.119), the variation of amplitude ratio (Z/Y) and phase of the response (ϕ) with the frequency ratio for different values of damping ratio can be plotted as shown in Figs. 4.22 and 4.23, respectively.

According to their useful operating range, there are two common types of vibration measuring instruments: seismometer and accelerometer.

4.7.1 Seismometer

It can be noted from Eq. (4.120) that when ω/ω_n tends to infinity, Z/Y becomes almost equal to 1, i.e., the relative displacement Z becomes equal to Y. Under this condition, the seismic mass m then remains stationary while the supporting case

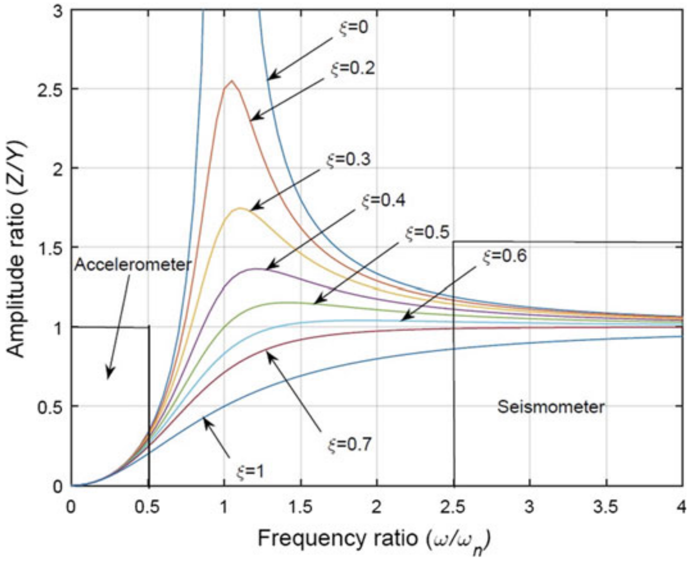


Fig. 4.22 Amplitude response of a vibration measuring instrument

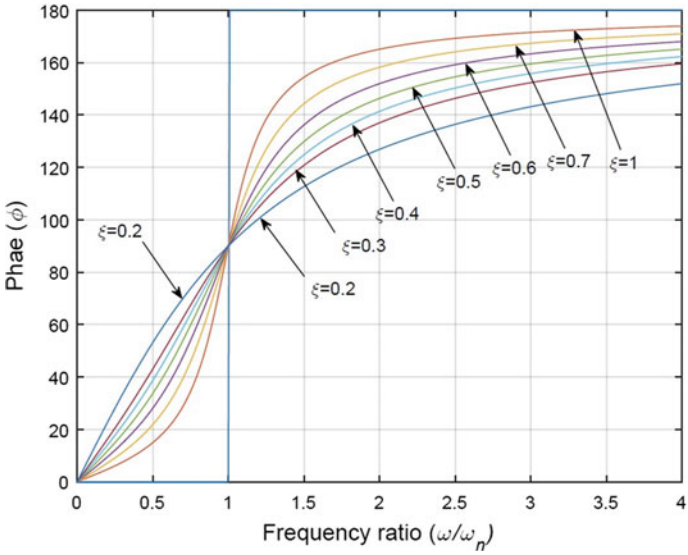


Fig. 4.23 Phase response of a vibration measuring instrument

moves with the vibrating system. Vibration measuring instrument used in this operating range is called a seismometer. Since it is useful at high frequency ratio, its natural frequency should be low.

The main limitation of the seismometer is the required large size. Since $Z = Y$, the relative motion of the seismic mass must be of the same order of magnitude as the vibration is to be measured.

The output reading of the instrument obtained as a relative motion (Z) can be converted to an electrical voltage by making the seismic mass (m) a magnet moving relative to coils fixed in the case. Since the voltage generated is proportional to the rate of cutting of the magnetic field, the output of the instrument will be proportional to the velocity of the vibrating body.

4.7.2 Accelerometer

It can be noted from Eq. (4.120) that when ω/ω_n is very low, denominator of Eq. (4.120) becomes almost equal to 1, i.e., then relative displacement Z can be related to Y as

$$Z = \left(\frac{\omega}{\omega_n}\right)^2 Y \tag{4.121}$$

Equation (4.121) shows that, under this condition, Z becomes proportional to the acceleration of the motion to be measured ($\omega^2 Y$). Vibration measuring instrument used in this operating range is called an accelerometer. Since it is useful at low frequency ratio, its natural frequency should be high.

Under this condition, the seismic mass m then remains stationary while the supporting case moves with the vibrating system. Vibration measuring instrument used in this operating range is called a seismometer. Since it is useful at high frequency ratio, its natural frequency should be low.

Accelerometers are preferred as vibration measuring instruments because of their small size and high sensitivity. Earthquakes are recorded by accelerometers, and the velocity and displacement are obtained by successive integrations.

4.8 Response to Multi-Frequency and General Periodic Excitations

4.8.1 Response to Multi-Frequency Excitation

Equation of motion of a SDOF system subjected to multi-frequency excitation is given by

$$m\ddot{x} + c\dot{x} + kx = \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \tag{4.122}$$

The governing equation of motion is a linear differential equation, principle of superposition can be applied. Hence the total response of the system is the sum of responses due to each harmonics of the external excitation, i.e.,

$$x(t) = \sum_{i=1}^{n} X_i \sin(\omega_i t + \psi_i - \phi_i) \tag{4.123}$$

where

$$X_i = \frac{F_i/k}{\sqrt{\left\{1 - \left(\frac{\omega_i}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega_i}{\omega_n}\right)^2}} \tag{4.124}$$

and

$$\phi_i = \tan^{-1}\left\{\frac{2\xi\frac{\omega_i}{\omega_n}}{1 - \left(\frac{\omega_i}{\omega_n}\right)^2}\right\} \tag{4.125}$$

4.8.2 Response to a General Periodic Excitation

If the external force imposed on the system is periodic (with a period of T) but non-harmonic, then it can be converted into harmonic terms by using Fourier series expansion as

$$F(t) = a_0 + \sum_{i=1}^{n}[a_i \cos(\omega_i t) + b_i \sin(\omega_i t)] \tag{4.126}$$

where

$$\omega_i = i\omega_1 = \frac{2\pi i}{T} \tag{4.127}$$

$$a_0 = \frac{1}{T}\int_0^T F(t)\mathrm{d}t \tag{4.128}$$

$$a_i = \frac{2}{T}\int_0^T F(t)\cos(\omega_i t)\mathrm{d}t \tag{4.129}$$

$$b_i = \frac{2}{T}\int_0^T F(t)\sin(\omega_i t)\mathrm{d}t \tag{4.130}$$

Equation (4.126) can also be expressed in terms of sine components only as

$$F(t) = a_0 + \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \tag{4.131}$$

where

$$F_i = \sqrt{a_i^2 + b_i^2} \tag{4.132}$$

and

$$\psi_i = \tan^{-1}\left(\frac{a_i}{b_i}\right) \tag{4.133}$$

If $F(t)$ defined by Eq. (4.131) is applied to a SDOF system, the governing differential equation of motion of the system can be given as

$$m\ddot{x} + c\dot{x} + kx = a_0 + \sum_{i=1}^{n} F_i \sin(\omega_i t + \psi_i) \tag{4.134}$$

The principle of linear superposition can be applied to determine the response of the system as

$$x(t) = \frac{a_0}{k} + \sum_{i=1}^{n} X_i \sin(\omega_i t + \psi_i - \phi_i) \tag{4.135}$$

where X_i and ϕ_i are defined in Eqs. (4.124) and (4.125), respectively.

4.9 Response to Transient Input Forces

When a system is imposed to a harmonic excitation, there exists a response which has a combination of both the free response and the forced response. Free response under-damped condition, which is also called the transient response, decays with time and die out after certain interval whereas the forced response which is also called a steady state response continues as long as the external excitation exists. But when a system is imposed to a transient forces, the effects of both the free response and the force response may exist for a significant interval depending upon the interval to which the transient input force is active. In many cases, the response of the system due to a transient force of a certain interval may exist for an interval greater that the duration of the transient force.

Equation of motion of a SDOF system subjected to a transient force $F(t)$ can be expressed as

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{4.136}$$

The initial values required to determine the complete solution of the system can be defined as

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{4.137}$$

Transient response of any system defined by Eqs. (4.136) and (4.137) can be determined by using two common methods: the convolution integral and the Laplace transform. While using the convolution integral method, the effect of transient force is included in the initial conditions by applying principle of impulse and momentum whereas in the method of Laplace transform, initial conditions are applied during the transform procedure.

4.9.1 Response Due to a Unit Impulse

An impulse is a force with a very high magnitude that exists for a very short time interval. If a force of magnitude $F(t)$ is applied to a system between time instants t_1 and t_2, then the impulse is defined as

$$I = \int_{t_1}^{t_2} F(\eta)\mathrm{d}\eta \tag{4.138}$$

Substituting

$$F(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta}(mv)$$

and integrating for the given time interval t_1 to t_2, we get

$$I = mv(t_2) - mv(t_1)$$

$$\therefore \quad mv(t_2) = I + mv(t_1) \tag{4.139}$$

Equation (4.139) is the expression for principle of impulse and momentum.

If the impulse of magnitude I is applied to a system which is initially at rest, Eq. (4.139) reduces to

$$I = mv(t_2)$$
$$\therefore \quad v(t_2) = \frac{I}{m} \tag{4.140}$$

Equation (4.139) shows that the velocity of the system immediately after the application of impulse I is I/m. Hence the initial value problem defined by Eqs. (4.136) and (4.137) can be expressed in equivalent form for the response due to applied impulse as

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.141}$$

$$x(0) = x_0 = 0 \quad \text{and} \quad \dot{x}(0) = v_0 = \frac{I}{m} \tag{4.142}$$

Response of an Un-damped System to the Impulse

Equation (4.141) can be reduced for an un-damped system as

$$\ddot{x} + \omega_n^2 x = 0 \tag{4.143}$$

The solution of Eq. (4.142) with initial values given by Eq. (4.141) can be determined as

$$x(t) = \frac{I}{m\omega_n} \sin \omega_n t \tag{4.144}$$

Equation (4.143) can also be expressed as

$$x(t) = Ih(t) \tag{4.145}$$

where

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t \tag{4.146}$$

is the response due to a unit impulse applied at $t = 0$.

Response of a Damped System to the Impulse

Equation (4.141) can be reduced for a damped system as

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{4.147}$$

The solution of Eq. (4.146) for an over-damped with initial values given by Eq. (4.141) can be determined as

$$x(t) = \frac{I}{2\left(\sqrt{\xi^2 - 1}\right)m\omega_n}\left[\mathrm{e}^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} - \mathrm{e}^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}\right] \tag{4.148}$$

From which response of an over-damped system due to a unit impulse applied at $t = 0$ can be expressed as

$$h(t) = \frac{1}{2\left(\sqrt{\xi^2 - 1}\right)m\omega_n}\left[\mathrm{e}^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} - \mathrm{e}^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}\right] \tag{4.149}$$

The solution of Eq. (4.147) for a critically damped with initial values given by Eq. (4.142) can be determined as

$$x(t) = \frac{I}{m}te^{-\omega_n t} \tag{4.150}$$

From which response of a critically damped system due to a unit impulse applied at $t = 0$ can be expressed as

$$h(t) = \frac{1}{m}te^{-\omega_n t} \tag{4.151}$$

Similarly, the solution of Eq. (4.147) for an under-damped system with initial values given by Eq. (4.142) can be determined as

$$x(t) = \frac{I}{m\omega_d}\mathrm{e}^{-\xi\omega_n t}\sin\omega_d t \tag{4.152}$$

From which response of an under-damped system due to a unit impulse applied at $t = 0$ can be expressed as

$$h(t) = \frac{1}{m\omega_d}\mathrm{e}^{-\xi\omega_n t}\sin\omega_d t \tag{4.153}$$

If the unit impulse is applied at $t = t_0$ instead of $t = 0$, then the response of the system due to the delayed unit impulse is given by

$$h(t) = h(t - t_0)u(t - t_0) \tag{4.154}$$

4.9.2 Response Due to a General Excitation

Response of a system due to any arbitrary transient force can be determined by dividing the force into a number of impulses and determining the response of the system to each impulse and then applying the principle of superposition.

Consider an arbitrary transient force shown in Fig. 4.24 applied to a SDOF system. Divide the force into n number of impulses each having width of $\Delta\eta$ as shown in Fig. 4.25.

The strength of the impulse that exists between the time instants $i\Delta\eta$ and $(i+1)\Delta\eta$ is given by

$$I_i = \int\limits_{i\Delta\eta}^{(i+1)\Delta\eta} F(\eta)\mathrm{d}\eta \tag{4.155}$$

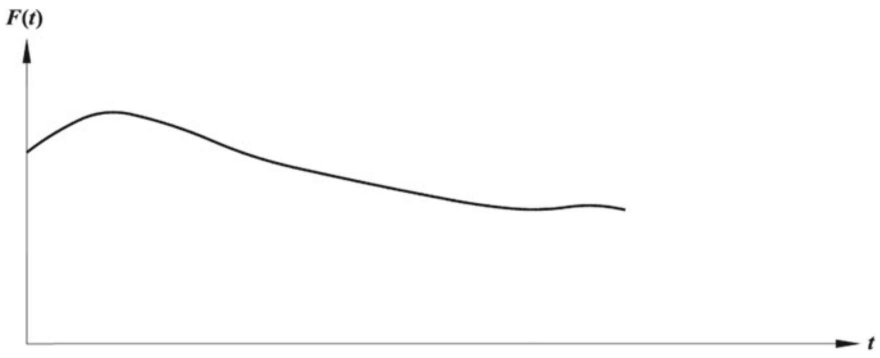


Fig. 4.24 Arbitrary transient force

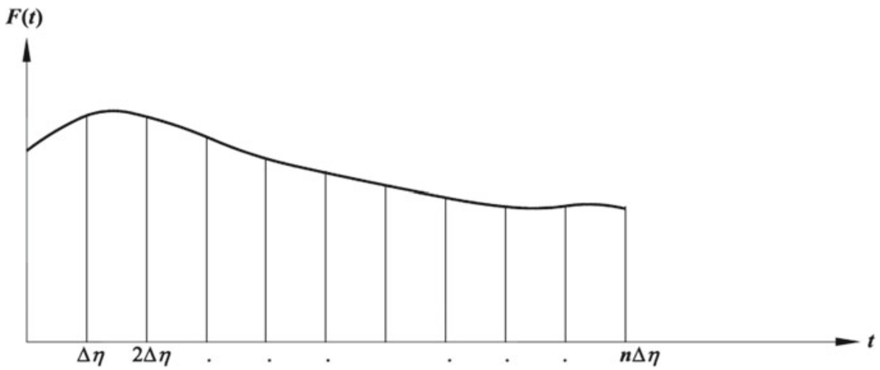


Fig. 4.25 Arbitrary transient force divided into a number of impulses

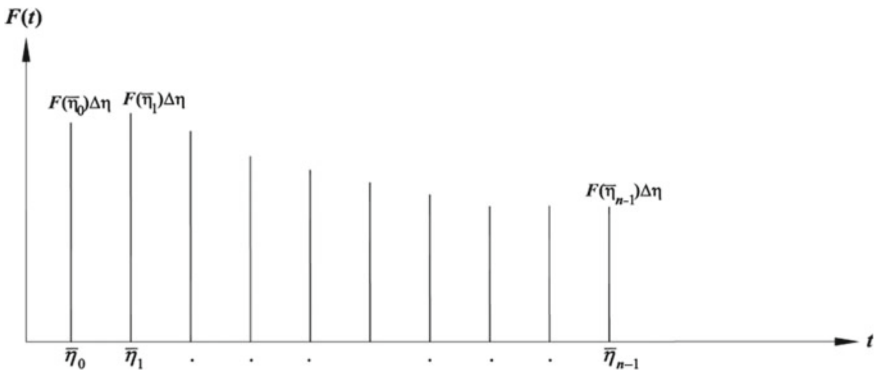


Fig. 4.26 Approximation for strength of impulse

Then the approximate strength of each impulse can be determined by applying mean value theorem of integral calculus as shown in Fig. 4.26.

$$I_i = F(\overline{\eta}_i)\Delta\eta \tag{4.156}$$

where

$$\overline{\eta}_i = \frac{1}{2}[i\Delta\eta + (i+1)\Delta\eta]$$

Transient force shown in Fig. 4.24 can be approximated as

$$F(\eta) = \sum_{i=0}^{n} I_i \delta(\eta - \overline{\eta}_i) \tag{4.157}$$

Then using Eq. (4.154) the response of a SDOF system due to the delayed unit impulse of Eq. (4.156) is given by

$$x_i(t) = I_i h(t - \overline{\eta}_i) u(t - \overline{\eta}_i) \tag{4.158}$$

Applying principle of superposition, the response of the system due all impulses shown in Fig. 4.25 can be determined as

$$x(t) = \sum_{i=0}^{n} x_i(t) = \sum_{i=0}^{n} I_i h(t - \overline{\eta}_i) u(t - \overline{\eta}_i) \tag{4.159}$$

Substituting I_i from Eq. (4.156) into Eq. (4.159), we get

$$x(t) = \sum_{i=0}^{n} F\left(\overline{\eta}_i\right) h\left(t - \overline{\eta}_i\right) u\left(t - \overline{\eta}_i\right) \Delta\eta \tag{4.160}$$

The approximated response determined by Eq. (4.160) will be close to exact response when $n \to \infty$. Then using first principle of integration, Eq. (4.160) can be expressed as

$$x(t) = \int_0^t F(\eta) h(t - \eta) \mathrm{d}\eta \tag{4.161}$$

Equation (4.161) is a form of convolution integral which can be used to determine the response of a system subjected to any arbitrary excitation.

If the transient force is applied at $t = t_0$ instead of $t = 0$, then the response of the system due to the delayed transient force is given by

$$\begin{aligned} x(t) &= \int_0^t F(\eta) h(t - \eta) u(\eta - t_0) \mathrm{d}\eta \\ &= u(t - t_0) \int_{t_0}^t F(\eta) h(t - \eta) \mathrm{d}\eta \end{aligned} \tag{4.162}$$

4.10 Solution Using the Method of Laplace Transform

The method of Laplace transform is one of the common tool used for solving ordinary differential equations with time as an independent variable. It can be used for any kind of excitation: periodic, transient and even excitations whose forms change at discrete times.

While using this method the given ordinary differential equation is transformed into an algebraic equation, called the subsidiary equation. The subsidiary equation is simplified by purely algebraic manipulations. Then the simplified algebraic expression is transformed back by taking the inverse of the Laplace transform which gives the solution of the system in time domain.

To demonstrate the method, consider an equation of motion of a SDOF system subjected to a transient force $F(t)$ given by

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \tag{4.163}$$

Taking the Laplace transform of Eq. (4.163), we get

$$\mathcal{L}\{\ddot{x}\} + 2\xi\omega_n\mathcal{L}\{\dot{x}\} + \omega_n^2\mathcal{L}\{x\} = \frac{F(s)}{m} \tag{4.164}$$

Using Laplace transforms of first and second derivates, Eq. (4.163) gives

$$s^2\mathrm{X}(s) - sx(0) - \dot{x}(0) + 2\xi\omega_n[sX(s) - x(0)] + \omega_n^2X(s) = \frac{F(s)}{m} \tag{4.165}$$

Rearranging Eq. (4.164) for $X(s)$, we get

$$X(s) = \frac{1}{m}\frac{F(S)}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{(s + 2\xi\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{4.166}$$

Then taking inverse Laplace transform, we get the solution for the differential equation given by Eq. (4.162) as

$$x(t) = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{F(S)}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 2\xi\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \tag{4.167}$$

Transfer Function

The transfer function of a dynamic system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable under the assumption that all initial condition are zero. In case of forced response of a system, if the Laplace transform of the response $x(t)$ is $X(s)$ and the Laplace transform of the externally imposed force $F(t)$ is $F(s)$, then its transfer function $G(s)$ is given by

$$G(s) = \frac{X(s)}{F(s)} \tag{4.168}$$

Substituting initial values $x(0) = 0$ and $\dot{x}(0) = 0$, into Eq. (4.166) and rearranging, we get an expression for the transfer function as

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{m\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)} \tag{4.169}$$

It can be noted from Eq. (4.169) that the transfer function is dependent only on the system parameters. For a SDOF system, it is dependent upon the mass, damping ratio and natural frequency.

4.11 Energy Dissipated in Viscous Damper

Damping removes energy from the system. Due to this loss of energy the amplitude of free vibration decrease continuously and it comes to equilibrium position after certain interval whereas during steady state vibration of a forced system the amplitude of vibration remains constant because the loss of energy is balanced by the energy which is supplied by the external excitation.

Energy lost by a damped system during a cycle can be determined as

$$W_d = \oint F \mathrm{d}x \tag{4.170}$$

Substituting $F = c\dot{x}$, for a system with viscous damping, we get

$$W_d = \oint c\dot{x}\mathrm{d}x = \oint c(\dot{x})^2 \mathrm{d}t \tag{4.171}$$

The amplitude of steady state oscillation under harmonic excitation is given by

$$x = X\sin(\omega t - \phi) \tag{4.172}$$

Differentiating Eq. (4.171) with respect to time, we get

$$\dot{x} = \omega X\cos(\omega t - \phi) \tag{4.173}$$

Substituting $\dot{x}$ from Eq. (4.173) into Eq. (4.171), we get

$$W_d = \oint c\{\omega X\cos(\omega t - \phi)\}^2 \mathrm{d}t = \frac{1}{2}c\omega^2 X^2 \oint [1 + \cos 2(\omega t - \phi)] \tag{4.174}$$

Energy dissipated during a cycle can then be determined as

$$W_d = \frac{1}{2}c\omega^2 X^2 \int_0^{\frac{2\pi}{\omega}} [1 + \cos 2(\omega t - \phi)]\mathrm{d}t = \frac{1}{2}c\omega^2 X^2 \left[\frac{2\pi}{\omega}\right]$$

$$\therefore \quad W_d = \pi c\omega X^2 \tag{4.175}$$

Substituting $c\omega = \xi c_c = 2\xi\sqrt{km}$, Eq. (4.175) can be expressed as

$$W_d = 2\pi\xi X^2\sqrt{km} \tag{4.176}$$

Using $\omega_n = \sqrt{k/m}$, energy dissipated during resonance can be determined as

$$W_d = 2\pi\xi k\omega_n X^2 \tag{4.177}$$

4.12 Response of a System with Coulomb Damping

4.12.1 Free Response for a System with Coulomb Damping

Consider a SDOF system with Coulomb damping as shown in Fig. 4.27. Equation of motion for the system can expressed as

$$m\ddot{x} + kx = -\mu mg\frac{|\dot{x}|}{\dot{x}} \tag{4.178}$$

Equation (4.178) can also be expressed as

$$m\ddot{x} + kx = -\mu mg \quad \text{if } \dot{x} > 0 \tag{4.179a}$$

$$m\ddot{x} + kx = \mu mg \quad \text{if } \dot{x} < 0 \tag{4.179b}$$

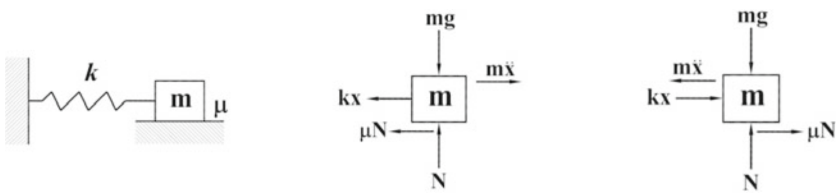
Assume that the block is moved toward right by x_0 and released. This gives initial condition as $x(0) = x_0$ and $\dot{x}(0) = 0$. As soon as block is released it moves toward left, during which $\dot{x} < 0$ and hence Eq. (4.179b) should be used. The general solution of Eq. (4.179b) can determined as

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{\mu mg}{k} \tag{4.181}$$

Applying initial condition of $x(0) = x_0$ and $\dot{x}(0) = 0$, we get arbitrary constants A and B as

$$A = 0 \quad \text{and} \quad B = x_0 - \frac{\mu mg}{k}$$

Substituting A and B into Eq. (4.181), we get the solution for the first half of the first cycle as



(a) Block Sliding on a Rough Surface **(b) Free body Diagrams showing Direction of Frictional Force**

Fig. 4.27 Free vibration of a system with coulomb damping

$$x(t) = \left(x_0 - \frac{\mu mg}{k}\right)\cos\omega_n t + \frac{\mu mg}{k} \tag{4.182}$$

Then the velocity for the first half of the first cycle can be obtained by the differentiation of Eqs. (4.179a, b) as

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{\mu mg}{k}\right)\sin\omega_n t \tag{4.183}$$

Equation (4.183) shows that $\dot{x} < 0$ for $0 < t < \pi/\omega_n$ and $\dot{x} > 0$ for $\pi/\omega_n < t < 2\pi/\omega_n$.

For the second half of the first cycle, $\dot{x} > 0$, hence Eq. (4.179a) should be used. The general solution of Eq. (4.179a) can determined as

$$x(t) = C\sin\omega_n t + D\cos\omega_n t - \frac{\mu mg}{k} \tag{4.184}$$

Initial condition for the second half of the first cycle can be taken as

$$x\left(\frac{\pi}{\omega_n}\right) = -x_0 + \frac{2\mu mg}{k} \quad \text{and} \quad \dot{x}\left(\frac{\pi}{\omega_n}\right) = 0$$

Substituting these initial conditions into Eq. (4.184), we get arbitrary constants C and D as

$$C = 0 \quad \text{and} \quad D = x_0 - \frac{3\mu mg}{k}$$

Substituting C and D into Eq. (4.184), we get the solution for the second half of the first cycle as

$$x(t) = \left(x_0 - \frac{3\mu mg}{k}\right)\cos\omega_n t - \frac{\mu mg}{k} \tag{4.185}$$

Similarly, the velocity for the second half of the first cycle can be obtained by the differentiation of Eq. (4.185) as

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{3\mu mg}{k}\right)\sin\omega_n t \tag{4.186}$$

Hence the sign of velocity again changes when $t = 2\pi/\omega_n$. Then displacement and velocity for this instant are determined as

$$x\left(\frac{2\pi}{\omega_n}\right) = x_0 - \frac{4\mu mg}{k} \quad \text{and} \quad \dot{x}\left(\frac{2\pi}{\omega_n}\right) = 0$$

Equations (4.182) and (4.184) can be generalized for the solution of successive half cycles as

$$x(t) = \left\{x_0 - (4n-3)\frac{\mu mg}{k}\right\}\cos\omega_n t + \frac{\mu mg}{k} \tag{4.187}$$

$$\text{for} \quad 2(n-1)\frac{\pi}{\omega_n} \leq t \leq 2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n}$$

$$x(t) = \left\{x_0 - (4n-1)\frac{\mu mg}{k}\right\}\cos\omega_n t + \frac{\mu mg}{k} \tag{4.188}$$

$$\text{for} \quad 2\left(n - \frac{1}{2}\right)\frac{\pi}{\omega_n} \leq t \leq 2n\frac{\pi}{\omega_n}$$

The solutions show that in the presence of Coulomb damping, vibration amplitude decreases linearly as shown in Fig. 4.28, and the difference between two successive amplitudes can be given as

$$x_n - x_{n+1} = \frac{4\mu mg}{k} \tag{4.189}$$

It can also be noted that system comes to rest after n cycles when the restoring force becomes less than the friction force, i.e.,

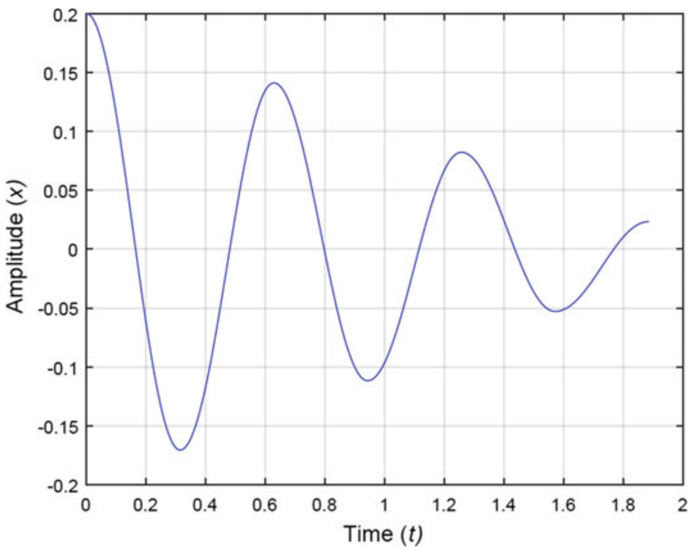


Fig. 4.28 Free response of a system with coulomb damping

$$x_0 - n\frac{4\mu mg}{k} \leq \frac{\mu mg}{k}$$

$$\therefore \quad n \geq \frac{x_0 - \frac{\mu mg}{k}}{\frac{4\mu mg}{k}} \tag{4.190}$$

Comparison between Free Response due to Viscous Damping and Coulomb Damping

Following features can be noted from the response of a system with viscous damping with that with Coulomb damping:

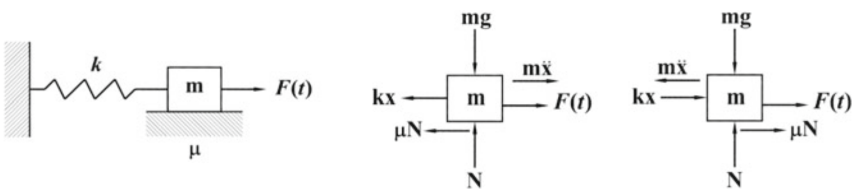
- Equation of motion of a viscously damped system is linear whereas it is nonlinear for a system with Coulomb damping.
- Response of a viscously damped system may be periodic (in case of an under-damped system) or aperiodic (in case of a critically damped or an over-damped system) whereas it is always periodic in a system with Coulomb damping.
- The natural frequency of oscillation of a viscously damped system (when it is under-damped) is less than the corresponding un-damped natural frequency of the system whereas the natural frequency of oscillation of a system with Coulomb damping is always equal to the corresponding un-damped natural frequency of the system.
- The amplitude of vibration decays exponentially in a viscously damped system whereas it decays linearly in a system with Coulomb damping.

4.12.2 Forced Response for a System with Coulomb Damping

Consider a SDOF system with Coulomb damping subjected to an external excitation $F(t) = F_0 \sin \omega t$ as shown in Fig. 4.29a.

With reference to free-body diagrams shown in Fig. 4.29b, equation of motion for the system can expressed as

$$m\ddot{x} + kx + \mu mg = F_0 \sin \omega t \quad \text{if } \dot{x} > 0 \tag{4.191a}$$



(a) Block Sliding on a Rough Surface with an External Force $F(t)$

(b) Free body Diagrams showing Direction of Frictional Force and External Force $F(t)$

Fig. 4.29 Forced vibration of a system with coulomb damping

$$m\ddot{x} + kx - \mu mg = F_0 \sin \omega t \quad \text{if } \dot{x} < 0 \tag{4.191b}$$

If the magnitude of frictional force is relatively larger, the response of the system will be nonlinear whereas if the frictional force is very small in comparison to the externally imposed force $F_0 \sin \omega t$, the response can be approximated as steady state harmonic response of the form

$$x = X \sin(\omega t - \phi) \tag{4.192}$$

To determine the steady state harmonic response, analogous approach can be used by determining equivalent viscous damping of the system. The energy dissipated during a cycle due to Coulomb damping can be expressed as

$$W_{\text{cd}} = 4\mu mgX \tag{4.193}$$

Using Eq. (1.174), energy dissipated during a cycle due to viscous damping can be expressed as

$$W_d = \pi c_{\text{eq}} \omega X^2 \tag{4.194}$$

Comparing Eqs. (4.194) and (4.195), equivalent viscous damping constant c_{eq} can be determined as

$$\begin{aligned} c_{\text{eq}} \pi \omega X^2 &= 4\mu mgX \\ \therefore \quad c_{\text{eq}} &= \frac{4\mu mg}{\pi \omega X} \end{aligned} \tag{4.195}$$

Then equivalent damping ratio ξ_{eq} can also be defined as

$$\xi_{\text{eq}} = \frac{c_{\text{eq}}}{c_c} = \frac{1}{2m\omega_n} \frac{4\mu mg}{\pi \omega X} = \frac{2\mu g}{\pi \omega \omega_n X} = \frac{2\mu g \omega_n}{\pi \omega \omega_n^2 X} = \frac{2\mu mg \omega_n}{\pi \omega k X} \tag{4.196}$$

Now substituting ξ_{eq} into Eqs. (4.88) and (4.89), expressions for amplitude and phase response of the system with Coulomb damping can be given as

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(\frac{4\mu mg}{\pi k X}\right)^2}} \tag{4.197}$$

and

$$\phi = \tan^{-1}\left\{\frac{\frac{4\mu mg}{\pi kX}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.198}$$

Solving Eq. (4.197) for X, we get

$$X = \frac{F_0}{k}\left[\frac{\sqrt{1-\left(\frac{4\mu mg}{\pi F_0}\right)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.199}$$

It can be noted from Eq. (4.199), there exists a steady state oscillation only when it gives real value of X, for which the following condition should be satisfied

$$\frac{4\mu mg}{\pi F_0} < 1$$

$$\therefore \quad F_0 > \frac{4\mu mg}{\pi} \tag{4.200}$$

Substituting X from Eq. (4.199) into Eq. (4.198), we get

$$\phi = \tan^{-1}\left\{\frac{\frac{4\mu mg}{\pi kX}}{1-\left(\frac{4\mu mg}{\pi F_0}\right)^2}\right\} \tag{4.201}$$

It can be noted from Eq. (4.201) that the phase of the response is constant in the presence of the Coulomb damping.

4.13 Response of a System with Hysteretic Damping

4.13.1 Free Response for a System with Hysteretic Damping

The energy dissipated during a cycle due to hysteretic damping is given by,

$$W_d = \pi k\beta X^2 \tag{4.202}$$

where k is the stiffness, X is the amplitude of vibration, π is any convenient proportionality constant and β is called the hysteretic damping constant.

If W_d is small then the free response is almost sinusoidal. If X_1 be the first peak amplitude, $X_{1/2}$ be the peak after first half cycle and X_2 be the peak amplitude after a cycle, then energy lost during the first of the cycle is given by

$$\frac{1}{2}\pi k\beta X_1^2 = \frac{1}{2}kX_1^2 - \frac{1}{2}kX_{1/2}^2 \tag{4.203}$$

Then the relationship between X_1 and $X_{1/2}$ is given by

$$\frac{X_{1/2}}{X_1} = \sqrt{1 - \pi\beta} \tag{4.204}$$

Similarly, applying energy equation for the second half of the first cycle, the relationship between $X_{1/2}$ and X_2 is given by

$$\frac{X_2}{X_{1/2}} = \sqrt{1 - \pi\beta} \tag{4.205}$$

Multiplying Eqs. (4.204) and (4.205), the ratio of successive amplitudes is given by

$$\frac{X_2}{X_1} = 1 - \pi\beta \tag{4.206}$$

Then equivalent logarithmic decrement δ_{eq} for hysteretic damping is given by

$$\delta_{\text{eq}} = \ln\left(\frac{X_1}{X_2}\right) = \ln(1 - \pi\beta)^{-1} = \ln\left(\frac{1}{1 - \pi\beta}\right) \tag{4.207}$$

For small β,

$$\delta_{\text{eq}} = \pi\beta \tag{4.208}$$

Then equivalent damping ratio for hysteretic damping is given by

$$\xi_{\text{eq}} = \frac{\delta_{\text{eq}}}{2\pi} = \frac{\beta}{2} \tag{4.209}$$

Then equivalent damping constant for hysteretic damping is given by

$$c_{\text{eq}} = \xi_{\text{eq}} c_c = \xi_{\text{eq}} 2m\omega_n = \beta m\omega_n = \frac{k\beta}{\omega_n} \tag{4.210}$$

Then the response of a system with hysteretic damping can be determined by using the equivalent damping ratio (ξ_{eq}) or equivalent damping constant (c_{eq}) and applying the expressions developed for viscously damped system.

4.13.2 *Forced Response for a System with Hysteretic Damping*

Using equivalent damping constant ($c_{\mathrm{eq}} = k\beta/\omega$), equation of motion for forced harmonic vibration of a system with hysteretic damping can be expressed as

$$m\ddot{x} + \frac{k\beta}{\omega}\dot{x} + kx = F_0 \sin\omega t \tag{4.211}$$

Now using (4.88) and (4.89), expressions for amplitude and phase response of the system with hysteretic damping can be given as

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \beta^2}} \tag{4.212}$$

and

$$\phi = \tan^{-1}\left\{\frac{\beta}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \tag{4.213}$$

With the help of Eqs. (4.212) and (4.213), the variation of amplitude ratio (kX/F_0) and phase of the response (ϕ) with the frequency ratio for different values of β can be plotted as shown in Figs. 4.30 and 4.31, respectively.

The forced response due to hysteretic damping is somehow similar to that of a system with viscous damping.

Solved Examples

Example 4.1

A spring-mass ($k - m$) system has a natural frequency of 10 rad/s. When 5 kg mass is attached with m, the natural frequency of the system is lowered by 25%. Determine the mass (m) and stiffness (k) of the system.

Solution

If mass of the system is k and stiffness is m, its natural frequency is given by

$$\omega_n = \sqrt{\frac{k}{m}}$$
$$\therefore \quad k = m\omega_n^2 = m \times (10)^2 = 100m \qquad \textbf{(a)}$$

When 5 kg mass is added to the system, its natural frequency reduces by 25%, i.e.,

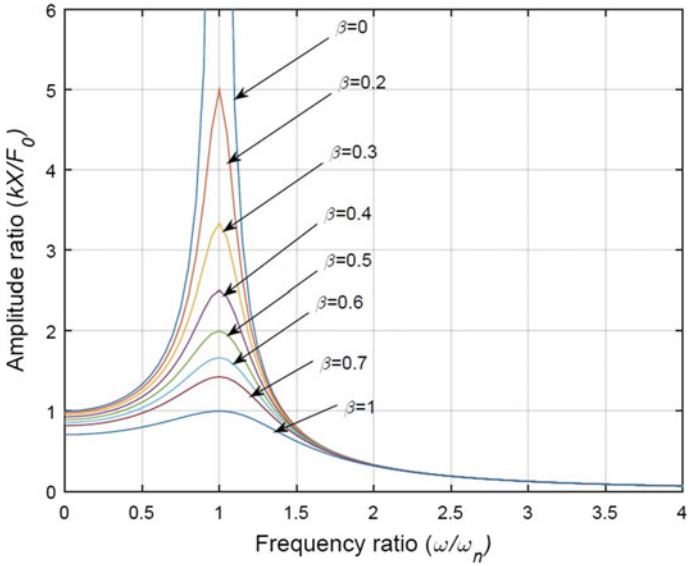


Fig. 4.30 Amplitude response of a system with hysteretic damping for forced harmonic vibration

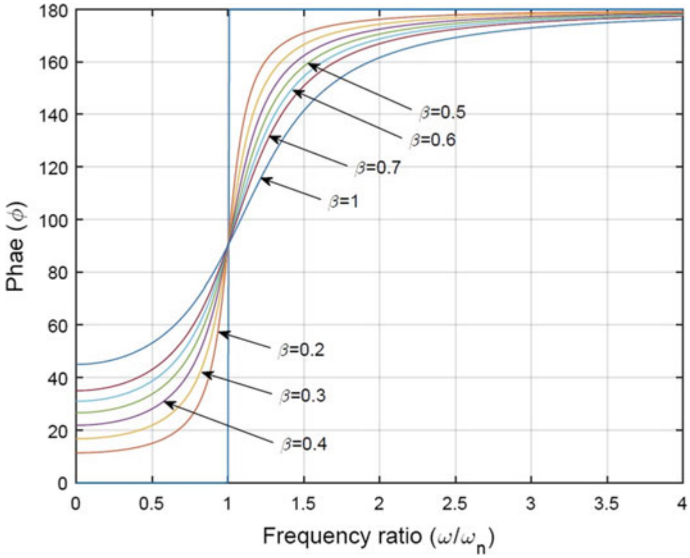


Fig. 4.31 Phase response of a system with hysteretic damping for forced harmonic vibration

$$
\begin{aligned}
0.75\omega_n &= \sqrt{\frac{k}{m+5}} \\
\therefore \quad k &= (m+5)(0.75\omega_n)^2 \\
&= (m+5)(0.75 \times 10)^2 \\
&= 56.25m + 281.25 \qquad (\mathbf{b})
\end{aligned}
$$

Equating Eqs. (**a**) and (**b**),

$$100m = 56.25m + 281.25$$

or,

$$43.75m = 281.25$$

$$\therefore \quad m = \frac{281.25}{43.75} = 6.4286\,\text{kg}$$

Substituting m into Eq. (**a**),

$$k = 100 \times 6.4286 = 642.86\,\text{N/m}$$

Example 4.2

A spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Figure E4.2(a). The natural frequency of this system is found to be 8 rad/s. If this spring is cut so that one part is one-third and the other part two-thirds of the original length and the same mass m is connected to the two parts as shown in Figure E4.2(b), what would be the new natural frequency of the system?

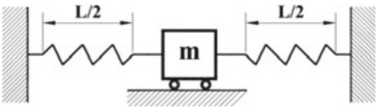


Figure E4.2(a)

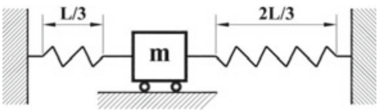


Figure E4.2(b)

Solution

When the spring of stiffness k is divided into two equal parts, its stiffness will be doubled, i.e., $2k$. Then the equivalent stiffness for the system shown in **Figure E4.2(c)** is

$$\left(k_{\text{eq}}\right)_1 = 2k + 2k = 4k$$

Then the natural frequency of the system is given by

$$(\omega_n)_1 = \sqrt{\frac{(k_{\text{eq}})_1}{m}} = \sqrt{\frac{4k}{m}} \qquad (\mathbf{a})$$

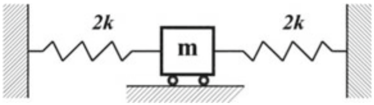


Figure E4.2(c)

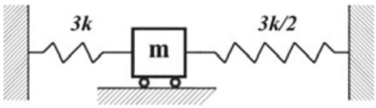


Figure E4.2(d)

If the same spring is cut so that one part is one-third and the other part two-thirds of the original length, the smaller part will have stiffness three times that of the larger part. Let us assume that smaller part has a stiffness of $2k'$ and larger part have a stiffness of k'. These two springs when connected in series should have an equivalent stiffness of k, i.e.,

$$\frac{1}{k} = \frac{1}{2k'} + \frac{1}{k'} = \frac{3}{2k'}$$

$$\therefore \quad k' = \frac{3}{2}k$$

Hence the stiffness of the part of the spring with length $L/3$ is $3k$ and that of the part of the spring with length $2L/3$ is $3k/2$. When these springs are connected to opposite sides of the mass as shown in **Figure E4.2(d)**, then the equivalent stiffness of the system is given by

$$(k_{\text{eq}})_2 = 3k + \frac{3}{2}k = \frac{9}{2}k$$

Then the natural frequency of the system is given by

$$(\omega_n)_2 = \sqrt{\frac{(k_{\text{eq}})_2}{m}} = \sqrt{\frac{9k}{2m}} \qquad (\mathbf{b})$$

Dividing Eq. (**b**) by Eq. (**a**), we get

$$\frac{(\omega_n)_2}{(\omega_n)_1} = \sqrt{\frac{9k}{2m} \times \frac{m}{4k}} = \sqrt{\frac{9}{8}} = 1.0607$$

$$\therefore \quad (\omega_n)_2 = 1.0607 \times (\omega_n)_1 = 1.0607 \times 8 = 8.4853 \text{ rad/s}$$

Example 4.3

Determine the natural frequency of the system shown in Figure E4.3. Assume that the weight of the bar is negligible.

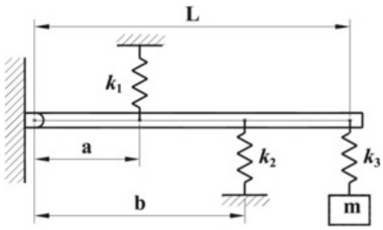


Figure E4.3

Solution

If the angular displacement of the bar (θ) is taken as a generalized coordinate, then equivalent stiffness (k_{12}) at right end of the bar due to springs with stiffness values k_1 and k_2 can be determined as

$$\frac{1}{2}k_{12}(L\theta)^2 = \frac{1}{2}k_1(a\theta)^2 + \frac{1}{2}k_2(b\theta)^2$$

$$\therefore \quad k_{12} = \frac{k_1a^2 + k_2b^2}{L^2}$$

This equivalent spring k_{12} acts in series with the spring with stiffness k_3. Therefore equivalent stiffness of the system can be determined as

$$k_{eq} = \frac{k_{12} \times k_3}{k_{12} + k_3} = \frac{k_3\left(k_1a^2 + k_2b^2\right)}{k_1a^2 + k_2b^2 + k_3L^2}$$

Then the natural frequency of the system can be determined as,

$$\omega_n = \sqrt{\frac{k_{\mathrm{eq}}}{m}} = \sqrt{\frac{1}{m}\left\{\frac{k_3\left(k_1a^2 + k_2b^2\right)}{k_1a^2 + k_2b^2 + k_3L^2}\right\}}$$

Example 4.4

Determine the natural frequency of the system shown in Figure E4.4. Mass of the bar is *M* and mass of each spring is *m*.

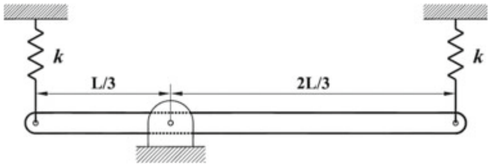


Figure E4.4

Solution

Equivalent model for the given system by considering the inertia effect of the springs is shown in **Figure E4.4(a)**.

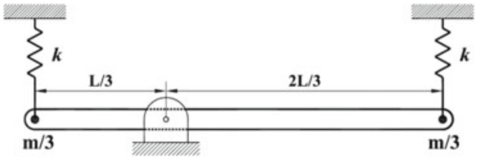


Figure E4.4(a)

If the angular displacement of the bar (θ) is taken as a generalized coordinate, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}I_0\dot{\theta}^2 + \frac{1}{2}\frac{m}{3}\left(\frac{L}{3}\dot{\theta}\right)^2 + \frac{1}{2}\frac{m}{3}\left(\frac{2L}{3}\dot{\theta}\right)^2 = \frac{1}{2}\left(\frac{1}{9}ML^2 + \frac{5}{27}mL^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}k\left(\frac{L}{3}\theta\right)^2 + \frac{1}{2}k\left(\frac{2L}{3}\theta\right)^2 = \frac{1}{2}\left(\frac{5}{9}kL^2\right)\theta^2$$

Then equivalent inertia and equivalent stiffness of the system can be determined as

$$I_{\text{eq}} = \frac{1}{9}ML^2 + \frac{5}{27}mL^2 \quad \text{and} \quad k_{\text{eq}} = \frac{5}{9}kL^2$$

Then the natural frequency of the system can be determined as,

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{I_{\text{eq}}}} = \sqrt{\frac{15k}{3M + 5m}}$$

Example 4.5

Determine the natural frequency of the system shown in Figure E4.5.

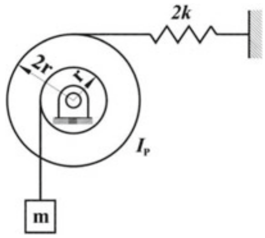


Figure E4.5

Solution

If the vertical displacement of the block with mass $m(x)$ is taken as a generalized coordinate, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}^2$$
$$V = \frac{1}{2}(2k)(2x)^2 = \frac{1}{2}(8k)x^2$$

Then equivalent mass and equivalent stiffness of the system can be determined as

$$m_{\text{eq}} = m + \frac{I_P}{4r^2} \quad \text{and} \quad k_{\text{eq}} = 8k$$

Then the natural frequency of the system can be determined as,

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{8k}{m + \frac{I_P}{4r^2}}} = \sqrt{\frac{32kr^2}{4mr^2 + I_P}}$$

Example 4.6

Determine the natural frequency of the system shown in Figure E4.6. The lower spring is attached to the bar midway between the points of attachment of the upper springs. The weight of the bar AB is negligible.

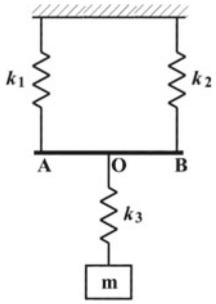


Figure E4.6

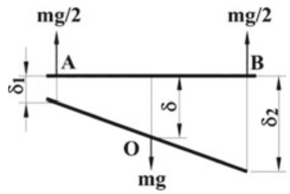


Figure E4.6(a)

Solution

When mass m is suspended on the end of the spring k_3, forces acting on each spring k_1 and k_2 will be $mg/2$, as shown in **Figure E4.6(a)**.

Then the deflections of these two springs are

$$\delta_1 = \frac{mg}{2k_1} \quad \text{and} \quad \delta_2 = \frac{mg}{2k_2}$$

Then the deflection at the middle of the bar is

$$\delta = \frac{1}{2}(\delta_1 + \delta_2) = \frac{mg}{4}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$$

Hence the total deflection (static deflection) of the mass m is given by

$$\begin{aligned}\Delta &= \frac{mg}{4}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) + \frac{mg}{k_3} \\ &= \frac{mg}{4}\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{4}{k_3}\right) \\ &= \frac{mg}{4}\left(\frac{k_2k_3 + k_3k_1 + 4k_1k_2}{k_1k_2k_3}\right)\end{aligned}$$

Then the natural frequency of the system can be determined as,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta}} == \sqrt{\frac{1}{m}\left\{\frac{4k_1k_2k_3}{k_2k_3 + k_3k_1 + 4k_1k_2}\right\}}$$

Example 4.7

Determine the natural frequency of the system shown in Figure E4.7. Neglect the inertia effects of the pulleys.

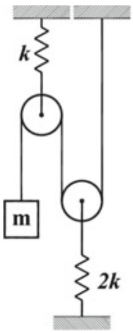


Figure E4.7

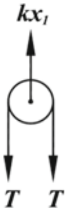


Figure E4.7(a) Free-body diagram of upper pulley

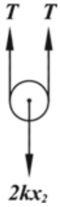


Figure E4.7(b) Free-body diagram of lower pulley

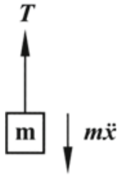


Figure E4.2(c) Free-body diagram of block

Solution

If the displacement of the upper pulley is x_1 when the lower pulley is fixed, then the displacement of the block due to this displacement is $2x_1$. Similarly, if the displacement of the lower pulley is x_2 when the upper pulley is fixed, then the displacement of the block due to this displacement is $2x_2$. When there are simultaneous displacements of both the pulleys by x_1 and x_1, then the total displacement of the block is given by

$$x = 2(x_1 + x_2) \qquad (\mathbf{a})$$

If T be the tension in the cord, then referring to the free-body diagram of the upper pulley [**Figure E4.7(a)**], we can apply the dynamic equilibrium equation as

$$kx_1 = 2T$$

$$\therefore \quad x_1 = \frac{2T}{k} \qquad (\mathbf{b})$$

Similarly, referring to the free-body diagram of the lower pulley [**Figure E4.7(b)**], we can apply the dynamic equilibrium equation as

$$2kx_2 = 2T$$

$$\therefore \quad x_2 = \frac{T}{k} \qquad (\mathbf{c})$$

Substituting x_1 and x_2 from Eqs. (**b**) and (**c**) into Eq. (**a**), we get

$$x = 2\left(\frac{2T}{k} + \frac{T}{k}\right) = \frac{6T}{k}$$

$$\therefore \quad T = \frac{kx}{6} \qquad (\mathbf{d})$$

Again, referring to the free-body diagram of the block [**Figure E4.7(c)**], we can apply the dynamic equilibrium equation as

$$-T = m\ddot{x}$$

$$\therefore \quad m\ddot{x} + T = 0 \qquad (\mathbf{e})$$

Substituting T from Eq. (**c**) into Eq. (**e**), we get

$$m\ddot{x} + \frac{kx}{6} = 0 \qquad (\mathbf{e})$$

Now comparing Eq. (**f**) with standard equation of motion of an un-damped single degree of freedom system, natural frequency of the system can be expressed as

$$\omega_n = \sqrt{\frac{k}{6m}}$$

Example 4.8

A cylinder of mass *m* and radius *r* rolls without slipping on a curved surface of radius *R* as shown in Figure E4.8. Determine the natural frequency of oscillation of the cylinder.

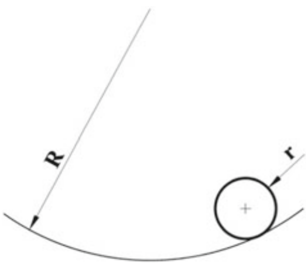


Figure E4.8

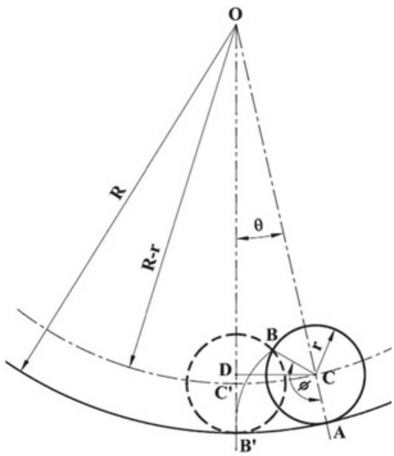


Figure E4.8(a)

Solution

Method 1 (Lagrange Equation)

When the cylinder of radius r rolls on the curved surface of radius R as shown in **Figure E4.8(a)**, length of the path BB' traced by the point B on the surface of the cylinder will be equal to the arc length AB', i.e.,

$$\widehat{BB'} = \widehat{AB'}$$

or,

$$r\phi = R\theta$$
$$\therefore \quad \phi = \frac{R}{r}\theta$$

Kinetic energy of the rolling cylinder is then given by

$$T = T_{\text{translational}} + T_{\text{rotational}}$$

or,

$$T = \frac{1}{2}m\left\{(R-r)\dot{\theta}\right\}^2 + \frac{1}{2}I\left(\dot{\phi} - \dot{\theta}\right)^2$$

or,

$$T = \frac{1}{2}m\left\{(R-r)\dot{\theta}\right\}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{R}{r}\dot{\theta} - \dot{\theta}\right)^2$$
$$\therefore \quad T = \frac{1}{2}\left[\frac{3}{2}m(R-r)^2\right]\dot{\theta}^2$$

Similarly, potential energy of the rolling cylinder with reference to equilibrium position (axis OB') is then given by

$$V = mg\left(C'D\right)$$

or,

$$V = mg\left(OC' - OD\right)$$

or,

$$V = mg[(R-r) - (R-r)\cos\theta]$$
$$\therefore \quad V = mg(R-r)(1-\cos\theta)$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}m\left\{(R-r)\dot{\theta}\right\}^2 - mg(R-r)(1-\cos\theta)$$

Now, using Lagrange' equation,

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

or,

$$-mg(R-r)\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{3}{2}m(R-r)^2\dot{\theta}\right] = 0$$

or,

$$mg(R-r)\sin\theta + \frac{3}{2}m(R-r)^2\ddot{\theta} = 0$$

$$\therefore \quad \frac{3}{2}(R-r)^2\ddot{\theta} + g(R-r)\sin\theta = 0$$

Method 2 (Newton's second law of motion)

Referring to the free-body diagram of the rolling cylinder shown in **Figure E4.8(b)** and applying Newton's second law of motion for transverse direction

$$\sum F_\theta = ma_\theta$$

or,

$$-F - mg\sin\theta = ma_\theta$$

or,

$$-F - mg\sin\theta = m(R-r)\ddot{\theta}$$

$$\therefore \quad F = -mg\sin\theta - m(R-r)\ddot{\theta} \qquad (\mathbf{a})$$

Similarly, applying Newton's second law of motion for rotational effect by taking moment about mass center of the rolling cylinder,

$$\sum M_c = I\left(\ddot{\phi} - \ddot{\theta}\right)$$

$$\therefore \quad Fr = \frac{1}{2}mr^2\left(\ddot{\phi} - \ddot{\theta}\right) \qquad (\mathbf{b})$$

Substituting F from Eq. (**a**) into Eq. (**b**),

$$\left[-mg\sin\theta - m(R-r)\ddot{\theta}\right]r = \frac{1}{2}mr^2\left(\ddot{\phi}-\ddot{\theta}\right)$$

$$\therefore \quad \frac{3}{2}(R-r)^2\ddot{\theta} + g(R-r)\sin\theta = 0$$

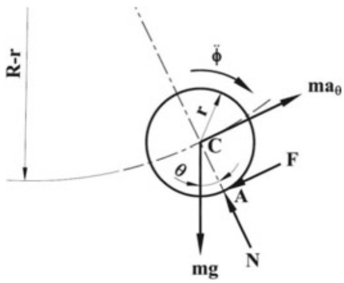


Figure E4.8(b)

For small amplitude oscillation, $\sin\theta \approx \theta$; equation of motion reduces to

$$\frac{3}{2}(R-r)^2\ddot{\theta} + g(R-r)\theta = 0$$

Now comparing the equation of motion with standard equation of motion of an un-damped single degree of freedom system, natural frequency of the system can be expressed as

$$\omega_n = \sqrt{\frac{2g}{3(R-r)}}$$

Example 4.9

A rigid rotor of mass 400 kg is attached at mid-length of a steel shaft of length 1.6 m and diameter of 200 mm supported by two bearings at its ends. Determine the natural frequency for the transverse vibration of the shaft assuming simply supported end conditions. Neglect the mass of the shaft. Take $E = 210$ GPa for steel.

Solution

Given: Mass of the rotor, $m = 400$ kg.
Length of the shaft, $L = 1.6$ m.
Diameter of the shaft, $D = 200$ mm.
Second moment of area of the section of the beam is then given by

$$I = \frac{\pi}{32}D^4 = \frac{\pi}{32} \times (0.2)^4 = 157.0796 \times 10^{-6}\,\text{m}^4$$

Then equivalent stiffness of the shaft with simply supported end conditions with concentrated load at mid-span is given by

$$k = \frac{48EI}{L^3} = \frac{48 \times 210 \times 10^9 \times 157.0796 \times 10^{-6}}{(1.6)^3} = 386.56\,\text{MN/m}$$

Natural frequency for transverse vibration of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{386.56 \times 10^6}{400}} = 983.06\,\text{rad/s}$$

Example 4.10

Determine the natural frequency of a system consisting of a beam and spring and mass assembly shown in Figure E4.10 by modeling it as a single degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. The beam material has a modulus of elasticity of *E* and moment of inertia of section of *I*.

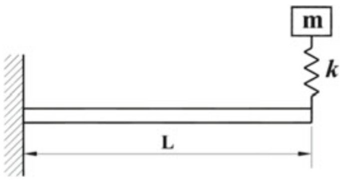


Figure E4.10

Solution

Equivalent stiffness of a cantilever beam when the concentrated mass is located at its free end is given by

$$k_b = \frac{3EI}{L^3}$$

The equivalent stiffness of the beam *kb* acts in series with the spring with the stiffness *k*. The equivalent stiffness of the system is given by

$$k_{eq} = \frac{k_b \times k}{k_b + k} = \frac{\frac{3EI}{L^3} \times k}{\frac{3EI}{L^3} + k} = \frac{3EIk}{3EI + kL^3}$$

Then the natural frequency of the system can be determined as,

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{3EIk}{m(3EI + kL^3)}}$$

Example 4.11

A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. Derive the expression for the response $x(t)$ of the system when

(a) **It is displaced by 0.1 m from equilibrium position and released.**
(b) **It is subjected to initial velocity of 1 m/s.**
(c) **It is displaced by 0.08 m from the equilibrium position and released with a velocity of 0.8 m/s.**

Also compare the responses with plots.

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.
Stiffness of the system, $k = 1$ kN/m.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\,\text{rad/s}$$

Then response of a SDOF system with initial condition is given by

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t)$$

(a) When the system is subjected to an initial condition of $x_0 = 0.1$ and $v_0 = 0$, its response is given by

$$x_a(t) = 0.1\cos(10t)$$

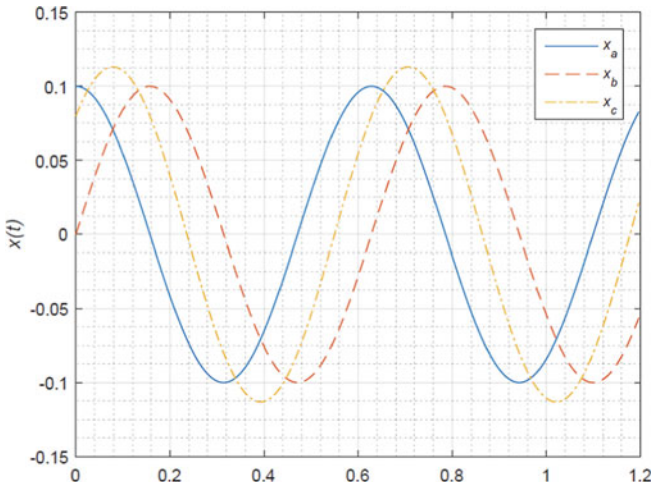
(b) When the system is subjected to an initial condition of $x_0 = 0$ and $v_0 = 1$, its response is given by

$$x_b(t) = \frac{1}{10}\sin(10t) = 0.1\sin(10t)$$

(c) When the system is subjected to an initial condition of $x_0 = 0.08$ and $v_0 = 0.8$, its response is given by

$$\begin{aligned} x_b(t) &= \frac{0.8}{10}\sin(10t) + 0.08\cos(10t) \\ &= 0.08\sin(10t) + 0.08\cos(10t) \end{aligned}$$

Plots of response for each case are shown in **Figure E4.11**.



Figuree E4.11 Response of a SDOF system subjected to different initial conditions

Example 4.12

A spring-mass-damper system consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. It is displaced by 0.05 m from the equilibrium position and released with a velocity of 0.5 m/s. Derive the expression for the response $x(t)$ of the system when the damping constant is

(a) **50 Ns/m,**
(b) **200 Ns/m and**
(c) **300 Ns/m**.

Also compare the responses for each case with plots.

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.
Stiffness of the system, $k = 1\,\text{kN/m}$.
Initial displacement, $x_0 = 0.05\,\text{m}$.
Initial velocity, $v_0 = 0.5\,\text{m/s}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\,\text{rad/s}$$

Critical damping constant of the system is given by

$$c_c = 2\sqrt{km} = 2\sqrt{1000 \times 10} = 200\,\text{N s/m}$$

(a) When $c = 50$ N s/m, damping ratio is given by

$$\xi = \frac{c}{c_c} = \frac{50}{200} = 0.25$$

Hence the system is under-damped and its damped natural frequency is given by

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.25)^2} = 9.6825\,\text{rad/s}$$

Then the response of an under-damped system is given by

$$x_a(t) = e^{-\xi\omega_n t}\left[\frac{v_0 + \xi\omega_n x_0}{\omega_d}\sin(\omega_d t) + x_0\cos(\omega_d t)\right]$$
$$= e^{-0.25\times 10t}\left[\frac{0.5 + 0.25\times 10\times 0.05}{9.6825}\sin(9.6825t) + 0.05\cos(9.6825t)\right]$$
$$\therefore\quad x_a(t) = e^{-2.5t}[0.06454\sin(9.6825t) + 0.05\cos(9.6825t)]$$

(b) When $c = 200$ N s/m, damping ratio is given by

$$\xi = \frac{c}{c_c} = \frac{200}{200} = 1$$

Hence the system is critically damped and its response is given by

$$x_b(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}$$
$$= [0.05 + (0.5 + 10\times 0.05)t]e^{-10t}$$
$$\therefore\quad x_b(t) = (0.05 + t)e^{-10t}$$

(c) When $c = 300$ N s/m, damping ratio is given by

$$\xi = \frac{c}{c_c} = \frac{300}{200} = 1.5$$

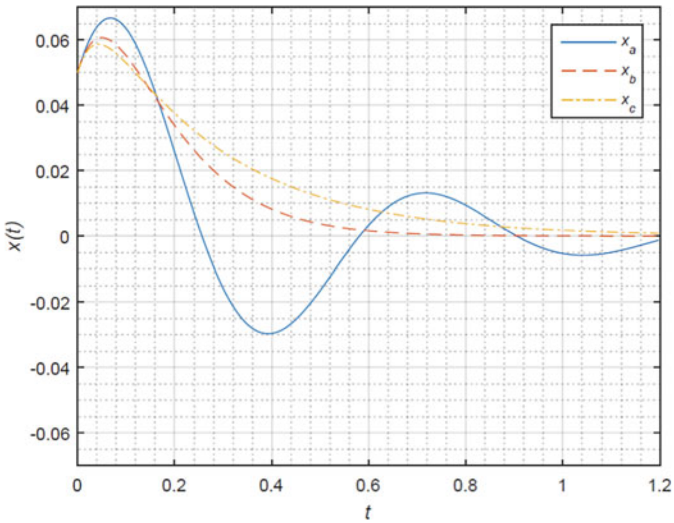
Hence the system is over-damped and its response is given by

$$x_c(t) = \frac{1}{2\left(\sqrt{\xi^2-1}\right)\omega_n}\left[\left\{v_0 + \left(\xi + \sqrt{\xi^2-1}\right)\omega_n x_0\right\}e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}\right.$$
$$\left. - \left\{v_0 + \left(\xi - \sqrt{\xi^2-1}\right)\omega_n x_0\right\}e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}\right]$$
$$= \frac{1}{2\left(\sqrt{(1.5)^2-1}\right)\times 10}$$

$$\left[\left\{0.5 + \left(1.5 + \sqrt{(1.5)^2 - 1}\right) \times 10 \times 0.05\right\} e^{\left(-1.5+\sqrt{(1.5)^2-1}\right)\times 10t} - \left\{0.5 + \left(1.5 - \sqrt{(1.5)^2 - 1}\right) \times 10 \times 0.05\right\} e^{\left(-1.5-\sqrt{(1.5)^2-1}\right)\times 10t}\right]$$

$$\therefore \quad x_c(t) = 0.0809e^{-3.8197t} - 0.0309e^{-26.1803t}$$

Plots of response for each case are shown in **Figure E4.12**.



Figuree E4.12 Response of a damped SDOF system

Example 4.13

Determine the damping constant for the system shown in Figure E4.13 such that it is critically damped. Given $k = 1000\,\mathrm{N/m}$ and $m = 20\,\mathrm{kg}$. Assume that the disk is thin and rolls without slip.

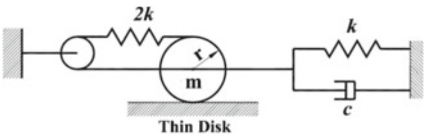


Figure E4.13

Solution

If the disk is displaced toward right by x, then right end of the left spring will be displaced by $2x$ and the left end of the same spring will be displaced by x. Therefore, the relative displacement between two ends of the left spring will be $3x$. Then the compression in the right spring will be x. Then total potential energy of the system can be determined as

$$V = \frac{1}{2}(2k)(3x)^2 + \frac{1}{2}(k)(x)^2 = \frac{1}{2}(19k)x^2$$

Therefore, equivalent stiffness of the system is given by

$$k_{eq} = 19k$$

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{\dot{x}}{r}\right)^2 = \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}^2$$

Therefore, equivalent mass of the system is given by

$$m_{\text{eq}} = \frac{3}{2}m$$

Therefore, critical damping constant of the system can be determined as

$$c = c_c = 2\sqrt{k_{\text{eq}}m_{\text{eq}}} = 2\sqrt{19k \times \frac{3}{2}m}$$

$$= 2\sqrt{19 \times 1000 \times \frac{3}{2} \times 20} = 1509.9668\,\text{N s/m}$$

Example 4.14

Determine the damping constant for the system shown in Figure E4.14 such that it is critically damped. Given $k = 1500\,\text{N/m}$, $m = 5\,\text{kg}$, $a = 0.2\,\text{m}$, $b = 0.4\,\text{m}$ and $L = 0.6\,\text{m}$. Assume that weight of the bar is negligible.

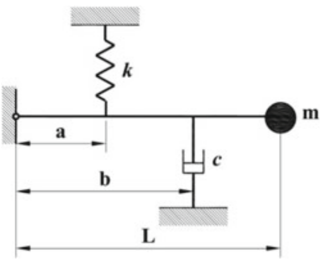


Figure E4.14

Solution

When the bar is rotated in clockwise direction θ, transverse displacement of the mass will be $(L\theta)$, elongation in the spring will be $(a\theta)$, and the velocity at the upper end of the damper is $\left(b\dot{\theta}\right)$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}m\left(L\dot{\theta}\right)^2 = \frac{1}{2}\left(mL^2\right)\dot{\theta}^2$$

Therefore, equivalent inertia of the system is given by

$$I_{\text{eq}} = mL^2$$

Then total potential energy of the system can be determined as

$$V = \frac{1}{2}(k)(a\theta)^2 = \frac{1}{2}\left(ka^2\right)\theta^2$$

Therefore, equivalent stiffness of the system is given by

$$k_{\text{eq}} = ka^2$$

Work done against the damping is given by

$$W_d = \int -c\left(b\dot{\theta}\right)d(b\theta) = \int -\left(cb^2\right)\dot{\theta}\mathrm{d}\theta$$

Therefore, equivalent damping of the system can be determined as

$$c_{\text{eq}} = cb^2$$

Therefore, critical damping constant of the system can be determined as

$$(c_c)_{\text{eq}} = 2\sqrt{k_{\text{eq}}I_{\text{eq}}} = 2\sqrt{ka^2 \times mL^2} = 2La\sqrt{km}$$

Then critical damping constant of the system can be determined as

$$c_c b^2 = 2La\sqrt{km}$$

$$\therefore \quad c_c = 2\frac{La}{b^2}\sqrt{km}$$

Substituting $k = 1500\,\text{N/m}$, $m = 5\,\text{kg}$, $a = 0.2\,\text{m}$, $b = 0.4\,\text{m}$ and $L = 0.6\,\text{m}$, we get

$$c_c = 2 \times \frac{0.6 \times 0.2}{(0.4)^2}\sqrt{1500 \times 5} = 129.904\,\text{N s/m}$$

Example 4.15

A block of mass 2 m shown in Figure E4.15 is initially displaced by 20 mm and released. Given $k = 1000\,\text{N/m}$, $r = 12\,\text{cm}$, $I_p = 1\,\text{kgm}^2$, $c = 200\,\text{Ns/m}$ and $m = 10\,\text{kg}$. How many cycles will be executed before the amplitude of vibration is reduced to 1 mm?

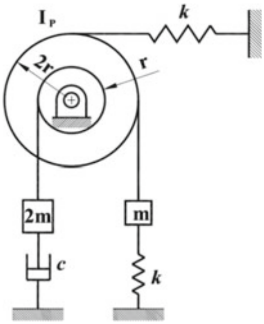


Figure E4.15

Solution

When the block of mass $2m$ is displaced downward by x, rotation of the pulley will be x/r, displacement of the block of mass m will be $2x$, elongation in the lower spring with stiffness k will be $2x$, elongation in the upper spring with stiffness k will be $2x$ and the velocity of upper end of the damper will be $\dot{x}$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m(2\dot{x})^2 = \frac{1}{2}\left(6m + \frac{I_P}{r^2}\right)\dot{x}^2$$

Therefore, equivalent mass of the system can be determined as

$$m_{\text{eq}} = 6m + \frac{I_P}{r^2}$$

Then total potential energy of the system can be determined as

$$V = \frac{1}{2}(k)(2x)^2 + \frac{1}{2}(k)(2x)^2 = \frac{1}{2}(8k)x^2$$

Therefore, equivalent stiffness of the system is given by

$$k_{\text{eq}} = 8k$$

Work done against the damping is given by

$$W_d = \int -c(\dot{x})\mathrm{d}(x) = \int -(c)\dot{x}\mathrm{d}x$$

Therefore, equivalent damping of the system can be determined as

$$c_{\text{eq}} = c$$

Substituting $k = 1000\,\text{N/m}$, $r = 12\,\text{cm}$, $I_p = 1\,\text{kg m}^2$, $c = 200\,\text{N s/m}$ and $m = 10\,\text{kg}$, we get the values of equivalent system parameters as

$$\begin{aligned}
m_{\text{eq}} &= 6 \times 10 + \frac{1}{(0.12)^2} = 129.444\,\text{kg} \\
k_{\text{eq}} &= 8 \times 1000 = 8000\,\text{N/m} \\
c_{\text{eq}} &= 200\,\text{N s/m}
\end{aligned}$$

Critical damping constant of the system is then given by

$$\begin{aligned}
(c_c)_{\text{eq}} &= 2\sqrt{k_{\text{eq}} m_{\text{eq}}} = 2\sqrt{8000 \times 129.444} \\
&= 2035.245\,\text{N s/m}
\end{aligned}$$

Then damping ratio of the system is then given by

$$\xi = \frac{c_{\text{eq}}}{(c_c)_{\text{eq}}} = \frac{200}{2035.245} = 0.09827$$

Logarithmic decrement of the system then can be determined as

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2 \times \pi \times 0.09827}{\sqrt{1-(0.09827)^2}} = 0.62044$$

Then, the number of cycles when the amplitude reduces from 20 to 1 mm can be determined as

$$n = \frac{1}{\delta}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{1}{0.62044}\ln\left(\frac{20}{1}\right) = 4.8284 \approx 5\ \text{cycles}$$

Example 4.16

A vibrating system consists of a 10 kg mass, a spring with stiffness 4 kN/m and a viscous damper with damping constant 100 Ns/m. Determine:

(a) **the damping ratio**,
(b) **the damped natural frequency**,
(c) **the logarithmic decrement and**
(d) **the ratio of two successive amplitudes**.

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.
Stiffness of the system, $k = 4\,\text{kN/m}$.
Damping constant of the system, $c = 100\,\text{N s/m}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\,\text{rad/s}$$

Critical damping constant of the system is given by

$$c_c = 2\sqrt{km} = 2\sqrt{4000 \times 10} = 400\,\text{N s/m}$$

(a) The damping ratio of the system is given by

$$\xi = \frac{c}{c_c} = \frac{100}{400} = 0.25$$

(b) The damped natural frequency of the system is given by

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 20\sqrt{1-(0.25)^2} = 19.3649\,\text{rad/s}$$

(c) The logarithmic decrement of the system is given by

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2 \times \pi \times 0.25}{\sqrt{1-(0.25)^2}} = 1.6233$$

(d) The ratio of successive amplitudes is given by

$$\frac{x_1}{x_2} = \mathrm{e}^{\delta} = \mathrm{e}^{1.6233} = 5.0698$$

Example 4.17

The system shown in Figure E4.17 has a damped natural frequency of 8 Hz, and the amplitude of its free vibration is reduced to 40% in 8 cycles. Given $r = 12$ cm, $I_p = 1$ kgm^2 and $m = 10$ kg. Determine the values of stiffness k and damping constant c.

Solution

When the block of mass m is displaced downward by x, rotation of the pulley will be $x/2r$, compression in the lower spring with stiffness k will be x, and the velocity of left end of the damper will be $\dot{x}/2$.

Total kinetic energy of the system can be determined as

$$T = \frac{1}{2}(m)\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{2r}\right)^2 = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}^2$$

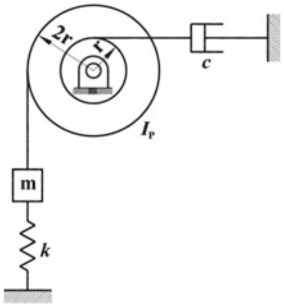


Figure E4.17

Therefore, equivalent mass of the system can be determined as

$$m_{\text{eq}} = m + \frac{I_P}{4r^2}$$

Then total potential energy of the system can be determined as

$$V = \frac{1}{2}(k)(x)^2 = \frac{1}{2}(k)x^2$$

Therefore, equivalent stiffness of the system is given by

$$k_{\text{eq}} = k$$

Work done against the damping is given by

$$W_d = \int -c\left(\frac{\dot{x}}{2}\right)\mathrm{d}\left(\frac{x}{2}\right) = \int -\left(\frac{c}{4}\right)\dot{x}\,\mathrm{d}x$$

Therefore, equivalent damping of the system can be determined as

$$c_{\text{eq}} = \frac{c}{4}$$

Substituting $r = 12\,\text{cm}$, $I_p = 1\,\text{kg m}^2$ and $m = 10\,\text{kg}$, we get the value of equivalent mass of the system as

$$m_{\text{eq}} = 10 + \frac{1}{4 \times (0.12)^2} = 27.3611\,\text{kg}$$

Logarithmic decrement of the system then can be determined as

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{1}{8}\ln\left(\frac{x_1}{x_9}\right) = \frac{1}{8}\ln\left(\frac{1}{0.4}\right) = 0.1145$$

Logarithmic decrement of the system can be expressed in terms of damping ratio as

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

or,

$$\delta^2\left(1-\xi^2\right) = 4\pi^2\xi^2$$

or,

$$\xi^2 = \frac{\delta^2}{4\pi^2+\delta^2}$$

$$\therefore \quad \xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.1145}{\sqrt{4\times\pi^2+(0.1145)^2}} = 0.01823$$

Natural frequency of the system can be determined as

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{2\pi f_d}{\sqrt{1-\xi^2}} = \frac{2\times\pi\times 8}{\sqrt{1-(0.01823)^2}} = 50.2738\,\text{rad/s}$$

Then equivalent stiffness of the system is given by

$$k_{\text{eq}} = \omega_n^2 m_{\text{eq}} = (50.2738)^2 \times 27.3611 = 69154\,\text{N/m}$$

The value of stiffness k is then given by

$$k = k_{\text{eq}} = 69.154\,\text{kN/m}$$

Equivalent critical damping constant of the system is then given by

$$\begin{aligned}(c_c)_{\text{eq}} &= 2\sqrt{k_{\text{eq}} m_{\text{eq}}} = 2\sqrt{69154 \times 27.3611} \\ &= 2751.0959\,\text{N s/m}\end{aligned}$$

Then equivalent damping constant of the system is then given by

$$\begin{aligned}c_{\text{eq}} &= \xi (c_c)_{\text{eq}} = 0.01823 \times 2751.0959 \\ &= 50.1415\,\text{N s/m}\end{aligned}$$

The value of damping constant c is then given by

$$c = 4c_{\text{eq}} = 4 \times 50.1415 = 200.5659\,\text{N s/m}$$

Example 4.18

A block of mass 5 kg is traveling on a frictionless horizontal surface with a velocity of 16 m/s and engages a spring ($k = 35\,\text{kN/m}$) and viscous damper ($c = 200\,\text{Ns/m}$), as shown in Figure E4.18. Determine:

(a) **the maximum displacement of the block after engaging the springs and damper and**
(b) **the time taken to reach the maximum displacement**.

Figure E4.18

Solution

Given: Mass of the system, $m = 5\,\text{kg}$.
Stiffness of the system, $k = 35\,\text{kN/m}$.
Damping constant of the system, $c = 200\,\text{N s/m}$.
Initial displacement, $x_0 = 0$.
Initial velocity, $v_0 = 16\,\text{m/s}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{35 \times 10^3}{5}} = 83.667\,\text{rad/s}$$

Critical damping constant of the system is given by

$$c_c = 2\sqrt{km} = 2\sqrt{35 \times 10^3 \times 5} = 836.66\,\text{N s/m}$$

The damping ratio of the system is given by

$$\xi = \frac{c}{c_c} = \frac{20}{836.66} = 0.2390$$

Hence the system is under-damped and its damped natural frequency is given by

$$\begin{aligned}\omega_d &= \omega_n\sqrt{1-\xi^2} = 836.66\sqrt{1-(0.2390)^2}\\ &= 81.2404\,\text{rad/s}\end{aligned}$$

Then the response of an under-damped system is given by

$$\begin{aligned}x(t) &= \mathrm{e}^{-\xi\omega_n t}\left[\frac{v_0 + \xi\omega_n x_0}{\omega_d}\sin(\omega_d t) + x_0\cos(\omega_d t)\right]\\ &= \mathrm{e}^{-0.2390\times 83.667t}\left[\frac{16}{81.2404}\sin(81.2404t)\right]\\ \therefore \quad x(t) &= \mathrm{e}^{-20t}[0.1969\sin(81.2404t)]\end{aligned}$$

For the peak amplitude,

$$\frac{\mathrm{d}}{\mathrm{d}t}\{x(t)\} = 0$$

or,

$$\mathrm{e}^{-20t}[-3.9389\sin(81.2404t) + 16\cos(81.2404t)] = 0$$

Since $\mathrm{e}^{-20t} \neq 0$,

$$-3.9389\sin(81.2404t) + 16\cos(81.2404t) = 0$$

or,

$$\tan(81.2404t) = \frac{16}{3.9389} = 4.0620$$

or,

$$81.2404t = \tan^{-1}(4.0620) = 1.3294$$

The instant at which peak amplitude occurs can be determined as

$$t_p = \frac{1.3294}{81.2404} = 0.016364\,\mathrm{s}$$

Then the corresponding peak amplitude can be determined as

$$\begin{aligned} x_p &= x(t = t_p) \\ &= \mathrm{e}^{-20\times 0.016364t}[0.1969\sin(81.2404 \times 0.016364)] = 0.1379\,\mathrm{m} \end{aligned}$$

Example 4.19

A machine with a mass of 250 kg is placed on two different isolators and the corresponding free vibration records as shown in Figure E4.19(a) and (b). Determine in each case the type of damping and its stiffness and damping constant.

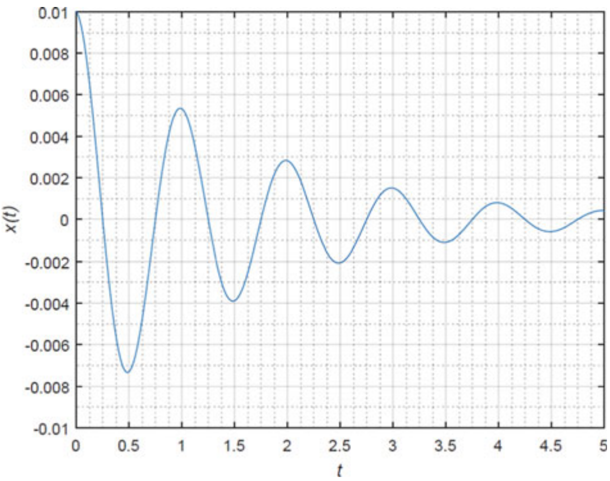


Figure E4.19(a)

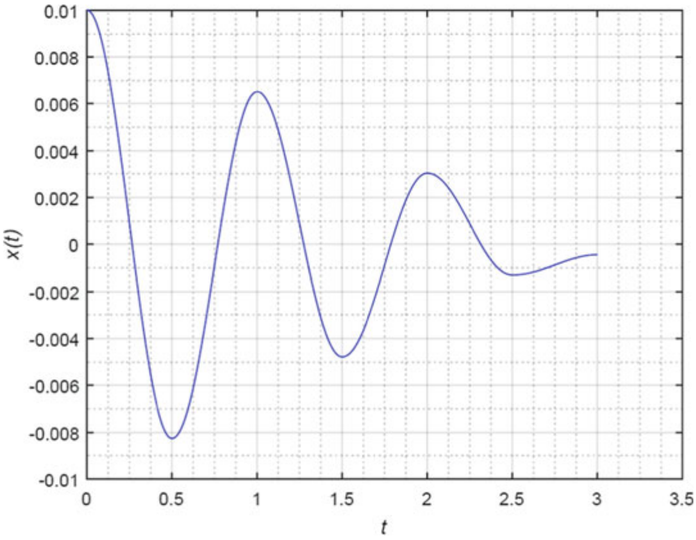


Figure E4.19(b)

Solution

(a) It can be observed from the response plot of **Figure E4.19(a)**, the successive amplitudes of damped oscillation are $x_1 = 0.01$, $x_2 = 0.0053$, $x_3 = 0.0029$, $x_4 = 0.0015$, $x_5 = 0.0009$ and the time period of damped oscillation is $T_d = 1$ s. The decrease in amplitudes occurs exponentially; hence the isolator is viscously damped.

Logarithmic decrement of the system considering first four cycles can be determined as

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{1}{4}\ln\left(\frac{x_1}{x_5}\right)$$
$$= \frac{1}{4}\ln\left(\frac{0.01}{0.0009}\right) = 0.601986$$

Then the damping ratio of the system can be determined as

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.601986}{\sqrt{4 \times \pi^2 + (0.601986)^2}} = 0.0954$$

Damped natural frequency of the system can be determined as

$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{1} = 2\pi = 6.2832\,\text{rad/s}$$

Natural frequency of the system can be determined as

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{6.2832}{\sqrt{1-(0.0954)^2}} = 6.31196\,\text{rad/s}$$

Then stiffness of the system is given by

$$k = \omega_n^2 m = (6.31196)^2 \times 250 = 9960.201\,\text{N/m}$$

Critical damping constant of the system is given by

$$c_c = 2\sqrt{km} = 2\sqrt{9960.201 \times 250} = 3155.9787\,\text{N s/m}$$

Then, the damping constant of the system is given by

$$c = \xi c_c = 0.0954 \times 3155.9787 = 300.993\,\text{N s/m}$$

(b) It can be observed from the response plot of **Figure E4.19(b)**, the successive amplitudes of damped oscillation are $x_1 = 0.01$, $x_2 = 0.0065$, $x_3 = 0.003$ and the time period of damped oscillation is $T = 1$ s. The decrease in amplitudes occurs linearly; hence the isolator has Coulomb damping.

Natural frequency of the system can be determined as

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi = 6.2832\,\text{rad/s}$$

Then stiffness of the system is given by

$$k = \omega_n^2 m = (6.2832)^2 \times 250 = 9869.604\,\text{N/m}$$

For a system with Coulomb damping, successive amplitudes are related by

$$\begin{aligned} x_1 - x_2 &= \frac{4\mu m g}{k} \\ \therefore \quad \mu &= \frac{k(x_1 - x_2)}{4mg} \\ &= \frac{9869.604 \times (0.01 - 0.0065)}{4 \times 250 \times 9.81} = 0.00352 \end{aligned}$$

Example 4.20

A spring-mass system consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. It is excited by a harmonic force 300 sin 20*t*. Derive the expression for the total response $x(t)$ of the system when

(a) **It is displaced by 0.1 m from equilibrium position and released.**
(b) **It is subjected to initial velocity of 1 m/s.**
(c) **It is displaced by 0.08 m from the equilibrium position and released with a velocity of 0.8 m/s.**

Also compare the responses for each case with plots.

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.
Stiffness of the system, $k = 1\,\text{kN/m}$.
Magnitude of external force, $F_0 = 300\,\text{N}$.
Frequency of external force, $\omega = 20\,\text{rad/s}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\,\text{rad/s}$$

Forced harmonic response of an un-damped SDOF system can be expressed as

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0/k}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}\sin\omega t$$

Substituting $k = 1\,\text{kN/m}$, $F_0 = 300\,\text{N}$, $\omega_n = 10\,\text{rad/s}$ and $\omega = 20\,\text{rad/s}$, we get

$$x(t) = A\sin 10t + B\cos 10t + \frac{300/1000}{\left(\frac{20}{10}\right)^2 - 1}\sin 20t$$

$$\therefore \quad x(t) = A\sin 10t + B\cos 10t + 0.1\sin 20t \qquad (\mathbf{a})$$

(a) Substituting the given initial condition for displacement $x(0) = 0.1$ into Eq. (**a**), we get

$$B = 0.1$$

Similarly, substituting the given initial condition for velocity $\dot{x}(0) = 0$ into Eq. (**a**), we get

$$10A + 2 = 0$$

$$\therefore \quad A = -0.2$$

Substituting A and B, into Eq. (**a**), we get the response of the system for an initial condition of $x(0) = 0.1$ and $\dot{x}(0) = 0$ as

$$x_a(t) = -0.2 \sin 10t + 0.1 \cos 10t + 0.1 \sin 20t$$

(b) Substituting the given initial condition for displacement $x(0) = 0$ into Eq. (**a**), we get

$$B = 0$$

Similarly, substituting the given initial condition for velocity $\dot{x}(0) = 1$ into Eq. (**a**), we get

$$10A + 2 = 1$$
$$\therefore \quad A = -0.1$$

Substituting A and B, into Eq. (**a**), we get the response of the system for an initial condition of $x(0) = 0$ and $\dot{x}(0) = 1$ as

$$x_b(t) = -0.1 \sin 10t + 0.1 \sin 20t$$

(c) Substituting the given initial condition for displacement $x(0) = 0.08$ into Eq. (**a**), we get

$$B = 0.08$$

Similarly, substituting the given initial condition for velocity $\dot{x}(0) = 0.8$ into Eq. (**a**), we get

$$10A + 2 = 0.8$$
$$\therefore \quad A = -0.12$$

Substituting A and B, into Eq. (a), we get the response of the system for an initial condition of $x(0) = 0.08$ and $\dot{x}(0) = 0.8$ as

$$x_c(t) = -0.12 \sin 10t + 0.08 \cos 10t + 0.1 \sin 20t$$

Plots of response for each case are shown in **Figure E4.20**.

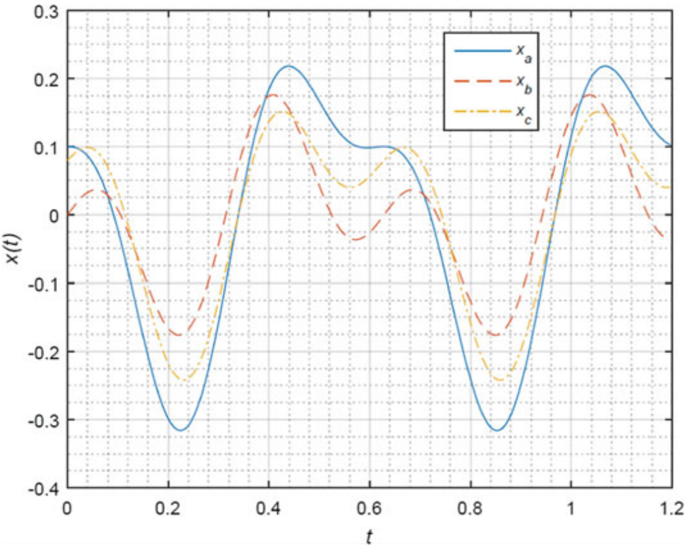


Figure E4.20

Example 4.21

Spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 4 kN/m and damping constant of 40 Ns/m. It is excited by an external harmonic force 200 sin ωt. Derive the expression for the steady state response $x(t)$ of the system when the frequency of external excitation is

(a) **19 rad/s,**
(b) **20 rad/s and**
(c) **21 rad/s.**

Also compare the responses for each case with plots.

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.
Stiffness of the system, $k = 4\,\text{kN/m}$.
Damping constant of the system, $c = 40\,\text{N s/m}$.
Magnitude of external force, $F_0 = 200\,\text{N}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\,\text{rad/s}$$

Critical damping constant of the system is given by

$$c_c = 2\sqrt{km} = 2\sqrt{4000 \times 10} = 400\,\text{N s/m}$$

The damping ratio of the system is given by

$$\xi = \frac{c}{c_c} = \frac{40}{400} = 0.1$$

Then the forced harmonic steady state response of a damped SDOF system can be expressed as

$$x(t) = \frac{F_0/k}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \sin\left[\omega t - \tan^{-1}\left\{\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\}\right]$$

Substituting $k = 4\,\text{kN/m}$, $F_0 = 200\,\text{N}$ and $\omega_n = 20\,\text{rad/s}$, we get

$$x(t) = \frac{200/4000}{\sqrt{\left\{1 - \left(\frac{\omega}{20}\right)^2\right\}^2 + \left(2 \times 0.1 \times \frac{\omega}{20}\right)^2}} \sin\left[\omega t - \tan^{-1}\left\{\frac{2 \times 0.1 \times \frac{\omega}{20}}{1 - \left(\frac{\omega}{20}\right)^2}\right\}\right]$$

$$\therefore \quad x(t) = \frac{0.05}{\sqrt{\left\{1 - (0.05\omega)^2\right\}^2 + (0.01\omega)^2}} \sin\left[\omega t - \tan^{-1}\left\{\frac{0.01\omega}{1 - (0.05\omega)^2}\right\}\right] \quad (\mathbf{a})$$

(a) Substituting $\omega = 19\,\text{rad/s}$ into Eq. (**a**), we get the expression for steady state response as

$$x_a(t) = \frac{0.05}{\sqrt{\left\{1 - (0.05 \times 19)^2\right\}^2 + (0.01 \times 19)^2}}$$
$$\sin\left[19t - \tan^{-1}\left\{\frac{0.01 \times 19}{1 - (0.05 \times 19)^2}\right\}\right]$$
$$\therefore \quad x_a(t) = 0.1081 \sin[19t - 0.3303]$$

(b) Substituting $\omega = 20\,\text{rad/s}$ into Eq. (**a**), we get the expression for steady state response as

$$x_b(t) = \frac{0.05}{\sqrt{\left\{1 - (0.05 \times 20)^2\right\}^2 + (0.01 \times 20)^2}}$$
$$\sin\left[20t - \tan^{-1}\left\{\frac{0.01 \times 20}{1 - (0.05 \times 20)^2}\right\}\right]$$
$$\therefore \quad x_b(t) = 0.25 \sin[20t - 1.5708]$$

(c) Substituting $\omega = 21\,\text{rad/s}$ into Eq. (**a**), we get the expression for steady state response as

$$x_c(t) = \frac{0.05}{\sqrt{\left\{1-(0.05\times 21)^2\right\}^2+(0.01\times 21)^2}}$$
$$\sin\left[21t-\tan^{-1}\left\{\frac{0.01\times 21}{1-(0.05\times 21)^2}\right\}\right]$$
$$\therefore\quad x_c(t) = 0.0812\sin[21t-0.4182]$$

Plots of response for each case are shown in **Figure E4.21**.

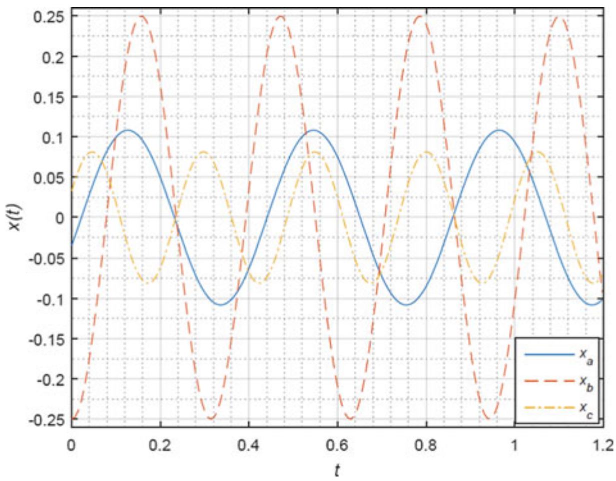


Figure E4.21

Example 4.22

During a vibration test under harmonic excitation, it was noted that the amplitude of vibration at resonance was exactly 1.5 times the amplitude at an excitation frequency 25 % greater than resonance. Determine the damping ratio of the system.

Solution

Amplitude of steady state vibration due to harmonic excitation can be expressed as

$$X = \frac{F_0/k}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Amplitude of steady state vibration at resonance ($\omega = \omega_n$) can be expressed as

$$X_r = \frac{F_0/k}{2\xi \times 1} = \frac{F_0/k}{2\xi} \qquad (\mathbf{a})$$

Amplitude of steady state vibration at an excitation frequency 25 % greater than resonance ($\omega = 1.25\omega_n$) can be expressed as

$$\begin{aligned} X_1 &= \frac{F_0/k}{\sqrt{\left\{1-(1.25)^2\right\}^2 + (2\xi \times 1.25)^2}} \\ &= \frac{F_0/k}{\sqrt{0.31640625 + 6.25\xi^2}} \qquad (\mathbf{b}) \end{aligned}$$

Dividing Eq. (**a**) by Eq. (**b**), we get

$$\begin{aligned} \frac{X_r}{X_1} &= \frac{F_0/k}{2\xi} \times \frac{\sqrt{0.31640625 + 6.25\xi^2}}{F_0/k} \\ &= \frac{\sqrt{0.31640625 + 6.25\xi^2}}{2\xi} \qquad (\mathbf{c}) \end{aligned}$$

The amplitude of vibration at resonance was exactly 1.5 times the amplitude at an excitation frequency 25% greater than resonance. Therefore, substituting $X_r = 1.5X_1$ into Eq. (**c**), we get

$$1.5 = \frac{\sqrt{0.31640625 + 6.25\xi^2}}{2\xi}$$

or,

$$3\xi = \sqrt{0.31640625 + 6.25\xi^2}$$

or,

$$9\xi^2 = 0.31640625 + 6.25\xi^2$$

or,

$$2.75\xi^2 = 0.31640625$$

or,

$$\begin{aligned} \xi^2 &= 0.1150568 \\ \therefore \quad \xi &= 0.3392 \end{aligned}$$

Example 4.23

A spring-mass-damper system consists of a machine of mass of 100 kg, a spring with a stiffness of 300 kN/m and damping constant of 1 kNs/m. It is excited by an external harmonic force of magnitude 450 N. For what excitation frequencies will the steady state amplitude of the machine be less than 1.8 mm?

Solution

Given: Mass of the system, $m = 500\,\text{kg}$.
Stiffness of the system, $m_u e = 0.6\,\text{kg m}$.
Damping constant of the system, $c = 1000\,\text{N s/m}$.
Magnitude of external force, $F_0 = 450\,\text{N}$.
Amplitude of steady state vibration, $X < 1.8\,\text{mm}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300 \times 10^3}{100}} = 54.7723\,\text{rad/s}$$

The damping ratio of the system is given by

$$\xi = \frac{c}{2\sqrt{km}} = \frac{1000}{2\sqrt{300 \times 10^3 \times 100}} = 0.09129$$

Amplitude of steady state vibration due to harmonic excitation can be expressed as

$$X = \frac{F_0/k}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

or,

$$\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2} = \frac{F_0}{kX}$$

or,

$$\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2 = \left(\frac{F_0}{kX}\right)^2$$

or,

$$1 - (2 - 4\xi^2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4 = \left(\frac{F_0}{kX}\right)^2$$

or,

$$1 - \left\{2 - 4 \times (0.09129)^2\right\}\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4$$
$$= \left(\frac{450}{300 \times 10^3 \times 1.8 \times 10^{-3}}\right)^2$$

or,

$$1 - 1.9667\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4 = 0.6944$$

or,

$$\left(\frac{\omega}{\omega_n}\right)^4 - 1.9667\left(\frac{\omega}{\omega_n}\right)^2 + 0.3056 = 0$$

or,

$$\frac{\omega_1}{\omega_n} = 0.4124, \quad \frac{\omega_2}{\omega_n} = 1.3404$$
$$\therefore \quad \omega_1 = 0.4124\omega_n = 0.4124 \times 54.7723$$
$$= 22.5882\,\text{rad/s}$$
$$\omega_2 = 1.3404\omega_n = 1.3404 \times 54.7723$$
$$= 73.4151\,\text{rad/s}$$

Hence, the steady state amplitude of the machine be less than 1.8 mm when the excitation frequency is less than 22.5882 rad/s or higher than 73.4151 rad/s. [Refer **Figure E4.23**.]

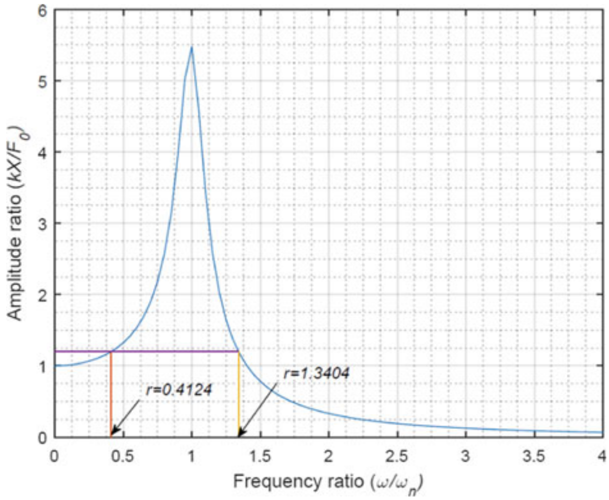


Figure E4.23

Example 4.24

A block of mass 2 m shown in Figure E4.24 is excited by a harmonic force $F(t) = 200 \sin 20t$ N. Given $k = 800$ N/m, $r = 10$ cm, $I_P = 1$ kgm^2, $m = 8$ kg and $m_d = 6$ kg. Determine the amplitude of steady state vibration of the block.

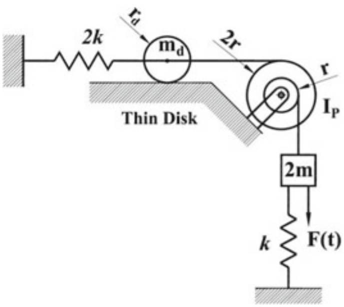


Figure E4.24

Solution

When the block is displaced downward by x, clockwise rotation of the pulley will be x/r and linear displacement of the center of the disk is $2x$ and compression in the spring with stiffness k will be x and that in the spring with stiffness $2k$ will be $2x$.

Total kinetic energy of the system can be determined as

$$\begin{aligned}T &= \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m_d\dot{x}^2 + \frac{1}{2}I_d\left(\frac{\dot{x}}{r_d}\right)^2 \\ &= \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I_P\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}m_d\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}m_d r_d^2\right)\left(\frac{\dot{x}}{r_d}\right)^2 \\ &= \frac{1}{2}\left(2m + \frac{3}{2}m_d + \frac{I_P}{r^2}\right)\dot{x}^2\end{aligned}$$

Therefore, equivalent mass of the system can be determined as

$$m_{\text{eq}} = 2m + \frac{3}{2}m_d + \frac{I_P}{r^2}$$

Similarly, total potential energy of the system can be determined as

$$V = \frac{1}{2}kx^2 + \frac{1}{2}(2k)(2x)^2 = \frac{1}{2}(9k)x^2$$

Therefore, equivalent stiffness of the system can be determined as

$$k_{\text{eq}} = 9k$$

Substituting $k = 800\,\text{N/m}$, $r = 10\,\text{cm}$, $I_P = 1\,\text{kg m}^2$, $m = 8\,\text{kg}$ and $m_d = 6\,\text{kg}$, we get the values of equivalent system parameters as

$$\begin{aligned}m_{\text{eq}} &= 2 \times 8 + \frac{3}{2} \times 6 + \frac{1}{(0.1)^2} = 125\,\text{kg} \\ k_{\text{eq}} &= 9 \times 800 = 7200\,\text{N/m}\end{aligned}$$

Then, the natural frequency of the system is given as

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{7200}{125}} = 7.5895\,\text{rad/s}$$

Then, amplitude of steady state vibration of the system at an excitation frequency $\omega = 20\,\text{rad/s}$ can be determined as

$$\begin{aligned}X &= \frac{F_0/k_{\text{eq}}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{200}{7200} \times \frac{1}{\left(\frac{20}{7.5895}\right)^2 - 1} \\ &= 0.004672\,\text{m} = 4.672\,\text{mm}\end{aligned}$$

Example 4.25

A 500 kg machine with a rotating unbalance of 0.5 kgm is placed on an elastic foundation of stiffness 1 MN/m, for what speed will the steady state amplitude of the tumbler be less than 2 mm? Assume that the damping effect is negligible.

Solution

Given: Mass of the system, $m = 500\,\text{kg}$.
Stiffness of the system, $k = 1\,\text{MN/m}$.
Magnitude of unbalance, $m_u e = 0.5\,\text{kg m}$
Amplitude of steady state vibration, $X < 2\,\text{mm}$.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1 \times 10^6}{500}} = 44.7214\,\text{rad/s}$$

Amplitude of steady state vibration due to rotating unbalance for an un-damped system can be expressed as

$$X = \frac{\frac{m_u e}{m}\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2}} = \pm \frac{\frac{m_u e}{m}\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Taking positive sign

$$X\left\{1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right\}^2 = \frac{m_u e}{m}\left(\frac{\omega_1}{\omega_n}\right)^2$$

or,

$$2 \times 10^{-3} \times \left\{1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right\}^2 = \frac{0.5}{500}\left(\frac{\omega_1}{\omega_n}\right)^2$$

or,

$$2 - 2\left(\frac{\omega_1}{\omega_n}\right)^2 = \left(\frac{\omega_1}{\omega_n}\right)^2$$

or,

$$3\left(\frac{\omega_1}{\omega_n}\right)^2 = 2$$

or,

$$\left(\frac{\omega_1}{\omega_n}\right)^2 = 0.6667$$

or,

$$\frac{\omega_1}{\omega_n} = 0.8164$$
$$\therefore \quad \omega_1 = 0.8164\omega_n = 0.8164 \times 44.7214$$
$$= 36.5148\,\text{rad/s}$$

Taking negative sign

$$X\left\{1 - \left(\frac{\omega_2}{\omega_n}\right)^2\right\}^2 = -\frac{m_u e}{m}\left(\frac{\omega_2}{\omega_n}\right)^2$$

or,

$$2 \times 10^{-3} \times \left\{1 - \left(\frac{\omega_2}{\omega_n}\right)^2\right\}^2 = -\frac{0.5}{500}\left(\frac{\omega_2}{\omega_n}\right)^2$$

or,

$$2 - 2\left(\frac{\omega_2}{\omega_n}\right)^2 = -\left(\frac{\omega_2}{\omega_n}\right)^2$$

or,

$$\left(\frac{\omega_2}{\omega_n}\right)^2 = 2$$

or,

$$\frac{\omega_2}{\omega_n} = 1.4142$$
$$\therefore \quad \omega_2 = 1.4142\omega_n = 1.4142 \times 44.7214$$
$$= 63.2456\,\text{rad/s}$$

Hence, the steady state amplitude of the machine be less than 2 mm when the excitation frequency is less than 36.5148 rad/s or higher than 63.2456 rad/s. [Refer **Figure E4.25**.]

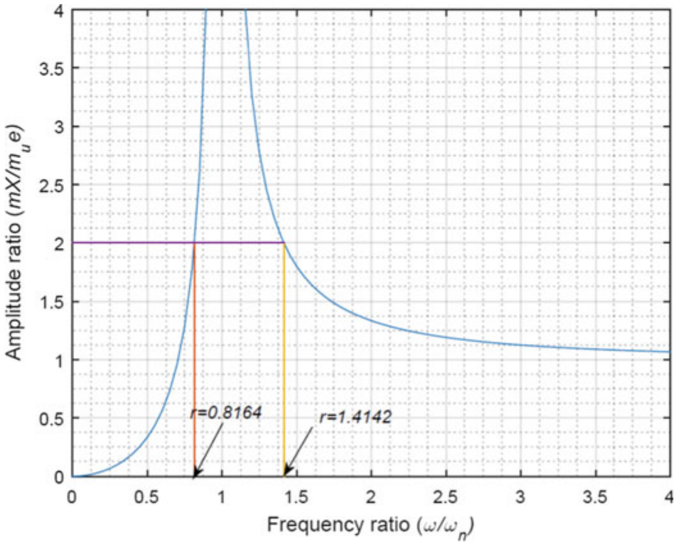


Figure E4.25

Example 4.26

A rotating machine of 500 kg mass has a rotating unbalance of 0.6 kgm and operates between 300 and 500 rpm. Determine the equivalent stiffness of the elastic foundation having damping ratio of 0.1 such that the amplitude of steady state vibration does not exceed 2 mm.

Solution

Given: Mass of the system, $m = 500\,\text{kg}$.

Magnitude of unbalance, $m_u e = 0.6\,\text{kg m}$

Damping ratio of the system, $\xi = 0.1$

Lower operating speed, $\omega_1 = 300\,\text{rpm} = (2\pi \times 300)/60 = 31.4259\,\text{rad/s}$.

Lower operating speed, $\omega_2 = 500\,\text{rpm} = (2\pi \times 500)/60 = 52.3599\,\text{rad/s}$.

Amplitude of steady state vibration, $X < 2\,\text{mm}$.

Amplitude of steady state vibration due to rotating unbalance can be expressed as

$$X = \frac{\frac{m_u e}{m}\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

or,

$$\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2} = \frac{m_u e}{mX}\left(\frac{\omega}{\omega_n}\right)^2$$

or,

$$\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2=\left(\frac{m_u e}{mX}\right)^2\left(\frac{\omega}{\omega_n}\right)^4$$

or,

$$\left[1-\left(\frac{m_u e}{mX}\right)^2\right]\left(\frac{\omega}{\omega_n}\right)^4-\left(2-4\xi^2\right)\left(\frac{\omega}{\omega_n}\right)^2+1=0$$

or,

$$\left[1-\left(\frac{0.6}{500\times 2\times 10^{-3}}\right)^2\right]\left(\frac{\omega}{\omega_n}\right)^4-\left[2-4\times(0.1)^2\right]\left(\frac{\omega}{\omega_n}\right)^2+1=0$$

or,

$$0.64\left(\frac{\omega}{\omega_n}\right)^4-1.96\left(\frac{\omega}{\omega_n}\right)^2+1=0$$

or,

$$\frac{\omega_1}{(\omega_n)_1}=0.8968 \quad \text{and} \quad \frac{\omega_2}{(\omega_n)_2}=1.2467$$

$$\therefore \quad (\omega_n)_1=\frac{\omega_1}{0.8968}=\frac{31.4259}{0.8968}=35.0312\,\text{rad/s}$$

and

$$(\omega_n)_2=\frac{\omega_2}{1.2467}=\frac{52.3599}{1.2467}=41.9989\,\text{rad/s}$$

Then the equivalent stiffness of the foundation can be determined as

$$k_1=m\{(\omega_n)_1\}^2=500\times(35.0312)^2=613.5924\,\text{kN/m}$$

and

$$k_2=m\{(\omega_n)_2\}^2=500\times(41.9989)^2=881.9562\,\text{kN/m}$$

Hence, the steady state amplitude of the machine be less than 2 mm when the stiffness of the foundation is higher than 613.5924 kN/m and less than 881.9562 kN/m.

Example 4.27

A rotating machine of mass 80 kg operates at 750 rpm. Determine the equivalent stiffness and damping ratio of an isolator if when it passes through the resonance the transmissibility ratio should not exceed 2.4 and 85% isolation should be provided at the operating speed.

Solution

Given: Mass of the system, $m = 80\,\text{kg}$.

Operating speed, $\omega = 750\,\text{rpm} = (2\pi \times 750)/60 = 78.5398\,\text{rad/s}$.

Transmissibility ratio at resonance, $\text{TR}_1 = T(\omega = \omega_n) < 2.4$

Transmissibility ratio at operating speed, $\text{TR}_2 = T(\omega) = 0.15$.

Transmissibility ratio for an isolator of a SDOF system is given by

$$\text{TR} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

Substituting $\omega = \omega_n$ for the resonance condition, we get

$$\text{TR}_1 = \sqrt{1 + \frac{1}{4\xi^2}}$$

or,

$$(\text{TR}_1)^2 = 1 + \frac{1}{4\xi^2}$$

or,

$$\frac{1}{4\xi^2} = (\text{TR}_1)^2 - 1$$

$$\therefore \quad \xi = \frac{1}{2\sqrt{(\text{TR}_1)^2 - 1}} = \frac{1}{2\sqrt{(2.4)^2 - 1}} = 0.2292$$

Again, transmissibility ratio for the operating speed is given by

$$\text{TR}_2 = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

or,

$$(\mathrm{TR}_2)^2\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2\right]=\left[1+\left(2\xi\frac{\omega}{\omega_n}\right)^2\right]$$

or,

$$(0.15)^2\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\times 0.2292\times\frac{\omega}{\omega_n}\right)^2\right]$$
$$=\left[1+\left(2\times 0.2292\times\frac{\omega}{\omega_n}\right)^2\right]$$

or,

$$0.0225-0.045\left(\frac{\omega}{\omega_n}\right)^2+0.0225\left(\frac{\omega}{\omega_n}\right)^4$$
$$+0.00473\left(\frac{\omega}{\omega_n}\right)^2-1-0.21\left(\frac{\omega}{\omega_n}\right)^2=0$$

or,

$$0.0225\left(\frac{\omega}{\omega_n}\right)^4-0.25036\left(\frac{\omega}{\omega_n}\right)^2-0.9775=0$$

Solving quadratic equation for $(\omega/\omega_n)^2$, we get

$$\left(\frac{\omega}{\omega_n}\right)^2=14.1889,\ -3.06187$$

Considering only positive root

$$\left(\frac{\omega}{\omega_n}\right)^2=14.1889$$
$$\therefore\quad \omega_n^2=\frac{\omega^2}{14.1889}=\frac{(78.5398)^2}{14.1889}=434.7428$$

Then equivalent stiffness of the isolator can be determine as

$$k=m\omega_n^2=80\times 434.7428=34.779\,\mathrm{kN/m}$$

Example 4.28

A machine of mass 150 kg supported on springs of equivalent stiffness 1000 kN/m has a rotating unbalance force of 6000 N at a speed of 2000 rpm

If the damping ratio is 0.2, determine (a) the amplitude caused by the unbalance and its phase angle, (b) the transmissibility ratio and (c) the actual force transmitted.

Solution

Given: Mass of the system, $m = 150\,\text{kg}$.
Stiffness of the system, $k = 1000\,\text{kN/m}$.
Operating speed, $\omega = 2000\,\text{rpm} = (2\pi \times 2000)/60 = 209.4395\,\text{rad/s}$.
Damping ratio of the system, $\xi = 0.2$.
Rotating unbalance force, $F_0 = 6000\,\text{N}$.
Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \times 10^3}{150}} = 81.6497\,\text{rad/s}$$

(a) Amplitude of steady state vibration due to rotating unbalance can be expressed as

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \\ &= \frac{6000/\left(1000 \times 10^3\right)}{\sqrt{\left\{1 - \left(\frac{209.4395}{81.6497}\right)^2\right\}^2 + \left(2 \times 0.2 \times \frac{209.4395}{81.6497}\right)^2}} \\ &= 0.0010576\,\text{m} = 1.0576\,\text{mm} \end{aligned}$$

Similarly, the phase of the response due to rotating unbalance is given by

$$\begin{aligned} \phi &= \tan^{-1}\left\{\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left\{\frac{2 \times 0.2 \times \frac{209.4395}{81.6497}}{1 - \left(\frac{209.4395}{81.6497}\right)^2}\right\} \\ &= -0.1818\,\text{rad} = 100.42^\circ \end{aligned}$$

(b) Transmissibility ratio for an isolator of a SDOF system is given by

$$\text{TR} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{\sqrt{1 + \left(2 \times 0.2 \times \frac{209.4395}{81.6497}\right)^2}}{\sqrt{\left\{1 - \left(\frac{209.4395}{81.6497}\right)^2\right\}^2 + \left(2 \times 0.2 \times \frac{209.4395}{81.6497}\right)^2}} = 0.2525$$

(c) The actual force transmitted to the foundation can be determined as

$$F_T = F_0 \mathrm{TR} = 6000 \times 0.2525 = 1515.253\,\mathrm{N}$$

Example 4.29

A rotating machine of mass 150 kg has a rotating unbalance of 0.25 kgm. If an isolator of stiffness 600 kN/m and damping ratio of 0.1 is used, determine the range of operating speeds of the machine over which the force transmitted to the foundation will be less than 3000 N.

Solution

Given: Mass of the system, $m = 150\,\mathrm{kg}$.
Stiffness of the system, $k = 600\,\mathrm{kN/m}$.
Magnitude of unbalance, $m_u e = 0.25\,\mathrm{kg\ m}$
Damping ratio of the system, $\xi = 0.1$
Magnitude of the transmitted force, $F_T < 3000\,\mathrm{N}$.
Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600 \times 10^3}{150}} = 63.2456\,\mathrm{rad/s}$$

Magnitude of force produced due to rotating unbalance is given by

$$F_0 = m_u e \omega^2 = 0.25\omega^2$$

Then the transmissibility ratio can be determined as

$$\mathrm{TR} = \frac{F_T}{F_0} = \frac{3000}{0.25\omega^2} = \frac{12{,}000}{\omega^2}$$

Transmissibility ratio for an isolator of a SDOF system is given by

$$\mathrm{TR} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

or,

$$(\mathrm{TR})^2\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2\right]=1+\left(2\xi\frac{\omega}{\omega_n}\right)^2$$

or,

$$\left(\frac{12000}{\omega^2}\right)^2\left[\left\{1-\left(\frac{\omega}{62.2456}\right)^2\right\}^2+\left(2\times 0.1\times\frac{\omega}{63.2456}\right)^2\right]$$
$$=1+\left(2\times 0.1\times\frac{\omega}{63.2456}\right)^2$$

or,

$$144\times 10^6\left(1-0.0005\omega^2+6.25\times 10^{-8}\omega^4+0.00001\omega^2\right)$$
$$-\omega^4-0.00001\omega^6=0$$

or,

$$0.00001\omega^6+8\omega^4-70560\omega^2+144\times 10^6=0$$
$$\therefore\quad \omega=56.4791\,\text{rpm},\,75.5399\,\text{rpm},\,889.4402\,\text{rpm}$$

Hence, the force transmitted to the foundation will be less than 3000 N if the operating speed is less than 56.4791 rpm or lies between 75.5399 and 889.4402 rpm. [Refer **Figure E4.29**.]

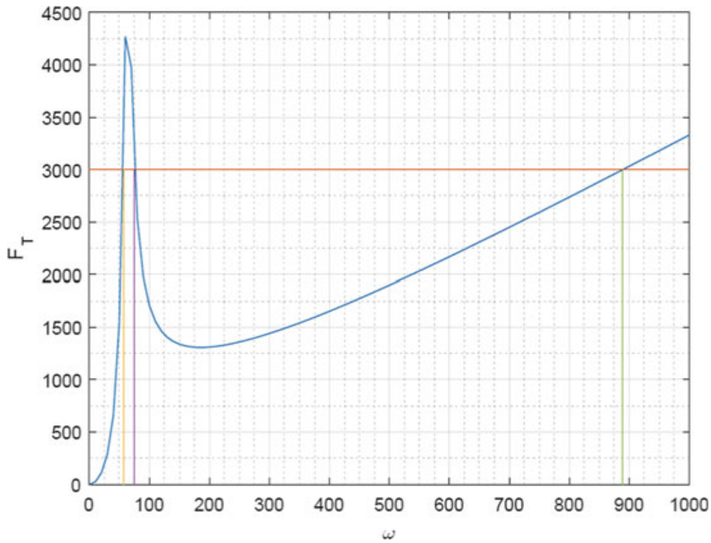


Figure E4.29

Example 4.30

A vehicle can be modeled as a single degree of freedom system with $m = 400$ kg, $k = 320$ kN/m and $c = 3$ kNs/m. The road surface varies sinusoidally with an amplitude of 0.02 m and a wavelength of 4 m. Determine the displacement amplitude of the vehicle when it travels through the road at a speed of 50 km/h.

Solution

Given: Mass of the vehicle, $m = 400$ kg.
Stiffness of the system, $k = 320$ kN/m.
Damping constant, $c = 3$ kN s/m.
Amplitude of the road surface, $Y = 0.02$ m.
Wavelength of the road surface, $\lambda = 4m$.
Vehicle speed, $V_h = 50$ km/h $= 13.8889$ m/s.
Natural frequency of the system is then given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{320 \times 10^3}{400}} = 28.2843 \text{ rad/s}$$

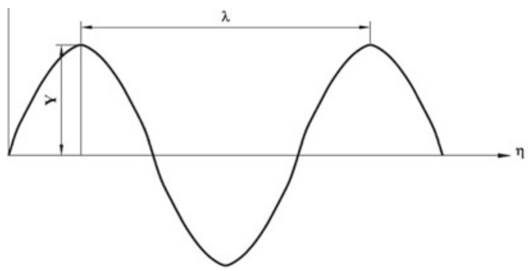


Figure E4.30

The damping ratio of the system is given by

$$\xi = \frac{c}{2\sqrt{km}} = \frac{3 \times 10^3}{2\sqrt{320 \times 10^3 \times 400}} = 0.1326$$

With reference to **Figure E4.30**, contour of road surface can be defined as

$$y(\eta) = Y \sin\left(\frac{2\pi}{\lambda}\eta\right) = 0.02 \sin\left(\frac{2\pi}{4}\eta\right) = 0.02 \sin(0.5\pi\eta)$$

Distance traveled can be related to velocity of the vehicle as

$$\eta = V_h t = 13.8889t$$

Then the base motion imposed upon the vehicle can be expressed as

$$y(\eta) = 0.02\sin(0.5\pi \times 13.8889t) = 0.02\sin(21.8167t)$$

Then the displacement amplitude of the vehicle due to the base motion can be determined as,

$$X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{0.02\sqrt{1+\left(2\times 0.1326\times\frac{21.8167}{28.2843}\right)^2}}{\sqrt{\left\{1-\left(\frac{21.8167}{28.2843}\right)^2\right\}^2+\left(2\times 0.1326\times\frac{21.8167}{28.2843}\right)^2}} = 0.0449\,\text{m}$$

Example 4.31

A vibrometer indicates 2% error in measurement and its natural frequency is 5 Hz. If the lowest frequency that can be measured is 30 Hz, determine the value of damping ratio.

Solution

Given: Natural frequency of the vibrometer, $f_n = 5\,\text{Hz}$.
Lowest measurable frequency, $f = 30\,\text{Hz}$.
Error percentage, $e = 2\,\%$
Output reading Z and the input motion Y are related as

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Error of 2 % for a vibrometer means $Z = 1.02Y$, i.e.,

$$1.02 = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} = \frac{\left(\frac{f}{f_n}\right)^2}{\sqrt{\left\{1-\left(\frac{f}{f_n}\right)^2\right\}^2+\left(2\xi\frac{f}{f_n}\right)^2}}$$

or,

$$(1.02)^2\left[\left\{1-\left(\frac{f}{f_n}\right)^2\right\}^2+\left(2\xi\frac{f}{f_n}\right)^2\right]=\left(\frac{f}{f_n}\right)^4$$

or,

$$1.0404\left[\left\{1-\left(\frac{30}{5}\right)^2\right\}^2+\left(2\times\xi\times\frac{30}{5}\right)^2\right]=\left(\frac{30}{5}\right)^4$$

or,

$$1.0404\left[1225+144\xi^2\right]=1296$$

or,

$$1225+144\xi^2=1245.6747$$

or,

$$144\xi^2=20.6747$$

or,

$$\xi^2=0.1436$$
$$\therefore\quad \xi=0.3789$$

Example 4.32

A 500 kg machine is attached to a spring of stiffness 3 MN/m in parallel with a viscous damper such that the system's damping ratio is 0.2. The block is excited by the periodic excitation shown in Figure E4.32. Determine the approximate maximum displacement of steady state vibration of the machine by considering

(a) **the first odd harmonics only**,
(b) **the first two odd harmonics, and**
(c) **the first three odd harmonics**.

Also compare the responses for each case with plots.

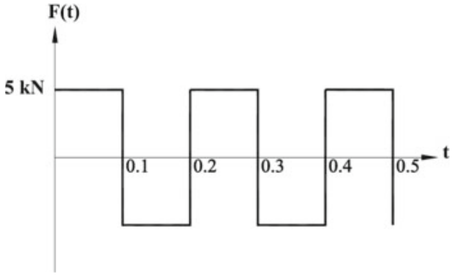


Figure E4.32

Solution

Given: Mass of the system, $m = 500\,\text{kg}$.
Stiffness of the system, $k = 3\,\text{MN/m}$.
Damping ratio of the system, $\xi = 0.2$.
Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3 \times 10^6}{500}} = 77.4957\,\text{rad/s}$$

Given periodic force has a time period $T = 0.2$ s and can be define mathematically as

$$f(t) = \begin{cases} 5000 & 0 < t < 0.1 \\ -5000 & 0.1 < t < 0.2 \end{cases}$$

Coefficients for Fourier series expansion for the given periodic force are given by

$$a_0 = \frac{1}{T}\int_0^T F(t)\mathrm{d}t = \frac{1}{0.2}\left[\int_0^{0.1} 5000\ \mathrm{d}t - \int_{0.1}^{0.2} 5000\ \mathrm{d}t\right] = 0$$

$$a_i = \frac{2}{T}\int_0^T F(t)\cos(\omega_i t)\mathrm{d}t$$

$$= \frac{2}{0.2}\left[\int_0^{0.1} 5000\cos\left(\frac{2\pi i}{0.2}t\right)\mathrm{d}t - \int_{0.1}^{0.2} 5000\cos\left(\frac{2\pi i}{0.2}t\right)\ \mathrm{d}t\right] = 0$$

$$b_i = \frac{2}{T}\int_0^T F(t)\sin(\omega_i t)\mathrm{d}t$$

$$= \frac{2}{0.2}\left[\int_0^{0.1} 5000\sin\left(\frac{2\pi i}{0.2}t\right)\mathrm{d}t - \int_{0.1}^{0.2} 5000\sin\left(\frac{2\pi i}{0.2}t\right)\mathrm{d}t\right]$$
$$= -\frac{50000}{10\pi i}\left[\cos(10\pi i t)|_0^{0.1} - \cos(10\pi i t)|_{0.1}^{0.2}\right]$$
$$= -\frac{5000}{\pi i}[\cos(\pi i) - 1 - \cos(2\pi i) + \cos(\pi i)]$$
$$= \frac{5000}{\pi i}[1 - 2\cos(\pi i) + \cos(2\pi i)]$$
$$= \frac{10000}{\pi i}\left[1 - (-1)^i\right]$$

Substituting $i = 1, 2, 3, \ldots$, we get harmonic coefficients b_i as

$$b_1 = \frac{20000}{\pi} = 6366.1977$$

$$b_2 = 0$$

$$b_3 = \frac{20000}{3\pi} = 2122.0659$$

$$b_4 = 0$$

$$b_5 = \frac{20000}{5\pi} = 1273.2395$$

Hence, the given periodic force can be represented as a Fourier series as

$$\begin{aligned} F(t) = {} & 6366.1977\sin(31.4159t) \\ & + 2122.0659\sin(94.2477t) \\ & + 1273.2395\sin(157.0796t) \end{aligned}$$

(a) Amplitude of the steady state response of the system considering only the first odd harmonics is given by

$$X_1 = \frac{b_1}{k}\frac{1}{\sqrt{\left\{1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega_1}{\omega_n}\right)^2}}$$
$$= \frac{6366.1977}{3\times 10^6}\frac{1}{\sqrt{\left\{1 - \left(\frac{31.4159}{77.4957}\right)^2\right\}^2 + \left(2\times 0.2\times\frac{31.4159}{77.4957}\right)^2}}$$

$$= 0.002493\,\text{m}$$

Similarly phase of the steady state response of the system considering only the first odd harmonics is given by

$$\phi_1 = \tan^{-1}\left\{\frac{2\xi\frac{\omega_1}{\omega_n}}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}\right\} = \tan^{-1}\left\{\frac{2\times 0.2\times\frac{31.4159}{77.4957}}{1-\left(\frac{31.4159}{77.4957}\right)^2}\right\} = 0.19178\,\text{rad}$$

Therefore, the response of the system due to first odd harmonics can be expressed as

$$x_a(t) = 0.002493\sin(31.4159t - 0.1178)$$

(b) Similarly, the amplitude of the steady state response of the system considering only the second odd harmonics is given by

$$\begin{aligned} X_3 &= \frac{b_3}{k}\frac{1}{\sqrt{\left\{1-\left(\frac{\omega_3}{\omega_n}\right)^2\right\}^2+\left(2\xi\frac{\omega_3}{\omega_n}\right)^2}} \\ &= \frac{2122.0659}{3\times 10^6}\frac{1}{\sqrt{\left\{1-\left(\frac{94.2477}{77.4957}\right)^2\right\}^2+\left(2\times 0.2\times\frac{94.2477}{77.4957}\right)^2}} \\ &= 0.001034\,\text{m} \end{aligned}$$

Similarly phase of the steady state response of the system considering only the second odd harmonics is given by

$$\begin{aligned} \phi_3 &= \tan^{-1}\left\{\frac{2\xi\frac{\omega_3}{\omega_n}}{1-\left(\frac{\omega_3}{\omega_n}\right)^2}\right\} = \tan^{-1}\left\{\frac{2\times 0.2\times\frac{94.2477}{77.4957}}{1-\left(\frac{94.2477}{77.4957}\right)^2}\right\} \\ &= -0.7919\,\text{rad} \end{aligned}$$

Therefore, the response of the system due to first two odd harmonics can be expressed as

$$\begin{aligned} x_b(t) &= 0.002493\sin(31.4159t - 0.1178) \\ &\quad + 0.001034\sin(94.2477t + 0.7919) \end{aligned}$$

(c) Similarly, the amplitude of the steady state response of the system considering only the third odd harmonics is given by

$$
\begin{aligned}
X_5 &= \frac{b_5}{k} \frac{1}{\sqrt{\left\{1 - \left(\frac{\omega_5}{\omega_n}\right)^2\right\}^2 + \left(2\xi \frac{\omega_5}{\omega_n}\right)^2}} \\
&= \frac{1273.2395}{3 \times 10^6} \frac{1}{\sqrt{\left\{1 - \left(\frac{157.0796}{77.4957}\right)^2\right\}^2 + \left(2 \times 0.2 \times \frac{157.0796}{77.4957}\right)^2}} \\
&= 0.000131\,\mathrm{m}
\end{aligned}
$$

Similarly phase of the steady state response of the system considering only the second odd harmonics is given by

$$
\begin{aligned}
\phi_5 &= \tan^{-1}\left\{\frac{2\xi \frac{\omega_5}{\omega_n}}{1 - \left(\frac{\omega_5}{\omega_n}\right)^2}\right\} = \tan^{-1}\left\{\frac{2 \times 0.2 \times \frac{157.0796}{77.4957}}{1 - \left(\frac{157.0796}{77.4957}\right)^2}\right\} \\
&= -0.2549\,\mathrm{rad}
\end{aligned}
$$

Therefore, the response of the system due to first three odd harmonics can be expressed as

$$
\begin{aligned}
x_c(t) = {} & 0.002493 \sin(31.4159t - 0.1178) \\
& + 0.001034 \sin(94.2477t + 0.7919) \\
& + 0.000131 \sin(157.0796t + 0.2549)
\end{aligned}
$$

Plots of steady state response for each case are shown in **Figure E4.32(a)**.

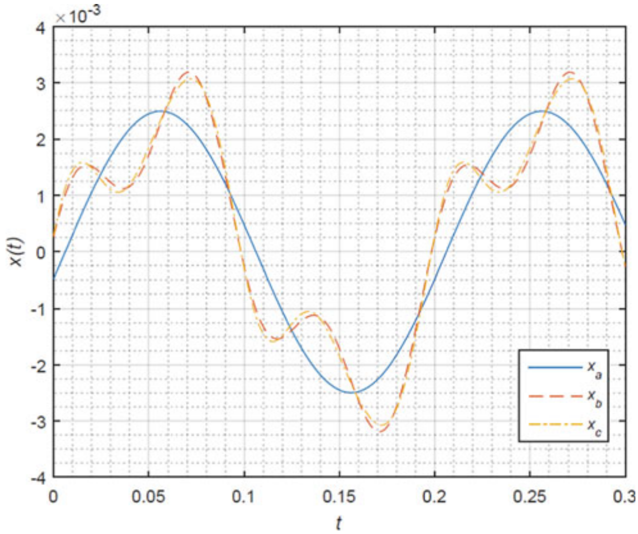


Figure E4.32(a)

Example 4.33

Use the convolution integral to determine the response of an un-damped single degree of freedom system with the natural frequency of ω_n and mass of m when it is subject to

(a) **a delayed impulse shown in Figure E4.33(a),**
(b) **a delayed step function shown in Figure E4.33(b) and**
(c) **a delayed ramp function as shown in Figure E4.33(c), for which $\dot{F} = A$.**

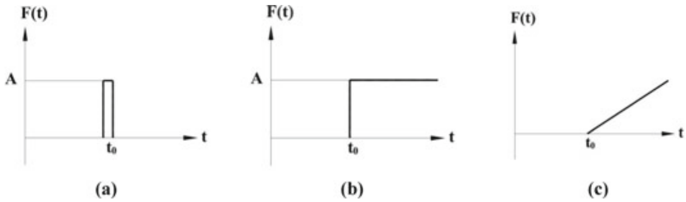


Figure E4.33

Solution

The response of an un-damped single degree of freedom system with the natural frequency of ω_n and mass of m subjected to an unit impulse at $t = 0$ is given by

$$x(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

(a) Given delayed impulse can be defined mathematically as

$$F(t) = A\delta(t - t_0)$$

Then the response of the system due to delayed impulse can be determined by applying convolution integral as

$$\begin{aligned} x(t) &= u(t - t_0)\int_{t_0}^{t} F(\eta)h(t-\eta)\mathrm{d}\eta \\ &= u(t - t_0)\int_{t_0}^{t} A\delta(\eta - t_0)\frac{1}{m\omega_n}\sin\omega_n(t-\eta)\mathrm{d}\eta \\ \therefore \quad x(t) &= u(t - t_0)\left[\frac{A}{m\omega_n}\sin\omega_n(t - t_0)\right] \end{aligned}$$

(b) Given delayed step function can be defined mathematically as

$$F(t) = Au(t - t_0)$$

Then the response of the system due to delayed step function can be determined by applying convolution integral as

$$\begin{aligned} x(t) &= u(t - t_0)\int_{t_0}^{t} F(\eta)h(t-\eta)\mathrm{d}\eta \\ &= u(t - t_0)\int_{t_0}^{t} Au(\eta - t_0)\frac{1}{m\omega_n}\sin\omega_n(t-\eta)\mathrm{d}\eta \\ &= u(t - t_0)\left[\frac{A}{m\omega_n^2}\cos\omega_n(t-\eta)\right]_{t_0}^{t} \\ \therefore \quad x(t) &= u(t - t_0)\left[\frac{A}{m\omega_n^2}\{1 - \cos\omega_n(t - t_0)\}\right] \end{aligned}$$

(c) Given delayed ramp function can be defined mathematically as

$$F(t) = A(t - t_0)u(t - t_0)$$

Then the response of the system due to delayed ramp function can be determined by applying convolution integral as

$$x(t) = u(t - t_0) \int_{t_0}^{t} F(\eta)h(t - \eta)\mathrm{d}\eta$$

$$= u(t - t_0) \int_{t_0}^{t} A(\eta - t_0)\frac{1}{m\omega_n} \sin \omega_n(t - \eta)\mathrm{d}\eta$$

$$= u(t - t_0)\left[\frac{A}{m\omega_n^3}\{\sin \omega_n(t - \eta) + \omega_n(\eta - t_0)\cos \omega_n(t - \eta)\}\right]_{t_0}^{t}$$

$$\therefore \quad x(t) = u(t - t_0)\left[\frac{A}{m\omega_n^3}\{\omega_n(t - t_0) - \sin \omega_n(t - t_0)\}\right]$$

Example 4.34

Determine the response of an un-damped single degree of freedom system when subject to the excitation of Figure E4.34 by using

(a) **the convolution integral, and**
(b) **the Laplace transform method**.

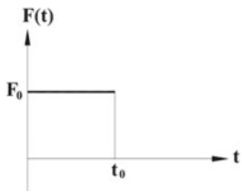


Figure E4.34

Solution

(a) The response of an un-damped single degree of freedom system with the natural frequency of ω_n and mass of m subjected to an unit impulse at $t = 0$ is given by

$$x(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

Given force can be defined mathematically as

$$F(t) = F_0 - F_0 u(t - t_0)$$

Then the response of the system due to the given force can be determined by applying convolution integral as

$$
\begin{aligned}
x(t) &= \int_0^t F(\eta)h(t-\eta)\mathrm{d}\eta \\
&= \int_0^t \frac{F_0}{m\omega_n}\sin\omega_n(t-\eta)\mathrm{d}\eta - u(t-t_0)\int_{t_0}^t \frac{F_0}{m\omega_n}\sin\omega_n(t-\eta)\mathrm{d}\eta \\
&= \left[\frac{F_0}{m\omega_n^2}\cos\omega_n(t-\eta)\right]_0^t - u(t-t_0)\left[\frac{F_0}{m\omega_n^2}\cos\omega_n(t-\eta)\right]_{t_0}^t \\
\therefore\quad x(t) &= \frac{F_0}{m\omega_n^2}[1-\cos\omega_n t] - u(t-t_0)\frac{F_0}{m\omega_n^2}[1-\cos\omega_n(t-t_0)]
\end{aligned}
$$

(b) Equation of motion of an un-damped SDOF system subjected to force shown in **Figure E4.34** can be expressed as

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} - \frac{F_0}{m}u(t-t_0)$$

Taking Laplace transform of both sides,

$$\mathcal{L}\{\ddot{x}\} + \omega_n^2\mathcal{L}\{x\} = \mathcal{L}\left\{\frac{F_0}{m}\right\} - \mathcal{L}\left\{\frac{F_0}{m}u(t-t_0)\right\}$$

or,

$$s^2\mathrm{X}(s) - sx(0) - \dot{x}(0) + \omega_n^2 X(s) = \frac{F_0}{m}\frac{1}{s} - \frac{F_0}{m}\frac{\mathrm{e}^{-t_0 s}}{s}$$

For the particular solution only, $x(0) = 0$ and $\dot{x}(0) = 0$

$$s^2\mathrm{X}(s) + \omega_n^2 X(s) = \frac{F_0}{m}\frac{1}{s} - \frac{F_0}{m}\frac{\mathrm{e}^{-t_0 s}}{s}$$

or,

$$\left(s^2 + \omega_n^2\right)X(s) = \frac{F_0}{m}\frac{1}{s} - \frac{F_0}{m}\frac{\mathrm{e}^{-t_0 s}}{s}$$

or,

$$X(s) = \frac{F_0}{m}\frac{1}{s\left(s^2+\omega_n^2\right)} - \frac{F_0}{m}\frac{\mathrm{e}^{-t_0 s}}{s\left(s^2+\omega_n^2\right)}$$

Taking inverse Laplace transform of both sides,

$$x(t) = \frac{F_0}{m\omega_n}\int_0^t \sin\omega_n t\,\mathrm{d}t - u(t-t_0)\frac{F_0}{m\omega_n}\int_{t_0}^t \sin\omega_n(t-t_0)\mathrm{d}t$$

$$\therefore \quad x(t) = \frac{F_0}{m\omega_n^2}[1-\cos\omega_n t] - u(t-t_0)\frac{F_0}{m\omega_n^2}[1-\cos\omega_n(t-t_0)]$$

Example 4.35

A machine tool with a mass of 10 kg is mounted on an un-damped foundation of stiffness 1000 N/m. During operation, it is subject to a transient force shown in Figure E4.35. Determine the response of the system to this force.

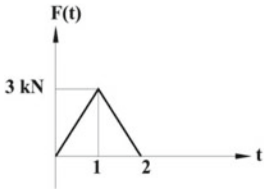


Figure E4.35

Solution

Given: Mass of the system, $m = 10\,\text{kg}$.

Stiffness of the system, $k = 1000\,\text{N/m}$.

Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\,\text{rad/s}$$

The response of an un-damped single degree of freedom system with the natural frequency of ω_n and mass of m subjected to an unit impulse at $t = 0$ is given by

$$x(t) = \frac{1}{m\omega_n}\sin\omega_n t = \frac{1}{10\times 10}\sin 10t = 0.01\sin 10t$$

Given force can be defined mathematically as

$$F(t) = 3000t - 3000tu(t-1) + 3000(2-t)u(t-1) - 3000(2-t)u(t-2)$$

Then the response of the system due to the given force can be determined by applying convolution integral as

$$
\begin{aligned}
x(t) &= \int_0^t F(\eta)h(t-\eta)\mathrm{d}\eta \\
&= \int_0^t 3000\eta \times 0.01 \sin 10(t-\eta)\mathrm{d}\eta - u(t-1)\int_1^t 3000\eta \\
&\quad \times 0.01 \sin 10(t-\eta)\mathrm{d}\eta \\
&\quad + u(t-1)\int_2^t 3000(1-\eta) \times 0.01 \sin 10(t-\eta)\mathrm{d}\eta \\
&\quad - u(t-2)\int_2^t 3000(2-\eta) \times 0.01 \sin 10(t-\eta)\mathrm{d}\eta \\
&= [3t - 0.3\sin 10t] - u(t-1)[3t - 0.3\sin 10(t-1) - 3\cos 10(t-1)] \\
&\quad + u(t-1)[6 - 3t + 0.3\sin 10(t-1) - 3\cos 10(t-1)] \\
&\quad - u(t-2)[6 - 3t + 0.3\sin 10(t-2)] \\
\therefore \quad x(t) &= [3t - 0.3\sin 10t] + u(t-1)[6 - 6t + 0.6\sin 10(t-1)] \\
&\quad - u(t-2)[6 - 3t + 0.3\sin 10(t-2)]
\end{aligned}
$$

Example 4.36

A block with a spring is placed on a rough surface and is given an initial displacement of 12 cm from its equilibrium position. After four cycles of oscillation in 2 s, the final position of the metal block is found to be 2 cm from its equilibrium position. Determine the coefficient of friction between the surface and the block.

Solution

Given: Initial displacement, $x_1 = 12\,\text{cm}$.
Final displacement, $x_5 = 12\,\text{cm}$.
Time period, $T = 2/4 = 0.5\,\text{s}$.
Natural frequency of the system can be determined as

$$\omega_n = \frac{2\pi}{T} = \frac{2 \times \pi}{0.5} = 12.5663\,\text{rad/s}$$

The difference between two successive amplitudes for a system with Coulomb damping is given as

$$x_{n+1} - x_1 = \frac{4\mu m g}{k}$$

Therefore, the difference between four successive amplitudes for a system with Coulomb damping is given as

$$x_5 - x_1 = 4 \times \frac{4\mu m g}{k} = \frac{16\mu g}{\omega_n^2}$$

$$\therefore \quad \mu = \frac{(x_5 - x_1)\omega_n^2}{16g} = \frac{(0.12 - 0.02) \times (12.5663)^2}{16 \times 9.81} = 0.1$$

Example 4.37

A spring-mass system, having a mass of 15 kg and a spring of stiffness of 3 kN/m, vibrates on a rough surface. The coefficient of friction is 0.12. When subjected to a harmonic force of frequency 4 Hz, the mass is found to vibrate with an amplitude of 42 mm. Determine the amplitude of the harmonic force applied to the mass.

Solution

Given: Mass of the system, $m = 15\,\text{kg}$.
Stiffness of the system, $k = 3000\,\text{N/m}$.
Coefficient of friction, $\mu = 0.12$.
Frequency of external excitation, $f = 4\,\text{Hz}$.
Amplitude of steady state vibration, $X = 42 \times 10^{-3}\,\text{m}$.
Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000}{15}} = 14.1421\,\text{rad/s}$$

Frequency of the external excitation in rad/s is given by

$$\omega = 2\pi f = 2 \times \pi \times 4 = 25.1327\,\text{rad/s}$$

Amplitude of steady state vibration of a system with Coulomb damping is given by

$$X = \frac{F_0}{k}\left[\frac{\sqrt{1 - \left(\frac{4\mu m g}{\pi F_0}\right)^2}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

or,

$$\left(\frac{kX}{F_0}\right)^2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 = 1 - \left(\frac{4\mu m g}{\pi F_0}\right)^2$$

or,

$$\left(\frac{3000 \times 4.2 \times 10^{-3}}{F_0}\right)^2 \left[1 - \left(\frac{25.1327}{14.1421}\right)^2\right]^2 = 1 - \left(\frac{4 \times 0.12 \times 15 \times 9.81}{\pi F_0}\right)^2$$

or,

$$\frac{15876 \times 4.6581}{F_0^2} = 1 - \frac{505.4792}{F_0^2}$$

or,

$$\frac{74458.1751}{F_0^2} = 1$$

$$\therefore \quad F_0 = \sqrt{74458.1751} = 272.8703\,\text{N}$$

Example 4.38

A panel made of a composite material can be modeled as a single degree of freedom system with a mass of 5 kg and a stiffness of 10 N/m. The ratio of successive amplitudes is found to be 1.2. Determine the value of the hysteresis damping constant β, the equivalent viscous damping constant c_{eq} and the energy loss per cycle for an amplitude of 12 mm.

Solution

Given: Mass of the system, $m = 5\,\text{kg}$.
Stiffness of the system, $k = 10\,\text{N/m}$.
Ration of successive amplitudes, $X_n/X_{n+1} = 1.2$
Amplitude of steady state vibration, $X = 12 \times 10^{-3}$ m.
The ratio of successive amplitudes for a system with hysteretic damping is given by

$$\frac{X_n}{X_{n+1}} = 1 - \pi\beta$$

or,

$$\frac{1}{1.2} = 1 - \pi\beta$$

$$\therefore \quad \beta = \frac{1.2 - 1}{1.2\pi} = 0.0531$$

Natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{5}} = 1.4142\,\text{rad/s}$$

Then equivalent damping constant for hysteretic damping is given by

$$c_{\text{eq}} = \beta m \omega_n = 0.0531 \times 5 \times 1.4142 = 0.3751\,\text{N s/m}$$

Energy dissipated during a cycle can be determined as

$$\begin{aligned} W_d = \pi c_{\text{eq}} \omega X^2 &= \pi \times 0.3751 \\ &\times 1.4142 \times \left(12 \times 10^{-3}\right)^2 = 24 \times 10^{-3}\,\text{J} \end{aligned}$$

Review Questions

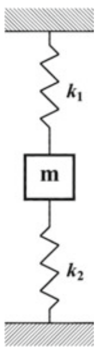
1. Define the spring stiffness and damping constant.
2. How do you connect several springs to increase the overall stiffness?
3. How do you connect several springs to decrease the overall stiffness?
4. What effect does a decrease in mass have on the frequency of a system?
5. What effect does a decrease in the stiffness have on the natural period?
6. How can you find the natural frequency of a system by measuring its static deflection?
7. Derive the response of an un-damped vibration of single degree of freedom system.
8. What is critical damping and what is its importance?
9. Define logarithmic decrement. Derive expression for logarithmic decrement in terms of successive amplitudes.
10. Derive the response of damped vibration of single degree of freedom system when it is (**a**) over-damped, (**b**) critically damped and (**c**) under-damped.
11. Sketch time response for single degree of freedom system when it is (**a**) un-damped, (**b**) over-damped, (**c**) critically damped and (**d**) under-damped.
12. Define the term magnification factor or amplitude ratio. Derive the relation for magnification factor and phase angle in terms of frequency ratio.
13. Derive the expressions for frequency at which peak amplitude occurs and corresponding peak amplitude
14. Show the various forces considered for forced harmonic vibration in vector (phasor) diagram.
15. Show the various forces considered for forced harmonic vibration in vector (phasor) diagram when (a) $\omega/\omega_n \ll 1$, (b) $\omega/\omega_n = 1$ and (c) $\omega/\omega_n \gg 1$.
16. Why damping is considered only in the neighborhood of the resonance in most cases?
17. What is the difference between peak amplitude and resonant amplitude?
18. Derive the relation for amplitude ratio of a viscously damped system under rotating unbalance.

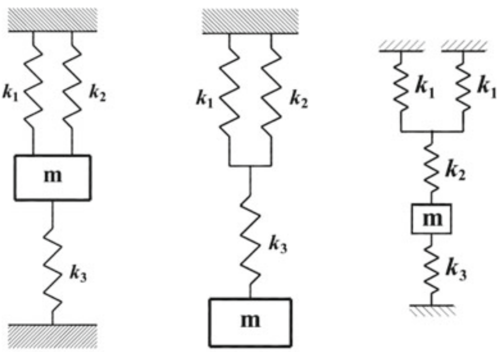
19. What is the function of a vibration isolator?
20. Define transmissibility. Derive the relation for transmissibility for a viscously damped system in terms of frequency ratio.
21. Explain the working of a vibration measuring instrument with necessary expression.
22. Explain the working of a seismometer with necessary expression.
23. Explain the working of an accelerometer with necessary expression.
24. Differentiate between the seismometer and accelerometer.
25. Prove that an un-damped measuring instrument will show a true response for frequency ratio of $1/\sqrt{2}$.
26. Derive an expression for energy dissipated in viscously damped system.
27. Write down the methods for finding the response of a system under periodic but non-harmonic forces.
28. What is the convolution integral? What is its use?
29. Compare the free response of systems with viscous damping, Coulomb damping and hysteretic damping.
30. Compare the forced harmonic response of systems with viscous damping, Coulomb damping and hysteretic damping.

Exercise

1. An unknown mass m is attached to the lower free end of a light spring and causes it to elongate 5 mm from its un-stretched length. Determine the natural frequency of the system.
2. A spring-mass $(k - m)$ system has a natural time period of 0.1 s. What will be the new period if the spring constant is **(a)** increased by 20% and **(b)** decreased by 20%?
3. A spring-mass $(k - m)$ system has a natural frequency of 20 rad/s. When the spring constant is reduced by 500 N/m, the natural frequency of the system is lowered by 20%. Determine the mass (m) and stiffness (k) of the system.
4. A spring-mass $(k - m)$ system has a natural frequency of 10 Hz. When 5 kg mass is attached with m, the natural frequency of the system is lowered by 6 Hz. Determine the mass (m) and stiffness (k) of the system.
5. A vehicle is found to have a natural frequency of 40 rad/s without passengers and 30 rad/s with passengers of mass 350 kg. Find the mass and stiffness of the vehicle by modeling it as a single degree of freedom system.
6. A table supported by flat steel legs is modeled as single degree of freedom system. Its natural time period in horizontal motion is 0.25 s. When a 20 kg mass is place on its surface, the natural period in horizontal motion is increased to 0.3 s. Determine the mass of the table and the effective spring constant.
7. A spring-mass $(k_1 - m)$ system has a natural frequency of ω rad/s. When another spring with a spring constant of k_2 is connected in parallel with k_1, the natural frequency of the system is increased by 40%. Determine the value of k_2 in terms of k_1.

8. A spring-mass ($k_1 - m$) system has a natural frequency of rad/s. When another spring with a spring constant of k_2 is connected in series with k_1, the natural frequency of the system is decreased by 40%. Determine the value of k_2 in terms of k_1.
9. A linear spring requires a force of 50 N to produce an elongation of 5 mm. Two ends of the spring are fixed and a mass of 20 kg is attached at the middle point of its length. Determine the natural frequency of the system.
10. A mass m attached to the lower end of a spring, whose upper end is fixed, vibrates with a natural time period of 0.5 s. Determine the natural time period when the same mass is attached to the middle point of the same spring with the upper and lower ends fixed.
11. When two springs of spring constants k_1 and k_2 are connected in series to a block, it vibrates with a natural frequency of 8 rad/s. When the same two springs are connected in parallel to the same block, the block vibrates with natural frequency of 25 rad/s. Determine the ratio k_1/k_2.
12. The time period of free vibration of a system shown in **Figure P4.12** is found to be $0.8s$. When the spring of spring constant $k_2 = 1000$ N/m is removed from the system, the time period is increased to 1 s. Determine **(a)** the spring constant k_1 and **(b)** the mass (m) of the block.
13. Determine natural frequency of the system shown in **Figure P4.13**. Given $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 2000$ N/m and $m = 20$ kg.





(a) (b) (c)

Figure P4.12 **Figure P4.13**

14. Determine natural frequency of the system shown in **Figure P4.14**. Given $k = 1000$ N/m and $m = 20$ kg.

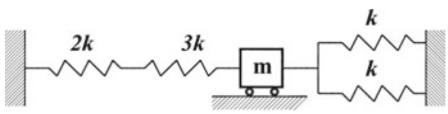


Figure P4.14

15. Determine the natural frequency of the system shown in **Figure P4.15**. Assume that the weight of the bar is negligible.

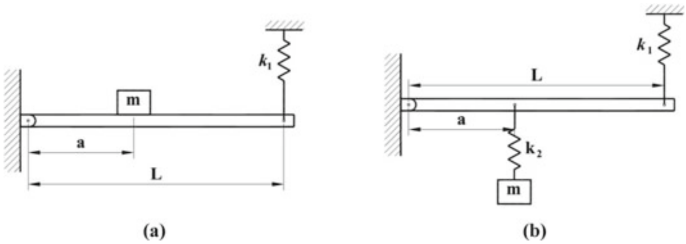


Figure P4.15

16. Determine the natural frequency of the system shown in **Figure P4.16**. Mass of the bar is M.
    (a) Assume ideal springs.
    (b) Assume each spring has a mass of m.

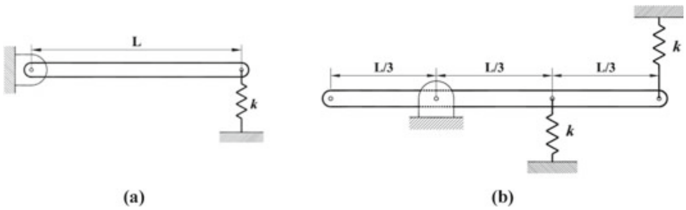


Figure P4.16

17. Determine the natural frequency of the system shown in **Figure P4.17**. Mass of the bar is 25 kg and the spring stiffness is 4 kN/m.

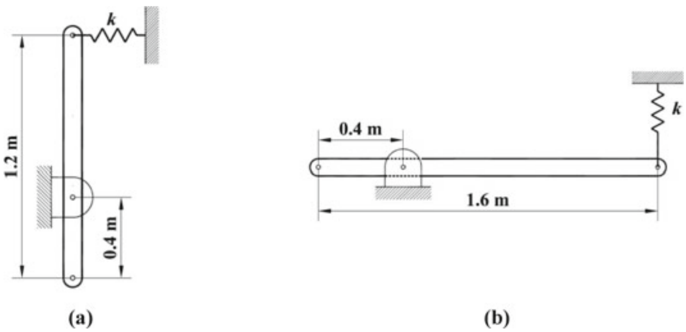


Figure P4.17

18. Determine the natural frequency of the system shown in **Figure P4.18**. Mass of the bar is M.
19. Determine the natural frequency of the bell crank system shown in **Figure P4.19**. Neglect the weight of the crank.

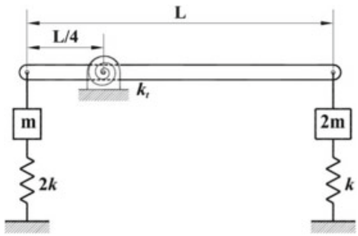


Figure P4.18

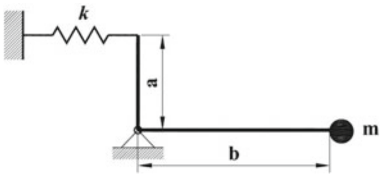


Figure P4.19

20. A thin disk of mass 25 kg and radius of 20 cm is connected by a spring of stiffness 1000 N/m as shown in **Figure P4.20**. It is free to roll on horizontal surface without slipping, determine the natural frequency of the system.
21. Determine the natural frequency of the system shown in **Figure P4.21**. The cylinder rolls on horizontal surface without slipping.

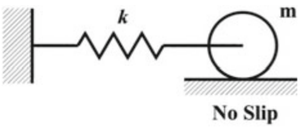


Figure P4.20

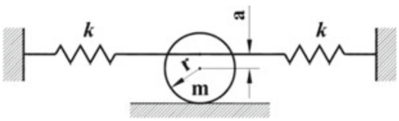


Figure P4.21

22. Determine natural frequency of the system shown in **Figure P4.22**. Given $k = 1000$ N/m, $r = 12$ cm, $I_P = 1$ kgm^2 and $m = 10$ kg.

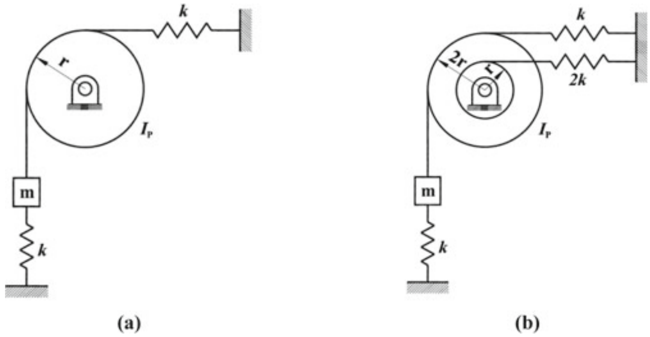


Figure P4.22

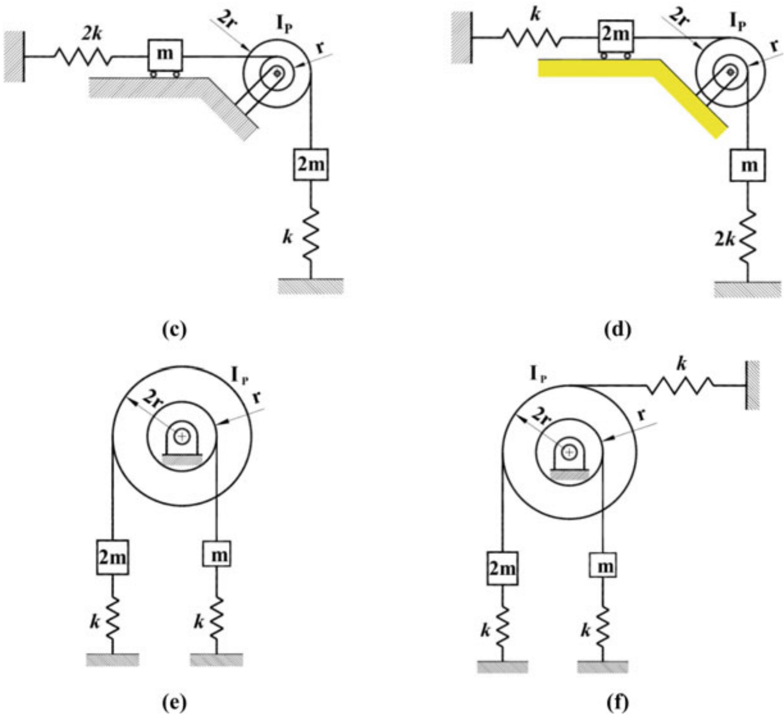


Figure P4.22

23. Determine natural frequency of the system shown in **Figure P4.23**. Given $k = 1200\,\text{N/m}$, $r = 10\,\text{cm}$, $I_P = 1\,\text{kgm}^2$, $m = 10\,\text{kg}$, $m_d = 4\,\text{kg}$ and $r_d = 6\,\text{cm}$. Assume that the disk is thin and rolls without slip.
24. A small mass m is attached to a horizontal wire which is under tension T as shown in **Figure P4.24**. What will be the natural frequency of vibration of the mass if it is displaced laterally a slight distance and then released?

25. Determine the natural frequency of the system shown in **Figure P4.25**. The weight of the bar is negligible.
26. Determine the natural frequency of the system shown in **Figure P4.26**. Neglect the inertia effect of the pulley.
27. Determine the natural frequency of oscillation of a semi-cylinder of mass m and radius r as shown in **Figure P4.27**, when is slightly displaced from its equilibrium position.

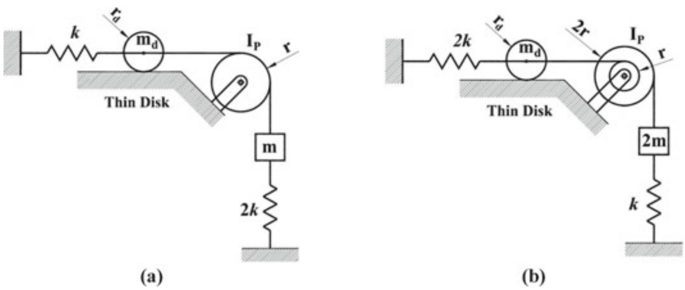


Figure P4.23

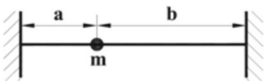


Figure P4.24

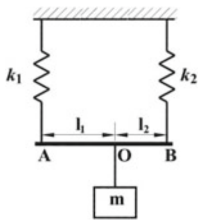


Figure P4.25

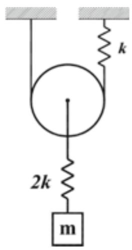


Figure P4.26

28. A thin disk of mass m and radius r is suspended from a point on its circumference as shown in **Figure P4.28**. Determine the natural frequency of oscillation when is slightly displaced from its equilibrium position.
29. A semi-circular disk of mass m and radius r is suspended about its center as shown in **Figure P4.29**. Determine the natural frequency of oscillation when is slightly displaced from its equilibrium position.

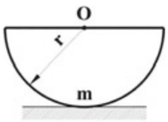


Figure P4.27

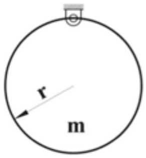


Figure P4.28

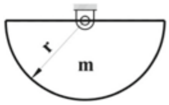


Figure P4.29

30. Determine the natural frequency of the system shown in **Figure P4.30**. Neglect the inertia effect of the pulley.
31. A steel bar is fixed at the upper end and carried a concentrated mass of 20 kg at its lower end. The bar has a cross-section of 10 mm × 15 mm and a length of 10 m. Determine the natural frequency for the longitudinal vibration neglecting the mass of the bar. Take $E = 210$ GPa for steel.

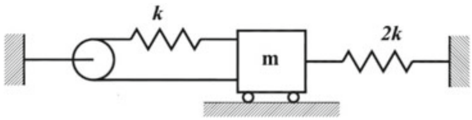


Figure P4.30

32. A steel wire is of 25 mm diameter and is 20 m long. It is fixed at the upper end and a concentrated mass m is attached at its lower end. Determine the value of m so that the frequency of longitudinal vibration is 60 Hz. Take $E = 210$ GPa for steel.
33. Determine the natural frequency of the system shown in **Figure P4.33**. Take $A = 2 \times 10^{-6}\,\text{m}^2$ $E = 210$ GPa for the bar, $m = 15$ kg, $k_1 = 100$ kN/m and $k_2 = 150$ kN/m.

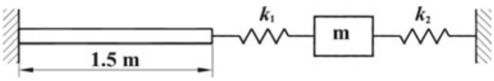


Figure P4.33

34. A steel cantilever beam of length 1.2 m carries a mass of 60 kg at its free end. The beam has a cross-section of 40 mm (depth) × 30 mm (width). Determine the natural frequency of transverse vibration of the mass by modeling it as a single degree of freedom system. Neglect the mass of the beam. Take $E = 210$ GPa for steel.
35. A cantilever beam of rectangular Sect. 50 mm (depth) × 40 mm (width) has a mass fixed at its free end. Determine the ratio of the frequency of the lateral vibration in vertical plane to that in the horizontal plane. Neglect the effect of self-weight of the beam.

36. A simply supported beam of a cross-section of 30 mm (depth) × 20 mm (width) and a length of 1 m carries a mass of 20 kg at its middle. The natural frequency of transverse vibration of beam is found to be 150 rad/s. Determine the Young's modulus of elasticity of the beam material.
37. A beam fixed at both ends has a cross-section of 25 mm (depth) × 20 mm (width) and a length of 0.8 m. It carries a mass of 25 kg at its middle. The natural frequency of transverse vibration of beam is found to be 286 rad/s. Determine the Young's modulus of elasticity of the beam material.
38. A cantilever beam is used to carry a machine at its free end. Determine the ratio of natural frequency of transverse vibration of beam if it is made of a steel to that if it is made of an aluminum. Take Young's modulus of elasticity of steel as 210 GPa and that of aluminum as 95 GPa.
39. A steel cantilever beam of length 1 m carries a mass of 25 kg at its free end. The natural frequency of transverse vibration of the mass is found to be 2.5 Hz. Determine the moment of inertia of the section of the beam about its neutral axis. Neglect the mass of the beam. Take $E = 210$ GPa for steel.
40. A rigid disk is mounted at the free end of a steel shaft of length 1.5 m and radius 8 mm. The time period of its torsional vibration is found to be 2 s. Determine the mass moment of inertia of the disk. Neglect the mass of the shaft. Take $G = 84$ GPa for steel.
41. Determine the natural frequency of a system consisting of a beam and spring and mass assembly shown in **Figure P4.41** by modeling it as a single degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. The beam material has a modulus of elasticity of E and moment of inertia of section of I.

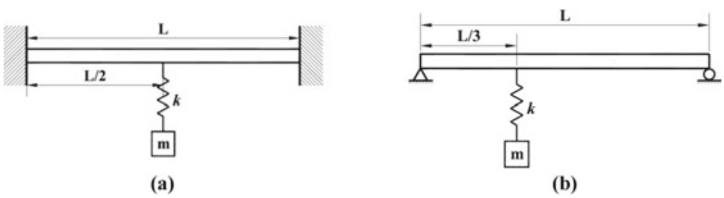


Figure P4.41

42. Determine the natural frequency of a system consisting of a beam and spring and mass assembly shown in **Figure P4.42** by modeling it as a single degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. Take $E = 210$ GPa and $I = 1 \times 10^{-5}$ m^4 for beam; $m = 40$ kg, $k_1 = 1$ MN/m and $k_2 = 2$ MN/m.

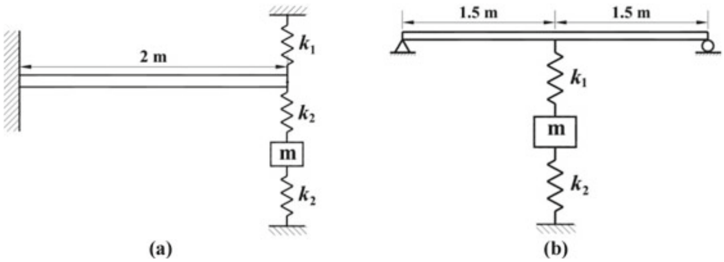


Figure P4.42

43. Determine the natural frequency of a system consisting of a beam and spring and mass assembly shown in **Figure P4.43** by modeling it as a single degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. Take $E = 210\,\text{GPa}$, $L = 1.5\,\text{m}$ and $I = 2 \times 10^{-6}\,\text{m}^4$ for beam, $k = 1\,\text{MN/m}$ and $m = 40\,\text{kg}$. Determine the new natural frequency when the stiffness value (k) of spring is (**i**) reduced by 25% and (**ii**) increased by 25%.

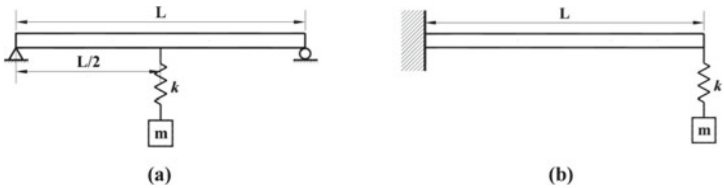


Figure P4.43

44. A single degree of freedom consists of a mass of 5 kg and a spring with a stiffness of 100 N/m. Derive the expression for its displacement, velocity and acceleration when it is displaced by 0.1 m from equilibrium position and released. Also plot displacement, velocity and acceleration for $0 \leq t \leq 5s$.
45. A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 150 N/m. Derive the expression for its displacement, velocity and acceleration when it is subjected to an initial velocity of 1 m/s. Also plot displacement, velocity and acceleration for $0 \leq t \leq 5$ s.
46. A single degree of freedom consists of a mass of 15 kg and a spring with a stiffness of 100 N/m. Derive the expression for its displacement, velocity and acceleration when it is displaced by 0.05 m from the equilibrium position and released with an initial velocity of 0.5 m/s. Also plot displacement, velocity and acceleration for $0 \leq t \leq 5$ s.
47. A single degree of freedom consists of a mass of 5 kg and a spring with a stiffness of 1 kN/m. It is set into oscillation by applying initial displacement of 0.1 m from the equilibrium position. What should be the maximum initial velocity such that the maximum amplitude of free vibration does not exceed 0.15 m?

48. A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. It is set into oscillation by applying initial velocity of 1 m/s. What should be the maximum initial displacement such that the maximum amplitude of free vibration does not exceed 0.12 m?
49. A rigid block of mass M is supported by a spring with stiffness k as shown in **Figure P4.49**. A mass m drops from a height h and adheres to the rigid block without rebounding. Derive an expression for the free vibration response of the system.
50. A block of mass 8 kg falls from a height 0.5 m onto the end of a cantilever beam as shown in **Figure P4.50** and adheres to it without rebounding. Determine the resulting transverse vibration of the beam. Take $E = 210\,\text{GPa}$ and $I = 1 \times 10^{-5}\,\text{m}^4$ for beam.
51. A single degree of freedom consists of a mass of 5 kg and a spring with a stiffness of 100 N/m. Derive the expression for its displacement, velocity and acceleration when it is displaced by 0.1 m from equilibrium position and released. Also plot displacement, velocity and acceleration for $0 \le t \le 5$ s. The damping constant of the system is

    (a) 5 N s/m,
    (b) $20\sqrt{5}$ N s/m and
    (c) 50 N s/m.

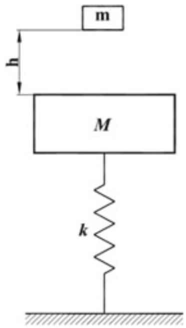


Figure P4.49

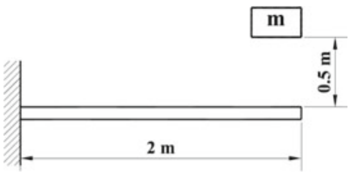


Figure P4.50

52. A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 150 N/m. Derive the expression for its displacement, velocity and acceleration when it is subjected to an initial velocity of 1 m/s. Also plot displacement, velocity and acceleration for $0 \le t \le 5$ s. The damping constant of the system is

    (a) 5 N s/m,
    (b) $20\sqrt{15}$ N s/m and
    (c) 100 N s/m.

53. A single degree of freedom consists of a mass of 15 kg and a spring with a stiffness of 100 N/m. Derive the expression for its displacement, velocity and acceleration when it is displaced by 0.05 m from the equilibrium position and released with an initial velocity of 0.5 m/s. Also plot displacement, velocity and acceleration for $0 \leq t \leq 5$ s.

    (a) 5 N s/m,
    (b) $20\sqrt{15}$ N s/m and
    (c) 100 N s/m.

54. Determine the damping constant for the system shown in **Figure P4.54** such that it is critically damped. Given $k = 1000$ N/m and $m = 20$ kg.

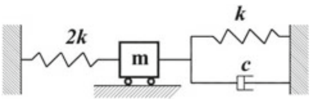


Figure P4.54

55. Determine the damping constant for the system shown in **Figure P4.55** such that it is critically damped. Assume that the disk is thin and rolls without slip.
56. Determine the damping constant for the system shown in **Figure P4.56** such that it is critically damped. Given $k = 1500$ N/m, $m = 5$ kg, $a = 0.2$ m, $b = 0.4$ m and $L = 0.6$ m. Assume that weight of the bar is negligible.

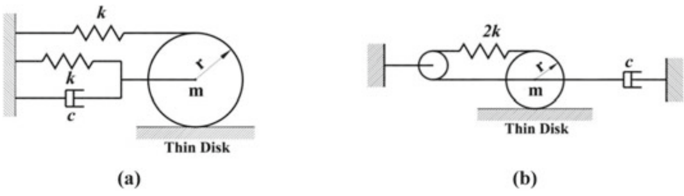


Figure P4.55

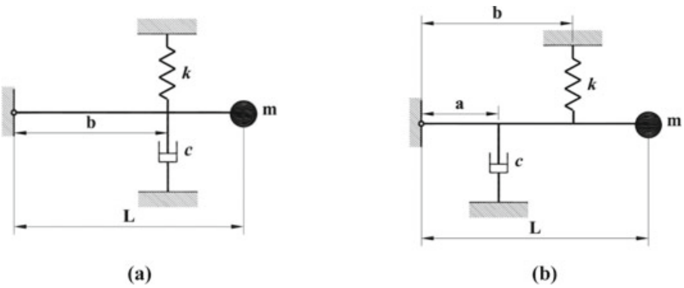


Figure P4.56

57. Determine the damping constant for the system shown in Figure P4.57 such that it is critically damped. Mass of the bar is 20 kg, length of the bar is 1.2 m and stiffness of the spring is 3 kN/m.
58. Determine the damping constant for the system shown in Figure P4.58 such that damping ratio of the system is 0.25. Mass of the bar is 50 kg, length of the bar is 1*m* and stiffness of the spring is 2.5 kN/m.

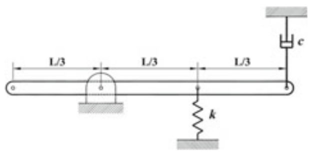


Figure P4.57

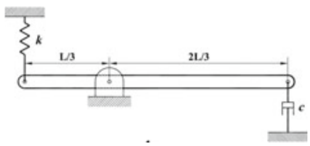


Figure P4.58

59. Determine whether the system shown in Figure P4.59 is under-damped, critically damped or over-damped. Mass of the bar is 20 kg, length of the bar is 1.2 m, damping constant is 50 N s/m and stiffness of the spring is 4 kN/m. Also determine the response $\theta(t)$ of the system when it is subjected to the initial conditions of $\theta(0) = 0.1$ rad and $\dot{\theta}(0) = 1$ rad/s.
60. A bar shown in **Figure P4.60** is initially displaced by 0.15 rad and released. Mass of the bar is 20 kg, length of the bar is 1.2 m, damping constant is 50 N s/m and stiffness of the spring is 4 kN/m. How many cycles will be executed before the amplitude of vibration will be less than 0.01 rad?

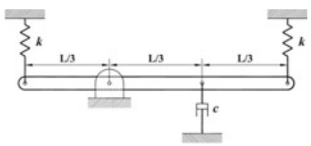


Figure P4.59

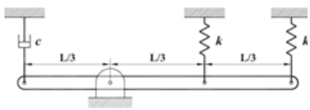


Figure P4.60

61. Determine the damping constant for the system shown in **Figure P4.61** such that it is critically damped. Given $k = 1000$ N/m, $r = 12$ cm, $I_P = 1$ kgm^2 and $m = 10$ kg.

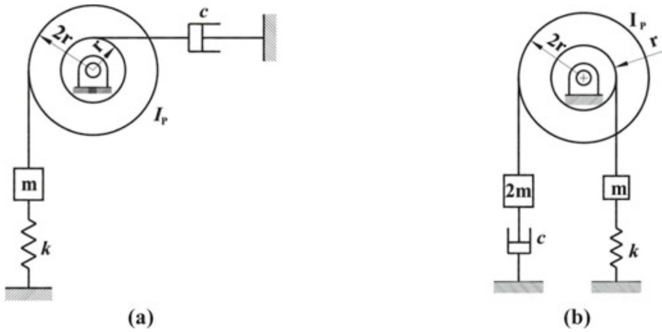


Figure P4.61

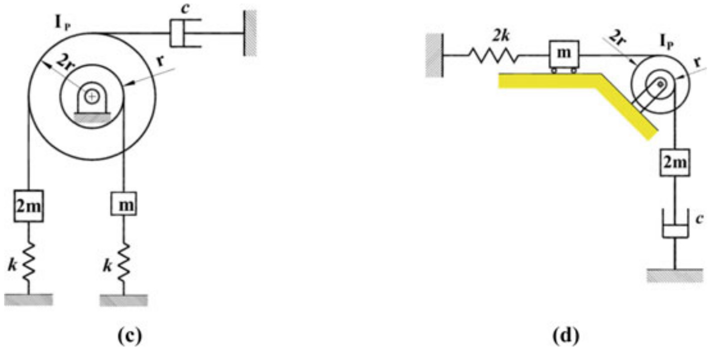


Figure P4.61

62. Determine the damping constant for the system shown in Figure P4.62 such that damping ratio of the system is 0.2. Given $k = 1200\,\text{N/m}$, $r = 15\,\text{cm}$, $I_P = 1\,\text{kgm}^2$ and $m = 10\,\text{kg}$.

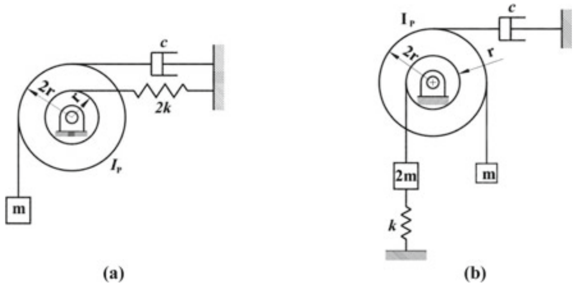
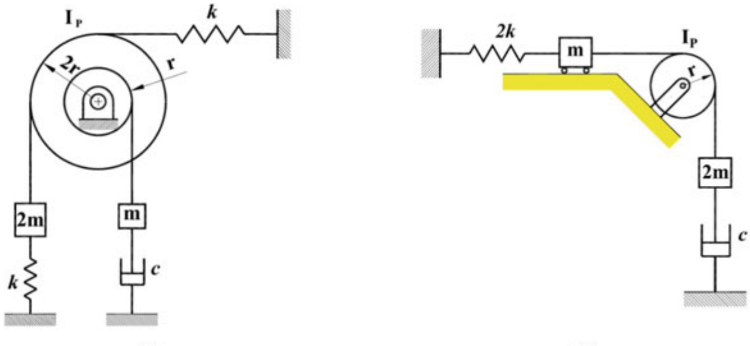


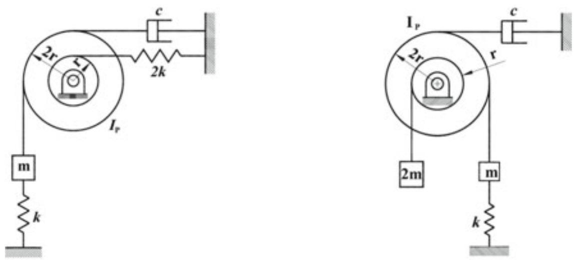
Figure P4.62



(c) (d)

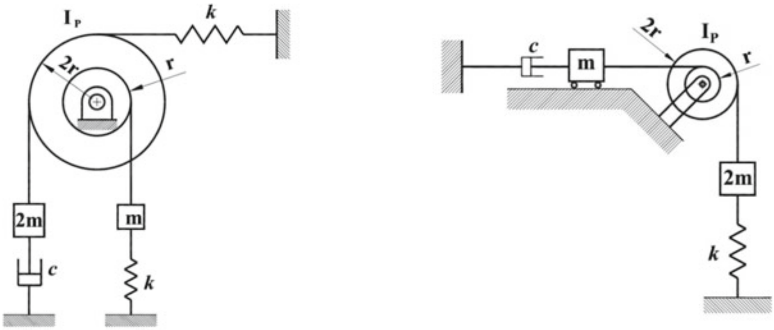
Figure P4.62

63. Determine whether the system shown in **Figure P4.63** is under-damped, critically damped or over-damped. Given $k = 1000$ N/m, $r = 12$ cm, $I_P = 1$ kgm^2, $c = 120$ N s/m and $m = 10$ kg. Use displacement (x) of the block with mass m as a generalized coordinate and determine the response $x(t)$ of the system when it is subjected to the initial conditions of $x(0) = 0.1$ m and $\dot{x}(0) = 1$ m/s.
64. A block with mass m shown in **Figure P4.64** is initially displaced by 20 mm and released. Given $k = 1200$ N/m, $r = 15$ cm, $I_P = 1.2$ kgm^2, $c = 60$ N s/m and $m = 10$ kg. How many cycles will be executed before the amplitude of vibration will be less than 1 mm?
65. The response of a damped single degree of freedom system with $m = 10$ kg and $k = 1000$ N/m is found to be $x(t) = 0.1414\mathrm{e}^{-2t}\sin(\omega_d t + 0.785)$. Determine damping constant and damped natural frequency (ω_d) of the system. Also determine the initial condition to which the system is subjected?
66. The response of a damped single degree of freedom system with $m = 15$ kg and $k = 1000$ N/m is found to be $x(t) = \mathrm{e}^{-\omega_n t}(0.1 + 1.8165t)$. Determine damping constant and natural frequency (ω_n) of the system. Also determine the initial condition to which the system is subjected?



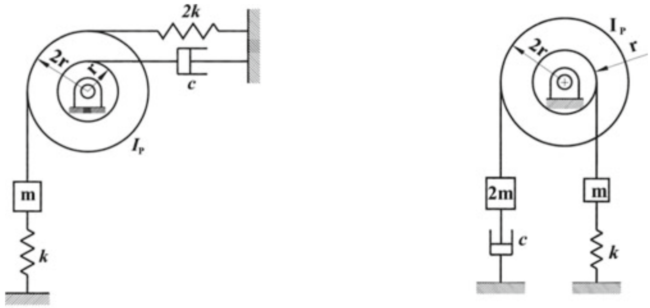
(a) (b)

Figure P4.63



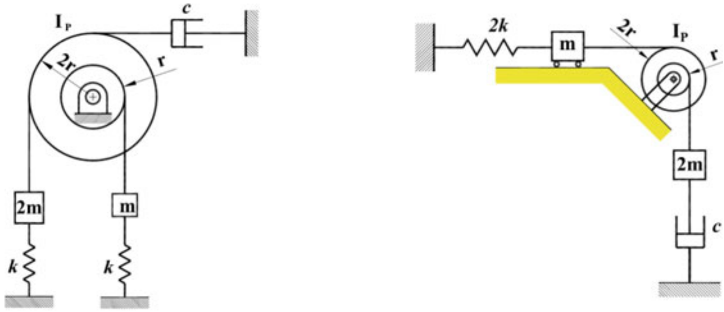
(c) (d)

Figure P4.63



(a) (b)

Figure P4.64



(c) (d)

Figure P4.64

67. The response of a damped single degree of freedom system with $m = 10$ kg and stiffness k is found to be $x(t) = 0.08942e^{-5.367t} + 0.07133e^{-18.633t}$. Determine

the stiffness and damping constant of the system. Also determine the initial condition to which the system is subjected.

68. A critically damped system subjected to an initial displacement of 0.02 m and an initial velocity of -1 m/s crosses the equilibrium position at $t = 0.025$ s before coming to rest. Determine the natural frequency of the system.
69. It is found from the response of an under-damped single degree of freedom system with $m = 10$ kg, $c = 40$ N s/m and $k = 1000$ N/m that the highest peak of magnitude 0.1205 m occurs at $t = 0.08$ s. Determine the initial condition to which the system is subjected.
70. Determine the ratio between the damped and un-damped natural frequencies of a system if the damping ratio is **(a)** 0.1; **(b)** 0.25; **(c)** 0.5.
71. If the ratio of successive amplitudes of damped free vibration is 2, determine the logarithmic decrement and damping ratio of the system.
72. A single degree of freedom system with $m = 10$ kg and $k = 1000$ N/m has a logarithmic decrement of 2.5. If the system is given an initial velocity of 1 m/s when it is at an equilibrium position, determine the maximum displacement of the system.
73. A 5 kg piston on a helical spring vibrates with a natural frequency of 10 rad/s. When oscillating within an oil-filled cylinder, the frequency of free oscillation is reduced to 9.8 rad/s. Determine the damping constant of the system.
74. A 20 kg block is attached to a spring of stiffness 4 kN/m in parallel with a viscous damper. The period of free vibration of this system is observed as 0.5 s. Determine the value of damping constant.
75. A 100 kg machine is placed on a vibration isolator of stiffness 100 kN/m. The machine is given an initial displacement of 10 cm and released. After 5 cycles the machine's amplitude is 1 cm. What is the damping constant of the system?
76. A 100 kg block is attached to a spring of stiffness 2 NN/m in parallel with a viscous damper. The block is given an initial velocity of 5 m/s. Determine the maximum displacement if the damping constant is

    (a) 5000 N s/m
    (b) 30,000 N s/m

77. A single degree of freedom system has a mass of 5 kg and a stiffness of 300 N/m. The amplitude decreases 0.25 of the initial value after five consecutive cycles. Determine the damping constant of the system.
78. A vibrating system consists of a mass of 4 kg, stiffness of 150 N/m and damping constant of 5 N s/m. Determine:

    (a) the damping ratio,
    (b) the natural frequency of damped vibration,
    (c) logarithmic decrement,
    (d) the ratio of two consecutive amplitudes and
    (e) the number of cycles after which the original amplitude is reduced to 20%.

79. The mass of a single degree of damped vibrating system is 8 kg and oscillates with a frequency of 2 Hz when disturbed from its equilibrium position. The

amplitude of vibration reduces to 0.2 of its initial value after five oscillations. Determine:

(a) logarithmic decrement,
(b) damping ratio and
(c) stiffness of the spring.

80. When a 50 kg machine is supported by a spring and a viscous damper, free vibrations appear to decay exponentially with a frequency of 70 rad/s. When a 60 kg machine is placed on the same foundation, the frequency of the exponentially decaying oscillations is 65 rad/s. Determine the stiffness and equivalent viscous damping constant of the system.
81. The amplitude of vibration a single degree of freedom system with a viscous damper decays to half of its initial value in 5 cycles with a period of 0.25 s. If the mass of the system is 20 kg, determine the spring stiffness and the damping constant of the system.
82. A block with a mass of 40 kg is placed on two different isolators and the corresponding free vibration records as shown in **Figure P4.82(a)** and **(b)**. Determine in each case the type of damping and its stiffness and damping constant.
83. A spring-mass system consists of a mass of 10 kg and a spring with a stiffness of 4 kN/m. It is excited by an external harmonic force $400 \sin \omega t$. It is initially displaced by 0.05 m from the equilibrium position and released. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

(a) 19 rad/s,
(b) 20 rad/s and
(c) 21 rad/s.

Also compare the responses for each case with plots.

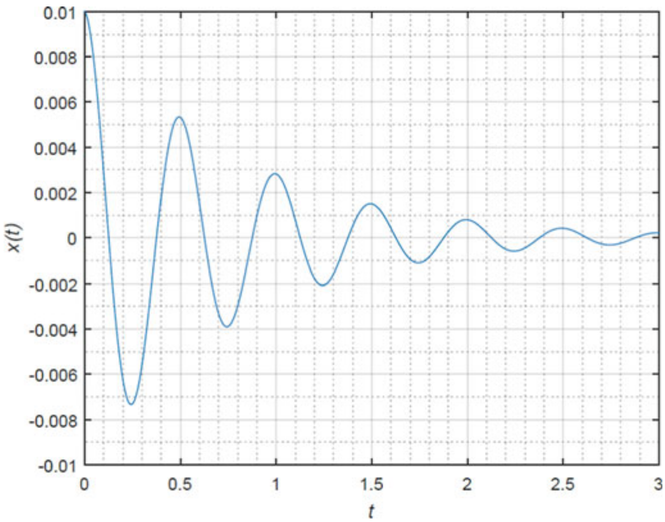


Figure P4.82(a)

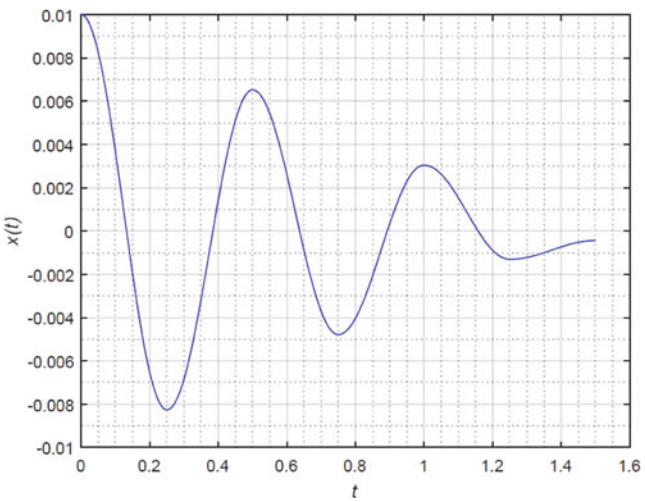


Figure P4.82(b)

84. A spring-mass system consists of a mass of 10 kg and a spring with a stiffness of 4 kN/m. It is excited by an external harmonic force $400 \sin \omega t$. It is initially released from the equilibrium position with a velocity of 0.5 m/s. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

(a) 19 rad/s,
(b) 20 rad/s and

(c) 21 rad/s.

Also compare the responses for each case with plots.

85. A spring-mass system consists of a mass of 10 kg and a spring with a stiffness of 4 kN/m. It is excited by an external harmonic force 400 $\sin \omega t$. It is initially displaced by 0.05 m from the equilibrium position and released with a velocity of 0.5 m/s. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

    (a) 19 rad/s,
    (b) 20 rad/s and
    (c) 21 rad/s.

    Also compare the responses for each case with plots.

86. A spring-mass system consists of a spring with a stiffness of 2 kN/m. It is subjected to a harmonic force having an amplitude of 150 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 16 mm. Determine the value of mass m of the system.
87. A spring-mass system consists of a mass of 10 kg and a spring with a stiffness of 4 kN/m. It is excited by an external harmonic force 400 $\sin \omega t$. If the vibration amplitude of the mass is observed to be 80 mm, determine the frequency of the external excitation.
88. A spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 1 kN/m and a damping constant of 15 N s/m. It is excited by an external harmonic force 200 $\sin \omega t$. It is initially displaced by 0.05 m from the equilibrium position and released. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

    (a) 9 rad/s,
    (b) 10 rad/s and
    (c) 11 rad/s.

    Also compare the responses for each case with plots.

89. A spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 1 kN/m and a damping constant of 15 N s/m. It is excited by an external harmonic force 200 $\sin \omega t$. It is initially released from the equilibrium position with a velocity of 0.5 m/s. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

    (a) 9 rad/s,
    (b) 10 rad/s and
    (c) 11 rad/s.

    Also compare the responses for each case with plots.

90. A spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 1 kN/m and a damping constant of 15 N s/m. It is excited by an

external harmonic force $200 \sin \omega t$. It is initially displaced by 0.05 m from the equilibrium position and released with a velocity of 0.5 m/s. Derive the expression for the total response $x(t)$ of the system when the frequency of external excitation is

(a) 9 rad/s,
(b) 10 rad/s and
(c) 11 rad/s

Also compare the responses for each case with plots.

91. A spring-mass-damper system consists of a mass of 15 kg, a spring with a stiffness of 2 kN/m and a damping constant of 50 N s/m. It is excited by an external harmonic force $30 \sin \omega t$. Assuming viscous damping, determine:

    (a) the resonant frequency,
    (b) damped frequency,
    (c) the phase angle at resonance,
    (d) the amplitude at resonance,
    (e) the frequency corresponding to the peak amplitude and
    (f) the peak amplitude.

92. A spring-mass-damper system consists of a mass of 10 kg and a spring with a stiffness of 2 kN/m. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. Determine the amplitude of steady state vibration when it is excited by an external harmonic force $250 \cos 25t$.
93. A spring-mass-damper system consists of a mass of 10 kg and a spring with a stiffness of 8 kN/m. It is subjected to a damping effect adjusted to a value 0.2 times that required for critical damping. Determine the natural frequency of the un-damped and damped vibrations and ratio of successive amplitudes for damped vibrations. If the body is subjected to a periodic disturbing force of amplitude 500 N and of frequency equal to 0.75 times the natural un-damped frequency, determine the amplitude of steady state vibration and the phase difference with respect to the disturbing force.
94. A machine of mass 80 kg is mounted on an elastic foundation. It is found that, when a harmonic force of amplitude 150 N is applied to the machine, the maximum steady state displacement of 15 mm occurred at a frequency of 500 rpm. Determine the equivalent stiffness and damping constant of the foundation.
95. A spring-mass-damper system is subjected to an external harmonic excitation. The amplitude is found to be 30 mm at resonance and 20 mm at a frequency times the resonant frequency. Determine the damping ratio of the system.
96. A block of mass 5 kg vibrates in a viscous medium. When it is excited by a harmonic force of magnitude 50 N, the steady state response at resonance is found to have an amplitude of 10 mm and a period of 0.1 s. Determine the damping constant of the system.

97. A machine of mass 40 kg is attached to a spring of stiffness 20 kN/m. When it is subjected to a harmonic excitation of magnitude 150 N, the maximum amplitude of steady state vibration is found to be 16 mm. Determine the damping constant of the system.
98. A machine of mass 35 kg is placed on an elastic foundation. A harmonic force of magnitude 40 N is applied to the machine. It is found that the maximum steady state amplitude of 1.5 mm occurs when the period of response is 0.2 s. Determine the equivalent stiffness and damping ratio of the foundation.
99. A machine of mass 50 kg is mounted on an elastic foundation which can be modeled as a spring and viscous damper in parallel. When a harmonic force with a magnitude of 5 kN is applied to the machine, it is observed that the maximum steady state amplitude is 2 mm occurs at 30 Hz. Determine the equivalent stiffness and equivalent damping coefficient of the foundation.
100. A machine of mass 120 kg is attached to a spring of stiffness 200 kN/m and is subjected to a harmonic excitation of 750 N and period of 0.1 s. Determine the amplitude of steady state vibration. What would be the amplitude of steady state vibration when a viscous damper of damping constant 1600 N s/m is attached in parallel with the spring?
101. The damped natural frequency of a system as obtained from a free vibration test is 10 Hz. During the forced vibration test with constant exciting force on the same system, the maximum amplitude of vibration is found to be at 9.8 Hz. Determine the damping ratio and natural frequency of the system in Hz.
102. A machine of mass 50 kg is supported by an assembly of springs with an equivalent stiffness of 220 kN/m. When the machine runs at an operating speed of 2500 rpm, the steady state amplitude of harmonic vibration is found to be 2 mm. Determine the magnitude of the excitation provided to the machine at the operating speed.
103. A motor of 30 kg operating at 25 Hz is mounted on an elastic foundation of stiffness 2.5 MN/m. If the phase difference between the excitation and the steady state response is found to be 20°. Determine the damping ratio of the system.
104. A machine of mass 150 kg is attached to the mid-span of a simply supported steel beam of length 1.6 m. The moment of area of the section of the beam is 2.8×10^{-6} m^4 and modulus of elasticity of 210 GPa. Determine the steady state amplitude of the machine when it is subjected to a harmonic excitation of magnitude 30 kN and frequency 120 rad/s. What would be the amplitude of steady state vibration if the equivalent viscous damping ratio of the beam ?
105. A machine of mass 180 kg is attached to the mid-span of a steel beam fixed at both ends. The beam has a length of 4 m, width of 0.1 m and thickness 0.15 m, At an operating speed of 1200 rpm, a rotating force of magnitude $F_0 =$ 4000 N is developed due to the unbalance in the rotor of the motor. Find the amplitude of steady state vibrations. Neglect the damping and inertia effects of the beam. Take $E = 210$ GPa for the beam.
106. A machine of mass 50 kg is placed at the end of a 2 m steel cantilever beam. The machine is to be subjected to a harmonic excitation of magnitude 1000 N

at 30 rad/s. For what values of the beam's cross-sectional moment of inertia, the steady state amplitude of vibration of the system will be limited to 20 mm? Neglect the damping and inertia effects of the beam. Take $E = 210$ GPa for the beam.

107. A motor of 60 kg is placed at the end of a 1.6 m steel cantilever beam. The moment of area of the section of the beam is 1.5×10^{-6} m^4 and modulus of elasticity of 210 GPa. When the motor operates at 180 rpm, the phase difference between the operation of motor and the response of the system is 10°. Determine the damping ratio of the beam.
108. A machine of mass 50 kg is placed at the end of a 2 m steel cantilever beam. The moment of area of the section of the beam is 2×10^{-6} m^4 and modulus of elasticity of 210 GPa. When the machine operates, it produces a harmonic force of magnitude 150 N. Determine the range of operating speed for which the steady state amplitude of vibration of the system will be less than 2 mm. Neglect the damping and inertia effects of the beam.
109. A 150 kg machine is mounted at the mid-span of a 1.5 m long simply supported steel beam. The moment of area of the section of the beam is 2×10^{-6} m^4 and modulus of elasticity of 210 GPa. When the system is subjected to a harmonic force of magnitude 2.5 kN, the largest steady state amplitude is found to be 2.2 mm. Determine the damping ratio of the system.
110. Determine the amplitude of steady state vibration of block of mass m of the system shown in **Figure P4.110** when it is excited by a harmonic force $F(t) = 300 \sin 40t$ N. Given $k = 1000$ N/m, $r = 12$ cm, $I_P = 1$ kgm^2 and $m = 10$ kg.

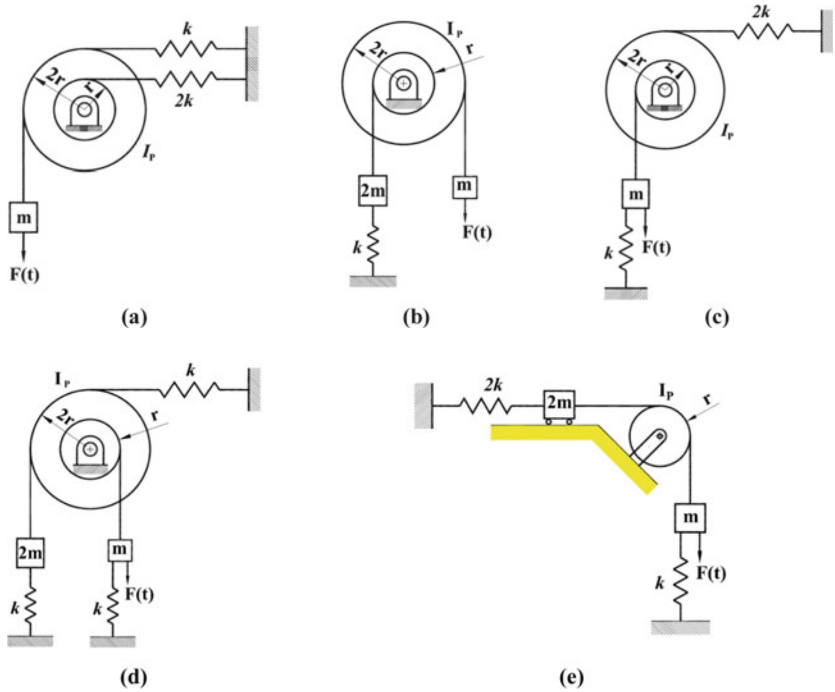


Figure P4.110

111. Determine the amplitude of steady state vibration of pulley of the system shown in **Figure P4.111** when it is excited by a harmonic moment $M(t) = 250 \sin 30t$ N m. Given $k = 1200$ N/m, $r = 15$ cm, $I_P = 1.2\,\text{kgm}^2$ and $m = 10$ kg.

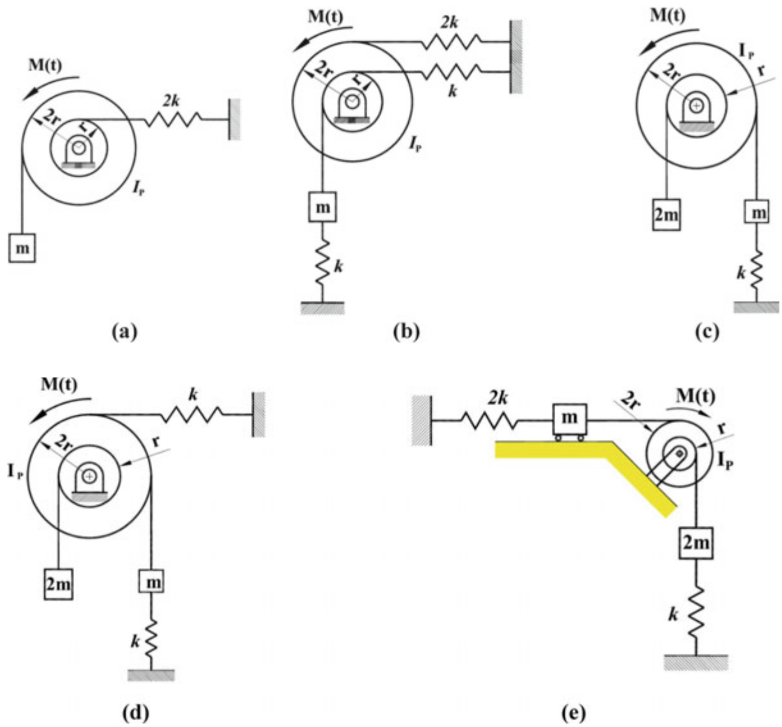
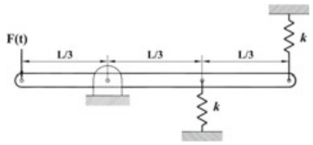


Figure P4.111

112. Determine the amplitude of steady state vibration of the bar shown in **Figure P4.112** when it is excited by a harmonic force $F(t) = 300 \sin 40t$ N. Mass of the bar is 20 kg, length of the bar is 1.2 m and stiffness of each spring is 3 kN/m.
113. Determine the amplitude of steady state vibration of the bar shown in **Figure P4.113** when it is excited by a harmonic moment $M(t) = 200 \sin 20t$ N m. Mass of the bar is 16 kg, length of the bar is 1 m and stiffness of the spring is 2.5 kN/m.
114. Determine the amplitude of steady state vibration of block of mass m of the system shown in **Figure P4.114** when it is excited by a harmonic force $F(t) = 300 \sin 40t$ N. Given $k = 1000$ N/m, $c = 100$ N s/m, $r = 12$ cm, $I_P = 1$ kgm^2 and $m = 10$ kg.
115. A block of mass m of the system shown in **Figure P4.115** is excited by a harmonic force $F(t) = F_0 \sin 30t$ N. Given $k = 1200$ N/m, $c = 120$ N s/m, $r = 15$ cm, $I_P = 1.2$ kgm^2 and $m = 10$ kg. Determine the maximum value of F_0 such that the amplitude of steady state vibration does not exceed 12 mm.



P4.112

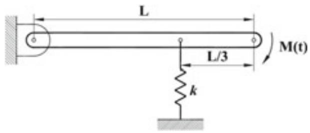


Figure P4.113

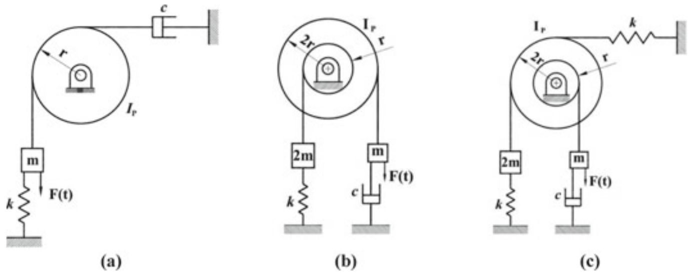
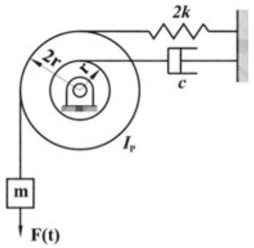


Figure P4.114

116. A block of mass m of the system shown in **Figure P4.116** is excited by a harmonic force $F(t) = 20 \sin \omega t$ N. Given $k = 8000$ N/m, $c = 25$ N s/m, $r = 10$ cm, $I_P = 0.5\,\text{kgm}^2$ and $m = 5$ kg. Determine the range of frequency for which the amplitude of steady state vibration does not exceed 20 mm.



P4.115

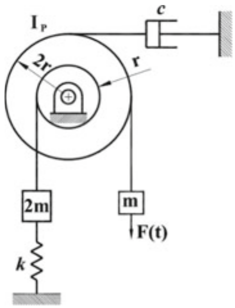


Figure P4.116

117. Determine the amplitude of steady state vibration of pulley of the system shown in **Figure P4.117** when it is excited by a harmonic moment $M(t) = 250 \sin 30t$ N m. Given $k = 1000$ N/m, $c = 80$ N s/m, $r = 12$ cm, $I_P = 1\,\text{kgm}^2$ and $m = 10$ kg.

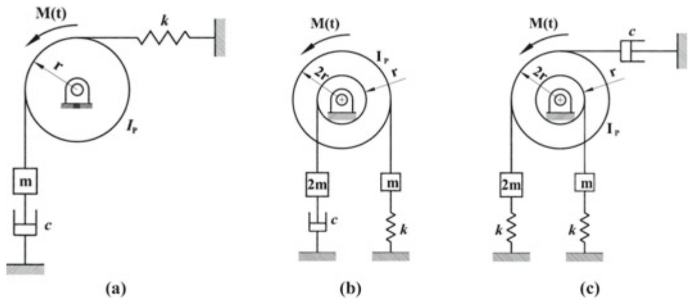


Figure P4.117

118. A system shown in **Figure P4.118** is excited by a harmonic moment $M(t) = M_0 \sin 30t$ N m. Given $k = 1000$ N/m, $c = 100$ N s/m, $r = 12$ cm, $I_P = 1$ kgm^2 and $m = 10$ kg. Determine the maximum value of M_0 such that the amplitude of steady state vibration of the block with mass m does not exceed 12 cm.
119. A system shown in **Figure P4.119** is excited by a harmonic moment $M(t) = 25 \sin \omega t$ N m. Given $= 10$ kN/m, $c = 100$ N s/m, $r = 15$ cm, $I_P = 0.6$ kgm^2 and $m = 6$ kg. Determine the range of frequency for which the amplitude of steady state vibration of the pulley does not exceed 0.03 rad.

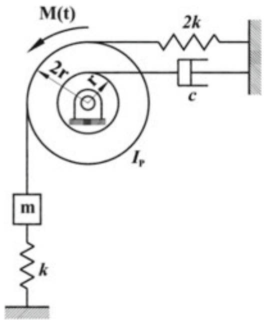


Figure P4.118

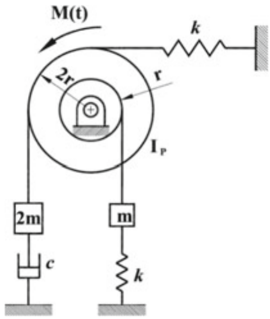


Figure P4.119

120. Determine the amplitude of steady state vibration of the bar shown in **Figure P4.120** when it is excited by a harmonic force $F(t) = 300 \sin 40t\ N$. Mass of the bar is 20 kg, length of the bar is 1.2 m, stiffness of each spring is 3 kN/m and damping constant is 10 N s/m.

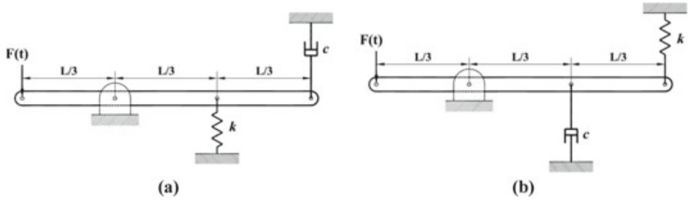


Figure P4.120

121. Determine the amplitude of steady state vibration of the bar shown in **Figure P4.121** when it is excited by a harmonic moment $M(t) = 200 \sin 20t$ N m. Mass of the bar is 16 kg, length of the bar is 1 m, stiffness of each spring is 4.5 kN/m and damping constant is 50 N s/m.

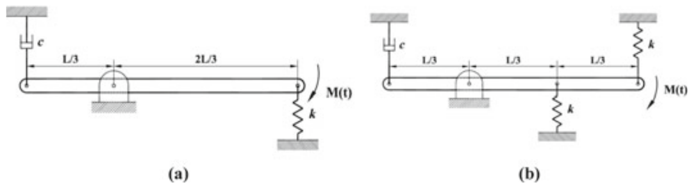


Figure P4.121

122. The amplitude of steady state vibration of a machine with a mass of 80 kg approaches 2 mm when its operating speed increases. Determine the magnitude of unbalance existing in the machine.
123. A machine of mass 75 kg is placed on a thin massless beam. During vibration test of the system it is found that the maximum steady state amplitude occurs for a speed less than 75 rad/s and the steady state amplitude of the machine at a speed of 75 rad/s is 8 mm. As the speed becomes very high, the steady state amplitude approaches 3 mm. Determine the magnitude of unbalance existing in the machine and the equivalent stiffness of the beam. Assume that the damping effect is negligible.
124. A machine of mass 75 kg has a rotating unbalance of 0.2 kg m. It is mounted on a foundation of equivalent stiffness 2 MN/m and damping ratio 0.16. Determine the amplitude of steady state vibration due to unbalance when the machine operates at 7500 rpm.
125. A machine of mass 65 kg operates at 100 rpm and is mounted on an elastic foundation with equivalent stiffness of 36 kN/m. Determine the maximum permissible unbalance in order to limit the amplitude of steady state vibration to 3 mm. Neglect the damping effect.
126. A machine of mass 350 kg with a rotating unbalance of 0.4 kg m is placed on an elastic foundation of stiffness 1 MN/m and damping ratio 0.1. Determine the maximum steady state amplitude of the machine and the speed at which this occurs.

127. A rotating machine of 80 kg mass has a rotating unbalance of 0.16 kg m and operates between 300 and 500 rpm. Determine the equivalent stiffness of the elastic foundation such that the amplitude of steady state vibration does not exceed 2.5 mm, if

    (a) the damping effect of the isolator is negligible,
    (b) the damping ratio of the isolator .

128. A machine of mass 1000 kg has an unbalanced mass of 15 kg located at 10 cm radius. It is found that the resonance occurs at 2200 rpm. Determine the amplitude of steady state vibration when the machine operates at 1500 rpm, if damping ratio is .
129. The total mass of a machine is 30 kg. The eccentric unbalanced mass of 1 kg has a radius of rotation 4 cm. At speed of 100 rpm, the amplitude of steady state vibration is and the corresponding phase is 90°. Determine the natural frequency of the system and the damping ratio. Also determine the amplitude of steady state vibration and the corresponding phase when the machine operates at 1200 rpm.
130. An electric motor of mass 80 kg mounted on an elastic foundation. The unbalanced mass of the motor is 5% of the mass of the rotor. It is found that the amplitude of steady state vibration at resonance is 30 mm. If the damping ratio of the foundation is . Determine:

    (a) the eccentricity of the unbalanced mass, and
    (b) the additional mass to be added uniformly to the motor if the deflection of the motor at resonance is to be reduced to 20 mm.

131. A fan with a mass of 50 kg and a rotating unbalance of magnitude 0.12 kg m is attached at the free end of a cantilever steel beam. The beam has a length of 2 m, the moment of area of the section of the beam is 2×10^{-6} m^4 and the modulus of elasticity of 210 GPa. The maximum amplitude of steady state vibration is found to be 20 mm. Determine the amplitude of steady state vibration when the fan operates at 1200 rpm.
132. A machine of mass 165 kg has a rotating unbalance of 0.45 kg at 0.18 m from the center of rotation. The motor is to be mounted at the end of a steel cantilever beam of length 1 m. The operating range of the motor is from 500 to 1200 rpm. Determine the values of the beam's cross-sectional moment of inertia for which the steady state amplitude of vibration of the system will be less than 1.5 mm. Assume that the equivalent damping ratio is . Take $E = 210$ GPa for the beam.
133. A machine of mass 120 kg operates at 1500 rpm. What percent isolation can be achieved if the machine is mounted on an un-damped isolator consisting four identical springs in parallel, each having stiffness of 200 kN/m?
134. A machine of mass 100 kg operates at 800 rpm. Determine the equivalent stiffness of an un-damped isolator that provides 80% isolation.

135. Determine the equivalent stiffness of an un-damped isolator to provide 90% isolation to a machine of mass of 80 kg that operates at speeds between 1000 and 1500 rpm.
136. A 120 kg machine operates at 1000 rpm and has a rotating unbalance of 0.4 kg m. Determine the stiffness of an un-damped isolator such that the force transmitted to the foundation is less than 2000 N.
137. A 200 kg machine operates at speeds between 1200 and 1800 rpm. The turbine has a rotating unbalance of 0.2 kg m. Determine the stiffness of an un-damped isolator such that the maximum force transmitted to the turbine's foundation is 800 N.
138. An exhaust fan of mass 50 kg has an operating speed of 1200 rpm is supported by four identical springs. Determine the stiffness of each spring if 90% isolation should be provided.
139. A machine of mass 120 kg operates at 1000 rpm. Determine the equivalent of an isolator that provides 85% isolation. Assume that the damping ratio of the isolator is .
140. A machine of mass 120 kg supported on springs of equivalent stiffness 800 kN/m has a rotating unbalance force of 4000 N at a speed of 4000 rpm. If the damping ratio is 0.16, determine **(a)** the amplitude caused by the unbalance and its phase angle, **(b)** the transmissibility and **(c)** the actual force transmitted.
141. A vibrating system is to be isolated from its foundation. Determine the required damping ratio that must be achieved by the isolator to limit the transmissibility ratio of 3 at the resonance.
142. The mass of a variable speed motor has a mass of 180 kg. The motor is mounted on an isolator having an equivalent stiffness of 8 kN/m and an equivalent damping ratio of 0.1. Determine the speed range over which the amplitude of the fluctuating force transmitted to the foundation will be larger than the exciting force. Also determine the speed range over which the transmitted force will be less than 15 percent of the exciting force amplitude.
143. A machine of mass 400 kg operating at 1500 rpm produces a repeating force of 25 kN when attached to a rigid foundation. Determine the damping constant of a viscously damped isolator such that the amplitude of steady state vibration at the operating speed should be less than 3 mm and the amplitude of steady state vibration during start up should not exceed 15 mm.
144. A rotating machine with a mass of 180 kg has some rotating unbalance and operates between 1000 and 2000 rpm. When it is placed directly on the foundation, the force transmitted to the foundation at 1000 rpm is 5 kN and at 2000 rpm is 20 kN. Determine the equivalent stiffness of the isolator having a damping ratio of 0.1 to reduce the force transmitted to the floor to 4 kN over the operating speed range of the engine.
145. An electronic device of mass 1.5 kg is supported to the base through an un-damped isolator. During shipping, the base is subjected to a harmonic disturbance of amplitude of 2.5 mm and frequency 4 Hz. Design the equivalent stiffness of an isolator so that the displacement transmitted to the device is less than 10% of the base motion.

146. A block of mass 40 kg is attached to an isolator which has an equivalent stiffness 1 MN/m in and a damping coefficient 2 kN s/m. The base is given a harmonic displacement of amplitude 8 mm at a frequency of 40 Hz. Determine the steady state amplitude of the absolute displacement of the block.
147. A vehicle can be modeled as a single degree of freedom system with $m = 1000$ kg, $k = 500$ kN/m and $\xi = 0.1$. The road surface varies sinusoidally with an amplitude of 0.01 m and a wavelength of2 m. Determine the displacement amplitude of the vehicle when it travels through the road at a speed of40 km/h.
148. A seismic instrument with a natural frequency of 5 Hz and negligible damping is used to measure the vibration of a machine operating at 100 rpm. The displacement of the seismic mass as read from the instrument is 0.05 mm. Determine the amplitude of vibration of the machine.
149. A vibrometer has period of free vibration of 2 s. It is attached to a machine with a harmonic frequency of 1 Hz. If the vibrometer mass has an amplitude of 3 mm relative to the vibrometer frame, determine the amplitude of vibration of the machine.
150. A vibration pick-up has a natural frequency of 6 Hz, and a damping factor of 0.65. What is lowest frequency beyond which the amplitude can be measured within **(a)** one percent error, and **(b)** two percent error?
151. A vibration measuring device is used to measure vibration of a machine running at 100 rpm. If the natural frequency of the instrument is 4.5 Hz and it shows 0.05 mm. If the damping of the device is negligible, determine the displacement, velocity and acceleration amplitude of the machine.
152. Determine the limiting frequency for an accelerometer with 2% error having damping ratio of 0.70 and natural frequency of 72 Hz.
153. Determine the smallest natural frequency of an accelerometer of damping ratio 0.25 that measures to vibrations of a body vibrating at 180 Hz with an error of a 2%.
154. A spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 2 kN/m and a viscous damper with a damping ratio of 0.2. Determine the expression steady state vibration when it is excited by an external harmonic force

    (a) $250 \sin 25t + 100 \sin 50t$.
    (b) $250 \sin 25t + 100 \cos 50t$.

155. A 500 kg machine is attached to a spring of stiffness 3 MN/m. The block is excited by the periodic excitation shown in **Figure P4.155**. Determine the approximate maximum displacement of steady state vibration of the machine by considering

    (a) the first harmonic only,
    (b) the first two harmonics and
    (c) the first three harmonics.

Also compare the responses for each case with plots.

156. A 750 kg machine is attached to a spring of stiffness 4 MN/m in parallel with a viscous damper such that the system's damping ratio is 10.15. The block is excited by the periodic excitation shown in **Figure P4.156**. Determine the approximate maximum displacement of steady state vibration of the machine by considering

(a) the first harmonic only,
(b) the first two harmonics and
(c) the first three harmonics.

Also compare the responses for each case with plots.

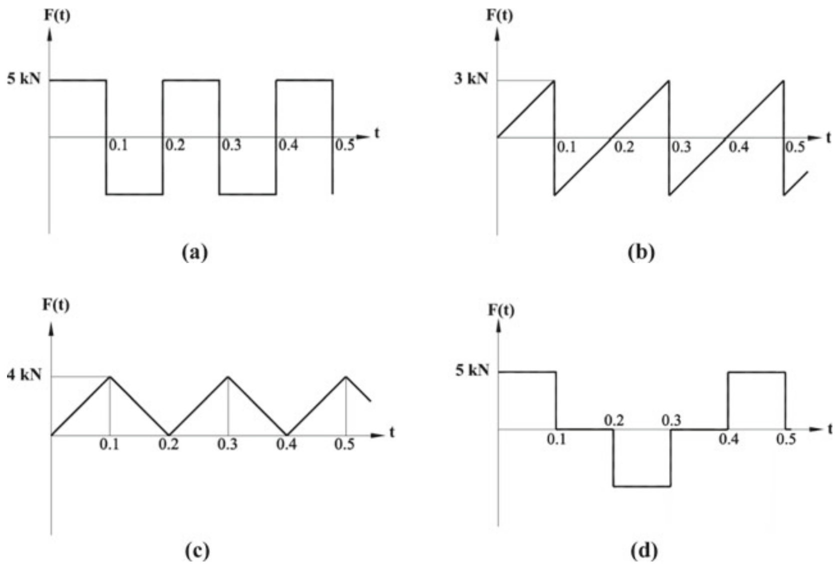


Figure P4.155

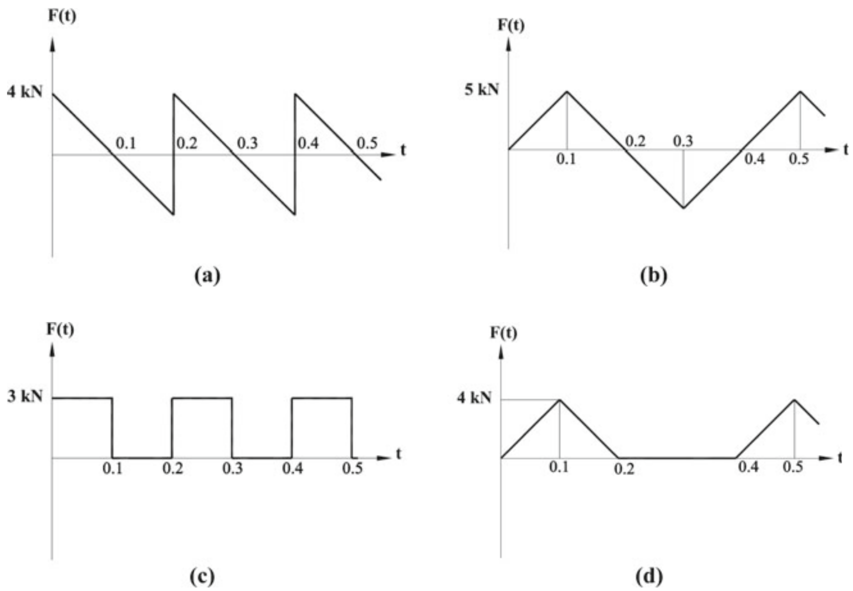


Figure P4.156

157. Use the convolution integral to determine the response of a single degree of freedom system with the natural frequency of and mass of when it is subject to a constant force of magnitude . The system is at rest in equilibrium at . Assume that

   (a) the system is un-damped,
   (b) the system is under-damped with a damping ratio of ,
   (c) the system is over-damped with a damping ratio of and
   (d) the system is critically damped.

158. Use the convolution integral to determine the response of a single degree of freedom system with the natural frequency of and mass of when it is subject to a time-dependent force . The system is at rest in equilibrium at . Assume that

   (a) the system is un-damped,
   (b) the system is under-damped with a damping ratio of ,
   (c) the system is over-damped with a damping ratio of and
   (d) the system is critically damped.

159. Use the convolution integral to determine the response of a single degree of freedom system with the natural frequency of and mass of when it is subject to a delayed impulse shown in **Figure P4.159**. Assume that

   (a) the system is under-damped with a damping ratio of ,

(b) the system is over-damped with a damping ratio of and
(c) the system is critically damped.

160. Use the convolution integral to determine the response of a single degree of freedom system with the natural frequency of and mass of when it is subject to a delayed step function shown in **Figure P4.160**. Assume that

(a) the system is under-damped with a damping ratio of ,
(b) the system is over-damped with a damping ratio of , and
(c) the system is critically damped.

161. A punch press is subject to impulses of magnitude 8 N s at and at $t = 2$ s. The mass of the press is 12 kg, and it is mounted on an elastic foundation with a stiffness of 16 kN/m and damping ratio of 0.1. Determine the response of the press.
162. Determine the response of an under-damped single degree of freedom system when subject to the excitation of **Figure P4.162** by using

(a) the convolution integral and
(b) the Laplace transform method.

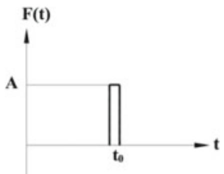


Figure P4.159

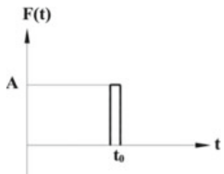


Figure P4.160

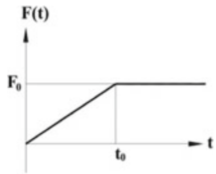


Figure P4.162

163. A machine tool with a mass of 20 kg is mounted on an un-damped foundation of stiffness 2000 N/m. During operation, it is subject to transient force shown in **Figure P4.163**. Determine the response of the system to each force.

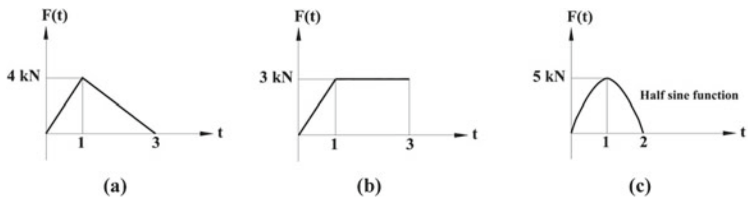


Figure P4.163

164. Determine the power required to vibrate a spring-mass system with an amplitude of 10 cm and at a frequency of 120 Hz. The system has a damping factor 0.1 and a damped natural frequency of 30 Hz. The mass of the system is 1 kg.

165. A single degree of freedom system consists of a mass of 25 kg and a spring of stiffness 5 kN/m. The amplitudes of successive cycles are found to be40, 35, 30 mm, …. Determine the nature of damping, damping constant and the frequency of vibration.
166. A block with a spring is placed on a rough surface and initially displaced 10 cm from equilibrium. It is observed that the period of motion is 0.5 s and that the amplitude decreases by 1 cm on successive cycles. Determine the coefficient of friction and how many cycles of motion the block executes before motion ceases.
167. A mass of 16 kg vibrates on a rough surface due to the action of a spring having a stiffness of 10 kN/m. When it is initially displaced by 15 cm the amplitude of vibration reduces to 7 cm after four complete cycles. Determine the coefficient of friction between the two surfaces and the time elapsed during the four cycles.
168. A block of mass 10 kg is attached to a spring of stiffness 4 kN/m and is released after giving an initial displacement of 120 mm. The coefficient of friction between the block and the surface is 0.1. Determine the position at which the block comes to rest.
169. A spring-mass system, having a mass of 15 kg and a spring of stiffness of 3 kN/m, vibrates on a rough surface. The coefficient of friction is 0.12. Determine amplitude of steady state vibration when the system is subjected to a harmonic excitation of 200 sin $10t$.
170. A cantilever beam can be modeled as a single degree of freedom system with a mass of 5 kg and a stiffness of 500 N/m. The mass is displaced initially by 25 mm and released. If the amplitude is found to be 15 mm after 80 cycles, determine the hysteretic damping constant of the beam.
171. A 120 kg machine is mounted at the mid-span of a 2 m simply supported beam. The moment of area of the section of the beam is 2×10^{-6} m^4 and modulus of elasticity of 210 GPa. The machine has a rotating unbalance of 0.5 kg m and operates at 2000 rpm. It is found from the free vibration test that the ratio of amplitudes on successive cycles is 1.8. Determine the steady state amplitude of vibration induced by the rotating unbalance. Assume the damping is hysteretic
172. A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. Determine the expression for its response using the method of Laplace transform when it is displaced by 0.05 m from the equilibrium position and released with an initial velocity of 0.5 m/s.
173. A single degree of freedom consists of a mass of 10 kg and a spring with a stiffness of 1 kN/m. Determine the expression for its response using the method of Laplace transform when it is displaced by 0.05 m from the equilibrium position and released with an initial velocity of 0.5 m/s.

   (a) 120 N s/m,
   (b) 200 N s/mand
   (c) 300 N s/m.

174. A spring-mass-damper system consists of a mass of 10 kg, a spring with a stiffness of 1 kN/m and a damping constant of 15 N s/m. It is excited by an external harmonic force $200 \sin 20t$. It is initially displaced by from the equilibrium position and released. Derive the expression for the total response (t) of the system using the method of Laplace transform.
175. A spring-mass-damper system consists of a mass of 15 kg, a spring with a stiffness of 1 kN/m and a damping constant of 20 N s/m. It is excited by an external harmonic force $300 \sin 40t$. Derive the expression for the steady state response of the system using the method of Laplace transform.
176. A machine tool with a mass of 10 kg is mounted on an un-damped foundation of stiffness 1000 N/m. During operation, it is subject to transient force shown in **Figure P4.176**. Determine the response of the system using the method of Laplace transform.

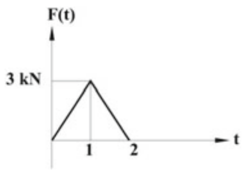


Figure P4.176

Answers

1. 44.2945 rad/s
2. 0.0913 s, 0.1118 s
3. 3.4722 kg, 1388.89 N/m
4. 2.8125 kg, 11.103 kN/m
5. 450 kg, 720 kN/m
6. 45.45 kg, 28.711 kN/m
7. $0.96k_1$
8. $0.5625k_1$
9. 44.7214 rad/s
10. 0.25 s
11. 0.13098, 7.6346
12. 1777.78 N/m, 45.0316 kg
13. 15 rad/s, 7.454 rad/s, 11.952 rad/s
14. 12.649 rad/s
15.
    (a). $\frac{L}{a}\sqrt{\frac{k}{m}}$
    (b). $\sqrt{\frac{L^2 k_1 k_2}{m\left(L^2 k_1 + a^2 k_2\right)}}$

16.

(a). $\sqrt{\frac{3k}{M}}; \sqrt{\frac{3k}{M+m}}$
(b). $\sqrt{\frac{5k}{M}}; \sqrt{\frac{15k}{3M+5m}}$

17.

(a). 25.298 rad/s
(b). 24.842 rad/s

18. $\sqrt{\frac{3\left(11kL^2+16k_t\right)}{L^2(7M+57m)}}$
19. $\frac{a}{b}\sqrt{\frac{k}{m}}$
20. 5.164 rad/s
21. $2\left(1+\frac{a}{r}\right)\sqrt{\frac{k}{3m}}$
22.

(a). 5.018 rad/s
(b). 9.559 rad/s
(c). 6.134 rad/s
(d). 6.134 rad/s
(e). 5.599 rad/s
(f). 7.513 rad/s

23.

(a). 5.571 rad/s
(b). 6.222 rad/s

24. $\sqrt{\frac{T}{m}\left(\frac{1}{a}+\frac{1}{b}\right)}$
25. $\sqrt{\frac{(l_1+l_2)^2k_1k_2}{m\left(l_1^2k_1+l_2^2k_2\right)}}$
26. $2\sqrt{\frac{k}{3m}}$
27. $\sqrt{\frac{8g}{r(9\pi-16)}}$
28. $\sqrt{\frac{2g}{3r}}$
29. $\sqrt{\frac{8g}{3\pi r}}$
30. $\sqrt{\frac{6k}{m}}$
31. 396.8627 rad/s
32. 36.2658 kg
33. 122.1158 rad/s
34. 31.1805 rad/s
35. 1.25
36. 208.33 GPa
37. 209.398 GPa

38. 1.4868
39. $9.7913 \times 10^{-9} \text{m}^4$
40. 36.5063 kgm^2
41.
    (a). $\sqrt{\frac{192EIk}{m(192EI+kL^3)}}$
    (b). $\sqrt{\frac{243EIk}{m(243EI+4kL^3)}}$
42.
    (a). 271.2883 rad/s
    (b). 264.0423 rad/s
43.
    (a). 146.3384 rad/s, 129.0674 rad/s, 160.7550 rad/s
    (b). 82.4386 rad/s, 78.9395 rad/s, 84.7753 rad/s
44. $0.1\cos(4.4721t)$, $-0.4472\sin(4.4721t)$, $-2\cos(4.4721t)$
45. $0.2582\sin(3.8729t)$, $\cos(3.8729t)$, $-3.8729\sin(3.8729t)$
46. $0.1936\sin(2.5819t) + 0.05\cos(2.5819t)$, $0.5\cos(2.5819t) - 0.1291\sin(2.5819t)$,$-1.2910\sin(2.5819t) - 0.3333\cos(2.5819t)$
47. 1.5811 m/s
48. 0.0663 m
49. $m\sqrt{\frac{2gh}{k(M+m)}}\sin\left\{\left(\sqrt{\frac{k}{M+m}}\right)t\right\} - \frac{mg}{k}\cos\left\{\left(\sqrt{\frac{k}{M+m}}\right)t\right\}$
50. $0.009983\sin(313.7475t) - 0.000099\cos(313.7475t)$
51.
    (a). $e^{-0.5t}(0.01125\sin 4.4441t + 0.1\cos 4.4441t)$, $e^{-0.5t}(-0.4500\sin 4.4441t)$, $e^{-0.5t}(0.2251\sin 4.4441t - 2\cos 4.4441t)$
    (b). $e^{-4.4721t}(0.1 + 0.4472t)$, $-2te^{-4.4721t}$, $e^{-4.4721t}(-2 + 8.9443t)$
    (c). $0.1618e^{-2.7639t} - 0.0618e^{-7.2361t}$, $-0.4472e^{-2.7639t} + 0.4472e^{-7.2361t}$, $1.2361e^{-2.7639t} - 3.2361e^{-7.2361t}$
52.
    (a). $e^{-0.25t}(0.2587\sin 3.8649t)$, $e^{-0.25t}(-0.0647\sin 3.86493.8649t + \cos 3.86493.8649t)$, $e^{-0.25t}(-3.8487\sin 3.8649t - 0.5\cos 3.8649t)$
    (b). $te^{-3.8729t}$, $e^{-3.8729t}(1 - 3.8729t)$, $e^{-3.8729t}(-7.7459 + 15t)$
    (c). $0.1581e^{-1.8377t} - 0.1581e^{-8.1623t}$, $-0.2906e^{-1.8377t} + 1.2906e^{-8.1623t}$, $0.5339e^{-1.8377t} - 10.53391e^{-8.1623t}$
53.
    (a). $e^{-0.1667t}(0.1973\sin 2.5766t + 0.05\cos 2.5766t)$, $e^{-0.1667t}(-0.1617\sin 2.5766t + 0.05\cos 2.5766t)$, $e^{-0.1667t}(-1.2614\sin 2.5766t - 0.05\cos 2.5766t)$

(b). $e^{-2.5819t}(0.05 + 0.6291t)$, $e^{-2.5819t}(0.5 - 1.6243t)$,
$e^{-2.5819t}(-2.9153 + 4.1940t)$
(c). $0.1831e^{-1.2251t} - 0.1331e^{-5.4415t}$, $-0.2243e^{-1.2251t} + 0.7243e^{-5.4415t}$,
$0.2749e^{-1.2251t} - 3.9415e^{-5.4415t}$

54. 489.8979 N s/m
55.
    (a). $5.4772\sqrt{km}$
    (b). $10.3923\sqrt{km}$
56.
    (a). 259.8076Ns/m
    (b). 1039.2305 Ns/m
57. 122.4745Ns/m
58. 44.1942Ns/m
59. $e^{-1.25t}(0.0356 \sin 31.5981t + 0.1 \cos 31.5981t)$
60. $11 cycles$
61.
    (a). 1323.2955Ns/m
    (b). 199.6525 Ns/m
    (c). 446.4365 Ns/m
    (d). 2035.2450 Ns/m
62.
    (a). 45.0185 Ns/m
    (b). 35.4024 Ns/m
    (c). 454.4300 Ns/m
    (d). 169.0759 Ns/m
63.
    (a). $e^{-2.1929t}(0.1724 \sin 7.0720t + 0.1 \cos 7.0720t)$
    (b). $e^{-1.8541t}(0.2262 \sin 5.2406t + 0.1 \cos 5.2406t)$
    (c). $e^{-1.5052t}(0.2133 \sin 5.3938t + 0.1 \cos 5.3938t)$
    (d). $e^{-0.3763t}(0.2078 \sin 4.9945t + 0.1 \cos 4.9945t)$
64.
    (a). 19 cycles
    (b). 2 cycles
    (c). 4 cycles
    (d). 17 cycles
65. 40 Ns/m, 9.7979 rad/s, 0.09995 m, 0.78015 m/s
66. 244.949 Ns/m, 8.165 rad/s, 0.1 m, 1 m/s
67. 1000 N/m, 240 Ns/m, 0.16075 m, -1.80901 m/s

68. 10 rad/s
69. 0.0798 m, 1.0189 m/s
70. 0.9949, 0.9682, 0.8660
71. 0.6931, 0.1097
72. 0.1283 s, 0.0622 m
73. 19.8997 Ns/m
74. 259.4959 N s/m
75. 462.3091 Ns/m
76.
    (a). 0.01 s, 0.0275 m
    (b). 0.00693 s, 0.0125 m
77. 3.4147 Ns/m
78. 0.1021, 6.0917 rad/s, 0.6446, 1.9053, 3 cycles
79. 0.3219, 0.0512, 1266.6249 N/m
80. 296 kN/m, 3.1937 kNs/m
81. 12639.2434 kN/m, 22.1807 Ns/m
82.
    (a). Viscous, 6377.3142 kN/m, 98.6042 Ns/m
    (b). Coloumb, 6316.5468 kN/m, 0.01409
83.
    (a). $-0.9744\ \sin\ 20t + 0.05\ \cos 20t + 1.0256 \sin 19t$
    (b). $0.05\ \sin\ 20t + 0.05 \cos 20t - t \cos 20t$
    (c). $1.0244 \sin 20t + 0.05 \cos 20t - 0.9756 \sin 21t$
84.
    (a). $-0.9494 \sin 20t + 1.0256 \sin 19t$
    (b). $0.075 \sin 20t - t \cos 20t$
    (c). $1.0494 \sin 20t - 0.9756 \sin 21t$
85.
    (a). $-0.9494 \sin 20t + 0.05 \cos 20t + 1.0256 \sin 19t$
    (b). $0.075 \sin 20t + 0.05 \cos 20t - t \cos 20t$
    (c). $1.0494 \sin 20t + 0.05 \cos 20t - 0.9756 \sin 21t$
86. 11.5253 kg
87. 30 rad/s
88.
    (a). $e^{-0.75t}(-0.5902 \sin 9.9718t + 0.5470 \cos 9.9718t) +$
    $0.8581 \sin(9t - 0.6178)$
    (b). $e^{-0.75t}(0.1040 \sin 9.9718t + 1.3833 \cos 9.9718t) - 1.3333 \cos 10t$

(c). $e^{-0.75t}(0.6881 \sin 9.9718t + 0.5127 \cos 9.9718t) +$
$0.7488 \sin(11t + 0.6659)$

89.

(a). $e^{-0.75t}(-0.5438 \sin 9.9718t + 0.4970 \cos 9.9718t) +$
$0.8581 \sin(9t - 0.6178)$
(b). $e^{-0.75t}(0.1504 \sin 9.9718t + 1.3333 \cos 9.9718t) - 1.3333 \cos 10t$
(c). $e^{-0.75t}(0.7345 \sin 9.9718t + 0.4627 \cos 9.9718t) +$
$0.7488 \sin(11t + 0.6659)$

90.

(a). $e^{-0.75t}(-0.5400 \sin 9.9718t + 0.5471 \cos 9.9718t) +$
$0.8581 \sin(9t - 0.6178)$
(b). $e^{-0.75t}(0.1541 \sin 9.9718t + 1.3833 \cos 9.9718t) - 1.3333 \cos 10t$
(c). $e^{-0.75t}(0.7382 \sin 9.9718t + 0.5127 \cos 9.9718t) +$
$0.7488 \sin(11t + 0.6659)$

91. 11.5470 rad/s, 11.4261 rad/s, 90°, 0.05196 m,11.3039 rad/s, 0.05251 m
92. 0.05816 m
93. 28.2843 rad/s, 27.7128 rad/s, 3.6058, 0.1178 m,34.4389°
94. 219.5524 kN/m, 190.9363 Ns/m
95. 0.1419
96. 79.5774 Ns/m
97. 432.0539 Ns/m
98. 43.6191 kN/m, 0.3228
99. 3.0669 MN/m, 1.1359 kNs/m
100. 0.00274 m, 0.00257 m
101. 0.1952
102. 6.4139 kN
103. 0.2354
104. 0.00633 m
105. 0.000269 m
106. 1.2063×10^{-6} m^4
107. 0.2632
108. $\omega < 40.620$ rad/s,$\omega > 68.191$ rad/s
109. 0.0956
110.

(a). 0.007096 m
(b). 0.005822 m
(c). 0.002539 m
(d). 0.002220 m
(e). 0.001922 m

111.

(a). 7.8017°

(b). $14.1471°$
(c). $6.5496°$
(d). $6.8898°$
(e). $6.9805°$

112. $2.5278°$
113. $11.2100°$
114.
    (a). 0.002378 m
    (b). 0.005804 m
    (c). 0.001214 m
115. 223.4611 N
116. $\omega < 7.1274$ rad/s and $\omega > 12.1506$ rad/s
117.
    (a). $14.1013°$
    (b). $8.8399°$
    (c). $7.1649°$
118. 623.1745 N
119. $\omega < 12.8507$ rad/s and $\omega > 32.4028$ rad/s
120.
    (a). $1.4795°$
    (b). $2.1482°$
121.
    (a). $8.8579°$
    (b). $6.3934°$
122. 0.16 kgm
123. 0.225 kgm, 263.6719 kN/m
124. 0.002780 m
125. 0.7898 kgm
126. 0.005743 m, 515.6136 rpm
127.
    (a). $k < 15.7914$ kN/m and $k > 394.7842$ kN/m
    (b). $k < 16.0487$ kN/m and $k > 388.4547$ kN/m
128. 0.001263 m
129. 10.4719 rad/s, 0.1333, 0.001342 m, $-1.2819°$
130. 24 mm, 40 kg
131. 0.002991 m
132. $I < 0.51711 \times 10^{-6}\ \mathrm{m}^4$ and $I > 5.1277 \times 10^{-6}\ \mathrm{m}^4$
133. 62.9781%
134. 116.9731 kN/m

135. $k < 79.7544$ kN/m
136. 412.1034 kN/m
137. 319.5778 kN/m
138. 17.9447 kN/m
139. 150.9121 kN/m
140. 0.000197 m, −3.7098°; 0.0758, 303.0499 N
141. 0.1768
142. 187.9911 rpm
143. 0.1810
144. 869.4659 kN/m
145. 86.1347 N/m
146. 5.5709 mm
147. 0.007525 m
148. 0.4 mm
149. 2.25 mm
150. 20.0821 Hz, 9.3652 Hz
151. 0.3145 mm, 3.2934 mm/s, 34.4887 mm/s^2
152. 152.1138 Hz
153. 26.9868 Hz
154.
    (a). $0.0558\sin(25t + 0.3212) + 0.0043\sin(50t + 0.1224)$
    (b). $0.0558\sin(25t + 0.3212) + 0.0043\cos(50t + 0.1224)$
155.
    (a). $0.002539\sin(31.4159t) + 0.001472\sin(94.2478t) + 0.000136\sin(157.0796t)$
    (b). $0.000762\sin(31.4159t) - 0.000931\sin(62.8319t) + 0.000442\sin(94.2478t)$
    (c). $0.000667 - 0.000647\cos(31.4159t) - 0.000125\cos(92.2478t) - 0.000007\cos(157.0796t)$
    (d). $0.001565\sin(15.7079t + 0.7854) + 0.000794\sin(47.1239t - 0.7854) + 0.007556\sin(78.5398t + 0.7854)$
156.
    (a). $0.000772\sin(31.4159t - 0.1571) + 0.000794\sin(94.2478t + 0.5269) + 0.000035\sin(157.0796t + 0.1761)$
    (b). $0.00106\sin(15.7079t - 0.0676) - 0.000183\sin(47.1239t - 0.3203) + 0.000113\sin(78.5398t + 1.1189)$
    (c). $0.000375 + 0.000579\sin(31.4159t - 0.1571) + 0.000207\sin(94.2478t + 0.5269) + 0.000026\sin(157.0796t + 0.1761)$
    (d). $0.00025 + 0.000424\sin(15.7079t - 0.0676) - 0.000246\cos(31.4159t - 0.1571) - 0.000073\sin(47.1239t - 0.3202) + 0.000045\sin(78.5398t + 1.1189)$

157.

(a). $\frac{1}{m\omega_n^2}[1-\cos(\omega_n t)]$

(b). $\frac{F}{m\omega_n^2}\left[1-\mathrm{e}^{-\omega_n t}\left\{\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)+\cos(\omega_d t)\right\}\right]$

(c). $$\frac{F}{2m\omega_n^2\sqrt{\xi^2-1}}\left[\left(-\xi-\sqrt{\xi^2-1}\right)\mathrm{e}^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}-\left(-\xi+\sqrt{\xi^2-1}\right)\right.$$
$$\left.\mathrm{e}^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}+2\sqrt{\xi^2-1}\right]$$

(d). $\frac{1}{m\omega_n^2}\left[1-\mathrm{e}^{-\omega_n t}(1+\omega_n t)\right]$

158.

(a). $\frac{F_0}{m\left(\alpha^2+\omega_n^2\right)}\left[\mathrm{e}^{-\alpha t}+\frac{\alpha}{\omega_n}\sin(\omega_n t)-\omega_n\cos(\omega_n t)\right]$

(b). $\frac{F_0}{m\left(\alpha^2-2\alpha\xi\omega_n+\xi^2\omega_n^2+\omega_d^2\right)}\left[\mathrm{e}^{-\alpha t}+\mathrm{e}^{-\xi\omega_n t}\left(\frac{\alpha-\xi\omega_n}{\omega_d}\sin(\omega_d t)-\cos(\omega_d t)\right)\right]$

(c). $$\frac{F_0\mathrm{e}^{\left\{-\alpha+\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n\right\}t}}{2m\omega_n^2\left(\alpha^2-2\alpha\xi\omega_n+\xi^2\omega_n^2\right)\sqrt{\xi^2-1}}$$
$$\left[\left(\alpha-\xi-\sqrt{\xi^2-1}\right)\omega_n\mathrm{e}^{\left(\alpha+2\omega_n\sqrt{\xi^2-1}\right)t}\right.$$
$$\left.+2\omega_n\sqrt{\xi^2-1}\mathrm{e}^{\left(\xi+\sqrt{\xi^2-1}\right)\omega_n t}-\left(\alpha-\xi+\sqrt{\xi^2-1}\right)\mathrm{e}^{-\alpha t}\right]$$

(d). $\frac{F_0}{m\left(\alpha^2-2\alpha\omega_n+\omega_n^2\right)}\left[\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\omega_n t}\{1-(\alpha-\omega_n)t\}\right]$

159.

(a). $u(t-t_0)\frac{A}{m\omega_d}\mathrm{e}^{-\xi\omega_n(t-t_0)}\sin\{\omega_d(t-t_0)\}$

(b). $u(t-t_0)\frac{A}{2m\omega_n^2\sqrt{\xi^2-1}}\left[\mathrm{e}^{\left\{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n(t-t_0)\right\}}-\mathrm{e}^{\left\{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n(t-t_0)\right\}}\right]$

(c). $u(t-t_0)\frac{A}{m}(t-t_0)\mathrm{e}^{-\omega_n(t-t_0)}$

160.

(a). $u(t-t_0)\frac{1}{m\omega_n^2}\left[1-\mathrm{e}^{-\omega_n(t-t_0)}\left\{\frac{\xi\omega_n}{\omega_d}\sin\omega_d(t-t_0)+\cos\omega_d(t-t_0)\right\}\right]$

(b). $$u(t-t_0)\frac{A}{2m\omega_n^2\sqrt{\xi^2-1}}$$
$$\left[\left(-\xi-\sqrt{\xi^2-1}\right)\mathrm{e}^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n(t-t_0)}\right.$$
$$\left.-\left(-\xi+\sqrt{\xi^2-1}\right)\mathrm{e}^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n(t-t_0)}+2\sqrt{\xi^2-1}\right]$$

(c). $u(t-t_0)\frac{1}{m\omega_n^2}\left[1-\mathrm{e}^{-\omega_n(t-t_0)}\{1+\omega_n(t-t_0)\}\right]$

161. $$0.0183\mathrm{e}^{-3.6515t}\sin(36.3318t)+u(t-2)0.0183$$
$$\mathrm{e}^{-3.6515(t-2)}\sin(36.3318t-72.6636)$$

$$\frac{F_0}{t_0 m\omega_n^3}[\omega_n t - \sin(\omega_n t)]$$

162. $$- u(t-t_0)\frac{F_0}{t_0 m\omega_n^3}[\omega_n t - \sin\{\omega_n(t-t_0)\} - \omega_n t_0 \cos\{\omega_n(t-t_0)\}]$$
$$- u(t-t_0)\frac{F_0}{m\omega_n^2}[1-\cos\{\omega_n(t-t_0)\}]$$

163.

(a). $$\begin{aligned}&[2t-0.2\sin(10t)] - u(t-1)[2t-0.2\sin(10t-10)-2\cos(10t-10)]\\ &\quad + u(t-1)[3-t+0.1\sin(10t-10)-2\cos(10t-10)]\\ &\quad - u(t-3)[3-t+0.1\sin(10t-30)]\end{aligned}$$

(b). $$\begin{aligned}&[1.5t-0.15\sin(10t)]\\ &\quad - u(t-1)[1.5t-0.15\sin(10t-10)-1.5\cos(10t-10)]\\ &\quad + u(t-1)[1.5-1.5\cos(10t-10)]\\ &\quad - u(t-3)[1.5-1.5\cos(10t-30)]\end{aligned}$$

(c). $$\begin{aligned}&\frac{50}{\pi^2-400}\left[\pi\sin(10t) - 20\sin\left(\frac{\pi}{2}t\right)\right]\\ &\quad - u(t-2)\frac{50}{\pi^2-400}\left[\pi\sin(10t-20) - 20\sin\left(\frac{\pi}{2}t\right)\right]\end{aligned}$$

164. 223.2452 W
165. Coulomb, 0.02548, 14.1421 rad/s
166. 0.04024, 10 cycles
167. 0.3186, 1.0053 s
168. 2.28 mm
169. 0.1325 m
170. 0.002033
171. 0.007878 m
172. $0.05\sin 20t + 0.05\cos 20t$
173.

(a). $e^{-6t}(0.1\sin 8t + 0.05\cos 8t)$
(b). $e^{-10t}(0.05 + t)$
(c). $0.08090e^{-3.8197t} - 0.03090e^{-26.1803t}$

174. $e^{-0.75t}(0.1366\sin 9.9718t + 0.0566\cos 9.9718t) - 0.0066\sin 20t - 0.0066\cos 20t$
175. $-0.01303\sin 40t - 0.00045\cos 40t$
176. $[3t-0.3\sin(10t)] - u(t-1)[3t-0.3\sin(10t-10)-3\cos(10t-10)] + u(t-1)[6-3t+0.3\sin(10t-10)-3\cos(10t-10)] - u(t-2)[3-3t+0.3\sin(10t-20)]$

Chapter 5
Response of a Two Degree of Freedom System

5.1 Introduction

In the previous chapter, basic procedure of vibration analysis and associated phenomenon and terminologies were explained with reference to a single degree of freedom system model. However a single degree of freedom model will not give satisfactory result for all real physical systems undergoing vibrations. In such cases, a two degree of freedom system model can be used as a general extension of a single degree of freedom system.

A system is said to be a two degree of freedom system, when two independent coordinates are required to completely describe the motion of the system. Consider few examples shown in Fig. 5.1.

Motion of a spring-mass assembly shown in Fig. 5.1a can be defined by the displacements of each block x_1 and x_2. Likewise, a double pendulum shown in Fig. 5.1b requires two angular displacements θ_1 and θ_2 to describe it motion. Similarly, a bar supported by two springs shown in Fig. 5.1c requires either the displacement of center of gravity x and rotation of the bar about its c. g. θ or the vertical displacements of two ends of the bar x_1 and x_2 to define its motion. Hence the systems shown above have two degree of freedom.

In the following section we will show that a two degree of freedom system will have two natural frequencies. When the system vibrates at any one of the natural frequency, a definite relationship exists between two vibration amplitudes, which is defined as the mode shape of the system. Two degree of freedom system will therefore have two mode shapes. Free response of any two degree freedom of system due to any arbitrary initial disturbances will be superposition of these two modes.

Forced vibration of a two degree of freedom system occurs at the frequency equal to the frequency of the excitation force and the system undergoes resonance when the excitation frequency becomes equal to any one of the natural frequency of the system.

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_5

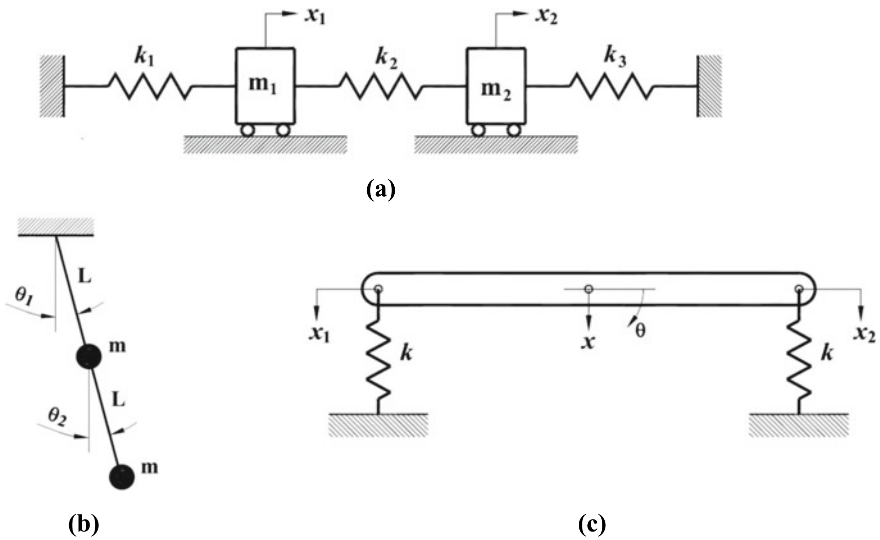


Fig. 5.1 Examples of two degree of freedom systems

5.2 Free Response of an Undamped Two Degree of Freedom System

Consider a spring-mass assembly system shown in Fig. 5.1a, which is the simplest model of a two degree of freedom system. Applying any one method explained in Chap. 3, equation of motion of the system can be derived as

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 &= 0 \end{aligned} \tag{5.1}$$

Equation (5.1) shows that equation of motion of a two degree of freedom system is a coupled linear differential equations, which is also called a system of linear differential equations. We can assume the solution for a system of linear differential equations as

$$\begin{aligned} x_1 &= A_1 \sin \omega_n t \\ x_2 &= A_2 \sin \omega_n t \end{aligned} \tag{5.2}$$

where A_1 and A_2 are the amplitudes of vibration of masses m_1 and m_2, respectively, and ω_n is the natural frequency of free vibration.

Substituting Eq. (5.2) into Eq. (5.1), we get a system of algebraic equations as

$$\left[(k_1 + k_2) - m_1\omega_n^2\right]A_1 - k_2A_2 = 0$$

$$-k_2A_1 + \left[(k_2 + k_3) - m_2\omega_n^2\right]A_2 = 0 \tag{5.3}$$

Rearranging both equations of Eq. (5.3) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{k_2}{(k_1 + k_2) - m_1\omega_n^2} \tag{5.4}$$

$$\frac{A_1}{A_2} = \frac{(k_2 + k_3) - m_2\omega_n^2}{k_2} \tag{5.5}$$

Equating Eqs. (5.4) and (5.5), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{k_2}{(k_1 + k_2) - m_1\lambda} = \frac{(k_2 + k_3) - m_2\lambda}{k_2}$$

$$\text{or,}\quad [\{(k_1 + k_2) - m_1\lambda\}\{(k_2 + k_3) - m_2\lambda\}] - k_2^2 = 0$$

$$\text{or,}\quad m_1m_2\lambda^2 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\lambda + (k_1k_2 + k_2k_3 + k_3k_1) = 0$$

$$\therefore\quad \lambda^2 - \left\{\left(\frac{k_1 + k_2}{m_1}\right) + \left(\frac{k_2 + k_3}{m_2}\right)\right\}\lambda + \left(\frac{k_1k_2 + k_2k_3 + k_3k_1}{m_1m_2}\right) = 0 \tag{5.6}$$

Equation (5.6) is quadratic on λ and hence gives two roots λ_1 and λ_2, from which two natural frequencies of the system can be determined as $\omega_1 = \sqrt{\lambda_1}$ and $\omega_2 = \sqrt{\lambda_2}$. Hence Eq. (5.6) is called a frequency equation or a characteristic equation of the system.

Substituting λ_1 and λ_2 into Eq. (5.4) or Eq. (5.5), we get the mode shapes corresponding to each natural frequency.

To have more specific interpretation, let us assume that $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$. Then Eq. (5.6) reduces to

$$\lambda^2 - 4\frac{k}{m}\lambda + 3\frac{k^2}{m^2} = 0$$

$$\text{or,}\quad \lambda_{1,2} = \frac{1}{2}\left[\frac{4k}{m} \pm \sqrt{\left(\frac{4k}{m}\right)^2 - \frac{12k}{m}}\right] = \frac{1}{2}\left[\frac{4k}{m} \pm \frac{2k}{m}\right]$$

$$\therefore\quad \lambda_1 = \frac{k}{m} \tag{5.7}$$

and

$$\lambda_2 = \frac{3k}{m} \tag{5.8}$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}} \tag{5.9}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3k}{m}} \tag{5.10}$$

Substituting $\lambda = \lambda_1 (= k/m)$ into Eq. (5.4), we get mode shape corresponding to the first natural frequency as

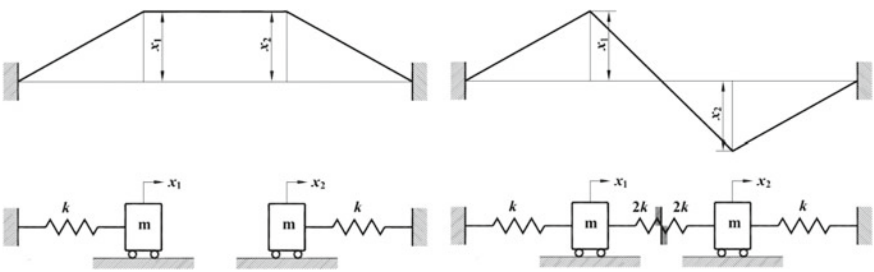
$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{2k - m\lambda_1} = 1 \tag{5.11}$$

Similarly, substituting $\lambda = \lambda_2 (= 3k/m)$ into Eq. (5.5), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k}{2k - m\lambda_2} = -1 \tag{5.12}$$

Mode shape $(A_1/A_2)_1 = 1$ means that when the system vibrates with a frequency of $\omega_1 = \sqrt{k/m}$, both masses move in same direction (in phase) with same amplitudes. Similarly, mode shape $(A_1/A_2)_2 = -1$ means that when the system vibrates with a frequency of $\omega_2 = \sqrt{3k/m}$, both masses move in opposite direction (out of phase) with same amplitudes. Mode shapes can also be demonstrated graphically as shown in Fig. 5.2.

It can be noted from the mode shape of Fig. 5.2a that when the both masses move in phase with each other with the same amplitudes, there will be no effect on the spring that is connected between two masses. Hence the equivalent system for this mode will be a system with a spring of stiffness k and mass m and which will have a natural frequency of $\sqrt{k/m}$.



(a) Mode Shape Corresponding to $\omega_1 = \sqrt{k/m}$

(b) Mode Shape Corresponding to $\omega_2 = \sqrt{3k/m}$

Fig. 5.2 Mode shapes for an un-damped two degree of freedom system

Similarly, it can also be noted from the mode shape of Fig. 5.2b that when the both masses move out of phase with each other with the same amplitudes, there will be no effect at the middle of the spring that is connected between two masses. Therefore the middle spring is symmetrically compressed and elongated alternately and be considered as two springs of stiffness $2k$. Hence the equivalent system for this mode will be a system with springs of stiffness values k and $2k$ attached to two opposite sides of the block with mass m and which will have a natural frequency of $\sqrt{3k/m}$.

Comparing two mode shapes, it can be noted that the first mode has two stationary points, i.e., left end and right end of the system, whereas the second mode has three stationary points, i.e., the left end point, the right end point and the middle point of the system. Such stationary points on a vibrating system are called nodes. The mode corresponding to minimum number of nodes is called a fundamental mode of the system.

Mode shapes of a vibrating system can also be understood in reverse way. When both the masses are shifted by same amount from the equilibrium position in the same direction and released then the system will vibrate with a first natural frequency ($\omega_1 = \sqrt{k/m}$). Similarly, when both the masses are shifted by same amount from the equilibrium position in the opposite direction and released then the system will vibrate with a second natural frequency ($\omega_2 = \sqrt{3k/m}$) whereas if two masses are subjected to any arbitrary initial disturbances, then the response of the system will be superposition of both modes, i.e.,

$$x_1 = A_{11} \sin(\omega_1 t + \psi_1) + A_{12} \sin(\omega_2 t + \psi_2) \tag{5.13}$$

$$x_2 = A_{21} \sin(\omega_1 t + \psi_1) + A_{22} \sin(\omega_2 t + \psi_2) \tag{5.14}$$

Since A_{21} is dependent on A_{11} and A_{22} is dependent on A_{12}, Eqs. (5.13) and (5.14) have four unknown arbitrary constants A_{11}, A_{12}, ψ_1 and ψ_2. These constants can be determined by applying initial conditions for the displacements and the velocities of each mass, i.e., $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$.

5.3 Free Response of a Damped Two Degree of Freedom System

Consider a damped two degree of freedom system shown in Fig. 5.3. Applying any one method explained in Chap. 3, equation of motion of the system can be derived as

$$\begin{aligned} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2 + k_3)x_2 - k_2x_1 = 0 \end{aligned} \tag{5.15}$$

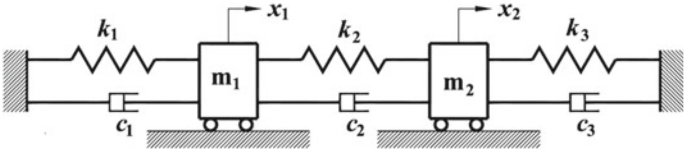


Fig. 5.3 Damped two degree of freedom system

Equation (5.15) shows that equation of motion of a damped two degree of freedom system is also a coupled linear differential equations or a system of linear differential equations. We can assume the solution for a system of linear differential equations as

$$x_1 = A_1 e^{st} \tag{5.16}$$

$$x_2 = A_2 e^{st}$$

where A_1 and A_2 are the amplitudes of vibration of masses m_1 and m_2, respectively.

Substituting Eq. (5.16) into Eq. (5.15), we get a system of algebraic equations as

$$\begin{aligned}&\left[m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)\right]A_1 - [c_2 s + k_2]A_2 = 0\\ &-[c_2 s + k_2]A_1 + \left[m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)\right]A_2 = 0\end{aligned} \tag{5.17}$$

Rearranging both equations of Eq. (5.17) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{c_2 s + k_2}{m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)} \tag{5.18}$$

$$\frac{A_1}{A_2} = \frac{m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)}{c_2 s + k_2} \tag{5.19}$$

Equating Eqs. (5.18) and (5.19), we get

$$\begin{aligned}&\frac{c_2 s + k_2}{m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)} = \frac{m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)}{c_2 s + k_2}\\ \text{or,}\quad &\left[\left\{m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)\right\}\left\{m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)\right\}\right]\\ &\quad - (c_2 s + k_2)^2 = 0\\ \text{or,}\quad &m_1 m_2 s^4 + \{(c_1 + c_2)m_2 + (c_2 + c_3)m_1\}s^3\\ &\quad + \left\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1 + (c_1 + c_2)(c_2 + c_3) - c_2^2\right\}s^2\\ &\quad + \{(k_1 + k_2)(c_2 + c_3) + (k_2 + k_3)(c_1 + c_2) - 2c_2 k_2\}s\\ &\quad + \left\{(k_1 + k_2)(k_2 + k_3) - k_2^2\right\} = 0\end{aligned}$$

$$\begin{aligned}\therefore \quad & s^4 + \left\{\left(\frac{c_1 + c_2}{m_1}\right) + \left(\frac{c_2 + c_3}{m_2}\right)\right\} s^3 \\ & + \left\{\left(\frac{k_1 + k_2}{m_1}\right) + \left(\frac{k_2 + k_3}{m_2}\right) + \frac{c_1 c_2 + c_2 c_3 + c_3 c_1}{m_1 m_2}\right\} s^2 \\ & + \left\{\frac{k_1(c_2 + c_3)}{m_1 m_2} + \frac{k_2(c_3 + c_1)}{m_1 m_2} + \frac{k_3(c_1 + c_2)}{m_1 m_2}\right\} s \\ & + \left(\frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{m_1 m_2}\right) = 0 \end{aligned} \tag{5.20}$$

Equation (5.20) is the frequency equation or a characteristic equation of the system shown in Fig. 5.3. This equation gives four roots of s as s_1, s_2, s_3 and s_4. Substituting each root into Eq. (5.18) or (5.19), we get amplitude ratios respectively as $((A_1/A_2)_1$, $(A_1/A_2)_2$, $(A_1/A_2)_3$ and $(A_1/A_2)_4$. Then damped free response of the two degree of freedom system is then given by

$$x_1 = A_{11}e^{s_1 t} + A_{12}e^{s_2 t} + A_{13}e^{s_3 t} + A_{14}e^{s_4 t} \tag{5.21}$$

$$x_2 = A_{21}e^{s_1 t} + A_{22}e^{s_2 t} + A_{23}e^{s_3 t} + A_{24}e^{s_4 t} \tag{5.22}$$

Since A_{2i} is dependent on A_{1i}, Eqs. (5.21) and (5.22) have four unknown arbitrary constants and these constants can be determined by applying initial conditions for the displacements and the velocities of each mass, i.e., $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$.

Since all coefficients of dependent variables (m_i, k_i and c_i) in the system of linear differential equation of Eq. (5.15) are positive, roots s_i of characteristic equation will be either negative real numbers or complex numbers with negative real parts or combination of these. If the roots are negative real numbers, the response will be a decaying aperiodic motion whereas if the roots are complex conjugates with the negative real part, then the response will be sinusoidal with decaying amplitude. If the roots are combination of negative real numbers and complex conjugates, then the response will be the superposition of aperiodic motion and sinusoidal oscillation with decaying amplitude.

5.4 Forced Harmonic Response of a Two Degree of Freedom System

5.4.1 Forced Harmonic Response of an Un-damped Two Degree of Freedom System

Consider an un-damped two degree of freedom system with an external harmonic force $F(t) = F_1 \sin \omega t$ acting upon the mass m_1 as shown in Fig. 5.4.

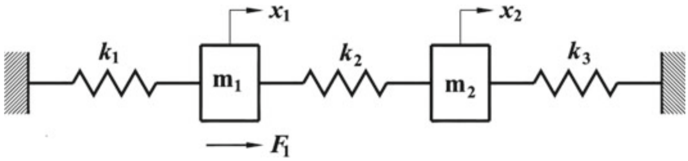


Fig. 5.4 Undamped two degree of freedom system subjected to an external harmonic force

The equations of motion of the system can be written as

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= F_1 \sin \omega t \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 &= 0 \end{aligned} \tag{5.23}$$

In the absence of damping, steady state response of the system can be assumed as

$$\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned} \tag{5.24}$$

where X_1 and X_2 are the amplitudes of steady state vibration of masses m_1 and m_2, respectively

Substituting Eq. (5.24) into Eq. (5.23), we get a system of algebraic equations as

$$\begin{aligned} \left[(k_1 + k_2) - m_1\omega^2\right]X_1 - k_2X_2 &= F_1 \\ -k_2X_1 + \left[(k_2 + k_3) - m_2\omega^2\right]X_2 &= 0 \end{aligned} \tag{5.25}$$

Solving simultaneous Eq. (5.25) for X_1 and X_2, we get

$$\begin{aligned} X_1 &= \frac{\left\{(k_2 + k_3) - m_2\omega^2\right\}F_1}{m_1m_2\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + (k_1k_2 + k_2k_3 + k_3k_1)} \\ X_2 &= \frac{k_2F_1}{m_1m_2\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + (k_1k_2 + k_2k_3 + k_3k_1)} \end{aligned} \tag{5.26}$$

If we assume that $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, then Eq. (5.26) reduces to

$$\begin{aligned} X_1 &= \frac{\left(2k - m\omega^2\right)F_1}{m^2\omega^4 - 4km\omega^2 + 3k^2} \\ X_2 &= \frac{kF_1}{m^2\omega^4 - 4km\omega^2 + 3k^2} \end{aligned} \tag{5.27}$$

Equation (5.27) can be expressed by factorizing the denominators as

$$
\begin{aligned}
X_1 &= \frac{\left(2k - m\omega^2\right)F_1}{m^2\left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - \frac{3k}{m}\right)} \\
X_2 &= \frac{kF_1}{m^2\left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - \frac{3k}{m}\right)}
\end{aligned} \tag{5.28}
$$

As we know $\omega_1^2 = k/m$ and $\omega_2^2 = 3k/m$, Eq. (5.28) can be expressed in terms of two natural frequencies ω_1 and ω_2 as

$$
\begin{aligned}
X_1 &= \frac{\left(2k - m\omega^2\right)F_1}{m^2\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right)} \\
X_2 &= \frac{kF_1}{m^2\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right)}
\end{aligned} \tag{5.29}
$$

Equation (5.29) shows that both steady state amplitudes X_1 and X_2 tend to infinity when the frequency of the external excitation approaches any one natural frequency. It is also shown graphically in Fig. 5.5.

5.4.2 *Forced Harmonic Response of a Damped Two Degree of Freedom System*

Consider a damped two degree of freedom system with an external harmonic force $F(t) = F_1 \sin \omega t$ acting upon the mass m_1 as shown Fig. 5.6.

The equation of motion of the system can be written as

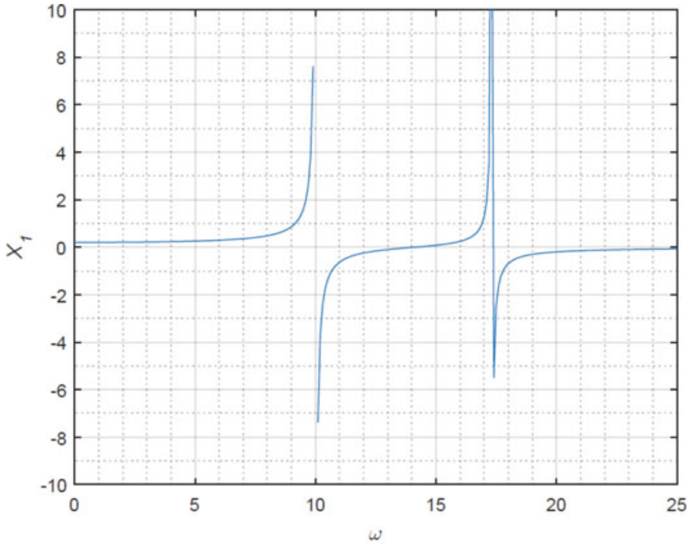
$$
\begin{aligned}
&m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \sin \omega t \\
&m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2 + k_3)x_2 - k_2x_1 = 0
\end{aligned} \tag{5.30}
$$

In the presence of damping, there will be phase difference between the excitation and the response of the system, then the steady state response of the system can be assumed as

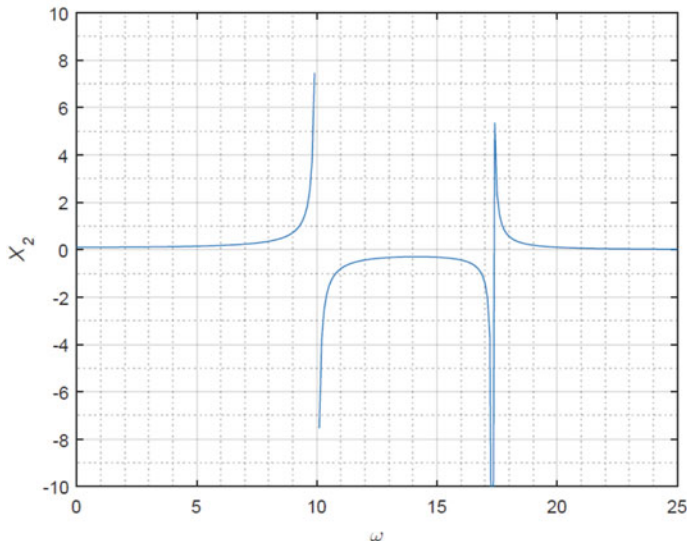
$$
\begin{aligned}
x_1 &= X_1 \sin \omega t + Y_1 \cos \omega t \\
x_2 &= X_2 \sin \omega t + Y_2 \cos \omega t
\end{aligned} \tag{5.31}
$$

Substituting Eq. (5.31) into Eq. (5.30), we get a system of algebraic equations as

$$
\begin{aligned}
&\left[\left\{(k_1 + k_2) - m_1\omega^2\right\}X_1 - (c_1 + c_2)\omega Y_1 - k_2X_2 + c_2\omega Y_2 - F_1\right] \sin \omega t \\
&\quad + \left[\{(c_1 + c_2)\omega\}X_1 + \left\{(k_1 + k_2) - m_1\omega^2\right\}Y_1 - c_2\omega X_2 - k_2Y_2\right] \cos \omega t = 0
\end{aligned} \tag{5.32}
$$



(a) Steady State Response of Mass m_1



(b) Steady State Response of Mass m_2

Fig. 5.5 Steady state response of the system of Fig. 5.4 for $k_1 = k_2 = k_3 = k = 1000\,\text{N/m}$, $m_1 = m_2 = m = 10\,\text{kg}$ and $F_1 = 300\,\text{N}$

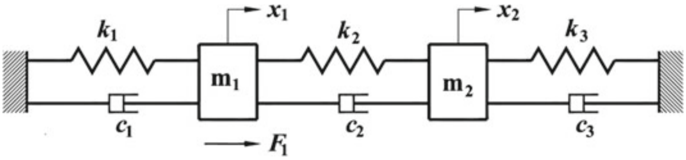


Fig. 5.6 Damped two degree of freedom system subjected to an external harmonic force

$$\begin{aligned}&\left[-k_2X_1+\left\{(k_2+k_2)-m_2\omega^2\right\}X_2-(c_2+c_3)\omega Y_2+c_2\omega Y_1\right]\sin\omega t\\&\quad+\left[-c_2\omega X_1+\{(c_2+c_3)\omega\}X_2+\left\{(k_2+k_3)-m_2\omega^2\right\}Y_2-k_2Y_1\right]\cos\omega t=0\end{aligned} \tag{5.33}$$

Equating coefficients of $\sin\omega t$ and $\cos\omega t$ from Eqs. (5.32) and (5.33) to zero, we get simultaneous algebraic equations as

$$\left\{(k_1+k_2)-m_1\omega^2\right\}X_1-(c_1+c_2)\omega Y_1-k_2X_2+c_2\omega Y_2-F_1=0 \tag{5.34}$$

$$\{(c_1+c_2)\omega\}X_1+\left\{(k_1+k_2)-m_1\omega^2\right\}Y_1-c_2\omega X_2-k_2Y_2=0 \tag{5.35}$$

$$-k_2X_1+c_2\omega Y_1+\left\{(k_2+k_2)-m_2\omega^2\right\}X_2-(c_2+c_3)\omega Y_2=0 \tag{5.36}$$

$$-c_2\omega X_1-k_2Y_1+\{(c_2+c_3)\omega\}X_2+\left\{(k_2+k_3)-m_2\omega^2\right\}Y_2=0 \tag{5.37}$$

Simultaneous Eqs. (5.34)–(5.37) can be solved for X_1, Y_1, X_2 and Y_2 for given m_i, k_i and c_i, F_1 and ω. Then we can determine the steady state response by substituting X_1, Y_1, X_2 and Y_2 into Eq. (5.31).

5.5 Transfer Functions

As defined earlier, transfer function of a dynamic system can be defined as the of the Laplace transform of the output to the Laplace transform of the input under the assumption that all initial conditions are zero.

If a system has multiple inputs and multiple outputs, its inputs and outputs are related by a matrix of transfer functions. For example, if a two degree of freedom system shown in Fig. 5.7 is excited by external forces $F_1(t)$ and $F_2(t)$, then transfer function matrix of the system can be defined as

$$G(s)=\begin{bmatrix}G_{11}(s) & G_{12}(s)\\ G_{21}(s) & G_{22}(s)\end{bmatrix} \tag{5.38}$$

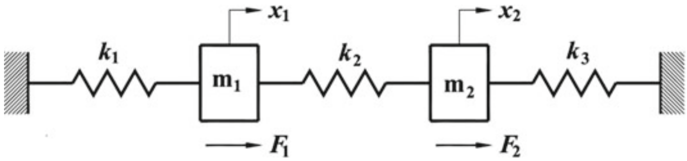


Fig. 5.7 Two degree of freedom system subjected to two external forces $F_1(t)$ and $F_2(t)$

Each element of the transfer function matrix $G_{ij}(s)$ is the transfer function for the displacement x_i at point i due to a force F_j applied at the point j. Hence, the transfer function matrix $G_{ij}(s)$ can also defined as the Laplace transform of the response of x_i due to a unit impulse applied at the point j, i.e.,

$$X_i(s) = F_j(s)G_{ij}(s) \tag{5.39}$$

The response $x_i(t)$ can be determined by taking inverse Laplace transform of Eq. (5.39) as

$$x_i(t) = \mathcal{L}^{-1}\left\{F_j(s)G_{ij}(s)\right\} \tag{5.40}$$

Inverse Laplace transform of Eq. (5.40) is given by convolution integral as

$$x_i(t) = \int_0^t F_j(\eta)h_{ij}(t-\eta)\mathrm{d}\eta \tag{5.41}$$

where

$$h_{ij} = \mathcal{L}^{-1}\left\{G_{ij}(s)\right\} \tag{5.42}$$

is the response of the system due to a unit impulse.

5.6 Vibration Absorber

Consider a machine of mass m_1 supported by a spring assembly with an equivalent stiffness of k_1 and excited by the external force $F(t)$ as shown in Fig. 5.8. The amplitude of steady state vibration of the machine will be very high when the frequency of the excitation corresponding to the operating speed of the machine approaches the natural frequency of the system $\omega_1 = \sqrt{k_1/m_1}$.

Vibration amplitude of the machine can be significantly reduced by adding an auxiliary spring-mass system $(k_2 - m_2)$ as shown in Fig. 5.9.

Fig. 5.8 Machine excited by an external force $F(t)$

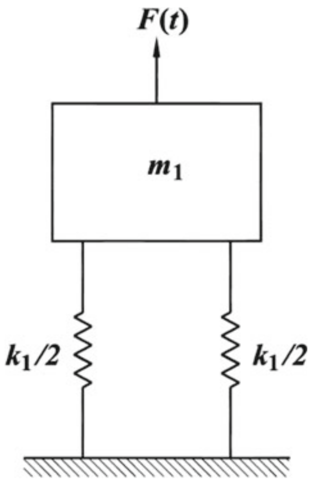
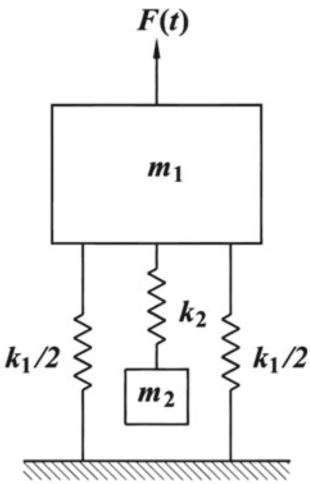


Fig. 5.9 Vibration absorber



The amplitudes of steady state vibration of this system can be directly expressed by substituting $k_3 = 0$ into Eq. (5.26) as

$$X_1 = \frac{\left(k_2 - m_2\omega^2\right)F_1}{m_1 m_2 \omega^4 - \{(k_1 + k_2)m_2 + k_2 m_1\}\omega^2 + k_1 k_2}$$
$$X_2 = \frac{k_2 F_1}{m_1 m_2 \omega^4 - \{(k_1 + k_2)m_2 + k_2 m_1\}\omega^2 + k_1 k_2} \tag{5.43}$$

If the auxiliary spring-mass system $(k_2 - m_2)$ is chosen such that $k_2 = m_2\omega^2$, the vibration amplitude of the machine X_1 becomes almost zero whereas the auxiliary mass keeps on vibrating. In this way vibration of the machine can be reduced even under the excitation by adding an auxiliary system and such auxiliary system is called a vibration absorber.

To study the behavior of the vibration absorber, we can convert Eq. (5.43) by defining the following parameters.

$\omega_{11} = \sqrt{\frac{k_1}{m_1}}$ as the natural frequency of the primary system,
$\omega_{22} = \sqrt{\frac{k_2}{m_2}}$ as the natural frequency of the auxiliary system and
$\mu = \frac{m_2}{m_1}$ as the mass ratio of the auxiliary system to the primary system.

Introducing theses parameters, Eq. (5.43) reduces to

$$\begin{aligned}
\frac{k_1 X_1}{F_1} &= \frac{\left\{1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right\}}{\frac{\omega^4}{\omega_{11}^2 \omega_{22}^2} - \left\{(1+\mu)\left(\frac{\omega}{\omega_{11}}\right)^2 + \left(\frac{\omega}{\omega_{22}}\right)^2\right\} + 1} \\
\frac{k_1 X_2}{F_1} &= \frac{1}{\frac{\omega^4}{\omega_{11}^2 \omega_{22}^2} - \left\{(1+\mu)\left(\frac{\omega}{\omega_{11}}\right)^2 + \left(\frac{\omega}{\omega_{22}}\right)^2\right\} + 1}
\end{aligned} \tag{5.44}$$

It can also be noted from the first part of Eq. (5.44) that $X_1 = 0$ when $\omega = \omega_{22}$, i.e., when the frequency of the external excitation is equal to the natural frequency of the absorber, the primary system remains motionless.

Similarly, when $\omega = \omega_{22}$, the second part of Eq. (5.44) reduces to

$$\begin{aligned}
\frac{k_1 X_2}{F_1} &= -\frac{1}{\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2} = -\frac{1}{\frac{m_2}{m_1}\frac{k_2}{m_2}\frac{m_1}{k_1}} = -\frac{k_1}{k_2} \\
\therefore \quad F_1 &= -k_2 X_2
\end{aligned} \tag{5.45}$$

Equation (5.45) shows that the external exciting force applied to the primary system is transferred to the auxiliary mass through the spring (k_2) of the absorber system.

Vibration absorber will be effective if it reduces the steady state vibration of the main system even if it operates near the resonance for which ω should be equal to ω_{11}. But the vibration absorber works when $\omega = \omega_{22}$. If both the stated conditions are satisfied,

$$\begin{aligned}
\omega_{11} &= \omega_{22} \\
\therefore \quad \frac{k_1}{m_1} &= \frac{k_2}{m_2}
\end{aligned} \tag{5.46}$$

Any vibration absorber which satisfies Eq. (5.46) is called a tuned vibration absorber. For the tuned vibration absorber, Eq. (5.44) reduces to

$$\frac{k_1X_1}{F_1}=\frac{\left\{1-\left(\frac{\omega}{\omega_{22}}\right)^2\right\}}{\left(\frac{\omega}{\omega_{22}}\right)^4-\left\{(2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2\right\}+1}$$
$$\frac{k_1X_2}{F_1}=\frac{1}{\left(\frac{\omega}{\omega_{22}}\right)^4-\left\{(2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2\right\}+1} \tag{5.47}$$

Any combination of k_2 and m_2 which satisfy Eq. (5.46) can be used to design a tuned vibration absorber. However, it can be noted from Eq. (5.45) small value of k_2 results into large amplitude of the auxiliary mass m_2 and the vibration amplitude of auxiliary mass will be low if relatively large value of k_2 is chosen. However, for given parameters of primary system (k_1 and m_1), if high value of k_2 is selected mass of the auxiliary system m_2 should also be high to satisfy Eq. (5.46). Usually high value of m_2 is not desirable because it increases the inertia effect of the system, therefore selection of parameters of the auxiliary system is made by making compromise between the vibration amplitude X_2 and mass ratio μ. The mass ratio is commonly taken between 0.05 and 0.25.

Variation of amplitude ratios of both primary and auxiliary systems with the frequency ratio for a typical mass ratio of 0.15 is shown in Figs. 5.10 and 5.11. A proper selection of the stiffness of the absorber system can be made to limit the amplitude ratio at desired level.

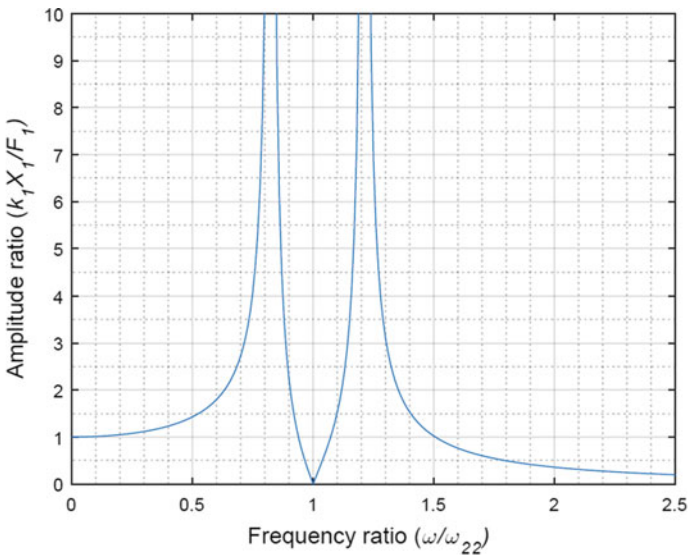


Fig. 5.10 Variation of amplitude ratio of primary mass with frequency ratio for $\mu = 0.15$

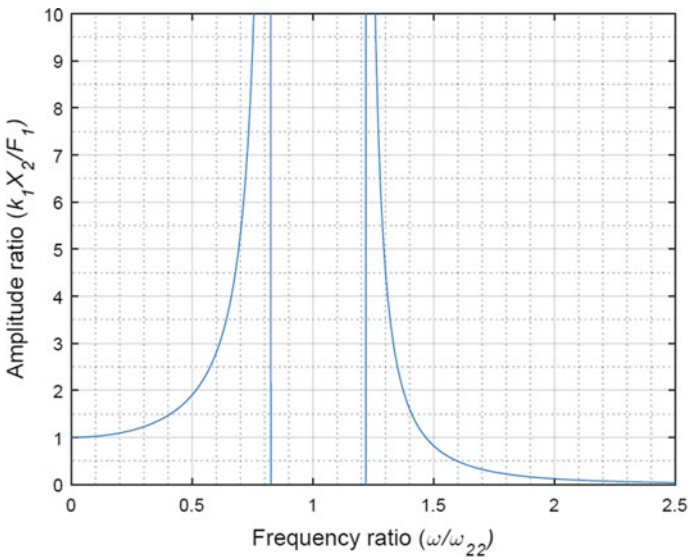


Fig. 5.11 Variation of amplitude ratio of auxiliary mass with frequency ratio for $\mu = 0.15$

Both parts of Eq. (5.47) have same denominator. Hence both masses undergo resonance when the denominators become zero, i.e.

$$\left(\frac{\omega}{\omega_{22}}\right)^4 - \left\{(2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2\right\} + 1 = 0 \tag{5.48}$$

Solving Eq. (5.48) for frequency ratio (ω/ω_{22}), we get

$$\frac{\omega}{\omega_{22}} = \sqrt{\left(1+\frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}} \tag{5.49}$$

Using Eq. (5.49), variation of frequency ratio (ω/ω_{22}) with the mass ratio μ is shown in Fig. 5.12.

The primary system without vibration absorber have only one resonant frequency at $\omega = \omega_{11}$. If the frequency of external excitation is constant and tuned vibration absorber is used, the vibration amplitude of the primary mass will be negligible even near the resonance condition. However if the excitation frequency undergo variation during operating condition, then the amplitude of vibration of the primary mass remains no longer negligible. It should also be noted that due to the addition of vibration absorber to the primary system, it becomes a two degree of freedom system and the overall system will have two resonant condition. These two resonant points are spread on either side of the original resonant point corresponding to the primary system alone. If the variation of the frequency of the external excitation is

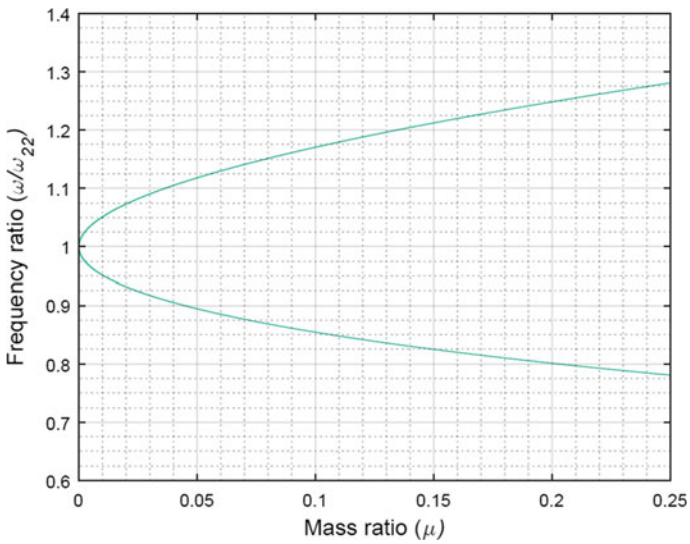


Fig. 5.12 Variation of frequency ratio with mass ratio

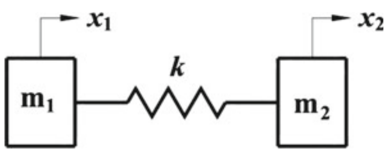
such that the operating point shifts near one of the new resonant points, then the amplitudes of vibration of both primary and auxiliary mass become very high. This is also undesirable and to avoid such resonance the spread between the two resonant frequencies has to be decided depending upon the variation of the frequency of the external excitation. After deciding the spread between the resonant frequencies, the proper value of mass ratio μ can be obtained from the curve of Fig. 5.12.

5.7 Semi-definite System

Consider two blocks with masses m_1 and m_2 connected by a spring with stiffness k as shown in Fig. 5.13. Substituting $k_1 = 0$, $k_2 = k$ and $k_3 = 0$ into Eq. (5.1), equations of motion of the system can be derived as

$$\begin{aligned} m_1\ddot{x}_1 + kx_1 - kx_2 &= 0 \\ m_2\ddot{x}_2 + kx_2 - kx_1 &= 0 \end{aligned} \tag{5.50}$$

Fig. 5.13 Two blocks connected by a spring



We can assume the solution for a system of linear differential equations as

$$
\begin{aligned}
x_1 &= A_1 \sin \omega_n t \\
x_2 &= A_2 \sin \omega_n t
\end{aligned}
\tag{5.51}
$$

where A_1 and A_2 are the amplitudes of vibration of masses m_1 and m_2, respectively, and ω_n is the natural frequency of free vibration.

Substituting Eq. (5.51) into Eq. (5.50), we get a system of algebraic equations as

$$
\begin{aligned}
\left[k - m_1\omega_n^2\right]A_1 - kA_2 &= 0 \\
-kA_1 + \left[k - m_2\omega_n^2\right]A_2 &= 0
\end{aligned}
\tag{5.52}
$$

Rearranging both equations of Eq. (5.52) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{k}{k - m_1\omega_n^2} \tag{5.53}$$

$$\frac{A_1}{A_2} = \frac{k - m_2\omega_n^2}{k} \tag{5.54}$$

Equating Eqs. (5.53) and (5.54), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$
\begin{aligned}
& \frac{k}{k - m_1\omega_n^2} = \frac{k - m_2\omega_n^2}{k} \\
\text{or,}\quad & \left[\left(k - m_1\omega_n^2\right)\left(k - m_2\omega_n^2\right)\right] - k^2 = 0 \\
\text{or,}\quad & m_1 m_2 \lambda^2 - \{k(m_1 + m_2)\}\lambda = 0 \\
\text{or,}\quad & \lambda^2 - \left(\frac{k}{m_1} + \frac{k}{m_2}\right)\lambda = 0 \\
\therefore\quad & \lambda_1 = 0, \quad \lambda_2 = \frac{k}{m_1} + \frac{k}{m_2}
\end{aligned}
\tag{5.55}
$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = 0 \tag{5.56}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{k}{m_1} + \frac{k}{m_2}} \tag{5.57}$$

Substituting $\lambda = \lambda_1 (= 0)$ into Eq. (5.53), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{k - m_1\lambda_1} = 1 \tag{5.58}$$

Similarly, substituting $\lambda = \lambda_2(= k/m_1 + k/m_2)$ into Eq. (5.53), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k}{k - m_2\lambda_2} = -\frac{m_2}{m_1} \tag{5.59}$$

It can be noted from the first mode defined by $\omega_1 = 0$ and $(A_1/A_2)_1 = 1$ that there is no oscillation and it represents rigid body motion of the overall system.

Hence the system has only oscillatory mode defined by $\omega_2 = \sqrt{k(m_1 + m_2)/m_1m_2}$ and $(A_1/A_2)_2 = -m_2/m_1$.

Any system which has one of the natural frequency equal to zero is called a semi-definite system or a degenerated system. The number of natural frequencies of a semi-definite or a degenerated system will less than the degree of freedom of the system.

5.8 Coordinate Coupling and Principal Coordinates

Consider a rigid bar of mass M and length L supported by two springs k_1 and k_2 at its two ends as shown in Fig. 5.14. This model can be used to study vibration of different system such as vehicles, lathe machines, etc. The system has two degree of freedom and a set of generalized coordinates can be taken as either vertical displacements of end points of the bar (x_1 and x_2) or vertical displacement of C. G. of the bar (x) and rotation of bar about C. G. (θ).

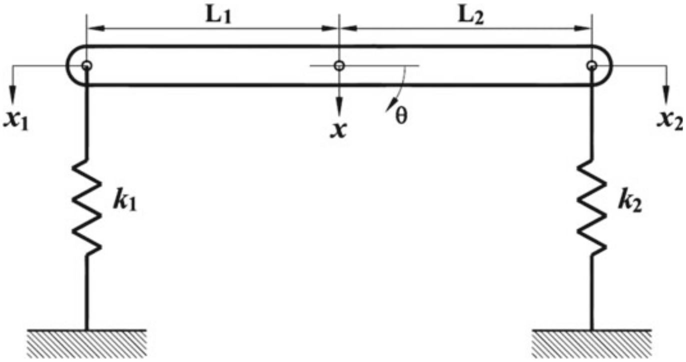


Fig. 5.14 Rigid bar supported by two springs

5.8.1 Equation of Motion Using x_1 and x_2 as Generalized Coordinates

Referring to the free-body diagram of the bar shown in Fig. 5.15, transverse displacement of C.G. of the bar (x) and angular displacement of the bar (θ) can be related to the transverse displacements of left and right ends of the bar (x_1 and x_2) as

$$x = \frac{L_2 x_1 + L_1 x_2}{L_1 + L_2} \tag{5.60}$$

$$\theta = \frac{x_2 - x_1}{L_1 + L_2} \tag{5.61}$$

Then the kinetic energy of the bar can be expressed as

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} \overline{I} \dot{\theta}^2 \tag{5.62}$$

Substituting x and θ for Eqs. (5.60) and (5.61) into Eq. (5.62), we get

$$T = \frac{1}{2} M \left(\frac{L_2 \dot{x}_1 + L_1 \dot{x}_2}{L_1 + L_2} \right)^2 + \frac{1}{2} \overline{I} \left(\frac{\dot{x}_2 - \dot{x}_1}{L_1 + L_2} \right)^2$$

$$\therefore \quad T = \frac{1}{2} M \left(\frac{L_2 \dot{x}_1 + L_1 \dot{x}_2}{L_1 + L_2} \right)^2 + \frac{1}{2} \left(\frac{1}{12} M \right) (\dot{x}_2 - \dot{x}_1)^2 \tag{5.63}$$

Potential energy of the system can be expressed as

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 \tag{5.64}$$

Then Lagrangian functional is given by

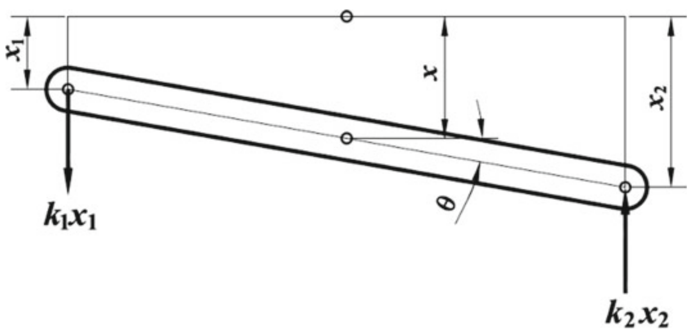


Fig. 5.15 Free-body diagram of the rigid bar with forces in terms of x_1 and x_2

$$L = T - V$$
$$= \frac{1}{2}M\left(\frac{L_2\dot{x}_1 + L_1\dot{x}_2}{L_1 + L_2}\right)^2 + +\frac{1}{2}\left(\frac{1}{12}M\right)(\dot{x}_2 - \dot{x}_1)^2 - \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \quad (5.65)$$

Applying Lagrange equation for displacement variables x_1 and x_2, respectively, we get the equations of motion of the system as

$$M\left[\frac{L_2^2}{(L_1 + L_2)^2} + \frac{1}{12}\right]\ddot{x}_1 + M\left[\frac{L_2^2}{(L_1 + L_2)^2} - \frac{1}{12}\right]\ddot{x}_2 + k_1x_1 = 0 \quad (5.66)$$

$$M\left[\frac{L_2^2}{(L_1 + L_2)^2} - \frac{1}{12}\right]\ddot{x}_1 + M\left[\frac{L_2^2}{(L_1 + L_2)^2} - \frac{1}{12}\right]\ddot{x}_2 + k_2x_2 = 0 \quad (5.67)$$

In all systems discussed earlier, the coupling of displacement variables occur only in stiffness terms. However, it can be noted from Eqs. (5.66) and (5.67) that coupling of displacements also occur in mass or inertia terms. A system is said to be statically coupled if the coupling appears in stiffness terms only whereas the system is said to be dynamically coupled if the coupling appears in the mass or inertia terms.

5.8.2 Equation of Motion Using x and θ as Generalized Coordinates

Referring to the free-body diagram of the bar shown in Fig. 5.16, transverse displacements of left and right ends of the bar (x_1 and x_2) can be related to the transverse displacement of C.G. of the bar (x) and angular displacement of the bar (θ) as

$$x_1 = x - L_1\theta \quad (5.68)$$

$$x_2 = x + L_2\theta \quad (5.69)$$

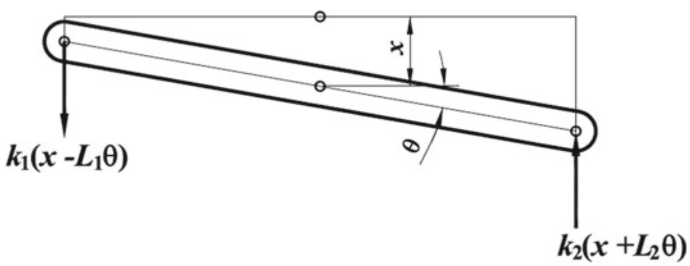


Fig. 5.16 Free-body diagram of the rigid bar with forces in terms of x and θ

Then the kinetic energy of the bar can be expressed as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\overline{I}\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 \tag{5.70}$$

Potential energy of the system can be expressed as

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \tag{5.71}$$

Substituting x_1 and x_2 from Eqs. (5.68) and (5.69) into Eq. (5.71),

$$V = \frac{1}{2}k_1(x - L_1\theta)^2 + \frac{1}{2}k_2(x + L_2\theta)^2 \tag{5.72}$$

Then Lagrangian functional is given by

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 - \frac{1}{2}k_1(x - L_1\theta)^2 + \frac{1}{2}k_2(x + L_2\theta)^2 \end{aligned} \tag{5.73}$$

Applying Lagrange equation for displacement variables x and θ, respectively, we get the equation of motion of the system as

$$M\ddot{x} + (k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta = 0 \tag{5.74}$$

$$\left(\frac{1}{12}ML^2\right)\ddot{\theta} + \left(k_1L_1^2 + k_2L_2^2\right)\theta - (k_1L_1 - k_2L_2)x = 0 \tag{5.75}$$

It can be noted from Eqs. (5.74) and (5.75), coupling occurs only in stiffness terms; hence a system defined by theses equations is a statically coupled system.

If $k_1L_1 - k_2L_2 = 0$, then Eqs. (5.74) and (5.75) reduce to

$$M\ddot{x} + (k_1 + k_2)x = 0 \tag{5.76}$$

$$\left(\frac{1}{12}ML^2\right)\ddot{\theta} + \left(k_1L_1^2 + k_2L_2^2\right)\theta = 0 \tag{5.77}$$

Equations (5.75) and (5.76) are uncoupled and can be solved independently for x and θ. A specific set of generalized coordinates which gives a system of equations in which all equations are uncoupled is called a set of principal coordinates. Therefore, if $k_1L_1 - k_2L_2 = 0$, x and θ are the principal coordinates for the system shown in Fig. 5.14.

Solved Examples

Example 5.1

Determine the natural frequencies and mode shapes a two degree of freedom system shown in Figure E5.1.

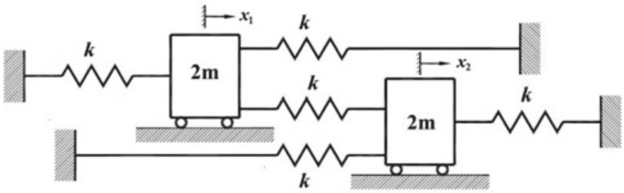


Figure E5.1

Solution

With reference to the free-body diagrams of both masses shown in **Figure E5.1(a)**, equations of motion for the given system can be written as

$$\begin{aligned} 2m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \\ 2m\ddot{x}_2 + 3kx_2 - kx_1 = 0 \end{aligned} \quad (\mathbf{a})$$

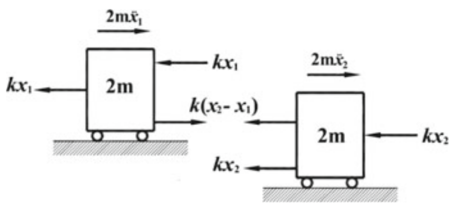


Figure E5.1(a) Free-body diagrams of two masses of system of Figure E5.1

We can assume the solution for a system of linear differential equations as

$$\begin{aligned} x_1 = A_1 \sin \omega_n t \\ x_2 = A_2 \sin \omega_n t \end{aligned} \quad (\mathbf{b})$$

Substituting Eq. (**b**) into Eq. (**a**), we get a system of algebraic equations as

$$\begin{aligned} \left[3k - 2m\omega_n^2\right]A_1 - kA_2 &= 0 \\ -kA_1 + \left[3k - 2m\omega_n^2\right]A_2 &= 0 \end{aligned} \qquad (\mathbf{c})$$

Rearranging both equations of Eq. (**c**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{k}{3k - 2m\omega_n^2} \qquad (\mathbf{d})$$

$$\frac{A_1}{A_2} = \frac{3k - 2m\omega_n^2}{k} \qquad (\mathbf{e})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\begin{aligned} &\frac{k}{3k - 2m\lambda} = \frac{3k - 2m\lambda}{k} \\ \text{or,}\quad &(3k - 2m\lambda)^2 = k^2 \\ \text{or,}\quad &3k - 2m\lambda = \pm k \\ \text{or,}\quad &\lambda = \frac{3k \pm k}{2m} \\ \therefore\quad &\lambda_1 = \frac{k}{m}, \quad \lambda_2 = \frac{2k}{m} \end{aligned} \qquad (\mathbf{f})$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{2k}{m}}$$

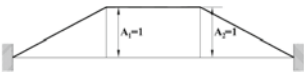
Substituting $\lambda = \lambda_1 (= k/m)$ into Eq. (**d**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{3k - 2m\lambda_1} = 1$$

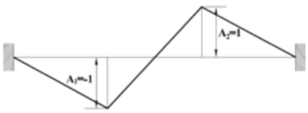
Similarly, substituting $\lambda = \lambda_2 (= 2k/m)$ into Eq. (**d**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k}{3k - 2m\lambda_2} = -1$$

Mode shapes are shown graphically as shown in **Figure E5.1(b)**.



Mode Shape Corresponding to
$\omega_1 = \sqrt{k/m}$



Mode Shape Corresponding to
$\omega_2 = \sqrt{2k/m}$

Figure E5.1(b) Mode shapes for a system of Figure E5.1

Example 5.2

Determine the two natural frequencies and the modes of vibration of the system shown in Figure E5.2. Determine the response of the system when it is subject to the following initial conditions:

$$x_1(0) = 0.1m, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0$$

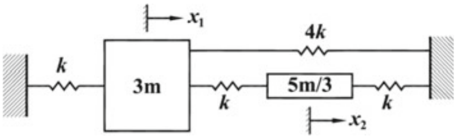


Figure E5.2

Solution

With reference to the free-body diagrams of both masses shown in **Figure E5.2(a)**, equations of motion for the given system can be written as

$$\begin{aligned} &3m\ddot{x}_1 + 6kx_1 - kx_2 = 0 \\ &\frac{5}{3}m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \qquad \textbf{(a)} \end{aligned}$$

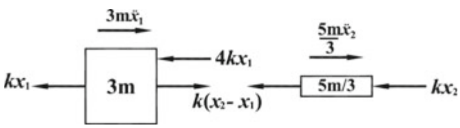


Figure E5.2(a) Free-body diagrams of two masses of system of Figure E5.2

We can assume the solution for a system of linear differential equations as

$$\begin{aligned} x_1 &= A_1 \sin \omega_n t \\ x_2 &= A_2 \sin \omega_n t \qquad (\mathbf{b}) \end{aligned}$$

Substituting Eq. (**b**) into Eq. (**a**), we get a system of algebraic equations as

$$\begin{aligned} &\left[6k - 3m\omega_n^2\right]A_1 - kA_2 = 0 \\ &-kA_1 + \left[2k - \frac{5}{3}m\omega_n^2\right]A_2 = 0 \qquad (\mathbf{c}) \end{aligned}$$

Rearranging both equations of Eq. (**c**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{k}{6k - 3m\omega_n^2} \qquad (\mathbf{d})$$

$$\frac{A_1}{A_2} = \frac{2k - \frac{5}{3}m\omega_n^2}{k} \qquad (\mathbf{e})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\begin{aligned} &\frac{k}{6k - 3m\lambda} = \frac{2k - \frac{5}{3}m\lambda}{k} \\ \text{or,}\quad &\left\{(6k - 3m\lambda)\left(2k - \frac{5}{3}m\lambda\right)\right\} - k^2 = 0 \\ \text{or,}\quad &5m^2\lambda^2 - 16km\lambda + 11k^2 = 0 \\ \text{or,}\quad &5\lambda^2 - 16\frac{k}{m}\lambda + 11\frac{k^2}{m^2} = 0 \\ \text{or,}\quad &\lambda = \frac{16k \pm 6k}{10m} = \frac{8k \pm 3k}{5m} \\ \therefore\quad &\lambda_1 = \frac{k}{m}, \lambda_2 = \frac{11k}{5m} \qquad (\mathbf{f}) \end{aligned}$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{11k}{5m}} = 1.4832\sqrt{\frac{k}{m}}$$

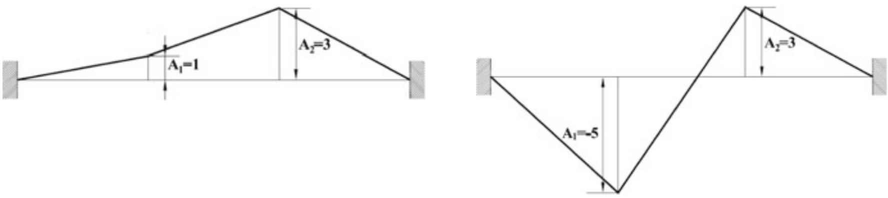
Substituting $\lambda = \lambda_1 (= k/m)$ into Eq. (**d**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{6k - 3m\lambda_1} = \frac{1}{3}$$

Similarly, substituting $\lambda = \lambda_2 (= 11k/5m)$ into Eq. (**d**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k}{6k - 3m\lambda_2} = -\frac{5}{3}$$

Mode shapes are shown graphically as shown in **Figure E5.2(b)**.



Mode Shape Corresponding to $\omega_1 = \sqrt{k/m}$ **Mode Shape Corresponding to $\omega_2 = \sqrt{11k/5m}$**

Figure E5.2(b) Mode shapes for a system of Figure E5.2

The response of the system is given by

$$x_1 = A_{11}\sin(\omega_1 t + \psi_1) + A_{12}\sin(\omega_2 t + \psi_2) \qquad (\mathbf{g})$$

$$x_2 = A_{21}\sin(\omega_1 t + \psi_1) + A_{22}\sin(\omega_2 t + \psi_2) \qquad (\mathbf{h})$$

where

$$\frac{A_{11}}{A_{21}} = \left(\frac{A_1}{A_2}\right)_1 = \frac{1}{3} = \frac{A}{3A} \qquad (\mathbf{i})$$

$$\frac{A_{12}}{A_{22}} = \left(\frac{A_1}{A_2}\right)_2 = -\frac{5}{3} = -\frac{5B}{3B} \qquad (\mathbf{j})$$

Substituting A_{11}, A_{21}, A_{12} and A_{22} from Eqs. (**i**) and (**j**) into Eqs. (**g**) and (**h**), we get

$$x_1 = A\sin(\omega_1 t + \psi_1) - 5B\sin(\omega_2 t + \psi_2) \qquad (\mathbf{k})$$

$$x_2 = 3A\sin(\omega_1 t + \psi_1) + 3B\sin(\omega_2 t + \psi_2) \qquad (\mathbf{l})$$

Substituting given initial conditions, $x_1(0) = 0.1m, \dot{x}_1(0) = 0, x_2(0) = 0, \dot{x}_2(0) = 0$, we get

$$A\sin\psi_1 - 5B\sin\psi_2 = 0.1 \qquad (\mathbf{m})$$

$$A\omega_1 \cos\psi_1 - 5B\omega_2 \cos\psi_2 = 0 \qquad (\mathbf{n})$$

$$3A \sin\psi_1 + 3B \sin\psi_2 = 0 \qquad (\mathbf{o})$$

$$3A\omega_1 \cos\psi_1 + 3BB\omega_2 \cos\psi_2 = 0 \qquad (\mathbf{p})$$

Solving simultaneous Eqs. (**n**) and (**p**),

$$\cos\psi_1 = 0 \quad \therefore \quad \psi_1 = \frac{\pi}{2} \qquad (\mathbf{q})$$

$$\cos\psi_2 = 0 \quad \therefore \quad \psi_2 = \frac{\pi}{2} \qquad (\mathbf{r})$$

Substituting ψ_1 and ψ_2, Eqs. (**m**) and (**o**) reduce to

$$A - 5B = 0.1 \qquad (\mathbf{s})$$

$$3A + 3B = 0 \qquad (\mathbf{t})$$

Solving simultaneous Eqs. (**s**) and (**t**), for A and B, we get

$$A = \frac{1}{60} \quad \text{and } B = -\frac{1}{60} \qquad (\mathbf{q})$$

Substituting A, B, ψ_1 and ψ_2 into Eqs. (**k**) and (**l**), we get the response of the system as

$$x_1 = \frac{1}{60}\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{12}\cos\left(\sqrt{\frac{11k}{5m}}t\right) \qquad (\mathbf{k})$$

$$x_2 = \frac{1}{20}\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{1}{20}\cos\left(\sqrt{\frac{11k}{5m}}t\right) \qquad (\mathbf{l})$$

Example 5.3

Determine the natural frequencies of a two degree of freedom system shown in Figure E5.3.

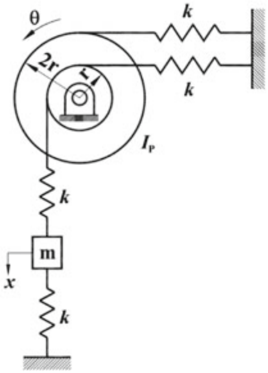


Figure E5.3

Solution

If the displacement of block m (x) and the rotation of the pulley (θ) as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\dot{\theta}\right)^2$$
$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x - r\theta)^2 + \frac{1}{2}(k)(r\theta)^2 + \frac{1}{2}(k)(2r\theta)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_P\left(\dot{\theta}\right)^2 - \frac{1}{2}kx^2 - \frac{1}{2}k(x - r\theta)^2 - \frac{1}{2}(k)(r\theta)^2 - \frac{1}{2}(k)(2r\theta)^2$$

Now, using Lagrange' equation for the generalized coordinate x,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$
$$\text{or,} \quad -kx - k(x - r\theta) - \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}] = 0$$
$$\text{or,} \quad -2kx + kr\theta - m\ddot{x} = 0$$
$$\therefore \quad m\ddot{x} + 2kx - kr\theta = 0 \qquad \textbf{(a)}$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\text{or,}\quad kr(x - r\theta) - kr^2\theta - 4kr^2\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left[I_P\dot{\theta}\right] = 0$$

$$\text{or,}\quad krx - 6kr^2\theta - I_P\,\ddot{\theta} = 0$$

$$\therefore\quad I_P\,\ddot{\theta} + 6kr^2\theta - krx = 0 \qquad (\mathbf{b})$$

We can assume the solution for a system of linear differential equations as

$$x = A\sin\omega_n t$$
$$\theta = \overline{\theta}\sin\omega_n t \qquad (\mathbf{c})$$

Substituting Eq. (**c**) into Eqs. (**a**) and (**b**), we get a system of algebraic equations as

$$\left[2k - m\omega_n^2\right]A - kr\overline{\theta} = 0$$
$$-krA + \left[6kr^2 - I_P\omega_n^2\right]\overline{\theta} = 0 \qquad (\mathbf{d})$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A}{\overline{\theta}} = \frac{kr}{2k - m\omega_n^2} \qquad (\mathbf{e})$$

$$\frac{A}{\overline{\theta}} = \frac{6kr^2 - I_P\omega_n^2}{kr} \qquad (\mathbf{f})$$

Equating Eqs. (**e**) and (**f**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{kr}{2k - m\lambda} = \frac{6kr^2 - I_P\lambda}{kr}$$

$$\text{or,}\quad \left\{(2k - m\lambda)\left(6kr^2 - I_P\lambda\right)\right\} - k^2r^2 = 0$$

$$\text{or,}\quad I_P m\lambda^2 - \left(6kmr^2 + 2I_Pk\right)\lambda + 11k^2r^2 = 0$$

$$\text{or,}\quad \lambda^2 - \left(\frac{6kr^2}{I_P} + \frac{2k}{m}\right)\lambda + 11\frac{k^2r^2}{I_Pm} = 0$$

$$\text{or,}\quad \lambda_{1,2} = \left(\frac{3kr^2}{I_P} + \frac{k}{m}\right) \pm \frac{k}{I_Pm}\sqrt{9m^2r^4 - 5I_pmr^2 + I_p^2}$$

$$\therefore\quad \lambda_1 = \left(\frac{3kr^2}{I_P} + \frac{k}{m}\right) - \frac{k}{I_Pm}\sqrt{9m^2r^4 - 5I_pmr^2 + I_p^2}$$

and

$$\lambda_2 = \left(\frac{3kr^2}{I_P} + \frac{k}{m}\right) + \frac{k}{I_Pm}\sqrt{9m^2r^4 - 5I_pmr^2 + I_p^2} \qquad (\mathbf{g})$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\left(\frac{3kr^2}{I_P} + \frac{k}{m}\right) - \frac{k}{I_P m}\sqrt{9m^2r^4 - 5I_p mr^2 + I_p^2}}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\left(\frac{3kr^2}{I_P} + \frac{k}{m}\right) + \frac{k}{I_P m}\sqrt{9m^2r^4 - 5I_p mr^2 + I_p^2}}$$

Example 5.4

Determine the natural frequencies and mode shapes a two degree of freedom system shown in Figure E5.4. Given $k = 1000\,\text{N/m}$, $r = 10\,\text{cm}$, $I_p = 1\,\text{kg ms}$ and $m = 10$ kg.

Solution

If the displacement of block m (x_1) and displacement of block $2m$ (x_2) as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}I_P\left(\frac{\dot{x}_1}{2r}\right)^2 + \frac{1}{2}(2m)\dot{x}_2^2 = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2$$
$$V = \frac{1}{2}(2k)(x_1)^2 + \frac{1}{2}k\left(x_2 - \frac{x_1}{2}\right)^2 + \frac{1}{2}(k)(x_2)^2$$

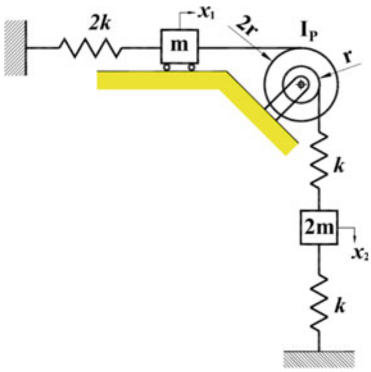


Figure E5.4

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\left(m + \frac{I_P}{4r^2}\right)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 - \frac{1}{2}(2k)(x_1)^2 - \frac{1}{2}k\left(x_2 - \frac{x_1}{2}\right)^2$$

$$-\frac{1}{2}(k)(x_2)^2$$

Now, using Lagrange' equation for the generalized coordinate x_1,

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$$

$$\text{or,} \quad -2kx_1 + \frac{k}{2}\left(x_2 - \frac{x_1}{2}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(m + \frac{I_P}{4r^2}\right)\dot{x}_1\right] = 0$$

$$\text{or,} \quad -\frac{9k}{4}x_1 + \frac{k}{2}x_2 - \left(m + \frac{I_P}{4r^2}\right)\ddot{x}_1 = 0$$

$$\therefore \quad \left(m + \frac{I_P}{4r^2}\right)\ddot{x}_1 + \frac{9k}{4}x_1 - \frac{k}{2}x_2 = 0 \qquad (\mathbf{a})$$

Again, using Lagrange' equation for the generalized coordinate x_2,

$$\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0$$

$$\text{or,} \quad -k\left(x_2 - \frac{x_1}{2}\right) - kx_2 - \frac{\mathrm{d}}{\mathrm{d}t}[(2m)\dot{x}_2] = 0$$

$$\text{or,} \quad -2kx_2 + \frac{k}{2}x_1 - 2m\ddot{x}_2 = 0$$

$$\therefore \quad 2m\ddot{x}_2 - \frac{k}{2}x_1 + 2kx_2 = 0 \qquad (\mathbf{b})$$

Substituting $k = 1000\,\text{N/m}$, $r = 10\,\text{cm}$, $I_p = 1\,\text{kg m}^2$ and $m = 10\,\text{kg}$, equations of motion can be expressed as

$$35\ddot{x}_1 + 2250x_1 - 500x_2 = 0$$
$$20\ddot{x}_2 + 2000x_2 - 500x_1 = 0 \qquad (\mathbf{c})$$

We can assume the solution for a system of linear differential equations as

$$x_1 = A_1 \sin \omega_n t$$
$$x_2 = A_2 \sin \omega_n t \qquad (\mathbf{b})$$

Substituting Eq. (**b**) into Eq. (**a**), we get a system of algebraic equations as

$$\left[2250 - 35\omega_n^2\right]A_1 - 500A_2 = 0$$
$$-500A_1 + \left[2000 - 20\omega_n^2\right]A_2 = 0 \qquad (\mathbf{c})$$

Rearranging both equations of Eq. (**c**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{500}{2250 - 35\omega_n^2} = \frac{100}{450 - 7\omega_n^2} \qquad (\mathbf{d})$$

$$\frac{A_1}{A_2} = \frac{2000 - 20\lambda}{500} = \frac{100 - \omega_n^2}{25} \qquad (\mathbf{e})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{100}{450 - 7\lambda} = \frac{100 - \lambda}{25}$$

$$\text{or,} \quad \{(450 - 7\lambda)(100 - \lambda)\} - 2500 = 0$$

$$\text{or,} \quad 7\lambda^2 - 1150\lambda + 42500 = 0$$

$$\text{or,} \quad \lambda = \frac{1150 \pm \sqrt{132500}}{14} = \frac{1150 \pm 364.0055}{14}$$

$$\therefore \quad \lambda_1 = 56.1425, \quad \lambda_2 = 108.1432 \qquad (\mathbf{f})$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{56.1425} = 7.4928\,\text{rad/s}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{108.1432} = 10.3992\,\text{rad/s}$$

Substituting $\lambda = \lambda_1 (= 56.1425)$ into Eq. (**d**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{100}{450 - 7\lambda_1} = \frac{100}{450 - 7 \times 56.1425} = 1.7543$$

Similarly, substituting $\lambda = \lambda_2 (= 108.1432)$ into Eq. (**d**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{100}{450 - 7\lambda_2} = \frac{100}{450 - 7 \times 108.1432} = -0.3257$$

Example 5.5

Determine the natural frequencies of a two degree of freedom system shown in Figure E5.5. The uniform beam has a mass of *M*.

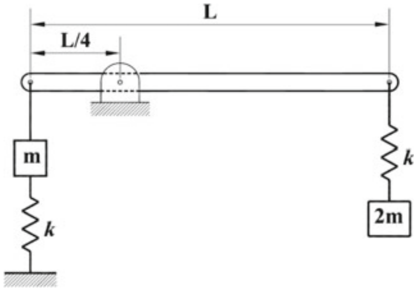


Figure E5.5

Solution

If the displacement of block $2m$ (x) and the rotation of the bar (θ) are taken as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I_0(\dot{\theta})^2 + \frac{1}{2}(2m)\left(\frac{L}{4}\dot{\theta}\right)^2$$

$$= \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}k\left(x - \frac{3L}{4}\theta\right)^2 + \frac{1}{2}(k)\left(\frac{L}{4}\theta\right)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}\left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\dot{\theta}^2 - \frac{1}{2}k\left(x - \frac{3L}{4}\theta\right)^2 - \frac{1}{2}(k)\left(\frac{L}{4}\theta\right)^2$$

Now, using Lagrange' equation for the generalized coordinate x,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\text{or,} \quad -k\left(x - \frac{3L}{4}\theta\right) - \frac{\mathrm{d}}{\mathrm{d}t}[2m\dot{x}] = 0$$

$$\text{or,} \quad -kx + \frac{3kL}{4}\theta - 2m\ddot{x} = 0$$

$$\therefore \quad 2m\ddot{x} + kx - \frac{3kL}{4}\theta = 0 \qquad (\mathbf{a})$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\text{or,} \quad \frac{3kL}{4}\left(x - \frac{3L}{4}\theta\right) - \frac{kL}{4}\left(\frac{L}{4}\theta\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\dot{\theta}\right] = 0$$

$$\text{or,} \quad \frac{5kL^2}{8}x - 6kr^2\theta - \left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\ddot{\theta} = 0$$

$$\therefore \quad \left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\ddot{\theta} + \frac{5kL^2}{8}\theta - \frac{3kL}{4}x = 0 \qquad (\mathbf{b})$$

We can assume the solution for a system of linear differential equations as

$$x = A \sin \omega_n t$$
$$\theta = \overline{\theta} \sin \omega_n t \qquad (\mathbf{c})$$

Substituting Eq. (**c**) into Eqs. (**a**) and (**b**), we get a system of algebraic equations as

$$\left[k - 2m\omega_n^2\right]A - \frac{3kL}{4}\overline{\theta} = 0$$
$$-\frac{3kL}{4}A + \left[\frac{5kL^2}{8} - \left(\frac{7}{48}ML^2 + \frac{1}{16}mL^2\right)\omega_n^2\right]\overline{\theta} = 0 \qquad (\mathbf{d})$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A}{\overline{\theta}} = \frac{3kL}{4\left(k - 2m\omega_n^2\right)} \qquad (\mathbf{e})$$

$$\frac{A}{\overline{\theta}} = \frac{L\left[30k - (7M + 3m)\omega_n^2\right]}{36k} \qquad (\mathbf{f})$$

Equating Eqs. (**e**) and (**f**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{3kL}{4(k - 2m\lambda)} = \frac{L[30k - (7M + 3m)\lambda]}{36k}$$

$$\text{or,} \quad [(k - 2m\lambda)\{30k - (7M + 3m)\lambda\}] - 27k^2 = 0$$

$$\text{or,} \quad 2m(7M + 3m)\lambda^2 - 7k(M + 9m)\lambda + 3k^2 = 0$$

$$\text{or,} \quad \lambda^2 - \frac{7}{2}\frac{k}{m}\frac{(M + 9m)}{(7M + 3m)}\lambda + \frac{3}{2}\frac{k^2}{m(7M + 3m)} = 0$$

$$\text{or,} \quad \lambda_{1,2} = \frac{k}{4m(7M + 3m)}\left[7(M + 9m) \pm \sqrt{49M^2 + 714Mm + 3897m^2}\right]$$

$$\therefore \quad \lambda_1 = \frac{k}{4m(7M + 3m)}\left[7(M + 9m) - \sqrt{49M^2 + 714Mm + 3897m^2}\right]$$

and

$$\lambda_2 = \frac{k}{4m(7M + 3m)}\left[7(M + 9m) + \sqrt{49M^2 + 714Mm + 3897m^2}\right] \qquad \textbf{(g)}$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{4m(7M + 3m)}\left[7(M + 9m) - \sqrt{49M^2 + 714Mm + 3897m^2}\right]}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{k}{4m(7M + 3m)}\left[7(M + 9m) + \sqrt{49M^2 + 714Mm + 3897m^2}\right]}$$

Example 5.6

Determine the natural frequencies and mode shapes of a system consisting of a beam with a concentrated mass at its mid-span and spring and mass assembly shown in Figure E5.6 by modeling it as a two degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. Take $E = 210$ GPa and $I = 1 \times 10^{-5}$ m^4; $k_1 = 1$ MN/m, $k_2 = 2$ MN/m and $m = 40$ kg.

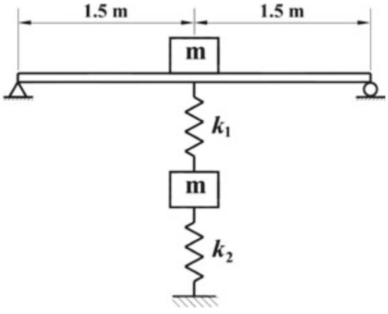


Figure E5.6

Solution

Equivalent stiffness of a simply supported beam with a concentrated mass is located at its mid-span is given by

$$k_b = \frac{48EI}{L^3} = \frac{48 \times 210 \times 10^9 \times 1 \times 10^{-5}}{3^3} = 3.7333\,\text{MN/m}$$

The equivalent stiffness of the beam k_b acts in series with the spring with the stiffness k_1. The equivalent two degree of freedom model of the given system will be as shown in **Figure E5.6(a)**. The equivalent stiffness for the series connection of k_b and k_1 can be determined as

$$k_{b1} = \frac{k_b \times k_1}{k_b + k_1} = \frac{3.7333 \times 1}{3.7333 + 1} = 0.7877\,\text{MN/m}$$

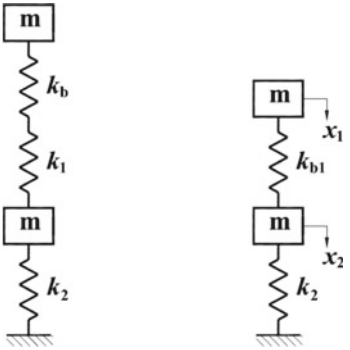


Figure E5.6(a) Equivalent two degree of freedom model for a system shown in Figure E5.6

With reference to the equivalent model, equations of motion for the given system can be written as

$$m\ddot{x}_1 + k_{b1} \times 10^6 x_1 - k_{b1} x_2 = 0$$
$$\therefore\ 40\ddot{x}_1 + 0.7877 \times 10^6 x_1 - 0.7877 \times 10^6 x_2 = 0 \qquad \textbf{(a)}$$

$$m\ddot{x}_2 + (k_{b1} + k_2) \times 10^6 x_2 - k_{b1} x_1 = 0$$
$$\therefore\ 40\ddot{x}_2 + 2.7877 \times 10^6 x_2 - 0.7877 \times 10^6 x_1 = 0 \quad \textbf{(b)}$$

We can assume the solution for a system of linear differential equations as

$$\left.\begin{aligned} x_1 &= A_1 \sin \omega_n t \\ x_2 &= A_2 \sin \omega_n t \end{aligned}\right. \textbf{(c)}$$

Substituting Eq. (**c**) into Eqs. (**a**) and (**b**), we get a system of algebraic equations as

$$\left.\begin{aligned} &\left[0.7877 \times 10^6 - 40\omega_n^2\right]A_1 - 0.7877 \times 10^6 A_2 = 0 \\ &-0.7877 \times 10^6 A_1 + \left[2.7877 \times 10^6 - 40\omega_n^2\right]A_2 = 0 \end{aligned}\right. \textbf{(d)}$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40\omega_n^2} \qquad (\mathbf{e})$$

$$\frac{A_1}{A_2} = \frac{2.7877 \times 10^6 - 40\omega_n^2}{0.7877 \times 10^6} \qquad (\mathbf{f})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40\lambda} = \frac{2.7877 \times 10^6 - 40\lambda}{0.7877 \times 10^6}$$

or, $\{(0.7877 \times 10^6 - 40\lambda)(2.7877 \times 10^6 - 40\lambda)\} - (0.7877 \times 10^6)^2 = 0$

or, $\lambda^2 - 89436.6197\lambda + 9.8592 \times 10^8 = 0$

$\therefore \quad \lambda_1 = 12877.8993, \lambda_2 = 76558.7203 \qquad (\mathbf{g})$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{12877.8993} = 113.4808 \text{ rad/s}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{76558.7203} = 276.6925 \text{ rad/s}$$

Substituting $\lambda = \lambda_1 (= 12877.8993)$ into Eq. (**e**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40\lambda_1} = \frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40 \times 12877.8993} = 2.8826$$

Similarly, substituting $\lambda = \lambda_2 (= 76558.7203)$ into Eq. (**e**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40\lambda_2} = \frac{0.7877 \times 10^6}{0.7877 \times 10^6 - 40 \times 76558.7203} = -0.3469$$

Example 5.7

Determine the two natural frequencies and the modes of vibration of the system shown in Figure E5.7. The two equal masses are under a very large tension *T*.

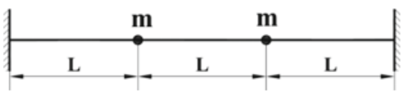


Figure E5.7

Solution

Assume that transverse displacements of two masses are x_1 and x_2 as shown in **Figure E5.7(a)**.

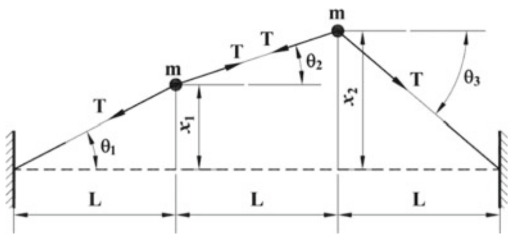


Figure E5.7(a)

Referring to the free-body diagram of the first mass and applying Newton second law of motion,

$$\sum F_1 = m\ddot{x}_1$$

$$\text{or,} \quad -T \ \sin\theta_1 + T \ \sin\theta_2 = m\ddot{x}_1$$

For small transverse displacements,

$$\sin\theta_1 \approx \theta_1 = \frac{x_1}{L} \quad \text{and} \quad \sin\theta_2 \approx \theta_2 = \frac{x_2 - x_1}{L}$$

Then equation of motion reduces to

$$-T\frac{x_1}{L} + T\left(\frac{x_2 - x_1}{L}\right) = m\ddot{x}_1$$

$$\therefore \ m\,\ddot{x}_1 + \frac{2T}{L}x_1 - \frac{T}{L}x_2 = 0 \qquad \textbf{(a)}$$

Again referring to the free-body diagram of the second mass and applying Newton second law of motion,

$$\sum F_2 = m\ddot{x}_2$$

$$-T\sin\theta_2 - T\sin\theta_3 = m\ddot{x}_2$$

For small transverse displacements,

$$\sin\theta_3 \approx \theta_3 = \frac{x_2}{L}$$

Then equation of motion reduces to

$$-T\left(\frac{x_2 - x_1}{L}\right) - T\frac{x_2}{L} = m\ddot{x}_2$$

$$\therefore \quad m\ddot{x}_2 + \frac{2T}{L}x_2 - \frac{T}{L}x_1 = 0 \qquad (\mathbf{b})$$

We can assume the solution for a system of linear differential equations as

$$x_1 = A_1 \sin \omega_n t$$
$$x_2 = A_2 \sin \omega_n t \qquad (\mathbf{c})$$

Substituting Eq. (**b**) into Eq. (**a**), we get a system of algebraic equations as

$$\left[\frac{2T}{L} - m\omega_n^2\right]A_1 - \frac{T}{L}A_2 = 0$$
$$-\frac{T}{L}A_1 + \left[\frac{2T}{L} - m\omega_n^2\right]A_2 = 0 \quad (\mathbf{d})$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{T}{2T - mL\omega_n^2} \quad (\mathbf{e})$$

$$\frac{A_1}{A_2} = \frac{2T - mL\omega_n^2}{T} \quad (\mathbf{f})$$

Equating Eqs. (**e**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{T}{2T - mL\lambda} = \frac{2T - mL\lambda}{T}$$
$$\text{or,} \quad (2T - mL\lambda)^2 = T^2$$
$$\text{or,} \quad 2T - mL\lambda = \pm T$$
$$\text{or,} \quad \lambda = \frac{2T \pm T}{mL}$$
$$\therefore \quad \lambda_1 = \frac{T}{mL}, \quad \lambda_2 = \frac{3T}{mL} \qquad (\mathbf{g})$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{T}{mL}}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3T}{mL}}$$

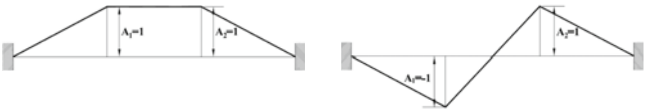
Substituting $\lambda = \lambda_1 (= T/mL)$ into Eq. (**e**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{T}{2T - mL\lambda_1} = 1$$

Similarly, substituting $\lambda = \lambda_2 (= 3T/mL)$ into Eq. (**e**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{T}{2T - mL\lambda_2} = -1$$

Mode shapes are shown graphically as shown in **Figure E5.7(b)**.



Mode Shape Corresponding to $\omega_1 = \sqrt{T/mL}$

Mode Shape Corresponding to $\omega_2 = \sqrt{3T/mL}$

Figure E5.7(b) Mode shapes for a system of Figure E5.7

Example 5.8

Determine the natural frequencies of a two degree of freedom system shown in Figure E5.8.

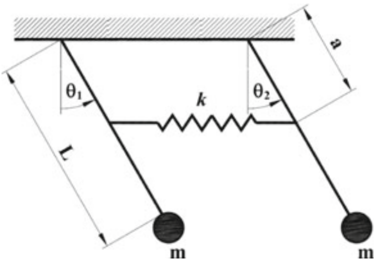


Figure E5.8

Solution

Referring to the free-body diagram shown in **Figure E5.8a** of the first mass and applying Newton's second law of motion,

$$\sum M_1 = I_0\ddot{\theta}_1$$

$$\text{or,} \quad -mg \times L\theta_1 + ka(\theta_2 - \theta_1) \times a = mL^2\ddot{\theta}_1$$

$$\therefore \quad mL^2\ddot{\theta}_1 + \left(mgL + ka^2\right)\theta_1 - ka^2\theta_2 = 0 \qquad (\mathbf{a})$$

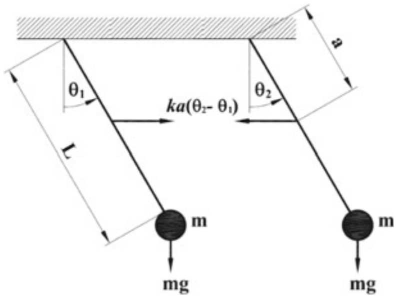


Figure E5.8(a)

Again applying Newton's second law of motion for the second mass

$$\sum M_2 = I_0\ddot{\theta}_2$$

$$\text{or,} \quad -mg \times L\theta_2 - ka(\theta_2 - \theta_1) \times a = mL^2\ddot{\theta}_2$$

$$\therefore \quad mL^2\ddot{\theta}_2 + \left(mgL + ka^2\right)\theta_2 - ka^2\theta_1 = 0 \qquad (\mathbf{b})$$

We can assume the solution for a system of linear differential equations as

$$\theta_1 = \overline{\theta}_1 \sin\omega_n t$$
$$\theta_2 = \overline{\theta}_2 \sin\omega_n t \qquad (\mathbf{c})$$

Substituting Eq. (**b**) into Eq. (**a**), we get a system of algebraic equations as

$$\left[\left(mgL + ka^2\right) - mL^2\omega_n^2\right]\overline{\theta}_1 - ka^2\overline{\theta}_2 = 0$$
$$-\frac{T}{L}\overline{\theta}_1 + \left[\frac{2T}{L} - m\omega_n^2\right]\overline{\theta}_2 = 0 \qquad (\mathbf{d})$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{\overline{\theta}_1}{\overline{\theta}_2} = \frac{ka^2}{\left(mgL + ka^2\right) - mL^2\omega_n^2} \quad \textbf{(e)}$$

$$\frac{\overline{\theta}_1}{\overline{\theta}_2} = \frac{\left(mgL + ka^2\right) - mL^2\omega_n^2}{ka^2} \quad \textbf{(f)}$$

Equating Eqs. (**e**) and (**f**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{ka^2}{\left(mgL + ka^2\right) - mL^2\lambda} = \frac{\left(mgL + ka^2\right) - mL^2\lambda}{ka^2}$$

$$\text{or,} \quad \left[\left(mgL + ka^2\right) - mL^2\lambda\right]^2 = k^2a^4$$

$$\text{or,} \quad \left(mgL + ka^2\right) - mL^2\lambda = \pm ka^2$$

$$\text{or,} \quad \lambda = \frac{mgL + ka^2 \pm ka^2}{mL^2}$$

$$\therefore \quad \lambda_1 = \frac{g}{L}, \lambda_2 = \frac{g}{L} + \frac{2k}{m}\left(\frac{a}{L}\right)^2 \quad \textbf{(g)}$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{g}{L}}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{g}{L} + \frac{2k}{m}\left(\frac{a}{L}\right)^2}$$

Example 5.9

The mass matrix, stiffness matrix and the mode shapes of a two degree of freedom system are given by

$$[\boldsymbol{M}] = \begin{bmatrix} \boldsymbol{m_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{m_2} \end{bmatrix}, \quad [\boldsymbol{K}] = \begin{bmatrix} \mathbf{2000} & \mathbf{-1000} \\ \mathbf{-1000} & \mathbf{3000} \end{bmatrix} \mathbf{N/m},$$

$$\left(\frac{\boldsymbol{A_1}}{\boldsymbol{A_2}}\right)_{\mathbf{1}} = \mathbf{1}, \quad \left(\frac{\boldsymbol{A_1}}{\boldsymbol{A_2}}\right)_{\mathbf{2}} = \mathbf{-2}$$

If the first natural frequency is given by $\boldsymbol{\omega_1 = 10}$ rad/s, determine the masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ and the second natural frequency of vibration $\boldsymbol{\omega_2}$.

Solution

From the given mass and stiffness matrices, equation of motion for the system can be written as

$$m_1\ddot{x}_1 + 2000x_1 - 1000x_2 = 0 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + 3000x_2 - 1000x_1 = 0 \quad (\mathbf{b})$$

We can assume the solution for a system of linear differential equations as

$$\left.\begin{aligned} x_1 &= A_1 \sin\omega_n t \\ x_2 &= A_2 \sin\omega_n t \end{aligned}\right. \quad (\mathbf{c})$$

Substituting Eq. (**c**) into Eqs. (**a**) and (**b**), we get a system of algebraic equations as

$$\begin{aligned} &\left[2000 - m_1\omega_n^2\right]A_1 - 1000A_2 = 0 \\ &-1000A_1 + \left[3000 - m_2\omega_n^2\right]A_2 = 0 \quad (\mathbf{d}) \end{aligned}$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{1000}{2000 - m_1\omega_n^2} \quad (\mathbf{e})$$

$$\frac{A_1}{A_2} = \frac{3000 - m_2\omega_n^2}{1000} \quad (\mathbf{f})$$

Using Eq. (**e**) for the first natural frequency, we get

$$\begin{aligned} &\left(\frac{A_1}{A_2}\right)_1 = \frac{1000}{2000 - m_1\omega_1^2} \\ \text{or,}\quad &\frac{1000}{2000 - m_1 \times 10^2} = 1 \\ \text{or,}\quad &2000 - 100m_1 = 1000 \\ \text{or,}\quad &100m_1 = 1000 \\ \therefore\quad &m_1 = 10\,\text{kg} \end{aligned}$$

Again using Eq. (**f**) for the first natural frequency, we get

$$\begin{aligned} &\left(\frac{A_1}{A_2}\right)_1 = \frac{3000 - m_2\omega_1^2}{1000} \\ \text{or,}\quad &\frac{3000 - m_2 \times 10^2}{1000} = 1 \end{aligned}$$

$$\text{or,} \quad 3000 - 100m_2 = 1000$$
$$\text{or,} \quad 100m_2 = 2000$$
$$\therefore \quad m_2 = 20\,\text{kg}$$

Using Eq. (**e**) for the second natural frequency, we get

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{1000}{2000 - m_1\omega_2^2}$$
$$\text{or,} \quad \frac{1000}{2000 - m_1 \times \omega_2^2} = -2$$
$$\text{or,} \quad 2000 - 10 \times \omega_2^2 = -500$$
$$\text{or,} \quad 10 \times \omega_2^2 = 2500$$
$$\text{or,} \quad \omega_2^2 = 250$$
$$\therefore \quad \omega_2 = 15.8114\ \text{rad/s}$$

Example 5.10

Determine the free vibration response of a damped two degree of freedom system shown in Figure E5.10. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$ and $c = 50\,\text{N s/m}$. Use initial conditions: $x_1(0) = 0.1$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1$, $\dot{x}_2(0) = 0$.

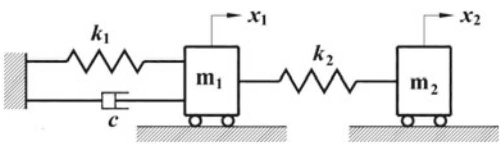


Figure E5.10

Solution

With reference to the free-body diagrams of both masses shown in **Figure E5.10(a)**, equations of motion for the given system can be written as

$$m_1\ddot{x}_1 + c\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

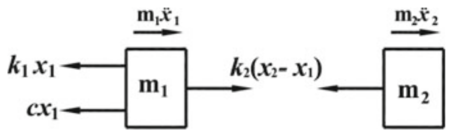


Figure E5.10(a) Free-body diagrams of two mass system of Figure E5.10

Substituting $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $m_1 = 10$ kg, $m_2 = 15$ kg and $c = 50$ N s/m into Eqs. (**a**) and (**b**), we get

$$10\ddot{x}_1 + 50\dot{x}_1 + 2500x_1 - 1500x_2 = 0 \quad (\mathbf{c})$$

$$15\ddot{x}_2 + 1500x_2 - 1500x_1 = 0 \quad (\mathbf{d})$$

We can assume the solution for a system of linear differential equations as

$$x_1 = A_1e^{st} \quad (\mathbf{e})$$

$$x_2 = A_2e^{st}$$

Substituting Eq. (**e**) into Eqs. (**c**) and (**d**), we get a system of algebraic equations as

$$\begin{aligned}\left(10s^2 + 50s + 2500\right)A_1 - 1500A_2 = 0\\ -1500A_1 + \left(15s^2 + 1500\right)A_2 = 0 \quad (\mathbf{f})\end{aligned}$$

Rearranging both equations of Eq. (**f**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{1500}{10s^2 + 50s + 2500} = \frac{150}{s^2 + 5s + 250} \quad (\mathbf{g})$$

$$\frac{A_1}{A_2} = \frac{15s^2 + 1500}{1500} = \frac{s^2 + 100}{100} \quad (\mathbf{h})$$

Equating Eqs. (**g**) and (**h**), we get

$$\begin{aligned}&\frac{150}{s^2 + 5s + 250} = \frac{s^2 + 100}{100}\\ \text{or,}\quad &\left(s^2 + 5s + 250\right)(c_1 + c_2) - \left(s^2 + 100\right)^2 - 15000 = 0\\ \therefore\quad &s^4 + 5s^3 + 350s^2 + 500s + 10000 = 0\end{aligned}$$

The roots of the characteristic equation are obtained as

$$s_{1,2} = -0.6046 \pm i5.6152$$

$$s_{3,4} = -1.8954 \pm i17.6048$$

Substituting s_1, s_2, s_3 and s_4 into Eq. (**h**), we get amplitude ratios corresponding to each root as

$$\left(\frac{A_1}{A_2}\right)_1 = 0.6884 \pm i0.0679$$

$$\left(\frac{A_1}{A_2}\right)_2 = -2.0634 \pm i0.6674$$

Then the expressions for free vibration response of each mass can be expressed as

$$\begin{aligned} x_1 &= A(0.6884 + i0.0679)e^{(-0.6046+i5.6152)t} \\ &\quad + B(0.6884 - i0.0679)e^{(-0.6046-i5.6152)t} \\ &\quad + C(-2.0634 + i0.6674)e^{(-1.8954+i17.6048)t} \\ &\quad + D(-2.0634 - i0.6674)e^{(-1.8954-i17.6048)t} \qquad (\mathbf{i}) \end{aligned}$$

$$\begin{aligned} x_2 &= Ae^{(-0.6046+i5.6152)t} + Be^{(-0.6046-i5.6152)t} \\ &\quad + Ce^{(-1.8954+i17.6048)t} + De^{(-1.8954-i17.6048)t} \qquad (\mathbf{j}) \end{aligned}$$

Equations (**i**) and (**j**) can also be expressed in terms of trigonometric functions as

$$\begin{aligned} x_1 &= e^{-0.6046t}[\{0.6884(A+B) + i0.0679(A-B)\}\cos(5.6153t) \\ &\quad + \{-0.0679(A+B) + i0.6884(A-B)\}\sin(5.6153t)] \\ &\quad + e^{-1.8954t}[\{-2.0634(C+D) + i0.6674(C-D)\}\cos(17.6048t) \\ &\quad + \{-2.0634(C+D) + i0.6884(C-D)\}\sin(17.6048t)] \qquad (\mathbf{k}) \end{aligned}$$

$$\begin{aligned} x_2 &= e^{-0.6046t}[(A+B)\cos(5.6153t) + i(A-B)\sin(5.6153t)] \\ &\quad + e^{-1.8954t}[(C+D)\cos(17.6048t) + i(C-D)\sin(17.6048t)] \qquad (\mathbf{l}) \end{aligned}$$

Using the given initial conditions, $x_1(0) = 0.1$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1$, $\dot{x}_2(0) = 0$, we get

$$\begin{aligned} &(0.6884 + i0.0679)A + (0.6884 - i0.0679)B \\ &\quad + (-2.0634 + i0.6884)C + (-2.0634 - i0.6884)D = 0.1 \qquad (\mathbf{m}) \end{aligned}$$

$$\begin{aligned} &(-0.0349 - i3.9063)A + (-0.0349 + i3.9063)B \\ &\quad + (15.6599 + i35.0599)C + (15.6599 - i35.0599)D = 0 \qquad (\mathbf{n}) \end{aligned}$$

$$A + B + C + D = 0 \quad (\mathbf{o})$$

$$\begin{aligned}&(-0.6046 - i5.6152)A + (-0.6046 + i5.6152)B \\ &\quad + (-1.8954 - i17.6048)C + (-1.8954 + i17.6048)D = 0 \quad (\mathbf{p})\end{aligned}$$

Solving simultaneous Eqs. (**m**) to (**p**), we get

$$\begin{aligned}A &= 0.0171 + i0.0149 \quad B = 0.0171 - i0.0149 \\ C &= -0.0171 - i0.0060 \; D = -0.0171 + i0.0060\end{aligned}$$

Substituting values of A, B, C and D into Eqs. (**k**) and (**l**), we get the expressions for free response of the system as

$$\begin{aligned}x_1 &= e^{-0.6046t}[0.0215\cos(5.6153t) + 0.0227\sin(5.6153t)] \\ &\quad + e^{-1.8954t}[0.0785\cos(17.6048t) + 0.0785\sin(17.6048t)]\end{aligned}$$

$$\begin{aligned}x_2 &= e^{-0.6046t}[0.0342\cos(5.6153t) + 0.0297\sin(5.6153t)] \\ &\quad + e^{-1.8954t}[-0.0342\cos(17.6048t) - 0.0120\sin(17.6048t)]\end{aligned}$$

The plots of response of both masses are also shown in **Figure E5.10(b)**.

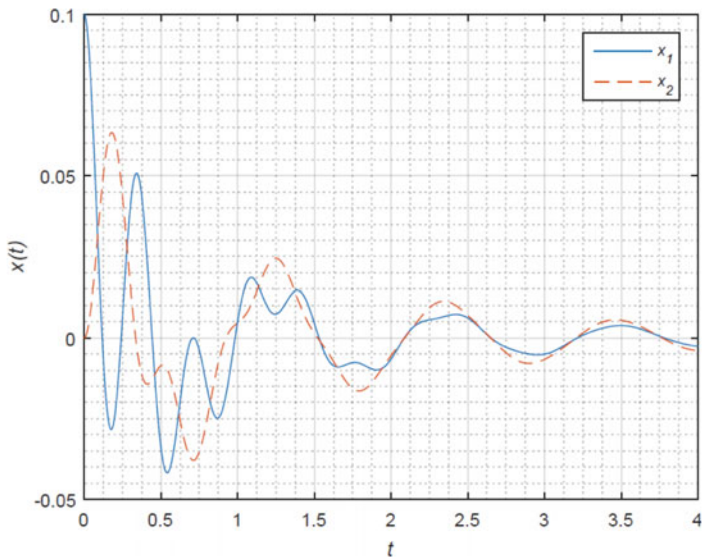


Figure E5.10(b) Response of the system shown in Figure E5.10

Example 5.11

Determine the steady state response of the system shown in Figure E5.11. Take $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $k_3 = 1000\,\mathrm{N/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 15\,\mathrm{kg}$ and $F_1 = 300 \sin 50t$ N.

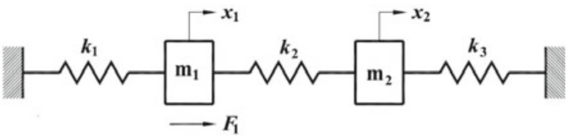


Figure E5.11

Solution

The equation of motions of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 1000$ N/m, $m_1 = 10$ kg, $m_2 = 15$ kg and $F_1 = 300 \sin 50t$ N into Eqs. (**a**) and (**b**), we get

$$10\ddot{x}_1 + 2500x_1 - 1500x_2 = 300 \sin 50t \quad (\mathbf{c})$$

$$15\ddot{x}_2 + 2500x_2 - 1500x_1 = 0 \quad (\mathbf{d})$$

The steady state response of the system can be assumed as

$$x_1 = X_1 \sin 50t \quad (\mathbf{e})$$

$$x_2 = X_2 \sin 50t \quad (\mathbf{f})$$

Substituting Eqs. (**e**) and (**f**) into Eqs. (**c**) and (**d**), we get a system of algebraic equations as

$$\begin{aligned} &\left(2500 - 10 \times 50^2\right)X_1 - 1500X_2 = 300 \\ \therefore \quad &-22500X_1 - 1500X_2 = 300 \qquad (\mathbf{g}) \end{aligned}$$

$$\begin{aligned} &-1500X_1 + \left(2500 - 15 \times 50^2\right)X_2 = 0 \\ \therefore \quad &-1500X_1 - 35000X_2 = 0 \qquad (\mathbf{h}) \end{aligned}$$

Solving simultaneous Eqs. (**g**) and (**h**) for X_1 and X_2, we get

$$X_1 = -0.0134 \quad \text{and} \quad X_2 = 0.006$$

Substituting X_1 and X_2 into Eqs. (**e**) and (**f**), we get the steady state response of the system as

$$x_1 = -0.0134 \sin 50t$$

$$x_2 = 0.006 \sin 50t$$

Example 5.12

Determine the steady state response of the system shown in Figure E5.12. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 200\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $F_1 = 20 \sin 50t$ N and $F_2 = 30 \sin 30t$ N.

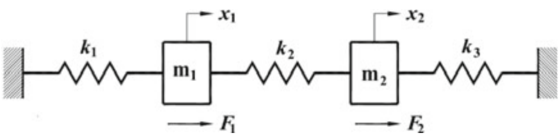


Figure E5.12

Solution

The equations of motion of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = F_2 \quad (\mathbf{b})$$

Substituting $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 200\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $F_1 = 20 \sin 50t$ N and $F_2 = 30 \sin 30t$ N into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 250x_1 - 150x_2 = 20 \sin 50t \quad (\mathbf{c})$$

$$2\ddot{x}_2 + 350x_2 - 150x_1 = 30 \sin 30t \quad (\mathbf{d})$$

The steady state response of the system can be assumed as

$$x_1 = (X_1)_1 \sin 50t + (X_1)_2 \sin 30t \quad (\mathbf{e})$$

$$x_2 = (X_2)_1 \sin 50t + (X_2)_2 \sin 30t \quad (\mathbf{f})$$

Substituting Eqs. (**e**) and (**f**) into Eqs. (**c**) and (**d**) and equating coefficients of $\sin 50t$ and $\sin 30t$ of each equation, we get a system of algebraic equations as

$$-2250(X_1)_1 - 150(X_2)_1 = 20 \quad (\mathbf{g})$$

$$-650(X_1)_2 - 150(X_2)_2 = 0 \quad (\mathbf{h})$$

$$-150(X_1)_1 - 4650(X_2)_1 = 0 \quad (\mathbf{i})$$

$$-150(X_1)_2 - 1450(X_2)_2 = 30 \quad (\mathbf{j})$$

Solving simultaneous Eqs. (**g**) and (**i**) for $(X_1)_1$ and $(X_2)_1$, we get

$$(X_1)_1 = -0.0089 \quad \text{and} \quad (X_2)_1 = 0.0049$$

Solving simultaneous Eqs. (**h**) and (**j**) for $(X_1)_2$ and X_2, we get

$$(X_1)_2 = 0.0003 \quad \text{and} \quad (X_2)_2 = -0.0212$$

Substituting $(X_1)_1$, $(X_2)_1$, $(X_1)_2$ and $(X_2)_2$ into Eqs. (**e**) and (**f**), we get the steady state response of the system as

$$x_1 = -0.0089 \sin 50t + 0.0003 \sin 30t$$

$$x_2 = 0.0049 \sin 50t - 0.0212 \sin 30t$$

Example 5.13

Determine the range of frequency of the external excitation for which the amplitude of steady state vibration of both masses of the system shown in Figure E5.13 does not exceed 15 mm. Take $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 2000$ N/m, $m_1 = 5$ kg, $m_2 = 10$ kg and $F_1 = 20 \sin \omega t$ N.

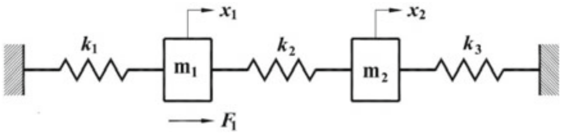


Figure E5.13

Solution

The equations of motion of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 2000\,\text{N/m}$, $m_1 = 5\,\text{kg}$, $m_2 = 10\,\text{kg}$ and $F_1 = 20\sin\omega t$ N into Eqs. (**a**) and (**b**), we get

$$5\ddot{x}_1 + 2500x_1 - 1500x_2 = 20\sin\omega t \quad (\mathbf{c})$$

$$10\ddot{x}_3 + 3500x_2 - 1500x_1 = 0 \quad (\mathbf{d})$$

The steady state response of the system can be assumed as

$$x_1 = X_1\sin\omega t \quad (\mathbf{e})$$

$$x_2 = X_2\sin\omega t \quad (\mathbf{f})$$

Substituting Eqs. (**e**) and (**f**) into Eqs. (**c**) and (**d**), we get a system of algebraic equations as

$$\begin{aligned} &\left(2500 - 5\times\omega^2\right)X_1 - 1500X_2 = 20 \\ \therefore\quad &\left(500 - \omega^2\right)X_1 - 300X_2 = 4 \qquad (\mathbf{g}) \end{aligned}$$

$$\begin{aligned} &-1500X_1 + \left(3500 - 10\times\omega^2\right)X_2 = 0 \\ \therefore\quad &-150X_1 + \left(350 - \omega^2\right)X_2 = 0 \qquad (\mathbf{h}) \end{aligned}$$

Solving simultaneous Eqs. (**g**) and (**h**) for X_1 and X_2, we get

$$X_1 = \frac{4\left(350 - \omega^2\right)}{\omega^4 - 850\omega^2 + 130000} \quad (\mathbf{i})$$

$$X_2 = \frac{600}{\omega^4 - 850\omega^2 + 130000} \quad \textbf{(j)}$$

Variations of vibration amplitudes X_1 and X_2 with the frequency of external excitation are shown in **Figure E5.13(a)** and **Figure E5.13(b)**.

Substituting $X_1 = 15 \times 10^{-3}$ m into Eq. (**i**),

$$\frac{4(350 - \omega^2)}{\omega^4 - 850\omega^2 + 130000} = 15 \times 10^{-3}$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 130000 = -266.667\omega^2 + 93333.333$$

$$\text{or,} \quad \omega^4 - 583.333\omega^2 + 36666.667 = 0$$

$$\therefore \quad \omega_1 = 8.4652\,\text{rad/s}, \quad \omega_2 = 22.6201\,\text{rad/s}$$

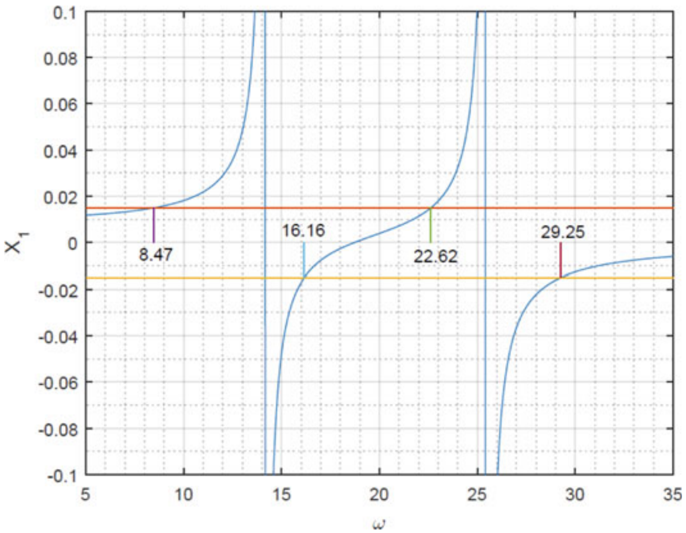


Figure E5.13 (a) Variation of steady state amplitude X_1 with the frequency ω

Substituting $X_1 = -15 \times 10^{-3}$ m into Eq. (**i**),

$$\frac{4(350 - \omega^2)}{\omega^4 - 850\omega^2 + 130000} = -15 \times 10^{-3}$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 130000 = 266.667\omega^2 - 93333.333$$

$$\text{or,} \quad \omega^4 - 1116.667\omega^2 + 223333.333 = 0$$

$$\therefore \quad \omega_1 = 16.1557\,\text{rad/s}, \quad \omega_2 = 29.2517\,\text{rad/s}$$

It can be noted from **Figure E5.13(a)** that the amplitude of steady state vibration X_1 will be less than 15 mm if the frequency of the external excitation is less than 8.4652 rad/s, lies between 16.1557 rad/s rad/s and 22.6201 rad/s or greater than 29.2517 rad/s.

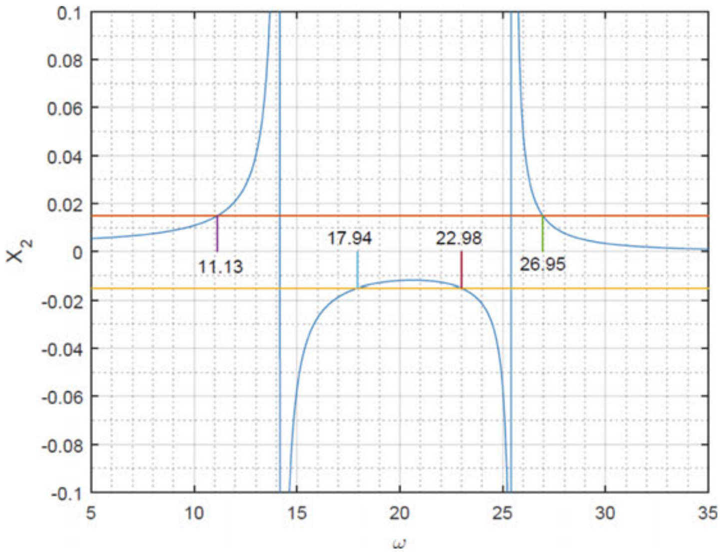


Figure E55.13 (b) Variation of steady state amplitude X_2 with the frequency ω

Substituting $X_2 = 15 \times 10^{-3}$ m into Eq. (**j**),

$$\frac{600}{\omega^4 - 850\omega^2 + 130000} = 15 \times 10^{-3}$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 130000 = 40000$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 90000 = 0$$

$$\therefore \quad \omega_1 = 11.1337\,\text{rad/s}, \quad \omega_2 = 26.9451\,\text{rad/s}$$

Again substituting $X_2 = -15 \times 10^{-3}$ m into Eq. (**j**),

$$\frac{600}{\omega^4 - 850\omega^2 + 130000} = -15 \times 10^{-3}$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 130000 = -40000$$

$$\text{or,} \quad \omega^4 - 850\omega^2 + 170000 = 0$$

$$\therefore \quad \omega_1 = 17.9422\,\text{rad/s}, \quad \omega_2 = 22.9799\,\text{rad/s}$$

It can be noted from **Figure E5.13(b)** that the amplitude of steady state vibration X_2 will be less than 15 mm if the frequency of the external excitation is less than 11.1337 rad/s, lies between 17.9422 rad/s and 22.9799 rad/s or greater than 26.9451 rad/s.

Hence the steady state amplitudes of both masses will be less than 15*mm* if the frequency of the external excitation is less than 11.1337 rad/s, lies between 17.9422 rad/s and 22.9799 rad/s or greater than 29.2517 rad/s.

Example 5.14

Determine the steady state response of the system shown in Figure E5.14. Take $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $c = 80\,\mathrm{N\,s/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 15\,\mathrm{kg}$ and $F_1 = 200\sin 40t$ N.

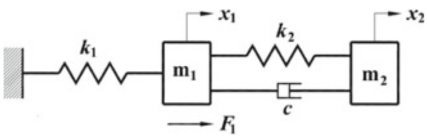


Figure E5.14

Solution

The equations of motion of the system can be written as

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + c\dot{x}_2 - c\dot{x}_1 + k_2x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $c = 80$ N s/m, $m_1 = 10$ kg, $m_2 = 15$ kg and $F_1 = 200\sin 40t$ N into Eqs. (**a**) and (**b**), we get

$$10\ddot{x}_1 + 80\dot{x}_1 - 80\dot{x}_2 + 2500x_1 - 1500x_2 = 200\sin 40t \quad (\mathbf{c})$$

$$15\ddot{x}_2 + 80\dot{x}_2 - 80\dot{x}_1 + 1500x_2 - 1500x_1 = 0 \quad (\mathbf{d})$$

The steady state response of the system can be assumed as

$$x_1 = X_1\sin 40t + Y_1\cos 40t \quad (\mathbf{e})$$

$$x_2 = X_2\sin 40t + Y_2\cos 40t \quad (\mathbf{f})$$

Substituting Eqs. (**e**) and (**f**) into Eqs. (**c**) and (**d**) and equating coefficients of $\sin 40t$ and $\cos 40t$ of each equation, we get a system of algebraic equations as

$$-13500X_1 - 3200Y_1 - 1500X_2 + 3200Y_2 = 200 \quad (\mathbf{g})$$

$$3200X_1 - 13500Y_1 - 3200X_2 - 1500Y_2 = 0 \quad (\mathbf{h})$$

$$-1500X_1 + 3200Y_1 - 22500X_2 - 3200Y_2 = 0 \quad (\mathbf{i})$$

$$-3200X_1 - 1500Y_1 + 3200X_2 - 22500Y_2 = 0 \quad (\mathbf{j})$$

Solving simultaneous Eqs. (**g**), (**h**), (**i**) and (**j**) for X_1, Y_1, X_2 and Y_2, we get

$$X_1 = -0.0135, \quad Y_1 = 0.0035, \quad X_2 = 0.0001 \quad \text{and} \quad Y_2 = 0.0022$$

Substituting X_1, Y_1, X_2 and Y_2 into Eqs. (**e**) and (**f**), we get the steady state response of the system as

$$x_1 = -0.0135 \sin 40t + 0.0035 \cos 40t$$

$$x_2 = 0.0001 \sin 40t + 0.0022 \cos 40t$$

Example 5.15

A machine has a mass of 50 kg and runs at a speed of 4000 rpm. The frequency of the external excitation is very near to its natural frequency. Determine the mass and spring constant of the absorber system to be added if the natural frequencies of the system are to be at least 20% from the frequency of the external excitation.

Solution

Mass of the machine, $m_1 = 50\,\text{kg}$.

Operating speed of the machine in rpm, $N = 4000\,\text{rpm}$.

The operating machine of the machine in rad/s can be determined as

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 4000}{60} = 418.879\,\text{rad/s}$$

For a tuned vibration absorber

$$\omega_n = \omega = 418.879\,\text{rad/s}$$

Then the stiffness of the primary system can be determined as

$$k_1 = m_1\omega_n^2 = 50 \times (418.879)^2 = 8.7730\,\text{MN/m}$$

Relationship between the frequency ratio and mass ratio for a tuned vibration absorber is given by

$$\left(\frac{\omega}{\omega_{22}}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \tag{a}$$

Substituting $\omega/\omega_{22} = 0.8$, we get

$$\begin{aligned}
&(0.8)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & 0.64 - 1 - \frac{\mu}{2} = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & \left(-0.36 - \frac{\mu}{2}\right)^2 = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & 0.25\mu^2 + 0.36\mu + 0.1296 = \mu + 0.25\mu^2 \\
\text{or,}\quad & 0.64\mu = 0.1296 \\
\therefore\quad & \mu = 0.2025
\end{aligned}$$

Similarly, substituting $\omega/\omega_{22} = 1.2$, we get

$$\begin{aligned}
&(1.2)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & 1.44 - 1 - \frac{\mu}{2} = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & \left(0.44 - \frac{\mu}{2}\right)^2 = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
\text{or,}\quad & 0.25\mu^2 - 0.44\mu + 0.1936 = \mu + 0.25\mu^2 \\
\text{or,}\quad & 1.44\mu = 0.1936 \\
\therefore\quad & \mu = 0.1344
\end{aligned}$$

The larger value of μ will give wider range of operating frequency and therefore $\mu = 0.2025$ is chosen. Then mass of the absorber system can be determined as

$$m_2 = \mu m_1 = 0.2025 \times 50 = 10.125\,\text{kg}$$

Similarly, the stiffness of the absorber system can be determined as

$$k_2 = \frac{m_2}{m_1} k_1 = \mu k_1 = 0.2025 \times 8.7730 \times 10^6 = 1.777\,\text{MN/m}$$

Example 5.16

A machine is found undergoing violent vibration at an operating speed of 2000 rpm. To avoid this difficulty, it was proposed to attach a spring- mass assembly to act as an absorber. When a trial mass of 1.5 kg is attached to the

machine, it resulted in two natural frequencies of 1700 and 2300 rpm. If the absorber system is to be designed so that the natural frequencies lie outside the region 1500 and 2500 rpm, determine the required mass and stiffness of the absorber system.

Solution

Operating speed of the machine in rpm, $m = 2000$ rpm.

The operating machine of the machine in rad/s can be determined as

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 2000}{60} = 209.4395\,\text{rad/s}$$

Frequency ratios for given trial mass of the absorber system can be determined as

$$\left(\frac{\omega}{\omega_{22}}\right)_{t1} = \frac{1700}{2000} = 0.85 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right)_{t2} = \frac{2300}{2000} = 1.15$$

Relationship between the frequency ratio and mass ratio for a tuned vibration absorber is given by

$$\left(\frac{\omega}{\omega_{22}}\right)^2 = \left(1 + \frac{\mu_t}{2}\right) \pm \sqrt{\mu_t + \frac{\mu_t^2}{4}} \quad (\mathbf{a})$$

Using Eq. (**a**) when the trail mass is used, we get

$$\left(\frac{\omega_{t1}}{\omega_{22}}\right)^2 = \left(1 + \frac{\mu_t}{2}\right) - \sqrt{\mu_t + \frac{\mu_t^2}{4}} \quad (\mathbf{b})$$

$$\left(\frac{\omega_{t2}}{\omega_{22}}\right)^2 = \left(1 + \frac{\mu_t}{2}\right) + \sqrt{\mu_t + \frac{\mu_t^2}{4}} \quad (\mathbf{c})$$

Adding Eqs. (**b**) and (**c**), we get

$$\left(\frac{\omega_{t1}}{\omega_{22}}\right)^2 + \left(\frac{\omega_{t2}}{\omega_{22}}\right)^2 = 2 + \mu_t$$

$$\text{or,} \quad (0.85)^2 + (1.15)^2 = 2 + \mu_t$$

$$\text{or,} \quad 2 + \mu_t = 2.045$$

$$\therefore \quad \mu_t = 0.045$$

Mass of the machine can be then determined as

$$m_1 = \frac{m_t}{\mu_t} = \frac{1.5}{0.045} = 33.33\,\text{kg}$$

For a tuned vibration absorber

$$\omega_n = \omega = 209.4395\,\text{rad/s}$$

Then the stiffness of the primary system can be determined as

$$k_1 = m_1\omega_n^2 = 33.33 \times (209.4395)^2 = 1.4622\,\text{MN/m}$$

Frequency ratios for the required absorber system can be determined as

$$\left(\frac{\omega}{\omega_{22}}\right)_1 = \frac{1500}{2000} = 0.75 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right)_2 = \frac{2500}{2000} = 1.25$$

Substituting $\omega/\omega_{22} = 0.75$ into Eq. (**a**), we get

$$\begin{aligned}
&(0.75)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } 0.5625 - 1 - \frac{\mu}{2} = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } \left(-0.4375 - \frac{\mu}{2}\right)^2 = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } 0.25\mu^2 + 0.4375\mu + 0.1914 = \mu + 0.25\mu^2 \\
&\text{or, } 0.5625\mu = 0.1914 \\
&\therefore \mu = 0.3402
\end{aligned}$$

Similarly, substituting $\omega/\omega_{22} = 1.25$, we get

$$\begin{aligned}
&(1.25)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } 1.5625 - 1 - \frac{\mu}{2} = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } \left(0.5625 - \frac{\mu}{2}\right)^2 = \pm\sqrt{\mu + \frac{\mu^2}{4}} \\
&\text{or, } 0.25\mu^2 - 0.5625\mu + 0.3164 = \mu + 0.25\mu^2 \\
&\text{or, } 1.5625\mu = 0.3164 \\
&\therefore \mu = 0.2025
\end{aligned}$$

The larger value of μ will give wider range of operating frequency and therefore $\mu = 0.3402$ is chosen. Then mass of the absorber system can be determined as

$$m_2 = \mu m_1 = 0.3402 \times 33.33 = 11.3426\,\text{kg}$$

Similarly, the stiffness of the absorber system can be determined as

$$k_2 = \frac{m_2}{m_1} k_1 = \mu k_1 = 0.3402 \times 1.4622 \times 10^6 = 0.4975\,\text{MN/m}$$

Example 5.17

Determine the free response of a system shown in Figure E5.17 using Laplace transform approach. The system is subjected to an initial conditions $x_1(0) = 0.1$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1$, $\dot{x}_2(0) = 1\,\text{m/s}$. Take $k_1 = 100\,\text{N/m}$, $k_2 = 200\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1\,\text{kg}$.

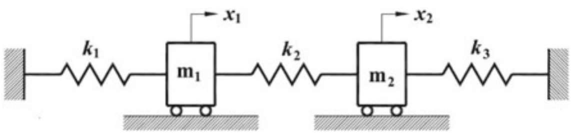


Figure E5.17

Solution

The equations of motion of the system can be written as

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (\mathbf{a})$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = 100\,\text{N/m}$, $k_2 = 200\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1\,\text{kg}$ into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 300x_1 - 200x_2 = 0 \quad (\mathbf{c})$$

$$\ddot{x}_2 + 300x_2 - 200x_1 = 0 \quad (\mathbf{d})$$

Taking Laplace transform of Eqs. (**c**) and (**d**), we get

$$\left(s^2 + 300\right)X_1(s) - 200X_2(s) = sx_1(0) + \dot{x}_1(0) \quad (\mathbf{e})$$

$$-200X_1(s) + \left(s^2 + 300\right)X_2(s) = sx_2(0) + \dot{x}_2(0) \quad (\mathbf{f})$$

Substituting given initial conditions, Eqs. (**e**) and (**f**) reduce to

$$\left(s^2 + 300\right)X_1(s) - 200X_2(s) = 0.1s \quad (\mathbf{g})$$

$$-200X_1(s) + \left(s^2 + 300\right)X_2(s) = 1 \quad (\mathbf{h})$$

Solving Eqs. (**g**) and (**h**) for $X_1(s)$ and $X_2(s)$, we get

$$X_1(s) = \frac{0.1\left(s^3 + 300s + 2000\right)}{\left(s^2 + 100\right)\left(s^2 + 500\right)} \quad (\mathbf{i})$$

$$X_2(s) = \frac{0.1\left(s^2 + 20s + 300\right)}{\left(s^2 + 100\right)\left(s^2 + 500\right)} \quad (\mathbf{j})$$

Equations (**i**) and (**j**) can be expressed in terms of their partial fractions as

$$X_1(s) = \frac{0.05s}{\left(s^2 + 100\right)} + \frac{0.5}{\left(s^2 + 100\right)} + \frac{0.5s}{\left(s^2 + 500\right)} - \frac{0.5}{\left(s^2 + 500\right)} \quad (\mathbf{k})$$

$$X_2(s) = \frac{0.05s}{\left(s^2 + 100\right)} + \frac{0.5}{\left(s^2 + 100\right)} - \frac{0.5s}{\left(s^2 + 500\right)} + \frac{0.5}{\left(s^2 + 500\right)} \quad (\mathbf{l})$$

Taking inverse Laplace transform of Eqs. (**k**) and (**l**), we get the response of the system as

$$x_1(t) = 0.05\cos(10t) + 0.05\sin(10t) + 0.05\cos(22.3607t) - 0.0223\sin(22.3607t)$$

$$x_2(t) = 0.05\cos(10t) + 0.05\sin(10t) - 0.05\cos(22.3607t) + 0.0223\sin(22.3607t)$$

Example 5.18

Determine the free response of a system shown in Figure E5.18 using Laplace transform approach. The system is subjected to an initial conditions $x_1(0) = 0.1$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1$, $\dot{x}_2(0) = 1\,\mathrm{m/s}$. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 100\,\mathrm{N/m}$, $c_1 = 4\,\mathrm{N\,s/m}$, $c_2 = 8\,\mathrm{N\,s/m}$, $c_3 = 4\,\mathrm{N\,s/m}$, $m_1 = 1\,\mathrm{kg}$ and $m_2 = 1\,\mathrm{kg}$.

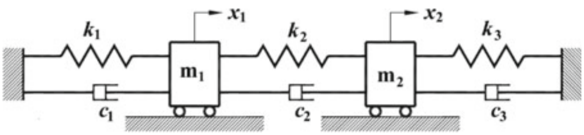


Figure E5.18

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2 + k_3)x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $c_1 = 4\,\text{N s/m}$, $c_2 = 8\,\text{N s/m}$, $c_3 = 4\,\text{N s/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1\,\text{kg}$ into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 12\dot{x}_1 - 8\dot{x}_2 + 250x_1 - 150x_2 = 0 \quad (\mathbf{c})$$

$$\ddot{x}_2 + 12\dot{x}_2 - 8\dot{x}_1 + 250x_2 - 150x_1 = 0 \quad (\mathbf{d})$$

Taking Laplace transform of Eqs. (**c**) and (**d**), we get

$$\begin{aligned}&\left(s^2 + 12s + 250\right)X_1(s) - (8s + 150)X_2(s)\\ &\quad = 12x_1(0) + sx_1(0) + \dot{x}_1(0) - 8x_2(0) \quad (\mathbf{e})\end{aligned}$$

$$\begin{aligned}&-(8s + 150)X_1(s) + \left(s^2 + 12s + 250\right)X_2(s)\\ &\quad = 12x_2(0) + sx_2(0) + \dot{x}_2(0) - 8x_1(0) \qquad (\mathbf{f})\end{aligned}$$

Substituting the given initial conditions, Eqs. (**e**) and (**f**) reduce to

$$\left(s^2 + 12s + 250\right)X_1(s) - (8s + 200)X_2(s) = 1.2 + 0.1s \quad (\mathbf{g})$$

$$-(8s + 200)X_1(s) + \left(s^2 + 12s + 250\right)X_2(s) = 0.2 \quad (\mathbf{h})$$

Solving Eqs. (**g**) and (**h**) for $X_1(s)$ and $X_2(s)$, we get

$$X_1(s) = \frac{0.1s^3 + 2.4s^2 + 41s + 330}{\left(s^2 + 4s + 100\right)\left(s^2 + 20s + 400\right)} \quad (\mathbf{i})$$

$$X_2(s) = \frac{s^2 + 27s + 230}{\left(s^2 + 4s + 100\right)\left(s^2 + 20s + 400\right)} \quad (\mathbf{j})$$

Equations (**i**) and (**j**) can be expressed in terms of their partial fractions as

$$\begin{aligned}X_1(s) = &\frac{0.05(s + 2)}{(s + 2)^2 + (9.7979)^2} + \frac{0.6}{(s + 2)^2 + (9.7979)^2}\\ &+ \frac{0.05(s + 10)}{(s + 10)^2 + (17.3205)^2} \qquad (\mathbf{k})\end{aligned}$$

$$X_2(s) = \frac{0.05(s+2)}{(s+2)^2 + (9.7979)^2} + \frac{0.6}{(s+2)^2 + (9.7979)^2} - \frac{0.05(s+10)}{(s+10)^2 + (17.3205)^2} \quad \textbf{(l)}$$

Taking inverse Laplace transform of Eqs. (**k**) and (**l**), we get the response of the system as

$$x_1(t) = 0.05e^{-2t}\cos(9.7979t) + 0.0612e^{-2t}\sin(9.7979t) + 0.05e^{-10t}\cos(17.3205t)$$

$$x_2(t) = 0.05e^{-2t}\cos(9.7979t) + 0.0612e^{-2t}\sin(9.7979t) - 0.05e^{-10t}\cos(17.3205t)$$

Example 5.19

Use method of Laplace transform to determine the steady state response of the system shown in Figure E5.19. Take $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $k_3 = 1000\,\mathrm{N/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 10\,\mathrm{kg}$ and $F_2 = 200\sin 40t$ N.

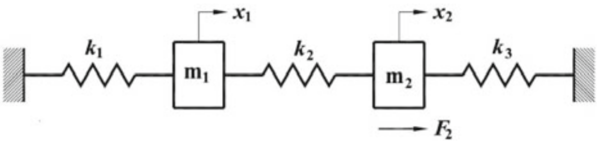


Figure E5.19

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \quad \textbf{(a)}$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = F_2 \quad \textbf{(b)}$$

Substituting $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $k_3 = 1000\,\mathrm{N/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 10\,\mathrm{kg}$ and $F_2 = 200\sin 40t$ into Eqs. (**a**) and (**b**), we get

$$10\ddot{x}_1 + 2500x_1 - 1500x_2 = 0 \quad \textbf{(c)}$$

$$10\ddot{x}_2 + 2500x_2 - 1500x_1 = 200\sin 40t \quad \textbf{(d)}$$

Taking Laplace transform of Eqs. (**c**) and (**d**), we get

$$\left(10s^2 + 300\right)X_1(s) - 1500X_2(s) = 10sx_1(0) + 10\dot{x}_1(0) \quad \textbf{(e)}$$

$$-1500X_1(s) + \left(10s^2 + 2500\right)X_2(s) = 10sx_2(0) + 10\dot{x}_2(0) + \frac{8000}{s^2 + 1600} \quad \textbf{(f)}$$

For the steady state response, substituting values of all initial conditions as zero, Eqs. (**e**) and (**f**) reduce to

$$\left(10s^2 + 300\right)X_1(s) - 1500X_2(s) = 0 \quad \textbf{(g)}$$

$$-1500X_1(s) + \left(10s^2 + 2500\right)X_2(s) = \frac{8000}{s^2 + 1600} \quad \textbf{(h)}$$

Solving Eqs. (**g**) and (**h**) for $X_1(s)$ and $X_2(s)$, we get

$$X_1(s) = \frac{120000}{\left(s^2 + 1600\right)\left(s^4 + 500s^2 + 4000\right)} \quad \textbf{(i)}$$

$$X_2(s) = \frac{800\left(s^2 + 250\right)}{\left(s^2 + 1600\right)\left(s^4 + 500s^2 + 4000\right)} \quad \textbf{(j)}$$

Equations (**i**) and (**j**) can be expressed in terms of their partial fractions as

$$X_1(s) = \frac{1}{15\left(s^2 + 1600\right)} + \frac{s^2 - 1100}{\left(s^4 + 500s^2 + 4000\right)} \quad \textbf{(k)}$$

$$X_2(s) = \frac{1}{5\left(s^2 + 1600\right)} + \frac{3s^2 + 700}{\left(s^4 + 500s^2 + 4000\right)} \quad \textbf{(l)}$$

Denominator of second term of both Eqs. (**k**) and (**l**) is the characteristic equation of the system. For the steady state response, we can consider the first term of both equations, i.e.

$$X_1(s) = \frac{1}{15\left(s^2 + 1600\right)} \quad \textbf{(m)}$$

$$X_2(s) = \frac{1}{5\left(s^2 + 1600\right)} \quad \textbf{(n)}$$

Taking inverse Laplace transform of Eqs. (**k**) and (**l**), we get the response of the system as

$$x_1(t) = \frac{1}{600}\sin(40t)$$

$$x_2(t) = -\frac{3}{200}\sin(40t)$$

Example 5.20

Use method of Laplace transform to determine the steady state response of the system shown in Figure E5.20. Take $k_1 = k_2 = k_3 = 100\,\mathrm{N/m}$, $c_1 = c_2 = c_3 = 5\,\mathrm{N\,s/m}$, $m_1 = m_2 = 1\,\mathrm{kg}$ and $F_1 = 500 \sin 30t$ N.

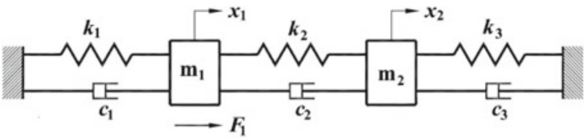


Figure E5.20

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2 + k_3)x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = k_2 = k_3 = 100\,\mathrm{N/m}, c_1 = c_2 = c_3 = 5\,\mathrm{N\,s/m}, m_1 = m_2 = 1\,\mathrm{kg}$ and $F_1 = 500 \sin 30t$ into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 10\dot{x}_1 - 5\dot{x}_2 + 200x_1 - 100x_2 = 500 \sin 30t \quad (\mathbf{c})$$

$$\ddot{x}_2 + 10\dot{x}_2 - 5\dot{x}_1 + 200x_2 - 100x_1 = 0 \quad (\mathbf{d})$$

Taking Laplace transform of Eqs. (**c**) and (**d**), we get

$$\begin{aligned}&\left(s^2 + 10s + 200\right)X_1(s) - (5s + 100)X_2(s)\\ &= sx_1(0) + \dot{x}_1(0) + x_2(0) + \frac{15000}{s^2 + 900} \qquad (\mathbf{e})\end{aligned}$$

$$\begin{aligned}&-(5s + 100)X_1(s) + \left(s^2 + 10s + 200\right)X_2(s)\\ &= sx_2(0) + \dot{x}_2(0) + x_2(0) \qquad (\mathbf{f})\end{aligned}$$

For the steady state response, substituting values all initial conditions as zero, Eqs. (**e**) and (**f**) reduce to

$$\left(s^2 + 10s + 200\right)X_1(s) - (5s + 100)X_2(s) = \frac{15000}{s^2 + 900} \quad (\mathbf{g})$$

$$-(5s + 100)X_1(s) + \left(s^2 + 10s + 200\right)X_2(s) = 0 \quad (\mathbf{h})$$

Solving Eqs. (**g**) and (**h**) for $X_1(s)$ and $X_2(s)$, we get

$$X_1(s) = \frac{15000\left(s^2 + 10s + 200\right)}{\left(s^2 + 900\right)\left(s^4 + 20s^3 + 475s^2 + 3000s + 30000\right)} \quad (\mathbf{i})$$

$$X_2(s) = \frac{75000(s + 20)}{\left(s^2 + 900\right)\left(s^4 + 20s^3 + 475s^2 + 3000s + 30000\right)} \quad (\mathbf{j})$$

Equations (**i**) and (**j**) can be expressed in terms of their partial fractions as

$$X_1(s) = \frac{-(68s + 4520)}{265\left(s^2 + 1600\right)} + \frac{4\left(17s^3 + 1470s^2 + 15375s + 228500\right)}{265\left(s^4 + 20s^3 + 475s^2 + 3000s + 30000\right)} \quad (\mathbf{k})$$

$$X_2(s) = \frac{(38s - 280)}{265\left(s^2 + 1600\right)} - \frac{2\left(19s^3 + 240s^2 + 10875s - 225500\right)}{265\left(s^4 + 20s^3 + 475s^2 + 3000s + 30000\right)} \quad (\mathbf{l})$$

Denominator of second term of both Eqs. (**k**) and (**l**) is the characteristic equation of the system. For the steady state response, we can consider the first term of both equations, i.e.

$$X_1(s) = \frac{-(68s + 4520)}{265\left(s^2 + 1600\right)} \quad (\mathbf{m})$$

$$X_2(s) = \frac{(38s - 280)}{265\left(s^2 + 1600\right)} \quad (\mathbf{n})$$

Taking inverse Laplace transform of Eqs. (**k**) and (**l**), we get the response of the system as

$$x_1(t) = -\frac{68}{265}\cos(30t) - \frac{452}{795}\sin(30t)$$

$$x_2(t) = \frac{38}{265}\cos(30t) - \frac{28}{795}\sin(30t)$$

Example 5.21

Determine the response of m_1 of the system shown in Figure E5.21(a) when it is subjected to a transient force shown in Figure E5.21(b) using Laplace transform. Take $k_1 = k_2 = k_3 = 100\,\mathrm{N/m}$, and $m_1 = m_2 = 1\,\mathrm{kg}$.

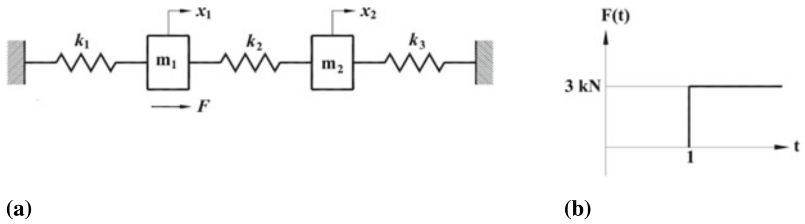


Figure E5.21

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F(t) \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0 \quad (\mathbf{b})$$

Substituting $k_1 = k_2 = k_3 = 100\,\text{N/m}$, $m_1 = m_2 = 1\,\text{kg}$ and $F(t) = 3000u(t-1)$ into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 200x_1 - 100x_2 = 3000u(t-1) \quad (\mathbf{c})$$

$$\ddot{x}_2 + 200x_2 - 100x_1 = 0 \quad (\mathbf{d})$$

Taking Laplace transform of Eqs. (**c**) and (**d**) with zero initial conditions, we get

$$\left(s^2 + 200\right)X_1(s) - 100X_2(s) = 3000\frac{e^{-s}}{s} \quad (\mathbf{e})$$

$$-100X_1(s) + \left(s^2 + 200\right)X_2(s) = 0 \quad (\mathbf{f})$$

Equations (**e**) and (**f**) can be expressed in matrix form as

$$\begin{bmatrix} s^2 + 200 & -100 \\ -100 & s^2 + 200 \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{Bmatrix} 3000\frac{e^{-s}}{s} \\ 0 \end{Bmatrix} \quad (\mathbf{g})$$

Taking inverse of the coefficient matrix of Eq. (**g**), we get the matrix of transfer functions as

$$G(s) = \begin{bmatrix} \frac{s^2+200}{s^2+400s+30000} & \frac{100}{s^2+400s+30000} \\ \frac{100}{s^2+400s+30000} & \frac{s^2+200}{s^2+400s+30000} \end{bmatrix} \quad (\mathbf{h})$$

Then the response of the first mass x_1 due to a force impulse applied to the first mass can be determined by

$$X_1(s) = F_1(s)G_{11}(s) = 3000\frac{e^{-s}}{s}\left(\frac{s^2+200}{s^2+400s+30000}\right) \quad (\mathbf{i})$$

Equations (**i**) can be expressed in terms of partial fractions as

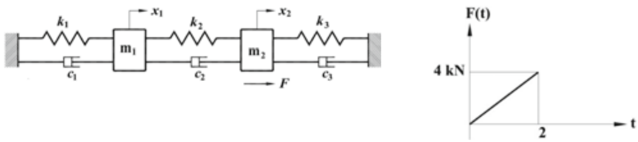
$$X_1(s) = e^{-s}\left(\frac{20}{s} - \frac{15s}{s^2+100} - \frac{5s}{s^2+300}\right) \quad (\mathbf{j})$$

Taking inverse Laplace transform of Eq. (**j**), we get the response of the first mass of the system as

$$x_1(t) = u(t-1)[20 - 15\cos 10(t-1) - 5\cos 17.3205(t-1)]$$

Example 5.22

Determine the response of m_1 of the system shown in Figure E5.22(a) when it is subjected to a transient force shown in Figure E5.22(b) using Laplace transform. Take $k_1 = k_2 = k_3 = 100\,\text{N/m}$, $c_1 = c_2 = c_3 = 5\,\text{N s/m}$ and $m_1 = m_2 = 1\,\text{kg}$.



(a) **(b)**

Figure E5.22

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = 0 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2+k_3)x_2 - k_2x_1 = F \quad (\mathbf{b})$$

Substituting $k_1 = k_2 = k_3 = 100\,\text{N/m}, c_1 = c_2 = c_3 = 5\,\text{N s/m}, m_1 = m_2 = 1\,\text{kg}$ and $F(t) = 2000t - 2000tu(t-1)$ into Eqs. (**a**) and (**b**), we get

$$\ddot{x}_1 + 10\dot{x}_1 - 5\dot{x}_2 + 200x_1 - 100x_2 = 0 \quad (\mathbf{c})$$

$$\ddot{x}_2 + 10\dot{x}_2 - 5\dot{x}_1 + 200x_2 - 100x_1 = 2000t - 2000tu(t-2) \quad (\mathbf{d})$$

Taking Laplace transform of Eqs. (**c**) and (**d**), we get

$$\left(s^2+10s+200\right)X_1(s)-(5s+100)X_2(s)=0 \quad (\mathbf{e})$$

$$-(5s+100)X_1(s)+\left(s^2+10s+200\right)X_2(s)=\frac{2000}{s^2}\left(1-e^{-2s}\right) \quad (\mathbf{f})$$

Equations (**e**) and (**f**) can be expressed in matrix form as

$$\begin{bmatrix} s^2+10s+200 & -(5s+100) \\ -(5s+100) & s^2+10s+200 \end{bmatrix}\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix}=\begin{Bmatrix} 0 \\ \frac{2000}{s^2}\left(1-e^{-2s}\right) \end{Bmatrix} \quad (\mathbf{g})$$

Taking inverse of the coefficient matrix of Eq. (**g**), we get the matrix of transfer functions as

$$G(s)=\begin{bmatrix} \frac{s^2+10s+200}{\left(s^2+5s+100\right)\left(s^2+15s+300\right)} & \frac{5s+100}{\left(s^2+5s+100\right)\left(s^2+15s+300\right)} \\ \frac{5s+100}{\left(s^2+5s+100\right)\left(s^2+15s+300\right)} & \frac{s^2+10s+200}{\left(s^2+5s+100\right)\left(s^2+15s+300\right)} \end{bmatrix} \quad (\mathbf{h})$$

Then the response of the first mass x_1 due to a force applied to the second mass can be determined by

$$\begin{aligned} X_1(s) &= F_2(s)G_{12}(s) \\ &= \frac{2000}{s^2}\left(1-e^{-2s}\right)\left(\frac{5s+100}{\left(s^2+5s+100\right)\left(s^2+15s+300\right)}\right) \quad (\mathbf{i}) \end{aligned}$$

Equation (**i**) can be expressed in terms of partial fractions as

$$\begin{aligned} X_1(s) = &\left[-\frac{1}{3s}+\frac{20}{3s^2}-\frac{s-15}{2\left(s^2+5s+100\right)}-\frac{s-5}{2\left(s^2+15s+300\right)}\right] \\ &- e^{-2s}\left[-\frac{1}{3s}+\frac{20}{3s^2}-\frac{s-15}{2\left(s^2+5s+100\right)}-\frac{s-5}{2\left(s^2+15s+300\right)}\right] \quad (\mathbf{j}) \end{aligned}$$

Taking inverse Laplace transform of Eq. (**j**), we get the response of the first mass of the system as

$$\begin{aligned} x_1(t) = &[-0.3333+6.6667t+e^{-2.5t}\{0.5\cos(9.6825t)-0.9037\sin(9.6825t)\} \\ &- e^{-7.5t}\{0.1667\cos(15.6125t)-0.1334\sin(15.6125t)\}] \\ &- u(t-2)[-0.3333+6.6667(t-2) \\ &+ e^{-2.5(t-2)}\{0.5\cos(9.6825t-19.3649) \\ &- 0.9037\sin(9.6825t-319.3649)\}-e^{-7.5(t-2)}\{0.1667\cos(15.6125t-31.2250) \\ &- 0.1334\sin(15.6125t-31.2250)\}] \end{aligned}$$

Example 5.23

Determine the natural frequencies and mode shapes a two degree of freedom system shown in Figure E5.23. Take $k = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 20\,\text{kg}$.

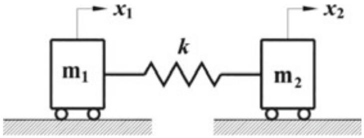


Figure E5.23

Solution

With reference to the free-body diagrams of both masses shown in **Figure E5.23(a)**, equation of motion for the given system can be written as

$$m_1\ddot{x}_1 + kx_1 - kx_2 = 0 \quad (\mathbf{a})$$

$$m_2\ddot{x}_2 + kx_2 - kx_1 = 0 \quad (\mathbf{b})$$

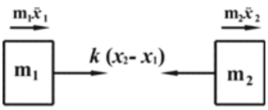


Figure E5.23(a) Free-body diagrams of two masses of system of Figure E5.23

Substituting $k = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 20\,\text{kg}$ into Eqs. (**a**) and (**b**), we get

$$10\ddot{x}_1 + 1000x_1 - 1000x_2 = 0$$
$$\therefore\ \ddot{x}_1 + 100x_1 - 100x_2 = 0 \qquad (\mathbf{c})$$

$$20\ddot{x}_2 + 1000x_2 - 1000x_1 = 0$$
$$\therefore\ 20\ddot{x}_2 + 50x_2 - 50x_1 = 0 \qquad (\mathbf{d})$$

We can assume the solution for a system of linear differential equations as

$$x_1 = A_1 \sin \omega_n t$$

$$x_2 = A_2 \sin \omega_n t\, (\mathbf{e})$$

Substituting Eq. (**e**) into Eq. (**d**), we get a system of algebraic equations as

$$\begin{aligned} &\left[100 - \omega_n^2\right]A_1 - 100A_2 = 0 \\ &-50A_1 + \left[50 - \omega_n^2\right]A_2 = 0 \qquad (\mathbf{f}) \end{aligned}$$

Rearranging both equations of Eq. (**c**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{100}{100 - \omega_n^2} \quad (\mathbf{d})$$

$$\frac{A_1}{A_2} = \frac{50 - \omega_n^2}{50} \quad (\mathbf{e})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\begin{aligned} &\frac{100}{100 - \lambda} = \frac{50 - \lambda}{50} \\ \text{or,}\quad &(100 - \lambda)(50 - \lambda) = 5000 \\ \text{or,}\quad &\lambda^2 - 150\lambda = 0 \\ \text{or,}\quad &\lambda(\lambda - 150) = 0 \\ \therefore\quad &\lambda_1 = 0, \quad \lambda_2 = 150 \qquad (\mathbf{f}) \end{aligned}$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = 0$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{150} = 12.2474\,\text{rad/s}.$$

Substituting $\lambda = \lambda_1 (= 0)$ into Eq. (**d**), we get mode shape corresponding to the first natural frequency as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{100}{100 - \lambda_1} = 1$$

Similarly, substituting $\lambda = \lambda_2 (= 150)$ into Eq. (**d**), we get mode shape corresponding to the second natural frequency as

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{100}{100 - \lambda_2} = -1$$

Example 5.24

Determine the natural frequencies of a system consisting of a uniform bar of mass M and length L shown in Figure E5.24.

(a) **Use vertical displacements of end points of the bar (x_1 and x_2) as a set of generalized coordinates.**
(b) **Use vertical displacement of C. G. of the bar (x) and rotation of bar about C. G (θ) as a set of generalized coordinates.**

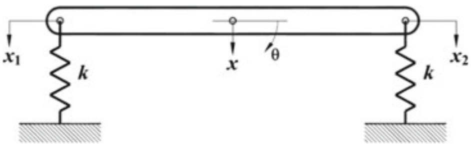


Figure E5.24

Solution

(a) If x_1 and x_2 are taken as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2 = \frac{1}{2}M\left\{\frac{1}{2}(\dot{x}_1 + \dot{x}_2)\right\}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\left\{\frac{1}{2}\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)\right\}^2$$

$$= \frac{1}{2}\left\{\frac{M}{4}(\dot{x}_1 + \dot{x}_2)^2\right\} + \frac{1}{2}\left\{\frac{M}{12}(\dot{x}_2 - \dot{x}_1)^2\right\}$$

$$V = \frac{1}{2}(k)(x_1)^2 + \frac{1}{2}(k)(x_2)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\left\{\frac{M}{4}(\dot{x}_1 + \dot{x}_2)^2\right\} + \frac{1}{2}\left\{\frac{M}{12}(\dot{x}_2 - \dot{x}_1)^2\right\} - \frac{1}{2}(k)(x_1)^2 - \frac{1}{2}(k)(x_2)^2$$

Now, using Lagrange' equation for the generalized coordinate x_1,

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$$

$$\text{or,} \quad -kx_1 - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{M}{4}(\dot{x}_1 + \dot{x}_2) - \frac{M}{12}(\dot{x}_2 - \dot{x}_1)\right] = 0$$

$$\text{or,} \quad -kx_1 - \frac{M}{3}\ddot{x}_1 - \frac{M}{6}\ddot{x}_2 = 0$$

$$\therefore \quad \frac{M}{3}\ddot{x}_1 + \frac{M}{6}\ddot{x}_2 + kx_1 = 0 \qquad \textbf{(a)}$$

Again, using Lagrange' equation for the generalized coordinate x_2,

$$\begin{aligned}
&\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0 \\
\text{or,}\quad &-kx_2 - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{M}{4}(\dot{x}_1 + \dot{x}_2) + \frac{M}{12}(\dot{x}_2 - \dot{x}_1)\right] = 0 \\
\text{or,}\quad &-kx_2 - \frac{M}{6}\ddot{x}_1 - \frac{M}{3}\ddot{x}_2 = 0 \\
\therefore\quad &\frac{M}{6}\ddot{x}_1 + \frac{M}{3}\ddot{x}_2 + kx_2 = 0 \qquad (\mathbf{b})
\end{aligned}$$

We can assume the solution for a system of linear differential equations as

$$\begin{aligned}
x_1 &= A_1 \sin\omega_n t \\
x_2 &= A_2 \sin\omega_n t \qquad (\mathbf{c})
\end{aligned}$$

Substituting Eq. (**c**) into Eq. (**b**), we get a system of algebraic equations as

$$\begin{aligned}
\left[k - \frac{M}{6}\omega_n^2\right]A_1 - \frac{M}{3}\omega_n^2 A_2 &= 0 \\
-\frac{M}{3}\omega_n^2 A_1 + \left[k - \frac{M}{6}\omega_n^2\right]A_2 &= 0 \quad (\mathbf{d})
\end{aligned}$$

Rearranging both equations of Eq. (**d**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{M}{2(3k - M\lambda)} \quad (\mathbf{d})$$

$$\frac{A_1}{A_2} = \frac{2(3k - M\lambda)}{M} \quad (\mathbf{e})$$

Equating Eqs. (**d**) and (**e**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\begin{aligned}
&\frac{M}{2(3k - M\lambda)} = \frac{2(3k - M\lambda)}{M} \\
\text{or,}\quad &(3k - M\lambda)^2 = \frac{(M\lambda)^2}{4} \\
\text{or,}\quad &3k - M\lambda = \pm\frac{M\lambda}{2} \\
\text{or,}\quad &M\lambda\left(1 \pm \frac{1}{2}\right) = 3k \\
\text{or,}\quad &\lambda = \frac{6k}{(2 \pm 1)M}
\end{aligned}$$

$$\therefore \quad \lambda_1 = \frac{2k}{M}, \quad \lambda_2 = \frac{6k}{M} \quad (\mathbf{f})$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{2k}{M}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{6k}{M}}$$

(b) If x and θ are taken as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\overline{I}\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}(k)\left(x - \frac{L}{2}\theta\right)^2 + \frac{1}{2}(k)\left(x + \frac{L}{2}\theta\right)^2$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 - \frac{1}{2}(k)\left(x - \frac{L}{2}\theta\right)^2 - \frac{1}{2}(k)\left(x + \frac{L}{2}\theta\right)^2$$

Now, using Lagrange' equation for the generalized coordinate x,

$$\begin{aligned}
&\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \\
\text{or,}\quad & -k\left(x - \frac{L}{2}\theta\right) - k\left(x + \frac{L}{2}\theta\right) - \frac{\mathrm{d}}{\mathrm{d}t}[M\dot{x}] = 0 \\
\text{or,}\quad & -2kx - M\ddot{x} = 0 \\
\therefore\quad & M\ddot{x} + 2kx_1 = 0 \qquad (\mathbf{a})
\end{aligned}$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \\
\text{or,}\quad & -k\frac{L}{2}\left(x - \frac{L}{2}\theta\right) + k\frac{L}{2}\left(x + \frac{L}{2}\theta\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{12}ML^2\right)\dot{\theta}\right] = 0 \\
\text{or,}\quad & -\frac{kL^2}{2}\theta - \left(\frac{1}{12}ML^2\right)\ddot{\theta} = 0 \\
\therefore\quad & \left(\frac{1}{12}ML^2\right)\ddot{\theta} + \frac{kL^2}{2}\theta = 0 \qquad (\mathbf{b})
\end{aligned}$$

From Eqs. (**a**) and (**b**), we get the natural frequencies of the system as

$$\omega_1 = \sqrt{\frac{2k}{M}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{6k}{M}}$$

Example 5.25

Consider schematic diagram for power transmission unit of an IC engine as shown in Figure E5.25. The pinion with a mass moment of inertia of $0.02\,\text{kg m}^2$ is directly coupled the engine with a mass moment of inertia of 1 kg m^2. The mass moment of inertia of driven gear is $0.15\,\text{kg m}^2$ and the equivalent mass moment of inertia of rotating parts of the system is $18\,\text{kg m}^2$. The flywheel has very large mass moment of inertia as compared to other parts of the system can be assumed as fixed. The shaft connecting the flywheel and the engine has a length of 0.75 m and a diameter of 30 mm and that for the shaft connecting the gear and the rotating parts is 1 m and 30 mm respectively. The speed reduction from the engine to the rotating parts is 3:1. Determine the natural frequencies for torsional vibration of the system. Take $G = 84$ GPa for both shafts.

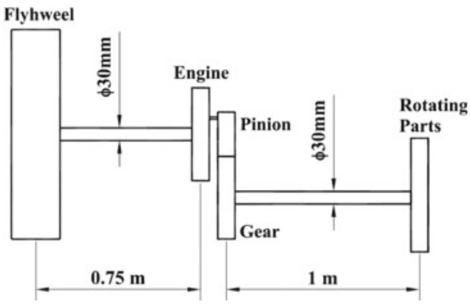


Figure E5.25

Solution

Given transmission system can be converted into an equivalent two degree of freedom system as shown in **Figure E5.25(a)**. In the equivalent model I_1 represents the equivalent mass moment of inertia of the engine, pinion and gear, I_2 represents the equivalent mass moment of inertia of rotating parts of the system, k_{t1} represents the equivalent torsional stiffness of the shaft connected between the flywheel and the engine and k_{t2} represents the equivalent torsional stiffness of the shaft connected between the gear and the rotating parts.

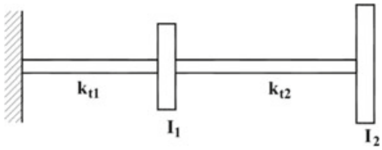


Figure E5.25(a)

These parameters of the equivalent model can be determined as

$$I_1 = 1 + 0.02 + \frac{0.15}{3^2} = 1.0367\,\text{kg m}^2$$

$$I_2 = \frac{18}{3^2} = 2\,\text{kg m}^2$$

$$\begin{aligned} k_{t1} &= \frac{J_1 G}{L_1} = \frac{\pi}{32} \times \frac{\left(30 \times 10^{-3}\right)^4 \times 84 \times 10^9}{0.75} \\ &= 8906.4152\,\text{N m/rad} \end{aligned}$$

$$\begin{aligned} k_{t2} &= \frac{1}{3^2} \times \frac{J_2 G}{L_2} = \frac{1}{3^2} \times \frac{\pi}{32} \times \frac{\left(30 \times 10^{-3}\right)^4 \times 84 \times 10^9}{1} \\ &= 742.2013\,\text{N m/rad} \end{aligned}$$

Then equations of motion for the equivalent system can be expressed as

$$I_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = 0 \quad (\mathbf{a})$$

$$I_2\ddot{\theta}_2 + k_{t2}\theta_2 - k_{t2}\theta_1 = 0 \quad (\mathbf{b})$$

Substituting values of each parameters of the system, we get

$$1.0367\ddot{\theta}_1 + 9648.6164\theta_1 - 742.2013\theta_2 = 0 \quad (\mathbf{c})$$

$$2\ddot{\theta}_2 + 742.2013\theta_2 - 742.2013\theta_1 = 0 \quad (\mathbf{d})$$

We can assume the solution for a system of linear differential equations as

$$\theta_1 = A_1 \sin \omega_n t \quad (\mathbf{e})$$

$$\theta_2 = A_2 \sin \omega_n t \quad (\mathbf{f})$$

Substituting Eqs. (**e**) and (**f**) into Eqs. (**c**) and (**d**), we get a system of algebraic equations as

$$\left[9648.6164 - 1.0367\omega_n^2\right]A_1 - 742.2013A_2 = 0 \quad (\mathbf{g})$$

$$-742.2013A_1 + \left[742.2013 - 2\omega_n^2\right]A_2 = 0 \quad (\mathbf{h})$$

Rearranging Eqs. (**g**) and (**h**) for amplitude ratio, we get

$$\frac{A_1}{A_2} = \frac{742.2013}{9648.6164 - 1.0367\omega_n^2} \quad (\mathbf{i})$$

$$\frac{A_1}{A_2} = \frac{742.2013 - 2\omega_n^2}{742.2013} \quad (\mathbf{j})$$

Equating Eqs. (**i**) and (**j**), and replacing $\omega_n^2 = \lambda$ for simplicity, we get

$$\frac{742.2013}{9648.6164 - 1.0367\lambda} = \frac{742.2013 - 2\lambda}{742.2013}$$

$$\text{or,} \quad (9648.6164 - 1.0367\lambda)(742.2013 - 2\lambda) = (742.2013)^2$$

$$\text{or,} \quad \lambda^2 - 9678.4477\lambda + 3.18827 \times 10^6 = 0$$

$$\therefore \quad \lambda_1 = 341.4672, \quad \lambda_2 = 9336.9804$$

Then the natural frequencies of the system can be determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{341.4672} = 18.4788\,\text{rad/s}$$

and

$$\omega_2 = \sqrt{\lambda_2} = \sqrt{9336.9804} = 96.6281\,\text{rad/s}$$

Review Questions

1. Differentiate between a single degree of freedom system and a two degree of freedom system.
2. What do you mean by mode shapes of vibration? Also define node of a vibrating system.
3. Define fundamental mode of vibration of a system.
4. What are the possible types of roots of characteristics equation of a damped TDOF system and what are the corresponding response?
5. What do you mean by a transfer function? How can it be used to determine the forced response of a two degree of freedom system?
6. Explain the working of a vibration absorber with necessary derivation.
7. Differentiate between the vibration isolator and vibration absorber.
8. Define a semi-definite or a degenerated system.

9. Differentiate between static and dynamic coupling. Also define principal coordinates.

Exercise

1. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.1**. Also determine the nodes of the system. Take the values of masses and stiffness values of the springs as:

   (a) $m_1 = m, m_2 = m, k_1 = k, k_2 = 2k, k_3 = k.$
   (b) $m_1 = m, m_2 = 2m, k_1 = 3k, k_2 = 2k, k_3 = k.$
   (c) $m_1 = 2m, m_2 = 2m, k_1 = 2k, k_2 = 5k, k_3 = 2k.$
   (d) $m_1 = m, m_2 = m, k_1 = k, k_2 = k/2, k_3 = k.$

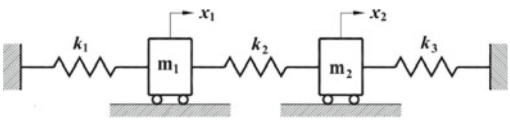


Figure P5.1

2. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.2**. Also determine the nodes of the system. Take the values of masses and stiffness values of the springs as:

   (a) $m_1 = m, m_2 = m, k_1 = 3k, k_2 = 2k.$
   (b) $m_1 = 2m, m_2 = m, k_1 = 2k, k_2 = k.$
   (c) $m_1 = m, m_2 = m, k_1 = 3k/2, k_2 = k.$

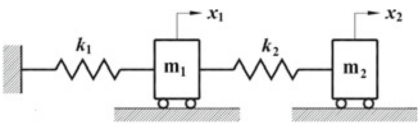
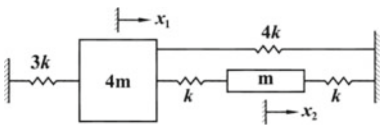


Figure P5.2

3. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.3**.



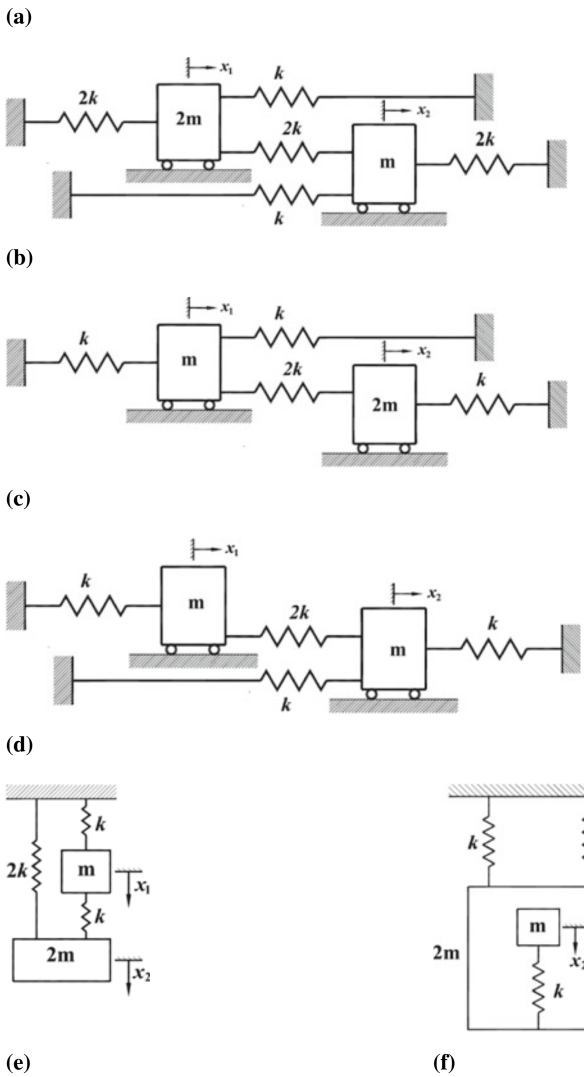


Figure P5.3

4. Determine the two natural frequencies and the modes of vibration of the system shown in **Figure P5.4**. Determine the response of the system when it is subject to the following initial conditions:

   (a) $x_1(0) = 0.1m$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
   (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

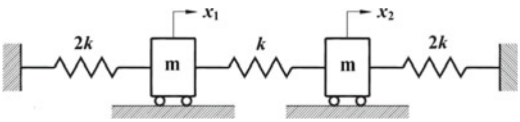


Figure P5.4

5. Determine the two natural frequencies and the modes of vibration of the system shown in **Figure P5.5**. Determine the response of the system when it is subject to the following initial conditions:

   (a) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1m$, $\dot{x}_2(0) = 0$.
   (b) $x_1(0) = 0$, $\dot{x}_1(0) = 1m/s$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.

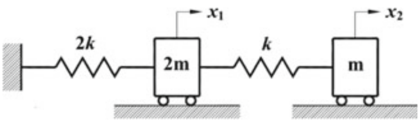


Figure P5.5

6. Determine the response of the system shown in **Figure P5.6**. Take $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 1000$ N/m, $m_1 = 10$ kg and $m_2 = 10$ kg. Use the following initial conditions:

   (a) $x_1(0) = 0.1m$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
   (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1$ m/s.

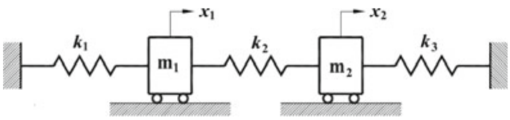


Figure P5.6

7. Determine the response of the system shown in **Figure P5.7**. Take $k = 1000$ N/m and $m = 20$ kg. Use the following initial conditions:

   (a) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1m$, $\dot{x}_2(0) = 0$.
   (b) $x_1(0) = 0$, $\dot{x}_1(0) = 1m/s$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.

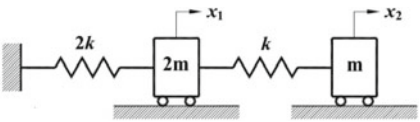


Figure P5.7

8. For a system shown in **Figure P5.8**, two natural frequencies and the corresponding mode shapes are found to be $\omega_1 = 9\,\text{rad/s}$, $\omega_2 = 16\,\text{rad/s}$ and $(A_1/A_2)_1 = 0.8$, $(A_1/A_2)_1 = -2.4$, respectively. If $m_1 = 1\,\text{kg}$, determine m_2, k_1, k_2 and k_3.

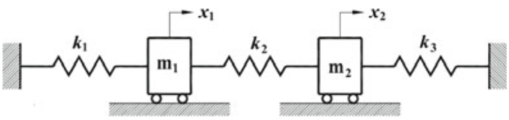


Figure P5.8

9. For a system shown in **Figure P5.9**, the natural frequency corresponding to the fundamental mode is found to be $\omega_1 = 8\,\text{rad/s}$. The mode shapes for two normal modes are found to be $(A_1/A_2)_1 = 1.14$ and $(A_1/A_2)_1 = -0.44$. If $m_1 = 2\,\text{kg}$ and $k_1 = 120\,\text{N/m}$, determine m_2, ω_2, k_2 and k_3.

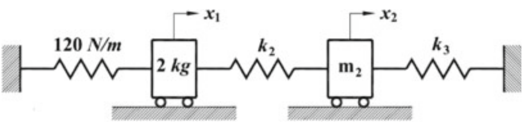


Figure P5.9

10. For a system shown in **Figure P5.10**, the natural frequency corresponding to the fundamental mode is found to be $\omega_1 = 4.1\,\text{rad/s}$. The mode shapes for two normal modes are found to be $(A_1/A_2)_1 = 2/3$ and $(A_1/A_2)_1 = -1$. If $k_1 = 100\,\text{N/m}$, determine m_1, m_2, ω_2 and k_2.

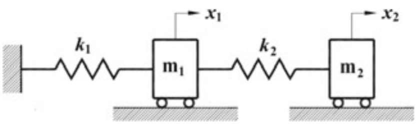


Figure P5.10

11. When both masses of a two degree of freedom system shown in **Figure P5.11** is displaced by 10 cm toward right from the equilibrium position and released, the system vibrates with a frequency of 10 rad/s. Similarly when one of the mass is displaced by 10 cm toward right and another mass is displaced by 10 cm toward left from the equilibrium position and released, the system vibrates with a frequency of $\sqrt{200}$ rad/s. Determine the response of the same system when it is subjected to the following initial conditions:

    (a) $x_1(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_2(0) = 0$.
    (c) $x_1(0) = 0$, $\dot{x}_1(0) = 1$ m/s, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
    (d) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1$ m/s.

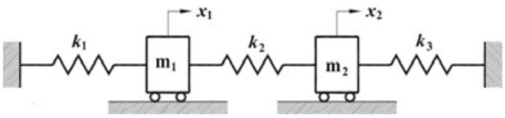


Figure P5.11

12. Derive the expression for the characteristics equation for the system shown in **Figure P5.12** in terms of system parameters m, I_p, r and k.

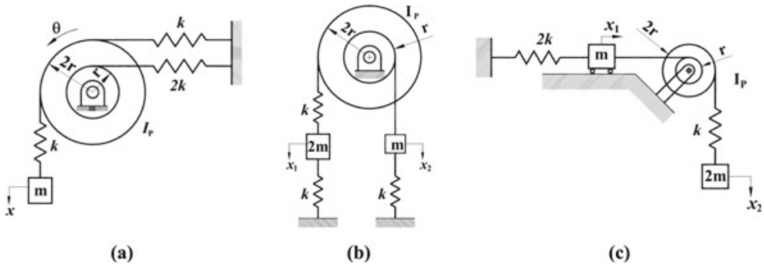


Figure P5.12

13. Derive the expression for the characteristics equation for the system shown in **Figure P5.12** in terms of system parameters m, m_d, I_p, r and k.

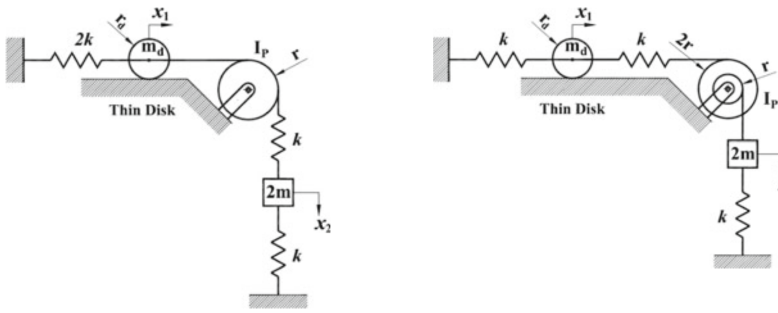


Figure P5.13

14. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.14**. Take $k = 1000\,\text{N/m}$, $r = 12\,\text{cm}$, $I_p = 1\,\text{kg}\,\text{m}^2$ and $m = 10\,\text{kg}$.

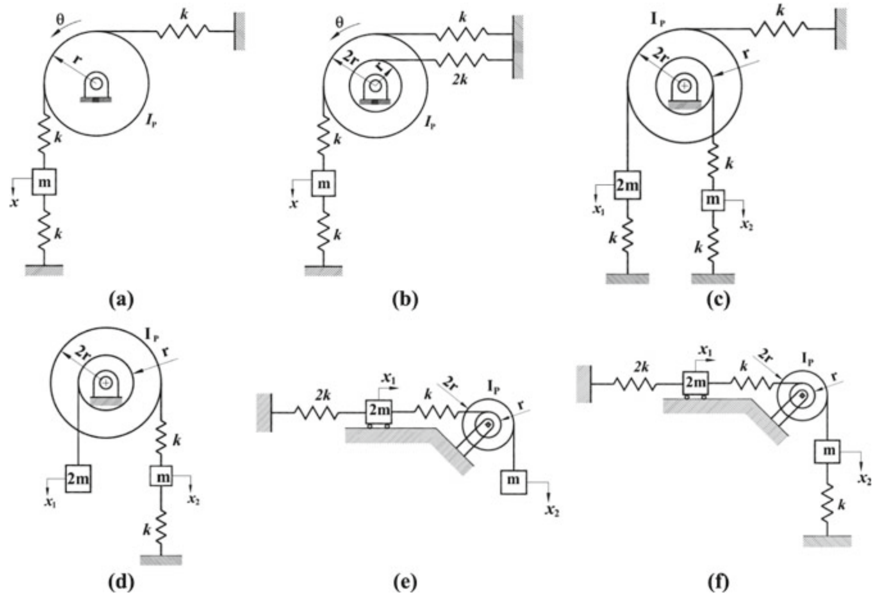


Figure P5.14

15. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.15**. Take $k = 1200\,\text{N/m}$, $r = 10\,\text{cm}$, $I_p = 1\,\text{kg}\,\text{m}^2$, $m = 10\,\text{kg}$, $m_d = 4\,\text{kg}$ and $r_d = 6\,\text{cm}$.

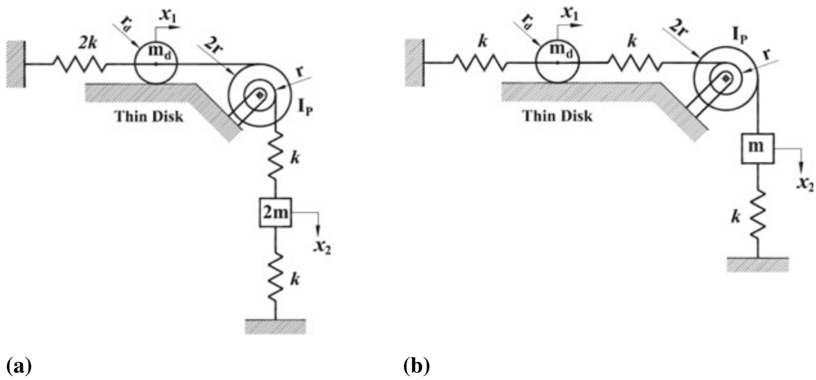


Figure P5.15

16. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P5.16**. Take $k = 1000$ N/m, $r = 10$ cm, and $m = 10$ kg.

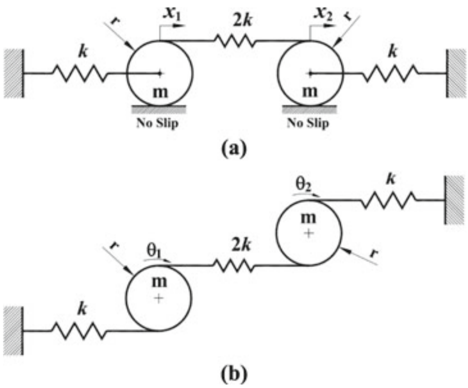


Figure P5.16

17. Determine the natural frequencies of the system shown in **Figure P5.17**. The uniform bar has a mass of M and length of L. Take $M = 20$ kg, $L = 1.6$ m, $m = 10$ kg and $k = 4$ kN/m.

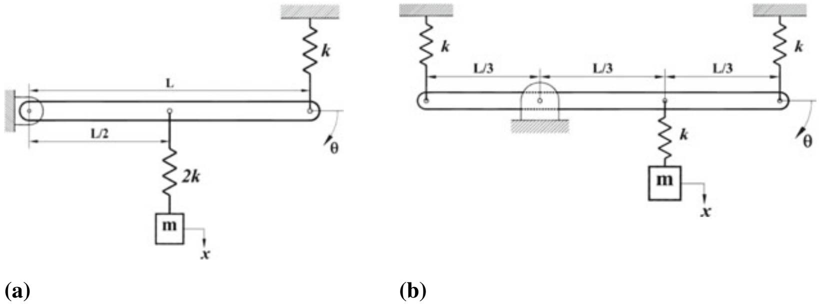


Figure P5.17

18. Determine the natural frequencies of the system shown in **Figure P5.18**. Take $I_1 = 1.2\,\text{kg m}^2$, $I_2 = 1\,\text{kg m}^2$, $r = 10\,\text{cm}$, $m = 8\,\text{kg}$ and $k = 2\,\text{kN/m}$.
19. Determine the natural frequencies and corresponding mode shapes of the double pendulum system shown in **Figure P5.19**.
20. Determine the two natural frequencies and the modes of vibration of the system shown in **Figure P5.20**. The two equal masses are under a very large tension T.
21. Determine the natural frequencies of the system costing of a pendulum attached to a spring-mass assembly as shown in **Figure P5.21**. Take $L = 1$ m, $m = 10$ kg and $k = 4\,\text{kN/m}$.
22. Determine the natural frequencies of the system costing of a vertical bar with a spring-mass assembly as shown in **Figure P5.22**. Take $M = 16\,\text{kg}$, $L = 1.2\,\text{m}$, $m = 8\,\text{kg}$ and $k = 2\,\text{kN/m}$.

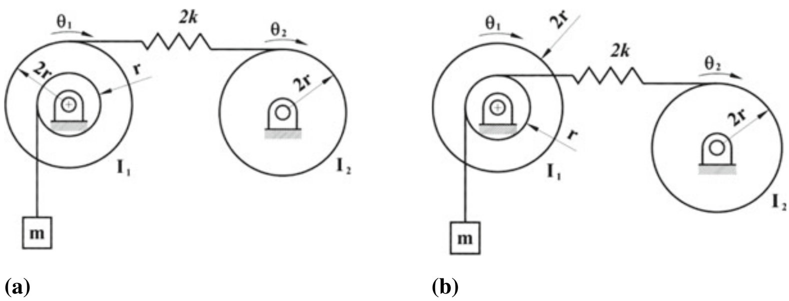


Figure P5.18

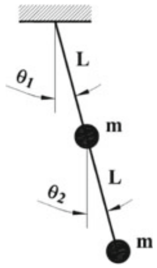


Figure P5.19

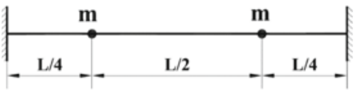


Figure P5.20

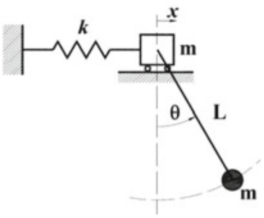


Figure P5.21

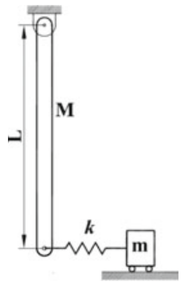


Figure P5.22

23. Determine the natural frequencies and mode shapes of a system consisting of a beam with a concentrated mass and spring and mass assembly shown in **Figure P5.23** by modeling it as a two degree of freedom system. Mass of the beam is negligible in comparison to that of the attached mass. Take $E = 210\,\text{GPa}$, $I = 1 \times 10^{-5}\,\text{m}^4$ for beam; $k = 1\text{MN/m}$ and $m = 40\,\text{kg}$.

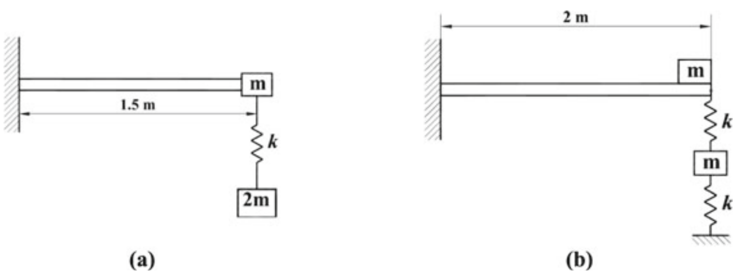


Figure P5.23

24. Determine the natural frequencies and corresponding mode shapes of the shaft-disk system shown in **Figure P5.24**. All shaft segments have a diameter of 200 mm. Total length of the shaft is 1.8 m and mass moment of inertia of disks are $I_1 = 80\,\text{kg}\,\text{m}^2$ and $I_2 = 100\,\text{kg}\,\text{m}^2$. Take $G = 84\,\text{GPa}$.

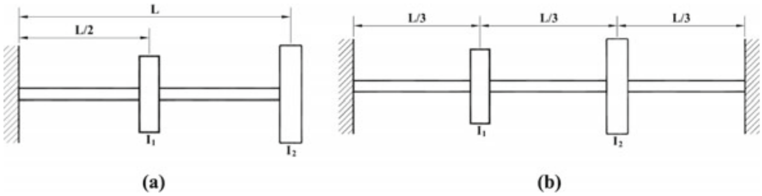


Figure P5.24

25. Determine free response of a damped two degree of freedom system shown in **Figure P5.25**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $c = 5\,\text{N s/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

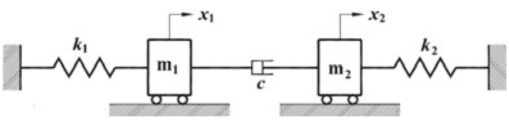


Figure P5.25

26. Determine free response of a damped two degree of freedom system shown in **Figure P5.26**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $c = 5\,\text{N s/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1m$, $\dot{x}_2(0) = 0$.

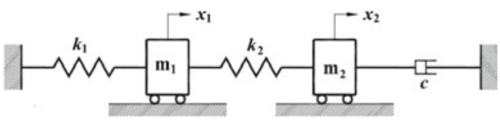


Figure P5.26

27. Determine free response of a damped two degree of freedom system shown in **Figure P5.27**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $c = 5\,\text{N s/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.

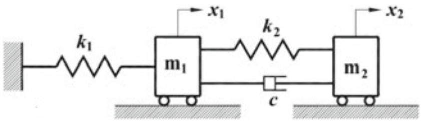


Figure P5.27

28. Determine free response of a damped two degree of freedom system shown in **Figure P5.28**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $c_1 = c_2 = 50\,\text{N s/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 15\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.
    (b) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.

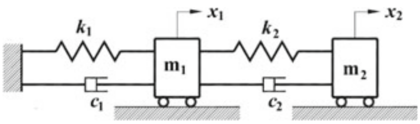


Figure P5.28

29. Determine free response of a damped two degree of freedom system shown in **Figure P5.29**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1000\,\text{N/m}$ $c_1 = c_2 = c_3 = 50\,\text{N s/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 10\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

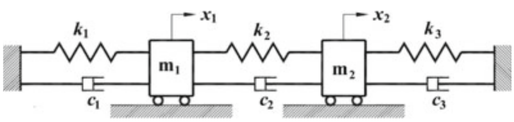


Figure P5.29

30. Determine the steady state response of the system shown in **Figure P5.30**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 200\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 1.5\,\text{kg}$ and $F_1 = 200 \sin 30t$ N.

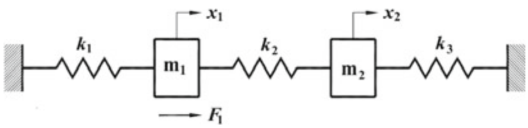


Figure P5.30

31. Determine the steady state response of the system shown in **Figure P5.31**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$ and $F_2 = 1000 \sin 40t$ N.

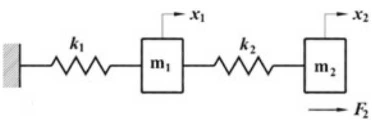


Figure P5.31

32. Determine the steady state response of the system shown in **Figure P5.32**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 200\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 1.5\,\text{kg}$, $F_1 = 200 \sin 30t$ N and $F_2 = 100 \sin 30t$ N.

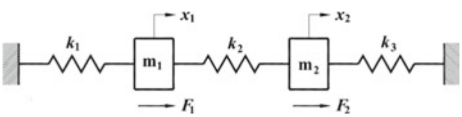


Figure P5.32

33. Determine the steady state response of the system shown in **Figure P5.33**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$, $F_1 = 2500 \sin 30t$ N and $F_2 = 1500 \sin 50t$ N.

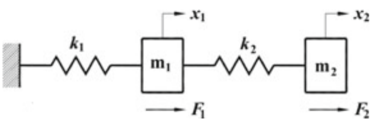


Figure P5.33

34. Determine the range of frequency of the external excitation for which the amplitude of steady state vibration of the first mass (m_1) of the system shown in **Figure P5.34** does not exceed 15 mm. Take $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 1000$ N/m, $m_1 = 5$ kg, $m_2 = 10$ kg and $F_2 = 20 \sin \omega t$ N.

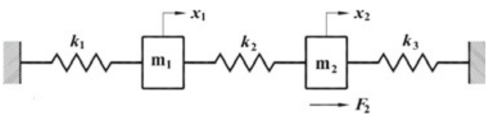


Figure P5.34

35. Determine the range of frequency of the external excitation for which the amplitude of steady state vibration of the second mass (m_2) of the system shown in **Figure P5.35** does not exceed 18 mm. Take $k_1 = 600$ N/m, $k_2 = 800$ N/m, $m_1 = 2$ kg, $m_2 = 4$ kg and $F_1 = 10 \sin \omega t$ N.

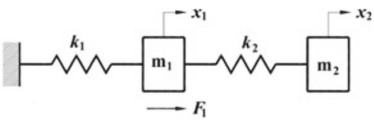


Figure P5.35

36. Determine the maximum amplitude (F_0) of the external excitation for which the amplitude of steady state vibration of the second mass (m_2) of the system shown in **Figure P5.36** does not exceed 18 mm. Take $k_1 = 100$ N/m, $k_2 = 150$ N/m, $k_3 = 100$ N/m, $m_1 = 2kg$, $m_2 = 1$ kg and $F_1 = F_0 \sin 30t$ N.

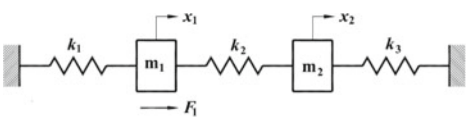


Figure P5.36

37. Determine the maximum amplitude (F_0) of the external excitation for which the amplitude of steady state vibration of the first mass (m_1) of the system shown in **Figure P5.37** does not exceed 18 mm. Take $k_1 = 1000$ N/m, $k_2 = 1200$ N/m $m_1 = 12$ kg, $m_2 = 15$ kg and $F_2 = F_0 \sin 40t\ N$.

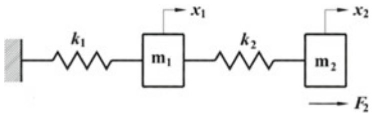
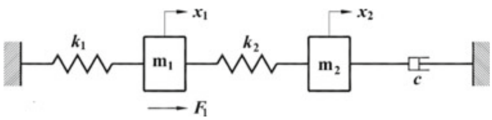
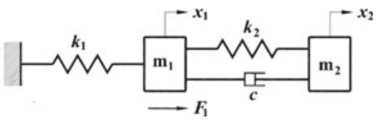


Figure P5.37

38. Determine the steady state response of a damped two degree of freedom system shown in **Figure P5.38**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $c = 5\,\text{N s/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $F_1 = 250 \sin 30t\,\text{N}$.



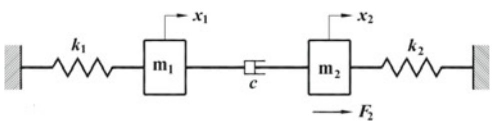
(a)



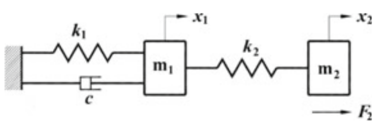
(b)

Figure P5.38

39. Determine the steady state response of a damped two degree of freedom system shown in **Figure P5.39**. Take $k_1 = 1500\,\text{N/m}$, $k_2 = 1200\,\text{N/m}$, $c = 10\,\text{N s/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$ and $F_2 = 3000 \sin 40t\,\text{N}$.



(a)



(b)

Figure P5.39

40. Determine the steady state response of a damped two degree of freedom system shown in **Figure P5.40**. Take $k_1 = 100$ N/m, $k_2 = 120$ N/m, $c_1 = c_2 = 5$ N s/m, $m_1 = 1$ kg, $m_2 = 1.5$ kg, $F_1 = 200 \cos 30t$ N and $F_2 = 150 \sin 50t$ N.

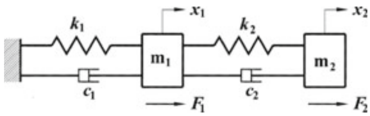


Figure P5.40

41. Determine the steady state response of a damped two degree of freedom system shown in **Figure P5.41**. Take $k_1 = 1000$ N/m, $k_2 = 1500$ N/m, $k_3 = 1000$ N/m, $c_1 = c_2 = c_3 = 10$ N s/m, $m_1 = m_2 = 10$ kg, $F_1 = 1500 \sin 30t$ N and $F_2 = 1000 \cos 50t$ N.

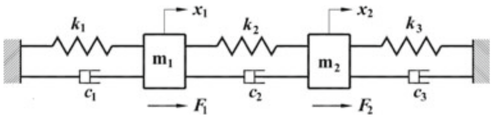


Figure P5.41

42. A reciprocating machine of mass 30kg running at 5600 rpm after installation has natural frequency very close to the forcing frequency of vibrating system. Design a dynamic absorber of the nearest frequency of the system which is to be at least 25% from the excitation frequency.
43. A machine of mass 50 kg is supported by an un-damped elastic foundation with an equivalent stiffness of 120 kN/m is subject to a harmonic force of amplitude 400 N. An absorber mass of 10 kg is attached to machine to make vibration amplitude of the machine zero. Determine the steady state amplitude of the absorber mass.
44. A machine of mass 160 kg is mounted on a table of stiffness 160 kN/m. During operation, it is subjected to a harmonic excitation of magnitude 2000 N at 60 rad/s. Determine the required stiffness of a 20 kg absorber to eliminate the steady state vibrations of the machine during the operation.
45. A petrol engine of mass 200 kg is supported by an un-damped elastic foundation. It is observed that at an operating of 5000 rpm, the engine vibrates violently. Determine the parameters of vibration absorber that will reduce the vibrations of the engine. The magnitude of exciting force is 300 N and the amplitude of motion of absorber mass should not exceed 3 mm.

46. A motor with a mass of 200 kg is mounted at the free end of a cantilever beam. The beam has a length of 1.5 m, and area moment of inertia of $1.5 \times 10^{-5}\,\text{m}^4$. Determine the parameters of a dynamic absorber of the nearest frequency of the system which is to be at least 20% from the excitation frequency. Take $E = 210\,\text{GPa}$.
47. Use Laplace transform method to determine the free response of a system shown in **Figure P5.47**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 100\,\text{N/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.

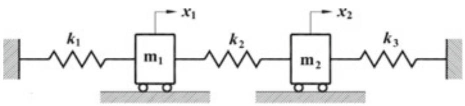


Figure P5.47

48. Use Laplace transform method to determine the free response of a system shown in **Figure P5.48**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1000\,\text{N/m}$, $c_1 = c_2 = c_3 = 50\,\text{N s/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 10\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.
    (b) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

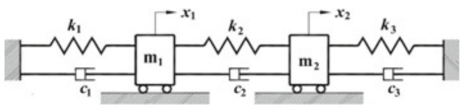


Figure P5.48

49. Use Laplace transform method to determine the steady state response of the system shown in **Figure P5.49**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 2000\,\text{N/m}$, $k_3 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$ and $F_1 = 2000 \sin 30t\,\text{N}$.

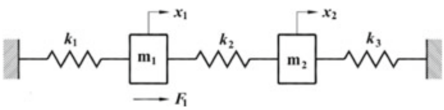


Figure P5.49

50. Use Laplace transform method to determine the steady state response of a system shown in **Figure P5.50**. Take $k_1 = 100$ N/m, $k_2 = 150$ N/m, $k_3 = 100$ N/m, $m_1 = 1$ kg, $m_2 = 2$ kg and $F_2 = 100 \sin 40t$ N.

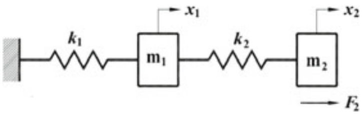


Figure P5.50

51. Use Laplace transform method to determine the steady state response of a system shown in **Figure P5.51**. Take $k_1 = 1000$ N/m, $k_2 = 2000$ N/m, $k_3 = 1000$ N/m, $m_1 = 10$ kg, $m_2 = 15$ kg, $F_1 = 2000 \sin 30t$ N and $F_2 = 1000 \sin 50t$ N.

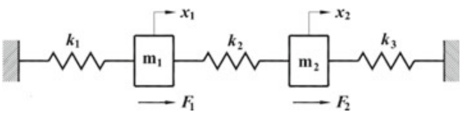


Figure P5.51

52. Use Laplace transform method to determine the steady state response of a system shown in **Figure P5.52**. Take $k_1 = 100$ N/m, $k_2 = 150$ N/m, $m_1 = 1$ kg, $m_2 = 2$ kg, $F_1 = 250 \sin 30t$ N and $F_2 = 150 \sin 50t$ N.

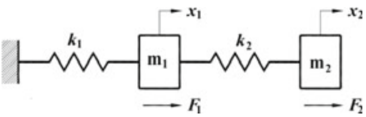


Figure P5.52

53. Use Laplace transform method to determine the steady state response of a damped two degree of freedom system shown in **Figure P5.53**. Take $k_1 = 1000$ N/m, $k_2 = 1200$ N/m, $c_1 = c_2 = 10$ N s/m, $m_1 = 10$ kg, $m_2 = 20$ kg and $F_1 = 1500 \sin 50t$ N.

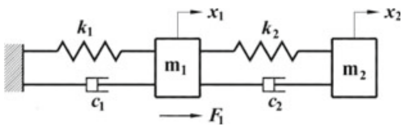


Figure P5.53

54. Use Laplace transform method to determine the steady state response of a damped two degree of freedom system shown in **Figure P5.54**. Take $k_1 = 100$ N/m, $k_2 = 150$ N/m, $k_3 = 100$ N/m, $c_1 = c_2 = c_3 = 8$ N s/m, $m_1 = m_2 = 1$ kg and $F_2 = 200 \sin 30t$ N.

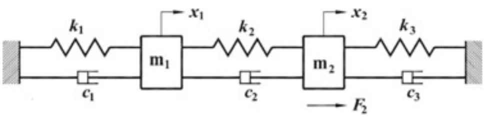
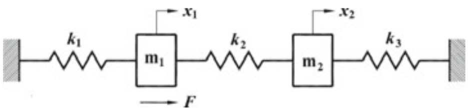
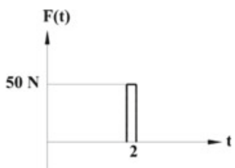


Figure P5.54

55. Determine the response of m_2 of the system shown in **Figure P5.55(a)** when it is subjected to a transient force shown in **Figure P5.55(b)** using Laplace transform. Take $k_1 = k_2 = k_3 = 1000$ N/m and $m_1 = m_2 = 10$ kg.



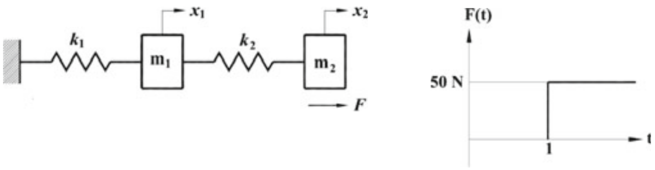
(a)



(b)

Figure P5.55

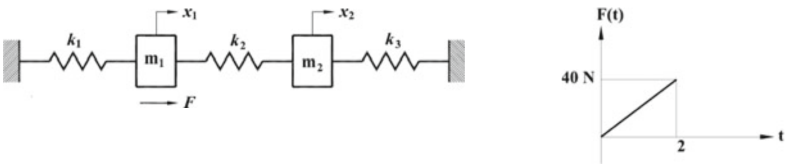
56. Determine the response of m_1 of the system shown in **Figure P5.56(a)** when it is subjected to a transient force shown in **Figure P5.56(b)** using Laplace transform. Take $k_1 = k_2 = 1000$ N/m and $m_1 = m_2 = 10$ kg.



(a) **(b)**

Figure P5.56

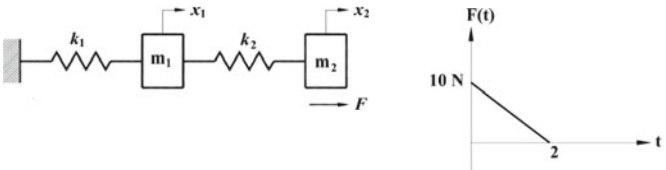
57. Determine the response of m_2 of the system shown in **Figure P5.57(a)** when it is subjected to a transient force shown in **Figure P5.57(b)** using Laplace transform. Take $k_1 = k_2 = k_3 = 1000$ N/m and $m_1 = m_2 = 10$ kg.



(a) **(b)**

Figure P5.57

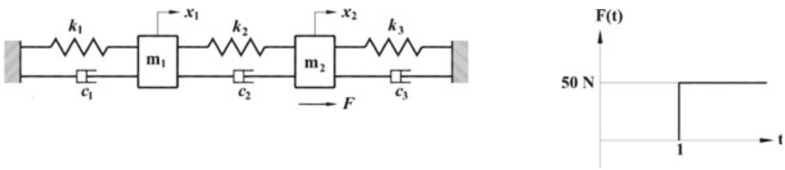
58. Determine the response of m_1 of the system shown in **Figure P5.58(a)** when it is subjected to a transient force shown in **Figure P5.58(b)** using Laplace transform. Take $k_1 = k_2 = 100$ N/m and $m_1 = m_2 = 1$ kg.



(a) **(b)**

Figure P5.58

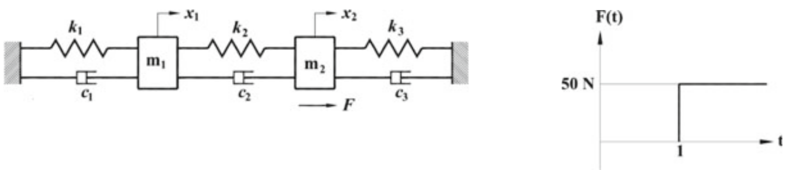
59. Determine the response of m_2 of the system shown in **Figure P5.59(a)** when it is subjected to a transient force shown in **Figure P5.59(b)** using Laplace transform. Take $k_1 = k_2 = 1000$ N/m, $c_1 = c_2 = 10$ N s/m and $m_1 = m_2 = 10$ kg.



(a) **(b)**

Figure P5.59

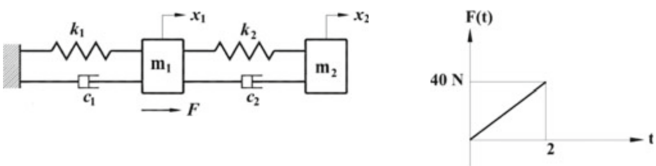
60. Determine the response of m_1 of the system shown in **Figure P5.60(a)** when it is subjected to a transient force shown in **Figure P5.60(b)** using Laplace transform. Take $k_1 = k_2 = k_3 = 100$ N/m, $c_1 = c_2 = c_3 = 5$ N s/m and $m_1 = m_2 = 1$ kg.



(a) **(b)**

Figure P5.60

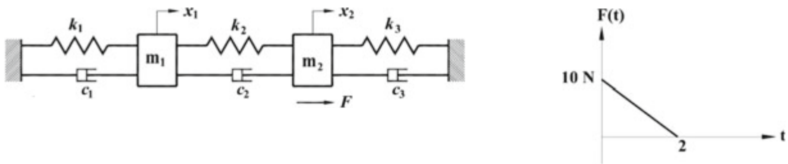
61. Determine the response of m_2 of the system shown in **Figure P5.61(a)** when it is subjected to a transient force shown in **Figure P5.61(b)** using Laplace transform. Take $k_1 = k_2 = 100$ N/m, $c_1 = c_2 = 5$ N s/m and $m_1 = m_2 = 1$ kg.



(a) **(b)**

Figure P5.61

62. Determine the response of m_1 of the system shown in **Figure P5.62(a)** when it is subjected to a transient force shown in **Figure P5.62(b)** using Laplace transform. Take $k_1 = k_2 = k_3 = 1000$ N/m, $c_1 = c_2 = c_3 = 10$ N s/m and $m_1 = m_2 = 10$ kg.



(a) (b)

Figure P5.62

63. Determine the natural frequencies and corresponding mode shapes of the system consisting of a block and a cylinder connected by a spring as shown in **Figure P5.63**. Take $m = 10\,\text{kg}$, $r = 12\,\text{cm}$ and $k = 2500\,\text{N/m}$.
64. Determine the natural frequencies and corresponding mode shapes of the system consisting of two identical cylinders connected by a spring as shown in **Figure P5.64**. Take $m = 10\,\text{kg}$, $r = 10\,\text{cm}$ and $k = 2000\,\text{N/m}$.
65. Determine the natural frequencies and corresponding mode shapes of the shaft-disk system shown in **Figure P5.65**. The shaft has a diameter of 120 mm and length of 1 m and mass moment of inertia of disks are $I_1 = 80\,\text{kg}\,\text{m}^2$ and $I_2 = 100\,\text{kg}\,\text{m}^2$. Take $G = 84\,\text{GPa}$.
66. Determine the natural frequencies of a system consisting of a uniform bar of mass M and length L shown in **Figure P5.66**. Take $M = 25\,\text{kg}$, $L = 2\,\text{m}$ and $k = 2000\,\text{N/m}$.

    (a) Use vertical displacements of the points of the bar (x_1 and x_2) at which springs are attached as a set of generalized coordinates.
    (b) Use vertical displacement of C. G. of the bar (x) and rotation of bar about C. G (θ) as a set of generalized coordinates.

67. A centrifugal pump is driven by a motor through a gear system as shown in **Figure P5.67**. The shaft transmitting power from the motor to the pinion has a diameter of 50 mm and a length of 250 mm, and the shaft between the pump and the driven gear has a diameter of 80 mm and a length of 500 mm. The mass moment of inertias of the rotor of the motor and the impeller of the pump are 300 kg m^2 and $1200\,\text{kg}\,\text{m}^2$, respectively. The speed reduction from the motor to the pump is 3:1. Neglecting the inertia effects of the gears and the shafts determine the natural frequency for the torsional vibration of the system. Take $G = 84\,\text{GPa}$ for both shafts.

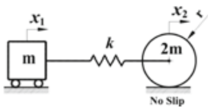


Figure P5.63

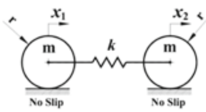


Figure P5.64

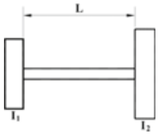


Figure P5.65

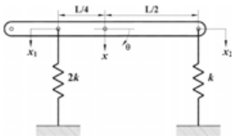


Figure P5.66

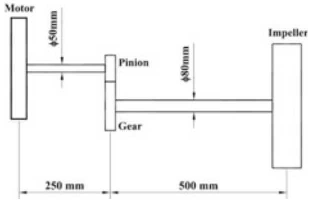


Figure P5.67

Answers

1.

(a) $\sqrt{\frac{k}{m}}, \sqrt{\frac{5k}{m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(b) $\sqrt{\frac{k}{m}}, \sqrt{\frac{11k}{2m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 4 \\ -1 \end{Bmatrix}$

(c) $\sqrt{\frac{k}{m}}, \sqrt{\frac{6k}{m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\sqrt{\frac{k}{m}}, \sqrt{\frac{6k}{m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(d) $\sqrt{\frac{k}{m}}, \sqrt{\frac{2k}{m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

2.

(a) $\sqrt{\frac{k}{m}}, \sqrt{\frac{6k}{m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$

(b) $\sqrt{\frac{k}{2m}}, \sqrt{\frac{2k}{m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(c) $\sqrt{\frac{k}{2m}}, \sqrt{\frac{3k}{m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$

3.

(a) $\sqrt{\frac{3k}{2m}}, \sqrt{\frac{5k}{2m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$

(b) $1.3647\sqrt{\frac{k}{m}}, 2.3743\sqrt{\frac{k}{m}}; \begin{Bmatrix} 1 \\ 0.6375 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -3.1375 \end{Bmatrix}$

(c) $0.9287\sqrt{\frac{k}{m}}, 2.1535\sqrt{\frac{k}{m}}; \begin{Bmatrix} 1 \\ 1.5687 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.3187 \end{Bmatrix}$

(d) $1.1993\sqrt{\frac{k}{m}}, 2.3583\sqrt{\frac{k}{m}}; \begin{Bmatrix} 1 \\ 0.7808 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.2808 \end{Bmatrix}$

(e) $\sqrt{\frac{k}{m}}, \sqrt{\frac{5k}{2m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$

(f) $\sqrt{\frac{k}{2m}}, \sqrt{\frac{2k}{m}}; \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

4.

(a) $0.05\cos\left(\sqrt{\frac{2k}{m}}t\right)+0.05\cos\left(2\sqrt{\frac{k}{m}}t\right), 0.05\cos\left(\sqrt{\frac{2k}{m}}t\right)-0.05\cos\left(2\sqrt{\frac{k}{m}}t\right)$

(b) $\sqrt{\frac{m}{k}}\left[0.3536\sin\left(\sqrt{\frac{2k}{m}}t\right) - 0.25\sin\left(2\sqrt{\frac{k}{m}}t\right)\right],$
$\sqrt{\frac{m}{k}}\left[0.3536\sin\left(\sqrt{\frac{2k}{m}}t\right) + 0.25\sin\left(2\sqrt{\frac{k}{m}}t\right)\right]$

5.

(a) $0.0333\cos\left(\sqrt{\frac{k}{2m}}t\right) - 0.0333\cos\left(\sqrt{\frac{2k}{m}}t\right), 0.0667\cos\left(\sqrt{\frac{k}{2m}}t\right) + 0.0333\cos\left(\sqrt{\frac{2k}{m}}t\right)$

(b) $\sqrt{\frac{m}{k}}\left[0.4714\sin\left(\sqrt{\frac{k}{2m}}t\right) + 0.4714\sin\left(\sqrt{\frac{2k}{m}}t\right)\right]$
$,\sqrt{\frac{m}{k}}\left[0.9428\sin\left(\sqrt{\frac{k}{2m}}t\right) - 0.4714\sin\left(\sqrt{\frac{2k}{m}}t\right)\right]$

6.

(a) $0.05\cos(10t) + 0.05\cos(20t), 0.05\cos(10t) - 0.05\cos(20t)$
(b) $0.05\sin(10t) - 0.025\sin(20t), 0.05\sin(10t) + 0.025\sin(20t)$

7.

(a) $0.0333\cos(5t) - 0.0333\cos(10t), 0.0667\cos(5t) + 0.0333\cos(10t)$
(b) $0.0667\sin(5t) + 0.0667\sin(10t), 0.1333\sin(5t) - 0.0667\sin(10t)$

8. 1.92 kg, 107.25 N/m, 105 N/m, 134.52 N/m
9. 1.0032 kg, 12.9073 rad/s, 65.1429 N/m, 73.3248 N/m
10. 2.9744 kg, 1.9829 kg, 10.0429 rad/s, 100 N/m
11.

(a) $0.05\cos(10t) + 0.05\cos\left(\sqrt{200}t\right), 0.05\cos(10t) - 0.05\cos\left(\sqrt{200}t\right)$
(b) $0.05\cos(10t) - 0.05\cos\left(\sqrt{200}t\right), 0.05\cos(10t) + 0.05\cos\left(\sqrt{200}t\right)$

(c) $0.05\sin(10t) + 0.0354\sin\left(\sqrt{200}t\right), 0.05\sin(10t) - 0.0354\sin\left(\sqrt{200}t\right)$

(d) $0.05\sin(10t) - 0.0354\sin\left(\sqrt{200}t\right), 0.05\sin(10t) + 0.0354\sin\left(\sqrt{200}t\right)$

12.

(a) $\lambda^2 - k\left(\frac{10mr^2+I_p}{mI_p}\right)\lambda + \frac{6k^2r^2}{mI_p} = 0$

(b) $\lambda^2 - \frac{k}{m}\left(\frac{6mr^2+I_p}{mr^2+I_p}\right)\lambda + \frac{3k^2r^2}{m\left(mr^2+I_p\right)} = 0$

(c) $\lambda^2 - \frac{k}{2m}\left(\frac{13mr^2+I_p}{mr^2+I_p}\right)\lambda + \frac{k^2r^2}{m\left(mr^2+I_p\right)} = 0$

13.

(a) $\lambda^2 - \frac{k}{m}\left(\frac{6mr^2+3m_dr^2+2I_p}{3m_dr^2+2I_p}\right)\lambda + \frac{5k^2r^2}{m\left(3m_dr^2+2I_p\right)} = 0$

(b) $\lambda^2 - \frac{k}{m_d}\left(\frac{8mr^2+15m_dr^2+4I_p}{2mr^2+I_p}\right)\lambda + \frac{4k^2r^2}{m_d\left(2mr^2+I_p\right)} = 0$

14.

(a) 4.5569 rad/s, 14.4234 rad/s; $\begin{Bmatrix} 1 \\ 14.9362 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.6695 \end{Bmatrix}$

(b) 9.5449 rad/s, 15.9027 rad/s; $\begin{Bmatrix} 1 \\ 4.5373 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2.2040 \end{Bmatrix}$

(c) 7.4558 rad/s, 14.3050 rad/s; $\begin{Bmatrix} 1 \\ 0.3462 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -10.7907 \end{Bmatrix}$

(d) 4.4599 rad/s, 14.9943 rad/s; $\begin{Bmatrix} 1 \\ 1.1104 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -8.0549 \end{Bmatrix}$

(e) 2.4424 rad/s, 12.3763 rad/s; $\begin{Bmatrix} 1 \\ 5.7614 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.1269 \end{Bmatrix}$

(f) 6.4401 rad/s, 12.4181 rad/s; $\begin{Bmatrix} 1 \\ 4.3409 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.1684 \end{Bmatrix}$

15.

(a) 8.6239 rad/s, 11.5207 rad/s; $\begin{Bmatrix} 1 \\ 0.6575 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2.3575 \end{Bmatrix}$

(b) 6.1741 rad/s, 20.1181 rad/s; $\begin{Bmatrix} 1 \\ 3.6188 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.0474 \end{Bmatrix}$

16.

(a) 8.1649 rad/s, 33.6650 rad/s; $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(b) 14.1421 rad/s, 31.6228 rad/s; $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

17.

(a) 18.9092 rad/s, 36.6393 rad/s; $\begin{Bmatrix} 1 \\ 0.6913 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.8476 \end{Bmatrix}$

(b) 17.6097 rad/s, 35.9151 rad/s; $\begin{Bmatrix} 1 \\ 0.4214 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -4.1714 \end{Bmatrix}$

18.

(a) 0, 16.8819 rad/s; $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.28 \end{Bmatrix}$

(b) 0, 13.8293 rad/s; $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2.56 \end{Bmatrix}$

19. $\sqrt{\left(2-\sqrt{2}\right)\frac{g}{L}}, \sqrt{\left(2+\sqrt{2}\right)\frac{g}{L}}; \begin{Bmatrix} \sqrt{2} \\ 1 \end{Bmatrix}, \begin{Bmatrix} \sqrt{2} \\ -1 \end{Bmatrix}$

20. $2\sqrt{\frac{T}{mL}}, 2\sqrt{2}\sqrt{\frac{T}{mL}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

21. 3.0935 rad/s, 20.2497 rad/s; $\begin{Bmatrix} 1 \\ 39.7993 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.9761 \end{Bmatrix}$

22. 2.2017 rad/s, 25.1479 rad/s; $\begin{Bmatrix} 1 \\ 1.2237 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.7845 \end{Bmatrix}$

23.

(a) 87.2966 rad/s, 276.6694 rad/s; $\begin{Bmatrix} 1 \\ 2.5618 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.1952 \end{Bmatrix}$

(b) 149.0068 rad/s, 269.2294 rad/s; $\begin{Bmatrix} 1 \\ 0.8994 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.1119 \end{Bmatrix}$

24.

(a) 243.2834 rad/s, 673.7507 rad/s; $\begin{Bmatrix} 1 \\ 1.6770 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.4770 \end{Bmatrix}$

(b) 492.8089 rad/s, 864.1418 rad/s; $\begin{Bmatrix} 1 \\ 1.1165 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.7165 \end{Bmatrix}$

25.

(a)
$$\begin{aligned} & e^{-0.1134t}(0.0372\cos 9.1162t - 0.0189\sin 9.1162t) \\ & + e^{-3.6366t}(0.0628\cos 8.7754t - 0.0461\sin 8.7754t), \\ & e^{-0.1134t}(0.0423\cos 9.1162t - 0.0042\sin 9.1162t) \\ & + e^{-3.6366t}(-0.0423\cos 8.7754t - 0.0013\sin 8.7754t) \end{aligned}$$

(b)
$$e^{-0.1134t}(0.0085\cos 9.1162t + 0.0770\sin 9.1162t) + e^{-3.6366t}(-0.0085\cos 8.7754t - 0.0834\sin 8.7754t),$$
$$e^{-0.1134t}(-0.0204\cos 9.1162t + 0.0763\sin 9.1162t) + e^{-3.6366t}(0.0204\cos 8.7754t + 0.00429\sin 8.7754t)$$

26.

(a)
$$e^{-0.2236t}(-0.0027\cos 17.2901t + 0.0476\sin 17.2901t) + e^{-1.0264t}(0.0027\cos 4.9021t + 0.0365\sin 4.9021t),$$
$$e^{-0.2236t}(-0.0016\cos 17.2901t - 0.0157\sin 17.2901t) + e^{-1.0264t}(0.0016\cos 4.9021t + 0.0555\sin 4.9021t)$$

(b)
$$e^{-0.2236t}(-0.0549\cos 17.2901t - 0.0016\sin 17.2901t) + e^{-1.0264t}(0.0549\cos 4.9021t + 0.0148\sin 4.9021t),$$
$$e^{-0.2236t}(0.0180\cos 17.2901t - 0.0023\sin 17.2901t) + e^{-1.0264t}(0.0820\cos 4.9021t + 0.0261\sin 4.9021t)$$

27.

(a)
$$e^{-0.1124t}(0.0543\cos 5.0137t + 0.0058\sin 5.0137t) + e^{-3.6376t}(-0.0543\cos 16.8814t - 0.0131\sin 16.8814t),$$
$$e^{-0.1124t}(0.0802\cos 5.0137t + 0.0135\sin 5.0137t) + e^{-3.6376t}(0.0198\cos 16.8814t + 0.0008\sin 16.8814t)$$

(b)
$$e^{-0.1124t}(0.0036\cos 5.0137t + 0.0368\sin 5.0137t) + e^{-3.6376t}(-0.0036\cos 16.8814t + 0.0476\sin 16.8814t),$$
$$e^{-0.1124t}(0.0020\cos 5.0137t + 0.0550\sin 5.0137t) + e^{-3.6376t}(-0.0020\cos 16.8814t - 0.0168\sin 16.8814t)$$

28.

(a)
$$e^{-0.7224t}(-0.0016\cos 5.5606t + 0.0940\sin 5.5606t) + e^{-5.9442t}(0.0016\cos 16.8841t - 0.0306\sin 16.8841t),$$
$$e^{-0.7224t}(0.0021\cos 5.5606t + 0.1373\sin 5.5606t) + e^{-5.9442t}(-0.0021\cos 16.8841t + 0.0134\sin 16.8841t)$$

(b)
$$e^{-0.7224t}(0.0224\cos 5.5606t + 0.0096\sin 5.5606t) + e^{-5.9442t}(0.0776\cos 16.8841t + 0.0252\sin 16.8841t),$$
$$e^{-0.7224t}(0.0331\cos 5.5606t + 0.0130\sin 5.5606t) + e^{-5.9442t}(-0.0331\cos 16.8841t - 0.0146\sin 16.8841t)$$

29.

(a)
$$e^{-2.5t}(0.05\cos 9.6825t + 0.0129\sin 9.6825t) + e^{-7,5t}(0.05\cos 18.5405t + 0.0202\sin 18.5405t),$$
$$e^{-2.5t}(0.05\cos 9.6825t + 0.0129\sin 9.6825t) + e^{-7.5t}(-0.05\cos 18.5405t - 0.0202\sin 18.5405t)$$

(b)
$$e^{-2.5t}(0.0516\sin 9.6825t) + e^{-7,5t}(-0.0270\sin 18.5405t),$$
$$e^{-2.5t}(0.0516\sin 9.6825t) + e^{-7.5t}(0.0270\sin 18.5405t)$$

30. $-0.3559\sin 30t$, $0.0678\sin 30t$
31. $0.0037\sin 40t$, $-0.0330\sin 340t$
32. $-0.3220\sin 30t$, $-0.0339\sin 30t$
33. $-0.3929\sin 30t + 0.0021\sin 50t$, $0.0357\sin 30t - 0.0310\sin 50t$
34. $\omega < 4.7369$ rad/s, $\omega > 26.3099$ rad/s
35. $\omega < 2.2284$ rad/s, $\omega > 29.9171$ rad/s
36. 26.2769 N
37. 304.8632 N
38.

(a) $-0.3928\sin 30t - 0.00076\cos 30t$, $0.0354\sin 30t + 0.00329\cos 30t$
(b) $-0.3569\sin 30t - 0.0968\cos 30t$, $0.0197\sin 30t + 0.0430\cos 30t$

39.

(a) $-0.0001\sin 40t + 0.0027\cos 40t$, $-0.0974\sin 40t - 0.0013\cos 40t$
(b) $0.0088\sin 40t + 0.00027\cos 40t$, $-0.0977\sin 40t - 0.00001\cos 40t$

40.
$$0.1124\sin 30t - 0.2364\cos 30t + 0.0007\sin 50t + 0.0048\cos 50t,$$
$$-0.0403\sin 30t + 0.0044\cos 30t - 0.0408\sin 50t - 0.0030\cos 50t$$

41.
$$-0.2389\sin 30t - 0.0297\cos 30t - 0.0013\sin 50t + 0.0029\cos 50t,$$
$$0.0517\sin 30t + 0.0226\cos 30t + 0.0021\sin 50t - 0.0445\cos 50t$$

42. 10.2083 kg, 3.51065 MN/m
43. $0.0167m$
44. 7.2 kN/m, 0.0056 m
45. 100 kN/m, 0.3648 kg
46. 5.67 MN/m, 40.5 kg
47.

(a)
$$0.05\sin 10t + 0.05\cos 10t - 0.025\sin 20t + 0.05\cos 20t,$$
$$0.05\sin 10t + 0.05\cos 10t + 0.025\sin 20t - 0.05\cos 20t$$

(b)
$$0.05\sin 10t + 0.05\cos 10t + 0.025\sin 20t - 0.05\cos 20t,$$
$$0.05\sin 10t + 0.05\cos 10t - 0.025\sin 20t + 0.05\cos 20t$$

48.

(a)
$$\begin{aligned}&e^{-2.5t}(0.0645\sin 9.6825t+0.05\cos 9.6825t)\\&+e^{-7.5t}(0.0067\sin 18.5405t-0.05\cos 18.5405t),\\&e^{-2.5t}(0.0645\sin 9.6825t+0.05\cos 9.6825t)\\&+e^{-7.5t}(-0.0067\sin 18.5405t+0.05\cos 18.5405t)\end{aligned}$$

(b)
$$\begin{aligned}&e^{-2.5t}(0.0645\sin 9.6825t+0.05\cos 9.6825t)\\&+e^{-7.5t}(-0.0067\sin 18.5405t+0.05\cos 18.5405t),\\&e^{-2.5t}(0.0645\sin 9.6825t+0.05\cos 9.6825t)\\&+e^{-7.5t}(0.0067\sin 18.5405t-0.05\cos 18.5405t)\end{aligned}$$

49. $-0.3559\sin 30t,\ 0.0678\sin 30t$
50. $0.0037\sin 40t,\ -0.0329\sin 40t$
51. $-0.3559\sin 30t+0.0026\sin 50t,\ 0.0678\sin 10t-0.0291\sin 50t$
52. $-0.3929\sin 30t+0.0021\sin 50t,\ 0.0357\sin 10t-0.0310\sin 50t$
53. $-0.0657\sin 50t-0.0030\cos 50t,\ 0.0016\sin 50t-0.00076\cos 50t$
54. $-0.0496\sin 30t-0.0593\cos 30t,\ -0.1797\sin 30t-0.1281\cos 30t$
55. $u(t-2)[0.25\sin\{10(t-2)\}-0.1443\sin\{17.3206(t-2)\}]$
56. $u(t-1)[0.05-0.0585\sin\{6.1803(t-1)\}+0.0085\sin\{16.1803(t-1)\}]$

57.
$$\begin{aligned}&0.0067t-0.001\sin 10t+0.00019\sin 17.3205t\\&-u(t-2)[-0.0067t+0.001\sin\{10(t-2)\}+0.002\cos\{10(t-2)\}\\&-0.00019\sin\{17.3205(t-2)\}-0.0067\cos\{17.3205(t-2)\}]\end{aligned}$$

58.
$$\begin{aligned}&-0.05(t-2)+0.0095\sin 6.1803t-0.1171\cos 6.1803t\\&-0.0005\sin 16.1803t-0.0171\cos 16.1803t\\&-u(t-2)[0.05(t-2)-0.0095\sin\{6.1803(t-2)\}\\&+0.0005\sin\{16.1803(t-2)\}]\end{aligned}$$

59.
$$\begin{aligned}&u(t-2)[0.3619e^{-0.1909(t-2)}\sin\{6.1774(t-2)\}\\&-0.1387e^{-1.3090(t-2)}\sin\{16.1273(t-2)\}]\end{aligned}$$

60.
$$\begin{aligned}&u(t-1)[0.1667+e^{-2.5(t-1)}\{-0.0645\sin 9.6825(t-1)\\&-0.25\cos 9.6825(t-1)\}+e^{-7.5(t-1)}\{-0.04\sin 15.6125(t-1)\\&-0.0833\cos 15.6125(t-1)]\end{aligned}$$

61.
$$\begin{aligned}&0.4t-0.02+e^{-0.9549t}(-0.0703\sin 6.1061t+0.0234\cos 6.1061t)\\&+e^{-6.5451t}(0.0031\sin 14.7975t-0.0034\cos 14.7975t)\\&+u(t-2)[0.4t-0.02+\ e^{-0.9549(t-2)}\{0.2195\sin 6.1061(t-2)\\&+0.9132\cos 6.1061(t-2)\}+e^{-6.5451(t-2)}\{-0.0636\sin 14.7975(t-2)\\&-0.1332\cos 14.7975(t-2)\}]\end{aligned}$$

62. $$\begin{aligned}&-0.0033t+0.0067+e^{-0.5t}(-0.000001\sin 9.9875t\\&-0.005\cos 9.9875t)+e^{-1.5t}(0.000097\sin 17.2554t\\&-0.0017\cos 17.2554t)+u(t-2)[0.0033t-0.0067\\&+\ e^{-0.5(t-2)}\{-0.000025\sin 9.98751(t-2)+0.000025\\&\times\cos 9.9875(t-2)\}+e^{-1.5(t-2)}\{-0.000048\sin 17.2554(t-2)\\&+0.000008\cos 17.2554(t-2)\}]\end{aligned}$$

63. 0, 20.4124 rad/s; $\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-0.6667\end{Bmatrix}$

64. 0, 16.3299 rad/s; $\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-1\end{Bmatrix}$

65. 0, 196.1523 rad/s; $\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-0.8\end{Bmatrix}$

66.

(a) 15.4919, 18.9737 rad/s; $\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-1\end{Bmatrix}$

(b) 15.4919, 18.9737 rad/s

67. 0, 15.1421 rad/s; $\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-0.25\end{Bmatrix}$

Chapter 6
Response of a Multi-Degree of Freedom System

6.1 Introduction

A system having a degree of freedom equal to or higher than two is called a multi-degree of freedom system. A multi-degree of freedom system will have a number of natural frequencies equal to degree of freedom of the system and will have the same number of mode shapes corresponding to each natural frequencies. Similarly, the system will undergo resonance if the operating frequency becomes equal to any one of the natural frequencies.

The procedure to determine the free and forced response of two degrees of freedom system was explained in the previous chapter. Therefore this chapter focuses mainly on the vibration analysis procedure of the systems having degrees of freedom higher than two.

It can be recalled from the previous chapter that equation of motion of a two degrees of freedom system consists of a two coupled linear differential equations, and the characteristics equation of the system can be derived by equating amplitude ratios obtained from equation of motion. However multi-degree of freedom systems will have large number of coupled differential equations, and derivation of characteristic equations by equating the proportions of amplitudes will be very tedious. Hence in this chapter we will discuss the vibration analysis procedure based on matrix formulation that can handle large number of coupled differential equations efficiently. However method discussed in this chapter can also be used to two degrees of freedom systems dealt in Chap. 5.

6.2 Formulation of Equation of Motion in Matrix Form

Consider an un-damped system with n degrees of freedom as shown in Fig. 6.1. The system consists of n number of masses and $(n + 1)$ number of springs. Free-body diagram of the system is also shown in Fig. 6.2.

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https://doi.org/10.1007/978-981-99-3614-4_6

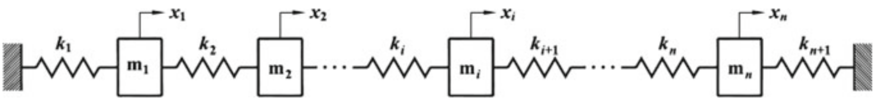


Fig. 6.1 Spring-mass system with ***n*** degree of freedom

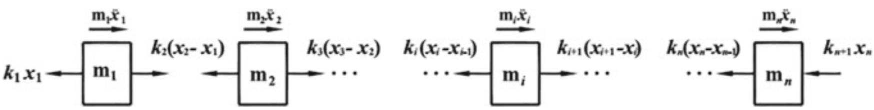


Fig. 6.2 Free-body diagram for a spring-mass system with ***n*** degree of freedom

With reference to the free-body diagram of the system, we can apply Newton's second law of motion for each mass as

$$
\begin{aligned}
&-k_1x_1 + k_2(x_2 - x_1) = m_1\ddot{x}_1\\
&-k_2(x_2 - x_1) + k_3(x_3 - x_2) = m_2\ddot{x}_2\\
&-k_3(x_3 - x_2) + k_4(x_4 - x_3) = m_3\ddot{x}_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) = m_i\ddot{x}_i\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-k_n(x_n - x_{n-1}) - k_{n+1}x_n = m_n\ddot{x}_n
\end{aligned} \tag{6.1}
$$

Rearranging Eq. (6.1), we get

$$
\begin{aligned}
&m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0\\
&m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0\\
&m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 - k_4x_4 = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_i\ddot{x}_i - k_ix_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_n\ddot{x}_n - k_nx_{n-1} + (k_n + k_{n+1})x_n = 0
\end{aligned} \tag{6.2}
$$

Equation (6.2) can also be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & m_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & m_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & m_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ .. \\ \ddot{x}_i \\ .. \\ \ddot{x}_n \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & .. & 0 & .. & 0 \\ -k_2 & k_2 + k_3 & -k_3 & .. & 0 & .. & 0 \\ 0 & -k_3 & k_3 + k_4 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & k_i + k_{i+1} & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & k_n + k_{n+1} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_i \\ .. \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \\ .. \\ 0 \end{Bmatrix} \quad (6.3)$$

Equation (6.3) then can be expressed in short from as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad (6.4)$$

where

- $[M]$ is the mass matrix. It is a square matrix and is a diagonal matrix for statically coupled system.
- $[K]$ is the stiffness matrix. It is also a symmetric square matrix with positive diagonal elements and negative non-diagonal elements,
- $\{\ddot{x}\}$ is the acceleration vector. It is a column vector consisting of accelerations of each mass.
- $\{x\}$ is the displacement vector. It is a column vector consisting of displacements of each mass.

6.3 Flexibility and Stiffness Matrices

6.3.1 Flexibility Influence Coefficients and Flexibility Matrix

Equation of motion of multi-degree of freedom systems can also be expressed in terms of a flexibility matrix $[A]$. . The elements of flexibility matrix (a_{ij}) are called flexibility influence coefficients. The flexibility influence coefficient (a_{ij}) can be defined as the displacement at point i due to a unit force or moment applied at point j.

Consider a linear elastic system subjected to the forces $f_1, f_2, f_3, \cdots\cdots ., f_n$ as shown in Fig. 6.3. Then by applying superposition principle, the displacements of each point of the system can be determined as

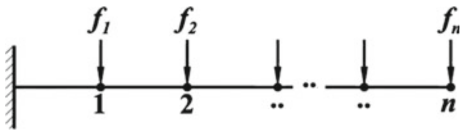


Fig. 6.3 Linear elastic system subjected to a number of forces

$$
\begin{aligned}
x_1 &= a_{11} f_1 + a_{12} f_2 + a_{13} f_3 + \cdots + a_{1i} f_i + \cdots a_{1n} f_n \\
x_2 &= a_{21} f_1 + a_{22} f_2 + a_{23} f_3 + \cdots + a_{2i} f_i + \cdots a_{2n} f_n \\
x_3 &= a_{31} f_1 + a_{32} f_2 + a_{33} f_3 + \cdots + a_{3i} f_i + \cdots a_{3n} f_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_i &= a_{i1} f_1 + a_{i2} f_2 + a_{i3} f_3 + \cdots + a_{ii} f_i + \cdots a_{in} f_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n &= a_{n1} f_1 + a_{n2} f_2 + a_{n3} f_3 + \cdots + a_{ni} f_i + \cdots a_{nn} f_n
\end{aligned}
\tag{6.5}
$$

Equation (6.5) can also be expressed in matrix form as

$$
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_i \\ .. \\ x_n \end{Bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & .. & a_{1i} & .. & a_{1n} \\
a_{21} & a_{22} & a_{23} & .. & a_{2i} & .. & a_{2n} \\
a_{31} & a_{32} & a_{33} & .. & a_{3i} & .. & a_{3n} \\
.. & .. & .. & .. & .. & .. & .. \\
a_{i1} & a_{i2} & a_{i3} & .. & a_{ii} & .. & a_{in} \\
.. & .. & .. & .. & .. & .. & .. \\
a_{n1} & a_{n2} & a_{n3} & .. & a_{ni} & .. & a_{nn}
\end{bmatrix}
\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ .. \\ f_i \\ .. \\ f_n \end{Bmatrix}
\tag{6.6}
$$

Equation (6.6) can be expressed in short form as

$$
\{x\} = [A]\{f\} \tag{6.7}
$$

6.3.2 *Stiffness Influence Coefficients and Stiffness Matrix*

Equation of motion of multi-degree of freedom systems can also be expressed directly in terms of a stiffness matrix$[K]$. The elements of stiffness matrix (k_{ij}) are called stiffness influence coefficients. The stiffness influence coefficient (k_{ij}) can be defined as the force or moment required at point i due to a unit displacement at point j with all other points of the system fixed.

Again applying superposition principle, the force required at each point of the system can be determined as

$$
\begin{aligned}
f_1 &= k_{11} x_1 + k_{12} x_2 + k_{13} x_3 + \cdots + k_{1i} x_i + \cdots k_{1n} x_n \\
f_2 &= k_{21} x_1 + k_{22} x_2 + k_{23} x_3 + \cdots + k_{2i} x_i + \cdots k_{2n} x_n
\end{aligned}
$$

$$\begin{aligned}
f_3 &= k_{31}x_1 + k_{32}x_2 + k_{33}x_3 + \cdots + k_{3i}x_i + \cdots k_{3n}x_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
f_i &= k_{i1}x_1 + k_{i2}x_2 + k_{i3}x_3 + \cdots + k_{ii}x_i + \cdots k_{in}x_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
f_n &= k_{n1}x_1 + k_{n2}x_2 + k_{n3}x_3 + \cdots + k_{ni}x_i + \cdots k_{nn}x_n
\end{aligned} \tag{6.8}$$

Equation (6.8) can also be expressed in matrix form as

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ .. \\ f_i \\ .. \\ f_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & .. & k_{1i} & .. & k_{1n} \\ k_{21} & k_{22} & k_{23} & .. & k_{2i} & .. & k_{2n} \\ k_{31} & k_{32} & k_{33} & .. & k_{3i} & .. & k_{3n} \\ .. & .. & .. & .. & .. & .. & .. \\ k_{i1} & k_{i2} & k_{i3} & .. & k_{ii} & .. & k_{in} \\ .. & .. & .. & .. & .. & .. & .. \\ k_{n1} & k_{n2} & k_{n3} & .. & k_{ni} & .. & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_i \\ .. \\ x_n \end{Bmatrix} \tag{6.9}$$

Equation (6.9) can be expressed in short form as

$$\{f\} = [K]\{x\} \tag{6.10}$$

6.3.3 Relationship Between Flexibility and Stiffness Matrix

Pre-multiplying Eq. (6.7) by the inverse of flexibility matrix $[A]^{-1}$, we get

$$[A]^{-1}\{x\} = \{f\} \tag{6.11}$$

Substituting $\{f\}$ from Eq. (6.9) into Eq. (6.11), we get

$$\begin{aligned}
&[A]^{-1}\{x\} = [K]\{x\} \\
\therefore \quad &[A]^{-1} = [K] \quad \text{or} \quad [K]^{-1} = [A]
\end{aligned} \tag{6.12}$$

It can be noted from Eq. (6.12) that flexibility and stiffness matrix are inverse to each other.

6.3.4 Reciprocity Theorem

A multi-degree of freedom system with n degrees of freedom will have a flexibility and stiffness matrix of size $n \times n$ and will have n^2 elements. All diagonal elements

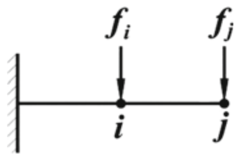


Fig. 6.4 Linear elastic system subjected to transverse loads f_i and f_j at points i and j

and half of the non-diagonal elements of the flexibility and stiffness matrix can be determined by applying static equilibrium equations, and the remaining elements can be determined by using reciprocity theorem.

Reciprocity theorem can be stated as:

The deflection at any point in a linear elastic system due to unit load acting at any other point of the same system is equal to the deflection at the second point due to unit load acting on the first point, i.e., $a_{ij} = a_{ji}$.

To prove this, consider an elastic system which is subjected to transverse loads f_i and f_j at points i and j of the system, respectively, as shown in Fig. 6.4. When only f_i is applied at point i of the system, the deflection produced at point i of the system will be $f_i a_{ii}$ and the work done due to this displacement will be $\frac{1}{2} f_i^2 a_{ii}$. When f_j is also applied at point j of the system, the deflections produced at point i and j of the system will be $f_j a_{ij}$ and $f_j a_{jj}$, respectively. Then the work done due to these displacements will be $f_i f_j a_{ij}$ and $\frac{1}{2} f_j^2 a_{jj}$, respectively. Therefore, the total work done due both loads is given by

$$W = \frac{1}{2} f_i^2 a_{ii} + \frac{1}{2} f_j^2 a_{jj} + f_i f_j a_{ij} \tag{6.13}$$

Similarly, the total work done due both loads, when the load f_j is first applied and then f_i, is given by

$$W = \frac{1}{2} f_j^2 a_{jj} + \frac{1}{2} f_i^2 a_{ii} + f_j f_i a_{ji} \tag{6.14}$$

Equating Eqs. (6.13) and (6.14), we get

$$a_{ij} = a_{ji} \tag{6.15}$$

Similar relationship can also established for the stiffness influence coefficients, i.e.,

$$k_{ij} = k_{ji} \tag{6.16}$$

6.4 Natural Frequencies and Mode Shapes of a MDOF System

Equation of motion can be expressed in matrix form as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \tag{6.17}$$

Pre-multiplying Eq. (6.16) by the inverse of mass matrix $[M]^{-1}$, we get

$$\begin{aligned} \{\ddot{x}\} + [M]^{-1}[K]\{x\} &= \{0\} \\ \therefore \quad \{\ddot{x}\} + [D]\{x\} &= \{0\} \end{aligned} \tag{6.18}$$

where $[D] = [M]^{-1}[K]$, is called a dynamic matrix.

We can assume the solution of system of linear differential Eq. (6.18) as

$$\{x\} = \{X\}\sin(\omega_n t) \tag{6.19}$$

Differentiating $\{x\}$ with respect to t twice, we get

$$\{\ddot{x}\} = -\omega_n^2\{X\}\sin(\omega_n t) = -\lambda[\{X\}\sin(\omega_n t)] = -\lambda\{x\} \tag{6.20}$$

where $\lambda = \omega_n^2$.

Substituting $\{x\}$ from Eq. (6.20) into Eq. (6.18), we get

$$\begin{aligned} -\lambda\{x\} + [D]\{x\} &= \{0\} \\ \therefore \quad [D - \lambda I]\{x\} &= \{0\} \end{aligned} \tag{6.21}$$

Equation (6.20) is the standard form of the eigen-value problem, and the non-trivial solution of the equation is given by

$$|D - \lambda I| = 0 \tag{6.22}$$

Equation (6.21) is called the characteristic determinant. When it is expanded, it gives a polynomial equation for λ and the expanded equation is called the characteristic equation of the system. The order of the polynomial equation will be equal to degree of freedom of the system. If the system has n degree of freedom, the characteristic equation gives n roots of λ. Each values of λ gives the natural frequencies of the system as

$$\omega_i = \sqrt{\lambda_i} \tag{6.23}$$

Substituting λ_i into Eq. (6.20), we get the eigen-vectors $\{X\}_i$ corresponding to each value of λ_i, i.e.,

$$[D - \lambda_i I]\{X\}_i = \{0\} \tag{6.24}$$

Eigen-vectors $\{X\}_i$ define mode shapes corresponding to each natural frequency.

6.5 Orthogonal Properties of the Eigen-Vectors

Substituting $\{\ddot{x}\} = -\lambda\{x\}$ into Eq. (6.17), we get

$$[K]\{X\} = \lambda[M]\{X\} \tag{6.25}$$

For the ith mode, Eq. (6.25) can be expressed as

$$[K]\{X\}_i = \lambda_i[M]\{X\}_i \tag{6.26}$$

Pre-multiplying Eq. (6.25) by the transpose of eigen-vector of jth mode, we get

$$\{X\}'_j[K]\{X\}_i = \lambda_i\{X\}'_j[M]\{X\}_i \tag{6.27}$$

Similarly, for the jth mode, Eq. (6.25) can be expressed as

$$[K]\{X\}_j = \lambda_j[M]\{X\}_j \tag{6.28}$$

Pre-multiplying Eq. (6.28) by the transpose of eigen-vector of ith mode, we get

$$\{X\}'_i[K]\{X\}_j = \lambda_j\{X\}'_i[M]\{X\}_j \tag{6.29}$$

Subtracting Eq. (6.29) from Eq. (6.27), we get

$$\begin{aligned}&\{X\}'_j[K]\{X\}_i - \{X\}'_i[K]\{X\}_j \\ &= \lambda_i\{X\}'_j[M]\{X\}_i - \lambda_j\{X\}'_i[M]\{X\}_j\end{aligned} \tag{6.30}$$

Since both stiffness matrix $[K]$ and mass matrix $[M]$ are symmetric, the following relationships exist

$$\{X\}'_i[K]\{X\}_j = \{X\}'_j[K]\{X\}_i \tag{6.31}$$

$$\{X\}'_i[M]\{X\}_j = \{X\}'_j[M]\{X\}_i \tag{6.32}$$

Applying Eqs. (6.30) and (6.31) into Eq. (6.29), it reduces to

$$0 = \left[\lambda_i - \lambda_j\right]\{X\}'_i[M]\{X\}_j \tag{6.33}$$

Equations (6.26) and (6.28) can also be expressed as

$$\{X\}_j'[M]\{X\}_i = \frac{1}{\lambda_i}\{X\}_j'[K]\{X\}_i \tag{6.34}$$

$$\{X\}_i'[M]\{X\}_j = \frac{1}{\lambda_j}\{X\}_i'[K]\{X\}_j \tag{6.35}$$

Again taking the difference of Eqs. (6.34) and (6.35) and applying Eqs. (6.30) and (6.31), we get

$$0 = \left[\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right]\{X\}_i'[K]\{X\}_j \tag{6.36}$$

If $\lambda_i \neq \lambda_j$, Eqs. (6.33) and (6.36) reduce to

$$\{X\}_i'[M]\{X\}_j = 0 \tag{6.37}$$

$$\{X\}_i'[K]\{X\}_j = 0 \tag{6.38}$$

Equations (6.37) and (6.38) show that eigen-vectors of different modes are orthogonal to each other.

If $\lambda_i = \lambda_j$, it can be noted from Eqs. (6.33) and (6.36) that both products $\{X\}_i'[M]\{X\}_i$ and $\{X\}_i'[K]\{X\}_i$ will have nonzero values. These nonzero values can be defined as

$$\{X\}_i'[M]\{X\}_j = [M]_i \tag{6.39}$$

$$\{X\}_i'[K]\{X\}_j = [K]_i \tag{6.40}$$

where $[M]_i$ and $[K]_i$ are called the generalized mass and the generalized stiffness, respectively.

6.6 Modal Analysis

As explained earlier, equation of motion of a multi-degree of freedom system is obtained in the form of coupled linear differential equations. These equations can be uncoupled by employing the orthogonal relationship between the eigen-vectors, and each mode can be solved independently.

6.6.1 Modal Analysis for Un-damped Free Response of a MDOF System

For modal analysis, let us define a modal matrix $[U]$ such that its columns are the eigen-vectors of the given MDOF system, i.e.,

$$[U] = \left[\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ .. \\ X_i \\ .. \\ X_n \end{Bmatrix}_1 \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ .. \\ X_i \\ .. \\ X_n \end{Bmatrix}_2 \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ .. \\ X_i \\ .. \\ X_n \end{Bmatrix}_3 \begin{Bmatrix} .. \\ .. \\ .. \\ .. \\ .. \\ .. \\ .. \end{Bmatrix}_{..} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ .. \\ X_i \\ .. \\ X_n \end{Bmatrix}_i \begin{Bmatrix} .. \\ .. \\ .. \\ .. \\ .. \\ .. \\ .. \end{Bmatrix}_{..} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ .. \\ X_i \\ .. \\ X_n \end{Bmatrix}_n \right] \tag{6.41}$$

Then the transpose of modal matrix $[U]'$ can be obtained as

$$[U]' = \begin{bmatrix} \{X_1 \; X_2 \; X_3 \; .. \; X_i \; .. \; X_n\}_1 \\ \{X_1 \; X_2 \; X_3 \; .. \; X_i \; .. \; X_n\}_2 \\ \{X_1 \; X_2 \; X_3 \; .. \; X_i \; .. \; X_n\}_3 \\ \{\ldots \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots\}_{..} \\ \{X_1 \; X_2 \; X_3 \; .. \; X_i \; .. \; X_n\}_i \\ \{\ldots \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots\}_{..} \\ \{X_1 \; X_2 \; X_3 \; .. \; X_i \; .. \; X_n\}_n \end{bmatrix} \tag{6.42}$$

To uncouple the equation, we can start with the equation of motion of a multi-degree of freedom system as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \tag{6.43}$$

Let us define a linear transformation

$$\{x\} = [U]\{y\} \tag{6.44}$$

where $\{y\}$ is the column vector of principal coordinates.

Pre-multiplying Eq. (6.44) by $[U]^{-1}$, we get

$$\{y\} = [U]^{-1}\{x\} \tag{6.45}$$

Now substituting Eq. (6.44) into Eq. (6.43), we get

$$[M][U]\{\ddot{y}\} + [K][U]\{y\} = \{0\} \tag{6.46}$$

Pre-multiplying Eq. (6.46) by $[U]'$, we get

$$[U]'[M][U]\{\ddot{y}\} + [U]'[K][U]\{y\} = \{0\} \tag{6.47}$$

It can be recalled from the orthogonal characteristic of the eigen-vectors that non-diagonal terms of both products $[U]'[M][U]$ and $[U]'[K][U]$ of Eq. (6.47) will be zero and the diagonal terms will be equal to generalized mass M_i and generalized stiffness K_i, respectively, i.e.,

$$\{X\}_i'[M]\{X\}_j = 0; \quad \{X\}_i'[K]\{X\}_j = 0 \tag{6.48}$$

$$\{X\}_i'[M]\{X\}_i = [M]_i; \quad \{X\}_i'[K]\{X\}_i = [K]_i \tag{6.49}$$

Using these relations into Eq. (6.47) reduces to

$$\begin{bmatrix} M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & M_n \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ .. \\ \ddot{y}_i \\ .. \\ \ddot{y}_n \end{Bmatrix} + \begin{bmatrix} K_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & K_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & K_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & K_i & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & K_n \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_i \\ .. \\ y_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \\ .. \\ 0 \end{Bmatrix} \tag{6.50}$$

Equation (6.50) gives n uncoupled differential equations for $y_1, y_2, \ldots$ and y_n. Equation of motion for ith mode can then be expressed as

$$M_i\ddot{y}_i + K_i y_i = 0 \tag{6.51}$$

Dividing Eq. (6.51) by M_i and substituting $K_i/M_i = \lambda_i = \omega_i^2$, we get

$$\ddot{y}_i + \omega_i^2 y_i = 0 \tag{6.52}$$

The general solution of Eq. (6.52) can be given as

$$y_i = A_i \sin(\omega_i t) + B_i \cos(\omega_i t) \tag{6.53}$$

The arbitrary constants A_i and B_i are determined from the initial conditions. For this the initial conditions given for $\{x_i\}(0)$ and $\{\dot{x}_i\}(0)$ should be transformed into $\{y_i\}(0)$ and $\{\dot{y}_i\}(0)$ by using Eq. (6.45).

Then the response vector $\{x\}$ can be determined by applying the linear transformation defined in Eq. (6.44).

6.6.2 *Modal Analysis for Damped Free Response of a MDOF System*

Equation of motion of a damped MDOF system is given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{6.54}$$

Substituting $\{x\} = [U]\{y\}$, into Eq. (6.54), we get

$$[M][U]\{\ddot{y}\} + [C][U]\{\dot{y}\} + [K][U]\{y\} = \{0\} \tag{6.55}$$

Pre-multiplying Eq. (6.46) by $[U]'$, we get

$$[U]'[M][U]\{\ddot{y}\} + [U]'[C][U]\{\dot{y}\} + [U]'[K][U]\{y\} = \{0\} \tag{6.56}$$

As explained in the previous section, the first and third products reduce to diagonal matrices as

$$[U]'[M][U] = \begin{bmatrix} M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & M_n \end{bmatrix} \tag{6.57}$$

and

$$[U]'[K][U] = \begin{bmatrix} K_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & K_3 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & K_4 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & K_i & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & K_n \end{bmatrix} \tag{6.58}$$

Damping present in the system is said to be proportional damping if $[C]$ is proportional to $[M]$ and $[K]$ and can be defined as a liner combination of $[M]$ and $[K]$ as

$$[C] = \alpha[M] + \beta[K] \tag{6.59}$$

where α and β are nonzero constants.

Then the second product of Eq. (6.56) can be expressed as

$$\begin{aligned}
&[U]'[C][U] = \alpha[U]'[M][U] + \beta[U]'[K][U] \\
&= \alpha\begin{bmatrix} M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & M_n \end{bmatrix} + \beta\begin{bmatrix} K_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & K_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & K_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & K_i & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & K_n \end{bmatrix} \\
&= \begin{bmatrix} \left(\alpha + \beta\omega_1^2\right)M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & \left(\alpha + \beta\omega_2^2\right)M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & \left(\alpha + \beta\omega_2^2\right)M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & \left(\alpha + \beta\omega_i^2\right)M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & \left(\alpha + \beta\omega_n^2\right)M_n \end{bmatrix}
\end{aligned} \tag{6.60}$$

The expression $\left(\alpha + \beta\omega_i^2\right)$ can be expressed in terms of modal damping ratio ξ_i as

$$\alpha + \beta\omega_i^2 = 2\xi_i\omega_i \tag{6.61}$$

Substituting Eq. (6.61) into Eq. (6.60), we get

$$[U]'[C][U] = \begin{bmatrix} 2\xi_1\omega_1 M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & 2\xi_2\omega_2 M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & 2\xi_3\omega_3 M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 2\xi_i\omega_i M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & 2\xi_n\omega_n M_n \end{bmatrix} \tag{6.62}$$

Substituting Eqs. (6.57), (6.58) and (6.62) into Eq. (6.56), we get

$$\begin{bmatrix} M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & M_n \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ .. \\ \ddot{y}_i \\ .. \\ \ddot{y}_n \end{Bmatrix}$$
$$+ \begin{bmatrix} 2\xi_1\omega_1 M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & 2\xi_2\omega_2 M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & 2\xi_3\omega_3 M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 2\xi_i\omega_i M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & 2\xi_n\omega_n M_n \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ .. \\ \dot{y}_i \\ .. \\ \dot{y}_n \end{Bmatrix}$$
$$+ \begin{bmatrix} K_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & K_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & K_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & K_i & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & K_n \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_i \\ .. \\ y_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \\ .. \\ 0 \end{Bmatrix} \tag{6.63}$$

Then equation of motion for i^{th} mode can then be expressed as

$$M_i\ddot{y}_i + 2\xi_i\omega_i M_i\dot{y}_i + K_i y_i = 0 \tag{6.64}$$

Dividing Eq. (6.51) by M_i and substituting $K_i/M_i = \lambda_i = \omega_i^2$, we get

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = 0 \tag{6.65}$$

The general solution of Eq. (6.65) can be given as

$$y_i = \mathrm{e}^{-\xi_i\omega_i t}\left[A_i \sin\left\{\left(\sqrt{1-\xi_i^2}\right)\omega_i t\right\} + B_i \cos\left\{\left(\sqrt{1-\xi_i^2}\right)\omega_i t\right\}\right] \tag{6.66}$$

The arbitrary constants A_i and B_i are determined from the initial conditions as explained in the previous section.

6.6.3 Modal Analysis for Forced Response of a MDOF System

Equation of motion of a MDOF system subject to external excitations is given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \tag{6.67}$$

where $\{F\} = \left\{ F_1 \; F_2 \; F_3 \; .. \; F_i \; .. \; F_n \right\}$ and $F_1,\ F_2, \cdots\cdots$ are the external forces applied to masses $m_1, m_2, \cdots$ of the system, respectively.

Substituting $\{x\} = [U]\{y\}$, into Eq. (6.66), we get

$$[M][U]\{\ddot{y}\} + [C][U]\{\dot{y}\} + [K][U]\{y\} = \{F\} \tag{6.68}$$

Pre-multiplying Eq. (6.67) by $[U]'$, we get

$$\begin{aligned} &[U]'[M][U]\{\ddot{y}\} + [U]'[C][U]\{\dot{y}\} \\ &+ [U]'[K][U]\{y\} = [U]'\{F\} = \left\{F_y\right\} \end{aligned} \tag{6.69}$$

Following the similar procedure explained in previous sections, we get

$$\begin{aligned} &\begin{bmatrix} M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & M_n \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ .. \\ \ddot{y}_i \\ .. \\ \ddot{y}_n \end{Bmatrix} \\ &+ \begin{bmatrix} 2\xi_1\omega_1 M_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & 2\xi_2\omega_2 M_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & 2\xi_3\omega_3 M_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 2\xi_i\omega_i M_i & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & 2\xi_n\omega_n M_n \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ .. \\ \dot{y}_i \\ .. \\ \dot{y}_n \end{Bmatrix} \\ &+ \begin{bmatrix} K_1 & 0 & 0 & .. & 0 & .. & 0 \\ 0 & K_2 & 0 & .. & 0 & .. & 0 \\ 0 & 0 & K_3 & .. & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & K_i & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & .. & K_n \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_i \\ .. \\ y_n \end{Bmatrix} = \begin{Bmatrix} \left(F_y\right)_1 \\ \left(F_y\right)_2 \\ \left(F_y\right)_3 \\ .. \\ \left(F_y\right)_i \\ .. \\ \left(F_y\right)_n \end{Bmatrix} \end{aligned} \tag{6.70}$$

Then equation of motion for i^{th} mode can then be expressed as

$$M_i\ddot{y}_i + 2\xi_i\omega_i M_i\dot{y}_i + K_i y_i = \left(F_y\right)_i \tag{6.71}$$

Dividing Eq. (6.51) by M_i and substituting $K_i/M_i = \lambda_i = \omega_i^2$, we get

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = \left(f_y\right)_i \tag{6.72}$$

where $\left(f_y\right)_i = \left(F_y\right)_i/M_i$.

Then the steady state response for ith mode can be determined by using the procedure explained for the single degree of freedom system in Chap. 4.

Solved Examples

Example 6.1

For a MDOF system shown in Figure E6.1

(a) **Determine the flexibility matrix**.
(b) **Determine the stiffness matrix**.
(c) **Show that the product of flexibility matrix and stiffness matrix is a unit matrix**.

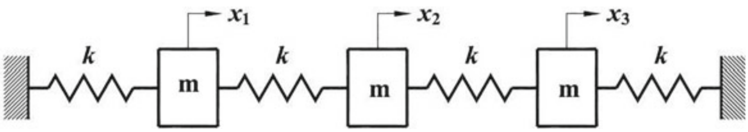


Figure E6.1

Solution

(a) If a unit force $(1N)$ is applied to the first mass, then the displacements produced on the first, second and third masses are a_{11}, a_{21} and a_{31}, respectively, as shown in **Figure E6.1(a)**.

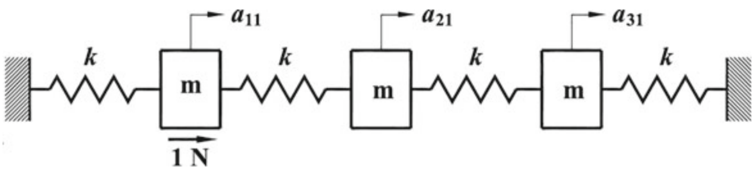


Figure E6.1(a)

Free-body diagram of the system under this condition is shown in **Figure E6.1(b)**.

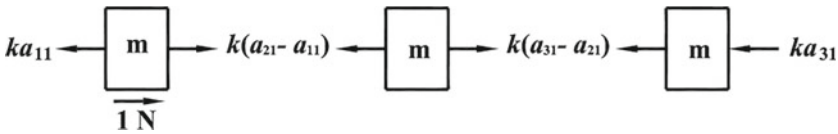


Figure E6.1(b)

Applying equilibrium equation for the third mass shown in **Figure E6.1(b)**, we get

$$-k(a_{31}-a_{21})-ka_{31}=0$$

$$\therefore \quad a_{31}=\frac{1}{2}a_{21} \tag{a}$$

Again applying equilibrium equation for the second mass shown in **Figure E6.1(b)**, we get

$$k(a_{31}-a_{21})-k(a_{21}-a_{11})=0$$

$$\therefore \quad ka_{31}-2ka_{21}+ka_{11}=0 \tag{b}$$

Substituting a_{31} from Equation (**a**) into Equation (**b**), we get

$$\left(2-\frac{1}{2}\right)a_{21}=a_{11}$$

$$\therefore \quad a_{21}=\frac{2}{3}a_{11} \tag{c}$$

Similarly applying equilibrium equation for the first mass shown in **Figure E6.1(b)**, we get

$$k(a_{21}-a_{11})-ka_{11}+1=0$$

$$\therefore \quad ka_{21}-2ka_{11}+1=0 \tag{d}$$

Substituting a_{21} from Equation (**c**) into Equation (**d**), we get

$$\left(\frac{2}{3}-2\right)ka_{11}+1=0$$

$$\therefore \quad a_{11}=\frac{3}{4k}$$

Substituting a_{11} into Equation (**c**), we get

$$a_{21} = \frac{2}{3} \times \frac{3}{4k} = \frac{1}{2k}$$

Substituting a_{21} into Equation (**a**), we get

$$a_{31} = \frac{1}{2} \times \frac{1}{2k} = \frac{1}{4k}$$

Then applying reciprocity theorem, we get other two coefficients as

$$a_{12} = a_{21} = \frac{1}{2k} \quad \text{and} \quad a_{13} = a_{31} = \frac{1}{4k}$$

If a unit force $(1N)$ is applied to the second mass, then the displacements produced on the first, second and third masses are a_{12}, a_{22} and a_{32}, respectively, as shown in **Figure E6.1(c)**.

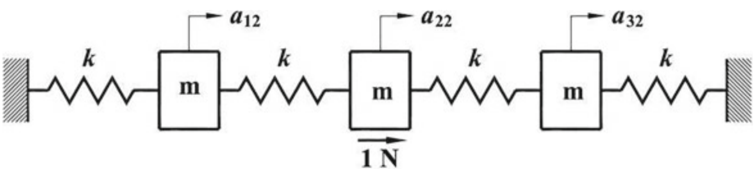


Figure E6.1(c)

Free-body diagram of the system under this condition is shown in **Figure E6.1(d)**.

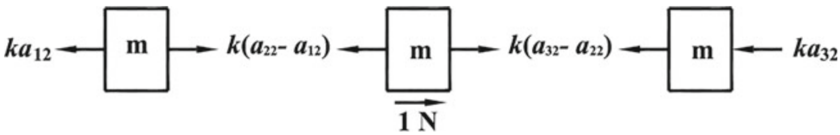


Figure E6.1(d)

Applying equilibrium equation for the first mass shown in **Figure E6.1(d)**, we get

$$k(a_{22} - a_{12}) - ka_{12} = 0$$
$$\therefore a_{22} = 2a_{12} = 2 \times \frac{1}{2k} = \frac{1}{k}$$

Again applying equilibrium equation for the third mass shown in **Figure E6.1(d)**, we get

$$-k(a_{32} - a_{22}) - ka_{22} = 0$$

$$\therefore \quad a_{32} = \frac{1}{2}a_{22} = \frac{1}{2} \times \frac{1}{k} = \frac{1}{2k}$$

Then applying reciprocity theorem, we get

$$a_{23} = a_{32} = \frac{1}{2k}$$

If a unit force $(1N)$ is applied to the third mass, then the displacements produced on the first, second and third masses are a_{13}, a_{23} and a_{33}, respectively, as shown in **Figure E6.1(e)**.

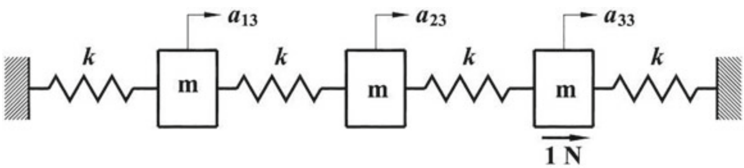


Figure E6.1(e)

Free-body diagram of the system under this condition is shown in **Figure E6.1(f)**.

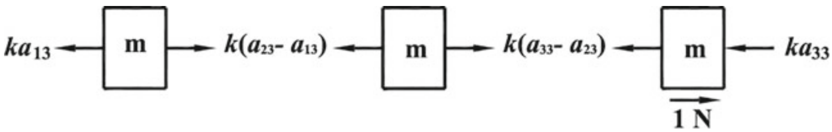


Figure E6.1(f)

Applying equilibrium equation for the third mass shown in **Figure E6.1(f)**, we get

$$-k(a_{33} - a_{23}) - ka_{33} + 1 = 0$$
$$\therefore \quad a_{33} = \frac{1}{2}a_{23} + \frac{1}{2k} = \frac{1}{2} \times \frac{1}{2k} + \frac{1}{2k} = \frac{3}{4k}$$

Then the flexibility matrix for the given system is obtained as

$$[A] = \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix}$$

(b) The forces required at the first, second and third masses to produce unit displacement only at the first mass are k_{11}, k_{21} and k_{31}, respectively, as shown in **Figure E6.1(g)**.

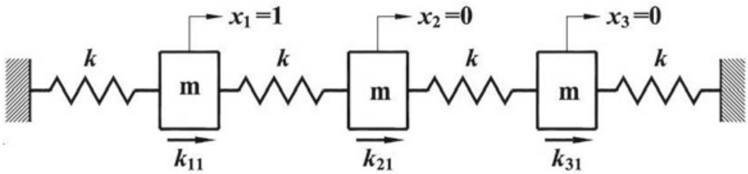


Figure E6.1(g)

Free-body diagram of the system under this condition is shown in **Figure E6.1(h)**.

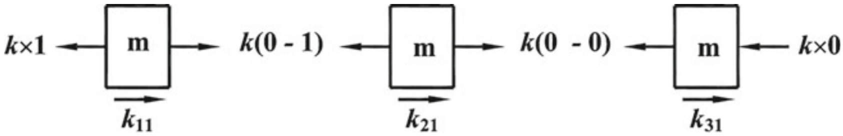


Figure E6.1(h)

Applying equilibrium equation for the first mass shown in **Figure E6.1(h)**, we get

$$k_{11} + (-k) - k = 0$$
$$\therefore k_{11} = 2k$$

Again, applying equilibrium equation for the second mass shown in **Figure E6.1(h)**, we get

$$k_{21} + 0 - (-k) = 0$$
$$\therefore k_{21} = -k$$

Similarly, applying equilibrium equation for the third mass shown in **Figure E6.1(h)**, we get

$$k_{31} - 0 - = 0$$
$$\therefore k_{31} = 0$$

Then applying reciprocity theorem, we get other two coefficients as

$$k_{12} = k_{21} = -k \quad \text{and} \quad k_{13} = k_{31} = 0$$

The forces required at the first, second and third masses to produce unit displacement only at the second mass are k_{12}, k_{22} and k_{32}, respectively, as shown in **Figure E6.1(i)**.

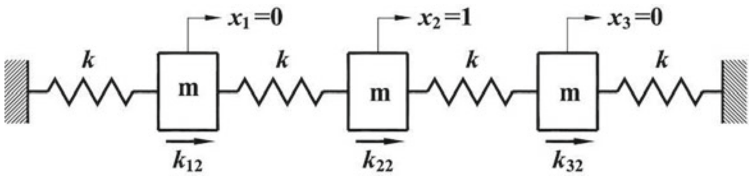


Figure E6.1(i)

Free-body diagram of the system under this condition is shown in **Figure E6.1(j)**.

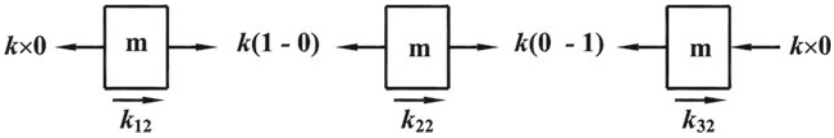


Figure E6.1(j)

Applying equilibrium equation for the second mass shown in **Figure E6.1(j)**, we get

$$k_{22} + (-k) - k = 0$$
$$\therefore k_{22} = 2k$$

Again, applying equilibrium equation for the second mass shown in **Figure E6.1(j)**, we get

$$k_{32} - (-k) - 0 = 0$$
$$\therefore k_{32} = -k$$

Then applying reciprocity theorem, we get

$$k_{23} = k_{32} = -k$$

The forces required at the first, second and third masses to produce unit displacement only at the third mass are k_{13}, k_{23} and k_{33}, respectively, as shown in **Figure E6.1(k)**.

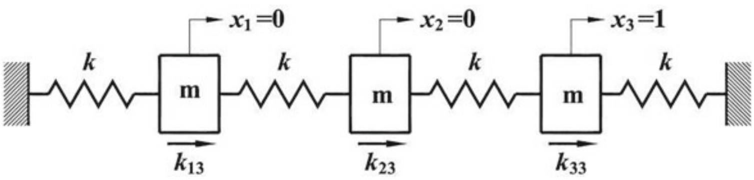


Figure E6.1(k)

Free-body diagram of the system under this condition is shown in **Figure E6.1(l)**.

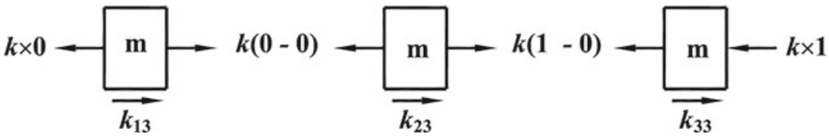


Figure E6.1(l)

Applying equilibrium equation for the third mass shown in **Figure E6.1(l)**, we get

$$k_{33} - k - k = 0$$
$$\therefore k_{33} = 2k$$

Then the stiffness matrix for the given system is obtained as

$$[K] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

(c) Then the product of flexibility matrix and stiffness matrix is given by

$$[A][K] = \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which shows that the product of flexibility matrix and stiffness matrix is a unit matrix.

Example 6.2
Determine the natural frequencies and mode shapes of a two degrees of freedom system shown in Figure E6.2 using matrix method.

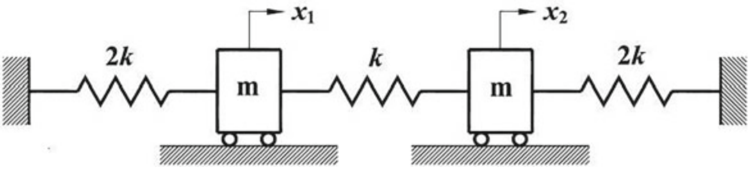


Figure E6.2

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 3k & -k \\ -k & 3k \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 3k & -k \\ -k & 3k \end{bmatrix} = \begin{bmatrix} \frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} \frac{3k}{m} - \lambda & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} - \lambda \end{vmatrix} = 0$$

$$\text{or,} \left(\frac{3k}{m} - \lambda\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

$$\text{or,} \frac{3k}{m} - \lambda = \pm\frac{k}{m}$$

$$\text{or,} \lambda = \frac{3k}{m} \pm \frac{k}{m}$$

$$\therefore \lambda_1 = \frac{2k}{m} \quad \text{and} \quad \lambda_2 = \frac{4k}{m}$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{2k}{m}} = 1.4142\sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} \frac{3k}{m} - \lambda_1 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} - \lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{3k}{m} - \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} - \frac{2k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get $(X_2)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\therefore \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Similarly, the mode shape corresponding to the second natural frequency is given by

$$\begin{bmatrix} \frac{3k}{m} - \lambda_2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} - \lambda_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} \frac{3k}{m} - \frac{4k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{3k}{m} - \frac{4k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_2 = 1$, we get $(X_2)_2 = -1$. Hence the mode shape corresponding to the second natural frequency is found to be

$$\therefore \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Example 6.3
Determine the natural frequencies and mode shapes of the system shown in Figure E6.3 using matrix method.

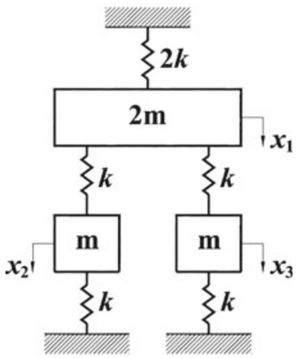


Figure E6.3

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 4k & -k & -k \\ -k & 2k & 0 \\ -k & 0 & 2k \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{2m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 4k & -k & -k \\ -k & 2k & 0 \\ -k & 0 & 2k \end{bmatrix} = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m} & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m} \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} \frac{2k}{m} - \lambda & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m} - \lambda & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m} - \lambda \end{vmatrix} = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right)^3 + \frac{k}{m}\left[-\frac{k}{2m}\left(\frac{2k}{m} - \lambda\right)\right] - \frac{k}{m}\left[\frac{k}{2m}\left(\frac{2k}{m} - \lambda\right)\right] = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right)^3 - \frac{k^2}{m^2}\left(\frac{2k}{m} - \lambda\right) = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right)\left[\left(\frac{2k}{m} - \lambda\right)^2 - \frac{k^2}{m^2}\right] = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right)\left(\frac{3k^2}{m^2} - \frac{4k}{m}\lambda + \lambda^2\right) = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right)\left[\left(\frac{k}{m} - \lambda\right)\left(\frac{3k}{m} - \lambda\right)\right] = 0$$

$$\text{or,} \left(\frac{k}{m} - \lambda\right)\left(\frac{2k}{m} - \lambda\right)\left(\frac{3k}{m} - \lambda\right) = 0$$

$$\therefore \lambda_1 = \frac{k}{m}; \quad \lambda_2 = \frac{2k}{m} \quad \text{and} \quad \lambda_3 = \frac{3k}{m}$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\frac{k}{m}}; \quad \omega_2 = \sqrt{\frac{2k}{m}} = 1.4142\sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{3k}{m}} = 1.7321\sqrt{\frac{k}{m}}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m}-\lambda_1 & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m}-\lambda_1 & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m}-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{2k}{m}-\frac{k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m}-\frac{k}{m} & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m}-\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{k}{m} & 0 \\ -\frac{k}{m} & 0 & \frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1$ and $(X_3)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Similarly, the mode shape corresponding to the second natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m}-\lambda_2 & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m}-\lambda_2 & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m}-\lambda_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{2k}{m}-\frac{2k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m}-\frac{2k}{m} & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m}-\frac{2k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 0 & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & 0 & 0 \\ -\frac{k}{m} & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_2)_2 = 1$, we get $(X_1)_2 = 0$ and $(X_1)_3 = -1$. Hence the mode shape corresponding to the second natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

Again, the mode shape corresponding to the third natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m} - \lambda_3 & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_3 & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m} - \lambda_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{2k}{m} - \frac{3k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m} & 0 \\ -\frac{k}{m} & 0 & \frac{2k}{m} - \frac{3k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} -\frac{k}{m} & -\frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & 0 & -\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_3 = 1$, we get $(X_2)_3 = -1$ and $(X_3)_3 = -1$. Hence the mode shape corresponding to the third natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix}$$

Example 6.4
Find the free response of the system shown in Figure E6.4 when it is subjected the initial conditions: $x_1(0) = 0.1m, \dot{x}_1(0) = 0, x_2(0) = 0, \dot{x}_2(0) = 0, x_3(0) = 0, \dot{x}_3(0) = 0$.

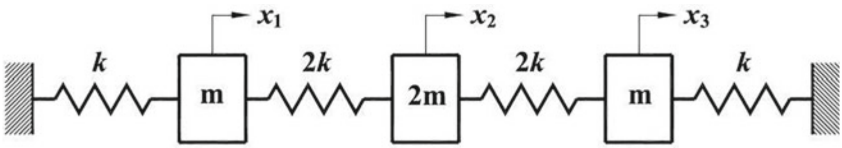


Figure E6.4

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{2m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{bmatrix} = \begin{bmatrix} \frac{3k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} \frac{3k}{m} - \lambda & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - \lambda \end{vmatrix} = 0$$

$$\text{or,} \left(\frac{3k}{m} - \lambda\right)\left[\left(\frac{2k}{m} - \lambda\right)\left(\frac{3k}{m} - \lambda\right) - 2\frac{k^2}{m^2}\right] + \frac{2k}{m}\left[-\frac{k}{m}\left(\frac{3k}{m} - \lambda\right)\right] = 0$$

$$\text{or,} \left(\frac{3k}{m} - \lambda\right)\left(4\frac{k^2}{m^2} - 5\frac{k}{m}\lambda + \lambda^2\right) - 2\frac{k^2}{m^2}\left(\frac{3k}{m} - \lambda\right) = 0$$

$$\text{or,} \left(\frac{3k}{m} - \lambda\right)\left[4\frac{k^2}{m^2} - 5\frac{k}{m}\lambda + \lambda^2 - 2\frac{k^2}{m^2}\right] = 0$$

$$\text{or,} \left(\frac{3k}{m} - \lambda\right)\left(2\frac{k^2}{m^2} - 5\frac{k}{m}\lambda + \lambda^2\right) = 0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 0.4384\frac{k}{m}; \quad \lambda_2 = \frac{3k}{m} \quad \text{and} \quad \lambda_3 = 4.5616\frac{3k}{m}$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = 0.6622\sqrt{\frac{k}{m}}; \quad \omega_2 = 1.7321\sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_3 = 2.1358\sqrt{\frac{k}{m}}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} \frac{3k}{m} - \lambda_1 & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_1 & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - \lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} \frac{3k}{m} - 0.4384\frac{k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - 0.4384\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - 0.4384\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 2.5616\frac{k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & 1.5616\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & 2.5616\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 2.5616 & -2 & 0 \\ -1 & 1.5616 & -1 \\ 0 & -2 & 2.5616 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.2808$ and $(X_3)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.2808 \\ 1 \end{Bmatrix}$$

Similarly, the mode shape corresponding to the second natural frequency is given by

$$\begin{bmatrix} \frac{3k}{m} - \lambda_2 & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_2 & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - \lambda_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} \frac{3k}{m} - \frac{3k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - \frac{3k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 0 & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & -\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 0 & -2 & 0 \\ -1 & -1 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_2 = 1$, we get $(X_2)_2 = 0$ and $(X_3)_2 = -1$. Hence the mode shape corresponding to the second natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

Again, the mode shape corresponding to the third natural frequency is given by

$$\begin{bmatrix} \frac{3k}{m} - \lambda_3 & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_3 & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - \lambda_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} \frac{3k}{m} - 4.5616\frac{3k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - 4.5616\frac{3k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{3k}{m} - 4.5616\frac{3k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} -1.5616\frac{k}{m} & -\frac{2k}{m} & 0 \\ -\frac{k}{m} & -2.5616\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & -1.5616\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} -1.5616 & -2 & 0 \\ -1 & -2.5616 & -1 \\ 0 & -2 & -1.5616 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_3 = 1$, we get $(X_2)_3 = -0.7808$ and $(X_3)_3 = -1$. Hence the mode shape corresponding to the third natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -0.7808 \\ 1 \end{Bmatrix}$$

With reference to the obtained mode shapes, the proportions of vibration of amplitudes of each mass can be assumed as

$$\begin{Bmatrix} X_{11} \\ X_{21} \\ X_{31} \end{Bmatrix} = \begin{Bmatrix} A \\ 1.2808A \\ A \end{Bmatrix}; \quad \begin{Bmatrix} X_{12} \\ X_{22} \\ X_{32} \end{Bmatrix} = \begin{Bmatrix} B \\ 0 \\ -B \end{Bmatrix} \text{ and } \begin{Bmatrix} X_{13} \\ X_{23} \\ X_{33} \end{Bmatrix} = \begin{Bmatrix} C \\ -0.7808C \\ C \end{Bmatrix}$$

Then, we can express the free response of the system as

$$x_1 = A\sin(\omega_1 t + \psi_1) + B\sin(\omega_2 t + \psi_2) + C\sin(\omega_3 t + \psi_3) \quad \text{(a)}$$

$$x_2 = 1.2808A\sin(\omega_1 t + \psi_1) - 0.7808C\sin(\omega_3 t + \psi_3) \quad \text{(b)}$$

$$x_3 = A\sin(\omega_1 t + \psi_1) - B\sin(\omega_2 t + \psi_2) + C\sin(\omega_3 t + \psi_3) \quad \text{(c)}$$

Substituting the given initial conditions of displacements $x_1(0) = 0.1m$, $x_2(0) = 0$, $x_3(0) = 0$, we get

$$A\sin(\psi_1) + B\sin(\psi_2) + C\sin(\psi_3) = 0.1 \tag{d}$$

$$1.2808A\sin(\psi_1) - 0.7808C\sin(\psi_3) = 0 \tag{e}$$

$$A\sin(\psi_1) - B\sin(\psi_2) + C\sin(\psi_3) = 0 \tag{f}$$

Similarly, substituting the given initial conditions of velocities $\dot{x}_1(0) = 0, \dot{x}_2(0) = 0$, $\dot{x}_3(0) = 0$, we get

$$\omega_1 A\cos(\psi_1) + \omega_2 B\cos(\psi_2) + \omega_3 C\cos(\psi_3) = 0.1 \tag{g}$$

$$1.2808\omega_1 A\cos(\psi_1) - 0.7808\omega_3 C\cos(\psi_3) = 0 \tag{h}$$

$$\omega_1 A\cos(\psi_1) - \omega_2 B\cos(\psi_2) + \omega_3 C\cos(\psi_3) = 0 \tag{i}$$

Solving Equations (**g**), (**h**) and (**i**), we get

$$\psi_1 = \psi_2 = \psi_3 = \frac{\pi}{2}$$

Substituting $\psi_1 = \psi_2 = \psi_3 = \pi/2$ into Equations (**d**), (**e**) and (**f**), we get

$$A + B + C = 0.1 \tag{j}$$

$$1.2808A - 0.7808C = 0 \tag{k}$$

$$A - B + C = 0 \tag{l}$$

Solving Equations (**j**), (**k**) and (**l**), we get

$$A = 0.0189; \quad B = 0.05 \quad \text{and} \quad C = 0.0310$$

Then substituting A, B, C, ψ_1, ψ_2, ψ_3, ω_1, ω_2 and ω_3 into Equations (**a**), (**b**) and (**c**), we get free response of the system as

$$x_1 = 0.0189\cos\left(0.6622\sqrt{\frac{k}{m}}t\right) + 0.05\cos\left(1.7321\sqrt{\frac{k}{m}}t\right)$$

$$+ 0.0310\cos\left(2.1358\sqrt{\frac{k}{m}}t\right)$$

$$x_2 = 0.0243\cos\left(0.6622\sqrt{\frac{k}{m}}t\right) - 0.0243\cos\left(2.1358\sqrt{\frac{k}{m}}t\right)$$

$$x_3 = 0.0189\cos\left(0.6622\sqrt{\frac{k}{m}}t\right) - 0.05\cos\left(1.7321\sqrt{\frac{k}{m}}t\right)$$

$$+ 0.0310\cos\left(2.1358\sqrt{\frac{k}{m}}t\right)$$

Example 6.5

For a system shown in Figure E6.5, the first two natural frequencies and the corresponding mode shapes are found to be $\omega_1 = \left(100 - 50\sqrt{2}\right)^{1/2}$ rad/s, $\omega_2 =$ 10 rad/s and $\{X\}_1 = \left\{1\ \sqrt{2}\ 1\right\}^T$, $\{X\}_2 = \left\{1\ 0\ -1\right\}^T$ respectively. If $m_1 = 2$ kg, determine $\{X\}_3$, k_1, k_2, k_3, k_4, m_2, m_3 and ω_3.

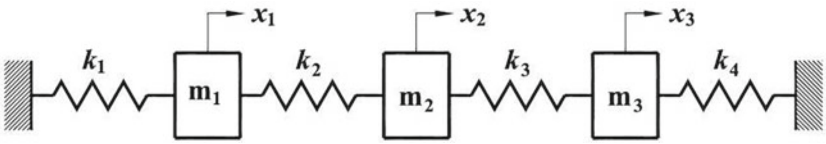


Figure E6.5

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} \end{bmatrix}$$

Eigen-values and eigen-vectors of the dynamic matrix can be related by

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} - \lambda_i & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} - \lambda_i & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} - \lambda_i \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_i = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{a}$$

Substituting $\lambda_1 = \omega_1^2 = \left(100 - 50\sqrt{2}\right)$, $\{X\}_1 = \left\{ 1 \ \sqrt{2} \ 1 \right\}^T$ and $m_1 = 2kg$ and expanding Equation (**a**), we get

$$\left(\frac{k_1 + k_2}{2} - 100 + 50\sqrt{2}\right) - \sqrt{2}\frac{k_2}{2} = 0 \tag{b}$$

$$-\frac{k_2}{m_2} + \sqrt{2}\left(\frac{k_2 + k_3}{m_2} - 100 + 50\sqrt{2}\right) - \frac{k_3}{m_2} = 0 \tag{c}$$

$$-\sqrt{2}\frac{k_3}{m_3} + \left(\frac{k_3 + k_4}{m_3} - 100 + 50\sqrt{2}\right) = 0 \tag{d}$$

Substituting $\lambda_1 = \omega_1^2 = 100$ and $\{X\}_2 = \left\{ 1 \ 0 \ -1 \right\}^T$ and expanding Equation (**a**), we get

$$\left(\frac{k_1 + k_2}{2} - 100\right) = 0 \tag{e}$$

$$-\frac{k_2}{m_2} + \frac{k_3}{m_2} = 0 \tag{f}$$

$$-\left(\frac{k_3 + k_4}{m_3} - 100\right) = 0 \tag{g}$$

Similarly, using equation (**a**) for the third mode of vibration assuming eigen-value λ_3 and corresponding eigen-vector $\{X\}_2 = \left\{ X_{13} \ X_{23} \ X_{33} \right\}^T$,

$$\left(\frac{k_1 + k_2}{2} - \lambda_3\right)X_{13} - \frac{k_2}{2}X_{23} = 0 \tag{h}$$

$$-\frac{k_2}{m_2}X_{13} + \left(\frac{k_2 + k_3}{m_2} - \lambda_3\right)X_{23} - \frac{k_3}{m_2}X_{33} = 0 \tag{i}$$

$$-\frac{k_3}{m_3}X_{23} + \left(\frac{k_3 + k_4}{m_3} - \lambda_3\right)X_{33} = 0 \tag{j}$$

Solving Equations (**b**) and (**e**) for k_1 and k_2, we get

$$k_1 = 100\,\text{N/m} \quad \text{and} \quad k_2 = 100\,\text{N/m}$$

Substituting k_2 into Equation (**f**), we get

$$k_3 = 100\,\text{N/m}$$

Substituting k_2 and k_3 into Equation (**c**), we get

$$m_2 = 2\,\text{kg}$$

Solving Equations (**d**) and (**g**) for k_4 and m_3, we get

$$k_4 = 100\,\text{N/m} \quad \text{and} \quad m_3 = 2kg$$

Substituting for k_1, k_2, k_3, k_4, m_1, m_2 and m_3 and assuming $X_{13} = 1$, Equations (**h**), (**i**) and (**j**) reduce to

$$(100 - \lambda_3) - 50X_{23} = 0 \tag{k}$$

$$-50 + (100 - \lambda_3)X_{23} - 50X_{33} = 0 \tag{l}$$

$$-50X_{23} + (100 - \lambda_3)X_{33} = 0 \tag{m}$$

Rearranging Equation (**m**) for X_{33}, we get

$$X_{33} = \frac{50X_{23}}{100 - \lambda_3} \tag{n}$$

Substituting X_{33} from Equation (**n**) into Equation (**l**), we get

$$X_{23} = \frac{50(100 - \lambda_3)}{(100 - \lambda_3)^2 - 2500} \tag{o}$$

Substituting X_{23} from Equation (**o**) into Equation (**k**), we get

$$(100 - \lambda_3) - \frac{2500(100 - \lambda_3)}{(100 - \lambda_3)^2 - 2500} = 0$$
$$\text{or, } (100 - \lambda_3)\left[1 - \frac{2500}{(100 - \lambda_3)^2 - 2500}\right] = 0$$
$$\text{or, } (100 - \lambda_3)\left[\frac{(100 - \lambda_3)^2 - 5000}{(100 - \lambda_3)^2 - 2500}\right] = 0$$

Since $100 \neq \lambda_3$, because $\lambda_2 = 100$,

$$(100 - \lambda_3)^2 - 5000 = 0$$
$$\text{or, } (100 - \lambda_3) = \pm 50\sqrt{2}$$
$$\text{or, } \lambda_3 = 100 \pm 50\sqrt{2}$$

Since $100 - 50\sqrt{2} \neq \lambda_3$, because $\lambda_1 = 100 - 50\sqrt{2}$,

$$\therefore \lambda_3 = 100 + 50\sqrt{2}$$

Substituting λ_3 into Equation (**o**), we get

$$X_{23} = -\sqrt{2}$$

Again substituting λ_3 and X_{23} into Equation (**n**), we get

$$X_{33} = 1$$

Hence the third natural frequency of the system is $\omega_3 = \left(100 + 50\sqrt{2}\right)^{1/2}$ rad/s and the corresponding mode shape is $\{X\}_3 = \left\{ 1 \;\; -\sqrt{2} \;\; 1 \right\}^T$.

Example 6.6
Determine the natural frequencies of a system shown in Figure E6.6. Given $k = 1000\,\mathrm{N/m}$, $r = 10\,\mathrm{cm}$, $I_p = 1\,\mathrm{kgm^2}$ and $m = 10\,\mathrm{kg}$.

Solution

If the displacement of block $m(x_1)$ and displacement of block $2m(x_2)$ and rotation of pulley (θ) are taken as a set of generalized coordinates, the total kinetic energy (T) and potential energy (V) of the system can be expressed as

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_p\dot{\theta}^2$$
$$V = \frac{1}{2}k(x_1 - r\theta)^2 + \frac{1}{2}k(x_2 - 2r\theta)^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(2r\theta)^2$$

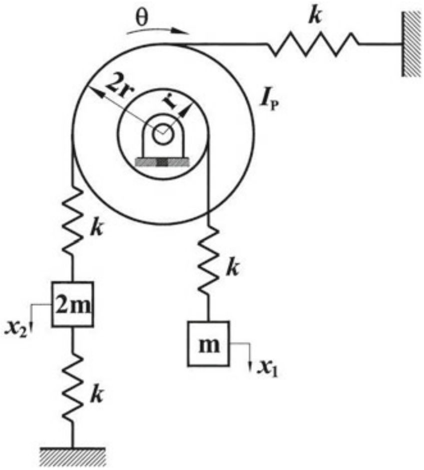


Figure E6.6

Then Lagrangian function for the system can be determined as

$$
\begin{aligned}
L = T - V &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_p\dot{\theta}^2 \\
&= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2}I_p\dot{\theta}^2 - \frac{1}{2}k(x_1 - r\theta)^2 \\
&\quad - \frac{1}{2}k(x_2 - 2r\theta)^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k(2r\theta)^2
\end{aligned}
$$

Now, using Lagrange' equation for the generalized coordinate x_1,

$$
\begin{aligned}
&\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0 \\
&\text{or, } -k(x_1 - r\theta) - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_1) = 0 \\
&\text{or, } -kx_1 + kr\theta - m\ddot{x}_1 = 0 \\
&\therefore m\ddot{x}_1 + kx_1 - kr\theta = 0 \qquad \text{(a)}
\end{aligned}
$$

Similarly, using Lagrange' equation for the generalized coordinate x_2,

$$
\begin{aligned}
&\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0 \\
&\text{or, } -k(x_2 - 2r\theta) - kx_2 - \frac{\mathrm{d}}{\mathrm{d}t}(2m\dot{x}_2) = 0 \\
&\text{or, } -kx_2 + 2kr\theta - kx_2 - 2m\ddot{x}_1 = 0 \\
&\therefore 2m\ddot{x}_1 + 2kx_2 - 2kr\theta = 0 \qquad \text{(b)}
\end{aligned}
$$

Again, using Lagrange' equation for the generalized coordinate θ,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\text{or, } kr(x_1 - r\theta) + 2kr(x_2 - 2r\theta) - 4kr^2\theta - \frac{d}{dt}\left(I_p\dot{\theta}\right) = 0$$

$$\text{or, } krx_1 + 2krx_2 - 9kr^2\theta - I_p\ddot{\theta} = 0$$

$$\therefore I_p\ddot{\theta} + 9kr^2\theta - krx_1 - 2krx_2 = 0 \tag{c}$$

Equations (**a**), (**b**) and (**c**) can be expressed in matrix form for the equation of motion of the system as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 & -kr \\ 0 & 2k & -2kr \\ -kr & -2kr & 9kr^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substituting $k = 1000\,\text{N/m}$, $r = 10\,\text{cm}$, $I_p = 1\,\text{kgm}^2$ and $m = 10\,\text{kg}$, we get mass and stiffness matrices of the system as

$$[M] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 1000 & 0 & -100 \\ 0 & 2000 & -200 \\ -100 & -200 & 90 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 & 0 & -100 \\ 0 & 2000 & -200 \\ -100 & -200 & 90 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 0 & -10 \\ 0 & 100 & -10 \\ -100 & -200 & 90 \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or, } \begin{vmatrix} 100-\lambda & 0 & -10 \\ 0 & 100-\lambda & -10 \\ -100 & -200 & 90-\lambda \end{vmatrix} = 0$$

$$\text{or, } (100-\lambda)[(100-\lambda)(90-\lambda) - 2000] - 100[10(100-\lambda)] = 0$$

$$\text{or, } (100-\lambda)[(100-\lambda)(90-\lambda) - 2000 - 1000] = 0$$

$$\text{or, } (100-\lambda)\left(9000 - 190\lambda + \lambda^2 - 3000\right) = 0$$

$$\text{or, } (100-\lambda)\left(6000-190\lambda+\lambda^2\right)=0$$
$$\text{or, } (40-\lambda)(100-\lambda)(150-\lambda)=0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 40; \quad \lambda_2 = 100 \quad \text{and} \quad \lambda_3 = 150$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{40} = 6.3246\text{rad/s}; \quad \omega_2 = \sqrt{100} = 10\text{rad/s}$$
$$\text{and} \quad \omega_3 = \sqrt{150} = 12.2474\text{rad/s}$$

Example 6.7
Use modal analysis to find the free vibration response of the un-damped system shown in Figure E6.7. Use the initial conditions: $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$, $x_3(0) = 0$, $\dot{x}_3(0) = 0$.

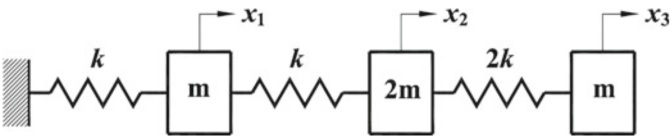


Figure E6.7

Solution
Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{2m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix} = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{2m} & \frac{3k}{2m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m} \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,}\begin{vmatrix} \frac{2k}{m}-\lambda & -\frac{k}{m} & 0 \\ -\frac{k}{2m} & \frac{3k}{2m}-\lambda & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m}-\lambda \end{vmatrix}=0$$

$$\text{or,}\left(\frac{2k}{m}-\lambda\right)\left[\left(\frac{2k}{m}-\lambda\right)\left(\frac{3k}{2m}-\lambda\right)-2\frac{k^2}{m^2}\right]+\frac{k}{2m}\left[-\frac{k}{m}\left(\frac{2k}{m}-\lambda\right)\right]=0$$

$$\text{or,}\left(\frac{2k}{m}-\lambda\right)\left[\left(\frac{2k}{m}-\lambda\right)\left(\frac{3k}{2m}-\lambda\right)-2\frac{k^2}{m^2}-\frac{k^2}{2m^2}\right]=0$$

$$\text{or,}\left(\frac{2k}{m}-\lambda\right)\left[\frac{3k^2}{m^2}-\frac{7k}{2m}\lambda+\lambda^2-\frac{5k^2}{2m^2}\right]=0$$

$$\text{or,}\left(\frac{2k}{m}-\lambda\right)\left(\frac{k^2}{2m^2}-\frac{7k}{2m}\lambda+\lambda^2\right)=0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1=0.1492\frac{k}{m};\quad \lambda_2=\frac{2k}{m}\quad\text{and}\quad \lambda_3=3.3508\frac{k}{m}$$

Therefore natural frequencies of the system are determined as

$$\omega_1=0.3863\sqrt{\frac{k}{m}};\quad \omega_2=1.4142\sqrt{\frac{k}{m}}\quad\text{and}\quad \omega_3=1.8305\sqrt{\frac{k}{m}}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m}-\lambda_1 & -\frac{k}{m} & 0 \\ -\frac{k}{2m} & \frac{3k}{2m}-\lambda_1 & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m}-\lambda_1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\begin{bmatrix} \frac{2k}{m}-0.1492\frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{2m} & \frac{3k}{2m}-0.1492\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m}-0.1492\frac{k}{m} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\begin{bmatrix} 1.8508\frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{2m} & 1.3508\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & 1.8508\frac{k}{m} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\begin{bmatrix} 1.8508 & -1 & 0 \\ -0.5 & 1..3508 & -1 \\ 0 & -2 & 1.8508 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.8508$ and $(X_3)_1 = 2$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.8508 \\ 2 \end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third root are

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -1.3508 \\ 2 \end{Bmatrix}$$

Then modal matrix is formed as

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1.8508 & 0 & -1.3508 \\ 2 & -0.5 & 2 \end{bmatrix}$$

Inverse of modal matrix is then given by

$$U^{-1} = \begin{bmatrix} 0.0844 & 0.3123 & 0.1688 \\ 0.8 & 0 & -0.4 \\ 0.1156 & -0.3123 & 0.2312 \end{bmatrix}$$

Given initial condition for displacements and velocities can be expressed in vector forms as

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Given initial conditions are transformed into y as

$$\begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{Bmatrix} = \left[U^{-1}\right] \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix}$$

$$= \begin{bmatrix} 0.0844 & 0.3123 & 0.1688 \\ 0.8 & 0 & -0.4 \\ 0.1156 & -0.3123 & 0.2312 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.00844 \\ 0.08 \\ 0.01156 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \\ \dot{y}_3(0) \end{Bmatrix} = \left[U^{-1}\right] \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix}$$

$$= \begin{bmatrix} 0.0844 & 0.3123 & 0.1688 \\ 0.8 & 0 & -0.4 \\ 0.1156 & -0.3123 & 0.2312 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.3123 \\ 0 \\ -0.3123 \end{Bmatrix}$$

Then the equations of motion or each mode with the corresponding initial conditions are given as

$$\begin{array}{ll} \ddot{y}_1 + 0.1492\frac{k}{m}y_1 = 0 & y_1(0) = 0.00844,\ \dot{y}_1(0) = 0.3123 \\ \ddot{y}_2 + 2\frac{k}{m}y_2 = 0 & y_2(0) = 0.08,\ \dot{y}_2(0) = 0 \\ \ddot{y}_3 + 3.3508\frac{k}{m}y_3 = 0 & y_3(0) = 0.01156,\ \dot{y}_3(0) = -0.3123 \end{array}$$

Then the solution for each mode can be determined as

$$\begin{aligned} y_1 &= \frac{\dot{y}_1(0)}{\omega_1}\sin(\omega_1 t) + y_1(0)\cos(\omega_1 t) \\ &= \frac{0.3123}{0.3863}\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right) + 0.00844\cos\left(0.3863\sqrt{\frac{k}{m}}t\right) \\ &= 0.8086\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right) + 0.00844\cos\left(0.3863\sqrt{\frac{k}{m}}t\right) \\ y_2 &= \frac{\dot{y}_2(0)}{\omega_2}\sin(\omega_2 t) + y_2(0)\cos(\omega_2 t) = 0.08\cos\left(1.4142\sqrt{\frac{k}{m}}t\right) \\ y_3 &= \frac{\dot{y}_3(0)}{\omega_3}\sin(\omega_3 t) + y_3(0)\cos(\omega_3 t) \\ &= \frac{-0.3123}{1.8305}\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right) + 0.01156\cos\left(1.8305\sqrt{\frac{k}{m}}t\right) \\ &= -0.17063\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right) + 0.01156\cos\left(1.8305\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

Then the response of the system is finally given as

$$\begin{aligned} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} &= [U]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1.8508 & 0 & -1.3508 \\ 2 & -0.5 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{Bmatrix} 0.8086\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right)+0.00844\cos\left(0.3863\sqrt{\frac{k}{m}}t\right) \\ 0.08\cos\left(1.4142\sqrt{\frac{k}{m}}t\right) \\ -0.17063\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right)+0.01156\cos\left(1.8305\sqrt{\frac{k}{m}}t\right) \end{Bmatrix}$$

$$\begin{aligned}
\therefore x_1 &= 0.8086\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right) \\
&+ 0.00844\cos\left(0.3863\sqrt{\frac{k}{m}}t\right)+0.08\cos\left(1.4142\sqrt{\frac{k}{m}}t\right) \\
&- 0.17063\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right)+0.01156\cos\left(1.8305\sqrt{\frac{k}{m}}t\right) \\
x_2 &= 1.4965\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right)+0.0156\cos\left(0.3863\sqrt{\frac{k}{m}}t\right) \\
&+ 0.2305\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right)-0.0156\cos\left(1.8305\sqrt{\frac{k}{m}}t\right) \\
x_3 &= 1.6172\sqrt{\frac{m}{k}}\sin\left(0.3863\sqrt{\frac{k}{m}}t\right)+0.0169\cos\left(0.3863\sqrt{\frac{k}{m}}t\right) \\
&- 0.04\cos\left(1.4142\sqrt{\frac{k}{m}}t\right)-0.3413\sqrt{\frac{m}{k}}\sin\left(1.8305\sqrt{\frac{k}{m}}t\right) \\
&+ 0.0231\cos\left(1.8305\sqrt{\frac{k}{m}}t\right)
\end{aligned}$$

Example 6.8

Use modal analysis to find the free vibration response of the damped system shown in Figure E6.8. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 150\,\text{N/m}$, $k_4 = 100\,\text{N/m}$, $c_1 = 4\,\text{N.s/m}$, $c_2 = 6\,\text{N.s/m}$, $c_3 = 6\,\text{N.s/m}$, $c_4 = 4\,\text{N.s/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 1\,\text{kg}$. Use the initial conditions: $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$, $x_3(0) = 0$, $\dot{x}_3(0) = 1m/s$.

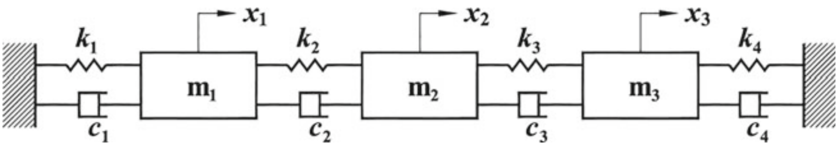


Figure E6.8

Solution

Mass, damping and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [C] = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 10 \end{bmatrix} \text{ and } [K] = \begin{bmatrix} 250 & -150 & 0 \\ -150 & 300 & -150 \\ 0 & -150 & 250 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 250 & -150 & 0 \\ -150 & 300 & -150 \\ 0 & -150 & 250 \end{bmatrix} = \begin{bmatrix} 250 & -150 & 0 \\ -75 & 150 & -75 \\ 0 & -150 & 250 \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or, } \begin{vmatrix} 250-\lambda & -150 & 0 \\ -75 & 150-\lambda & -75 \\ 0 & -150 & 250-\lambda \end{vmatrix} = 0$$

$$\text{or, } (250-\lambda)[(150-\lambda)(250-\lambda) - 11250] + 75[-150(250-\lambda)] = 0$$

$$\text{or, } (250-\lambda)[(150-\lambda)(250-\lambda) - 11250 - 11250] = 0$$

$$\text{or, } (250-\lambda)\left[37500 - 400\lambda + \lambda^2 - 22500\right] = 0$$

$$\text{or, } (250-\lambda)\left(15000 - 400\lambda + \lambda^2\right) = 0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 41.8861; \quad \lambda_2 = 250 \quad \text{and} \quad \lambda_3 = 358.1139$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = 6.4719\,\text{rad/s}, \ \omega_2 = \sqrt{\lambda_2} = 15.8114\,\text{rad/s}$$
$$\text{and } \omega_3 = \sqrt{\lambda_3} = 18.9239\,\text{rad/s}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} 250-\lambda_1 & -150 & 0 \\ -75 & 150-\lambda_1 & -75 \\ 0 & -150 & 250-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 208.1139 & -150 & 0 \\ -75 & 108.1139 & -75 \\ 0 & -150 & 208.1139 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.3874$ and $(X_3)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.3874 \\ 1 \end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third root are

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -0.7208 \\ 1 \end{Bmatrix}$$

Checking for the proportional damping,

$$\begin{aligned}
&[C] = \alpha[M] + \beta[K] \\
\text{or,}\ &\begin{bmatrix} 10 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 10 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 250 & -150 & 0 \\ -150 & 300 & -150 \\ 0 & -150 & 250 \end{bmatrix} \\
\text{or,}\ &\begin{bmatrix} 10 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 10 \end{bmatrix} = \begin{bmatrix} \alpha + 250\beta & -150\beta & 0 \\ -150\beta & 2\alpha + 300\beta & -150\beta \\ 0 & -150\beta & \alpha + 250\beta \end{bmatrix}
\end{aligned}$$

Given matrix equation is satisfied when $\alpha = 0$ and $\beta = 0.04$. Hence the given system is proportional damped, and the modal analysis can be used to determine its response.

Then modal matrix is formed as

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1.3874 & 0 & -0.7208 \\ 1 & -1 & 1 \end{bmatrix}$$

Inverse of modal matrix is then given by

$$U^{-1} = \begin{bmatrix} 0.1709 & 0.4743 & 0.1709 \\ 0.5 & 0 & -0.5 \\ 0.3291 & -0.4743 & 0.3291 \end{bmatrix}$$

Given initial condition for displacements and velocities can be expressed in vector forms as

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Given initial conditions are transformed into y as

$$\begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{Bmatrix} = \left[U^{-1}\right] \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix}$$

$$= \begin{bmatrix} 0.1709 & 0.4743 & 0.1709 \\ 0.5 & 0 & -0.5 \\ 0.3291 & -0.4743 & 0.3291 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.04743 \\ 0 \\ -0.04743 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \\ \dot{y}_3(0) \end{Bmatrix} = \left[U^{-1}\right] \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix}$$

$$= \begin{bmatrix} 0.1709 & 0.4743 & 0.1709 \\ 0.5 & 0 & -0.5 \\ 0.3291 & -0.4743 & 0.3291 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.1709 \\ -0.5 \\ -0.3291 \end{Bmatrix}$$

Then uncoupled equations of motion for each mode of the proportionally damped system can be determined from

$$\ddot{y}_i + 2\xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = 0$$

where

$$2\xi_i \omega_i = \alpha + \beta\omega_i^2 = 0.04\omega_i^2 \quad \therefore \quad \xi_i = 0.02\omega_i$$

Using above equations, equations of motion for each mode with the corresponding initial conditions are given as

$$\begin{aligned} &\ddot{y}_1 + 2(0.1294)(6.4719)\dot{y}_1 + 41.8861 y_1 = 0 && y_1(0) = 0.04743,\ \dot{y}_1(0) = 0.1709 \\ &\ddot{y}_2 + 2(0.3163)(15.8114)\dot{y}_2 + 250 y_2 = 0 && y_2(0) = 0,\ \dot{y}_2(0) = -0.5 \\ &\ddot{y}_3 + 2(0.3785)(18.9239)\dot{y}_3 + 358.1139 y_3 = 0 && y_3(0) = -0.04743,\ \dot{y}_3(0) = 0.3291 \end{aligned}$$

Then the solution for each mode can be determined as

$$y_1 = e^{-\xi_1\omega_1 t}\left[\left(\frac{\dot{y}_1(0) + \xi_1\omega_1 y_1(0)}{\omega_1\sqrt{1-\xi_1^2}}\right)\sin\left\{\left(\sqrt{1-\xi_1^2}\right)\omega_1 t\right\} + y_1(0)\cos\left\{\left(\sqrt{1-\xi_1^2}\right)\omega_1 t\right\}\right]$$

$$= e^{-(0.1294)(6.4719)t}[\left(\frac{0.1709 + (0.1294)(6.4719) \times 0.04743}{6.4719\sqrt{1-(0.1294)^2}}\right)$$

$$\sin\left\{\left(\sqrt{1-(0.1294)^2}\right)6.4719t\right\}$$
$$+\,0.04743\cos\left\{\left(\sqrt{1-(0.1294)^2}\right)6.4719t\right\}]$$
$$= e^{-0.8377t}[0.0328\ \sin(6.4175t) + 0.04743\cos(6.4175t)]$$

$$y_2 = e^{-\xi_2\omega_2 t}\left[\left(\frac{\dot{y}_2(0)+\xi_2\omega_2 y_2(0)}{\omega_2\sqrt{1-\xi_2^2}}\right)\sin\left\{\left(\sqrt{1-\xi_2^2}\right)\omega_2 t\right\} + y_2(0)\cos\left\{\left(\sqrt{1-\xi_2^2}\right)\omega_2 t\right\}\right]$$
$$= e^{-(0.3163)(15.8114)t}\left[\left(\frac{-0.5+(0.3163)(15.8114)\times 0}{15.8114\sqrt{1-(0.3163)^2}}\right)\sin\left\{\left(\sqrt{1-(0.3163)^2}\right)15.8114t\right\}\right]$$
$$= e^{-5t}[-0.0333\sin(15t)]$$

$$y_3 = e^{-\xi_3\omega_3 t}\left[\left(\frac{\dot{y}_3(0)+\xi_3\omega_3 y_3(0)}{\omega_3\sqrt{1-\xi_3^2}}\right)\sin\left\{\left(\sqrt{1-\xi_3^2}\right)\omega_3 t\right\} + y_3(0)\cos\left\{\left(\sqrt{1-\xi_3^2}\right)\omega_3 t\right\}\right]$$
$$= e^{-(0.3785)(18.9239)t}[\left(\frac{0.3291+(0.3785)(18.9239)\times(-0.04743)}{18.9239\sqrt{1-(0.3785)^2}}\right)$$
$$\sin\left\{\left(\sqrt{1-(0.3785)^2}\right)18.9239t\right\}$$
$$-\,0.04743\cos\left\{\left(\sqrt{1-(0.3785)^2}\right)18.9239t\right\}]$$
$$= e^{-7.1623t}[0.0007\sin(17.5162t) + 0.04743\cos(17.5162t)]$$

Then the response of the system is finally given as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [U]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1.3874 & 0 & -0.7208 \\ 1 & -1 & 1 \end{bmatrix}$$
$$\begin{Bmatrix} e^{-0.8377t}[0.0328\sin(6.4175t) + 0.04743\cos(6.4175t)] \\ e^{-5t}[-0.0333\ \sin(15t)] \\ e^{-7.1623t}[0.0007\sin(17.5162t) + 0.04743\cos(17.5162t)] \end{Bmatrix}$$
$$\therefore x_1 = e^{-0.8377t}[0.0328\sin(6.4175t) + 0.04743\cos(6.4175t)]$$
$$+\,e^{-5t}[-0.0333\sin(15t)] + e^{-7.1623t}[0.0007\sin(17.5162t)$$
$$+\,0.04743\cos(17.5162t)]$$
$$x_2 = e^{-0.8377t}[0.0455\sin(6.4175t) + 0.0658\cos(6.4175t)]$$
$$+\,e^{-7.1623t}[0.0004\sin(17.5162t) + 0.0342\cos(17.5162t)]$$
$$x_3 = e^{-0.8377t}[0.0328\sin(6.4175t) + 0.04743\cos(6.4175t)]$$

$$+ e^{-5t}[0.0333\sin(15t)] + e^{-7.1623t}[-0.0007\sin(17.5162t) - 0.04743\cos(17.5162t)]$$

Example 6.9

Determine the steady state response of the system shown in Figure E6.9. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1500\,\text{N/m}$, $k_4 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$, $m_3 = 10\,\text{kg}$ and $F_2 = 200\sin 50t$.

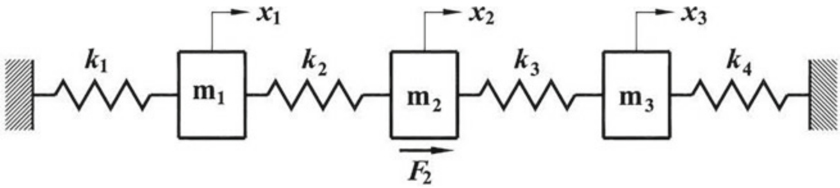


Figure E6.9

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \tag{a}$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = F_2 \tag{b}$$

$$m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 = 0 \tag{c}$$

Substituting $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1500\,\text{N/m}$, $k_4 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$, $m_3 = 10\,\text{kg}$ and $F_2 = 200\sin 50t$, we get

$$10\ddot{x}_1 + 2500x_1 - 1500x_2 = 0 \tag{d}$$

$$15\ddot{x}_2 - 1500x_1 + 3000x_2 - 1500x_3 = 200\sin 50t \tag{e}$$

$$10\ddot{x}_3 - 1500x_2 + 2500x_3 = 0 \tag{f}$$

The steady state response of the system can be assumed as

$$x_1 = X_1\sin 50t \tag{g}$$

$$x_2 = X_2\sin 50t \tag{h}$$

$$x_3 = X_3 \sin 50t \tag{i}$$

Substituting Equations (**g**), (**h**) and (**i**) into Equations (**d**), (**e**) and (**f**), we get a system of algebraic equations as

$$\begin{aligned}&\left(2500 - 10 \times 50^2\right)X_1 - 1500X_2 = 0\\ &-22500X_1 - 1500X_2 = 0\end{aligned} \tag{j}$$

$$\begin{aligned}&-1500X_1 + \left(3000 - 15 \times 50^2\right)X_2 - 1500X_3 = 200\\ &-1500X_1 + 34500X_2 - 1500X_3 = 200\end{aligned} \tag{k}$$

$$\begin{aligned}&-1500X_2 + \left(2500 - 10 \times 50^2\right)X_3 = 0\\ &-1500X_2 - 22500X_3 = 0\end{aligned} \tag{l}$$

Solving simultaneous Equations (**j**), (**k**) and (**l**) for X_1, X_2 and X_3, we get

$$X_1 = 0.0038; \quad X_2 = -0.0583 \quad \text{and} \quad X_3 = 0.0038$$

Substituting X_1, X_2 and X_3 into Equations (**g**), (**h**) and (**i**), we get the steady state response of the system as

$$x_1 = 0.0038 \sin 50t$$

$$x_2 = -0.0583 \sin 50t$$

$$x_3 = 0.0038 \sin 50t$$

Example 6.10

Determine the steady state of the response of the system shown in Figure E6.10. Take $k_1 = k_2 = k_3 = k_4 = 100$ N/m, $c_1 = c_2 = c_3 = c_4 = 10$ N.s/m, $m_1 = m_2 = m_3 = 1$ kg and $F_1 = 300 \sin 40t$.

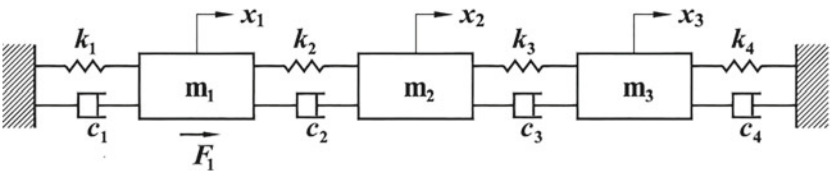


Figure E6.10

Solution

The equation of motion of the system can be written as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \quad \text{(a)}$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - c_3\dot{x}_3 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0 \quad \text{(b)}$$

$$m_3\ddot{x}_3 - c_3\dot{x}_2 + (c_3 + c_4)\dot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 = 0 \quad \text{(c)}$$

Substituting $k_1 = k_2 = k_3 = k_4 = 100$ N/m, $c_1 = c_2 = c_3 = c_4 = 10$ N.s/m, $m_1 = m_2 = m_3 = 1$ kg and $F_1 = 300\sin 40t$, we get

$$\ddot{x}_1 + 20\dot{x}_1 - 10\dot{x}_2 + 200x_1 - 100x_2 = 300\sin 40t \quad \text{(d)}$$

$$\ddot{x}_2 - 10\dot{x}_1 + 20\dot{x}_2 - 10\dot{x}_3 - 100x_1 + 200x_2 - 100x_3 = 0 \quad \text{(e)}$$

$$\ddot{x}_3 - 10\dot{x}_2 + 20\dot{x}_3 - 100x_2 + 200x_3 = 0 \quad \text{(f)}$$

The steady state response of the system can be assumed as

$$x_1 = X_1\sin 40t + Y_1\cos 40t \quad \text{(g)}$$

$$x_2 = X_2\sin 40t + Y_2\cos 40t \quad \text{(h)}$$

$$x_3 = X_3\sin 40t + Y_3\cos 40t \quad \text{(i)}$$

Substituting Equations (**g**), (**h**) and (**i**) into Equations (**d**), (**e**) and (**f**) and equating coefficients of $sin40t$ and $cos40t$ of each equation, we get a system of algebraic equations as

$$-1400X_1 - 800Y_1 - 100X_2 + 400Y_2 = 300 \quad \text{(j)}$$

$$800X_1 - 1400Y_1 - 400X_2 - 100Y_2 = 0 \quad \text{(k)}$$

$$-100X_1 + 400Y_1 - 1400X_2 - 800Y_2 - 100X_3 + 400Y_3 = 0 \quad \text{(l)}$$

$$-400X_1 - 100Y_1 + 800X_2 - 1400Y_2 - 400X_3 - 100Y_3 = 0 \quad \text{(m)}$$

$$-100X_2 + 400Y_2 - 1400X_3 - 800Y_3 = 0 \quad \text{(n)}$$

$$-400X_2 - 100Y_2 + 800X_3 - 1400Y_3 = 0 \tag{o}$$

Solving simultaneous Equations (**j**), (**k**), (**l**), (**m**), (**n**) and (**o**) for X_1, Y_1, X_2, Y_2, X_3 and Y_3, we get

$$\begin{aligned} X_1 &= -0.1557;\ Y_1 = -0.0831 \\ X_2 &= -0.0286;\ Y_2 = 0.0318; \\ X_3 &= 0.0058;\quad Y_3 = 0.0092 \end{aligned}$$

Substituting X_1, Y_1, X_2, Y_2, X_3 and Y_3 into Equations (**g**), (**h**) and (**i**), we get the steady state response of the system as

$$\begin{aligned} x_1 &= -0.1557 \sin 40t - 0.0831 \cos 40t \\ x_2 &= -0.0286 \sin 40t + 0.0318 \cos 40t \\ x_3 &= 0.0058 \sin 40t + 0.0092 \cos 40t \end{aligned}$$

Example 6.11
Use modal analysis to determine the steady state response of the system shown in Figure E6.11. Take $k_1 = 100\,\text{N/m}$, $k_2 = 200\,\text{N/m}$, $k_3 = 400\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $m_3 = 4\,\text{kg}$ and $F_3 = 150 \sin 30t$.

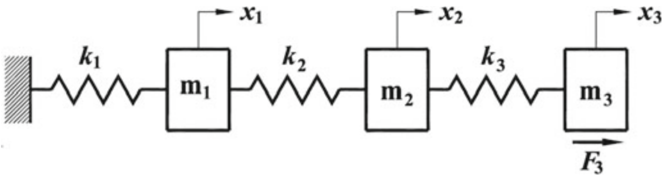


Figure E6.11

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 300 & -200 & 0 \\ -200 & 600 & -400 \\ 0 & -400 & 400 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \begin{bmatrix} 300 & -200 & 0 \\ -200 & 600 & -400 \\ 0 & -400 & 400 \end{bmatrix}$$

$$= \begin{bmatrix} 300 & -200 & 0 \\ -100 & 300 & -200 \\ 0 & -100 & 100 \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} 300-\lambda & -200 & 0 \\ -100 & 300-\lambda & -200 \\ 0 & -100 & 100-\lambda \end{vmatrix} = 0$$

$$\text{or, } (300-\lambda)[(300-\lambda)(100-\lambda) - 20000] + 200[-100(100-\lambda)] = 0$$

$$\text{or, } (300-\lambda)\left(\lambda^2 - 400\lambda + 10000\right) - 20000(100-\lambda) = 0$$

$$\text{or, } \lambda^3 - 700\lambda^2 + 110000\lambda - 100000 = 0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 9.6788; \quad \lambda_2 = 219.3936 \quad \text{and} \quad \lambda_3 = 470.9275$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = 3.1111 \text{ rad/s}; \quad \omega_2 = \sqrt{\lambda_2} = 14.8119 \text{ rad/s}$$
$$\text{and} \quad \omega_3 = \sqrt{\lambda_3} = 21.7008 \text{ rad/s}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} 300-\lambda_1 & -200 & 0 \\ -100 & 300-\lambda_1 & -200 \\ 0 & -100 & 100-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 300-9.6788 & -200 & 0 \\ -100 & 300-9.6788 & -200 \\ 0 & -100 & 100-9.6788 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 290.3212 & -200 & 0 \\ -100 & 290.3212 & -200 \\ 0 & -100 & 90.3212 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.4516$ and $(X_3)_1 = 1.6072$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.4516 \\ 1.6072 \end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third root are

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0.4030 \\ -0.3376 \end{Bmatrix} \text{ and } \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -0.8546 \\ 0.2304 \end{Bmatrix}$$

Then modal matrix is formed as

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1.4516 & 0.4030 & -0.8546 \\ 1.6072 & -0.3376 & 0.2304 \end{bmatrix}$$

Transpose of modal matrix is then given by

$$U' = \begin{bmatrix} 1 & 1.4516 & 1.6072 \\ 1 & 0.4030 & -0.3376 \\ 1 & -0.8546 & 0.2304 \end{bmatrix}$$

Then generalized mass for each mass can be determined as

$$M_1 = \{X_1\}'[M]\{X_1\} = \left\{ 1\ 1.4516\ 1.6072 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4516 \\ 1.6072 \end{Bmatrix} = 15.5461$$

$$M_2 = \{X_2\}'[M]\{X_2\} = \left\{ 1\ 0.4030\ -0.3376 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.4030 \\ -0.3376 \end{Bmatrix} = 1.7087$$

$$M_3 = \{X_3\}'[M]\{X_3\} = \left\{ 1\ -0.8546\ 0.2304 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.8546 \\ 0.2304 \end{Bmatrix} = 2.6732$$

Similarly transformed force vector is given by

$$\left\{F_y\right\} = \left[U'\right]\{F\} = \begin{bmatrix} 1 & 1.4516 & 1.6072 \\ 1 & 0.4030 & -0.3376 \\ 1 & -0.8546 & 0.2304 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 241.0739 \sin 30t \\ -50.6348 \sin 30t \\ 34.5608 \sin 30t \end{Bmatrix}$$

Magnitudes of modal forces for each mode can be then determined as

$$\left(f_y\right)_1 = \frac{\left(F_y\right)_1}{M_1} = \frac{241.0739 \sin 30t}{15.5461} = 15.5069 \sin 30t$$
$$\left(f_y\right)_2 = \frac{\left(F_y\right)_2}{M_2} = -\frac{50.6348 \sin 30t}{1.7087} = -28.4358 \sin 30t$$
$$\left(f_y\right)_3 = \frac{\left(F_y\right)_3}{M_3} = \frac{34.5608 \sin 30t}{2.6732} = 12.9288 \sin 30t$$

Then uncoupled equations of motion for each mode can be determined from

$$\ddot{y}_i + \omega_i^2 y_i = \left(f_y\right)_i = f_i \sin(\omega t)$$

Using the above equation, equations of motion for each mode can be expressed as

$$\ddot{y}_1 + (3.1111)^2 y_1 = 15.5069 \sin 30t$$
$$\ddot{y}_2 + (14.8119)^2 y_2 = -28.4358 \sin 30t$$
$$\ddot{y}_3 + (21.7008)^2 y_3 = 12.9288 \sin 30t$$

Then the steady state response to each mode can be determined as

$$y_1 = \frac{f_1}{\omega_1^2 - \omega^2} \sin(\omega t) = \frac{15.5069}{(3.1111)^2 - (30)^2} \sin 30t = -0.0174 \sin 30t$$
$$y_2 = \frac{f_2}{\omega_2^2 - \omega^2} \sin(\omega t) = \frac{-28.4358}{(14.8119)^2 - (30)^2} \sin 30t = 0.0418 \sin 30t$$
$$y_3 = \frac{f_1}{\omega_3^2 - \omega^2} \sin(\omega t) = \frac{12.9288}{(21.7008)^2 - (30)^2} \sin 30t = -0.0301 \sin 30t$$

Then the response of the system is finally given as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [U] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.4516 & 0.4030 & -0.8546 \\ 1.6072 & -0.3376 & 0.2304 \end{bmatrix}$$
$$\begin{Bmatrix} -0.0174 \sin 30t \\ 0.0418 \sin 30t \\ -0.0301 \sin 30t \end{Bmatrix} = \begin{Bmatrix} -0.0057 \sin 30t \\ 0.0173 \sin 30t \\ -0.0490 \sin 30t \end{Bmatrix}$$

Example 6.12
Use modal analysis to determine the steady state response of the system shown in Figure E6.12. Take $k_1 = 300$ N/m, $k_2 = 250$ N/m, $k_3 = 250$ N/m, $k_4 = 50$ N/m, $k_5 = 100$ N/m, $c_1 = c_2 = c_3 = c_4 = 5$ N.s/m, $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 1$ kg and $F_3 = 250 \sin 40t$.

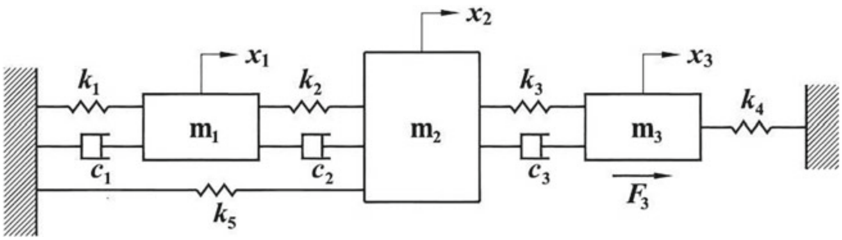


Figure E6.12

Solution

Mass, damping and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 10 & -5 \\ 0 & -5 & 10 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 550 & -250 & 0 \\ -250 & 600 & -250 \\ 0 & -250 & 300 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 550 & -250 & 0 \\ -250 & 600 & -250 \\ 0 & -250 & 300 \end{bmatrix} = \begin{bmatrix} 550 & -250 & 0 \\ -125 & 300 & -125 \\ 0 & -250 & 300 \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} 550-\lambda & -250 & 0 \\ -125 & 300-\lambda & -125 \\ 0 & -250 & 300-\lambda \end{vmatrix} = 0$$

$$\text{or, } (550-\lambda)[(300-\lambda)(300-\lambda) - 31250] + 250[-125(300-\lambda)] = 0$$

$$\text{or, } (550-\lambda)\left(90000 - 600\lambda + \lambda^2 - 31250\right) - 31250(300-\lambda) = 0$$

$$\text{or, } (550-\lambda)\left(58750 - 600\lambda + \lambda^2\right) - 31250(300-\lambda) = 0$$

$$\text{or, } 32312500 - 388750\lambda + 1150\lambda^2 - \lambda^3 - 9375000 + 31250\lambda = 0$$

$$\text{or, } \lambda^3 - 1150\lambda^2 + 357500\lambda - 22937500 = 0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 86.3406; \quad \lambda_2 = 400.7579 \quad \text{and} \quad \lambda_3 = 662.9015$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = 9.2919\,\text{rad/s}, \quad \omega_2 = \sqrt{\lambda_2} = 20.0189\,\text{rad/s}$$
$$\text{and} \quad \omega_3 = \sqrt{\lambda_3} = 25.7469\text{rad/s}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} 550-\lambda_1 & -250 & 0 \\ -125 & 300-\lambda_1 & -125 \\ 0 & -250 & 300-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 463.6594 & -250 & 0 \\ -125 & 213.6594 & -125 \\ 0 & -250 & 213.6594 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.8546$ and $(X_3)_1 = 2.1701$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third root are

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0.5969 \\ -1.4812 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -0.4516 \\ 0.3111 \end{Bmatrix}$$

Checking for the proportional damping,

$$[C] = \alpha[M] + \beta[K]$$

$$\text{or,} \begin{bmatrix} 10 & -5 & 0 \\ -5 & 10 & -5 \\ 0 & -5 & 10 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 550 & -250 & 0 \\ -250 & 600 & -250 \\ 0 & -250 & 300 \end{bmatrix}$$

$$\text{or,} \begin{bmatrix} 10 & -5 & 0 \\ -5 & 10 & -5 \\ 0 & -5 & 10 \end{bmatrix} = \begin{bmatrix} \alpha + 550\beta & -250\beta & 0 \\ -250\beta & 2\alpha + 600\beta & -250\beta \\ 0 & -250\beta & \alpha + 300\beta \end{bmatrix}$$

Given matrix equation is satisfied when $\alpha = -1$ and $\beta = 0.02$. Hence the given system is proportional damped, and the modal analysis can be used to determine its response.

Then modal matrix is formed as

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1.8546 & 0.5969 & -0.4516 \\ 2.1701 & -1.4812 & 0.3111 \end{bmatrix}$$

Transpose of modal matrix is then given by

$$U' = \begin{bmatrix} 1 & 1.8546 & 2.1701 \\ 1 & 0.5969 & -1.4812 \\ 1 & -0.4516 & 0.3111 \end{bmatrix}$$

Then generalized mass for each mass can be determined as

$$M_1 = \{X_1\}'[M]\{X_1\} = \left\{ 1 \ 1.8546 \ 2.1701 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.8546 \\ 2.1701 \end{Bmatrix} = 12.5886$$

$$M_2 = \{X_2\}'[M]\{X_2\} = \left\{ 1 \ 0.5969 \ -1.4812 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.5969 \\ -1.4812 \end{Bmatrix} = 3.9069$$

$$M_3 = \{X_3\}'[M]\{X_3\} = \left\{ 1 \ -0.4516 \ 0.3111 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.4516 \\ 0.3111 \end{Bmatrix} = 1.5047$$

Similarly transformed force vector is given by

$$\left\{F_y\right\} = \left[U'\right]\{F\} = \begin{bmatrix} 1 & 1.8546 & 2.1701 \\ 1 & 0.5969 & -1.4812 \\ 1 & -0.4516 & 0.3111 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 542.5216 \sin 40t \\ -370.2986 \sin 40t \\ 77.7769 \sin 40t \end{Bmatrix}$$

Magnitudes of modal forces for each mode can then be determined as

$$\left(f_y\right)_1 = \frac{\left(F_y\right)_1}{M_1} = \frac{542.5216 \sin 40t}{12.5886} = 43.0961 \sin 40t$$

$$\left(f_y\right)_2 = \frac{\left(F_y\right)_2}{M_2} = -\frac{370.2986 \sin 40t}{3.9069} = -94.7860 \sin 40t$$

$$\left(f_y\right)_3 = \frac{\left(F_y\right)_3}{M_3} = \frac{77.7769 \sin 40t}{1.5047} = 51.6899 \sin 40t$$

Then uncoupled equations of motion for each mode of the proportionally damped system can be determined from

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = \left(f_y\right)_i = f_i \sin(\omega t)$$

where

$$2\xi_i\omega_i = \alpha + \beta\omega_i^2 = -1 + 0.02\omega_i^2 \quad \therefore \quad \xi_i = \frac{-1 + 0.02\omega_i^2}{2\omega_i}$$

Using the above equation, equations of motion for each mode can be expressed as

$$\ddot{y}_1 + 2(0.0391)(9.2919)\dot{y}_1 + (9.2919)^2 y_1 = 43.0961 \sin 40t$$
$$\ddot{y}_2 + 2(0.1752)(20.0189)\dot{y}_2 + (20.0189)^2 y_2 = -94.7860 \sin 40t$$
$$\ddot{y}_3 + 2(0.2380)(25.7469)\dot{y}_3 + (25.7469)^2 y_3 = 51.6899 \sin 40t$$

The steady state response to each mode can be determined from

$$y_i = \frac{f_i}{\omega_i^2}\left[\frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_i}\right)^2\right\}^2 + \left(\frac{2\xi_i\omega}{\omega_i}\right)^2}}\right]\sin(\omega t - \phi_i)$$

where

$$\phi_i = \tan^{-1}\left\{\frac{\frac{2\xi_i\omega}{\omega_i}}{1 - \left(\frac{\omega}{\omega_i}\right)^2}\right\}$$

Substituting parameters of each mode, we get the steady state response to each mode as

$$\begin{aligned}
y_1 &= \frac{f_1}{\omega_1^2}\left[\frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_1}\right)^2\right\}^2 + \left(\frac{2\xi_1\omega}{\omega_1}\right)^2}}\right]\sin\left[\omega t - \tan^{-1}\left\{\frac{\frac{2\xi_1\omega}{\omega_1}}{1 - \left(\frac{\omega}{\omega_1}\right)^2}\right\}\right] \\
&= \frac{43.0961}{(9.2919)^2}\left[\frac{1}{\sqrt{\left\{1 - \left(\frac{40}{9.2919}\right)^2\right\}^2 + \left(\frac{2\times0.0391\times40}{9.2919}\right)^2}}\right]\sin\left[40t - \tan^{-1}\left\{\frac{\frac{2\times0.0391\times40}{9.2919}}{1 - \left(\frac{40}{9.2919}\right)^2}\right\}\right] \\
&= 0.0285\ \sin(40t + 0.0192) \\
y_2 &= \frac{f_2}{\omega_2^2}\left[\frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_2}\right)^2\right\}^2 + \left(\frac{2\xi_2\omega}{\omega_2}\right)^2}}\right]\sin\left[\omega t - \tan^{-1}\left\{\frac{\frac{2\xi_2\omega}{\omega_2}}{1 - \left(\frac{\omega}{\omega_2}\right)^2}\right\}\right]
\end{aligned}$$

$$= -\frac{94.7860}{(20.0189)^2}\left[\frac{1}{\sqrt{\left\{1-\left(\frac{40}{20.0189}\right)^2\right\}^2+\left(\frac{2\times0.1752\times40}{20.0189}\right)^2}}\right]\sin\left[40t-\tan^{-1}\left\{\frac{\frac{2\times0.1752\times40}{20.0189}}{1-\left(\frac{40}{20.0189}\right)^2}\right\}\right]$$

$$= -0.0769\sin(40t+0.2299)$$

$$y_3 = \frac{f_3}{\omega_3^2}\left[\frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_3}\right)^2\right\}^2+\left(\frac{2\xi_3\omega}{\omega_3}\right)^2}}\right]\sin\left[\omega t-\tan^{-1}\left\{\frac{\frac{2\xi_3\omega}{\omega_3}}{1-\left(\frac{\omega}{\omega_3}\right)^2}\right\}\right]$$

$$= \frac{51.6899}{(25.7469)^2}\left[\frac{1}{\sqrt{\left\{1-\left(\frac{40}{25.7469}\right)^2\right\}^2+\left(\frac{2\times0.2380\times40}{25.7469}\right)^2}}\right]\sin\left[40t-\tan^{-1}\left\{\frac{\frac{2\times0.2380\times40}{25.7469}}{1-\left(\frac{40}{25.7469}\right)^2}\right\}\right]$$

$$= 0.0489\sin(40t+0.4821)$$

Then the response of the system is finally given as

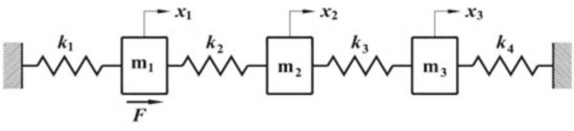
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [U]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.8546 & 0.5969 & -0.4516 \\ 2.1701 & -1.4812 & 0.3111 \end{bmatrix}\begin{Bmatrix} 0.0285\sin(40t+0.0192) \\ -0.0769\sin(40t+0.2299) \\ 0.0489\sin(40t+0.4821) \end{Bmatrix}$$

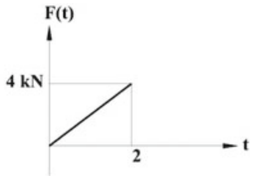
$$\therefore x_1 = 0.0285\sin(40t+0.0192) - 0.0769\sin(40t+0.2299) + 0.0489\sin(40t+0.4821)$$

$$x_2 = 0.0528\sin(40t+0.0192) - 0.0459\sin(40t+0.2299) - 0.0221\sin(40t+0.4821)$$

$$x_3 = 0.0618\sin(40t+0.0192) + 0.1139\sin(40t+0.2299) - 0.0152\sin(40t+0.4821)$$

Example 6.13
Use modal analysis to determine the response of the system shown in Figure E6.13(a) when it is subjected to a transient force shown in Figure E6.13(b). Take $k_1 = k_2 = k_3 = k_4 = 1000\,\mathrm{N/m}$ and $m_1 = m_2 = m_3 = 10\,\mathrm{kg}$.





(a) **(b)**

Figure E6.13

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 2000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 2000 \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 2000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 2000 \end{bmatrix} = \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 200 \end{bmatrix}$$

Then the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} 200-\lambda & -100 & 0 \\ -100 & 200-\lambda & -100 \\ 0 & -100 & 200-\lambda \end{vmatrix} = 0$$

$$\text{or, } (200-\lambda)[(200-\lambda)(200-\lambda) - 10000] + 100[-100(200-\lambda)] = 0$$

$$\text{or, } (200-\lambda)\left(40000 - 400\lambda + \lambda^2 - 10000 - 10000\right) = 0$$

$$\text{or, } (200-\lambda)\left(\lambda^2 - 400\lambda + 20000\right) = 0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1 = 58.5786; \quad \lambda_2 = 200 \quad \text{and} \quad \lambda_3 = 341.4214$$

Therefore natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = 7.6537\,\text{rad/s}; \quad \omega_2 = \sqrt{\lambda_2} = 14.1421\,\text{rad/s} \quad \text{and} \quad \omega_3 = \sqrt{\lambda_3} = 18.4776\,\text{rad/s}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} 200-\lambda_1 & -100 & 0 \\ -100 & 200-\lambda_1 & -100 \\ 0 & -100 & 200-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 200-58.5786 & -100 & 0 \\ -100 & 200-58.5786 & -100 \\ 0 & -100 & 200-58.5786 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\begin{bmatrix}141.4214 & -100 & 0\\ -100 & 141.4214 & -100\\ 0 & -100 & 141.4214\end{bmatrix}\begin{Bmatrix}X_1\\X_2\\X_3\end{Bmatrix}_1=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.4142$ and $(X_3)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix}X_1\\X_2\\X_3\end{Bmatrix}_1=\begin{Bmatrix}1\\1.4142\\1\end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third roots are

$$\begin{Bmatrix}X_1\\X_2\\X_3\end{Bmatrix}_2=\begin{Bmatrix}1\\0\\-1\end{Bmatrix}\quad\text{and}\begin{Bmatrix}X_1\\X_2\\X_3\end{Bmatrix}_3=\begin{Bmatrix}1\\-1.4142\\1\end{Bmatrix}$$

Then, the modal matrix is formed as

$$U=\begin{bmatrix}1 & 1 & 1\\ 1.4142 & 0 & -1.4142\\ 1 & -1 & 1\end{bmatrix}$$

Transpose of the modal matrix is then given by

$$U'=\begin{bmatrix}1 & 1.4142 & 1\\ 1 & 0 & -1\\ 1 & -1.4142 & 1\end{bmatrix}$$

Then, the generalized mass for each mass can be determined as

$$M_1=\{X_1\}'[M]\{X_1\}=\begin{Bmatrix}1 & 1.4142 & 1\end{Bmatrix}\begin{bmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{bmatrix}\begin{Bmatrix}1\\1.4142\\1\end{Bmatrix}=40$$

$$M_2=\{X_2\}'[M]\{X_2\}=\begin{Bmatrix}1 & 0 & -1\end{Bmatrix}\begin{bmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{bmatrix}\begin{Bmatrix}1\\0\\-1\end{Bmatrix}=20$$

$$M_3=\{X_3\}'[M]\{X_3\}=\begin{Bmatrix}1 & -1.4142 & 1\end{Bmatrix}\begin{bmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{bmatrix}\begin{Bmatrix}1\\-1.4142\\1\end{Bmatrix}=40$$

Similarly transformed force vector is given by

$$\{F_y\} = [U']\{F\} = \begin{bmatrix} 1 & 1.4142 & 1 \\ 1 & 0 & -1 \\ 1 & -1.4142 & 1 \end{bmatrix} \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F \\ F \\ F \end{Bmatrix}$$

Magnitudes of modal forces for each mode can then be determined as

$$\left(f_y\right)_1 = \frac{\left(F_y\right)_1}{M_1} = \frac{F}{40} = 0.025F$$
$$\left(f_y\right)_2 = \frac{\left(F_y\right)_2}{M_2} = \frac{F}{20} = 0.05F$$
$$\left(f_y\right)_3 = \frac{\left(F_y\right)_3}{M_3} = \frac{F}{40} = 0.025F$$

Then, the uncoupled equations of motion for each mode can be determined from

$$\ddot{y}_i + \omega_i^2 y_i = \left(f_y\right)_i$$

Using the above equation, equation of motion for each mode can be expressed as

$$\ddot{y}_1 + (7.6537)^2 y_1 = 0.025F$$
$$\ddot{y}_2 + (14.1421)^2 y_2 = 0.05F$$
$$\ddot{y}_3 + (18.4776)^2 y_3 = 0.025F$$

Given force F can be expressed as a time dependent force as

$$F = 2000t - 2000tu(t-2)$$

The response of each mode due to a unit impulse at $t = 0$ is given by

$$h(t) = \frac{1}{M_i \omega_i} \sin \omega_i t$$

Then, the response to each mode can be determined by using convolution integral as

$$y_i = \int_0^t \left(f_y\right)_i(\eta) h_i(t-\eta) d\eta$$
$$= \int_0^t \left(f_y\right)_i(\eta) \frac{1}{M_i \omega_i} \sin \omega_i (t-\eta) d\eta - u(t-2) \int_2^t \left(f_y\right)_i(\eta) \frac{1}{M_i \omega_i} \sin \omega_i (t-\eta) d\eta$$

Substituting parameters of each mode, we get the response for each mode as

$$\begin{aligned}
y_1 &= \int_0^t (f_y)_1(\eta)\frac{1}{M_1\omega_1}\sin\omega_1(t-\eta)d\eta - u(t-2)\int_2^t (f_y)_1(\eta)\frac{1}{M_1\omega_1}\sin\omega_1(t-\eta)d\eta \\
&= \int_0^t 0.025\times 2000\eta\times\frac{1}{40\times 7.6537}\sin\{7.6537(t-\eta)\}d\eta - u(t-2)\int_2^t 0.025 \\
&\quad \times 2000\eta\times\frac{1}{40\times 7.6537}\sin\{7.6537(t-\eta)\}d\eta \\
&= 0.0213t - 0.0028\sin(7.6537t) \\
&\quad - u(t-2)[0.0213t - 0.0028\sin\{7.6537(t-2)\} - 0.0427\cos\{7.6537(t-2)\}]
\end{aligned}$$

$$\begin{aligned}
y_2 &= \int_0^t (f_y)_2(\eta)\frac{1}{M_2\omega_2}\sin\omega_2(t-\eta)d\eta - u(t-2)\int_2^t (f_y)_2(\eta)\frac{1}{M_2\omega_2}\sin\omega_2(t-\eta)d\eta \\
&= \int_0^t 0.05\times 2000\eta\times\frac{1}{20\times 14.1421}\sin\{14.1421(t-\eta)\}d\eta \\
&\quad - u(t-2)\int_2^t 0.05\times 2000\eta\times\frac{1}{20\times 14.1421}\sin\{14.1421(t-\eta)\}d\eta \\
&= 0.025t - 0.0018\sin(14.1421t) \\
&\quad - u(t-2)[0.025t - 0.0018\sin\{14.1421(t-2)\} - 0.05\cos\{14.1421(t-2)\}]
\end{aligned}$$

$$\begin{aligned}
y_3 &= \int_0^t (f_y)_3(\eta)\frac{1}{M_3\omega_3}\sin\omega_3(t-\eta)d\eta - u(t-2)\int_2^t (f_y)_3(\eta)\frac{1}{M_3\omega_3}\sin\omega_3(t-\eta)d\eta \\
&= \int_0^t 0.025\times 2000\eta\times\frac{1}{40\times 18.4776}\sin\{18.4776(t-\eta)\}d\eta \\
&\quad - u(t-2)\int_2^t 0.025\times 2000\eta\times\frac{1}{40\times 18.4776}\sin\{18.4776(t-\eta)\}d\eta \\
&= 0.0037t - 0.0002\sin(18.4776t) \\
&\quad - u(t-2)[0.0037t - 0.0002\sin\{18.4776(t-2)\} - 0.073\cos\{18.4776(t-2)\}]
\end{aligned}$$

Then, the response of the system is finally given as

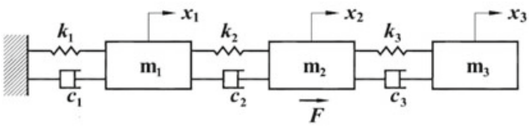
$$\begin{aligned}
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} &= [U]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1.4142 & 0 & -1.4142 \\ 1 & -1 & 1 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}\therefore x_1 &= 0.05t - 0.0028\sin(7.6537t) - 0.0018\sin(14.1421t)\\ &\quad - 0.0002\sin(18.4776t) - u(t-2)[0.05t\\ &\quad - 0.0028\sin\{7.6537(t-2)\} - 0.0427\cos\{7.6537(t-2)\}\\ &\quad - 0.0018\sin\{14.1421(t-2)\} - 0.05\cos\{14.1421(t-2)\}\\ &\quad - 0.0002\sin\{18.4776(t-2)\} - 0.073\cos\{18.4776(t-2)\}]\end{aligned}$$

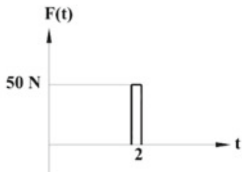
$$\begin{aligned}x_2 &= 0.025t - 0.0039\sin(7.6537t) + 0.0003\sin(18.4776t)\\ &\quad - u(t-2)[0.025t - 0.0039\sin\{7.6537(t-2)\} - 0.0603\cos\{7.6537(t-2)\}\\ &\quad + 0.0003\sin\{18.4776(t-2)\} + 0.0104\cos\{18.4776(t-2)\}]\end{aligned}$$

$$\begin{aligned}x_3 &= -0.0028\sin(7.6537t) + 0.0018\sin(14.1421t) - 0.0002\sin(18.4776t)\\ &\quad - u(t-2)[-0.0028\sin\{7.6537(t-2)\} - 0.0427\cos\{7.6537(t-2)\}\\ &\quad + 0.0018\sin\{14.1421(t-2)\} + 0.05\cos\{14.1421(t-2)\}\\ &\quad - 0.0002\sin\{18.4776(t-2)\} - 0.073\cos\{18.4776(t-2)\}]\end{aligned}$$

Example 6.14
Use modal analysis to determine the response of the system shown in Figure E6.14(a) when it is subjected to a transient force shown in Figure E6.1b(b). Take $k_1 = k_2 = k_3 = k_4 = 80\,\text{N/m}$, $c_1 = c_2 = c_3 = 4\,\text{kg}$ and $m_1 = m_2 = m_3 = 1\,\text{kg}$.



(a)



(b)

Figure E6.14

Solution

Mass, damping and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 160 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix}$$

Then, the dynamic matrix of the system is given by

$$[D] = [M]^{-1}[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 160 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix} = \begin{bmatrix} 160 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix}$$

Then, the characteristic equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or, } \begin{vmatrix} 160-\lambda & -80 & 0 \\ -80 & 160-\lambda & -80 \\ 0 & -80 & 80-\lambda \end{vmatrix} = 0$$

$$\text{or, } (160-\lambda)[(160-\lambda)(80-\lambda) - 6400] + 80[-80(80-\lambda)] = 0$$

$$\text{or, } (160-\lambda)\left(12800 - 240\lambda + \lambda^2 - 6400\right) - 6400(80-\lambda) = 0$$

$$\text{or, } \lambda^3 - 400\lambda^2 + 38400\lambda - 512000 = 0$$

Hence, the roots of the characteristic equation can be obtained as

$$\lambda_1 = 15.8449; \quad \lambda_2 = 124.3967 \quad \text{and} \quad \lambda_3 = 259.7584$$

Therefore, the natural frequencies of the system are determined as

$$\omega_1 = \sqrt{\lambda_1} = 3.9806\text{rad/s}, \quad \omega_2 = \sqrt{\lambda_2} = 11.1533\text{rad/s}$$
$$\text{and} \quad \omega_3 = \sqrt{\lambda_3} = 16.1170\text{rad/s}$$

Then, the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} 160-\lambda_1 & -80 & 0 \\ -80 & 160-\lambda_1 & -80 \\ 0 & -80 & 80-\lambda_1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or, } \begin{bmatrix} 144.1550 & -80 & 0 \\ -80 & 144.1550 & -80 \\ 0 & -80 & 64.1550 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1.8019$ and $(X_3)_1 = 2.2469$. Hence, the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix}$$

Similarly, eigen-vectors corresponding to second and third roots are

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix} \text{ and } \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -1.2469 \\ 0.5549 \end{Bmatrix}$$

Checking for the proportional damping,

$$[C] = \alpha[M] + \beta[K]$$

$$\text{or, } \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 160 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} \alpha + 160\beta & -80\beta & 0 \\ -80\beta & \alpha + 160\beta & -80\beta \\ 0 & -80\beta & \alpha + 80\beta \end{bmatrix}$$

Given matrix equation is satisfied when $\alpha = 0$ and $\beta = 0.05$. Hence, the given system is proportionally damped and the modal analysis can be used to determine its response.

Then, the modal matrix is formed as

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1.8019 & 0.4450 & -1.2469 \\ 2.2469 & -0.8019 & 0.5549 \end{bmatrix}$$

Transpose of the modal matrix is then given by

$$U' = \begin{bmatrix} 1 & 1.8019 & 2.2469 \\ 1 & 0.4450 & -0.8019 \\ 1 & -1.2469 & 0.5549 \end{bmatrix}$$

Then generalized mass for each mass can be determined as

$$M_1 = \{X_1\}'[M]\{X_1\} = \{1 \;\; 1.8019 \;\; 2.2469\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix} = 9.2959$$

$$M_2 = \{X_2\}'[M]\{X_2\} = \{1 \;\; 0.4450 \;\; -0.8019\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix} = 1.8412$$

$$M_3 = \{X_3\}'[M]\{X_3\} = \{1 \;\; -1.2469 \;\; 0.5549\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1.2469 \\ 0.5549 \end{Bmatrix} = 2.8629$$

Similarly, transformed force vector is given by

$$\{F_y\} = [U']\{F\} = \begin{bmatrix} 1 & 1.8019 & 2.2469 \\ 1 & 0.4450 & -0.8019 \\ 1 & -1.2469 & 0.5549 \end{bmatrix} \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.8019F \\ 0.4450F \\ -1.2469F \end{Bmatrix}$$

Magnitudes of modal forces for each mode can then be determined as

$$\left(f_y\right)_1 = \frac{\left(F_y\right)_1}{M_1} = \frac{1.8019F}{9.2959} = 0.1938F$$
$$\left(f_y\right)_2 = \frac{\left(F_y\right)_2}{M_2} = \frac{0.4450F}{1.8412} = 0.2417F$$
$$\left(f_y\right)_3 = \frac{\left(F_y\right)_3}{M_3} = -\frac{1.2469F}{2.8629} = -0.4356F$$

Then, the uncoupled equation of motion for each mode of the proportionally damped system can be determined from

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = \left(f_y\right)_i$$

where

$$2\xi_i\omega_i = \alpha + \beta\omega_i^2 = 0.05\omega_i^2 \quad \therefore \quad \xi_i = 0.025\omega_i$$

Using the above equation, equations of motion for each mode can be expressed as

$$\ddot{y}_1 + 2(0.0995)(3.9806)\dot{y}_1 + (3.9806)^2 y_1 = 0.1938F$$
$$\ddot{y}_2 + 2(0.2788)(11.1533)\dot{y}_2 + (11.1533)^2 y_2 = 0.2417F$$
$$\ddot{y}_3 + 2(0.4029)(16.1170)\dot{y}_3 + (16.1170)^2 y_3 = -0.4356F$$

Given force F can be expressed as a time dependent force as

$$F = 50\delta(t-2)$$

The response of each mode due to a unit impulse at $t = 0$ is given by

$$h(t) = \frac{1}{M_i\omega_i\sqrt{1-\xi_i^2}} e^{-\xi_i\omega_i t} \sin\left\{\left(\omega_i\sqrt{1-\xi_i^2}\right)t\right\}$$

Then, the response to each mode can be determined by using convolution integral as

$$y_i = \int_0^t \left(f_y\right)_i(\eta) h_i(t-\eta) d\eta$$

$$= u(t-2) \int_2^t \left(f_y\right)_i(\eta) \frac{1}{M_i \omega_i \sqrt{1-\xi_i^2}} \mathrm{e}^{-\xi_i \omega_i (t-\eta)} \sin\left\{\left(\omega_i \sqrt{1-\xi_i^2}\right)(t-\eta)\right\} d\eta$$

Substituting parameters of each mode, we get the response for each mode as

$$y_1 = u(t-2) \int_2^t \left(f_y\right)_1(\eta) \frac{1}{M_1 \omega_1 \sqrt{1-\xi_1^2}} \mathrm{e}^{-\xi_1 \omega_1 (t-\eta)} \sin\left\{\left(\omega_1 \sqrt{1-\xi_1^2}\right)(t-\eta)\right\} \mathrm{d}\eta$$

$$= \int_0^t 0.1938 \times 50 \times \delta(\eta - 2)$$

$$\times \frac{1}{9.2959 \times 3.9806 \times \sqrt{1-(0.0995)^2}} \mathrm{e}^{-(0.0995)\times 3.9806 \times (t-\eta)}$$

$$\sin\left\{3.9806\sqrt{1-(0.0995)^2} \times (t-\eta)\right\} \mathrm{d}\eta = 0.1938 \times 50$$

$$\times \frac{1}{9.2959 \times 3.9806 \times \sqrt{1-(0.0995)^2}} \mathrm{e}^{-(0.0995)\times 3.9806 \times (t-2)}$$

$$\sin\left\{3.9806\sqrt{1-(0.0995)^2} \times (t-2)\right\}$$

$$= u(t-2)\left[0.2632\mathrm{e}^{-0.3961(t-2)} \sin\{3.9608(t-2)\}\right]$$

$$y_2 = u(t-2) \int_2^t \left(f_y\right)_2(\eta) \frac{1}{M_2 \omega_2 \sqrt{1-\xi_2^2}} \mathrm{e}^{-\xi_2 \omega_2 (t-\eta)}$$

$$\sin\left\{\left(\omega_2 \sqrt{1-\xi_2^2}\right)(t-\eta)\right\} \mathrm{d}\eta$$

$$= \int_0^t 0.2417 \times 50 \times \delta(\eta - 2)$$

$$\times \frac{1}{1.8412 \times 11.1533 \times \sqrt{1-(0.2788)^2}} \mathrm{e}^{-(0.2788)\times 11.1533 \times (t-\eta)}$$

$$\sin\left\{11.1533\sqrt{1-(0.2788)^2} \times (t-\eta)\right\} \mathrm{d}\eta$$

$$= 0.2417 \times 50 \times \frac{1}{1.8412 \times 11.1533 \times \sqrt{1-(0.2788)^2}} \mathrm{e}^{-(0.2788)\times 11.1533 \times (t-2)}$$

$$\sin\left\{11.1533\sqrt{1-(0.2788)^2}\times(t-2)\right\}$$
$$=u(t-2)\left[0.6129\mathrm{e}^{-3.1099(t-2)}\sin\{10.7109(t-2)\}\right]$$

$$y_3=u(t-2)\int_2^t\left(f_y\right)_3(\eta)\frac{1}{M_3\omega_3\sqrt{1-\xi_3^2}}\mathrm{e}^{-\xi_3\omega_3(t-\eta)}$$
$$\sin\left\{\left(\omega_3\sqrt{1-\xi_3^2}\right)(t-\eta)\right\}\mathrm{d}\eta$$
$$=\int_0^t 0.4356\times 50\times\delta(\eta-2)$$
$$\times\frac{1}{2.8629\times 16.1170\times\sqrt{1-(0.4029)^2}}\mathrm{e}^{-(0.4029)\times 16.1170\times(t-\eta)}$$
$$\sin\left\{16.1170\sqrt{1-(0.4029)^2}\times(t-\eta)\right\}d\eta$$
$$=0.4356\times 50\times\frac{1}{2.8629\times 16.1170\times\sqrt{1-(0.4029)^2}}\mathrm{e}^{-(0.4029)\times 16.1170\times(t-2)}$$
$$\sin\left\{16.1170\sqrt{1-(0.4029)^2}\times(t-2)\right\}$$
$$=u(t-2)\left[-0.5157\mathrm{e}^{-6.4939(t-2)}\sin\{14.7508(t-2)\}\right]$$

Then, the response of the system is finally given as

$$\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}=[U]\begin{Bmatrix}y_1\\y_2\\y_3\end{Bmatrix}=\begin{bmatrix}1&1&1\\1.8019&0.4450&-1.2469\\2.2469&-0.8019&0.5549\end{bmatrix}\begin{Bmatrix}y_1\\y_2\\y_3\end{Bmatrix}$$

$$\begin{aligned}\therefore x_1&=u(t-2)[0.2632\,\mathrm{e}^{-0.3961(t-2)}\sin\{3.9608(t-2)\}\\&\quad+0.6129\,\mathrm{e}^{-3.1099(t-2)}\sin\{10.7109(t-2)\}\\&\quad-0.5157\,\mathrm{e}^{-6.4939(t-2)}\sin\{14.7508(t-2)\}]\\x_2&=u(t-2)[0.4743\,\mathrm{e}^{-0.3961(t-2)}\sin\{3.9608(t-2)\}\\&\quad+0.2727\,\mathrm{e}^{-3.1099(t-2)}\sin\{10.7109(t-2)\}\\&\quad+0.6431\,\mathrm{e}^{-6.4939(t-2)}\sin\{14.7508(t-2)\}]\\x_3&=u(t-2)[0.5915\,\mathrm{e}^{-0.3961(t-2)}\sin\{3.9608(t-2)\}\\&\quad-0.4915\,\mathrm{e}^{-3.1099(t-2)}\sin\{10.7109(t-2)\}\\&\quad-0.2862\,\mathrm{e}^{-6.4939(t-2)}\sin\{14.7508(t-2)\}]\end{aligned}$$

Example 6.15
Determine the natural frequencies and mode shapes of the system shown in Figure E6.15 using matrix method.

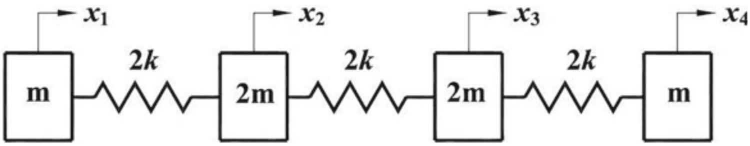


Figure E6.15

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 \\ 0 & 0 & 2m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 2k & -2k & 0 & 0 \\ -2k & 4k & -2k & 0 \\ 0 & -2k & 4k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}$$

Then the dynamic matrix of the system is given by,

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m} & 0 & 0 & 0 \\ 0 & \frac{1}{2m} & 0 & 0 \\ 0 & 0 & \frac{1}{2m} & 0 \\ 0 & 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 2k & -2k & 0 & 0 \\ -2k & 4k & -2k & 0 \\ 0 & -2k & 4k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} \end{bmatrix}$$

Then the characteristics equation of the system is given by

$$|D - \lambda I| = 0$$

$$\text{or,} \begin{vmatrix} \frac{2k}{m} - \lambda & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \lambda & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda \end{vmatrix} = 0$$

$$\text{or,} \left(\frac{2k}{m} - \lambda\right) \begin{vmatrix} \frac{2k}{m} - \lambda & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda \end{vmatrix} + \frac{k}{m} \begin{vmatrix} -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda & -\frac{k}{m} \\ 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda \end{vmatrix} = 0$$

$$\text{or, } \left(\frac{2k}{m}-\lambda\right)\left[\left(\frac{2k}{m}-\lambda\right)\left\{\left(\frac{2k}{m}-\lambda\right)^2-2\frac{k^2}{m^2}\right\}+\frac{k}{m}\left\{-\frac{k}{m}\left(\frac{2k}{m}-\lambda\right)\right\}\right]$$
$$+\frac{k}{m}\left[-\frac{2k}{m}\left\{\left(\frac{2k}{m}-\lambda\right)^2-2\frac{k^2}{m^2}\right\}\right]=0$$
$$\text{or, } \left(\frac{2k}{m}-\lambda\right)^2\left\{\left(\frac{2k}{m}-\lambda\right)^2-2\frac{k^2}{m^2}-\frac{k^2}{m^2}-2\frac{k^2}{m^2}\right\}+4\frac{k^4}{m^4}=0$$
$$\text{or, } \left(\frac{2k}{m}-\lambda\right)^2\left\{\left(\frac{2k}{m}-\lambda\right)^2-2\frac{k^2}{m^2}-\frac{k^2}{m^2}-2\frac{k^2}{m^2}\right\}+4\frac{k^4}{m^4}=0$$
$$\text{or, } \left(\frac{2k}{m}-\lambda\right)^2\left\{\left(\frac{2k}{m}-\lambda\right)^2-5\frac{k^2}{m^2}\right\}+4\frac{k^4}{m^4}=0$$
$$\text{or, } \left(\frac{2k}{m}-\lambda\right)^4-5\frac{k^2}{m^2}\left(\frac{2k}{m}-\lambda\right)^2+4\frac{k^4}{m^4}=0$$
$$\text{or, } \lambda^4-8\frac{k}{m}\lambda^3+24\frac{k^2}{m^2}\lambda^2-32\frac{k^3}{m^3}\lambda+16\frac{k^4}{m^4}-5\frac{k^2}{m^2}\lambda^2$$
$$+20\frac{k^3}{m^3}\lambda-20\frac{k^4}{m^4}+4\frac{k^4}{m^4}=0$$
$$\text{or, } \lambda^4-8\frac{k}{m}\lambda^3+19\frac{k^2}{m^2}\lambda^2-12\frac{k^3}{m^3}\lambda=0$$
$$\text{or, } \lambda\left(\lambda^3-8\frac{k}{m}\lambda^2+19\frac{k^2}{m^2}\lambda-12\frac{k^3}{m^3}\right)=0$$
$$\text{or, } \lambda\left(\lambda-\frac{k}{m}\right)\left(\lambda-3\frac{k}{m}\right)\left(\lambda-4\frac{k}{m}\right)=0$$

Hence the roots of the characteristic equation can be obtained as

$$\lambda_1=0;\quad \lambda_2=\frac{k}{m}\quad \lambda_3=\frac{3k}{m}\quad \text{and}\quad \lambda_4=\frac{4k}{m}$$

Therefore natural frequencies of the system are determined as

$$\omega_1=\sqrt{\lambda_1}=0;\quad \omega_2=\sqrt{\lambda_2}=\sqrt{\frac{k}{m}};\quad \omega_3=\sqrt{\lambda_3}=1.7321\sqrt{\frac{k}{m}}\quad \text{and}\quad \omega_4=\sqrt{\lambda_4}=2\sqrt{\frac{k}{m}}$$

Then the mode shape corresponding to the first natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m}-\lambda_1 & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m}-\lambda_1 & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m}-\lambda_1 & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m}-\lambda_1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_1=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_1 = 1$, we get the non-trivial solution as $(X_2)_1 = 1$, $(X_3)_1 = 1$ and $(X_4)_1 = 1$. Hence the mode shape corresponding to the first natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Similarly, the mode shape corresponding to the second natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m} - \lambda_2 & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_2 & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \lambda_2 & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} \frac{2k}{m} - \frac{k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \frac{k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \frac{k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} \frac{k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_2 = 1$, we get $(X_2)_2 = 0.5$, $(X_3)_2 = -0.5$ and $(X_4)_2 = -1$. Hence the mode shape corresponding to the second natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \\ -1 \end{Bmatrix}$$

Similarly, the mode shape corresponding to the third natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m} - \lambda_3 & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_3 & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \lambda_3 & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} \frac{2k}{m} - \frac{3k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \frac{3k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} -\frac{k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & -\frac{k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & -\frac{k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & -\frac{k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,} \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_3 = 1$, we get $(X_2)_3 = -0.5$, $(X_3)_3 = -0.5$ and $(X_4)_3 = -1$. Hence the mode shape corresponding to the third natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ -0.5 \\ -0.5 \\ 1 \end{Bmatrix}$$

Finally, the mode shape corresponding to the fourth natural frequency is given by

$$\begin{bmatrix} \frac{2k}{m} - \lambda_4 & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m} - \lambda_4 & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} - \lambda_4 & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m} - \lambda_4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} \frac{2k}{m}-\frac{4k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & \frac{2k}{m}-\frac{4k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m}-\frac{4k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & \frac{2k}{m}-\frac{4k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} & 0 & 0 \\ -\frac{k}{m} & -\frac{2k}{m} & -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} & -\frac{2k}{m} & -\frac{k}{m} \\ 0 & 0 & -\frac{2k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad \begin{bmatrix} -2 & -2 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $(X_1)_4 = 1$, we get $(X_2)_4 = -1$, $(X_3)_4 = 1$ and $(X_4)_4 = -1$. Hence the mode shape corresponding to the third natural frequency is found to be

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_4 = \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix}$$

6.7 Review Questions

1. Define the flexibility and stiffness influence coefficient.
2. Define flexibility and stiffness matrix. Establish the relationship between them.
3. State and prove reciprocity theorem.
4. What is a dynamic matrix? How it is calculated?
5. What do you mean by eigen-values and eigen-vectors with reference to the vibration of a discrete system?
6. State and prove the orthogonal characteristics of normal modes.
7. Explain the modal analysis procedure.

Exercise

1. Determine the flexibility and stiffness matrices of the two degree of freedom systems shown in **Figure P6.1**. Also verify that flexibility and stiffness matrices are inverse to each other.

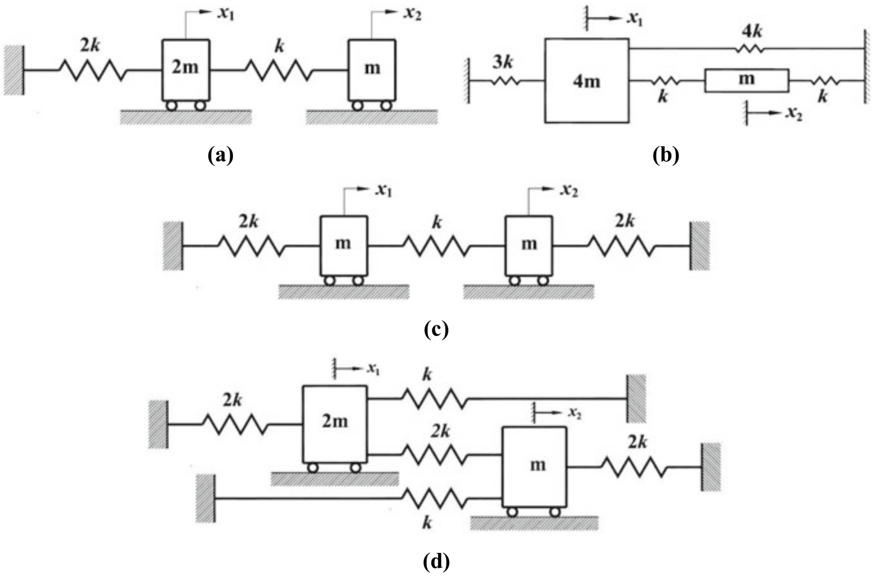


Figure P6.1

2. Determine the flexibility and stiffness matrices of the three degree of freedom systems shown in **Figure P6.2**. Also verify that flexibility and stiffness matrices are inverse to each other.

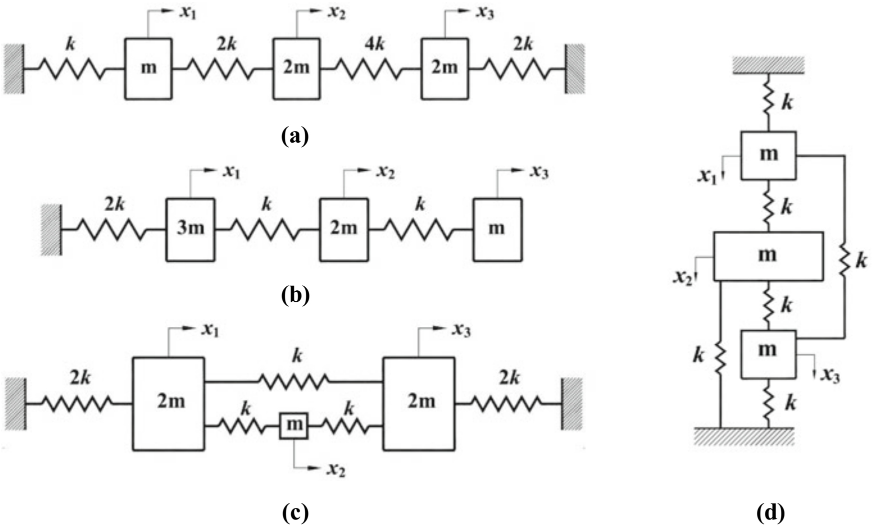
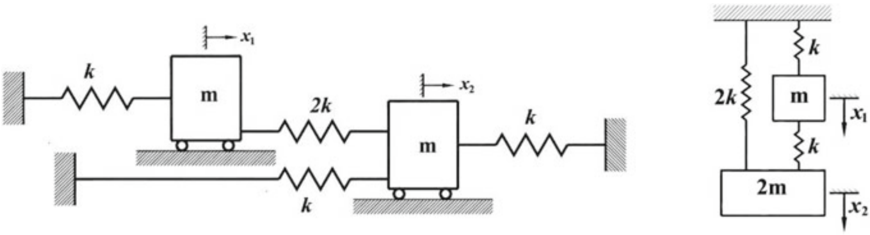


Figure P6.2

3. Determine the natural frequencies and the corresponding mode shapes of the two degrees of freedom systems shown in **Figure P6.3** using matrix method. Also determine the nodes of the system.
4. Determine the natural frequencies and the corresponding mode shapes of the three degree of freedom systems shown in **Figure P6.4**. Also determine the nodes of the system. Take the values of masses and stiffness values of the springs as:

   (a) $m_1 = 2m, m_2 = m, m_3 = 2m$ $k_1 = 3k, k_2 = k, k_3 = k, k_4 = 3k.$
   (b) $m_1 = m, m_2 = m, m_3 = m$ $k_1 = k, k_2 = k, k_3 = k, k_4 = k.$
   (c) $m_1 = 3m, m_2 = 2m, m_3 = 3m$ $k_1 = 2k, k_2 = k, k_3 = k, k_4 = 2k.$



(a) **(b)**

Figure P6.3

Figure P6.4

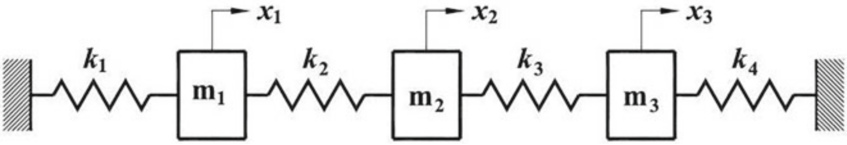


Figure P6.4

5. Determine the natural frequencies and the corresponding mode shapes of the three degree of freedom systems shown in **Figure P6.5**. Also determine the nodes of the system. Take the values of masses and stiffness values of the springs as:

   (a) $m_1 = 2m, m_2 = 2m, m_3 = m$ $k_1 = 3k, k_2 = k, k_3 = k.$
   (b) $m_1 = m, m_2 = m, m_3 = m$ $k_1 = k, k_2 = k, k_3 = k.$

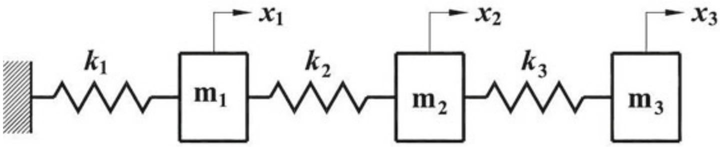


Figure P6.5

6. Determine the natural frequencies and the corresponding mode shapes of the three degree of freedom systems shown in **Figure P6.6**. Also determine the nodes of the system.

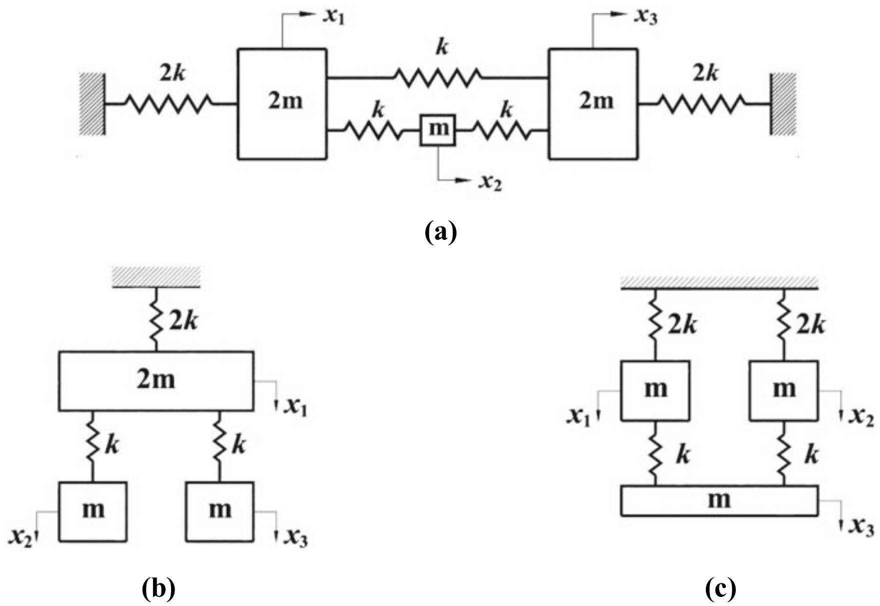


Figure P6.6

7. Determine the natural frequencies and the modes of vibration of the system shown in **Figure P6.7**. Determine the response of the system when it is subject to the following initial conditions:

(a) $x_1(0) = 0.1m$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, $x_3(0) = 0$, $\dot{x}_3(0) = 0$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1m/s$, $x_3(0) = 0$, $\dot{x}_3(0) = 0$.

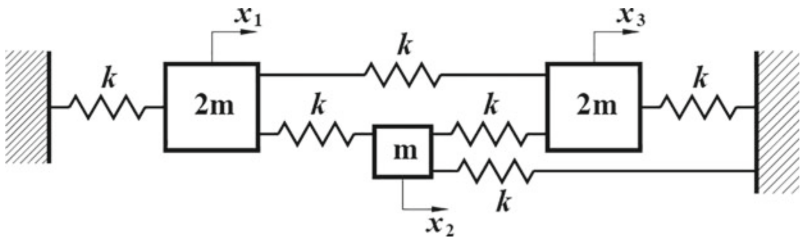


Figure P6.7

8. Determine the natural frequencies and the modes of vibration of the system shown in **Figure P6.8**. Determine the response of the system when it is subject to the following initial conditions:

(a) $x_1(0) = 0, \dot{x}_1(0) = 0, x_2(0) = 0.1m, \dot{x}_2(0) = 0, x_3(0) = 0, \dot{x}_3(0) = 0.$
(b) $x_1(0) = 0, \dot{x}_1(0) = 0, x_2(0) = 0, \dot{x}_2(0) = 0, x_3(0) = 0, \dot{x}_3(0) = 1m/s.$

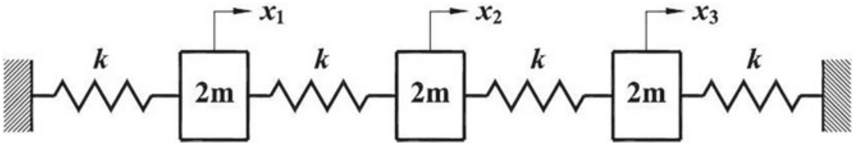


Figure P6.8

9. Determine the natural frequencies and the modes of vibration of the system shown in **Figure P6.9**. Determine the response of the system when it is subject to the following initial conditions:

(a) $x_1(0) = 0, \dot{x}_1(0) = 0, x_2(0) = 0, \dot{x}_2(0) = 0, x_3(0) = 0.1m, \dot{x}_3(0) = 0.$
(b) $x_1(0) = 0, \dot{x}_1(0) = 1m/s, x_2(0) = 0, \dot{x}_2(0) = 0, x_3(0) = 0, \dot{x}_3(0) = 0.$

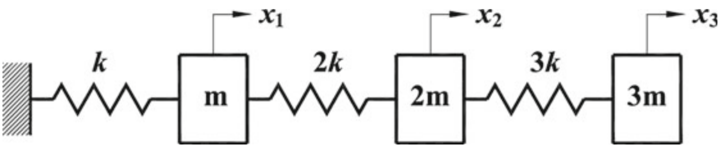


Figure P6.9

10. For a system shown in **Figure P6.10**, the higher two natural frequencies and the corresponding mode shapes are found to be $\omega_2 = 20\,\text{rad/s}$, $\omega_3 = \left(400 + 200\sqrt{2}\right)^{1/2}$ rad/s and $\{X\}_2 = \left\{1\ 0\ -1\right\}^T$, $\{X\}_3 = \left\{1\ -\sqrt{2}\ 1\right\}^T$ respectively. If $m_2 = 10\,\text{kg}$, determine $\{X\}_1, k_1, k_2, k_3, k_4, m_1, m_3$ and ω_1.

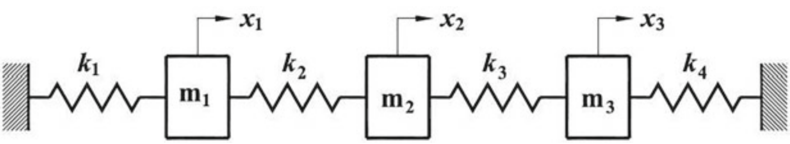


Figure P6.10

11. For a system shown in **Figure P6.11**, the lowest and highest natural frequencies and the corresponding mode shapes are found to be $\omega_1 = \left(100 - 50\sqrt{3}\right)^{1/2}$ rad/s, $\omega_3 = \left(100 + 50\sqrt{3}\right)^{1/2}$ rad/s and $\{X\}_1 = \left\{1\ \sqrt{3}\ 2\right\}^T$, $\{X\}_3 = \left\{1\ -\sqrt{3}\ 2\right\}^T$ respectively. If $m_3 = 1kg$, determine $\{X\}_2$, k_1, k_2, k_3, m_1, m_2 and ω_2.

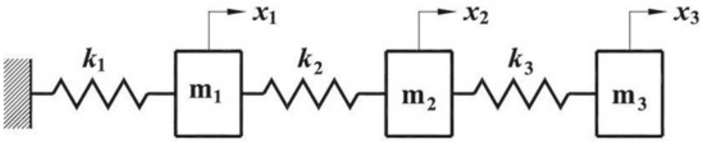


Figure P6.11

12. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P6.12**. Take $k = 1000\,\mathrm{N/m}$, $r = 12\,\mathrm{cm}$, $I_p = 1\,\mathrm{kgm}^2$ and $m = 5\,\mathrm{kg}$.

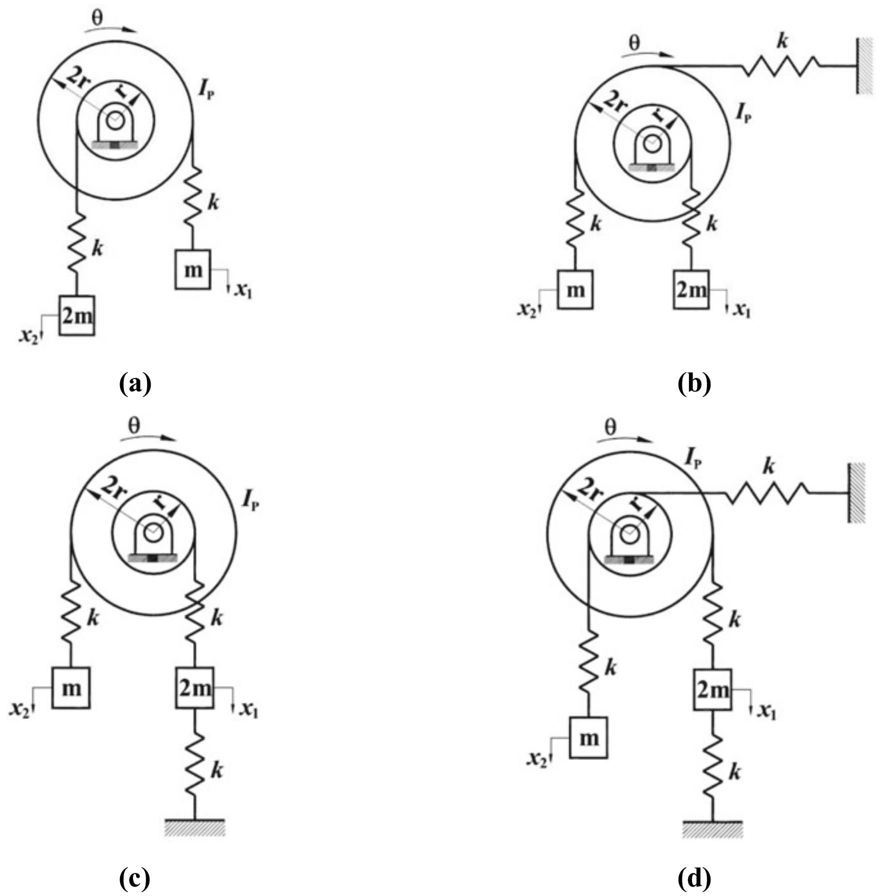


Figure P6.12

13. Determine the natural frequencies of the system shown in **Figure P6.13**. The uniform bar has a mass of M and length of L. Take $M = 20\,\text{kg}$, $L = 1.6\,\text{m}$, $m = 10\,\text{kg}$ and $k = 4\,\text{kN/m}$.
14. Determine the natural frequencies of the system shown in **Figure P6.14**. Take $I_1 = 1.2\,\text{kgm}^2$, $I_2 = 1\,\text{kgm}^2$, $r = 10\,\text{cm}$, $m = 5\,\text{kg}$ and $k = 2\,\text{kN/m}$.
15. Determine the natural frequencies and the corresponding mode shapes of the system shown in **Figure P6.15**. Take $k = 1000\,\text{N/m}$, $r = 12\,\text{cm}$ and $m = 5\,\text{kg}$. Both pulleys have same dimensions, and inner radius of the pulley is 0.75 times the outer radius. Neglect the inertia effects of the pulleys.
16. A string is stretched with a large tension T between two points and has three point masses fixed along its length as shown in **Figure P6.16**. The masses can vibrate freely in the lateral direction. Determine the three natural frequencies and the corresponding mode shapes.

17. Determine the steady state response of the systems shown in **Figure P6.17**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 150\,\mathrm{N/m}$, $k_4 = 100\,\mathrm{N/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$, $m_3 = 1\,\mathrm{kg}$ and $F = 200 \sin 50t$.

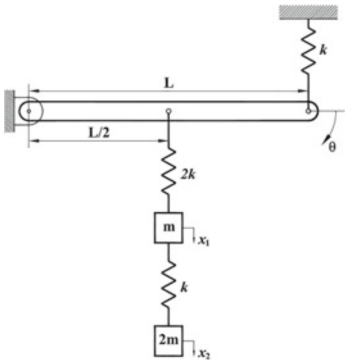


Figure P6.13

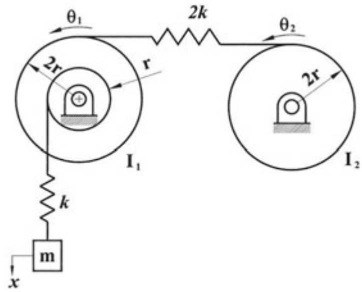


Figure P6.14

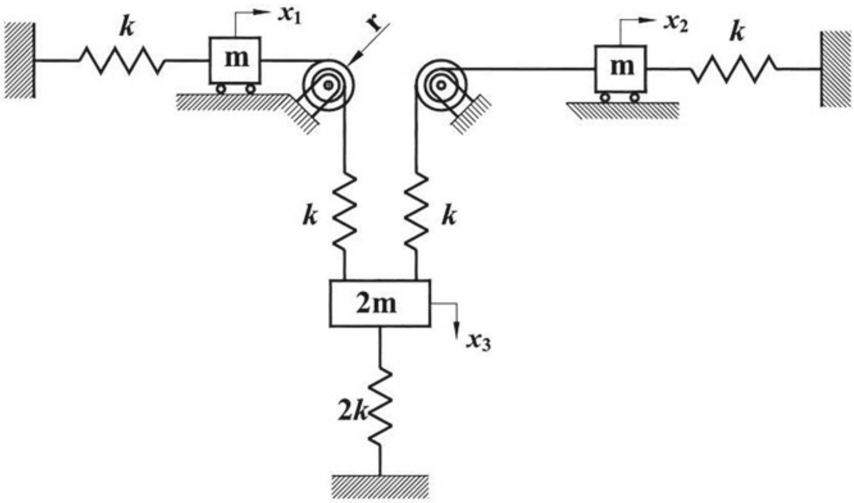
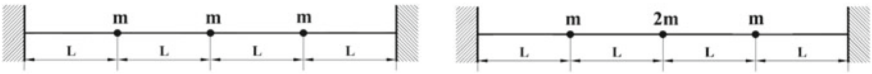


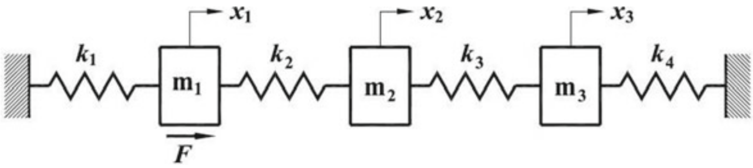
Figure P6.15

Figure P6.16

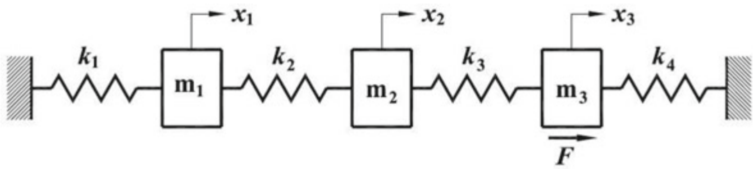


(a) (b)

Figure P6.16



(a)



(b)

Figure P6.17

18. Determine the steady state response of the systems shown in **Figure P6.17**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 2000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$, $m_3 = 20\,\text{kg}$ and $F = 250\sin 40t$.

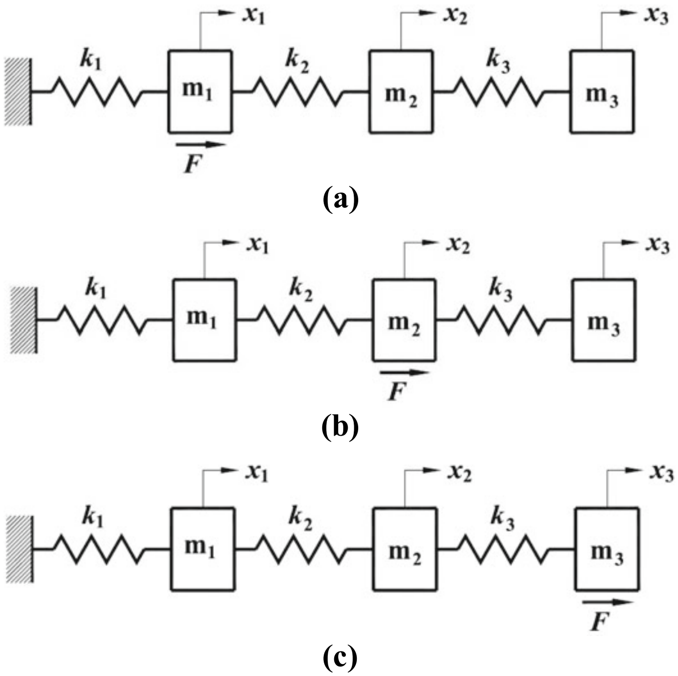


Figure P6.18

19. Determine the steady state of the response of the system shown in **Figure P6.19**. Take $k_1 = k_2 = k_3 = k_4 = 1000\,\text{N/m}$, $c_1 = c_2 = c_3 = c_4 = 100\,\text{N.s/m}$, $m_1 = m_2 = m_3 = 10\,\text{kg}$, and $F = 300 \sin 30t$.

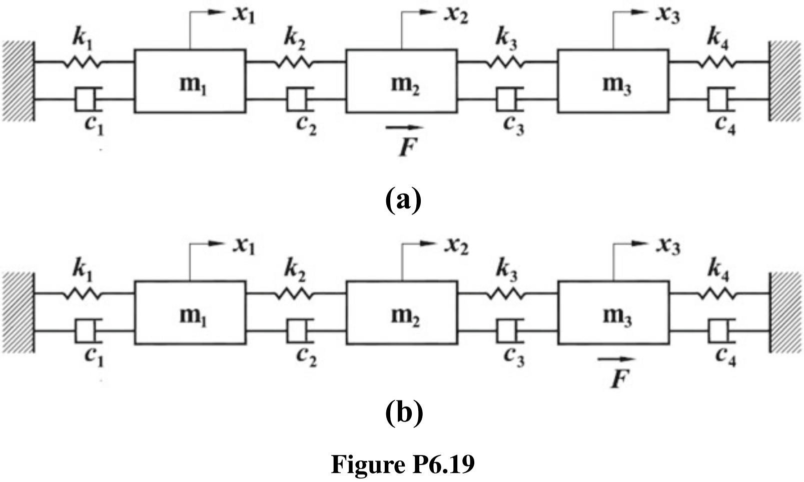


Figure P6.19

20. Determine the steady state of the response of the system shown in **Figure P6.20**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 200\,\text{N/m}$, $c_1 = 4N.s/m$, $c_2 = 6\,\text{N.s/m}$, $c_3 = 10\,\text{N.s/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 1.5\,\text{kg}$, $m_3 = 2\,\text{kg}$ and $F = 400\sin 40t$.

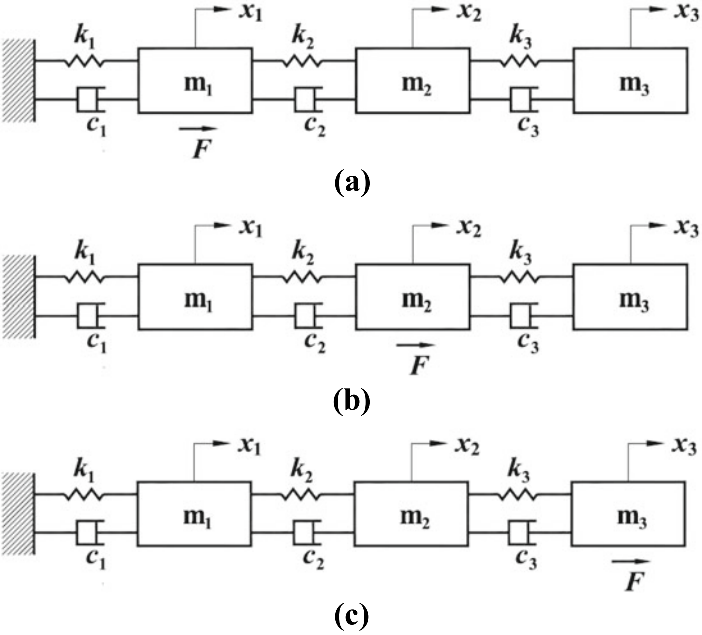


Figure P6.20

21. Use modal analysis to determine the free response of the system shown in **Figure P6.21**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$ and $m_2 = 10\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.

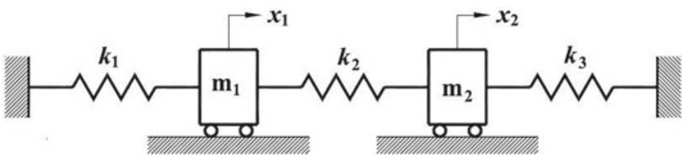


Figure P6.21

22. Use modal analysis to determine the free response of the system shown in **Figure P6.22**. Take $k = 1000\,\text{N/m}$ and $m = 20\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

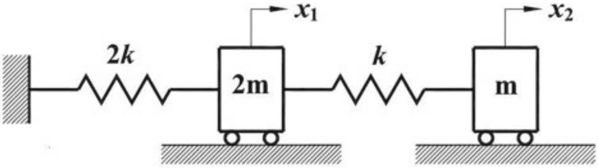


Figure P6.22

23. Use modal analysis to determine the free response of the system shown in **Figure P6.23**. Take $k_1 = 120\,\text{N/m}$, $k_2 = 100\,\text{N/m}$, $k_3 = 80\,\text{N/m}$, $c_1 = 2.5\,\text{N.s/m}$, $c_2 = 2\,\text{N.s/m}$, $c_3 = 1.7\,\text{N.s/m}$ and $m_1 = m_2 = 1\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.

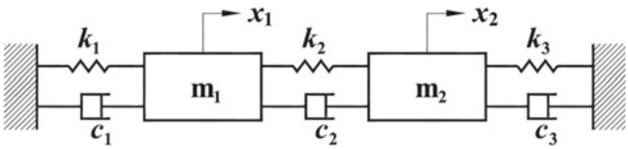


Figure P6.23

24. Use modal analysis to determine the free response of the system shown in **Figure P6.24**. Take $k_1 = 80\,\text{N/m}$, $k_2 = 120\,\text{N/m}$, $c_1 = 4\,\text{N.s/m}$, $c_2 = 6\,\text{N.s/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1.5\,\text{kg}$. Use the following initial conditions:

(a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$.
(b) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$.

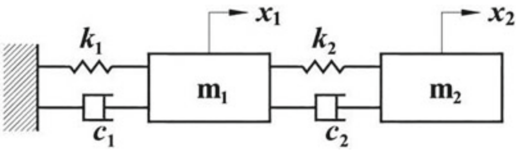


Figure P6.24

25. Use modal analysis to determine the free response of the system shown in **Figure P6.25**. Take $k_1 = 1000\,\text{N/m}$, $k_2 = 1500\,\text{N/m}$, $k_3 = 1500\,\text{N/m}$, $k_4 = 1000\,\text{N/m}$, $m_1 = 10\,\text{kg}$, $m_2 = 15\,\text{kg}$ and $m_3 = 10\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$, $x_3(0) = 0$, $\dot{x}_3(0) = 0$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$, $x_3(0) = 0$, $\dot{x}_3(0) = 1\,\text{m/s}$.
    (c) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, $x_3(0) = 0.1\,\text{m}$, $\dot{x}_3(0) = 0$.

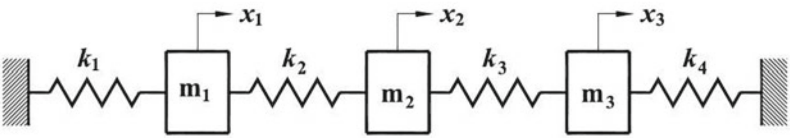


Figure P6.25

26. Use modal analysis to determine the free response of the system shown in **Figure P6.26**. Take $k_1 = k_2 = k_3 = k_4 = 1000\,\text{N/m}$, $c_1 = c_2 = c_3 = c_4 = 100\,\text{N.s/m}$, $m_1 = m_2 = m_3 = 10\,\text{kg}$. Use the following initial conditions:

    (a) $x_1(0) = 0.1\,\text{m}$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1\,\text{m/s}$, $x_3(0) = 0$, $\dot{x}_3(0) = 0$.
    (b) $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.1\,\text{m}$, $\dot{x}_2(0) = 0$, $x_3(0) = 0$, $\dot{x}_3(0) = 1\,\text{m/s}$.
    (c) $x_1(0) = 0$, $\dot{x}_1(0) = 1\,\text{m/s}$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, $x_3(0) = 0.1\,\text{m}$, $\dot{x}_3(0) = 0$.

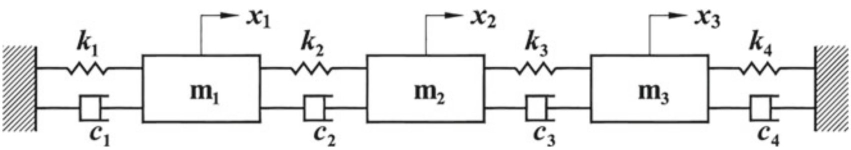


Figure P6.26

27. Use modal analysis to determine the steady state response of the system shown in **Figure P6.27**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 200\,\mathrm{N/m}$, $k_3 = 100\,\mathrm{N/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$ and $F_1 = 200\sin 30t$.

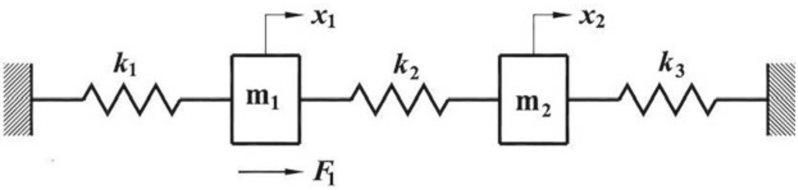


Figure P6.27

28. Use modal analysis to determine the steady state response of the system shown in **Figure P6.28**. Take $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 20\,\mathrm{kg}$ and $F_2 = 1000\sin 40t$.

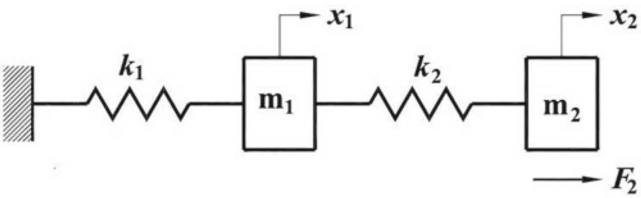


Figure P6.28

29. Use modal analysis to determine the steady state response of the damped two degrees of freedom systems shown in **Figure P6.29**. Take $k_1 = k_2 = 100\,\mathrm{N/m}$, $c_1 = c_2 = 5\,\mathrm{N.s/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$ and $F = 500\cos 50t$.

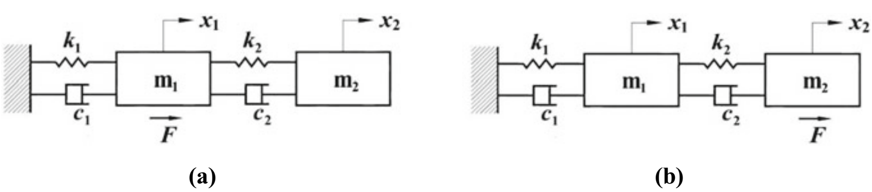
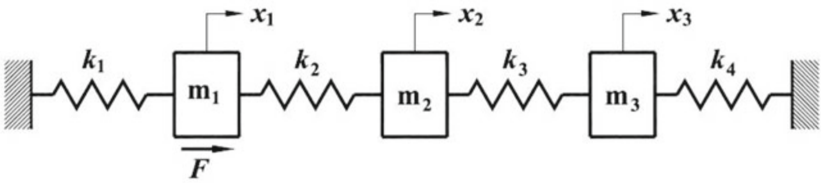
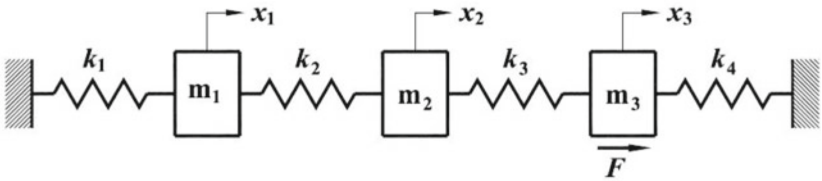


Figure P6.29

30. Use modal analysis to determine the steady state response of the system shown in **Figure P6.30**. Take $k_1 = 1000\,\mathrm{N/m}$, $k_2 = 1500\,\mathrm{N/m}$, $k_3 = 1500\,\mathrm{N/m}$, $k_4 = 1000\,\mathrm{N/m}$, $m_1 = 10\,\mathrm{kg}$, $m_2 = 15\,\mathrm{kg}$, $m_3 = 10\,\mathrm{kg}$ and $F = 2000\sin 40t$.



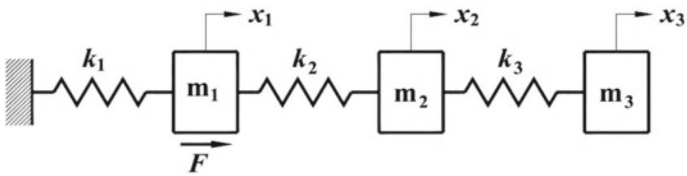
(a)



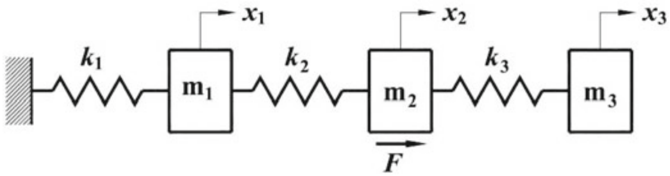
(b)

Figure P6.30

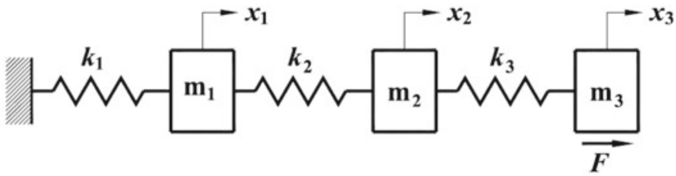
31. Use modal analysis to determine the steady state response of the systems shown in **Figure P6.31**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 200\,\mathrm{N/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$, $m_3 = 2\,\mathrm{kg}$ and $F = 250\cos 30t$.



(a)



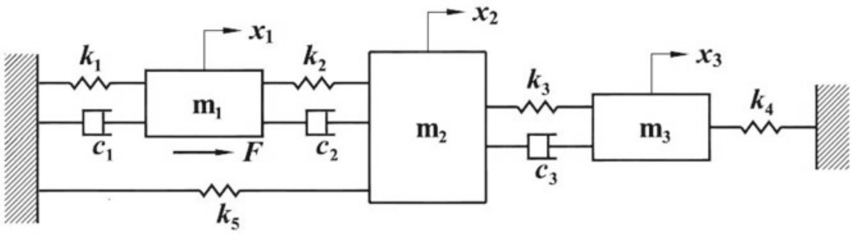
(b)



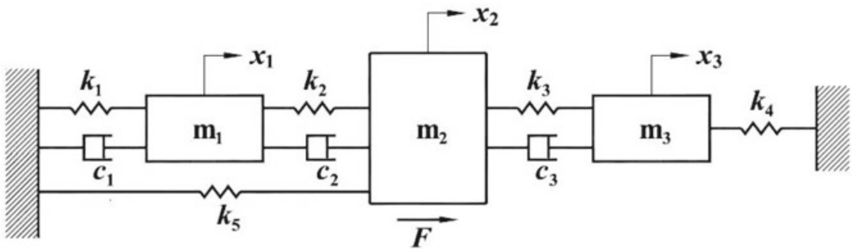
(c)

Figure P6.31

32. Use modal analysis to determine the steady state response of the systems shown in **Figure P6.32**. Take $k_1 = 300\,\text{N/m}$, $k_2 = 250\,\text{N/m}$, $k_3 = 250\,\text{N/m}$, $k_4 = 50\,\text{N/m}$, $k_5 = 100\,\text{N/m}$, $c_1 = c_2 = c_3 = 5\,\text{N.s/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $m_3 = 1\,\text{kg}$ and $F = 250\sin 40t$.



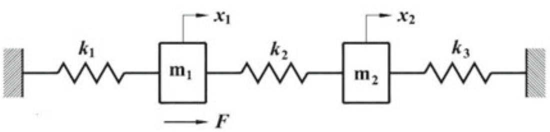
(a)

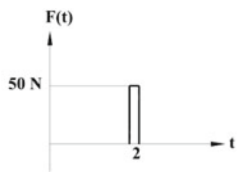


(b)

Figure P6.32

33. Use modal analysis to determine the response of the system shown in **Figure P6.33(a)** when it is subjected to a transient force shown in **Figure P6.33(b)**. Take $k_1 = k_2 = k_3 = 1000\,\text{N/m}$, $m_1 = m_2 = 10\,\text{kg}$.

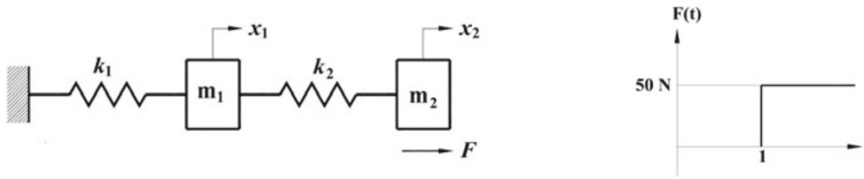




(a) **(b)**

Figure P6.33

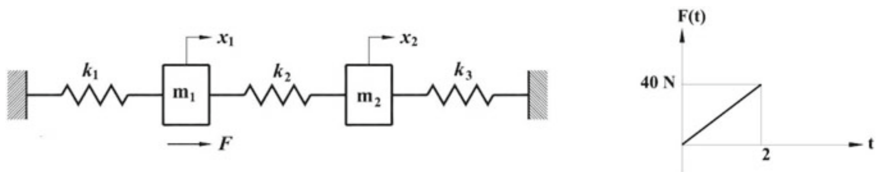
34. Use modal analysis to determine the response of m_1 of the system shown in **Figure P6.34(a)** when it is subjected to a transient force shown in **Figure P6.34(b)**. Take $k_1 = k_2 = k_3 = 1000\,\text{N/m}$, $m_1 = m_2 = 10\,\text{kg}$.



(a) (b)

Figure P6.34

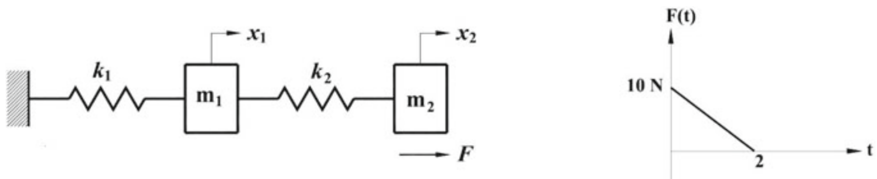
35. Determine the response of m_2 of the system shown in **Figure P6.35(a)** when it is subjected to a transient force shown in **Figure P6.35(b)**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 150\,\text{N/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1.5\,\text{kg}$.



(a) (b)

Figure P6.35

36. Use modal analysis to determine the response of the system shown in **Figure P6.36(a)** when it is subjected to a transient force shown in **Figure P6.35(b)**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $m_1 = 1\,\text{kg}$ and $m_2 = 1.5\,\text{kg}$.



(a) (b)

Figure P6.36

37. Use modal analysis to determine the response of the system shown in **Figure P6.37(a)** when it is subjected to a transient force shown in **Figure P6.37(b)**. Take $k_1 = k_2 = 1000\,\text{N/m}$, $c_1 = c_2 = 10\,\text{N.s/m}$ and $m_1 = m_2 = 10\,\text{kg}$.

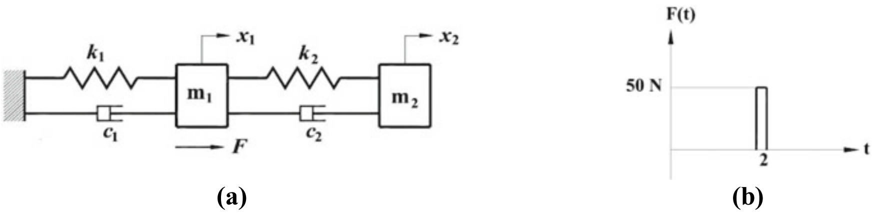


Figure P6.37

38. Use modal analysis to determine the response of the system shown in **Figure P6.38(a)** when it is subjected to a transient force shown in **Figure P6.38(b)**. Take $k_1 = k_2 = k_3 = 100\,\mathrm{N/m}, c_1 = c_2 = c_3 = 5\,\mathrm{N.s/m}$ and $m_1 = m_2 = 1\,\mathrm{kg}$.

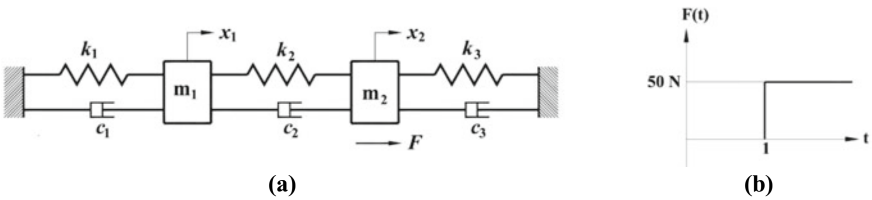


Figure P6.38

39. Use modal analysis to determine the response of the system shown in **Figure P6.39(a)** when it is subjected to a transient force shown in **Figure P6.39(b)**. Take $k_1 = k_2 = 100\,\mathrm{N/m}$, $c_1 = c_2 = 5\,\mathrm{N.s/m}$ and $m_1 = m_2 = 1\,\mathrm{kg}$.

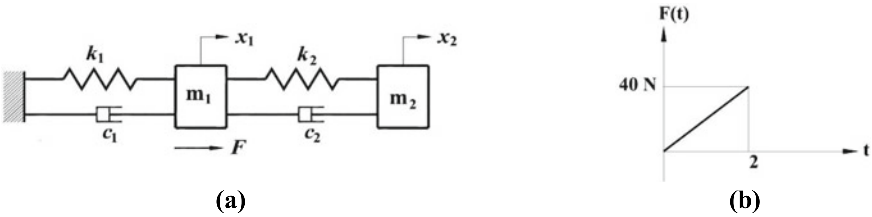
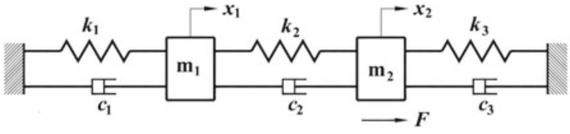
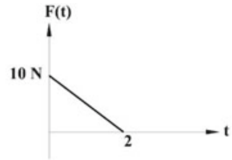


Figure P6.39

40. Use modal analysis to determine the response of the system shown in **Figure P6.39(a)** when it is subjected to a transient force shown in **Figure P6.39(b)**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 100\,\mathrm{N/m}$, $c_1 = 4\,\mathrm{N.s/m}$, $c_2 = 6\,\mathrm{N.s/m}$, $c_3 = 4\,\mathrm{N.s/m}$, $m_1 = 1\,\mathrm{kg}$ and $m_2 = 1\,\mathrm{kg}$.



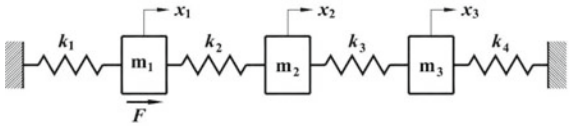
(a)



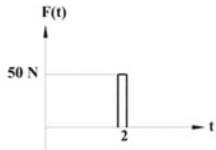
(b)

Figure P6.40

41. Use modal analysis to determine the response of the system shown in **Figure P6.41(a)** when it is subjected to a transient force shown in **Figure P6.41(b)**. Take $k_1 = k_2 = k_3 = k_4 = 1000\,\text{N/m}$ and $m_1 = m_2 = m_3 = 10\,\text{kg}$.



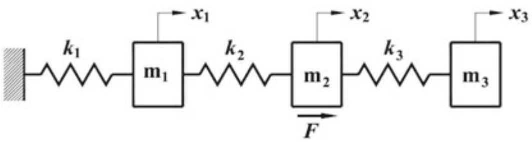
(a)



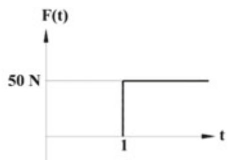
(b)

Figure P6.41

42. Use modal analysis to determine the response of m_1 of the system shown in **Figure P6.42(a)** when it is subjected to a transient force shown in **Figure P6.42(b)**. Take $k_1 = k_2 = k_3 = 1000\,\text{N/m}$ and $m_1 = m_2 = m_3 = 10\,\text{kg}$.



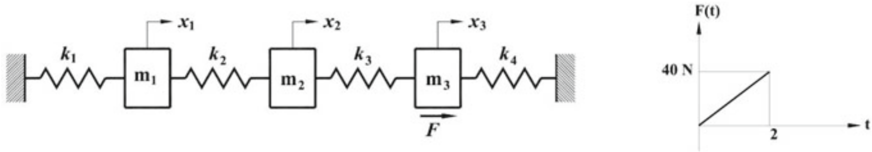
(a)



(b)

Figure P6.42

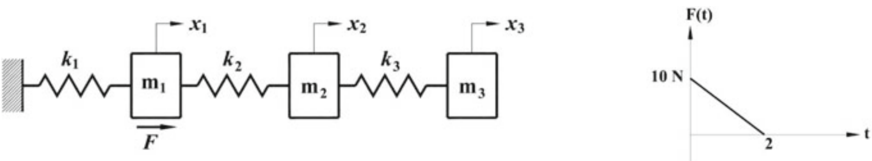
43. Use modal analysis to determine the response of m_2 of the system shown in **Figure P6.43(a)** when it is subjected to a transient force shown in **Figure P6.43(b)**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 150\,\mathrm{N/m}$, $k_4 = 100\,\mathrm{N/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$ and $m_3 = 1\,\mathrm{kg}$.



(a) (b)

Figure P6.43

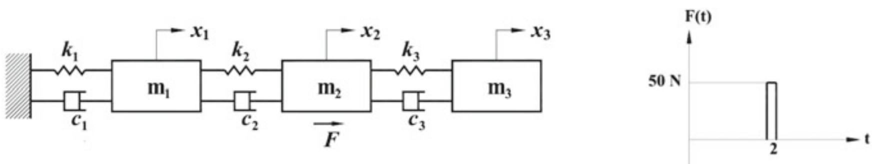
44. Use modal analysis to determine the response of the system shown in **Figure P6.44(a)** when it is subjected to a transient force shown in **Figure P6.44(b)**. Take $k_1 = 100\,\mathrm{N/m}$, $k_2 = 150\,\mathrm{N/m}$, $k_3 = 200\,\mathrm{N/m}$, $m_1 = 1\,\mathrm{kg}$, $m_2 = 1.5\,\mathrm{kg}$ and $m_3 = 2\,\mathrm{kg}$.



(a) (b)

Figure P6.44

45. Use modal analysis to determine the response of the system shown in **Figure P6.45(a)** when it is subjected to a transient force shown in **Figure P6.45(b)**. Take $k_1 = k_2 = k_3 = 1000\,\mathrm{N/m}$, $c_1 = c_2 = c_3 = 10\,\mathrm{N.s/m}$, $m_1 = m_2 = m_3 = 10\,\mathrm{kg}$.



(a) (b)

Figure P6.45

46. Use modal analysis to determine the response of the system shown in **Figure P6.46(a)** when it is subjected to a transient force shown in **Figure P6.46(b)**. Take $k_1 = k_2 = k_3 = k_4 = 100\,\text{N/m}$, $c_1 = c_2 = c_3 = c_4 = 5\,\text{N.s/m}$, $m_1 = m_2 = m_3 = 1\,\text{kg}$.

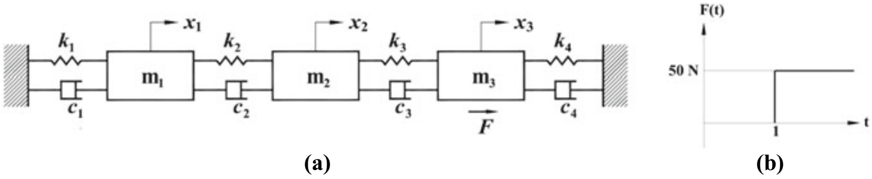


Figure P6.46

47. Use modal analysis to determine the response of the system shown in **Figure P6.47(a)** when it is subjected to a transient force shown in **Figure P6.47(b)**. Take $k_1 = k_2 = k_3 = 100\,\text{N/m}$, $c_1 = c_2 = c_3 = 5\,\text{N.s/m}$, $m_1 = m_2 = m_3 = 1\,\text{kg}$.

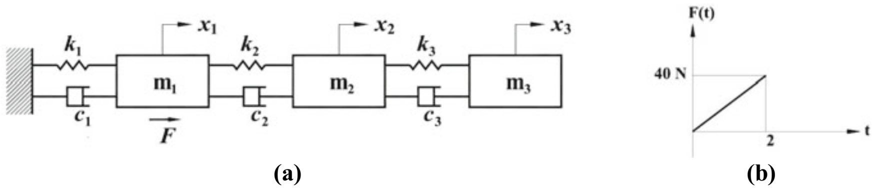
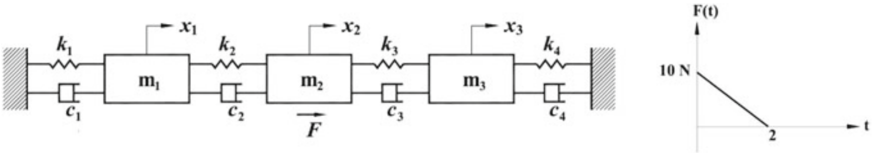


Figure P6.47

48. Use modal analysis to determine the response of the system shown in **Figure P6.48(a)** when it is subjected to a transient force shown in **Figure P6.48(b)**. Take $k_1 = 100\,\text{N/m}$, $k_2 = 150\,\text{N/m}$, $k_3 = 150\,\text{N/m}$, $k_4 = 100\,\text{N/m}$, $c_1 = 4\,\text{N.s/m}$, $c_2 = 6\,\text{N.s/m}$, $c_3 = 6\,\text{N.s/m}$, $c_4 = 4\,\text{N.s/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 1.5\,\text{kg}$ and $m_3 = 1\,\text{kg}$.



(a) (b)

Figure P6.48

49. Determine the natural frequencies and the corresponding mode shapes of a three degree of freedom system shown in **Figure P6.49**. Also determine the nodes of the system.

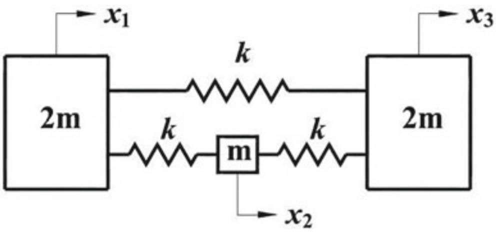


Figure P6.49

50. Determine the natural frequencies and the corresponding mode shapes of the four degrees of freedom systems shown in **Figure P6.50**. Also determine the nodes of the system.

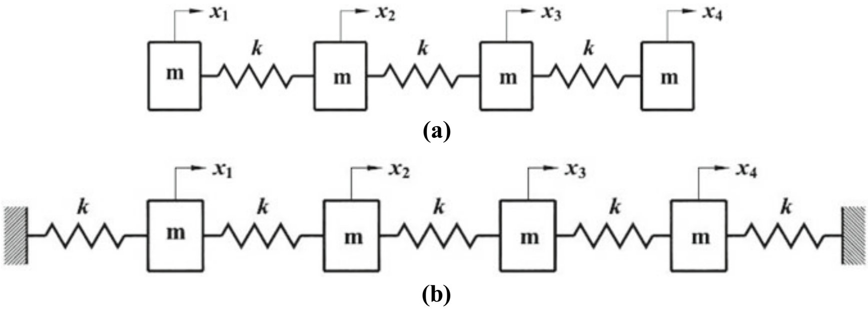


Figure P6.50

51. Determine the natural frequencies of the system shown in **Figure P6.51**. Take $I_1 = 1.2\,\text{kg.m}^2$ $I_2 = 1\,\text{kg.m}^2$, $r = 10\,\text{cm}$, $m = 8\,\text{kg}$ and $k = 2\,\text{kN/m}$.

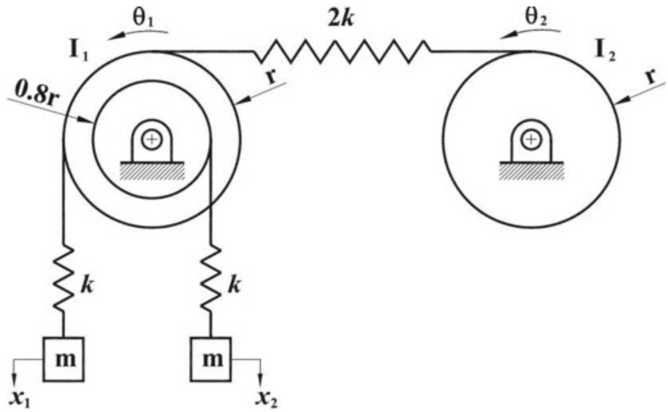


Figure P6.51

52. The layout for a compressor, turbine and generator of a gas turbine power plant is shown in **Figure P6.52**. The equivalent mass moment of inertias of the rotating parts of the compressor, turbine and generator is, respectively, $25\,\text{kg.m}^2$, $20\,\text{kg.m}^2$ and $15\,\text{kg.m}^2$. The torsional stiffness of the shaft between the compressor and the turbine is 10 MN.m/rad and that for the shaft between the turbine and the generator is 5 MN.m/rad. Determine the natural frequencies and the corresponding mode shapes for the torsional vibration of the system.

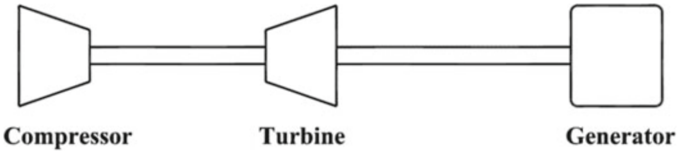


Figure P6.52

53. A pump is driven by an IC engine through a gearing system as shown in **Figure P6.53**. The equivalent mass moment of inertias of the rotating parts of the engine flywheel, gear, pinion and pump impeller is, respectively, 1000 kg.m^2, 25 kg.m^2, 12 kg.m^2 and 50 kg.m^2. The shaft connecting the engine flywheel and the gear has a length of 1 m and a diameter of 80 mm and that for the shaft connecting the pinion and the pump impeller is 0.4 m and 50 mm respectively. The speed ratio from the engine to the pump is 1 : 4. Determine the natural frequencies for torsional vibration of the system. Take $G = 84$ GPa for both shafts.

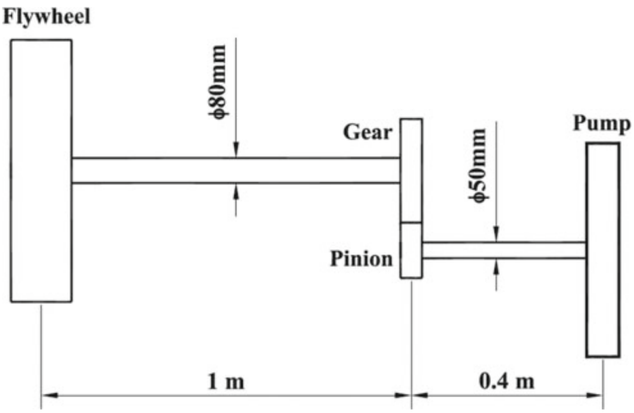


Figure P6.53

54. Repeat **Problem 53** for the following conditions:
    (a) Neglect the inertia effects of the gear and pinion.
    (b) Since flywheel has a very high moment of inertia, assume it as grounded (fixed).

55. **Figure P6.55** shows a three-cylinder engine with a flywheel. Determine the natural frequencies for the torsional vibration of the system. The mass moment of inertia of each engine is 0.04 kg.m^2 and that of the flywheel is 1.5 kg.m^2 and torsional stiffness of each shaft is 1.6 MN.m/rad.

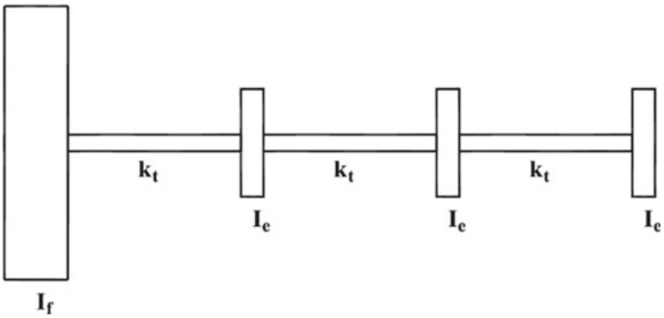


Figure P6.55

Answers

1.

(a) $\begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}, \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} \end{bmatrix}$

(b) $\begin{bmatrix} 8k & -k \\ -k & 2k \end{bmatrix}, \begin{bmatrix} \frac{2}{15k} & \frac{1}{15k} \\ \frac{1}{15k} & \frac{8}{15k} \end{bmatrix}$

(c) $\begin{bmatrix} 3k & -k \\ -k & 3k \end{bmatrix}, \begin{bmatrix} \frac{3}{8k} & \frac{1}{8k} \\ \frac{1}{8k} & \frac{3}{8k} \end{bmatrix}$

(d) $\begin{bmatrix} 5k & -2k \\ -2k & 5k \end{bmatrix}, \begin{bmatrix} \frac{5}{21k} & \frac{2}{21k} \\ \frac{2}{21k} & \frac{5}{21k} \end{bmatrix}$

2.

(a) $\begin{bmatrix} 3k & -2k & 0 \\ -2k & 6k & -4k \\ 0 & -4k & 6k \end{bmatrix}, \begin{bmatrix} \frac{5}{9k} & \frac{1}{3k} & \frac{2}{9k} \\ \frac{1}{3k} & \frac{1}{2k} & \frac{1}{3k} \\ \frac{2}{9k} & \frac{1}{3k} & \frac{7}{18k} \end{bmatrix}$

(b) $\begin{bmatrix} 3k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}, \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix}$

(c) $\begin{bmatrix} 4k & -k & -k \\ -k & 2k & -k \\ -k & -k & 4k \end{bmatrix}, \begin{bmatrix} \frac{7}{20k} & \frac{1}{4k} & \frac{3}{20k} \\ \frac{1}{4k} & \frac{3}{4k} & \frac{1}{4k} \\ \frac{3}{20k} & \frac{1}{4k} & \frac{7}{20k} \end{bmatrix}$

(d) $\begin{bmatrix} 3k & -k & -k \\ -k & 3k & -k \\ -k & -k & 3k \end{bmatrix}, \begin{bmatrix} \frac{1}{2k} & \frac{1}{4k} & \frac{1}{4k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{4k} & \frac{1}{4k} & \frac{1}{2k} \end{bmatrix}$

3.

(a) $1.1994\sqrt{\dfrac{k}{m}}, 2.3583\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 0.7808 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.2808 \end{Bmatrix}$

(b) $\sqrt{\dfrac{k}{m}}, 1.5811\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}$

4.

(a) $\sqrt{\dfrac{k}{m}}, 1.4142\sqrt{\dfrac{k}{m}}, 1.7321\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$

(b) $0.7654\sqrt{\dfrac{k}{m}}, 1.4142\sqrt{\dfrac{k}{m}}, 1.8478\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix},$
$\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.4142 \\ 1 \end{Bmatrix}$

(c) $0.6501\sqrt{\dfrac{k}{m}}, \sqrt{\dfrac{k}{m}}, 1.2559\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 1.7321 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.7321 \\ 1 \end{Bmatrix}$

5.

(a) $0.4682\sqrt{\dfrac{k}{m}}, 1.2247\sqrt{\dfrac{k}{m}}, 1.5102\sqrt{\dfrac{k}{m}}; \begin{Bmatrix} 1 \\ 3.5616 \\ 4.5616 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 1 \\ -2 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.5616 \\ 0.4385 \end{Bmatrix}$

(b) $0.4450\sqrt{\dfrac{k}{m}}, 1.2469\sqrt{\dfrac{k}{m}}, 1.8019\sqrt{\dfrac{k}{m}};$
$\begin{Bmatrix} 1 \\ 1.8019 \\ 2.2469 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.2469 \\ 0.5549 \end{Bmatrix}$

6.

(a)
$$0.8481\sqrt{\frac{k}{m}},\ 1.5811\sqrt{\frac{k}{m}},\ 1.6676\sqrt{\frac{k}{m}};\ \begin{Bmatrix} 1 \\ 1.5616 \\ 1 \end{Bmatrix},$$
$$\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix},\ \begin{Bmatrix} 1 \\ -2.5616 \\ 1 \end{Bmatrix}$$

(b) $0.6180\sqrt{\frac{k}{m}},\ \sqrt{\frac{k}{m}},\ 1.6180\sqrt{\frac{k}{m}};\ \begin{Bmatrix} 1 \\ 1.6180 \\ 1.6180 \end{Bmatrix},\ \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix},\ \begin{Bmatrix} 1 \\ -0.6180 \\ -0.6180 \end{Bmatrix}$

(c) $\sqrt{\frac{k}{m}},\ 1.7321\sqrt{\frac{k}{m}},\ 2\sqrt{\frac{k}{m}};\ \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix},\ \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix},\ \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}$

7.

(a)
$$0.0427\cos\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\} + 0.05\cos\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}$$
$$+ 0.0073\cos\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\},\ 0.0354\cos\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\}$$
$$- 0.0354\cos\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\},\ 0.0427\cos\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\}$$
$$- 0.05\cos\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\} + 0.0073\cos\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\}$$

(b)
$$0.2310\sqrt{\frac{m}{k}}\sin\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\} - 0.0957\sqrt{\frac{m}{k}}\sin\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\},$$
$$0.1913\sqrt{\frac{m}{k}}\sin\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\} + 0.4619\sqrt{\frac{m}{k}}\sin\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\},$$
$$0.2310\sqrt{\frac{m}{k}}\sin\left\{\left(0.7654\sqrt{\frac{k}{m}}\right)t\right\} - 0.0957\sqrt{\frac{m}{k}}\sin\left\{\left(1.8478\sqrt{\frac{k}{m}}\right)t\right\}$$

8.

$$0.0354\cos\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\} - 0.0354\cos\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\},$$

(a) $$0.05\cos\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\} + 0.05\cos\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\},$$

$$0.0354\cos\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\} - 0.0354\cos\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\},$$

(b) $$\begin{aligned}&0.4619\sqrt{\frac{m}{k}}\sin\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\} - 0.5\sqrt{\frac{m}{k}}\sin\left\{\left(\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad + 0.1913\sqrt{\frac{m}{k}}\sin\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\}, 0.6533\sqrt{\frac{m}{k}}\sin\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad - 0.2706\sqrt{\frac{m}{k}}\sin\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\}, 0.4619\sqrt{\frac{m}{k}}\sin\left\{\left(0.5412\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad + 0.5\sqrt{\frac{m}{k}}\sin\left\{\left(\sqrt{\frac{k}{m}}\right)t\right\} + 0.1913\sqrt{\frac{m}{k}}\sin\left\{\left(1.3065\sqrt{\frac{k}{m}}\right)t\right\},\end{aligned}$$

9.

(a) $$\begin{aligned}&0.0372\cos\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\} - 0.0667\cos\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad + 0.0294\cos\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\}, 0.0537\cos\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad - 0.0333\cos\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\} - 0.0204\cos\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\},\\ &0.0606\cos\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\} + 0.0333\cos\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}\\ &\quad + 0.0060\cos\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\}\end{aligned}$$

$$0.2257\sqrt{\frac{m}{k}}\sin\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\}+0.3143\sqrt{\frac{m}{k}}\sin\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}$$

$$+0.2289\sqrt{\frac{m}{k}}\sin\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\},\ 0.3257\sqrt{\frac{m}{k}}\sin\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\}$$

(b) $$+0.1571\sqrt{\frac{m}{k}}\sin\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}-0.1586\sqrt{\frac{m}{k}}\sin\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\},$$

$$0.3676\sqrt{\frac{m}{k}}\sin\left\{\left(0.3376\sqrt{\frac{k}{m}}\right)t\right\}-0.1571\sqrt{\frac{m}{k}}\sin\left\{\left(1.4142\sqrt{\frac{k}{m}}\right)t\right\}$$

$$+0.0468\sqrt{\frac{m}{k}}\sin\left\{\left(2.0943\sqrt{\frac{k}{m}}\right)t\right\}$$

10. $\left\{1\ \sqrt{2}\ 1\right\}^T$, 2000N/m, 2000N/m, 2000N/m, 2000N/m, 10kg, 10kg, $\left(400-200\sqrt{2}\right)^{1/2}$ rad/s

11. $\left\{1\ 0\ -1\right\}^T$, 100N/m, 100N/m, 100N/m, 2kg, 2kg, 10rad/s

12.

(a) $$0,\ 10.4319\,\text{rad/s},\ 16.2227\,\text{rad/s};\quad \begin{Bmatrix}1\\0.5\\4.1667\end{Bmatrix},\begin{Bmatrix}1\\-2.5829\\1.8995\end{Bmatrix},\begin{Bmatrix}1\\0.0968\\-1.3161\end{Bmatrix}$$

(b) $$6.0399\,\text{rad/s},\ 10.6092\,\text{rad/s},\ 16.7501\,\text{rad/s};\quad \begin{Bmatrix}1\\1.5538\\5.2933\end{Bmatrix},\begin{Bmatrix}1\\-0.5743\\-1.0463\end{Bmatrix},\begin{Bmatrix}1\\8.9649\\-15.0471\end{Bmatrix}$$

(c) $$2.3241\,\text{rad/s},\ 14.1421\,\text{rad/s},\ 16.3278\,\text{rad/s};\quad \begin{Bmatrix}1\\4\\16.2166\end{Bmatrix},\begin{Bmatrix}1\\-0.5\\0\end{Bmatrix},\begin{Bmatrix}1\\4\\-5.5499\end{Bmatrix}$$

(d) $$5.8539\,\text{rad/s},\ 14.1421\,\text{rad/s},\ 15.8787\,\text{rad/s};\quad \begin{Bmatrix}1\\1\\6.9055\end{Bmatrix},\begin{Bmatrix}1\\-2\\0\end{Bmatrix},\begin{Bmatrix}1\\1\\-2.1722\end{Bmatrix}$$

13. 10 rad/s, 24.4949 rad/s, 40 rad/s; $\begin{Bmatrix}1\\2\\0.4688\end{Bmatrix},\begin{Bmatrix}1\\-0.5\\1.25\end{Bmatrix},\begin{Bmatrix}1\\-0.1429\\-0.5357\end{Bmatrix}$

14. 0, 16.6547 rad/s, 20.7995 rad/s; $\begin{Bmatrix} 1 \\ 10 \\ 10 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 3.0655 \\ -4.1786 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.8155 \\ 0.4786 \end{Bmatrix}$

15. 14.1421 rad/s, 19.4074 rad/s, 26.2947 rad/s;
$$\begin{Bmatrix} 1 \\ 0.5625 \\ 0.75 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.6374 \\ -0.4276 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 4.9584 \\ -2.5261 \end{Bmatrix}$$

16.

(a) $$0.7654\sqrt{\frac{T}{mL}},\ 1.4142\sqrt{\frac{T}{mL}},\ 1.8478\sqrt{\frac{T}{mL}};$$
$$\begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.4142 \\ 1 \end{Bmatrix}$$

(b) $$0.6180\sqrt{\frac{T}{mL}},\ 1.4142\sqrt{\frac{T}{mL}},\ 1.6180\sqrt{\frac{T}{mL}};$$
$$\begin{Bmatrix} 1 \\ 1.6180 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.6180 \\ 1 \end{Bmatrix}$$

17.

(a) $-0.0891 \sin 50t,\ 0.0039 \sin 50t,\ -0.00026 \sin 50t$

(b) $-0.00026 \sin 50t,\ 0.0039 \sin 50t,\ -0.0891 \sin 50t$

18.

(a) $-0.0187 \sin 40t,\ -0.0014 \sin 40t,\ -0.00009 \sin 40t$

(b) $0014 \sin 40t,\ -0.0124 \sin 40,\ 0.0008 \sin 40tt$

(c) $-0.00009 \sin 40t,\ 0.0008 \sin 40t,\ -0.0084 \sin 40t$

19.

(a) $$-0.0077 \sin 30t + 0.0055 \cos 30t,\ -0.0232 \sin 30t \\ -0.0149 \cos 30t,\ -0.0077 \sin 30t + 0.0055 \cos 30t$$

(b) $0.0007\sin 30t + 0.0032\cos 30t, -0.0077\sin 30t + 0.0055\cos 30t,$
$\quad -0.0240\sin 30t - 0.0180\cos 30t$

20.

(a) $-0.2654\sin 40t - 0.0822\cos 40t, -0.0021\sin 40t$
$\quad + 0.0356\cos 40t, 0.0051\sin 40t - 0.0014\cos 40t$

(b) $-0.0021\sin 40t + 0.0356\cos 40t, -0.1678\sin 40t$
$\quad - 0.0573\cos 40t, 0.00005\sin 40t + 0.0262\cos 40t$

(c) $0.0051\sin 40t - 0.0014\cos 40t, 0.00005\sin 40t + 0.0262\cos 40t,$
$\quad -0.1273\sin 40t - 0.0187\cos 40t$

21.

(a) $0.05\sin 10t + 0.05\cos 10t - 0.025\sin 20t + 0.05\cos 20t,$
$0.05\sin 10t + 0.05\cos 10t + 0.025\sin 20t - 0.05\cos 20t$

(b) $0.05\sin 10t + 0.05\cos 10t + 0.025\sin 20t - 0.05\cos 20t,$
$0.05\sin 10t + 0.05\cos 10t - 0.025\sin 20t + 0.05\cos 20t$

22.

(a) $0.0667\cos 5t + 0.0333\cos 10t, 0.1333\cos 5t - 0.0333\cos 10t$

(b) $0.1333\sin 5t + 0.0333\sin 10t, 0.2667\sin 5t - 0.0333\sin 10t$

23.

(a)
$$\begin{aligned}
&e^{-1.0302t}[0.0540\sin(9.8467t) + 0.0402\cos(9.8467t)]\\
&\quad + e^{-3.0698t}[-0.0179\sin(17.1043t) + 0.0598\cos(17.1043t)],\\
&e^{-1.0302t}[0.0659\sin(9.8467t) + 0.0490\cos(9.8467t)]\\
&\quad + e^{-3.0698t}[0.0147\sin(17.1043t) - 0.0490\cos(17.1043t)]
\end{aligned}$$

(b)
$$\begin{aligned}
&e^{-1.0302t}[0.0459\sin(9.8467t) + 0.0490\cos(9.8467t)]\\
&\quad + e^{-3.0698t}[0.0261\sin(17.1043t) - 0.0490\cos(17.1043t)],\\
&e^{-1.0302t}[0.0560\sin(9.8467t) + 0.0598\cos(9.8467t)]\\
&\quad + e^{-3.0698t}[-0.0215\sin(17.1043t) + 0.0402\cos(17.1043t)]
\end{aligned}$$

24.

(a)
$$\begin{aligned}&e^{-0.6277t}[0.0096\sin(4.9714t)+0.0761\cos(4.9714t)]\\&\quad+e^{-6.3723t}[0.0104\sin(14.6385t)+0.0239\cos(14.6385t)],\\&e^{-0.6277t}[0.0140\sin(4.9714t)+0.1109\cos(4.9714t)]\\&\quad+e^{-6.3723t}[-0.0048\sin(14.6385t)-0.0109\cos(14.6385t)]\end{aligned}$$

(b)
$$\begin{aligned}&e^{-0.6277t}[0.1531\sin(4.9714t)]+e^{-6.3723t}[0.0163\sin(14.6385t)],\\&e^{-0.6277t}[0.2231\sin(4.9714t)]+e^{-6.3723t}[-0.0075\sin(14.6385t)]\end{aligned}$$

25.

(a)
$$\begin{aligned}&0.0606\sin(7.0711t)\\&\quad+0.0214\cos(7.0711t)+0.05\cos(15.8114t)-0.0214\sin(20t)\\&\quad+0.0286\cos(20t),\ 0.0808\ \sin(7.0711t)+0.0286\ \cos(7.0711t)\\&\quad+0.0214\ \sin(20t)-0.0286\ \cos(20t),\ 0.0606\sin(7.0711t)\\&\quad+0.0214\cos(7.0711t)-0.05\cos(15.8114t)-0.0214\sin(20t)\\&\quad+0.0286\cos(20t)\end{aligned}$$

(b)
$$\begin{aligned}&0.0303\sin(7.0711t)+0.0429\cos(7.0711t)\\&\quad-0.0316\sin(15.8114t)+0.0143\sin(20t)-0.0429\cos(20t),\\&0.0404\sin(7.0711t)+0.0571\cos(7.0711t)-0.0143\sin(20t)\\&\quad+0.0429\cos(20t),\ 0.0303\sin(7.0711t)+0.0429\cos(7.0711t)\\&\quad+0.0316\sin(15.8114t)+0.0143\sin(20t)-0.0429\cos(20t)\end{aligned}$$

(c)
$$\begin{aligned}&0.0303\sin(7.0711t)+0.0214\cos(7.0711t)+0.0316\sin(15.8114t)\\&\quad-0.05\cos(15.8114t)+0.0143\sin(20t)+0.0286\cos(20t),\\&0.0404\sin(7.0711t)+0.0286\cos(7.0711t)-0.0143\sin(20t)\\&\quad-0.0286\cos(20t),\ 0.0303\sin(7.0711t)+0.0214\cos(7.0711t)\\&\quad-0.0316\sin(15.8114t)+0.05\cos(15.8114t)\\&\quad+0.0143\sin(20t)+0.0286\cos(20t)\end{aligned}$$

26.

(a)

$$\begin{aligned}
&e^{-2.9290t}[0.0603\sin(7.0711t)+0.025\cos(7.0711t)]\\
&\quad+e^{-10t}[0.05\sin(10t)+0.05\cos(10t)]\\
&\quad+e^{-17.0711t}[0.0104\sin(7.0711t)+0.025\cos(7.0711t)],\\
&e^{-2.9290t}[0.0853\sin(7.0711t)+0.0354\cos(7.0711t)]\\
&\quad+e^{-17.0711t}[-0.0146\sin(7.0711t)-0.0354\cos(7.0711t)],\\
&e^{-2.9290t}[0.0603\sin(7.0711t)+0.025\cos(7.0711t)]\\
&\quad+e^{-10t}[-0.05\sin(10t)-0.05\cos(10t)]\\
&\quad+e^{-17.0711t}[0.0104\sin(7.0711t)+0.025\cos(7.0711t)]
\end{aligned}$$

(b)

$$\begin{aligned}
&e^{-2.9290t}[0.05\sin(7.0711t)+0.0354\cos(7.0711t)]\\
&\quad+e^{-10t}[-0.05\ \sin(10t)]\\
&\quad+e^{-17.0711t}[-0.05\sin(7.0711t)-0.0354\cos(7.0711t)],\\
&e^{-2.9290t}[0.0707\sin(7.0711t)+0.05\cos(7.0711t)]\\
&\quad+e^{-17.0711t}[0.0707\sin(7.0711t)+0.05\cos(7.0711t)],\\
&e^{-2.9290t}[0.05\sin(7.0711t)+0.0354\cos(7.0711t)]\\
&\quad+e^{-10t}[0.05\sin(10t)]\\
&\quad+e^{-17.0711t}[-0.05\sin(7.0711t)-0.0354\cos(7.0711t)]
\end{aligned}$$

(c)

$$\begin{aligned}
&e^{-2.9290t}[0.0457\sin(7.0711t)+0.0250\cos(7.0711t)]\\
&\quad+e^{-10t}[-0.05\cos(10t)]\\
&\quad+e^{-17.0711t}[0.0957\sin(7.0711t)+0.0250\cos(7.0711t)],\\
&e^{-2.9290t}[0.0646\sin(7.0711t)+0.0354\cos(7.0711t)]\\
&\quad+e^{-17.0711t}[-0.1354\sin(7.0711t)-0.0354\cos(7.0711t)],\\
&e^{-2.9290t}[0.0457\sin(7.0711t)+0.0250\cos(7.0711t)]\\
&\quad+e^{-10t}[0.05\cos(10t)]\\
&\quad+e^{-17.0711t}[0.0957\sin(7.0711t)+0.0250\cos(7.0711t)]
\end{aligned}$$

27. $-0.3559\sin 30t,\ 0.0678\sin 30t$

28. $0.0037\sin 30t,\ -0.0329\sin 30t$

29.

(a) $0.0456\sin 50t-0.2059\cos 50t,\ -0.0154\sin 50t+0.0015\cos 50t$

(b) $-0.0154\sin 50t+0.0015\cos 50t,\ 0.0098\sin 50t-0.1353\cos 50t$

30.

(a) $-0.1493 \sin 40t, 0.0108 \sin 40t, -0.0012 \sin 40t$

(b) $-0.0012 \sin 40t, 0.0108 \sin 40t, -0.1493 \sin 40t$

31.

(a) $-0.3988 \cos 30t, 0.0613 \cos 30t, -0.0077 \cos 30t$

(b) $0.0613 \cos 30t, -0.2658 \cos 30t, 0.0332 \cos 30t$

(c) $-0.0077 \cos 30t, 0.0332 \cos 30t, -0.1604 \cos 30t$

32.

(a) $$\begin{aligned}&-0.2029 \sin 40t - 0.0849 \cos 40t, 0.0083 \sin 40t\\&\quad + 0.0254 \cos 40t, 0.0032 \sin 40t - 0.0057 \cos 40t\end{aligned}$$

(b) $$\begin{aligned}&0.0083 \sin 40t + 0.0254 \cos 40t, -0.0915 \sin 40t\\&\quad - 0.0199 \cos 40t, 0.0115 \sin 40t - 0.0197 \cos 40t\end{aligned}$$

33. $$\begin{aligned}&u(t-2)[0.25 \sin\{10(t-2)\} + 0.1443 \sin\{17.3205(t-2)\}],\\&u(t-2)[0.25 \sin\{10(t-2)\} - 0.1443 \sin\{17.3205(t-2)\}]\end{aligned}$$

34. $$\begin{aligned}&u(t-1)[0.0585\ \{1 - \cos 6.1803(t-1)\}\\&\quad + 0.0085\ \{-1 + \cos 16.1803(t-1)\}],\\&u(t-1)[0.0947\ \{1 - \cos 6.1803(t-1)\}\\&\quad + 0.0053\ \{1 - \cos 16.1803(t-1)\}]\end{aligned}$$

35. $$\begin{aligned}&0.1143t - 0.08 \sin 10t - 0.0018 \sin 18.7083t - u(t-2)\\&[0.1143t - 0.08 \sin 10(t-2) - 0.16 \cos 10(t-2)\\&\quad - 0.0018 \sin 18.7083(t-2) - 0.0686 \cos 18.7083(t-2)],\\&0.0571t - 0.08 \sin 10t + 0.0012 \sin 18.7083t - u(t-2)\\&[0.0571t - 0.08 \sin 10(t-2) - 0.16 \cos 10(t-2)\\&\quad + 0.0012 \sin 18.7083(t-2) + 0.0457 \cos 18.7083(t-2)]\end{aligned}$$

$$0.1 - 0.05t + 0.01\sin 5.6023t - 0.1109\cos 5.6023t$$
$$- 0.0003\sin 17.8498t + 0.0109\cos 17.8498t$$
$$- u(t-2)[0.1 - 0.05t + 0.01\sin 5.6023(t-2)$$
$$- 0.0003\sin 17.8498(t-2)],$$

36. $$0.1667 - 0.08333t + 0.0144\sin 5.6023t$$
$$- 0.1617\cos 5.6023t + 0.0001\sin 17.8498t$$
$$+ 0.05\cos 17.8498t - u(t-2)$$
$$[0.1667 - 0.08333t + 0.0144\sin 5.6023(t-2)$$
$$+ 0.0001\sin 17.8498(t-2)]$$

37. $$u(t-2)\big[0.2237e^{-0.1909(t-2)}\sin\{6.1774(t-2)\}$$
$$+ 0.2243e^{-1.3090(t-2)}\sin\{16.1273(t-2)\}\big],$$
$$u(t-2)\big[0.3619e^{-0.1909(t-2)}\sin\{6.1774(t-2)\}$$
$$- 0.1387e^{-1.3090(t-2)}\sin\{16.1273(t-2)\}\big]$$

38. $$u(t-1)[0.1667 - 0.0645e^{-2.5(t-1)}\sin\{9.6825(t-1)\}$$
$$- 0.25e^{-2.5(t-1)}\cos\{9.6825(t-1)\} + 0.0400e^{-7.5(t-1)}\sin\{15.6125(t-1)\}$$
$$+ 0.0833e^{-7.5(t-1)}\cos\{15.6125(t-1)\}],\ u(t-1)[0.3333$$
$$- 0.0645e^{-2.5(t-1)}\sin\{9.6825(t-1)\} - 0.25e^{-2.5(t-1)}\cos\{9.6825(t-1)\}$$
$$- 0.0400\,e^{-7.5(t-1)}\sin\{15.6125(t-1)\} - 0.0833e^{-7.5(t-1)}\cos\{15.6125(t-1)\}]$$

39. $$0.2t - 0.01 - 0.0226e^{-0.9549t}\sin(6.1061t)$$
$$+ 0.0072e^{-0.9549t}\cos(6.1061t)$$
$$- 0.0025e^{-6.5451t}\sin(14.7975t) + 0.0028e^{-6.5451t}\cos(14.7975t) - u(t-2)$$
$$[0.2t - 0.01 - 0.0678e^{-0.9549(t-2)}\sin\{6.1061(t-2)\}$$
$$- 0.2822e^{-0.9549(t-2)}\cos\{6.1061(t-2)\}$$
$$- 0.0514\,e^{-6.5451(t-2)}\sin\{14.7975(t-2)\}$$
$$- 0.1078e^{-6.5451(t-2)}\cos\{14.7975(t-2)\}],$$
$$0.2t - 0.01 - 0.0365e^{-0.9549t}\sin(6.1061t) + 0.0117e^{-0.9549t}\cos(6.1061t)$$
$$+ 0.0016e^{-6.5451t}\sin(14.7975t) - 0.0017e^{-6.5451t}\cos(14.7975t) - u$$
$$(t-2)[0.2t - 0.01 - 0.1098e^{-0.9549(t-2)}\sin\{6.1061(t-2)\}$$
$$- 0.4566e^{-0.9549(t-2)}\cos\{6.1061(t-2)\} + 0.0318e^{-6.5451(t-2)}$$
$$\sin\{14.7975(t-2)\} + 0.0666e^{-6.5451(t-2)}\cos\{14.7975(t-2)\}]$$

40.
$$\begin{aligned}
&0.03825-0.01875t-0.0079\mathrm{e}^{-2t}\sin(9.7980t)-0.0510\mathrm{e}^{-2t}\cos(9.7980t)\\
&\quad+0.0052\mathrm{e}^{-8t}\sin(18.3303t)+0.01275\mathrm{e}^{-8t}\cos(18.3303t)-u(t-2)\\
&[0.03825-0.01875t+0.0023\mathrm{e}^{-2(t-2)}\sin\{9.79801(t-2)\}\\
&\quad-0.0010\mathrm{e}^{-2(t-2)}\cos\{9.7980(t-2)\}-0.0002\mathrm{e}^{-8(t-2)}\sin\{18.3303(t-2)\}\\
&\quad+0.0025\mathrm{e}^{-8(t-2)}\cos\{18.3303(t-2)\}],\ 0.06375-0.03125t\\
&\quad-0.0079\mathrm{e}^{-2t}\sin(9.7980t)-0.0510\mathrm{e}^{-2t}\cos(9.7980t)\\
&\quad-0.0052\mathrm{e}^{-8t}\sin(18.3303t)-0.01275\mathrm{e}^{-8t}\cos(18.3303t)-u(t-2)\\
&\quad[0.06375-0.03125t+0.0023\mathrm{e}^{-2(t-2)}\sin\{9.79801(t-2)\}\\
&\quad-0.0010\mathrm{e}^{-2(t-2)}\cos\{9.7980(t-2)\}\quad+0.0002\mathrm{e}^{-8(t-2)}\sin\{18.3303(t-2)\}\\
&\quad-0.0025\mathrm{e}^{-8(t-2)}\cos\{18.3303(t-2)\}]
\end{aligned}$$

41.
$$\begin{aligned}
&u(t-2)[0.1633\sin\{7.6537(t-2)\}+0.1768\sin\{14.1421(t-2)\}\\
&\quad+0.0676\sin\{18.4776(t-2)\}]\\
&u(t-2)[0.2309\sin\{7.6537(t-2)\}-0.0957\sin\{18.4776(t-2)\}],\\
&u(t-2)[0.1633\sin\{7.6537(t-2)\}-0.1768\sin\{14.1421(t-2)\}\\
&\quad+0.0676\sin\{18.4776(t-2)\}]
\end{aligned}$$

42.
$$\begin{aligned}
&u(t-1)[0.0489\{1-\cos 4.4504(t-1)\}\ +0.0078\{1-\cos 12.4698(t-1)\}\\
&\quad-0.0339\{1-\cos 18.0194(t-1)\}],\\
&u(t-1)[0.0882\,\{1-\cos 4.4504(t-1)\}\ +0.0035\ \{1-\cos 12.4698(t-1)\}\\
&\quad+0.0422\ \{1-\cos 18.0194(t-1)\}]\\
&u(t-1)[0.1100\,\{1-\cos 4.4504(t-1)\}-0.0062\ \{1-\cos 12.4698(t-1)\}\\
&\quad-0.0188\ \{1-\cos 18.0194(t-1)\}]
\end{aligned}$$

$$\begin{aligned}
&0.0592t - 0.0121\sin 7.0711t + 0.0025\sin 15.8114t\\
&\quad - 0.0007\sin 20t - u(t-2)\\
&[0.0592t - 0.0121\sin 7.0711(t-2) - 0.1714\cos 7.0711(t-2)\\
&\quad + 0.0025\sin 15.8114(t-2)\\
&+ 0.08\cos 15.8114(t-2) - 0.0007\sin 20(t-2)\\
&\quad - 0.0007\cos 20(t-2)],\\
&0.1008t - 0.0162\sin 7.0711t + 0.0007\sin 20t - u\\
43.\quad &(t-2)[0.1008t - 0.0162\sin 7.0711(t-2)\\
&\quad - 0.2286\cos 7.0711(t-2)\\
&\quad + 0.0007\sin 20(t-2) + 0.0007\cos 20(t-2),\\
&0.1392t - 0.0121\sin 7.0711t - 0.0025\sin 15.8114t\\
&\quad - 0.0007\sin 20t\\
&\quad - u(t-2)[0.1392t - 0.0121\sin 7.0711(t-2)\\
&\quad - 0.1714\cos 7.0711(t-2)]\\
&\quad - 0.0025\sin 15.8114(t-2) - 0.08\cos 15.8114(t-2)\\
&\quad - 0.0007\sin 20(t-2) - 0.0007\cos 20(t-2)
\end{aligned}$$

$$\begin{aligned}
&0.0975 - 0.0487t + 0.0081\sin 3.7734t - 0.0615\cos 3.7734t\\
&\quad + 0.0010\sin 13.4509t - 0.0281\cos 13.4509t\\
&\quad + 0.0002\sin 19.7020t - 0.0079\cos 19.7020t\\
&\quad - u(t-2)[0.0975 - 0.0487t + 0.0081\sin 3.7734(t-2)\\
&\quad + 0.0010\sin 13.4509(t-2) + 0.0002\sin 19.7020(t-2)],\\
&0.1023 - 0.0512t + 0.0128\sin 3.7734t - 0.0966\cos 3.7734t\\
&\quad + 0.0005\sin 13.4509t - 0.0129\cos 13.4509t\\
44.\quad &\quad - 0.0002\sin 19.7020t + 0.0073\cos 19.7020t - u(t-2)\\
&[0.1023 - 0.0512t + 0.0128\sin 3.7734(t-2)\\
&\quad + 0.0005\sin 13.4509(t-2) - 0.0002\sin 19.7020(t-2)],\\
&0.0992 - 0.0496t + 0.0149\sin 3.7734t - 0.1127\cos 3.7734t\\
&\quad - 0.0006\sin 13.4509t + 0.060\cos 13.4509t\\
&\quad + 0.00006\sin 19.7020t - 0.0025\cos 19.7020t - u(t-2)\\
&\quad [0.0992 - 0.0496t + 0.0149\sin 3.7734(t-2)\\
&\quad - 0.0006\sin 13.4509(t-2) + 0.00006\sin 19.7020(t-2)]
\end{aligned}$$

45.
$$u(t-2)[0.2178e^{-0.0990(t-2)}\sin\{6.1774(t-2)\} + 0.0971e^{-0.7775(t-2)}\sin\{12.4455(t-2)\} - 0.1214e^{-0.7775(t-2)}\sin\{17.9461(t-2)\}],$$
$$u(t-2)[0.3925e^{-0.0990(t-2)}\sin\{6.1774(t-2)\} + 0.0432e^{-0.7775(t-2)}\sin\{12.4455(t-2)\} + 0.1513e^{-0.7775(t-2)}\sin\{17.9461(t-2)\}],$$
$$u(t-2)[0.4895e^{-0.0990(t-2)}\sin\{6.1774(t-2)\} - 0.0779e^{-0.7775(t-2)}\sin\{12.4455(t-2)\} - 0.0673e^{-0.7775(t-2)}\sin\{17.9461(t-2)\}]$$

46.
$$u(t-1)[0.125 - 0.0416e^{-1.4645(t-1)}\sin\{7.5123(t-1)\} - 0.2133e^{-1.4645(t-1)}\cos\{7.51235(t-1)\} + 0.0472e^{-5(t-1)}\sin\{13.2288(t-1)\} + 0.125e^{-5(t-1)}\cos\{13.2288(t-1)\} - 0.0191e^{-8.5355(t-1)}\sin\{16.3880(t-1)\} - 0.0366e^{-8.5355(t-1)}\cos\{16.3880(t-1)\}],$$
$$u(t-1)[0.25 - 0.0588e^{-1.4645(t-1)}\sin\{7.5123(t-1)\} - 0.3018e^{-1.4645(t-1)}\cos\{7.51235(t-1)\} + 0.0269e^{-8.5355(t-1)}\sin\{16.3880(t-1)\} + 0.0518e^{-8.5355(t-1)}\cos\{16.3880(t-1)\}],$$
$$u(t-1)[0.375 - 0.0416e^{-1.4645(t-1)}\sin\{7.5123(t-1)\} - 0.2133e^{-1.4645(t-1)}\cos\{7.51235(t-1)\} - 0.0472e^{-5(t-1)}\sin\{13.2288(t-1)\} - 0.125e^{-5(t-1)}\cos\{13.2288(t-1)\} - 0.0191e^{-8.5355(t-1)}\sin\{16.3880(t-1)\} - 0.0366e^{-8.5355(t-1)}\cos\{16.3880(t-1)\}]$$

$$0.2t - 0.01 - 0.0239\mathrm{e}^{-0.4952t}\sin(4.2228t) + 0.0054\mathrm{e}^{-0.4952t}\cos(4.2228t)$$
$$- 0.00475\mathrm{e}^{-3.8874t}\sin(11.8484t) + 0.0035\mathrm{e}^{-3.8874t}\cos(11.8484t)$$
$$- 0.0008\mathrm{e}^{-8.1174t}\sin(16.0874t) + 0.0011\mathrm{e}^{-8.1174t}\cos(16.0874t)$$
$$- u(t-2)[0.2t - 0.01 - 0.0482\mathrm{e}^{-0.4952(t-2)}\sin\{4.2228(t-2)\}$$
$$- 0.2118\mathrm{e}^{-0.4952(t-2)}\cos\{4.2228(t-2)\} - 0.0506\mathrm{e}^{-3.8874(t-2)}$$
$$\sin\{11.8484(t-2)\} - 0.1362\mathrm{e}^{-3.8874(t-2)}\cos\{11.8484(t-2)\}$$
$$- 0.0225\mathrm{e}^{-8.1174(t-2)}\sin\{16.0874(t-2)\}$$
$$- 0.0420\mathrm{e}^{-8.1174(t-2)}\cos\{16.0874(t-2)\}],$$
$$0.2t - 0.01 - 0.0432\mathrm{e}^{-0.4952t}\sin(4.2228t)$$
$$+ 0.0098\mathrm{e}^{-0.4952t}\cos(4.2228t)$$
$$- 0.0021\mathrm{e}^{-3.8874t}\sin(11.8484t)$$
$$+ 0.0016\mathrm{e}^{-3.8874t}\cos(11.8484t)$$
$$+ 0.0010\mathrm{e}^{-8.1174t}\sin(16.0874t) - 0.0013\mathrm{e}^{-8.1174t}\cos(16.0874t) - u(t-2)$$
$$[0.2t - 0.01 - 0.0867\mathrm{e}^{-0.4952(t-2)}\sin\{4.2228(t-2)\}$$
$$- 0.3187\mathrm{e}^{-0.4952(t-2)}\cos\{4.2228(t-2)\}$$

47. $$- 0.0225\mathrm{e}^{-3.8874(t-2)}\sin\{11.8484(t-2)\}$$
$$- 0.0606\mathrm{e}^{-3.8874(t-2)}\cos\{11.8484(t-2)\}$$
$$+ 0.0281\mathrm{e}^{-8.1174(t-2)}\sin\{16.0874(t-2)\}$$
$$+ 0.0523\mathrm{e}^{-8.1174(t-2)}\cos\{16.0874(t-2)\}],$$
$$0.2t - 0.01 - 0.0538\mathrm{e}^{-0.4952t}\sin(4.2228t)$$
$$+ 0.0122\mathrm{e}^{-0.4952t}\cos(4.2228t)$$
$$+ 0.0038\mathrm{e}^{-3.8874t}\sin(11.8484t)$$
$$- 0.0028\mathrm{e}^{-3.8874t}\cos(11.8484t)$$
$$- 0.0004\mathrm{e}^{-8.1174t}\sin(16.0874t)$$
$$+ 0.0006\mathrm{e}^{-8.1174t}\cos(16.0874t) - u(t-2)$$
$$[0.2t - 0.01 - 0.1084e^{-0.4952(t-2)}\sin\{4.2228(t-2)\}$$
$$- 0.4760\mathrm{e}^{-0.4952(t-2)}\cos\{4.2228(t-2)\}$$
$$+ 0.0406\mathrm{e}^{-3.8874(t-2)}\sin\{11.8484(t-2)\}$$
$$+ 0.1092e^{-3.8874(t-2)}\cos\{11.8484(t-2)\}$$
$$- 0.0125\mathrm{e}^{-8.1174(t-2)}\sin\{16.0874(t-2)\}$$
$$- 0.0233\mathrm{e}^{-8.1174(t-2)}\cos\{16.0874(t-2)\}]$$

48.
$$
\begin{aligned}
&0.051 - 0.025t - 0.0042e^{-t}\sin(7t) - 0.0583e^{-t}\cos(7t)\\
&\quad + 0.0030e^{-8t}\sin(18.3303t) + 0.0073e^{-8t}\cos(18.3303t)\\
&\quad - u(t-2)[0.051 - 0.025t + 0.0039e^{-(t-2)}\sin\{7(t-2)\}\\
&\quad - 0.0011e^{-(t-2)}\cos\{7(t-2)\} - 0.0001e^{-8(t-2)}\sin\{18.3303(t-2)\}\\
&\quad + 0.0001e^{-8(t-2)}\cos\{18.3303(t-2)\}],\ 0.085 - 0.0417t\\
&\quad - 0.0057e^{-t}\sin(7t) - 0.0777e^{-t}\cos(7t) - 0.0030e^{-8t}\sin(18.3303t)\\
&\quad - 0.0073e^{-8t}\cos(18.3303t) - u(t-2)[0.085 - 0.0417t\\
&\quad + 0.0052e^{-(t-2)}\sin\{7(t-2)\} - 0.0015e^{-(t-2)}\cos\{7(t-2)\}\\
&\quad + 0.0001e^{-8(t-2)}\sin\{18.3303(t-2)\}\\
&\quad - 0.0001e^{-8(t-2)}\cos\{18.3303(t-2)\}],\\
&0.051 - 0.025t - 0.0042e^{-t}\sin(7t) - 0.0583e^{-t}\cos(7t)\\
&\quad + 0.0030e^{-8t}\sin(18.3303t) + 0.0073e^{-8t}\cos(18.3303t)\\
&\quad - u(t-2)[0.051 - 0.025t + 0.0039e^{-(t-2)}\sin\{7(t-2)\}\\
&\quad - 0.0011e^{-(t-2)}\cos\{7(t-2)\} - 0.0001e^{-8(t-2)}\sin\{18.3303(t-2)\}\\
&\quad + 0.0001e^{-8(t-2)}\cos\{18.3303(t-2)\}]
\end{aligned}
$$

49. $0, \sqrt{\dfrac{3k}{2m}}, \sqrt{\dfrac{5k}{2m}};\ \begin{Bmatrix} 1\\ 1\\ 1 \end{Bmatrix}, \begin{Bmatrix} 1\\ 0\\ -1 \end{Bmatrix}, \begin{Bmatrix} 1\\ -4\\ 1 \end{Bmatrix}$

50.

(a)
$$
0,\ 0.7654\sqrt{\frac{k}{m}},\ 1.4142\sqrt{\frac{k}{m}},\ 1.8478\sqrt{\frac{k}{m}};
$$
$$
\begin{Bmatrix} 1\\ 1\\ 1\\ 1 \end{Bmatrix}, \begin{Bmatrix} 1\\ 0.4142\\ -0.4142\\ -1 \end{Bmatrix}, \begin{Bmatrix} 1\\ -1\\ -1\\ 1 \end{Bmatrix}, \begin{Bmatrix} 1\\ -2.4142\\ 2.4142\\ -1 \end{Bmatrix}
$$

(b)
$$
0.6180\sqrt{\frac{k}{m}},\ 1.1756\sqrt{\frac{k}{m}},\ 1.6180\sqrt{\frac{k}{m}},\ 1.9021\sqrt{\frac{k}{m}};
$$
$$
\begin{Bmatrix} 1\\ 1.6180\\ 1.6180\\ 1 \end{Bmatrix}, \begin{Bmatrix} 1\\ 0.6180\\ -0.6180\\ -1 \end{Bmatrix}, \begin{Bmatrix} 1\\ -0.6180\\ -0.6180\\ 1 \end{Bmatrix}, \begin{Bmatrix} 1\\ -1.6180\\ 1.6180\\ -1 \end{Bmatrix}
$$

$0,\ 8.3043\,\text{rad/s},\ 15.8113\,\text{rad/s},\ 16.7841\,\text{rad/s};$

51. $$\begin{Bmatrix} 1 \\ 0.8 \\ 10 \\ 10 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0.8 \\ 7.2416 \\ -10 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.25 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0.8 \\ -1.2682 \\ 0.2099 \end{Bmatrix}$$

$0,\ 595.2131\,\text{rad/s},\ 1062.2698\,\text{rad/s};$

52. $$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0.1143 \\ -1.8191 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -1.8226 \\ 0.7635 \end{Bmatrix}$$

53. $0,\ 50.1483\,\text{rad/s},\ 117.6036\,\text{rad/s};\ \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -6.4452 \\ -266.8919 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -39.9451 \\ 9.1475 \end{Bmatrix}$

54.

(a) $0,\ 50.2488\,\text{rad}/s;\ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -320 \end{Bmatrix}$

(b) $50.0333\,\text{rad/s},\ 116.2082\,\text{rad/s};\ \begin{Bmatrix} 1 \\ 34.9389 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.2358 \end{Bmatrix}$

$0,\ 1151.4036\text{rad/s},\ 2573.9803\text{rad/s},\ 3621.5413\text{rad/s};$

55. $$\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -0.2429 \\ -1.4052 \\ -2.1019 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -5.2112 \\ -2.7909 \\ 4.2522 \end{Bmatrix}, \begin{Bmatrix} 1 \\ -11.2958 \\ 13.4461 \\ -5.9003 \end{Bmatrix}$$

Chapter 7
Modeling and Response of Continuous System

7.1 Introduction

The main assumption taken during the modeling of discrete system is that inertia (mass), stiffness and damping properties are concentrated at specific points of the system. However, all real systems cannot be effectively modeled as discrete systems because in real system usually the system properties are spatially distributed within the system. Any system in which its properties (inertia, stiffness and damping) are distributed within the system is called a distributed or continuous system.

If the instantaneous position or configuration of a continuous system can be described as a function of only one spatial coordinate, then it is called a one-dimensional continuous system. Longitudinal vibration of a rod, transverse vibration of beam, transverse vibration of a string, etc., are the examples of one-dimensional system. If two independent spatial coordinates are required to define the instantaneous configuration of the system then it is said be a two-dimensional continuous system. Transverse vibration of plate is the example of two-dimensional system. Similarly, if three independent spatial coordinates are required to define the instantaneous configuration of the system, then it is said be a three-dimensional continuous system. Vibration of any solid mass with finite dimensions is the example of three-dimensional system.

Instantaneous position or configuration of a continuous system can be defined by the instantaneous positions of infinite particles of the system; therefore a continuous system is said to have an infinite degree of freedom and will have infinite natural frequencies. When the system vibrates with any one of the natural frequencies, the system takes a specific deflection shape which is defined by a continuous spatial function and is called a principal mode of vibration corresponding to the considered natural frequency. The vibration response of a continuous system with any arbitrary disturbance will be a superposition of all the principal modes.

Governing equations of continuous system appear in the form of partial differential equations, and the response of the system can be determined by applying both the boundary values and the initial condition.

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_7

Main differences between the discrete and continuous system are presented in Table 7.1.

Modeling and vibration response analysis of common one-dimensional continuous systems are explained below.

Table 7.1 Differences between discrete and continuous system

S. No.	Feature	Discrete system	Continuous system
1.	Property distribution	System properties are concentrated at specific points of the system	System properties are distributed within the system
2.	Degree of freedom	Discrete systems have finite degrees of freedom	Continuous systems have infinite degree of freedom
3.	Classification	Discrete systems can be further classified as single degree of freedom system, two degree of freedom system, three degree of freedom system, etc.	Continuous systems can be further classified as one-dimensional continuous system, two-dimensional continuous system and three-dimensional continuous system
4.	Equation of motion	Equation of motion of a single degree of freedom system appears in the form of ordinary differential equation. Equation of motion of a two or higher degree of freedom system appear in the form system of ordinary differential equations	Equation of motion of a continuous system appears as a partial differential equation or a system of partial differential equations
5.	Conditions required to determine the response of the system	Only initial conditions are required to solve the equation of motion of the system	Both boundary conditions and initial conditions are required to solve the equation of motion of the system
6.	Natural frequency	Discrete systems have finite number of natural frequencies equal to the number of degree of freedom of the system	Continuous systems have infinite natural frequencies
7.	Mode shapes	Mode shapes of two or higher degree of freedom system are defined by the eigen-vectors. Number and dimensions of the eigen-vectors will be equal the number of degree of freedom of the system	Mode shapes of a continuous system are defined by a continuous function which is also called an eigen-function

7.2 Lateral Vibration of a String

As explained earlier in Chap. 3, equation of motion of a continuous system can be derived by applying Newton's second law of motion or any one form of the variational methods (Hamilton's principle or Lagrange equation).

7.2.1 *Derivation of Equation of Motion Using Newton's Second Law of Motion*

Consider a small segment of a stretched string subjected to a longitudinal tension of magnitude T and a distributed transverse load per unit length of $f(x,t)$ as shown in Fig. 7.1. The length of the string is L and its mass per unit length is ρ. If $w(x,t)$ is the transverse displacement of the string, then we can apply Newton's second law of motion as

$$T\sin(\theta+\mathrm{d}\theta)-T\sin\theta+f(x,t)\mathrm{d}x=\rho\mathrm{d}x\frac{\partial^2 w}{\partial t^2} \tag{7.1}$$

For a very small transverse displacement $w(x,t)$, slope of the string $\theta(x,t)$ will also be very small, i.e.,

$$\sin(\theta+\mathrm{d}\theta)\approx\theta+\mathrm{d}\theta \text{ and } \sin\theta\approx\theta \tag{7.2}$$

Substituting Eq. (7.2), into Eq. (7.1), we get

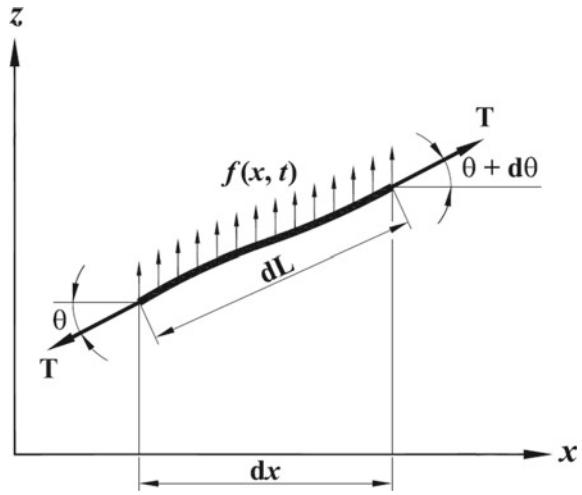


Fig. 7.1 A small segment of a stretched string undergoing transverse vibration

$$T\,\mathrm{d}\theta + f\mathrm{d}x = \rho\mathrm{d}x\frac{\partial^2 w}{\partial t^2} \tag{7.3}$$

Expressing the slope of the string θ in terms of the transverse displacement $w(x, t)$ as $\theta = \partial w/\partial x$, we get

$$\mathrm{d}\theta = \left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x \tag{7.4}$$

Substituting Eq. (7.4) into Eq. (7.3), we get the equation of motion for the transverse vibration of a string as

$$\rho\frac{\partial^2 w}{\partial t^2} - T\left(\frac{\partial^2 w}{\partial x^2}\right) = f \tag{7.5}$$

7.2.2 Derivation of Equation of Motion Using Hamilton's Principle

With reference to Fig. 7.1, the longitudinal displacement of differential segment of the string can be expressed as

$$\Delta = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}x\tan\theta)^2} - \mathrm{d}x \tag{7.6}$$

Again for small displacement,

$$\tan\theta \approx \theta = \frac{\partial w}{\partial x} \tag{7.7}$$

Substituting Eq. (7.7) into Eq. (7.6), we get

$$\Delta = \sqrt{(\mathrm{d}x)^2 + \left(\mathrm{d}x\frac{\partial w}{\partial x}\right)^2} - \mathrm{d}x = \left[\left\{1 + \left(\frac{\partial w}{\partial x}\right)^2\right\} - 1\right]\mathrm{d}x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \tag{7.8}$$

Then strain energy of the system due to longitudinal displacement of the string is given by

$$V = \int_0^L \frac{1}{2}T\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \tag{7.9}$$

Similarly, kinetic energy of the system due to transverse displacement of the string is given by

$$T = \int_0^L \frac{1}{2}\rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{7.10}$$

Work done by the external transverse load is given by

$$W_{nc} = \int_0^L f w \mathrm{d}x \tag{7.11}$$

Now, applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \delta \int_{t_1}^{t_2}\int_0^L \frac{1}{2}\rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \delta \int_{t_1}^{t_2}\int_0^L \frac{1}{2}T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t + \delta \int_{t_1}^{t_2}\int_0^L f w \mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)\delta\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L f\delta w \mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t - \int_{t_1} T\left(\frac{\partial w}{\partial x}\right)\delta(w)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L T\left(\frac{\partial^2 w}{\partial x^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L f\delta w \mathrm{d}x\,\mathrm{d}t = 0$$

Since $\delta\,(w)|_{t_1}^{t_2} = 0$

$$\int_{t_1}^{t_2}\int_0^L \left[\rho\left(\frac{\partial^2 w}{\partial t^2}\right) - T\left(\frac{\partial^2 w}{\partial x^2}\right) - f\right]\delta(w)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2}\left\{T\left(\frac{\partial w}{\partial x}\right)\delta(w)\Big|_{x=L}\right\}\mathrm{d}t$$
$$- \int_{t_1}^{t_2}\left\{T\left(\frac{\partial w}{\partial x}\right)\delta(w)\Big|_{x=0}\right\}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho\left(\frac{\partial^2 w}{\partial t^2}\right) - T\left(\frac{\partial^2 w}{\partial x^2}\right) = f \tag{7.12}$$

The associated boundary conditions are

$$\text{(a)} \quad \text{either } \frac{\partial w}{\partial x} = 0 \text{ or } \delta(w) = 0 \quad \text{at } x = 0. \tag{7.13}$$

$$\text{(b)} \quad \text{either } \frac{\partial w}{\partial x} = 0 \text{ or } \delta(w) = 0 \quad \text{at } x = L. \tag{7.14}$$

7.2.3 Free Response for Lateral Vibration of a String

Substituting $f = 0$ into Eqs. (7.5) or (7.12), we get the equation of motion for the lateral vibration of the string as

$$\rho\left(\frac{\partial^2 w}{\partial t^2}\right) - T\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{7.15}$$

Equation (7.15) can also expressed as

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 w}{\partial t^2}\right) \tag{7.16}$$

Equation (7.16) is the standard form of one-dimensional wave equation, and $c = \sqrt{T/\rho}$ is the velocity of wave propagation along the string.

General solution of Eq. (7.16) can be determined by applying the method of separation of variables. For this we can assume the solution as

$$w(x, t) = W(x)G(t) \tag{7.17}$$

where W is function of x only and G is function of t only.

Substituting Eq. (7.17) into Eq. (7.16), we get

$$G\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = \frac{1}{c^2} W \frac{\mathrm{d}^2 G}{\mathrm{d}t^2}$$

$$\therefore \quad \frac{1}{W}\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = \frac{1}{c^2}\frac{1}{G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} \tag{7.18}$$

It can be noted from Eq. (7.18) that the left-hand side term is independent of t and the right-hand side term is independent of x. Hence this equation will be satisfied only when both of these terms are equal to a constant value. If we assume this constant value be $-(\omega/c)^2$, then we can separate Eq. (7.18) as two independent ordinary differential equations as

$$\frac{1}{W}\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = -\left(\frac{\omega}{c}\right)^2 \tag{7.19}$$

$$\frac{1}{G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = -(\omega)^2 \tag{7.20}$$

Equations (7.19) and (7.20) can be also be expressed as

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + \left(\frac{\omega}{c}\right)^2 W = 0 \tag{7.21}$$

$$\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} + \omega^2 G = 0 \tag{7.22}$$

General solutions of Eqs. (7.21) and (7.22) can be given respectively as

$$W(x) = A\sin\left(\frac{\omega}{c}x\right) + B\cos\left(\frac{\omega}{c}x\right) \tag{7.23}$$

$$G(t) = C\sin(\omega t) + D\cos(\omega t) \tag{7.24}$$

It can be noted from Eqs. (7.23) and (7.24) that $W(x)$ represents the mode shape of the vibration and $G(t)$ is the time-dependent sinusoidal oscillation of the system.

Substituting $W(x)$ and $G(t)$ from Eqs. (7.23) and (7.24) into Eq. (7.17), we get the general solution

$$w(x,t) = \left[A\sin\left(\frac{\omega}{c}x\right) + B\cos\left(\frac{\omega}{c}x\right)\right][C\sin(\omega t) + D\cos(\omega t)] \tag{7.25}$$

where the arbitrary constants A and B are determined from the boundary conditions whereas constants C and D are determined from the initial conditions.

For an example, if the string of length L is fixed at both ends, its boundary conditions can be defined as

$$w(0,t) = 0 \text{ and } w(L,t) = 0 \tag{7.26}$$

Substituting the first boundary condition defined by Eq. (7.26) into Eq. (7.25), we get

$$B[C\sin(\omega t) + D\cos(\omega t)] = 0 \tag{7.27}$$

Since $C\sin(\omega t) + D\cos(\omega t) \neq 0$, Eq. (7.27) gives

$$B = 0 \tag{7.28}$$

Again substituting the second boundary condition defined by Eq. (7.26) and also substituting $B = 0$ into Eq. (7.25), we get

$$A\sin\left(\frac{\omega}{c}L\right)[C\sin(\omega t) + D\cos(\omega t)] = 0 \tag{7.29}$$

Since $C\sin(\omega t) + D\cos(\omega t) \neq 0$ and $A \neq 0$, Eq. (7.29) gives

$$\sin\left(\frac{\omega}{c}L\right) = 0 \tag{7.30}$$

Equation (7.30) is satisfied for multiple values of ω. Hence the roots are given by

$$\begin{aligned} &\frac{\omega_n}{c}L = n\pi \\ \therefore\quad &\omega_n = \frac{n\pi c}{L} \quad \text{where } n = 1, 2, 3, \ldots \end{aligned} \tag{7.31}$$

Substituting ω_n from Eq. (7.31) into Eq. (7.23), we get the expression for the mode shape as

$$W_n(x) = A_n\sin\left(\frac{\omega_n}{c}x\right) = A_n\sin\left(\frac{n\pi}{L}x\right) \quad \text{where } n = 1, 2, 3, \ldots \tag{7.32}$$

Mode shapes for the first four modes of a string fixed at both ends are shown in Fig. 7.2. As explained earlier for discrete system, number of nodes will be higher for higher modes. In the first mode there are two nodes (fixed points at both ends), in the second mode there are three nodes (additional node at mid-length) and so on.

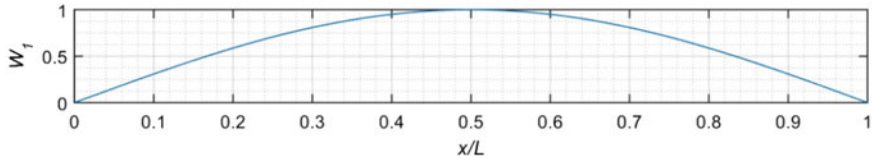
Response of any system due to any arbitrary initial disturbance is the sum of response due to all modes. Therefore the general response of the string undergoing transverse vibration can be determined by using mode shapes defined by Eq. (7.32) into Eq. (7.25) as

$$w(x,t) = \sum_{i=1}^{n}\sin\left(\frac{\omega_n}{c}x\right)[C_n\sin(\omega_n t) + D_n\cos(\omega_n t)] \tag{7.33}$$

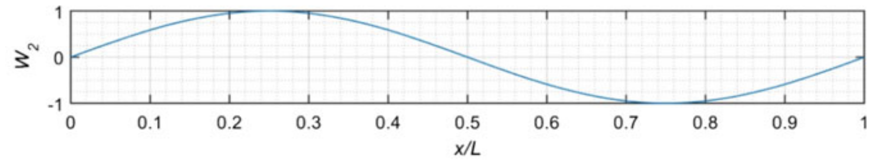
Arbitrary constants C_n and D_n are determined from the given initial conditions.

Orthogonality of Mode Shapes

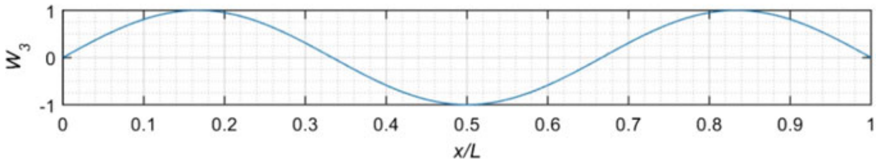
It can be noted from Eq. (7.32), continuous functions which define the mode shapes are also called eigen-functions. It can also be shown that eigen-functions are orthogonal to each other. For this using Eq. (7.21) for ith and jth modes respectively, we



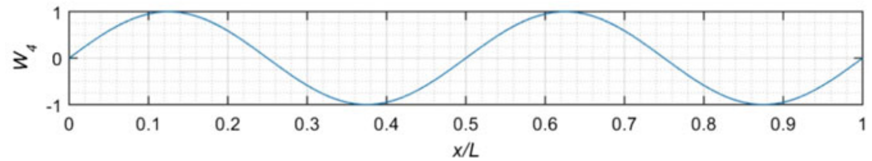
(a) First Mode



(b) Second Mode



(c) Third Mode



(d) Fourth Mode

Fig. 7.2 First four mode shapes of a string fixed at both ends

get

$$\frac{d^2 W_i}{dx^2} = -\frac{{\omega_i}^2}{c^2} W_i \tag{7.34}$$

$$\frac{d^2 W_j}{dx^2} = -\frac{{\omega_j}^2}{c^2} W_j \tag{7.35}$$

Multiplying Eq. (7.34) by W_j and integrating the left side over the domain length L, we get

$$\int_0^L \frac{d^2 W_i}{dx^2} W_j dx = -\frac{\omega_i^2}{c^2} \int_0^L W_i W_j dx$$

$$\therefore \quad \frac{\mathrm{d}W_i}{\mathrm{d}x}W_j\bigg|_{x=0}^{x=L} - \int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x = -\frac{\omega_i^2}{c^2}\int_0^L W_i W_j \mathrm{d}x \tag{7.36}$$

Either $W_j = 0$ and $\mathrm{d}W_i/\mathrm{d}x = 0$ at $x = 0$ and $x = L$, we get

$$\int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x = \frac{\omega_i^2}{c^2}\int_0^L W_i W_j \mathrm{d}x \tag{7.37}$$

Similarly, multiplying Eq. (7.35) by W_i and integrating the left side over the domain length L, we get

$$\int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x = \frac{\omega_j^2}{c^2}\int_0^L W_i W_j \mathrm{d}x \tag{7.38}$$

Subtracting Eq. (7.38) from Eq. (7.37), we get

$$0 = \frac{1}{c^2}\left(\omega_i^2 - \omega_j^2\right)\int_0^L W_i W_j \mathrm{d}x \tag{7.39}$$

Equation (7.39) can be re-expressed for two different conditions as

$$\int_0^L W_i W_j \mathrm{d}x = 0 \quad \text{if } i \neq j. \tag{7.40}$$

$$\int_0^L W_i W_j \mathrm{d}x \neq 0 \quad \text{if } i = j. \tag{7.41}$$

These conditions can be used to determine the arbitrary constants C_n and D_n of Eq. (7.33) to determine the free response of the string undergoing transverse vibration.

7.2.4 *Forced Harmonic Response for Lateral Vibration of a String*

Assume that the string fixed at both ends is subjected to a uniformly distributed harmonic force $f(x, t) = f_0 \sin \omega t$, where f_0 is the intensity of the uniformly distributed force, i.e., transverse force per unit length of the string. Substituting $f(x,t) = f_0 \sin \omega t$ into Eq. (7.12), we get

$$\left(\frac{\partial^2 w}{\partial t^2}\right) - c^2\left(\frac{\partial^2 w}{\partial x^2}\right) = \frac{f_0}{\rho} \sin \omega t = \overline{f}_0 \sin \omega t \tag{7.42}$$

where $\overline{f}_0 = f_0/\rho$.

For the steady state response of the system, the particular solution of Eq. (7.42) can be assumed as

$$w(x,t) = \overline{W}(x) \sin \omega t \tag{7.43}$$

Substituting Eq. (7.43) into Eq. (7.42), we get

$$\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}x^2} + \left(\frac{\omega^2}{c^2}\right)\overline{W} = -\frac{\overline{f}_0}{c^2} \tag{7.44}$$

We can assume the complementary solution of Eq. (7.44) as

$$\overline{W}_c(x) = \overline{A} \sin\left(\frac{\omega}{c}x\right) + \overline{B} \cos\left(\frac{\omega}{c}x\right) \tag{7.45}$$

Similarly, the particular solution of Eq. (7.44) due to constant force $-\left(f_0/c^2\right)$ can be assumed as

$$\overline{W}_p(x) = \overline{F} \tag{7.46}$$

Substituting Eq. (7.46) into Eq. (7.44), we get

$$\overline{W}_p(x) = -\frac{\overline{f}_0}{\omega^2} \tag{7.47}$$

Then the complete solution of Eq. (7.44) is given by the sum of complementary and particular solution as

$$\overline{W}(x) = \overline{A} \sin\left(\frac{\omega}{c}x\right) + \overline{B} \cos\left(\frac{\omega}{c}x\right) - \frac{\overline{f}_0}{\omega^2} \tag{7.48}$$

Since the string is fixed at both ends, we can apply the boundary conditions $\overline{W}(0) = \overline{W}(L) = 0$. Using $\overline{W}(0) = 0$, we get

$$\overline{B} = \frac{\overline{f}_0}{\omega^2} \tag{7.49}$$

Using $\overline{W}(L) = 0$ and substituting value of $\overline{B}$ into Eq. (7.48), we get

$$\overline{A} = \left(\frac{\overline{f}_0}{\omega^2}\right)\left[\frac{1 - \cos\left(\frac{\omega L}{c}\right)}{\sin\left(\frac{\omega L}{c}\right)}\right] = \left(\frac{\overline{f}_0}{\omega^2}\right)\tan\left(\frac{\omega L}{2c}\right) \tag{7.50}$$

Substituting the values $\overline{A}$ and $\overline{B}$ into Eq. (7.48), we get

$$\overline{W}(x) = \left(\frac{\overline{f}_0}{\omega^2}\right)\left[\tan\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega}{c}x\right) + \cos\left(\frac{\omega}{c}x\right) - 1\right] \tag{7.51}$$

Substituting Eq. (7.51) into Eq. (7.43), we get the steady state response of the system as

$$w(x,t) = \left(\frac{\overline{f}_0}{\omega^2}\right)\left[\tan\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega}{c}x\right) + \cos\left(\frac{\omega}{c}x\right) - 1\right] \sin\omega t \tag{7.52}$$

7.3 Longitudinal Vibration of a Bar

7.3.1 Derivation of Equation of Motion Using Newton's Second Law of Motion

Consider a differential segment of a bar subjected to a distributed longitudinal force per unit length of magnitude $f(x,t)$ as shown in Fig. 7.3.

The bar material has a density of ρ and its modulus of elasticity is E. The longitudinal displacement of left section of the differential element is u and that of the right section is $u + (\partial u/\partial x)\mathrm{d}x$ and the corresponding forces developed at those sections are respectively P and $P + (\partial P/\partial x)\mathrm{d}x$. If A is the cross-sectional area of the bar, then the longitudinal stain of the bar is given by

$$\frac{\partial u}{\partial x} = \frac{\sigma}{E} = \frac{P}{AE} \tag{7.53}$$

Then, differentiating Eq. (7.53) with respect to x, we get

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = \frac{\partial P}{\partial x} \tag{7.54}$$

Then applying Newton's second law of motion for the differential element undergoing longitudinal vibration, we get

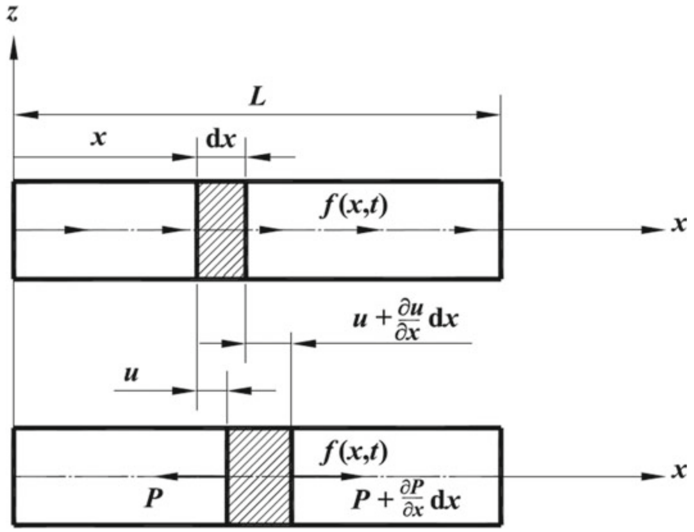


Fig. 7.3 A small segment of a bar undergoing longitudinal vibration

$$\left(P + \frac{\partial P}{\partial x}\mathrm{d}x\right) - P + f\mathrm{d}x = \rho A\mathrm{d}x\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \quad \frac{\partial P}{\partial x} + f = \rho A\frac{\partial^2 u}{\partial t^2} \tag{7.55}$$

Substituting $\partial P/\partial x$ from Eq. (7.54) into Eq. (7.55), we get the equation of motion for the longitudinal vibration of a bar as

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) + f = \rho A\frac{\partial^2 u}{\partial t^2} \tag{7.56}$$

Equation (7.56) can also be expressed for a uniform bar as

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{f}{A} = 0 \tag{7.57}$$

Equation (7.57) is one-dimensional non-homogeneous wave equation, and $c = \sqrt{E/\rho}$ is the velocity of wave propagation along the bar.

7.3.2 Derivation of Equation of Motion Using Hamilton's Principle

We can directly use the expressions developed in Chap. 2 for the kinetic and potential energy of the bar undergoing longitudinal vibration as

$$T = \int_0^L \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x \tag{7.58}$$

$$V = \int_0^L \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \tag{7.59}$$

Work done by the external longitudinal load is given by

$$W_{nc} = \int_0^L f u \,\mathrm{d}x \tag{7.60}$$

Now applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \delta\int_{t_1}^{t_2}\int_0^L \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \delta\int_{t_1}^{t_2}\int_0^L \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t + \delta\int_{t_1}^{t_2}\int_0^L f u \mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L f\delta u \mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta(u)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial^2 u}{\partial x^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L f\delta u \mathrm{d}x\,\mathrm{d}t = 0$$

Since $\delta\,(u)|_{t_1}^{t_2} = 0$

$$\int_{t_1}^{t_2}\int_0^L \left[\rho A\left(\frac{\partial^2 u}{\partial t^2}\right) - EA\left(\frac{\partial^2 u}{\partial x^2}\right) - f\right]\delta(u)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=L}\right\}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}\right\}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho\left(\frac{\partial^2 u}{\partial t^2}\right) - E\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{f}{A} \tag{7.61}$$

The associated boundary conditions are

$$\text{(a)}\quad \text{either } \frac{\partial u}{\partial x} = 0 \text{ or } \delta(u) = 0 \quad \text{at } x = 0. \tag{7.62}$$

$$\text{(b)}\quad \text{either } \frac{\partial u}{\partial x} = 0 \text{ or } \delta(u) = 0 \quad \text{at } x = L. \tag{7.63}$$

Since the equations of motion for the lateral vibration of a string and the longitudinal vibration of a bar are identical, free and forced vibration analysis procedure for both systems are also identical. Therefore, free and forced vibration analysis procedures are not repeated in this section.

7.4 Torsional Vibration of a Shaft

7.4.1 Derivation of Equation of Motion Using Newton's Second Law of Motion

Consider a differential segment of a shaft subjected to a distributed torque per unit length of magnitude $T(x, t)$ as shown in Fig. 7.4.

The shaft material has a density of ρ, and its shear modulus of elasticity is G. The torsional displacement of left section of the differential element is θ and that of the right section is $\theta + (\partial\theta/\partial x)\mathrm{d}x$ and the corresponding moments developed at those sections are respectively M and $M + (\partial M/\partial x)\mathrm{d}x$. If J is the polar moment of area of section of the shaft, then the angular displacement per unit length of the shaft is given by

$$\frac{\partial\theta}{\partial x} = \frac{M}{JG} \tag{7.64}$$

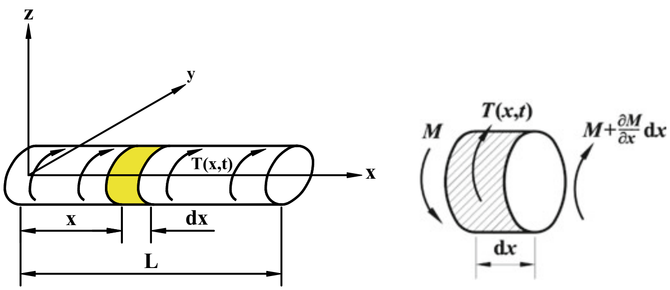


Fig. 7.4 A small segment of a shaft undergoing torsional vibration

Then differentiating Eq. (7.64) with respect to x, we get

$$\frac{\partial}{\partial x}\left(JG\frac{\partial \theta}{\partial x}\right) = \frac{\partial M}{\partial x} \tag{7.65}$$

Then applying Newton's second law of motion for the differential element undergoing torsional vibration, we get

$$\left(M + \frac{\partial M}{\partial x}\mathrm{d}x\right) - M + T\mathrm{d}x = \rho J \mathrm{d}x \frac{\partial^2 \theta}{\partial t^2}$$

$$\therefore \quad \frac{\partial M}{\partial x} + T = \rho J \frac{\partial^2 \theta}{\partial t^2} \tag{7.66}$$

Substituting $\partial M/\partial x$ from Eq. (7.65) into Eq. (7.66), we get the equation of motion for the torsional vibration of a shaft as

$$\frac{\partial}{\partial x}\left(JG\frac{\partial \theta}{\partial x}\right) + T = \rho J \frac{\partial^2 \theta}{\partial t^2} \tag{7.67}$$

Equation (7.67) can also be expressed for a uniform shaft as

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} + \frac{T}{GJ} = 0 \tag{7.68}$$

Equation (7.68) is one-dimensional non-homogeneous wave equation, and $c = \sqrt{G/\rho}$ is the velocity of wave propagation along the shaft.

7.4.2 Derivation of Equation of Motion Using Hamilton's Principle

We can directly use the expressions developed in Chap. 2 for the kinetic and potential energy of the shaft undergoing torsional vibration as

$$T = \int_0^L \frac{1}{2}\rho J \left(\frac{\partial\theta}{\partial t}\right)^2 \mathrm{d}x \tag{7.69}$$

$$V = \int_0^L \frac{1}{2}GJ \left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x \tag{7.70}$$

Work done by the external longitudinal load is given by

$$W_{nc} = \int_0^L T\theta \, \mathrm{d}x \tag{7.71}$$

Now, applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or,} \quad \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or,} \quad \delta \int_{t_1}^{t_2}\int_0^L \frac{1}{2}\rho J \left(\frac{\partial\theta}{\partial t}\right)^2 \mathrm{d}x \, \mathrm{d}t - \delta \int_{t_1}^{t_2}\int_0^L \frac{1}{2}GJ \left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x \, \mathrm{d}t + \delta \int_{t_1}^{t_2}\int_0^L T\theta \mathrm{d}x \, \mathrm{d}t = 0$$

$$\text{or,} \quad \int_{t_1}^{t_2}\int_0^L \rho J \left(\frac{\partial\theta}{\partial t}\right)\delta\left(\frac{\partial\theta}{\partial t}\right) \mathrm{d}x \, \mathrm{d}t - \int_{t_1}^{t_2}\int_0^L GJ \left(\frac{\partial\theta}{\partial x}\right)\delta\left(\frac{\partial\theta}{\partial x}\right) \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2}\int_0^L T\delta\theta \mathrm{d}x \, \mathrm{d}t = 0$$

$$\text{or,} \quad \int_0^L \rho J \left(\frac{\partial\theta}{\partial t}\right)\delta(\theta)\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho J \left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\mathrm{d}x \, \mathrm{d}t - \int_{t_1}^{t_2} GJ \left(\frac{\partial\theta}{\partial x}\right)\delta(\theta)\Big|_{x=0}^{x=L} \mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_0^L GJ \left(\frac{\partial^2\theta}{\partial x^2}\right)\delta(\theta)\mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2}\int_0^L T\delta\theta \mathrm{d}x \, \mathrm{d}t = 0$$

Since $\delta\,(\theta)|_{t_1}^{t_2} = 0$

$$\int_{t_1}^{t_2}\int_0^L\left[\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right)-GJ\left(\frac{\partial^2\theta}{\partial x^2}\right)-T\right]\delta(\theta)\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\delta(\theta)\Big|_{x=L}\right\}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\delta(\theta)\Big|_{x=0}\right\}\mathrm{d}t=0$$

Hence, equation of motion for the given system can be expressed as

$$\rho\left(\frac{\partial^2\theta}{\partial t^2}\right)-G\left(\frac{\partial^2\theta}{\partial x^2}\right)=\frac{T}{J} \tag{7.72}$$

The associated boundary conditions are

$$\text{(a)}\quad \text{either } \frac{\partial\theta}{\partial x}=0 \text{ or } \delta(\theta)=0 \quad \text{at } x=0. \tag{7.73}$$

$$\text{(b)}\quad \text{either } \frac{\partial\theta}{\partial x}=0 \text{ or } \delta(\theta)=0 \quad \text{at } x=L. \tag{7.74}$$

Since the equations of motion for the torsional vibration of shaft is also identical to the equation of motions for the lateral vibration of a string and the longitudinal vibration of a bar, free and forced vibration analysis procedure for all the systems are also identical. Therefore, free and forced vibration analysis procedures are also not repeated in this section.

7.5 Transverse Vibration of a Beam

7.5.1 Derivation of Equation of Motion Using Newton's Second Law of Motion

Consider a differential segment of a beam subjected to a distributed transverse load per unit length of magnitude $f(x,t)$ as shown in Fig. 7.5. The beam material has a density of ρ and its modulus of elasticity is E. The cross-sectional area and moment of inertia of the section of the beam are A and I, respectively. The transverse displacement of the is $w(x,t)$. The shear forces at the left section and the right section of the beam are F_s and $F_s+(\partial F_s/\partial x)\mathrm{d}x$, respectively. Similarly the corresponding bending moments at the left section and the right section of the beam are M and $M+(\partial M/\partial x)\mathrm{d}x$.

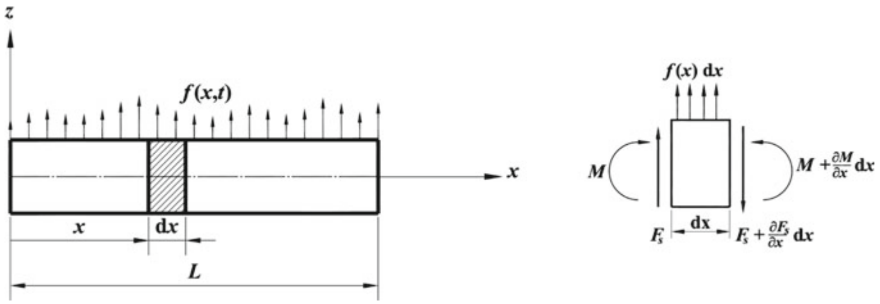


Fig. 7.5 A small segment of a beam undergoing transverse vibration

Then we can write the relationship between the load, shear force and bending moment as

$$\frac{\partial F_s}{\partial x} = f \tag{7.75}$$

$$\frac{\partial M}{\partial x} = F_s \tag{7.76}$$

Now applying Newton's second of motion,

$$F_s - \left(F_s + \frac{\partial F_s}{\partial x}\mathrm{d}x\right) + f\mathrm{d}x = \rho A dx\frac{\partial^2 w}{\partial t^2}$$

$$\therefore \quad -\frac{\partial F_s}{\partial x} + f = \rho A\frac{\partial^2 w}{\partial t^2} \tag{7.77}$$

Substituting F_s from Eq. (7.76) into Eq. (7.77), we get

$$-\frac{\partial^2 M}{\partial x^2} + f = \rho A\frac{\partial^2 w}{\partial t^2} \tag{7.78}$$

Again from the elementary strength of materials, bending moment can be related to the curvature as

$$M = EI\frac{\partial^2 w}{\partial x^2} \tag{7.79}$$

Substituting M from Eq. (7.79) into Eq. (7.78), we get

$$-\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + f = \rho A\frac{\partial^2 w}{\partial t^2} \tag{7.80}$$

If the beam is homogeneous and uniform, then Eq. (7.80) can also be expressed as

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = f \tag{7.81}$$

7.5.2 Derivation of Equation of Motion Using Hamilton's Principle

We can directly use the expressions developed in Chap. 2 for the kinetic and potential energy of the beam undergoing transverse vibration as

$$T = \int_0^L \frac{1}{2}\rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{7.82}$$

$$V = \int_0^L \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \tag{7.83}$$

Work done by the external longitudinal load is given by

$$W_{nc} = \int_0^L f u\,\mathrm{d}x \tag{7.84}$$

Now applying extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \delta\int_{t_1}^{t_2}\int_0^L \frac{1}{2}\rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \delta\int_{t_1}^{t_2}\int_0^L \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}t + \delta\int_{t_1}^{t_2}\int_0^L f w\mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L f\delta w\mathrm{d}x\,\mathrm{d}t = 0$$

or, $$\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}^{x=L}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L f\delta w\mathrm{d}x\,\mathrm{d}t = 0$$

Since $\delta\,(w)|_{t_1}^{t_2} = 0$

$$\int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - f\right]\delta(w)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L}\right\}\mathrm{d}t$$
$$- \int_{t_1}^{t_2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}\right\}\mathrm{d}t - \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)|_{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)|_{x=0}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - f = 0 \tag{7.85}$$

The associated boundary conditions are

$$\text{(a)}\quad \text{either } \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0 \text{ or } \delta(w) = 0 \quad \text{at } x = 0. \tag{7.86}$$

$$\text{(b)}\quad \text{either } \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0 \text{ or } \delta(w) = 0 \quad \text{at } x = L. \tag{7.87}$$

$$\text{(c)}\quad \text{either } EI\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \text{ or } \delta\left(\frac{\partial w}{\partial x}\right) = 0 \quad \text{at } x = 0. \tag{7.88}$$

$$\text{(d)}\quad \text{either } EI\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \text{ or } \delta\left(\frac{\partial w}{\partial x}\right) = 0 \quad \text{at } x = L. \tag{7.89}$$

7.5.3 Free Response for Transverse Vibration of a Beam

Substituting $f = 0$ into Eqs. (7.81) or (7.85), we get the equation of motion for the lateral vibration of a beam with uniform cross-section as

$$EI\left(\frac{\partial^4 w}{\partial x^4}\right) + \rho A\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{7.90}$$

Equation (7.90) can also be expressed as

$$\frac{\partial^4 w}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0 \tag{7.91}$$

where $c = \sqrt{EI/\rho A}$.

General solution of Eq. (7.91) can be determined by applying the method of separation of variables. For this we can assume the solution as

$$w(x, t) = W(x)G(t) \tag{7.92}$$

where W is function of x only and G is function of t only.

Substituting Eq. (7.92) into Eq. (7.91), we get

$$G\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} = -\frac{1}{c^2}W\frac{\mathrm{d}^2 G}{\mathrm{d}t^2}$$

$$\therefore \quad \frac{1}{W}\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} = -\frac{1}{c^2}\frac{1}{G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} \tag{7.93}$$

It can be noted from Eq. (7.93) that the left-hand side term is independent of t and the right-hand side term is independent of x. Hence this equation will be satisfied only when both of these terms are equal to a constant value. If we assume this constant value to be $(\omega/c)^2$, then we can separate Eq. (7.18) as two independent ordinary differential equations as

$$\frac{1}{W}\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} = \left(\frac{\omega}{c}\right)^2 \tag{7.94}$$

$$-\frac{1}{c^2}\frac{1}{G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = \left(\frac{\omega}{c}\right)^2 \tag{7.95}$$

Equation (7.94) can be also be expressed as

$$\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - \left(\frac{\omega}{c}\right)^2 W = 0$$

$$\text{or,}\quad \frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - \left(\frac{\omega^2 \rho A}{EI}\right)W = 0$$

$$\therefore \quad \frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - \beta^4 W = 0 \tag{7.96}$$

where $\beta^4 = \omega^2 \rho A / EI$.

Similarly, Eq. (7.95) can be also be expressed as

$$\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} + \omega^2 G = 0 \tag{7.97}$$

General solution of Eq. (7.96) can be assumed as

$$W(x) = \mathrm{e}^{sx} \tag{7.98}$$

Substituting Eq. (7.98) into Eq. (7.96),

$$s^4 - \beta^4 = 0 \tag{7.99}$$

Roots of Eq. (7.99) are given as

$$s_{1,2} = \pm i\beta; \quad s_{3,4} = \pm\beta \tag{7.100}$$

Then general solution of Eq. (7.96) can be expressed as

$$W(x) = C_1 \mathrm{e}^{-i\beta x} + C_2 \mathrm{e}^{i\beta x} + C_3 \mathrm{e}^{-\beta x} + C_4 \mathrm{e}^{-\beta x} \tag{7.101}$$

The general solution given by Eq. (7.101) can also be expressed in terms of trigonometric and hyperbolic functions as

$$W(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x \tag{7.102}$$

Similarly, general solution of Eq. (7.97) can be given as

$$F(t) = H \sin(\omega t) + K \cos(\omega t) \tag{7.103}$$

It can be noted from Eqs. (7.102) and (7.103) that $W(x)$ represents the mode shape of the vibration and $F(t)$ is the time-dependent sinusoidal oscillation of the system.

Substituting $W(x)$ and $F(t)$ from Eqs. (7.102) and (7.103) into Eq. (7.92), we get the general solution

$$\begin{aligned} w(x,t) = {} & [A \ \sin \beta x + B \ \cos \beta x + C \ \sinh \beta x + D \ \cosh \beta x] \\ & [H \ \sin(\omega t) + K \ \cos(\omega t)] \end{aligned} \tag{7.104}$$

where the arbitrary constants A, B, C and D are determined from the boundary conditions whereas constants H and K are determined from the initial conditions.

For an example, if the beam is pinned at both ends, its boundary conditions can be defined as

$$w(0,t) = 0;\ w''(0,t) = 0 \text{ and } w(L,t) = 0;\ w''(L,t) = 0 \tag{7.105}$$

Substituting the boundary condition $w(0,t) = 0$ into Eq. (7.104), we get

$$[B + D][H\ \sin(\omega t) + K\ \cos(\omega t)] = 0 \tag{7.106}$$

Since $H\ \sin(\omega t) + K\ \cos(\omega t) \neq 0$, Eq. (7.106) gives

$$B + D = 0 \tag{7.107}$$

Similarly, differentiating Eq. (7.104) twice with respect to x and substituting $w''(0,t) = 0$, we get

$$-B + D = 0 \tag{7.108}$$

Solving Eqs. (7.107) and (7.108), we get

$$B = D = 0 \tag{7.109}$$

Substituting $B = 0$ and $D = 0$ into Eq. (7.104), it reduces to

$$w(x,t) = [A\ \sin\beta x + C\ \sinh\beta x][H\ \sin(\omega t) + K\ \cos(\omega t)] \tag{7.110}$$

Substituting the boundary condition $w(L,t) = 0$ into Eq. (7.110), we get

$$A\ \sin\beta L + C\ \sinh\beta L = 0 \tag{7.111}$$

Similarly, differentiating Eq. (7.110) twice with respect to x and substituting $w''(L,t) = 0$, we get

$$-A\ \sin\beta L + C\ \sinh\beta L = 0 \tag{7.112}$$

Nontrivial solution for simultaneous Eqs. (7.111) and (7.112) can be determined as

$$\sin\beta L = 0 \text{ and } C = 0 \tag{7.113}$$

First part of Eq. (7.113) is satisfied for multiple values of β. Hence the roots are given by

$$\beta_n L = n\pi$$
$$\therefore \quad \beta_n = \frac{n\pi}{L} \quad \text{where } n = 1, 2, 3, \ldots \tag{7.114}$$

Then, the natural frequencies of the system can be determined as

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}} = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \quad \text{where } n = 1, 2, 3, \ldots \tag{7.115}$$

Substituting β_n from Eq. (7.114) and $B = C = D = 0$ into Eq. (7.102), we get the expression for the mode shape as

$$W_n(x) = \sin \beta_n x = \sin\left(\frac{n\pi}{L}x\right) \quad \text{where } n = 1, 2, 3, \ldots \tag{7.116}$$

Mode shapes for the first four modes of a beam pinned at both ends will be similar to those shown in Fig. 7.2.

Then, the general response of the beam undergoing transverse vibration can be determined by using mode shapes defined Eq. (7.116) into Eq. (7.110) as

$$w(x, t) = \sum_{i=1}^{n} \sin\left(\frac{n\pi}{L}x\right)[H_n \sin(\omega_n t) + K_n \cos(\omega_n t)] \tag{7.117}$$

Arbitrary constants G_n and H_n are determined from the given initial conditions.

Orthogonality of Mode Shapes

It can be shown that eigen-functions in this case are also orthogonal to each other. For this using Eq. (7.96) for ith and jth modes respectively, we get

$$\frac{\mathrm{d}^4 W_i}{\mathrm{d}x^4} = \beta_i{}^4 W_i = \left(\frac{\omega_i{}^2 \rho A}{EI}\right) W_i \tag{7.118}$$

$$\frac{\mathrm{d}^4 W_j}{\mathrm{d}x^4} = \beta_j{}^4 W_j = \left(\frac{\omega_j{}^2 \rho A}{EI}\right) W_j \tag{7.119}$$

Multiplying Eq. (7.118) by W_j and integrating the left side twice over the domain length L, we get

$$\int_0^L \frac{\mathrm{d}^4 W_i}{\mathrm{d}x^4} W_j \mathrm{d}x = \left(\frac{\omega_i^2 \rho A}{EI}\right) \int_0^L W_i W_j \mathrm{d}x$$

$$\therefore \quad \frac{\mathrm{d}^3 W_i}{\mathrm{d}x^3} W_j \bigg|_{x=0}^{x=L} - \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} \frac{\mathrm{d}W_j}{\mathrm{d}x} \bigg|_{x=0}^{x=L} + \int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} \frac{\mathrm{d}^2 W_j}{\mathrm{d}x^2} \mathrm{d}x = \left(\frac{\omega_i^2 \rho A}{EI} \right) \int_0^L W_i W_j \mathrm{d}x \tag{7.120}$$

Equating all boundary terms to zero, Eq. (7.120) reduces to

$$\int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} \frac{\mathrm{d}^2 W_j}{\mathrm{d}x^2} \mathrm{d}x = \left(\frac{\omega_i^2 \rho A}{EI} \right) \int_0^L W_i W_j \mathrm{d}x \tag{7.121}$$

Similarly, multiplying Eq. (7.119) by W_i and integrating the left side over the domain length L, we get

$$\int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} \frac{\mathrm{d}^2 W_j}{\mathrm{d}x^2} \mathrm{d}x = \left(\frac{\omega_j^2 \rho A}{EI} \right) \int_0^L W_i W_j \mathrm{d}x \tag{7.122}$$

Subtracting Eq. (7.122) from Eq. (7.121), we get

$$0 = \left(\omega_i^2 - \omega_j^2 \right) \left(\frac{\rho A}{EI} \right) \int_0^L W_i W_j \mathrm{d}x \tag{7.123}$$

Equation (7.123) can be re-expressed for two different conditions as

$$\int_0^L W_i W_j \mathrm{d}x = 0 \quad \text{if } i \neq j. \tag{7.124}$$

$$\int_0^L W_i W_j \mathrm{d}x \neq 0 \quad \text{if } i = j. \tag{7.125}$$

These conditions can be used to determine the arbitrary constants G_n and H_n of Eq. (7.117) to determine the free response of a beam undergoing transverse vibration.

7.5.4 *Forced Harmonic Response for Lateral Vibration of a Beam*

Assume that the beam pinned at both ends is subjected to a uniformly distributed harmonic force $f(x, t) = f_0 \sin \omega t$, where f_0 is the intensity of the uniformly distributed force, i.e., transverse force per unit length of the beam. Substituting $f(x,t) = f_0 \sin \omega t$ into Eq. (7.85), we get

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2} = \frac{f_0}{EI}\sin \omega t = \overline{f_0}\sin \omega t \tag{7.126}$$

where $\overline{f_0} = f_0/EI$.

For the steady state response of the system, the particular solution of Eq. (7.126) can be assumed as

$$w(x,t) = \overline{W}(x)\sin \omega t \tag{7.127}$$

Substituting Eq. (7.127) into Eq. (7.126), we get

$$\frac{\mathrm{d}^4\overline{W}}{\mathrm{d}x^4} - \left(\frac{\omega^2 \rho A}{EI}\right)\overline{W} = \overline{f}_0$$

$$\frac{\mathrm{d}^4\overline{W}}{\mathrm{d}x^4} - \beta^4\overline{W} = \overline{f}_0 \tag{7.128}$$

where $\beta^4 = \omega^2 \rho A/EI$.

We can assume the complementary solution of Eq. (7.126) as

$$\overline{W}_c(x) = \overline{A}\sin(\beta x) + \overline{B}\cos(\beta x) + \overline{C}\sinh(\beta x) + \overline{D}\cosh(\beta x) \tag{7.129}$$

Similarly, the particular solution of Eq. (7.128) due to constant force $-\left(f_0/c^2\right)$ can be assumed as

$$\overline{W}_p(x) = \overline{F} \tag{7.130}$$

Substituting Eq. (7.130) into Eq. (7.128), we get

$$\overline{W}_p(x) = -\frac{\overline{f}_0}{\beta^4} = -\frac{\overline{f}_0 EI}{\omega^2 \rho A} \tag{7.131}$$

Then the complete solution of Eq. (7.128) is given by the sum of complementary and particular solution as

$$\overline{W}(x) = \overline{A}\sin(\beta x) + \overline{B}\cos(\beta x) + \overline{C}\sinh(\beta x) + \overline{D}\cosh(\beta x) - \frac{\overline{f}_0 EI}{\omega^2 \rho A} \tag{7.132}$$

Applying the boundary conditions $\overline{W}(0) = 0$, we get

$$\overline{B} + \overline{D} = \frac{\overline{f}_0 EI}{\omega^2 \rho A} \tag{7.133}$$

Applying the boundary conditions $\overline{W''}(0) = 0$, we get

$$-\overline{B} + \overline{D} = 0 \tag{7.134}$$

Solving Eqs. (7.133) and (7.134) for $\overline{B}$ and $\overline{D}$, we get

$$\overline{B} = \frac{\overline{f}_0 EI}{2\omega^2 \rho A} \text{ and } \overline{D} = \frac{\overline{f}_0 EI}{2\omega^2 \rho A} \tag{7.135}$$

Again using $\overline{W}(L) = 0$ and substituting value of $\overline{B}$ and $\overline{D}$ into Eq. (7.132), we get

$$\overline{A}\sin(\beta L) + \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\cos(\beta L) + \overline{C}\sinh(\beta L) + \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\cosh(\beta L) = \frac{\overline{f}_0 EI}{\omega^2 \rho A} \tag{7.136}$$

Again using $\overline{W''}(L) = 0$ and substituting value of $\overline{B}$ and $\overline{D}$ into Eq. (7.132), we get

$$-\overline{A}\sin(\beta L) - \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\cos(\beta L) + \overline{C}\sinh(\beta L) + \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\cosh(\beta L) = \frac{\overline{f}_0 EI}{\omega^2 \rho A} \tag{7.137}$$

Solving Eqs. (7.133) and (7.134) for $\overline{A}$ and $\overline{C}$, we get

$$\overline{A} = \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\tanh\left(\frac{\beta}{2}L\right) \text{ and } \overline{C} = \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\tanh\left(\frac{\beta}{2}L\right) \tag{7.138}$$

Substituting the values $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$ into Eq. (7.132), we get

$$\overline{W}(x) = \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\left[\tanh\left(\frac{\beta}{2}L\right)\{\sin(\beta x) + \sinh(\beta x)\} + \{\cos(\beta x) + \cosh(\beta x)\} - 2\right] \tag{7.139}$$

Substituting Eq. (7.139) into Eq. (7.127), we get the steady state response of the system as

$$w(x,t) = \frac{\overline{f}_0 EI}{2\omega^2 \rho A}\left[\tanh\left(\frac{\beta}{2}L\right)\{\sin(\beta x) + \sinh(\beta x)\} + \{\cos(\beta x) + \cosh(\beta x)\} - 2\right]\sin\omega t \tag{7.140}$$

7.6 Modal Analysis for a Continuous System

7.6.1 Modal Analysis of a Continuous System Governed by Wave Equation

As explained earlier, lateral vibration of a string, longitudinal vibration of a bar and torsional vibration of a shaft are governed by one-dimensional wave equation. To explain the modal analysis, we consider the longitudinal vibration of a bar governed by the equation of motion

$$\left(\frac{\partial^2 w}{\partial t^2}\right) - c^2\left(\frac{\partial^2 w}{\partial x^2}\right) = \frac{f_0}{\rho}\sin\omega t = \overline{f}_0 \sin\omega t \tag{7.141}$$

Modal Analysis for Free Response

Substituting $f_0 = 0$ into Eq. (7.141) the governing equation for free vibration of the string becomes,

$$\left(\frac{\partial^2 w}{\partial t^2}\right) - c^2\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{7.142}$$

The initial conditions to which the system is subjected are given as

$$w(x,0) = w_0 \text{ and } \dot{w}(x,0) = \dot{w}_0 \tag{7.143}$$

As derived earlier, natural frequencies and the corresponding mode shapes or eigen-functions for a string fixed at both ends are respectively given as

$$\omega_n = \frac{n\pi c}{L} \quad \text{where } n = 1, 2, 3, \ldots \tag{7.144}$$

$$W_n(x) = \sin\left(\frac{\omega_n}{c}x\right) = \sin\left(\frac{n\pi}{L}x\right) \quad \text{where } n = 1, 2, 3, \ldots \tag{7.145}$$

Eigen-functions defined by Eq. (7.145) can be expressed in the form of normalized eigen-functions as

$$\widetilde{W}_n(x) = a_n \sin\left(\frac{n\pi}{L}x\right) \tag{7.146}$$

The constants a_n are determined by using definition of normalized eigen-functions as

$$\int_0^L \left[\widetilde{W}_n(x)\right]^2 \mathrm{d}x = 1 \tag{7.147}$$

Then for the modal analysis, we can assume solution of Eq. (7.142) as

$$w(x,t) = \sum_{i=1}^{\infty} \widetilde{W}_i(x) q_i(t) \tag{7.148}$$

Substituting Eq. (7.148) into Eq. (7.142), we get

$$\sum_{i=1}^{\infty} \widetilde{W}_i(x)\ddot{q}_i(t) - c^2 \sum_{i=1}^{\infty} \widetilde{W}_i''(x) q_i(t) = 0 \tag{7.149}$$

Multiplying Eq. (7.149) by $\widetilde{W}_j$ and integrating the left side twice over the domain length L, we get

$$\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)\int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x - c^2 q_i(t)\int_0^L \widetilde{W}_i''(x)\tilde{W}_j(x)\mathrm{d}x\right] = 0 \tag{7.150}$$

Substituting $\widetilde{W}_i''(x) = -(\omega_i/c)^2\widetilde{W}_i(x)$ and using orthogonality condition, Eq. (7.150) reduces for the ith mode as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \tag{7.151}$$

Then the general solution for free response of the ith mode is given by

$$q_i(t) = \frac{\dot{q}_{i0}}{\omega_i}\sin(\omega_i t) + q_{i0}\cos(\omega_i t) \tag{7.152}$$

Using (7.146) for initial conditions, we get

$$w(x,0) = w_0 = \sum_{i=1}^{\infty} \widetilde{W}_i(x) q_{i0} \tag{7.153}$$

$$\dot{w}(x,0) = \dot{w}_0 = \sum_{i=1}^{\infty} \widetilde{W}_i(x) \dot{q}_{i0} \tag{7.154}$$

Multiplying both sides of both Eqs. (7.153) and (7.154) by $\widetilde{W}_j$ and integrating the left side twice over the domain length L, we get

$$\int_0^L w_0 \widetilde{W}_j(x)\mathrm{d}x = \sum_{i=1}^{\infty} q_{i0} \int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x = q_{i0} \tag{7.155}$$

$$\int_0^L \dot{w}_0 \widetilde{W}_j(x)\mathrm{d}x = \sum_{i=1}^{\infty} \dot{q}_{i0} \int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x = \dot{q}_{i0} \tag{7.156}$$

Substituting initial conditions given by Eqs. (7.155) and (7.156) into Eq. (7.152), we get free response of each mode $q_i(t)$. Substituting $q_i(t)$ into Eq. (7.148), we get the complete solution for free response with the initial conditions.

Modal Analysis for Forced Response

Substituting Eq. (7.148) into Eq. (7.141), we get

$$\sum_{i=1}^{\infty} \widetilde{W}_i(x)\ddot{q}_i(t) - c^2 \sum_{i=1}^{\infty} \widetilde{W}_i''(x)q_i(t) = \overline{f}_0 \sin \omega t \tag{7.157}$$

Multiplying Eq. (7.157) by $\widetilde{W}_j$ and integrating the left side twice over the domain length L, we get

$$\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)\int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x - c^2 q_i(t)\int_0^L \widetilde{W}_i''(x)\widetilde{W}_j(x)\mathrm{d}x\right] = \overline{f}_0 \sin \omega t\left[\int_0^L \widetilde{W}_j(x)\mathrm{d}x\right] \tag{7.158}$$

Substituting $\widetilde{W}_i''(x) = -(\omega_i/c)^2\widetilde{W}_i(x)$ and using orthogonality condition, Eq. (7.158) reduces for the ith mode as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = F_i \sin \omega t \tag{7.159}$$

where

$$F_i = \overline{f}_0\left[\int_0^L \widetilde{W}_j(x)\mathrm{d}x\right] \tag{7.160}$$

Then the forced harmonic response for the ith mode is given by

$$Q_i = \frac{F_i}{\omega_i^2 - \omega^2} \sin \omega t \tag{7.161}$$

Then steady state response of the system is determined by using Eq. (7.148) as

$$W(x,t) = \sum_{i=1}^{\infty} \widetilde{W}_i(x) Q_i(t) \tag{7.162}$$

If the given force f_0 is transient, then the convolution integral should be used to determine the forced response $Q_i(t)$ of each mode.

7.6.2 Modal Analysis for Vibration Analysis of a Beam

To explain the modal analysis for the beam, we consider the transverse vibration of a beam governed by the equation of motion

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2} = \frac{f_0}{EI}\sin\omega t = \overline{f_0}\sin\omega t \tag{7.163}$$

Modal Analysis for Free Response

Substituting $f_0 = 0$ into Eq. (7.164) the governing equation for free vibration of the beam becomes,

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2} = 0 \tag{7.164}$$

The initial conditions to which the system is subjected are given as

$$w(x,0) = w_0 \text{ and } \dot{w}(x,0) = \dot{w}_0 \tag{7.165}$$

Then for the modal analysis, we can assume solution of Eq. (7.164) as

$$w(x,t) = \sum_{i=1}^{\infty} \widetilde{W}_i(x) q_i(t) \tag{7.166}$$

where $\widetilde{W}_i(x)$ is the normalized eigen-function for the given beam.

Substituting Eq. (7.166) into Eq. (7.164), we get

$$\sum_{i=1}^{\infty} \widetilde{W}_i^{iv}(x) q_i(t) + \frac{\rho A}{EI}\sum_{i=1}^{\infty} \widetilde{W}_i(x)\ddot{q}_i(t) = 0 \tag{7.167}$$

Multiplying Eq. (7.149) by $\widetilde{W}_j$ and integrating the left side twice over the domain length L, we get

$$\sum_{i=1}^{\infty}\left[q_i(t)\int_0^L \widetilde{W}_i^{iv}(x)\widetilde{W}_j(x)\mathrm{d}x + \frac{\rho A}{EI}\ddot{q}_i(t)\int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x\right] = 0 \tag{7.168}$$

Substituting $\widetilde{W}_i^{iv} = -(\beta_i)^4\widetilde{W}_i(x)$ and using orthogonality condition, Eq. (7.168) reduces for the ith mode as

$$\frac{\rho A}{EI}\ddot{q}_i(t) + (\beta_i)^4 q_i(t) = 0 \tag{7.169}$$

Substituting $(\beta_i)^4 = \omega_i^2\rho A/EI$, Eq. (7.169), we get

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \tag{7.170}$$

Then the general solution for free response of the ith mode can be determined by following the same procedure as explained for wave equation.

Modal Analysis for Forced Response

Substituting Eq. (7.166) into Eq. (7.163), we get

$$\sum_{i=1}^{\infty}\widetilde{W}_i^{iv}(x)q_i(t) + \frac{\rho A}{EI}\sum_{i=1}^{\infty}\widetilde{W}_i(x)\ddot{q}_i(t) = \overline{f}_0\sin\omega t \tag{7.171}$$

Multiplying Eq. (7.171) by $\widetilde{W}_j$ and integrating the left side twice over the domain length L, we get

$$\sum_{i=1}^{\infty}\left[q_i(t)\int_0^L \widetilde{W}_i^{iv}\widetilde{W}_j(x)\mathrm{d}x + \frac{\rho A}{EI}q_i(t)\int_0^L \widetilde{W}_i(x)\widetilde{W}_j(x)\mathrm{d}x\right]$$
$$= \overline{f}_0\sin\omega t\left[\int_0^L \widetilde{W}_j(x)\mathrm{d}x\right] \tag{7.172}$$

Substituting $\widetilde{W}_i^{iv} = -(\beta_i)^4\widetilde{W}_i(x)$ and using orthogonality condition, Eq. (7.172) reduces for the ith mode as

$$\frac{\rho A}{EI}\ddot{q}_i(t) + (\beta_i)^4 q_i(t) = F_i\sin\omega t \tag{7.173}$$

where

$$F_i = \overline{f}_0\left[\int_0^L \widetilde{W}_j(x)\mathrm{d}x\right] \tag{7.174}$$

Substituting $(\beta_i)^4 = \omega_i^2 \rho A/EI$, Eq. (7.173), we get

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \tag{7.175}$$

Then the steady state solution for forced harmonic response of the ith mode can be determined by following the same procedure as explained for wave equation.

Solved Examples

Example 7.1

A string of length L and mass per unit length ρ is stretched under a tension T. Its left end is fixed, and the right end is connected to a pin, which can move vertically in a frictionless slot. Determine the natural frequencies and corresponding mode shapes for the transverse vibration of the string.

Solution

Boundary conditions for the given string can be defined as

$$w(0, t) = 0 \tag{a}$$

and

$$w'(L, t) = 0 \tag{b}$$

The general solution for the transverse vibration of the string is given by

$$w(x, t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{c}$$

where $c = \sqrt{T/\rho}$.

Substituting the first boundary condition defined by Eq. (**a**) into Eq. (**c**), we get

$$\overline{B} = 0 \tag{d}$$

Differentiating Eq. (**c**) with respect to x and substituting the second boundary condition defined by Eq. (**b**) and also substituting $\overline{B} = 0$, we get

$$\cos\left(\frac{\omega}{c}L\right) = 0 \tag{e}$$

Equation (**e**) is satisfied for multiple values of ω. Hence the roots are given by

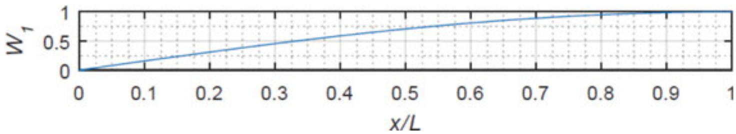
$$\frac{\omega_n}{c}L = \left(n - \frac{1}{2}\right)\pi$$

$$\therefore \quad \omega_n = \left(n - \frac{1}{2}\right)\frac{\pi c}{L} \quad \text{where } n = 1, 2, 3, \ldots \tag{f}$$

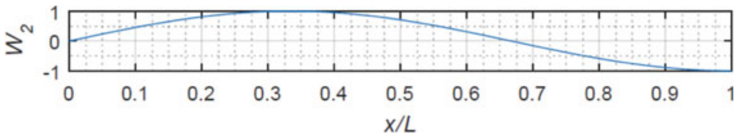
Substituting ω_n from Eq. (**f**) into spatial function of Eq. (**c**), we get the expression for the mode shape as

$$W_n(x) = \sin\left(\frac{\omega_n}{c}x\right) = \sin\left\{\left(n - \frac{1}{2}\right)\frac{\pi}{L}x\right\} \quad \text{where } n = 1, 2, 3, \ldots \tag{g}$$

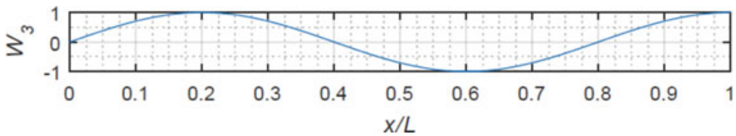
Mode shapes for the first four modes of a string fixed at the left end and pinned at the right end are shown in **Figure E7.1**.



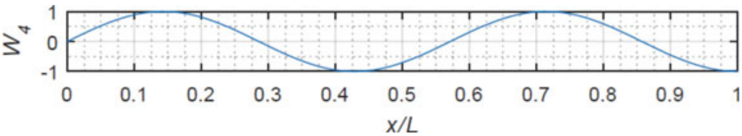
(a) First mode



(b) Second mode



(c) Third mode



(d) Fourth mode

Figure E7.1 First four mode shapes of a string fixed at the left end and pinned at the right end

Example 7.2

A string of length L and mass per unit length ρ is stretched under a tension T. It is fixed at the left end and is attached to a concentrated mass and a spring at its right end as shown in Figure E7.2. Derive an expression for the natural

frequencies of the system. Also derive the expression for orthogonality condition for the system.

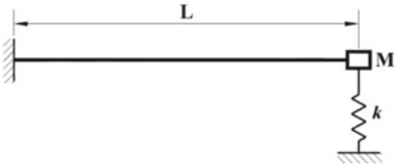


Figure E7.2

Solution

Let $w(x, t)$ be the transverse deformation of the string. Then kinetic energy of the string due to the transverse deformation is given by

$$T_s = \frac{1}{2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the concentrated mass is given by

$$T_M = \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_s + T_M = \frac{1}{2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\bigg|_{x=L}$$

The potential energy of the string is given by

$$V_s = \frac{1}{2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x$$

Similarly, the potential energy of the spring is given by

$$V_k = \frac{1}{2}k\,(w)^2\big|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$V = V_s + V_k = \frac{1}{2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}k\,(w)^2\Big|_{x=L}$$

Then, the Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\Bigg|_{x=L}$$
$$- \frac{1}{2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x - \frac{1}{2}k\,(w)^2\Big|_{x=L}$$

Now, applying Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\ \mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{M\left(\frac{\partial w}{\partial t}\right)^2\Bigg|_{x=L}\right\}\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\ \mathrm{d}t$$
$$- \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(w)^2\Big|_{x=L}\right\}\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2}\left\{M\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\Bigg|_{x=L}\right\}\mathrm{d}t$$
$$- \int_{t_1}^{t_2}\int_0^L T\left(\frac{\partial w}{\partial x}\right)\delta\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x\ \mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,w\delta(w)|_{x=L}\right\}\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho\left(\frac{\partial w}{\partial t}\right)\delta(w)\Bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\ \mathrm{d}t + \left\{M\left(\frac{\partial w}{\partial t}\right)\delta(w)\Bigg|_{x=L}\right\}_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\Bigg|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2} T\left(\frac{\partial w}{\partial x}\right)\delta(w)\Bigg|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{T\left(\frac{\partial w}{\partial x}\right)\right\}\delta(w)\mathrm{d}x\ \mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,w\delta(w)|_{x=L}\right\}\mathrm{d}t = 0$$

Since $\delta(w)|_{t_1}^{t_2} = 0$,

$$\text{or,}\ -\int_{t_1}^{t_2}\int_0^L \rho\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} T\left(\frac{\partial w}{\partial x}\right)\delta(w)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{T\left(\frac{\partial w}{\partial x}\right)\right\}\delta(w)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\ \int_{t_1}^{t_2}\int_0^L\left[\rho\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{T\left(\frac{\partial w}{\partial x}\right)\right\}\right]\delta(w)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right) + T\left(\frac{\partial w}{\partial x}\right) + kw\right\}\delta(w)\bigg|_{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left\{T\left(\frac{\partial w}{\partial x}\right)\right\}\delta(w)\bigg|_{x=0}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{T\left(\frac{\partial w}{\partial x}\right)\right\} = 0$$

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$c^2\left(\frac{\partial^2 w}{\partial t^2}\right) - \left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{a}$$

where $c = \sqrt{T/\rho}$.

The associated boundary conditions are

$$w = 0 \text{ at } x = 0. \tag{b}$$

$$M\left(\frac{\partial^2 w}{\partial t^2}\right) + T\left(\frac{\partial w}{\partial x}\right) + kw = 0 \text{ at } x = L. \tag{c}$$

The general solution for the transverse vibration of the string is given by

$$w(x,t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{d}$$

Substituting the first boundary condition defined by Eq. (**b**) into Eq. (**d**), we get

$$\overline{B} = 0 \tag{e}$$

Using the second boundary condition defined by Eq. (**c**) and also substituting $\overline{B} = 0$, we get frequency equation of the system as

$$\left(k - M\omega^2\right)\sin\left(\frac{\omega}{c}L\right) + \left(T\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right) = 0 \tag{f}$$

Equation (**f**) is satisfied for multiple values of ω. Hence the roots are given by

$$\begin{aligned} &\left(k - M\omega_n^2\right)\sin\left(\frac{\omega_n}{c}L\right) + \left(T\frac{\omega_n}{c}\right)\cos\left(\frac{\omega_n}{c}L\right) = 0 \\ \therefore\quad &\tan\left(\frac{\omega_n}{c}L\right) = -\frac{T\omega_n}{c\left(k - M\omega_n^2\right)} \end{aligned} \tag{g}$$

Substituting ω_n from Eq. (**g**) into spatial function of Eq. (**d**), we get the expression for the mode shape as

$$W_n(x) = \sin\left(\frac{\omega_n}{c}x\right) \tag{h}$$

With reference to Eqs. (**a**), (**b**) and (**c**), we can write governing equation and the associated boundary conditions for ith mode as

$$\frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} + \frac{\omega_i^2}{c^2}W_i = 0 \tag{i}$$

$$W_i(0) = 0 \tag{j}$$

$$\left(\frac{k - M\omega_i^2}{T}\right)W_i(L) + \frac{\mathrm{d}W_i}{\mathrm{d}x}(L) = 0 \tag{k}$$

Multiplying Eq. (**i**) by W_j and integrating the left side over the domain length L, we get

$$\int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2}W_j\mathrm{d}x + \frac{\omega_i^2}{c^2}\int_0^L W_i W_j\mathrm{d}x = 0 \tag{l}$$

Performing integration by part of the first term of Eq. (**l**), we get

$$\frac{\mathrm{d}W_i}{\mathrm{d}x}(L)W_j(L) - \frac{\mathrm{d}W_i}{\mathrm{d}x}(0)W_j(0) - \int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x + \frac{\omega_i^2}{c^2}\int_0^L W_i W_j \mathrm{d}x = 0 \quad \text{(m)}$$

Substituting Eqs. (**j**) and (**k**) into Eq. (**m**), we get

$$\int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x - \frac{\omega_i^2}{c^2}\int_0^L W_i W_j \mathrm{d}x + \left(\frac{k - M\omega_i^2}{T}\right)W_i(L)W_j(L) = 0 \quad \text{(n)}$$

Again starting with the governing equation for jth mode and multiplying by W_i and integrating over the domain length L, we get after simplification

$$\int_0^L \frac{\mathrm{d}W_i}{\mathrm{d}x}\frac{\mathrm{d}W_j}{\mathrm{d}x}\mathrm{d}x - \frac{\omega_j^2}{c^2}\int_0^L W_i W_j \mathrm{d}x + \left(\frac{k - M\omega_j^2}{T}\right)W_i(L)W_j(L) = 0 \quad \text{(o)}$$

Subtracting Eq. (**o**) from Eq. (**n**), we get the orthogonality condition as

$$\frac{1}{c^2}\left(\omega_i^2 - \omega_j^2\right)\int_0^L W_i W_j \mathrm{d}x + \frac{M}{T}\left(\omega_i^2 - \omega_j^2\right)W_i(L)W_j(L) = 0 \quad \text{(p)}$$

If $\omega_i \neq \omega_j$, Eq. (**p**) can also be expressed as

$$\frac{1}{c^2}\int_0^L W_i W_j \mathrm{d}x + \frac{M}{T}W_i(L)W_j(L) = 0 \quad \text{(q)}$$

Example 7.3

Determine the required tension in a transmission line of length 20 m and mass per unit length of 5 kg/m such that its fundamental natural frequency for transverse vibrations does not exceed 80 rad/s. Assume that the line is fixed at both ends.

Solution

Given, Length of the transmission line, $L = 20$ m.

Mass per unit length of the transmission line, $\rho = 5$ kg/m.

Fundamental natural frequency of transverse vibration, $\omega_1 = 80$ rad/s.

Fundamental frequency of a transmission line fixed at both ends is given by

$$\omega_1 = \frac{\pi c}{L}$$

$$\therefore \quad c = \frac{\omega_1 L}{\pi} = \frac{80 \times 20}{\pi} = 509.2958 \text{ m/s}$$

Then the required tension T can be determined as

$$T = \rho c^2 = 5 \times (509.2958)^2 = \mathbf{12.732\ kN}$$

Example 7.4

A string is initial deflected as shown in Figure E7.4 and released. Determine the resulting response of the system.

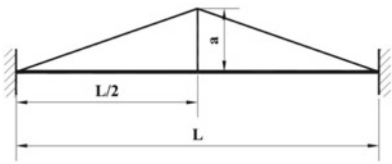


Figure E7.4

Solution

Referring to Eq. (7.33), the general response of the string fixed at both ends and undergoing transverse vibration can be expressed as

$$w(x,t) = \sum_{i=1}^{n} \sin\left(\frac{\omega_n}{c}x\right)[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)] \tag{a}$$

where $\omega_n = n\pi c/L$ and $c = \sqrt{T/\rho}$.

Initial conditions for the string shown in **Figure E7.4** can be defined as

$$w(x,0) = w_0(x) = \begin{cases} \frac{2ax}{L} & 0 \le x \le \frac{L}{2} \\ 2a\left(1 - \frac{x}{L}\right) & \frac{L}{2} \le x \le L \end{cases} \tag{b}$$

and

$$\dot{w}(x,0) = \dot{w}_0(x) = 0 \tag{c}$$

Substituting the initial condition given by Eq. (**b**) into Eq. (**a**) and using $\omega_n = n\pi c/L$, we get

$$\sum_{i=1}^{n} D_n \sin\left(\frac{n\pi}{L}x\right) = w_0(x) \tag{d}$$

Multiplying both sides of Eq. (**d**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$
\begin{aligned}
&D_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L w_0(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\
&\text{or, } D_n \int_0^L \frac{1}{2}\left[1-\cos\left(\frac{2n\pi}{L}x\right)\right]\mathrm{d}x = \int_0^{L/2} \frac{2ax}{L}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\
&\quad + \int_{L/2}^{L} 2a\left(1-\frac{x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\
&\text{or, } D_n\left(\frac{L}{2}\right) = \frac{aL}{n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - n\pi\cos\left(\frac{n\pi}{2}\right)\right] \\
&\quad + \frac{aL}{n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) + n\pi\cos\left(\frac{n\pi}{2}\right)\right] \\
&\text{or, } D_n = \frac{8a}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \\
&\therefore\ D_n = (-1)^{n+1}\frac{8a}{(2n+1)^2\pi^2} \qquad \text{(e)}
\end{aligned}
$$

Differentiating Eq. (**a**) with respect to t and substituting the initial condition given by Eq. (**c**), we get

$$
\sum_{i=1}^{n} \omega_n C_n \sin\left(\frac{n\pi}{L}x\right) = \dot{w}_0(x) \qquad \text{(f)}
$$

Multiplying both sides of Eq. (**f**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$
\begin{aligned}
&\omega_n C_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L \dot{w}_0(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0 \\
&\therefore\ C_n = 0 \qquad \text{(g)}
\end{aligned}
$$

Substituting C_n and D_n from Eqs. (**g**) and (**f**) into Eq. (**a**), we get response of the given string as

$$
w(x,t) = \sum_{i=1}^{n} (-1)^{n+1}\frac{8a}{(2n+1)^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t) \qquad \text{(h)}
$$

Example 7.5

A string of length L fixed at both ends is subjected to initial disturbance at $t = 0$ in the form of velocity distribution as shown in Figure E7.5. If the initial displacement distribution is zero, determine the resulting free vibration response of the string.

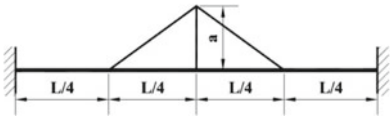


Figure E7.5

Solution

Referring to Eq. (7.33), the general response of the string fixed at both ends and undergoing transverse vibration can be expressed as

$$w(x,t) = \sum_{i=1}^{n} \sin\left(\frac{\omega_n}{c}x\right)[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)] \tag{a}$$

where $\omega_n = n\pi c/L$ and $c = \sqrt{T/\rho}$.

Initial conditions for the string shown in **Figure E7.5** can be defined as

$$w(x,0) = w_0(x) = 0 \tag{b}$$

and

$$\dot{w}(x,0) = \dot{w}_0(x) = \begin{cases} 0 & 0 \le x \le \frac{L}{4} \\ a\left(\frac{4x}{L} - 1\right) & \frac{L}{4} \le x \le \frac{L}{2} \\ a\left(3 - \frac{4x}{L}\right) & \frac{L}{2} \le x \le \frac{3L}{4} \\ 0 & \frac{3L}{4} \le x \le L \end{cases} \tag{c}$$

Substituting the initial condition given by Eq. (**b**) into Eq. (**a**) and using $\omega_n = n\pi c/L$, we get

$$\sum_{i=1}^{n} D_n \sin\left(\frac{n\pi}{L}x\right) = w_0(x) \tag{d}$$

Multiplying both sides of Eq. (**d**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$D_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L w_0(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

$$\therefore \quad D_n = 0 \tag{e}$$

Differentiating Eq. (**a**) with respect to t and substituting the initial condition given by Eq. (**c**), we get

$$\sum_{i=1}^{n} \omega_n C_n \sin\left(\frac{n\pi}{L}x\right) = \dot{w}_0(x) \tag{f}$$

Multiplying both sides of Eq. (**f**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$\omega_n C_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L \dot{w}_0(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } \omega_n C_n \int_0^L \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{L}x\right)\right]\mathrm{d}x$$

$$= \int_{L/4}^{L/2} a\left(\frac{4x}{L} - 1\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \int_{L/2}^{3L/4} a\left(3 - \frac{4x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } \omega_n C_n\left(\frac{L}{2}\right) = \frac{aL}{n^2\pi^2}\left[4\sin\left(\frac{n\pi}{2}\right) - 4\sin\left(\frac{n\pi}{4}\right) - n\pi\cos\left(\frac{n\pi}{2}\right)\right]$$

$$+ \frac{aL}{n^2\pi^2}\left[4\sin\left(\frac{n\pi}{2}\right) - 4\sin\left(\frac{3n\pi}{4}\right) + n\pi\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\text{or, } C_n = \frac{4a}{\omega_n n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{3n\pi}{4}\right)\right]$$

$$\text{or, } C_n = \frac{4a}{\omega_n n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - 2\sin(n\pi)\cos\left(\frac{3n\pi}{2}\right)\right]$$

$$\therefore \quad C_n = (-1)^{n+1}\frac{8a}{\omega_n(2n+1)^2\pi^2} \tag{g}$$

Substituting C_n and D_n from Eqs. (**f**) and (**g**) into Eq. (**a**), we get response of the given string as

$$w(x,t) = \sum_{i=1}^{n}(-1)^{n+1}\frac{8a}{\omega_n(2n+1)^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\sin(\omega_n t) \tag{h}$$

Example 7.6

A concentrated mass M is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in Figure E7.6. The bar material has a density of ρ, modulus of elasticity of E and its cross-sectional area is A. Determine the natural frequencies and mode shapes of the system. Also derive the expression for orthogonality condition for the system.

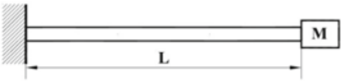


Figure E7.6

Solution

Let $u(x, t)$ be the longitudinal deformation of the continuous bar. Then kinetic energy of the bar due to longitudinal deformation is given by

$$T_b = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the concentrated mass is given by

$$T_M = \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\Bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_b + T_M = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\Bigg|_{x=L}$$

The potential energy of the bar is given by

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Then, the Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial u}{\partial t}\right)^2\Bigg|_{x=L} - \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Now, applying Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\delta \int_{t_1}^{t_2}\left\{ M\left(\frac{\partial u}{\partial t}\right)^2\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\frac{1}{2}\delta \int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\left\{ M\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta(u)\bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t + \left\{ M\left(\frac{\partial u}{\partial t}\right)\delta(u)\bigg|_{x=L}\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\left\{ M\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\bigg|_{x=L}\right\}\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\bigg|_{x=0}^{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t = 0$$

Since $\delta(u)|_{t_1}^{t_2} = 0$,

$$\text{or, } -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{ M\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \left[\rho A\left(\frac{\partial^2 u}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\right]\delta(u)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 u}{\partial t^2}\right) + EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\bigg|_{x=L}\mathrm{d}t - \int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\bigg|_{x=0}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 u}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\} = 0$$

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{a}$$

where $c = \sqrt{E/\rho}$ is the velocity of wave propagation along the bar.

The associated boundary conditions are

$$u = 0 \quad \text{at } x = 0. \tag{b}$$

and

$$M\left(\frac{\partial^2 u}{\partial t^2}\right) + EA\left(\frac{\partial u}{\partial x}\right) = 0 \quad \text{at } x = L. \tag{c}$$

The general solution for the longitudinal vibration of the bar is given by

$$u(x,t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{d}$$

Substituting the first boundary condition defined by Eq. (**b**) into Eq. (**d**), we get

$$\overline{B} = 0 \tag{e}$$

Using the second boundary condition defined by Eq. (**c**) and also substituting $\overline{B} = 0$, we get frequency equation of the system as

$$-M\omega^2 \sin\left(\frac{\omega}{c}L\right) + \left(EA\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right) = 0 \tag{f}$$

Equation (**f**) is satisfied for multiple values of ω. Hence the roots are given by

$$-M\omega_n^2 \sin\left(\frac{\omega_n}{c}L\right) + \left(EA\frac{\omega_n}{c}\right)\cos\left(\frac{\omega_n}{c}L\right) = 0$$

$$\therefore \quad \tan\left(\frac{\omega_n}{c}L\right) = \frac{EA}{Mc\omega_n} \tag{g}$$

Substituting ω_n from Eq. (**g**) into spatial function of Eq. (**d**), we get the expression for the mode shape as

$$U_n(x) = \sin\left(\frac{\omega_n}{c}x\right) \tag{h}$$

With reference to Eqs. (**a**), (**b**) and (**c**), we can write governing equation and the associated boundary conditions for *i*th mode as

$$\frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2} + \frac{\omega_i^2}{c^2}U_i = 0 \tag{i}$$

$$U_i(0) = 0 \tag{j}$$

$$-M\omega_i^2 U_i(L) + EA\frac{\mathrm{d}U_i}{\mathrm{d}x}(L) = 0 \tag{k}$$

Multiplying Eq. (**i**) by W_j and integrating the left side over the domain length L, we get

$$\int_0^L \frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2}U_j\mathrm{d}x + \frac{\omega_i^2}{c^2}\int_0^L U_i U_j \mathrm{d}x = 0 \tag{l}$$

Performing integration by part of the first term of Eq. (**l**), we get

$$\frac{\mathrm{d}U_i}{\mathrm{d}x}(L)U_j(L) - \frac{\mathrm{d}U_i}{\mathrm{d}x}(0)U_j(0) - \int_0^L \frac{\mathrm{d}U_i}{\mathrm{d}x}\frac{\mathrm{d}U_j}{\mathrm{d}x}\mathrm{d}x + \frac{\omega_i^2}{c^2}\int_0^L U_i U_j \mathrm{d}x = 0 \tag{m}$$

Substituting Eqs. (**j**) and (**k**) into Eq. (**m**), we get

$$\int_0^L \frac{\mathrm{d}U_i}{\mathrm{d}x}\frac{\mathrm{d}U_j}{\mathrm{d}x}\mathrm{d}x - \frac{\omega_i^2}{c^2}\int_0^L U_i U_j \mathrm{d}x - \frac{M}{EA}\omega_i^2 U_i(L)U_j(L) = 0 \tag{n}$$

Again starting with the governing equation for *j*th mode and multiplying by W_i and integrating over the domain length L, we get after simplification

$$\int_0^L \frac{\mathrm{d}U_i}{\mathrm{d}x}\frac{\mathrm{d}U_j}{\mathrm{d}x}\mathrm{d}x - \frac{\omega_j^2}{c^2}\int_0^L U_i U_j \mathrm{d}x - \frac{M}{EA}\omega_j^2 U_i(L)U_j(L) = 0 \tag{o}$$

Subtracting Eq. (**o**) from Eq. (**n**), we get the orthogonality condition as

$$\frac{1}{c^2}\left(\omega_i^2 - \omega_j^2\right)\int_0^L U_i U_j \mathrm{d}x + \frac{M}{EA}\left(\omega_i^2 - \omega_j^2\right)U_i(L)U_j(L) = 0 \tag{p}$$

If $\omega_i \neq \omega_j$, Eq. (**p**) can also be expressed as

$$\frac{1}{c^2}\int_0^L U_i U_j \mathrm{d}x + \frac{M}{EA}U_i(L)U_j(L) = 0 \tag{q}$$

Example 7.7

Two springs of stiffness k are attached to ends of a bar of length L undergoing longitudinal vibration as shown in Figure E7.7. The bar material has a density of ρ, modulus of elasticity of E and its cross-sectional area is A. Derive frequency equation of the system.

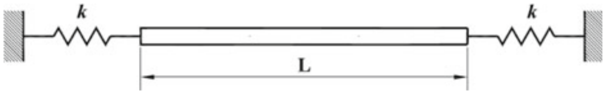


Figure E7.7

Solution

Let $u(x,t)$ be the longitudinal deformation of the continuous bar. Then kinetic energy of the bar due to longitudinal deformation is given by

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

The potential energy of the bar is given by

$$V_b = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Similarly, the potential energy of the spring is given by

$$V_S = \frac{1}{2}k\,(u)^2\big|_{x=0} + \frac{1}{2}k\,(u)^2\big|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$V = V_b + V_S = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}k\,(u)^2\big|_{x=0} + \frac{1}{2}k\,(u)^2\big|_{x=L}$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x - \frac{1}{2}k\,(u)^2\Big|_{x=0} - \frac{1}{2}k\,(u)^2\Big|_{x=L}$$

Now applying Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(u)^2\Big|_{x=0}\right\}\mathrm{d}t$$
$$- \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{k\,(u)^2\Big|_{x=L}\right\}\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,u\delta(u)|_{x=0}\right\}\mathrm{d}t$$
$$- \int_{t_1}^{t_2}\left\{k\,u\delta(u)|_{x=L}\right\}\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta(u)\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,u\delta(u)|_{x=0}\right\}\mathrm{d}t - \int_{t_1}^{t_2}\left\{k\,u\delta(u)|_{x=L}\right\}\mathrm{d}t = 0$$

Since $\delta(u)|_{t_1}^{t_2} = 0$,

$$\text{or, } -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\Big|_{x=0}^{x=L}\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t$$
$$- \int_{t_1}^{t_2}\{k\,u\delta(u)|_{x=0}\}\mathrm{d}t - \int_{t_1}^{t_2}\{k\,u\delta(u)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,} \int_{t_1}^{t_2}\int_0^L \left[\rho A\left(\frac{\partial^2 u}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\right]\delta(u)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right) + ku\right\}\delta(u)\bigg|_{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right) - ku\right\}\delta(u)\bigg|_{x=0}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 u}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\} = 0$$

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{a}$$

where $c = \sqrt{E/\rho}$ is the velocity of wave propagation along the bar.

The associated boundary conditions are

$$EA\left(\frac{\partial u}{\partial x}\right) - ku = 0 \quad \text{at } x = 0. \tag{b}$$

and

$$EA\left(\frac{\partial u}{\partial x}\right) + ku = 0 \quad \text{at } x = L. \tag{c}$$

The general solution for the longitudinal vibration of the bar is given by

$$u(x, t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{d}$$

Substituting the first boundary condition defined by Eq. (**b**) into Eq. (**d**), we get

$$\left(EA\frac{\omega}{c}\right)\overline{A} - k\overline{B} = 0$$

$$\therefore \quad \frac{\overline{A}}{\overline{B}} = \frac{kc}{EA\omega} \tag{e}$$

Using the second boundary condition defined by Eq. (**c**) and also substituting $\overline{B} = 0$, we get

$$\left(EA\frac{\omega}{c}\right)\overline{A}\cos\left(\frac{\omega}{c}L\right) - \left(EA\frac{\omega}{c}\right)\overline{B}\sin\left(\frac{\omega}{c}L\right) + k\left[\overline{A}\sin\left(\frac{\omega}{c}L\right) + \overline{B}\cos\left(\frac{\omega}{c}L\right)\right] = 0$$

$$\therefore \quad \frac{\overline{A}}{\overline{B}} = \frac{EA\omega\sin\left(\frac{\omega}{c}L\right) - kc\cos\left(\frac{\omega}{c}L\right)}{EA\omega\cos\left(\frac{\omega}{c}L\right) - kc\sin\left(\frac{\omega}{c}L\right)} \tag{f}$$

Equating Eqs. (**e**) and (**f**), we get the frequency equation of the system as

$$kc\left[EA\omega\cos\left(\frac{\omega}{c}L\right) - kc\sin\left(\frac{\omega}{c}L\right)\right] - EA\omega\left[EA\omega\sin\left(\frac{\omega}{c}L\right) - kc\cos\left(\frac{\omega}{c}L\right)\right] = 0$$

$$\therefore \quad \tan\left(\frac{\omega_n}{c}L\right) = \frac{2kcEA\omega}{(kc)^2 + (EA\omega)^2} \tag{g}$$

Example 7.8

A bar of length L is fixed at both ends as shown in Figure E7.8. The bar material has a density of ρ and its cross-sectional area is A. Determine the resulting free vibration response of the bar if it is subjected to an axial point load F_0 at the middle and removed at $t = 0$.

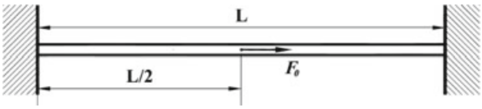


Figure E7.8

Solution

Referring to Eq. (7.33), the general response of the string fixed at both ends and undergoing transverse vibration can be expressed as

$$u(x,t) = \sum_{i=1}^{n}\sin\left(\frac{\omega_n}{c}x\right)[C_n\sin(\omega_n t) + D_n\cos(\omega_n t)] \tag{a}$$

where $\omega_n = n\pi c/L$ and $c = \sqrt{E/\rho}$.

Initial conditions for the string shown in **Figure E7.8** can be defined as

$$u(x,0) = u_0(x) = \begin{cases} \frac{F_0 x}{2AE} & 0 \le x \le \frac{L}{2} \\ \frac{F_0 L}{2AE}\left(1 - \frac{x}{L}\right) & \frac{L}{2} \le x \le L \end{cases} \tag{b}$$

and

$$\dot{u}(x,0) = \dot{u}_0(x) = 0 \tag{c}$$

Substituting the initial condition given by Eq. (**b**) into Eq. (**a**) and using $\omega_n = n\pi c/L$, we get

$$\sum_{i=1}^{n} D_n \sin\left(\frac{n\pi}{L}x\right) = u_0(x) \tag{d}$$

Multiplying both sides of Eq. (**d**) by $\sin(m\pi x/L)$ and integrating over the length of the bar L and using orthogonal properties of the eigen-function also, we get

$$D_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L u_0(x) \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } D_n \int_0^L \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{L}x\right)\right]\mathrm{d}x = \int_0^{L/2} \frac{F_0 x}{2AE} \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$+ \int_{L/2}^{L} \frac{F_0 L}{2AE}\left(1 - \frac{x}{L}\right) \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } D_n\left(\frac{L}{2}\right) = \frac{F_0 L^2}{4AEn^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - n\pi\cos\left(\frac{n\pi}{2}\right)\right]$$

$$+ \frac{F_0 L^2}{4AEn^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) + n\pi\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\text{or, } D_n = \frac{2F_0 L}{AE(2n+1)^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \quad D_n = (-1)^{n+1} \frac{2F_0 L}{AE(2n+1)^2\pi^2} \tag{e}$$

Differentiating Eq. (**a**) with respect to t and substituting the initial condition given by Eq. (**c**), we get

$$\sum_{i=1}^{n} \omega_n C_n \sin\left(\frac{n\pi}{L}x\right) = \dot{w}_0(x) \tag{f}$$

Multiplying both sides of Eq. (**f**) by $\sin(m\pi x/L)$ and integrating over the length of the bar L and using orthogonal properties of the eigen-function also, we get

$$\omega_n C_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L \dot{u}_0(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

$$\therefore \quad C_n = 0 \tag{g}$$

Substituting C_n and D_n from Eqs. (**g**) and (**f**) into Eq. (**a**), we get response of the given string as

$$w(x,t) = \sum_{i=1}^{n}(-1)^{n+1}\frac{2F_0L}{AE(2n+1)^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t) \tag{h}$$

Example 7.9

Determine the natural frequencies and corresponding mode shapes for torsional vibration of a uniform shaft of length L fixed at one end and free at the other end. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J.

Solution

The boundary conditions for the given shaft can be defined as

$$\theta(0,t) = 0 \tag{b}$$

and

$$\frac{\partial\theta}{\partial x}(L,t) = 0 \tag{c}$$

The general solution for the transverse vibration of the shaft is given by

$$\theta(x,t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{d}$$

where $c = \sqrt{T/\rho}$.

Substituting the first boundary condition defined by Eq. (**b**) into Eq. (**d**), we get

$$\overline{B} = 0 \tag{e}$$

Using the second boundary condition defined by Eq. (**c**) and also substituting $\overline{B} = 0$, we get frequency equation of the system as

$$\cos\left(\frac{\omega}{c}L\right) = 0 \tag{f}$$

Equation (**f**) is satisfied for multiple values of ω. Hence the roots are given by

$$\frac{\omega_n}{c}L = \left(n - \frac{1}{2}\right)\pi$$

$$\therefore \quad \omega_n = \left(n - \frac{1}{2}\right)\frac{\pi c}{L} \quad \text{where } n = 1, 2, 3, \ldots \tag{g}$$

Substituting ω_n from Eq. (**g**) into spatial function of Eq. (**d**), we get the expression for the mode shape as

$$\Theta_n(x) = \sin\left(\frac{\omega_n}{c}x\right) = \sin\left\{\left(n - \frac{1}{2}\right)\frac{\pi}{L}x\right\} \quad \text{where } n = 1, 2, 3, \ldots \tag{h}$$

Example 7.10

Two rigid disks with mass moment of inertia I_1 at and I_2 respectively are attached at the free ends of a shaft of length L as shown in Figure E7.10. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J. Derive frequency equation of the system.

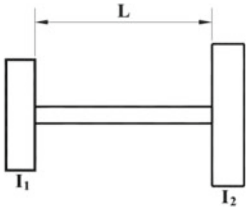


Figure E7.10

Solution

Let $\theta(x, t)$ be the torsional deformation of the continuous shaft. Then kinetic energy of the shaft due to torsional deformation is given by

$$T_b = \frac{1}{2}\int_0^L \rho J\left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energies of the rigid disks are given as

$$T_D = \frac{1}{2} I_1 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=0} + \frac{1}{2} I_2 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_b + T_M = \frac{1}{2} \int_0^L \rho J \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2} I_1 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=0} + \frac{1}{2} I_2 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L}$$

The potential energy of the shaft is given by

$$V = \frac{1}{2} \int_0^L GJ \left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x$$

Then, the Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2} \int_0^L \rho J \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2} I_1 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=0} + \frac{1}{2} I_2 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L} - \frac{1}{2} \int_0^L GJ \left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x$$

Now, applying Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2} \delta \int_{t_1}^{t_2} \int_0^L \rho J \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x \, \mathrm{d}t + \frac{1}{2} \delta \int_{t_1}^{t_2} \left\{ I_1 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=0} \right\} \mathrm{d}t + \frac{1}{2} \delta \int_{t_1}^{t_2} \left\{ I_2 \left(\frac{\partial \theta}{\partial t}\right)^2 \bigg|_{x=L} \right\} \mathrm{d}t - \frac{1}{2} \delta \int_{t_1}^{t_2} \int_0^L GJ \left(\frac{\partial \theta}{\partial x}\right)^2 \mathrm{d}x \, \mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2} \int_0^L \rho J \left(\frac{\partial \theta}{\partial t}\right) \delta \left(\frac{\partial \theta}{\partial t}\right) \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2} \left\{ I_1 \left(\frac{\partial \theta}{\partial t}\right) \delta \left(\frac{\partial \theta}{\partial t}\right) \bigg|_{x=0} \right\} \mathrm{d}t + \int_{t_1}^{t_2} \left\{ I_2 \left(\frac{\partial \theta}{\partial t}\right) \delta \left(\frac{\partial \theta}{\partial t}\right) \bigg|_{x=L} \right\} \mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L GJ\left(\frac{\partial\theta}{\partial x}\right)\delta\left(\frac{\partial\theta}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho J\left(\frac{\partial\theta}{\partial t}\right)\delta(\theta)\bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho J\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\mathrm{d}x\,\mathrm{d}t + \left\{I_1\left(\frac{\partial\theta}{\partial t}\right)\delta(\theta)\bigg|_{x=0}\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\left\{I_1\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=0}\right\}\mathrm{d}t + \left\{I_2\left(\frac{\partial\theta}{\partial t}\right)\delta(\theta)\bigg|_{x=L}\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\left\{I_2\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} GJ\left(\frac{\partial\theta}{\partial x}\right)\delta(\theta)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \frac{d}{dx}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\right\}\delta(\theta)\mathrm{d}x\,\mathrm{d}t = 0$$

Since $\delta(\theta)|_{t_1}^{t_2} = 0$,

$$\text{or, } -\int_{t_1}^{t_2}\int_0^L \rho J\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{I_1\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=0}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left\{I_2\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta(\theta)\bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2} GJ\left(\frac{\partial\theta}{\partial x}\right)\delta(\theta)\bigg|_{x=0}^{x=L}\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\right\}\delta(\theta)\mathrm{d}x\,\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L\left[\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\right\}\right]\delta(u)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{I_2\left(\frac{\partial^2\theta}{\partial t^2}\right) + GJ\left(\frac{\partial\theta}{\partial x}\right)\right\}\delta(\theta)\bigg|_{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{I_1\left(\frac{\partial^2\theta}{\partial t^2}\right) - GJ\left(\frac{\partial\theta}{\partial x}\right)\right\}\delta(\theta)\bigg|_{x=0}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{GJ\left(\frac{\partial\theta}{\partial x}\right)\right\} = 0$$

Since the shaft is uniform and homogeneous, equation of motion can also be expressed as

$$\left(\frac{\partial^2\theta}{\partial t^2}\right) - c^2\left(\frac{\partial^2\theta}{\partial x^2}\right) = 0 \tag{a}$$

where $c = \sqrt{G/\rho}$ is the velocity of wave propagation along the shaft.

The associated boundary conditions are

$$I_1\left(\frac{\partial^2\theta}{\partial t^2}\right) - GJ\left(\frac{\partial\theta}{\partial x}\right) = 0 \quad \text{at } x = 0 \tag{b}$$

and

$$I_2\left(\frac{\partial^2\theta}{\partial t^2}\right) + GJ\left(\frac{\partial\theta}{\partial x}\right) = 0 \quad \text{at } x = L. \tag{c}$$

The general solution for the torsional vibration of a shaft is given by

$$\theta(x,t) = \left[\overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right)\right]\left[\overline{C}\sin(\omega t) + \overline{D}\cos(\omega t)\right] \tag{d}$$

Substituting the first boundary condition defined by Eq. (**b**) into Eq. (**d**), we get

$$\omega^2 I_1\overline{B} + GJ\left(\frac{\omega}{c}\right)\overline{A} = 0$$

$$\therefore \quad \frac{\overline{A}}{\overline{B}} = -\frac{\omega c I_1}{GJ} \tag{e}$$

Using the second boundary condition defined by Eq. (**c**), we get

$$\begin{aligned}
&-\omega^2 I_2\left[\overline{A}\sin\left(\frac{\omega}{c}L\right) + \overline{B}\cos\left(\frac{\omega}{c}L\right)\right] \\
&\quad + GJ\left(\frac{\omega}{c}\right)\left[\overline{A}\cos\left(\frac{\omega}{c}L\right) - \overline{B}\sin\left(\frac{\omega}{c}L\right)\right] = 0 \\
\text{or, } &\left[-\omega^2 I_2\sin\left(\frac{\omega}{c}L\right) + GJ\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right)\right]\overline{A} \\
&\quad - \left[\omega^2 I_2\cos\left(\frac{\omega}{c}L\right) + GJ\left(\frac{\omega}{c}\right)\sin\left(\frac{\omega}{c}L\right)\right]\overline{B} = 0 \\
\therefore \quad &\frac{\overline{A}}{\overline{B}} = \frac{\omega^2 I_2\cos\left(\frac{\omega}{c}L\right) + GJ\left(\frac{\omega}{c}\right)\sin\left(\frac{\omega}{c}L\right)}{-\omega^2 I_2\sin\left(\frac{\omega}{c}L\right) + GJ\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right)} = \frac{\omega c I_2\cos\left(\frac{\omega}{c}L\right) + GJ\sin\left(\frac{\omega}{c}L\right)}{-\omega c I_2\sin\left(\frac{\omega}{c}L\right) + GJ\cos\left(\frac{\omega}{c}L\right)}
\end{aligned} \tag{f}$$

Equating Eqs. (**e**) and (**f**), we get the frequency equation of the system as

$$-\frac{\omega c I_1}{GJ} = \frac{\omega c I_2 \cos\left(\frac{\omega}{c}L\right) + GJ\sin\left(\frac{\omega}{c}L\right)}{-\omega c I_2 \sin\left(\frac{\omega}{c}L\right) + GJ\cos\left(\frac{\omega}{c}L\right)}$$

$$\text{or, } \omega^2 c^2 I_1 I_2 \sin\left(\frac{\omega}{c}L\right) - GJc\omega I_1 \cos\left(\frac{\omega}{c}L\right) = GJc\omega I_2 \cos\left(\frac{\omega}{c}L\right) + G^2J^2 \sin\left(\frac{\omega}{c}L\right) = 0$$

$$\text{or, } \left[\omega^2 c^2 I_1 I_2 - G^2 J^2\right]\tan\left(\frac{\omega}{c}L\right)$$

$$= GJ\omega c(I_1 + I_2)$$

$$\therefore \quad \tan\left(\frac{\omega_n}{c}L\right) = \frac{GJc\omega_n(I_1 + I_2)}{\omega_n^2 c^2 I_1 I_2 - G^2 J^2} \tag{g}$$

Example 7.11

A concentrated rigid disk of mass M and mass moment of inertia of I_D are attached at the free end of a shaft of length L and diameter d_s as shown in Figure E7.11. The shaft material has a density of ρ, modulus of elasticity of E and shear modulus of elasticity of G. Determine the fundamental frequencies for longitudinal and torsional vibration of the system. Take M= 120 kg, I_D= 15 kg m^2, L= 1.2 m, d_s= 8 cm, ρ= 7850 kg/m^3, E= 210 GPa and G= 84 GPa.

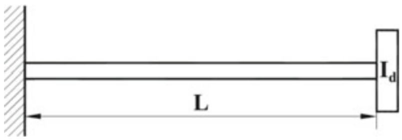


Figure E7.11

Solution

Given, Mass of the disk, $M = 120$ kg.

Mass moment of inertia of the disk, $I_D = 15$ kg m^2.

Length of the shaft, $L = 1.2$ m.

Diameter of the shaft, $d_s = 8$ cm.

Density of the shaft material, $\rho = 7850$ kg/m^3.

Modulus of elasticity of the shaft material, $E = 210$ GPa.

Shear modulus of elasticity of the shaft material, $G = 84$ GPa.

Then the cross-sectional area and polar moment of inertia of section of the shaft can be determined as

$$A = \frac{\pi}{4}d_s^2 = \frac{\pi}{4}\left(8\times 10^{-2}\right)^2 = 0.005027\ \text{m}^2$$

$$J_s = \frac{\pi}{32}d_s^4 = \frac{\pi}{32}\left(8\times 10^{-2}\right)^4 = 0.000004\ \text{m}^4$$

Then the frequency equation for the longitudinal vibration (refer Example 7.6) can be expressed as

$$\tan\left(\frac{\omega_n}{c}L\right) = \frac{EA}{Mc\omega_n} \tag{a}$$

where

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{210 \times 10^3}{7850}} = 5172.1942 \text{ m/s}$$

Substituting given parameters into Eq. (**a**), we get frequency equation as

$$\tan(0.000232\omega_1) = \frac{1700.7210}{\omega_1}$$

Roots of such transcendental equation can be determined by using numerical methods or by using software package. In this case, fundamental frequency of longitudinal vibration is given by the lowest root of the frequency equation as

$$\omega_1 = 2541.7115 \text{ rad/s}$$

Using analogous parameters, the frequency equation for the torsional vibration can be expressed as

$$\tan\left(\frac{\omega_n}{c}L\right) = \frac{GJ_s}{I_D c\omega_n} \tag{b}$$

where

$$c = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{84 \times 10^3}{7850}} = 3271.1828 \text{ m/s}$$

Substituting given parameters into Eq. (**b**), we get frequency equation as

$$\tan(0.0003668\omega_1) = \frac{6.8840}{\omega_1}$$

Then the fundamental frequency of torsional vibration is given by the lowest root of the frequency equation as

$$\omega_1 = 136.9306 \text{ rad/s}$$

Example 7.12

A shaft of length *L* is fixed at one end and is free at the other end. A moment *M* is statically applied to the free end of the shaft such that the angle of twist varies linearly over the length of the shaft. If the moment is suddenly removed,

determine the resulting free torsional vibration. The shaft material has a density of ρ, modulus of elasticity of E and shear modulus of elasticity of G.

Solution

Referring to Example 7.9, the general response for torsional vibration of a shaft fixed at one end and free at the other end can be expressed as

$$\theta(x,t) = \sum_{i=1}^{n} \sin\left(\frac{\omega_n}{c}x\right)[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)] \tag{a}$$

where $\omega_n = (2n-1)\pi c/2L$ and $c = \sqrt{G/\rho}$.

Angular displacement of the shaft due to twisting moment applied at its free end is given by

$$\theta(x) = \frac{M}{GJ}x \tag{b}$$

Then we can define the initial conditions for the shaft as

$$\theta(x,0) = \theta_0(x) = \frac{M}{GJ}x \tag{c}$$

and

$$\dot{\theta}(x,0) = \dot{\theta}_0(x) = 0 \tag{d}$$

Substituting the initial condition given by Eq. (**c**) into Eq. (**a**), we get

$$\sum_{i=1}^{n} D_n \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\} = \theta_0(x) \tag{e}$$

Multiplying both sides of Eq. (**d**) by $\sin\{(2m-1)\pi x/2L\}$ and integrating over the length of the bar L and using orthogonal properties of the eigen-function also, we get

$$D_n \int_0^L \left[\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\}\right]^2 \mathrm{d}x = \int_0^L \theta_0(x) \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\}\mathrm{d}x$$

$$\text{or, } D_n \int_0^L \frac{1}{2}\left[1 - \cos\left\{(2n-1)\frac{\pi x}{L}\right\}\right]\mathrm{d}x = \int_0^L \frac{M}{GJ}x \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\}\mathrm{d}x$$

$$\text{or, } D_n\left(\frac{L}{2}\right) = \frac{2ML^2}{\pi^2 GJ(2n-1)^2}\cos(n\pi)$$

$$\therefore \quad D_n = (-1)^{n+1} \frac{4ML}{(2n-1)^2 \pi^2 GJ} \tag{f}$$

Differentiating Eq. (**a**) with respect to t and substituting the initial condition given by Eq. (**d**), we get

$$\sum_{i=1}^{n} \omega_n C_n \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\} = \dot{\theta}_0(x) \tag{g}$$

Multiplying both sides of Eq. (**g**) by $\sin\{(2m-1)\pi x/2L\}$ and integrating over the length of the bar L and using orthogonal properties of the eigen-function also, we get

$$\omega_n C_n \int_0^L \left[\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\}\right]^2 \mathrm{d}x = \int_0^L \dot{\theta}_0(x) \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

$$\therefore \quad C_n = 0 \tag{h}$$

Substituting C_n and D_n from Eqs. (**f**) and (**h**) into Eq. (**a**), we get response of the given string as

$$\theta(x,t) = \sum_{i=1}^{n} (-1)^{n+1} \frac{4ML}{(2n-1)^2\pi^2 GJ} \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi x}{L}\right\} \cos\left\{\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}t\right\} \tag{i}$$

Example 7.13

Derive the frequency equation for a beam hinged at one end and free at the other end.

Solution

The general solution for the mode shape of beam transverse vibration is given by

$$W(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x \tag{a}$$

where $\beta = \sqrt[4]{\omega^2 \rho A/EI}$.

For the beam hinged at one end and free at the other end, boundary conditions can be defined as

$$W(0) = 0; \quad W''(0) = 0 \tag{b}$$

and

$$W''(L) = 0; \quad W'''(L) = 0 \tag{c}$$

Substituting first boundary condition defined by Eq. (**b**) into Eq. (**a**), we get

$$B + D = 0 \tag{d}$$

Substituting second boundary condition defined by Eq. (**b**) into Eq. (**a**), we get

$$\beta^2(-B + D) = 0 \tag{e}$$

Solving Eqs. (**d**) and (**e**), we get

$$B = 0; \quad D = 0 \tag{f}$$

Substituting $B = 0$ and $D = 0$ into Eq. (**a**), we get

$$W(x) = A \sin \beta x + C \sinh \beta x \tag{g}$$

Substituting first boundary condition defined by Eq. (**b**) into Eq. (**a**), we get

$$\begin{aligned} &\beta^2(-A \sin \beta L + C \sinh \beta L) = 0 \\ \therefore \quad &\frac{A}{C} = \frac{\sinh \beta L}{\sin \beta L} \end{aligned} \tag{h}$$

Substituting second boundary condition defined by Eq. (**b**) into Eq. (**a**), we get

$$\begin{aligned} &\beta^3(-A \cos \beta L + C \cosh \beta L) = 0 \\ \therefore \quad &\frac{A}{C} = \frac{\cosh \beta L}{\cos \beta L} \end{aligned} \tag{i}$$

Equating (**h**) and (**i**), we get

$$\begin{aligned} \frac{\sinh \beta L}{\sin \beta L} &= \frac{\cosh \beta L}{\cos \beta L} \\ \frac{\sin \beta L}{\cos \beta L} &= \frac{\sinh \beta L}{\cosh \beta L} \\ \therefore \quad \tan \beta L &= \tanh \beta L \end{aligned} \tag{j}$$

which is the required frequency equation.

Example 7.14

A beam of length L shown in Figure E7.14 undergoing traverse vibration is restrained by a spring of stiffness k. The beam material has a density of ρ,

modulus of elasticity of E and moment of inertia of section of I. Determine an expression for the frequency equation of the system.

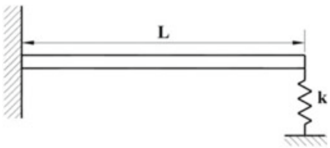


Figure E7.14

Solution

Let $w(x, t)$ be the transverse deformation of the beam due to bending about y-axis. Then kinetic energy of the beam due to bending is given by

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

The strain energy of the beam is given by

$$V_b = \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Similarly, the potential energy of the springs is given by

$$V_S = \frac{1}{2}k\,(w)^2\big|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$V = V_b + V_S = \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}k\,(w)^2\big|_{x=L}$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x - \frac{1}{2}k\,(w)^2\big|_{x=L}$$

Now applying Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta \int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \, \mathrm{d}t - \frac{1}{2}\delta \int_{t_1}^{t_2} \int_0^L E I_y \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \, \mathrm{d}t$$
$$- \frac{1}{2}\delta \int_{t_1}^{t_2} \left\{ k \, (w)^2 \big|_{x=L} \right\} \mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right) \delta \left(\frac{\partial w}{\partial t}\right) \mathrm{d}x \, \mathrm{d}t - \int_{t_1}^{t_2} \int_0^L E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \delta \left(\frac{\partial^2 w}{\partial x^2}\right) \mathrm{d}x \, \mathrm{d}t$$
$$- \int_{t_1}^{t_2} \{ k \, w \delta(w)|_{x=L} \} \mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right) \delta(w) \bigg|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial^2 w}{\partial t^2}\right) \delta(u) \mathrm{d}x \, \mathrm{d}t$$
$$- \int_{t_1}^{t_2} E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \delta \left(\frac{\partial w}{\partial x}\right) \bigg|_{x=0}^{x=L} \mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \int_0^L \frac{\mathrm{d}}{\mathrm{d}x} \left\{ E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \right\} \delta \left(\frac{\partial w}{\partial x}\right) \mathrm{d}x \, \mathrm{d}t - \int_{t_1}^{t_2} \{ k \, w \delta(w)|_{x=L} \} \mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right) \delta(w) \bigg|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial^2 w}{\partial t^2}\right) \delta(u) \mathrm{d}x \, \mathrm{d}t$$
$$- \int_{t_1}^{t_2} E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \delta \left(\frac{\partial w}{\partial x}\right) \bigg|_{x=0}^{x=L} \mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}x} \left\{ E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \right\} \delta(w) \bigg|_{x=0}^{x=L} \mathrm{d}t - \int_{t_1}^{t_2} \int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left\{ E I_y \left(\frac{\partial^2 w}{\partial x^2}\right) \right\} \delta(w) \mathrm{d}x \, \mathrm{d}t$$
$$- \int_{t_1}^{t_2} \{ k \, w \delta(w)|_{x=L} \} \mathrm{d}t = 0$$

Since $\delta(w)|_{t_1}^{t_2} = 0$,

$$\text{or,}\ -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=L}\right\}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}\right\}\mathrm{d}t + \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\bigg|_{x=L}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\bigg|_{x=0}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\{k\,w\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\ \int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]\delta(w)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=L}\right\}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=0}\right\}\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - kw\right]\delta(w)\bigg|_{x=L}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0$$

The associated boundary conditions are

(a) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0$ or $\delta(w) = 0$ at $x = 0$.

(b) either $\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} + kw = 0$ or $\delta(w) = 0$ at $x = L$.

(c) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = 0$.

(d) either $EI_y\left(\frac{\partial^2 w}{\partial x^2}\right) = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$ at $x = L$.

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + EI_y\left(\frac{\partial^4 w}{\partial x^4}\right) = 0 \tag{a}$$

Similarly, the appropriate boundary conditions for the system can be expressed as

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0. \tag{b}$$

and

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ and } EI_y\left(\frac{\partial^3 w}{\partial x^3}\right) + kw = 0 \quad \text{at } x = L. \tag{c}$$

Then, the general solution for the mode shape of beam transverse vibration is given by

$$W(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x \tag{d}$$

where $\beta = \sqrt[4]{\omega^2 \rho A / EI}$.

For the given beam, boundary conditions can be defined as

$$W(0) = 0; \quad W'(0) = 0 \tag{e}$$

and

$$W''(L) = 0; \quad EI_y W'''(L) + kW(L) = 0 \tag{f}$$

Substituting first boundary condition defined by Eq. (**e**) into Eq. (**d**), we get

$$\begin{aligned} &B + D = 0 \\ \therefore\ &D = -B \end{aligned} \tag{g}$$

Substituting second boundary condition defined by Eq. (**e**) into Eq. (**d**), we get

$$\begin{aligned} &A + C = 0 \\ \therefore\ &C = -A \end{aligned} \tag{h}$$

Substituting $C = -A$ and $D = -B$ into Eq. (**d**), we get

$$W(x) = A(\sin\beta x - \sinh\beta x) + B(\cos\beta x - \cosh\beta x) \tag{i}$$

Substituting first boundary condition defined by Eq. (**f**) into Eq. (**d**), we get

$$\begin{aligned} &A(-\sin\beta L - \sinh\beta L) + B(-\cos\beta L - \cosh\beta L) = 0 \\ \therefore\ &\frac{A}{B} = -\frac{\cos\beta L + \cosh\beta L}{\sin\beta L + \sinh\beta L} \end{aligned} \tag{j}$$

Substituting second boundary condition defined by Eq. (**f**) into Eq. (**d**), we get

$$\begin{aligned} &EI_y\beta^3[A(-\cos\beta L - \cosh\beta L) + B(\sin\beta L - \sinh\beta L)] \\ &\quad + k[A(\sin\beta L - \sinh\beta L) + B(\cos\beta L - \cosh\beta L)] = 0 \\ \text{or, } &A\left[EI_y\beta^3(-\cos\beta L - \cosh\beta L) + k(\sin\beta L - \sinh\beta L)\right] \\ &\quad + B\left[EI_y\beta^3(\sin\beta L - \sinh\beta L) + k(\cos\beta L - \cosh\beta L)\right] = 0 \\ \therefore\ &\frac{A}{B} = -\frac{EI_y\beta^3(\sin\beta L - \sinh\beta L) + k(\cos\beta L - \cosh\beta L)}{EI_y\beta^3(-\cos\beta L - \cosh\beta L) + k(\sin\beta L - \sinh\beta L)} \end{aligned} \tag{k}$$

Equating (**j**) and (**k**), we get

$$\begin{aligned} &-\frac{\cos\beta L + \cosh\beta L}{\sin\beta L + \sinh\beta L} = -\frac{EI_y\beta^3(\sin\beta L - \sinh\beta L) + k(\cos\beta L - \cosh\beta L)}{EI_y\beta^3(-\cos\beta L - \cosh\beta L) + k(\sin\beta L - \sinh\beta L)} \\ \text{or, } &EI_y\beta^3\left(\cos^2\beta L + \cosh^2\beta L + 2\cos\beta L\cosh\beta L\right) \\ &\quad - k(\cos\beta L\sin\beta L - \cos\beta L\sinh\beta L + \sin\beta L\cosh\beta L - \sinh\beta L\cosh\beta L) \\ &\quad + EI_y\beta^3\left(\sin^2\beta L - \sinh^2\beta L\right) + k(\cos\beta L\sin\beta L - \sin\beta L\cosh\beta L \\ &\quad + \cos\beta L\sinh\beta L - \sinh\beta L\cosh\beta L) = 0 \\ \therefore\ &EI_y\beta^3(1 + \cos\beta L\cosh\beta L) + k(\cos\beta L\sinh\beta L - \sin\beta L\cosh\beta L) = 0 \end{aligned} \tag{l}$$

which is the required frequency equation.

Example 7.15

A beam of length L shown in Figure E7.15 is initially at an inclined position. The beam is released from this position and adheres on the support at the right end after the impact. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the resulting vibration response of the beam assuming that there is no rebound after the impact and no loss of energy due to the impact.

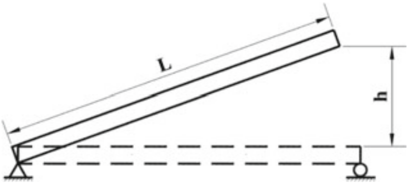


Figure E7.15

Solution

To determine the initial conditions for the problem, we should first determine the velocity of the beam with which it strikes the support. Applying energy principle for the system,

$$T_i + V_i = T_h + V_h \tag{a}$$

where T_i and V_i are the kinetic energy and potential energy of the beam when it is at an inclined position and T_h and V_h are the kinetic energy and potential energy of the beam when it is in horizontal position. Then the kinetic and potential energies for these positions can be determined as

$$\begin{aligned}
T_i &= 0 \\
V_i &= mg \times \frac{h}{2} = \frac{1}{2}mgh \\
T_h &= \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 = \frac{1}{6}mL^2\dot{\theta}^2 \\
V_h &= 0
\end{aligned}$$

Substituting these into Eq. (**a**), we get

$$0 + \frac{1}{2}mgh = \frac{1}{6}mL^2\dot{\theta}^2 + 0$$

$$\therefore \quad \dot{\theta} = \frac{\sqrt{3gh}}{L} \tag{b}$$

Then we can define the initial condition for the system as

$$w(x,0) = w_0(x) = 0 \tag{c}$$

and

$$\dot{w}(x,0) = \dot{w}_0(x) = \dot{\theta}x = \frac{\sqrt{3gh}}{L}x \tag{d}$$

Referring to Eq. (7.117), the general response of the simply supported beam undergoing transverse vibration can be expressed as

$$w(x,t) = \sum_{i=1}^{n} \sin\left(\frac{n\pi}{L}x\right)[G_n \sin(\omega_n t) + H_n \cos(\omega_n t)] \tag{e}$$

Substituting the initial condition given by Eq. (**c**) into Eq. (**e**), we get

$$\sum_{i=1}^{n} H_n \sin\left(\frac{n\pi}{L}x\right) = w_0(x) \tag{f}$$

Multiplying both sides of Eq. (**f**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$H_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L w_0(x) \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

$$\therefore \quad H_n = 0 \tag{g}$$

Differentiating Eq. (**e**) with respect to t and substituting the initial condition given by Eq. (**d**), we get

$$\sum_{i=1}^{n} \omega_n G_n \sin\left(\frac{n\pi}{L}x\right) = \dot{w}_0(x) \tag{h}$$

Multiplying both sides of Eq. (**h**) by $\sin(m\pi x/L)$ and integrating over the length of the string L and using orthogonal properties of the eigen-function also, we get

$$\omega_n G_n \int_0^L \left[\sin\left(\frac{n\pi}{L}x\right)\right]^2 \mathrm{d}x = \int_0^L \dot{w}_0(x) \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } \omega_n G_n \int_0^L \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{L}x\right)\right]\mathrm{d}x = \int_0^L \frac{\sqrt{3gh}}{L}x \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

$$\text{or, } \omega_n G_n \left(\frac{L}{2}\right) = \frac{L\sqrt{3gh}}{n^2\pi^2}[\sin(n\pi) - n\pi \cos(n\pi)]$$

$$\therefore \quad G_n = (-1)^{n-1} \frac{L\sqrt{3gh}}{\omega_n n\pi} \tag{i}$$

Substituting G_n and H_n from Eqs. (**h**) and (**i**) into Eq. (**e**), we get response of the given beam as

$$w(x,t) = \sum_{i=1}^{n} (-1)^{n-1} \frac{L\sqrt{3gh}}{\omega_n n\pi} \sin\left(\frac{n\pi}{L}x\right) \sin(\omega_n t) \tag{j}$$

Example 7.16

Determine the first three natural frequencies and the corresponding mode shapes of the transverse vibrations of a uniform beam of circular cross-section if it is used as:

(a) **a simply supported beam,**
(b) **a cantilever beam and**
(c) **a beam fixed at both ends.**

Use the following geometric and material properties of the beam: length of the beam, $L = 2$ m, radius of the section, $r = 0.3$ m, modulus of elasticity, $E = 210$ GPa and density, $\rho = 7850$ kg/m^3.

Solution

Given, Length of the beam, $L = 2$ m
Radius of the beam section, $r = 0.3$ m.
Density of the beam material, $\rho = 7850$ kg/m^3.
Modulus of elasticity of the shaft material, $E = 210$ GPa.
Then the cross-sectional area and moment of inertia of section of the beam can be determined as

$$A = \pi r^2 = \pi(0.3)^2 = 0.2827 \text{ m}^2$$
$$I = \frac{\pi}{4} r^4 = \frac{\pi}{4}(0.3)^4 = 0.0127 \text{ m}^4$$

Then natural frequencies for transverse vibration of the beam is given by

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}} = \beta_n^2 \sqrt{\frac{200 \times 10^9 \times 0.0127}{7850 \times 0.2827}} = 1097.1881 \beta_n^2 \tag{a}$$

where β_n can be determined from the frequency equation corresponding to the given end conditions.

(a) Referring to Eq. (7.114), frequency equation for a simply supported beam is given by

$$\beta_n = \frac{n\pi}{L} = \frac{n \times \pi}{2} = 1.5707n$$

For the first three modes,

$$\beta_1 = 1.5707; \quad \beta_2 = 3.1415; \quad \beta_3 = 4.7124$$

Now substituting β_1, β_2 and β_3 into Eq. (**a**), we get the first three natural frequencies as

$$\omega_1 = 1097.1881\ \beta_1^2 = 1097.1881 \times (1.5707)^2 = \mathbf{2707.2030\ rad/s}$$
$$\omega_2 = 1097.1881\ \beta_2^2 = 1097.1881 \times (3.1415)^2 = \mathbf{10{,}828.8122\ rad/s}$$
$$\omega_3 = 1097.1881\ \beta_3^2 = 1097.1881 \times (4.7124)^2 = \mathbf{24{,}364.8274\ rad/s}$$

(b) Substituting $k = 0$ into frequency equation of Example 7.14, frequency equation for a cantilever beam is given by

$$\cos \beta_n L \cosh \beta_n L + 1 = 0$$

For the first three modes,

$$\beta_1 = 0.9375; \quad \beta_2 = 2.3471; \quad \beta_3 = 3.9274$$

Now substituting β_1, β_2 and β_3 into Eq. (**a**), we get the first three natural frequencies as

$$\omega_1 = 1097.1881\ \beta_1^2 = 1097.1881 \times (0.9375)^2 = \mathbf{964.4325\ rad/s}$$
$$\omega_2 = 1097.1881\ \beta_2^2 = 1097.1881 \times (2.3471)^2 = \mathbf{6043.9953\ rad/s}$$
$$\omega_3 = 1097.1881\ \beta_3^2 = 1097.1881 \times (3.9274)^2 = \mathbf{16{,}923.3619\ rad/s}$$

(c) Now referring to the answer of **Exercise 7.35**, frequency equation for a beam fixed at both ends is given by

$$\cos \beta_n L \cosh \beta_n L = 1$$

For the first three modes,

$$\beta_1 = 2.3650; \quad \beta_2 = 3.9266; \quad \beta_3 = 5.4978$$

Now substituting β_1, β_2 and β_3 into Eq. (**a**), we get the first three natural frequencies as

$$\omega_1 = 1097.1881\ \beta_1^2 = 1097.1881 \times (2.3650)^2 = \mathbf{6136.9255\ rad/s}$$

$$\omega_2 = 1097.1881\ \beta_2^2 = 1097.1881 \times (3.9266)^2 = \mathbf{16{,}916.6714\ rad/s}$$
$$\omega_3 = 1097.1881\ \beta_3^2 = 1097.1881 \times (3.9274)^2 = \mathbf{33{,}163.4397\ rad/s}$$

Example 7.17

A rigid disk of mass moment of inertia I_d is attached to free end shaft of length L undergoing torsional vibration as shown in Figure E7.17. The shaft has a diameter of 120 mm and length of 1 m and mass moment of inertia of disk is $I_d = 40$ kg m^2. It is subjected to an external torque of $T(t) = 2500 \sin 500t$ N m. Determine steady state response for the torsional vibration of the shaft. Take $\rho = 7850$ kg/m^3 and $G = 84$ GPa.

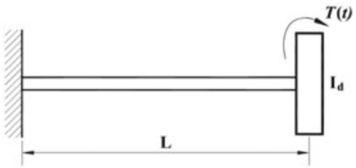


Figure E7.17

Solution

Given, Mass moment of inertia of the disk, $I_d = 40$ kg m^2.
Length of the shaft, $L = 1$ m.
Diameter of the shaft section, $d_s = 0.12$ m.
Density of the shaft material, $\rho = 7850$ kg/m^3.
Shear modulus of elasticity of the shaft material, $G = 84$ GPa.
Then the polar moment of inertia of section of the shaft can be determined as

$$J = \frac{\pi}{32} d_s^4 = \frac{\pi}{32}(0.12)^4 = 0.000020357\ \text{m}^4$$

Referring to Example 7.6 and using analogous parameters, equation of motion for the given system can be expressed as

$$\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) - GJ\left(\frac{\partial^2\theta}{\partial x^2}\right) = 0 \tag{a}$$

The associated boundary conditions are

$$\theta(0, t) = 0 \tag{b}$$

and

$$I_d\ddot{\theta}(L, t) + GJ\theta'(L, t) - T_0 \sin \omega t = 0 \tag{c}$$

Then the solution of Eq. (**a**) can be assumed as

$$\theta(x,t) = \overline{\theta}(x)\sin\omega t \tag{d}$$

Substituting Eq. (**d**) into Eq. (**a**), we get

$$\frac{\partial^2\overline{\theta}(x)}{\partial x^2} + \left(\frac{\omega}{c}\right)^2\overline{\theta}(x) = 0 \tag{e}$$

where $c = \sqrt{G/\rho}$.

Similarly, substituting Eq. (**d**) into Eqs. (**b**) and (**c**), we get

$$\overline{\theta}(0) = 0 \tag{f}$$

and

$$-\omega^2 I_d\overline{\theta}(0) + GJ\overline{\theta}'(L) - T_0 = 0 \tag{g}$$

The solution of Eq. (**d**) can be assumed as

$$\overline{\theta}(x) = \overline{A}\sin\left(\frac{\omega}{c}x\right) + \overline{B}\cos\left(\frac{\omega}{c}x\right) \tag{h}$$

Substituting the boundary condition defined by Eq. (**f**) into Eq. (**h**), we get

$$\overline{B} = 0 \tag{i}$$

Then Eq. (**f**) reduces to

$$\overline{\theta}(x) = \overline{A}\sin\left(\frac{\omega}{c}x\right) \tag{j}$$

Again, using the boundary condition defined by Eq. (**g**) into Eq. (**j**), we get

$$-\omega^2 I_d\overline{A}\sin\left(\frac{\omega}{c}L\right) + GJ\left(\frac{\omega}{c}\right)\overline{A}\cos\left(\frac{\omega}{c}L\right) - T_0 = 0$$

$$\therefore\ \ \overline{A} = \frac{T_0}{GJ\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right) - \omega^2 I_d\sin\left(\frac{\omega}{c}L\right)} \tag{k}$$

Substituting $\overline{A}$ and $\overline{B}$ into Eq. (**g**), we get

$$\overline{\theta}(x) = \frac{T_0}{GJ\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right) - \omega^2 I_d\sin\left(\frac{\omega}{c}L\right)}\sin\left(\frac{\omega}{c}x\right) \tag{l}$$

Again, substituting Eq. (**l**) into Eq. (**d**), we get response due to external torque as

$$\theta(x,t) = \frac{T_0}{GJ\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}L\right) - \omega^2 I_d \sin\left(\frac{\omega}{c}L\right)} \sin\left(\frac{\omega}{c}x\right)\sin\omega t \tag{m}$$

Substituting the values of given parameters $\rho = 7850$ kg/m^3, $G = 84$ GPa, $= 0.000020357$ m^4, $I_d = 40$ kg m^2, $L = 1$ m, $T_0 = 2500$ N m and $\omega = 500$ rad/s, we get response in terms of x and t as

$$\theta(x,t) = -0.001977\sin(0.15285x)\sin 500t \tag{n}$$

Alternative Method (Modal Analysis)

Governing equation and associated boundary condition for the given system can also be expressed as

$$\rho J\left(\frac{\partial^2\theta}{\partial t^2}\right) + I_d\left(\frac{\partial^2\theta}{\partial t^2}\right)\delta_d(x-L) - GJ\left(\frac{\partial^2\theta}{\partial x^2}\right) = T_0\delta_d(x-L)\sin\omega t \tag{o}$$

where δ_d is the Dirac delta function.

The associated boundary conditions are

$$\theta(0,t) = 0 \tag{p}$$

and

$$\theta'(L,t) = 0 \tag{q}$$

Equation (**o**) is a non-homogeneous linear differential equation, and its solution can be assumed as given by

$$\theta(x,t) = \left[\sum_i \{A_i \sin(\eta_i x) + B_i \cos(\eta_i x)\}\right]\sin\omega t \tag{r}$$

where A_i, B_i and η_i should be determined by using the given differential equation and the associated boundary conditions.

Substituting the boundary condition defined by Eq. (**p**) into Eq. (**r**), we get

$$\overline{B} = 0 \tag{s}$$

Then Eq. (**r**) reduces to

$$\theta(x,t) = \left[\sum_i A_i \sin(\eta_i x)\right]\sin\omega t \tag{t}$$

Again, using the boundary condition defined by Eq. (**q**) into Eq. (**t**), we get

$$\cos(\eta_i L) = 0$$
$$\therefore \quad \eta_i = \left(\frac{2i-1}{2}\right)\frac{\pi}{L} \tag{u}$$

Now substituting Eq. (**t**) into Eq. (**o**), we get

$$\begin{aligned} &-\omega^2 \rho J \sum_i A_i \sin(\eta_i x) - \omega^2 I_d \sum_i A_i \sin(\eta_i x)\delta_d(x-L) \\ &+ \eta_i^2 GJ \sum_i A_i \sin(\eta_i x) = T_0 \delta_d(x-L) \end{aligned} \tag{v}$$

Multiplying both sides of the Eq. (**v**) by $\sin(\eta_j x)$ and integrating over the length L of the shaft, we get

$$-\omega^2 \rho J A_i \left(\frac{L}{2}\right) - \omega^2 I_d A_i \sin^2(\eta_i L) + (\eta_i)^2 GJ A_i \left(\frac{L}{2}\right) = T_0 \sin(\eta_i L)$$
$$\begin{aligned} \therefore \quad A_i &= -\frac{2T_0 \sin(\eta_i L)}{\omega^2 \rho JL + 2\omega^2 I_d \sin^2(\eta_i L) - (\eta_i)^2 GJL} \\ &= \frac{2T_0(-1)^i}{\omega^2 \rho JL + 2\omega^2 I_d - (\eta_i)^2 GJL} \end{aligned} \tag{w}$$

Substituting Eq. (**w**) into Eq. (**t**), we get the steady state response of the system as

$$\theta(x,t) = \left[\sum_i \frac{2T_0(-1)^i}{\omega^2 \rho JL + 2\omega^2 I_d - (\eta_i)^2 GJL} \sin(\eta_i x)\right] \sin \omega t \tag{x}$$

Substituting the values of given parameters $\rho = 7850$ kg/m^3, $G = 84$ GPa, $= 0.000020357$ m^4, $I_d = 40$ kg m^2, $L = 1$ m, $T_0 = 2500$ N m and $\omega = 500$ rad/s, we get response in terms of x and t as

$$\theta(x,t) = \left[\sum_i \frac{5000(-1)^i}{2.004 \times 10^7 - (2i-1)^2 \times 4.2193 \times 10^6} \sin\{1.5708(2i-1)x\}\right] \sin 500t \tag{y}$$

Example 7.18

A uniform bar fixed at one end and free at the other end, as shown in Figure E7.18, is subjected to a uniformly distributed longitudinal load f_0 per unit length throughout its length and a concentrated axial load of F_0 at its middle. Determine its steady state response.

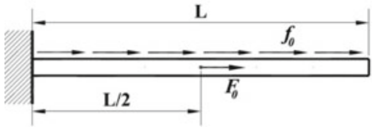


Figure E7.18

Solution

Let $u(x, t)$ be the longitudinal deformation of the continuous bar. Then kinetic energy of the bar due to longitudinal deformation is given by

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

The potential energy of the bar is given by

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Similarly, the work done by the external forces is given by

$$W_{nc} = \int_0^L f_0 u \mathrm{d}x + F_0\, u|_{x=L/2} = \int_0^L f_0 u \mathrm{d}x + \int_0^L F_0 \delta_d\left(x - \frac{L}{2}\right) u \mathrm{d}x$$

where δ_d is the Dirac delta function.

Now applying extended Hamilton's principle

$$\delta\int_{t_1}^{t_2} (L + W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\, \mathrm{d}t - \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\, \mathrm{d}t + \delta\int_{t_1}^{t_2}\int_0^L f_0 u \mathrm{d}x\, \mathrm{d}t$$

$$+\delta\int_{t_1}^{t_2}\int_0^L F_0\delta_d\left(x-\frac{L}{2}\right)u\mathrm{d}x\,\mathrm{d}t=0$$

$$\text{or, }\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta\left(\frac{\partial u}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\int_0^L f_0\delta u\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L F_0\delta_d\left(x-\frac{L}{2}\right)\delta u\mathrm{d}x\,\mathrm{d}t=0$$

$$\text{or, }\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)\delta(u)\bigg|_{t_1}^{t_2}\mathrm{d}x-\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\bigg|_{x=0}^{x=L}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\int_0^L f_0\delta u\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L F_0\delta_d\left(x-\frac{L}{2}\right)\delta u\mathrm{d}x\,\mathrm{d}t=0$$

Since $\delta(u)|_{t_1}^{t_2}=0$,

$$\text{or, }-\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 u}{\partial t^2}\right)\delta(u)\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2} EA\left(\frac{\partial u}{\partial x}\right)\delta(u)\bigg|_{x=0}^{x=L}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L f_0\delta u\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\int_0^L F_0\delta_d\left(x-\frac{L}{2}\right)\delta u\mathrm{d}x\,\mathrm{d}t=0$$

$$\text{or, }\int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 u}{\partial t^2}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}-f_0-F_0\delta_d\left(x-\frac{L}{2}\right)\right]\delta(u)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\bigg|_{x=L}\mathrm{d}t-\int_{t_1}^{t_2}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}\delta(u)\bigg|_{x=0}\mathrm{d}t=0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 u}{\partial t^2}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left\{EA\left(\frac{\partial u}{\partial x}\right)\right\}=f_0+F_0\delta_d\left(x-\frac{L}{2}\right)$$

Since the bar is uniform and homogeneous, equation of motion can also be expressed as

$$\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{f_0}{\rho A} + \frac{F_0}{\rho A}\delta_d\left(x - \frac{L}{2}\right) \tag{a}$$

where $c = \sqrt{E/\rho}$ is the velocity of wave propagation along the bar.

The associated boundary conditions are

$$u(0) = 0 \tag{b}$$

and

$$u'(L) = 0 \tag{c}$$

Mode shapes for the boundary conditions given by Eqs. (**b**) and (**c**) can be expressed as

$$U_i(x) = \sin\left(\frac{\omega_i}{c}x\right) \tag{d}$$

where

$$\omega_i = \left(\frac{2i-1}{2}\right)\frac{\pi c}{L}$$

Then the solution for the steady state response for the partial differential Eq. (**a**) can be assumed as

$$u(x,t) = \sum_i U_i(x)q_i(t) \tag{e}$$

Substituting $u(x,t)$ from Eq. (**e**) into Eq. (**a**), we get

$$\sum_i U_i\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} - c^2\sum_i \frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2}q_i = \frac{f_0}{\rho A} + \frac{F_0}{\rho A}\delta_d\left(x - \frac{L}{2}\right) \tag{f}$$

Multiplying both sides of Eq. (**f**) by U_j and integrating over the length of the bar and also using orthogonal properties of the eigen-function (mode shapes), we get

$$\left[\int_0^L U_i U_i \mathrm{d}x\right]\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} - c^2\left[\int_0^L \frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2}U_i \mathrm{d}x\right]q_i = \int_0^L \frac{f_0}{\rho A}U_i \mathrm{d}x + \int_0^L \frac{F_0}{\rho A}\delta_d\left(x - \frac{L}{2}\right)U_i \mathrm{d}x \tag{g}$$

Substituting U_i from Eq. (**d**) into Eq. (**g**) and integrating, we get the equation for ith mode as

$$\left(\frac{L}{2}\right)\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2}+\omega_i^2\left(\frac{L}{2}\right)q_i=\frac{f_0}{\rho A}\frac{c}{\omega_i}\left[1-\cos\left(\frac{\omega_i L}{c}\right)\right]+\frac{F_0}{\rho A}\sin\left(\frac{\omega_i L}{2c}\right)$$

$$\text{or,}\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2}+\omega_i^2 q_i=\frac{2f_0}{\rho AL}\frac{c}{\omega_i}\left[1-\cos\left\{\left(\frac{2i-1}{2}\right)\pi\right\}\right]+\frac{2F_0}{\rho AL}\sin\left\{\left(\frac{2i-1}{4}\right)\pi\right\}$$

$$\therefore\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2}+\omega_i^2 q_i=\frac{2f_0 c}{\rho AL\omega_i}+\frac{2F_0}{\rho AL}\sin\left\{\left(\frac{2i-1}{4}\right)\pi\right\} \tag{h}$$

Then the response for the ith mode can be determined by using convolution integral as

$$q_i(t)=\frac{1}{\omega_i}\int_0^t\left[\frac{2f_0 c}{\rho AL\omega_i}+\frac{2F_0}{\rho AL}\sin\left\{\left(\frac{2i-1}{4}\right)\pi\right\}\right]\sin\{\omega_i(t-\tau)\}d\tau$$

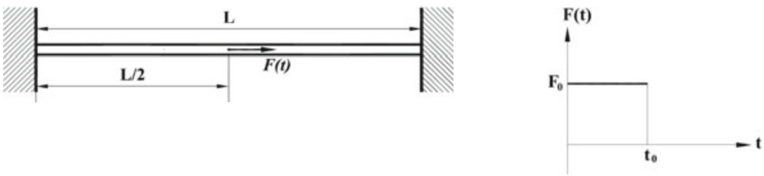
$$\therefore\quad q_i=\frac{1}{\omega_i^2}\left[\frac{2f_0 c}{\rho AL\omega_i}+\frac{2F_0}{\rho AL}\sin\left\{\left(\frac{2i-1}{4}\right)\pi\right\}\right][1-\cos(\omega_i t)] \tag{i}$$

Then substituting U_i from Eq. (**d**) and q_i from Eq. (**i**) into Eq. (**e**), we get the steady state response of the given system as

$$u(x,t)=\sum_i\frac{1}{\omega_i^2}\left[\frac{2f_0 c}{\rho AL\omega_i}+\frac{2F_0}{\rho AL}\sin\left\{\left(\frac{2i-1}{4}\right)\pi\right\}\right][1-\cos(\omega_i t)]\sin\left(\frac{\omega_i}{c}x\right) \tag{j}$$

Example 7.19

Consider a uniform bar fixed at both ends as shown in Figure E7.19(a). It is subjected to a time-dependent concentrated axial load of F_0, of the form shown in Figure E7.19(b), is applied at the mid-length of the bar. Determine the response of the bar.



(a) **(b)**

Figure E7.19

Solution

Following the procedure explained in the previous example, equation of motion for the given system can be derived as

$$\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{F}{\rho A}\delta_d\left(x - \frac{L}{2}\right) \tag{a}$$

where $c = \sqrt{E/\rho}$ is the velocity of wave propagation along the bar.

The associated boundary conditions are

$$u(0) = 0 \tag{b}$$

and

$$u(L) = 0 \tag{c}$$

Mode shapes for the boundary conditions given by Eqs. (**b**) and (**c**) can be expressed as

$$U_i(x) = \sin\left(\frac{\omega_i}{c}x\right) \tag{d}$$

where

$$\omega_i = \frac{i\pi c}{L}$$

Then the solution for the steady state response for the partial differential Eq. (**a**) can be assumed as

$$u(x,t) = \sum_i U_i(x)q_i(t) \tag{e}$$

Substituting $u(x,t)$ from Eq. (**e**) into Eq. (**a**), we get

$$\sum_i U_i\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} - c^2\sum_i \frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2}q_i = \frac{F}{\rho A}\delta_d\left(x - \frac{L}{2}\right) \tag{f}$$

Multiplying both sides of Eq. (**f**) by U_j and integrating over the length of the bar and also using orthogonal properties of the eigen-function (mode shapes), we get

$$\left[\int_0^L U_i U_i \mathrm{d}x\right]\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} - c^2\left[\int_0^L \frac{\mathrm{d}^2 U_i}{\mathrm{d}x^2}U_i \mathrm{d}x\right]q_i = \int_0^L \frac{F}{\rho A}\delta_d\left(x - \frac{L}{2}\right)U_i \mathrm{d}x \tag{g}$$

Substituting U_i from Eq. (**d**) into Eq. (**g**) and integrating, we get the equation for ith mode as

$$\left(\frac{L}{2}\right)\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2\left(\frac{L}{2}\right)q_i = \frac{F}{\rho A}\sin\left(\frac{\omega_i L}{2c}\right)$$

$$\text{or,}\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F}{\rho AL}\sin\left(i\frac{\pi}{2}\right)$$

$$\therefore\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F}{\rho AL}(-1)^{(i-1)/2} \quad (i = 1, 3, 5, \ldots) \tag{h}$$

Given time-dependent force can be defined as

$$F(t) = F - Fu(t - t_0) \tag{i}$$

Then the response for the ith mode can be determined by using convolution integral as

$$q_i(t) = \frac{1}{\omega_i}\int_0^t \left[\frac{2F}{\rho AL}(-1)^{\frac{(i-1)}{2}}\right]\sin\{\omega_i(t-\eta)\}\mathrm{d}\eta - \frac{u(t-t_0)}{\omega_i}\int_0^t \left[\frac{2F}{\rho AL}(-1)^{(i-1)/2}\right]\sin\{\omega_i(t-\eta)\}\mathrm{d}\eta$$

$$\therefore\quad q_i = \frac{1}{\omega_i^2}\left[\frac{2F}{\rho AL}(-1)^{(i-1)/2}\right][1-\cos(\omega_i t)] - \frac{u(t-t_0)}{\omega_i^2}\left[\frac{2F}{\rho AL}(-1)^{(i-1)/2}\right][1-\cos\{\omega_i(t-t_0)\}] \tag{j}$$

Then substituting U_i from Eq. (**d**) and q_i from Eq. (**j**) into Eq. (**e**), we get the steady state response of the given system as

$$u(x,t) = \sum_i \frac{1}{\omega_i^2}\left[\frac{2F}{\rho AL}(-1)^{(i-1)/2}\right][1-\cos(\omega_i t)]\sin\left(\frac{\omega_i}{c}x\right) - \sum_i \frac{u(t-t_0)}{\omega_i^2}\left[\frac{2F}{\rho AL}(-1)^{(i-1)/2}\right][1-\cos\{\omega_i(t-t_0)\}]\sin\left(\frac{\omega_i}{c}x\right)$$

$$(i = 1, 3, 5, \ldots) \tag{k}$$

Example 7.20

A machine with rotating parts is attached at the free end of a cantilever beam as shown in Figure E7.20. The machine has a mass of 15 kg, and it has a rotating unbalance is 0.2 kg m. Length of the beam is 1.2 m, cross-sectional area of

the beam is 3×10^{-3} m^2, second moment of inertia of the beam section is 5×10^{-6} m^4, modulus of elasticity of the beam material is 210 GPa and its density 7850 kg/m^3. Determine the steady state response the machine if the operating speed of the machine is 120 rad/s.

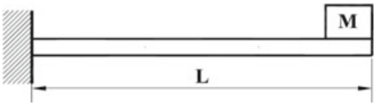


Figure E7.20

Solution

Rotating unbalance produces harmonic excitation and therefore the given system can also be modeled as a cantilever beam with concentrated mass and external harmonic force at its free end as shown in **Figure E7.20(a)**.

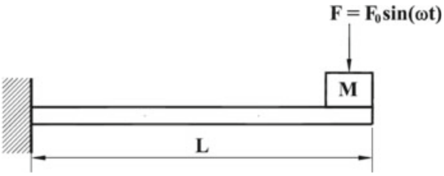


Figure E7.20(a)

Let $w(x, t)$ be the transverse deformation of the beam due to bending about y-axis. Then kinetic energy of the beam due to bending is given by

$$T_b = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x$$

Similarly, the kinetic energy of the concentrated mass M attached at the free end of the beam is given by

$$T_M = \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\bigg|_{x=L}$$

Then, the total kinetic energy of the system is given by

$$T = T_b + T_M = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\bigg|_{x=L}$$

The strain energy of the beam is given by

$$V = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Similarly, the work done by the external force is given by

$$W_{nc} = F\, w|_{x=L}$$

Then Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}M\left(\frac{\partial w}{\partial t}\right)^2\Bigg|_{x=L} - \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

Now applying extended Hamilton's principle

$$\delta\int_{t_1}^{t_2}(L + W_{nc})\mathrm{d}t = 0$$

$$\text{or, } \frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\delta\int_{t_1}^{t_2}\left\{M\left(\frac{\partial w}{\partial t}\right)^2\Bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\frac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}t + \delta\int_{t_1}^{t_2}\{F\, w|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or, } \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{t_1}^{t_2}\left\{M\left(\frac{\partial w}{\partial t}\right)\delta\left(\frac{\partial w}{\partial t}\right)\Bigg|_{x=L}\right\}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\{F\,\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or, } \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\Bigg|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t + \left\{M\left(\frac{\partial w}{\partial t}\right)\delta(w)\Bigg|_{x=L}\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\right\}\Bigg|_{x=L}\mathrm{d}t - \int_{t_1}^{t_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_{x=0}^{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\{F\,\delta(w)|_{x=L}\}\mathrm{d}t = 0$$

$$\text{or,}\ \int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t + \left\{M\left(\frac{\partial w}{\partial t}\right)\delta(w)\Big|_{x=L}\right\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\right\}\Big|_{x=L}\mathrm{d}t - \int_{t_1}^{t_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=L}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}^{x=L}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L \frac{d^2}{dx^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{F\,\delta(w)|_{x=L}\right\}\mathrm{d}t = 0$$

Since $\delta(w)|_{t_1}^{t_2} = 0$,

$$\text{or,}\ -\int_{t_1}^{t_2}\int_0^L \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left\{M\left(\frac{\partial^2 w}{\partial t^2}\right)\delta(w)\right\}\Big|_{x=L}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left\{EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L}\right\}\mathrm{d}t + \int_{t_1}^{t_2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}\right\}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=L}\mathrm{d}t - \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\{F\,\delta(w)|_{x=L}\}\mathrm{d}t = 0 = 0$$

$$\text{or,}\ \int_{t_1}^{t_2}\int_0^L\left[\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]\delta(w)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=0}\right\}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L}\right\}\mathrm{d}t + \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\delta(w)\Big|_{x=0}\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\left[M\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} - F\right]\delta(w)\Big|_{x=L}\mathrm{d}t = 0$$

Hence, equation of motion for the given system can be expressed as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{EI\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} = 0 \tag{a}$$

The associated boundary conditions are

$$w(0) = 0 \text{ and } w'(0) = 0 \tag{b}$$

$$w''(L) = 0 \text{ and } M\ddot{w}(L) - EIw'''(L) - F = 0 \tag{c}$$

Steady state response for Eq. (**a**) can be assumed as

$$w(x,t) = W(x)\sin\omega t \tag{d}$$

Substituting Eq. (**d**) into Eq. (**a**), we get

$$\begin{aligned} &\frac{\mathrm{d}^4 W(x)}{\mathrm{d}x^4} + \frac{\omega^2 \rho A}{EI} W(x) = 0 \\ \therefore\ &\frac{\mathrm{d}^4 W(x)}{\mathrm{d}x^4} + \beta^4 W(x) = 0 \end{aligned} \tag{e}$$

where $\beta = \sqrt[4]{\omega^2 \rho A/EI}$.

Solution of Eq. (**e**) for $W(x)$ can be determined as

$$W(x) = \overline{A}\sin(\beta x) + \overline{B}\cos(\beta x) + \overline{C}\sinh(\beta x) + \overline{D}\cosh(\beta x) \tag{f}$$

Substituting Eq. (**f**) into Eq. (**d**), we get

$$w(x,t) = \left[\overline{A}\sin(\beta x) + \overline{B}\cos(\beta x) + \overline{C}\sinh(\beta x) + \overline{D}\cosh(\beta x)\right]\sin\omega t \tag{g}$$

Applying first boundary condition defined by Eq. (**b**) into Eq. (**g**), we get

$$\overline{D} = -\overline{B} \tag{h}$$

Similarly, applying second boundary condition defined by Eq. (**b**) into Eq. (**g**), we get

$$\overline{C} = -\overline{A} \tag{i}$$

Substituting Eqs. (**h**) and (**i**) into Eq. (**g**), we get

$$w(x,t) = \left[\overline{A}(\sin\beta x - \sinh\beta x) + \overline{B}(\cos\beta x - \cosh\beta x)\right]\sin\omega t \tag{j}$$

Applying first boundary condition defined by Eq. (**c**) into Eq. (**j**), we get

$$\overline{A}(-\sin\beta L - \sinh\beta L) + \overline{B}(-\cos\beta L - \cosh\beta L) = 0$$
$$\therefore\quad \overline{B} = -\overline{A}\left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right) \tag{k}$$

Substituting $\overline{B}$ from Eqs. (**k**) into Eq. (**j**), we get

$$w(x,t) = \overline{A}\left[(\sin\beta x - \sinh\beta x) - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)(\cos\beta x - \cosh\beta x)\right]$$
$$\sin\omega t \tag{l}$$

Similarly, applying second boundary condition defined by Eq. (**c**) into Eq. (**l**) and also substituting $F = m_u e\omega^2 \sin\omega t$, we get

$$- M\omega^2\overline{A}\left[(\sin\beta L - \sinh\beta L) - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)(\cos\beta L - \cosh\beta L)\right]$$
$$- EI\beta^3\overline{A}\left[(-\cos\beta L - \cosh\beta L) - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)(\sin\beta L - \sinh\beta L)\right]$$
$$- m_u e\omega^2 = 0$$
$$\text{or, } \overline{A}\left[2M\omega^2(\cos\beta L\sinh\beta L - \sin\beta L\cosh\beta L) + EI\beta^3(1 + \cos\beta L\cosh\beta L)\right]$$
$$- m_u e\omega^2(\cos\beta L + \cosh\beta L) = 0$$
$$\therefore\quad \overline{A} = \frac{m_u e\omega^2(\cos\beta L + \cosh\beta L)}{2M\omega^2(\cos\beta L\sinh\beta L - \sin\beta L\cosh\beta L) + EI\beta^3(1 + \cos\beta L\cosh\beta L)} \tag{m}$$

Substituting $\overline{A}$ from Eqs. (**m**) into Eq. (**l**), we get the expression for the steady state response of the system as

$$w(x,t) = \frac{m_u e\omega^2(\cos\beta L + \cosh\beta L)}{2M\omega^2(\cos\beta L\sinh\beta L - \sin\beta L\cosh\beta L) + EI\beta^3(1 + \cos\beta L\cosh\beta L)}$$
$$\left[(\sin\beta x - \sinh\beta x) - \left(\frac{\sin\beta L + \sinh\beta L}{\cos\beta L + \cosh\beta L}\right)(\cos\beta x - \cosh\beta x)\right]\sin\omega t \tag{n}$$

Using the given system parameters: $\omega = 120$ rad/s, $\rho = 7850$ kg/m^3, $A = 3\times10^{-3}$ m^2, $E = 210$ GPa and $A = 5\times10^{-6}$ m^4, we get the value of parameter β as

$$\beta = \left(\frac{\omega^2\rho A}{EI}\right)^{1/4} = \left(\frac{120^2\times 7850\times 3\times 10^{-3}}{210\times 10^9\times 5\times 10^{-6}}\right)^{1/4} = 0.75386$$

Substituting $\beta = 0.75386$ and other given system parameters $M = 15$ kg, $m_u e = 0.2$ kg m and $L = 2$ m into Eq. (**n**), we get simplified expression for the steady state response as

$$w(x,t) = [0.003983\{\sin(0.75386x) - \sinh(0.75386x)\}$$
$$-0.0033525\{\cos(0.75386x) - \cosh(0.75386x)\}]\sin 120t \quad \text{(o)}$$

Substituting $x(= L) = 2$ m into Eq. (**o**), we get the steady state response of the machine as

$$w(x,t) = 0.001906\ \sin 120t \quad \text{(o)}$$

Example 7.21

A simply supported beam of length *L* shown in Figure E7.21 is subjected to a concentrated load *F* at its mid-span. The beam material has a density of ρ, modulus of elasticity of *E* and moment of inertia of section of *I*. Determine the steady state response of the system.

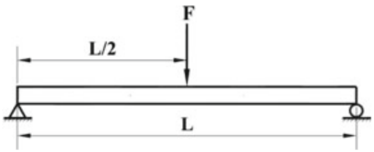


Figure E7.21

Solution

Referring to Example E3.14, equation of motion and the associated boundary conditions for the system are given as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + EI\left(\frac{\partial^4 w}{\partial x^4}\right) = F\delta_d\left(x - \frac{L}{2}\right) \quad \text{(a)}$$

$$w(0) = 0 \text{ and } w''(0) = 0 \quad \text{(b)}$$

$$w(L) = 0 \text{ and } w''(L) = 0 \quad \text{(c)}$$

Mode shapes for the boundary conditions given by Eqs. (**b**) and (**c**) can be expressed as

$$W_i(x) = \sin\left(\frac{i\pi}{L}x\right) \quad \text{(d)}$$

Then the solution for the steady state response for the partial differential Eq. (**a**) can be assumed as

$$w(x,t) = \sum_i W_i(x) q_i(t) \tag{e}$$

Substituting $w(x,t)$ from Eq. (**e**) into Eq. (**a**), we get

$$\sum_i W_i \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A}\sum_i \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} q_i = \frac{F}{\rho A}\delta_d\left(x - \frac{L}{2}\right) \tag{f}$$

Multiplying both sides of Eq. (**f**) by W_j and integrating over the length of the beam and also using orthogonal properties of the eigen-function (mode shapes), we get

$$\left[\int_0^L W_i W_i \mathrm{d}x\right]\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A}\left[\int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} W_i \mathrm{d}x\right] q_i = \int_0^L \frac{F}{\rho A}\delta_d\left(x - \frac{L}{2}\right) W_i \mathrm{d}x \tag{g}$$

Substituting W_i from Eq. (**d**) into Eq. (**g**) and integrating, we get the equation for ith mode as

$$\left(\frac{L}{2}\right)\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A}\left(\frac{i\pi}{L}\right)^4\left(\frac{L}{2}\right) q_i = \frac{F}{\rho A}\sin\left(i\frac{\pi}{2}\right)$$

$$\text{or,}\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F}{\rho A L}\sin\left(i\frac{\pi}{2}\right)$$

$$\therefore\quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F}{\rho A L}(-1)^{(i-1)/2} \quad (i = 1, 3, 5, \ldots) \tag{h}$$

where

$$\omega_i^2 = \frac{EI}{\rho A}\left(\frac{i\pi}{L}\right)^4 \tag{i}$$

Then the response for the ith mode can be determined by using convolution integral as

$$q_i(t) = \frac{1}{\omega_i}\int_0^t \left[\frac{2F}{\rho A L}(-1)^{\frac{(i-1)}{2}}\right]\sin\{\omega_i(t-\tau)\}\mathrm{d}\tau$$

$$\therefore\quad q_i = \frac{1}{\omega_i^2}\left[\frac{2F}{\rho A L}(-1)^{(i-1)/2}\right][1 - \cos(\omega_i t)] \tag{j}$$

Then substituting $W_i(x)$ from Eq. (**d**), q_i from Eq. (**j**) and ω_i^2 from Eq. (**i**) into Eq. (**e**), we get the steady state response of the given system as

$$w(x,t) = \sum_i \left(\frac{L}{i\pi}\right)^4 \left[\frac{2F}{EIL}(-1)^{(i-1)/2}\right][1-\cos(\omega_i t)]\sin\left(\frac{i\pi}{L}x\right) \quad (i = 1, 3, 5, \ldots) \tag{k}$$

Example 7.22

A simply supported beam of length L shown in Figure E7.22 is subjected to a distributed harmonic force within middle one-third span of the beam. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.

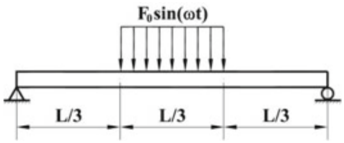


Figure E7.22

Solution

Following the procedure used in Example E3.14, equation of motion and the associated boundary conditions for the system are given as

$$\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) + EI\left(\frac{\partial^4 w}{\partial x^4}\right) = F_0 \sin \omega t\left[u\left(x - \frac{L}{3}\right) - u\left(x - \frac{2L}{3}\right)\right] \tag{a}$$

$$w(0) = 0 \text{ and } w''(0) = 0 \tag{b}$$

$$w(L) = 0 \text{ and } w''(L) = 0 \tag{c}$$

Mode shapes for the boundary conditions given by Eqs. (**b**) and (**c**) can be expressed as

$$W_i(x) = \sin\left(\frac{i\pi}{L}x\right) \tag{d}$$

Then the solution for the steady state response for the partial differential Eq. (**a**) can be assumed as

$$w(x,t) = \sum_i W_i(x) q_i(t) \tag{e}$$

Substituting $w(x,t)$ from Eq. (**e**) into Eq. (**a**), we get

$$\sum_i W_i \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A} \sum_i \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} q_i = \frac{F_0}{\rho A} \sin \omega t \left[u\left(x - \frac{L}{3}\right) - u\left(x - \frac{2L}{3}\right) \right] \tag{f}$$

Multiplying both sides of Eq. (**f**) by W_j and integrating over the length of the beam and also using orthogonal properties of the eigen-function (mode shapes), we get

$$\left[\int_0^L W_i W_i \mathrm{d}x \right] \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A} \left[\int_0^L \frac{\mathrm{d}^2 W_i}{\mathrm{d}x^2} W_i \mathrm{d}x \right] q_i = \frac{F_0}{\rho A} \sin \omega t \int_{L/3}^{2L/3} W_i \mathrm{d}x \tag{g}$$

Substituting W_i from Eq. (**d**) into Eq. (**g**) and integrating, we get the equation for ith mode as

$$\left(\frac{L}{2}\right) \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \frac{EI}{\rho A} \left(\frac{i\pi}{L}\right)^4 \left(\frac{L}{2}\right) q_i = \frac{F_0}{\rho A} \left(\frac{L}{i\pi}\right) \left[\cos\left(\frac{i\pi}{3}\right) - \cos\left(\frac{2i\pi}{3}\right) \right]$$

$$\text{or,} \quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F_0}{i\pi\rho A} \left[\cos\left(\frac{i\pi}{3}\right) - \cos\left(\frac{2i\pi}{3}\right) \right]$$

$$\therefore \quad \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \omega_i^2 q_i = \frac{2F_0}{i\pi\rho A} \left[\cos\left(\frac{i\pi}{3}\right) - \cos\left(\frac{2i\pi}{3}\right) \right] \tag{h}$$

where

$$\omega_i^2 = \frac{EI}{\rho A} \left(\frac{i\pi}{L}\right)^4 \tag{i}$$

Then the response for the ith mode can be determined as

$$q_i = \frac{2F_0}{i\pi\rho A \left(\omega_i^2 - \omega^2\right)} \sin \omega t \left[\cos\left(\frac{i\pi}{3}\right) - \cos\left(\frac{2i\pi}{3}\right) \right] \tag{j}$$

Then substituting q_i from Eq. (**j**) and ω_i^2 from Eq. (**i**) into Eq. (**e**), we get the steady state response of the given system as

$$w(x,t) = \sum_i \frac{2F_0}{i\pi\rho A \left(\omega_i^2 - \omega^2\right)} \left[\cos\left(\frac{i\pi}{3}\right) - \cos\left(\frac{2i\pi}{3}\right) \right] \sin\left(\frac{i\pi}{L} x\right) \sin \omega t \tag{k}$$

Review Questions

1. Differentiate between discrete system and continuous system.

2. Differentiate between boundary conditions and initial conditions.
3. State the possible boundary conditions at the ends of a string undergoing transverse vibration.
4. State the possible boundary conditions at the ends of a bar undergoing longitudinal vibration.
5. State the possible boundary conditions at the ends of a shaft undergoing torsional vibration.
6. State the possible boundary conditions at the ends of a beam undergoing transverse vibration.
7. Show how orthogonality condition can be derived for: a string, a bar, a shaft and a beam.

Exercise

1. A string of length L and mass per unit length ρ is stretched under a tension T. It is fixed at the left end and is attached to a spring of stiffness k at its right end as shown in **Figure P7.1**. Derive an expression for the natural frequencies of the system. Also derive the expression for the orthogonality condition for the system.
2. A string of length L and mass per unit length ρ is stretched under a tension T. One end of the cable is connected to a mass M, which can move in a frictionless vertical slot, and the other end is attached to a spring of stiffness k, as shown in **Figure P7.2**. Derive the frequency equation for the transverse vibration of the string. Also derive the expression for the orthogonality condition for the system.

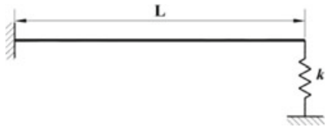


Figure P7.1

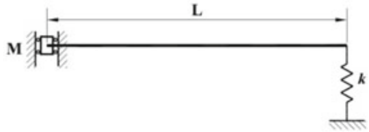


Figure P7.2

3. Determine the velocity of wave propagation in a cable of mass per unit length $\rho = 2$ kg/m when stretched by a tension $T = 30$ kN. Also determine the time taken by the wave to travel from one tower to another if the distance between them is 300 m.
4. A transmission line cable of mass per unit length $\rho = 4$ kg/m when stretched by a tension $T = 2000$ MN. Determine the first three natural frequencies and corresponding mode shapes if the cable is fixed between two towers which are at a distance of 1.8 km.
5. The string of a musical instrument fixed at both ends has a length 0.6 m, diameter 1 mm and density 7650 kg/m^3. Determine the fundamental natural frequency when it is stretched by a tensile force of 110 N.

6. A stretch cable of 2.5 m is fixed at both ends and has a fundamental frequency of 3200 Hz. Determine the frequencies for the second and third modes. What will be the changes in first three natural frequencies

   (a) if the length of the cable is increased by 20%?
   (b) if the tension in the cable is decreased by 20%?

7. The cord of a musical instrument fixed at both ends has a length 1 m, diameter 0.5 mm and density 7850 kg/m^3. Determine the tension required in order to have a fundamental frequency of 200 Hz.
8. A transmission line cable has a diameter of 10 mm and a density 8500 kg/m^3 and is clamped between two towers which are 2 km apart. Determine the tension required in the cable if the first three natural frequencies should be less than 24 Hz.
9. Determine the first three natural frequencies and corresponding the corresponding mode shapes for a system consisting of a string restrained by the springs as shown in **Figure P7.9**. Take $\rho = 0.2$ kg/m, $T = 200$ N, $L = 1$ m and $k = 40$ N/m.
10. A string of length L and mass per unit length ρ, stretched under a tension T and fixed at both ends is initial deflected as shown in **Figure P7.10** and released. Determine the resulting response of the system.

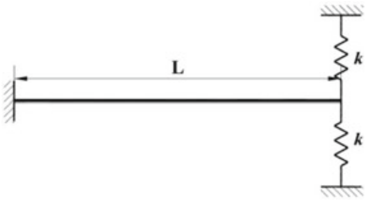


Figure P7.9

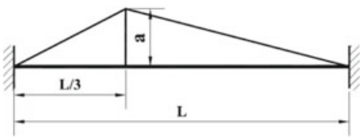


Figure P7.10

11. A string of length L and mass per unit length ρ, stretched under a tension T and fixed at both ends is initially deflected in the form of half sine wave of amplitude a as shown in **Figure P7.11** and released such that its initial conditions can be defined as $w(x, 0) = a \sin(\pi x/L)$ and $\dot{w}(x, 0) = 0$. Determine the resulting response of the system.
12. A string fixed at both ends has a mass per unit length $\rho = 4$ kg/m, length $L = 5$ m and stretched by a tension $T = 400$ N. It is initially deflected in the form defined by a function $f(x) = 0.1x(5 - x)$ and released from rest. Determine the resulting response of the system.
13. A string of length L and mass per unit length ρ, stretched under a tension T and fixed at both ends is initially deflected in the form defined by a function $f(x) = \sin(\pi x/L) + 0.5 \sin(2\pi x/L)$ and released such that its initial conditions can be defined as $w(x, 0) = f(x)$ and $\dot{w}(x, 0) = 0$. Determine the resulting response of the system.

14. A string of length L fixed at both ends is subjected to initial disturbance at $t = 0$ in the form of velocity distribution as shown in **Figure P7.14**. If the initial displacement distribution is zero, determine the resulting free vibration response of the string.

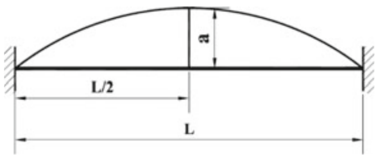


Figure P7.11

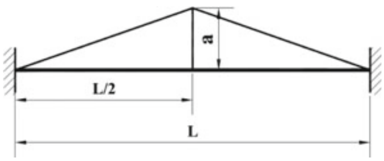
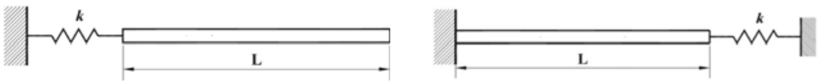


Figure P7.14

15. Determine the natural frequencies and corresponding mode shapes of a bar, when

    (a) its both ends are free,
    (b) its one end is fixed and other is free.

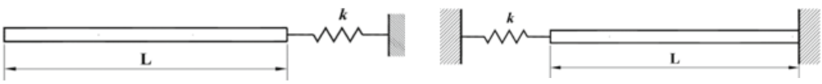
16. A spring of stiffness k is attached to one end of a bar of length L undergoing longitudinal vibration as shown in **Figure P7.16**. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.



(a) **(b)**

Figure P7.16

17. Determine the first three natural frequencies and corresponding mode shapes of a system consisting of a bar with the attached spring as shown in **Figure P7.17**. Take $\rho = 7850$ kg/m^3, $A = 2.5 \times 10^{-4}$ m^2, $L = 1.6$ m, $E = 210$ GPa and $k = 250$ kN/m.

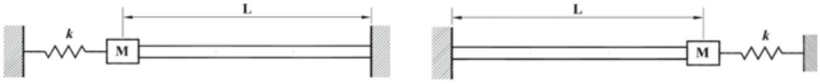


(a) **(b)**

Figure P7.17

18. A spring of stiffness k and a concentrated mass M is attached to one end of a bar of length L undergoing longitudinal vibration as shown in **Figure P7.18**. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional

area is A. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.



(a) (b)

Figure P7.18

19. Determine the first three natural frequencies and corresponding mode shapes of a system consisting of a bar with the attached spring and concentrated mass as shown in **Figure P7.19**. Take $\rho = 7850$ kg/m^3, $A = 2.5 \times 10^{-4}$ m^2, $L = 1.6$ m, $E = 210$ GPa, $k = 250$ kN/m and $M = 10$ kg.

Figure P7.19

20. Assemblies of a spring of stiffness k and a concentrated mass M are attached to both ends of a bar of length L undergoing longitudinal vibration as shown in **Figure P7.20**. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.

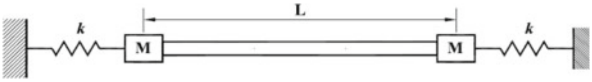


Figure P7.20

21. A uniform bar of length 1.6 m is made of a material which has a density of 7860 kg/m^3 and modulus of elasticity of 210 GPa. Determine the first three natural frequencies and the corresponding mode shapes when it is

    (a) Free at both ends,
    (b) Fixed at both ends and
    (c) Fixed at one end and free at the other end.

22. A concentrated mass M is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in **Figure P7.22**. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A. Determine its fundamental natural frequency. What will be the percentage change in fundamental natural frequency

    (a) if the mass M is increased by 25%?

(b) if the mass M is decreased by 25%?
(c) if the length L is increased by 20%?
(d) if the length L is decreased by 20%?

Take $\rho = 7850\ \text{kg/m}^3$, $A = 2.5 \times 10^{-4}\ \text{m}^2$, $L = 1.6\ \text{m}$, $E = 210$ GPa and $M = 10$ kg.

23. A bar shown in **Figure P7.23** is subjected to an axil force F at its free end such that it is stretched uniformly and its total length becomes $L + u_0$. Determine the resulting free response of the system when the force is removed. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A.

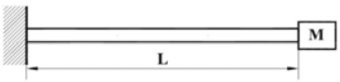


Figure P7.22

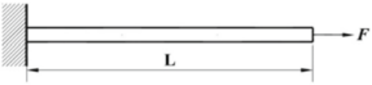


Figure P7.23

24. A uniform bar of length L and free at both ends is subjected to an axial compressive force F at both ends. Determine the resulting free response of the system when the force is removed. Assume initial strain at free ends of the bar as ε_0. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A.
25. A bar of length L fixed at both ends is subjected to an axial point load F_0 at the middle as shown in **Figure P7.25**. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A. Determine the resulting free vibration response of the bar when the force is removed at $t = 0$.

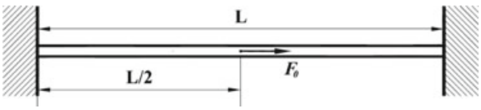


Figure P7.25

26. A uniform bar of length L fixed at both ends is subjected to longitudinal vibrations having a constant velocity V_0 at all points. The bar material has a density of ρ, modulus of elasticity E and its cross-sectional area is A. Determine the resulting free response of the system.
27. Determine the natural frequencies and corresponding mode shapes for torsional vibration of a shaft, when

(a) its both ends are fixed,
(b) its both ends are free,
(c) its one end is fixed and other is free.

28. Consider a flexible shaft and the rigid disk attached at its free end as shown in **Figure P7.28**. Determine the natural frequencies and mode shapes of the system. Also derive the expression for orthogonality condition for the system. The polar moment of inertia of its section is J_{ps}, density of the shaft material is ρ, shear modulus of shaft material is G and the mass moment of inertia of the rigid disk is I_d.
29. A rigid disk of mass moment of inertia I_d is attached to shaft of length L undergoing torsional vibration as shown in **Figure P7.29**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the natural frequencies and mode shapes of the system. Also derive the expression for orthogonality condition for the system.

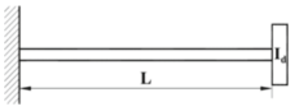


Figure P7.28

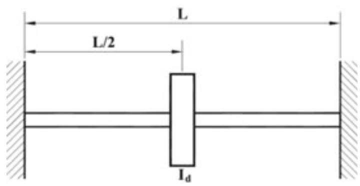


Figure P7.29

30. A torsional spring of stiffness k_t is attached to one end of a shaft of length L undergoing torsional vibration as shown in **Figure P7.30**. The polar moment of inertia of the shaft section is J_{ps}, density of the shaft material is ρ and shear modulus of shaft material is G. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.

(a) **(b)**

Figure P7.30

31. Determine the first three natural frequencies and corresponding mode shapes of a system consisting of a shaft with the attached spring as shown in **Figure P7.31**. The length of the shaft is 2 m, diameter of the shaft is 40 mm, density of the shaft material is 7850 kg/m^3, shear modulus of elasticity of shaft material is 84 GPa and stiffness of the torsional spring is 250 kN/m rad.

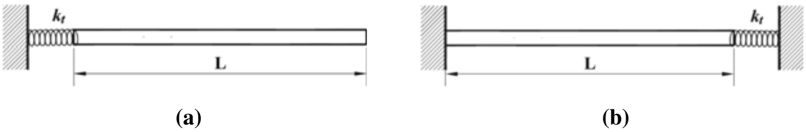


Figure P7.31

32. A uniform shaft free at both ends subjected to twisting torque such that its opposite ends undergo equal and opposite angular deformation of θ_0. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the resulting free response of the torsional vibration of the shaft when the torque is suddenly removed.
33. A uniform shaft fixed at one end and free at the other end is subjected to a twisting torque of magnitude T_0 at its free end. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the resulting free response of the torsional vibration of the shaft when the torque is suddenly removed.
34. A uniform shaft supported by the bearing at one end and free at the other end is rotating with a constant angular velocity ω. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the resulting free response of the torsional vibration of the shaft when the brake is suddenly applied at its bearing end.
35. Determine the natural frequencies and corresponding mode shapes for transverse vibration of a beam, when

    (a) its both ends are fixed,
    (b) its both ends are free,
    (c) its one end is fixed and the other is free,
    (d) its one end is fixed and the other is simply supported.

36. A concentrated mass M is attached to a beam of length L as shown in **Figure P7.36**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.
37. A beam of length L shown in **Figure P7.37** undergoing transverse vibration is restrained by two springs each having stiffness of k. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Derive frequency equation of the system. Also derive the expression for orthogonality condition for the system.

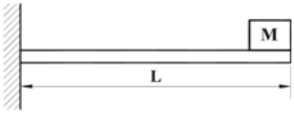


Figure P7.36

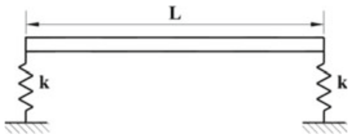


Figure P7.37

38. Determine the first three natural frequencies and corresponding mode shapes of a system consisting of a beam with the attached spring as shown in **Figure P7.38**. Take $\rho = 7850$ kg/m^3, $A = 5 \times 10^{-2}$ m^2, $I = 8 \times 10^{-4}$ m^4, $L = 1.5$ m, $E = 210$ GPa and $k = 500$ kN/m.
39. A simply supported beam of length L, is blown by a hammer at its mid-span such that the velocity distribution along beam at $t = 0$ is modeled by the profile shown in **Figure P.39**. Determine the resulting free response of the system.

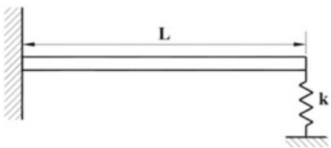


Figure P7.38

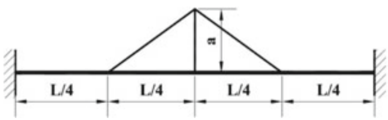


Figure P7.39

40. A simply supported beam is subjected to a uniformly distributed load of intensity f_0. Determine the free vibration response of the beam if the load is suddenly removed.
41. Determine the depth of cross-section h of a cantilever beam of length 2 m for which the first three natural frequencies fall in the range 2000-6000 Hz. Take $= 7850$ kg/m^3 and $E = 210$ GPa for the beam material.
42. A string of length L fixed at both ends is subjected to a concentrated harmonic force $F(t) = F_0 \sin \omega t$ as shown in **Figure P7.42**. Determine the steady state response of the system if (a) $x_0 = L/4$, (b) $x_0 = L/2$ and (c) $x_0 = 3L/4$.
43. A string of length L fixed at both ends is subjected to a concentrated force of magnitude F_0 at its mid-length as shown in **Figure P7.43**. Determine the steady state response of the system.

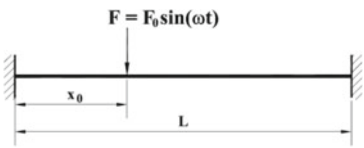


Figure P7.42

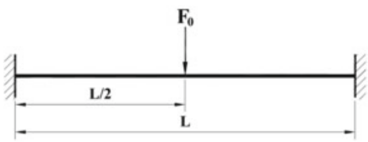


Figure P7.43

44. A string of length L fixed at both ends is subjected to a uniformly distributed force of intensity f_0 as shown in **Figure P7.44**. Determine the steady state response of the system.
45. A string of length L fixed at both ends is subjected to a time-dependent concentrated force $F(t)$ shown in **Figure P7.45** at its mid-length. Determine the steady state response of the system.

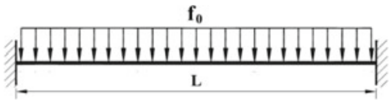


Figure P7.44

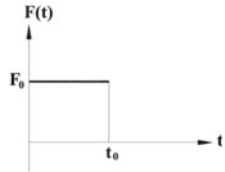


Figure P7.45

46. A uniform bar of length L fixed at one end and free at the other end is subjected to a concentrated force $F(t)$ at its free end as shown in **Figure P7.46**. Determine the steady state response of the system if

    (a) $F(t) = F_0 \sin \omega t$
    (b) $F(t) = F_0$.

47. If the force applied to the bar shown is **Figure P7.46** varies with time as shown in **Figure P7.47**. Determine the resulting steady state response.

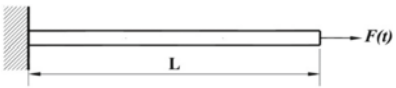


Figure P7.46

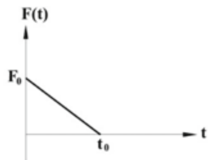


Figure P7.47

48. A uniform bar fixed both at ends, as shown in **Figure P7.48**, is subjected to a uniformly distributed longitudinal load f_0 per unit length throughout its length and a concentrated axial load of F_0 at its middle. Determine its steady state response.
49. A concentrated mass M is attached at the free end of a bar of length L and is subjected to a concentrated force $F(t)$ as shown in **Figure P7.49**. Determine the steady state response of the system if

    (a) $F(t) = F_0 \sin \omega t$
    (b) $F(t) = F_0$.

    Take $\rho = 7850$ kg/m^3, $A = 2.5 \times 10^{-4}$ m^2, $L = 2$ m, $E = 210$ GPa, $M = 10$ kg, $F_0 = 20$ kN and $\omega = 750$ rad/s.

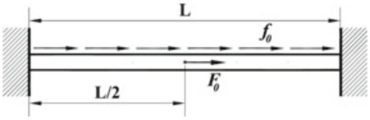


Figure P7.48

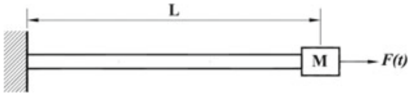


Figure P7.49

50. A uniform shaft of length L fixed at one end and free at the other end is subjected to a torque $T(t)$ at its free end as shown in **Figure P7.50**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the steady state response of the system if

   (a) $T(t) = T_0 \sin \omega t$
   (b) $T(t) = T_0$.

51. If the torque applied to the shaft shown in **Figure P7.50** varies with time as shown in **Figure P7.51**. The shaft material has a density of ρ, shear modulus of elasticity of G and polar moment of inertia of section of J_{ps}. Determine the resulting steady state response.

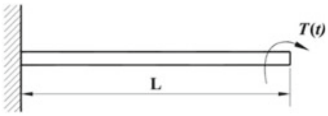


Figure P7.50

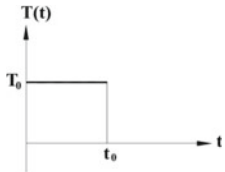


Figure P7.51

52. A simply supported beam of length L is subjected to a concentrated harmonic force $F(t) = F_0 \sin \omega t$ as shown in **Figure P7.52**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system if (a) $x_0 = L/4$, (b) $x_0 = L/2$ and (c) $x_0 = 3L/4$.
53. A cantilever beam of length L is subjected to a concentrated harmonic force $F(t) = F_0 \sin \omega t$ at its free end, as shown in **Figure P7.53**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.

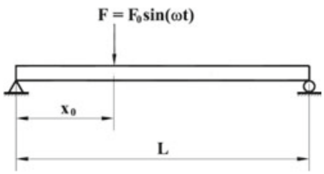


Figure P7.52

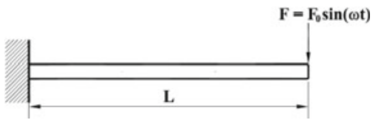
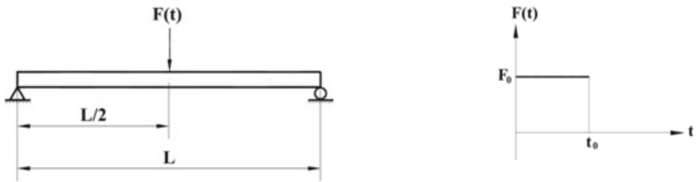


Figure P7.53

54. A simply supported beam of length L shown in **Figure P7.54(a)** is subjected to a time varying concentrated force $F(t)$ at its mid-length as shown in **Figure P7.54(b)**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.



(a) (b)

Figure P7.54

55. A simply supported beam of length L is subjected to a uniformly distributed harmonic force $F(x, t)$ as shown in **Figure P7.55**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.
56. A simply supported beam of length L is subjected to a uniformly varying harmonic force $F(x, t)$ as shown in **Figure P7.56**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.

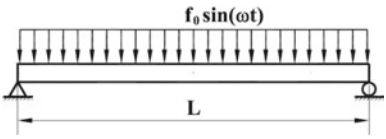


Figure P7.55

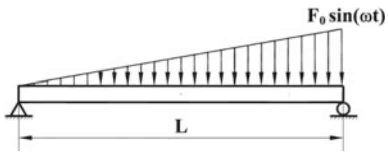


Figure P7.56

57. A simply supported beam of length L is subjected to a spatially varying harmonic force $F(x, t) = F_0 \sin(\pi x/L) \sin \omega t$ as shown in **Figure P7.57**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.

58. A machine with rotating parts is attached at the mid-span of a simply supported beam as shown in **Figure P7.58**. The machine has a mass of 25 kg and it has a rotating unbalance is 0.2 kg m. Length of the beam is 1.5 m, cross-sectional area of the beam is 3×10^{-3} m^2, second moment of inertia of the beam section is 5×10^{-6} m^4, modulus of elasticity of the beam material is 210 GPa and its density 7850 kg/m^3. Determine the steady state response the machine if the operating speed of the machine is 150 rad/s.

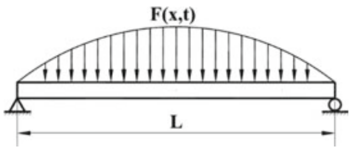


Figure P7.57

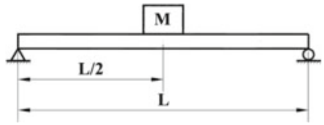
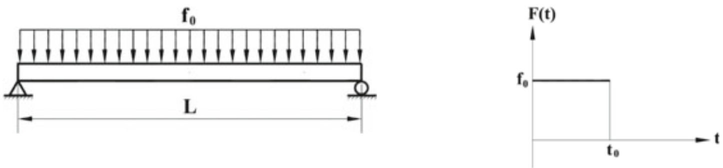


Figure P7.58

59. A simply supported beam of length L is subjected to a uniformly distributed force of intensity f_0 as shown in **Figure P7.59(a)**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. The variation of the applied force with time is shown in **Figure P7.59(b)**. Determine the steady state response of the system.



(a) **(b)**

Figure P7.59

60. A simply supported beam of length L is subjected to a uniformly distributed harmonic force within the right half span of the beam as shown in **Figure P7.60**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the steady state response of the system.

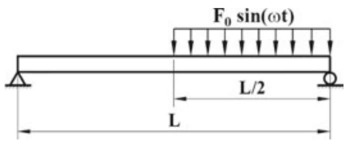


Figure P7.60

Answers

1. $\tan\left(\frac{\omega}{c}L\right) = \frac{Tc\left(k-M\omega^2\right)}{\omega\left(T^2+kMc^2\right)}, c = \sqrt{\frac{T}{\rho}}; \frac{1}{c^2}\int\limits_0^L W_i W_j \mathrm{d}x + \frac{M}{T} W_i(0) W_j(0) = 0$
2. $\tan\left(\frac{\omega}{c}L\right) = -\frac{T\omega}{kc}, c = \sqrt{\frac{T}{\rho}}; \int\limits_0^L W_i W_j \mathrm{d}x = 0$
3. 122.4744 m/s, 2.4495 s
4. 39.0267 rad/s, $\sin(0.001745x)$; 78.0535 rad/s, $\sin(0.003491x)$; 117.0802 rad/s, $\sin(0.005236x)$
5. 111.2394 Hz
6. 6400 Hz, 9600 Hz; 2666.67 Hz, 5333.33 Hz, 8000 Hz; 2862.17 Hz, 5724.33 Hz, 8586.50 Hz
7. 246.6150 N
8. 683.61 MN
9. 56.6231 rad/s, 151.6504 rad/s, 249.9638 rad/s
10. $\sum\limits_{n=1}^{\infty} \frac{9a}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right) \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi c}{L}t\right)$
11. $a \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c}{L}t\right)$
12. $\sum\limits_{n=1,3,5,\ldots}^{\infty} \frac{20}{n^3\pi^3} \sin\left(\frac{n\pi}{5}x\right) \cos(2n\pi t)$
13. $\sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c}{L}t\right) + \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi c}{L}t\right)$
14. $\sum\limits_{n=1}^{\infty} (-1)^{2n-1} \frac{8aL}{(2n-1)^3\pi^3 c} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi c}{L}t\right)$
15.
    (a) $\frac{n\pi c}{L}, \cos\left(\frac{n\pi}{L}x\right)$
    (b) $\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}, \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}$
16.
    (a) $\tan\left(\frac{\omega}{c}L\right) = \frac{kc}{EA\omega}; \int\limits_0^L U_i U_j \mathrm{d}x = 0$
    (b) $\tan\left(\frac{\omega}{c}L\right) = -\frac{EA\omega}{kc}; \int\limits_0^L U_i U_j \mathrm{d}x = 0$
17.
    (a) 281.8085 rad/s, $\cos(0.0545x)$; 10163.4132 rad/s, $\cos(1.9650x)$; 20315.0781 rad/s, $\cos(3.9277x)$
    (b) 5093.4211 rad/s, 0.0054 $\sin(0.9848x) + 1.1163 \cos(0.9848x)$; 15238.5940 rad/s, 0.0018 $\sin(2.9483x) + 1.1178 \cos(2.9463x)$; 25392.0842 rad/s, 0.0011 $\sin(4.9093x) + 1.1179 \cos(4.9093x)$
18.
    (a) $\tan\left(\frac{\omega}{c}L\right) = -\frac{EA\omega}{c\left(k-M\omega^2\right)}, c = \sqrt{\frac{E}{\rho}}; \frac{1}{c^2}\int\limits_0^L U_i U_j \mathrm{d}x + \frac{M}{EA} U_i(0) U_j(0) = 0$
    (b) $\tan\left(\frac{\omega}{c}L\right) = -\frac{EA\omega}{c\left(k-M\omega^2\right)}, c = \sqrt{\frac{E}{\rho}}; \frac{1}{c^2}\int\limits_0^L U_i U_j \mathrm{d}x + \frac{M}{EA} U_i(L) U_j(L) = 0$

19. 137.5260 rad/s, 0.1411 $\sin(0.0266x) + 0.7878 \cos(0.0266x)$; 5664.9788 rad/s, 0.0051 $\sin(1.0953x) + 1.1763 \cos(1.0953x)$; 15450.5881 rad/s, 0.0018 $\sin(2.9872x) + 1.1256 \cos(2.9872x)$

20. $\tan\left(\frac{\omega}{c}L\right) = -\frac{2EA\omega c\left(k-M\omega^2\right)}{\left[(EA\omega)^2-c^2\left(k-M\omega^2\right)^2\right]}$, $c = \sqrt{\frac{E}{\rho}}$; $\frac{1}{c^2}\int\limits_0^L U_iU_j\mathrm{d}x + \frac{M}{EA}U_i(0)U_j(0) + \frac{M}{EA}U_i(L)U_j(L) = 0$

21.

(a) 10,149.1171 rad/s, $\cos(1.9635x)$; 20,298.2342 rad/s, $\cos(3.9270x)$; 30,447.3514 rad/s, $\cos(5.8905x)$
(b) 10,149.1171 rad/s, $\sin(1.9635x)$; 20,298.2342 rad/s, $\sin(3.9270x)$; 30,447.3514 rad/s, $\sin(5.8905x)$
(c) 5074.5586 rad/s, $\sin(0.9817x)$; 15,223.6757 rad/s, $\sin(2.9452x)$; 25,372.7928 rad/s, $\sin(4.9087x)$

22. 4334.1185 rad/s, 4500.7630 rad/s, 4030.5762 rad/s, 3463.5167 rad/s, 5625.9537 rad/s

23. $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{8u_0}{(2n-1)^2\pi^2}\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}\cos\left\{\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}t\right\}$

24. $\sum\limits_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{2\varepsilon_0 L}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right)$

25. $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2F_0L}{(2n-1)^2\pi^2}\sin\left\{(2n-1)\frac{\pi}{L}x\right\}\cos\left\{(2n-1)\frac{\pi c}{L}t\right\}$

26. $\sum\limits_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{2V_0L}{n^2\pi^2 c}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right)$

27.

(a) $\frac{n\pi c}{L}$, $\sin\left(\frac{n\pi}{L}x\right)$
(b) $\frac{n\pi c}{L}$, $\cos\left(\frac{n\pi}{L}x\right)$
(c) $\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$, $\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}$

28. $\tan\left(\frac{\omega_n}{c}L\right) = \frac{GJ_{ps}}{I_d c\omega_n}$; $\frac{1}{c^2}\int\limits_0^L \Theta_i\Theta_j\mathrm{d}x + \frac{I_d}{GJ_{ps}}\Theta_i(L)\Theta_j(L) = 0$

29. $\tan\left(\frac{\omega_n}{2c}L\right) = \frac{2GJ_{ps}}{I_d c\omega_n}$; $\frac{1}{c^2}\int\limits_0^{L/2} \Theta_i\Theta_j\mathrm{d}x + \frac{I_d}{2GJ_{ps}}\Theta_i\left(\frac{L}{2}\right)\Theta_j\left(\frac{L}{2}\right) = 0$

30.

(a) $\tan\left(\frac{\omega}{c}L\right) = \frac{k_t c}{GJ_{ps}\omega}$; $\int\limits_0^L \Theta_i\Theta_j\mathrm{d}x = 0$

(b) $\tan\left(\frac{\omega}{c}L\right) = -\frac{GJ_{ps}\omega}{k_t c}$; $\int\limits_0^L \Theta_i\Theta_j\mathrm{d}x = 0$

31.

(a) 2465.2317 rad/s, 1.2341 $\sin(0.7536x) + 0.0786 \cos(0.7536x)$; 7398.8589 rad/s, 1.1471 $\sin(2.2618x) + 0.2191 \cos(2.2618x)$; 12,341.4414 rad/s, 1.0457 $\sin(3.7728x) + 0.3332 \cos(3.7728x)$

(b) 9015.7032 rad/s, $\sin(1.7431x)$; 18,242.1721 rad/s, $\sin(3.5270x)$; 27728.2493 rad/s, $\sin(5.3610x)$

32. $\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{2\theta_0}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right)$

33. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{8TL}{GJ(2n-1)^2\pi^2}\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}\cos\left\{\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}t\right\}$

34. $\sum_{n=1}^{\infty}\frac{8L\omega}{(2n-1)^2\pi^2 c}\sin\left\{(2n-1)\frac{\pi}{L}x\right\}\sin\left\{(2n-1)\frac{\pi c}{L}t\right\}$

35.

(a)
$$\cos(\beta L)\cosh(\beta L)=1;$$
$$\sin(\beta x)-\sinh(\beta x)-\frac{\sin(\beta L)-\sinh(\beta L)}{\cos(\beta L)-\cosh(\beta L)}[\cos(\beta x)-\cosh(\beta x)]$$

(b)
$$\cos(\beta L)\cosh(\beta L)=1;$$
$$\sin(\beta x)+\sinh(\beta x)-\frac{\sin(\beta L)-\sinh(\beta L)}{\cos(\beta L)-\cosh(\beta L)}[\cos(\beta x)+\cosh(\beta x)]$$

(c)
$$\cos(\beta L)\cosh(\beta L)=-1;$$
$$\sin(\beta x)-\sinh(\beta x)-\frac{\sin(\beta L)+\sinh(\beta L)}{\cos(\beta L)+\cosh(\beta L)}[\cos(\beta x)-\cosh(\beta x)]$$

(d)
$$\tan(\beta L)=\tanh(\beta L);$$
$$\sin(\beta x)-\sinh(\beta x)-\frac{\sin(\beta L)-\sinh(\beta L)}{\cos(\beta L)-\cosh(\beta L)}[\cos(\beta x)-\cosh(\beta x)]$$

36.
$$EI\beta^3[1+\cos(\beta L)\cosh(\beta L)] + \omega_n^2 M[\sin(\beta L)\cosh(\beta L)-\cos(\beta L)\sinh(\beta L)]=0;$$
$$\frac{1}{c^2}\int_0^L W_iW_j\mathrm{d}x+\frac{M}{EI}W_i(L)W_j(L)=0$$

37.
$$E^2I^2\beta^6[1-\cos(\beta L)\cosh(\beta L)] + 2EIk\beta^3[\cos(\beta L)\sinh(\beta L)-\sin(\beta L)\cosh(\beta L)] - 2k^2\beta\sin(\beta L)\sinh(\beta L)=0;$$
$$\int_0^L W_iW_j\mathrm{d}x=0$$

38. 1024.0179 rad/s, $\sin(1.2511x)-\sinh(1.2511x)-1.3622\cos(1.2511x)+1.3622\cosh(1.2511x)$; 6407.2740 rad/s, $\sin(3.1295x)-\sinh(3.1295x)-0.9819\cos(3.1295x)+0.9819\cosh(3.1295x)$; 17,939.9051 rad/s, $\sin(5.2365x)-\sinh(5.2365x)-1.0008\cos(5.2365x)+1.0008\cosh(5.2365x)$

39. $\sum_{n=1}^{\infty}\frac{8a}{n^2\pi^2\omega_n}\left[-\sin\left(\frac{n\pi}{4}\right)+2\sin\left(\frac{n\pi}{2}\right)-\sin\left(\frac{3n\pi}{4}\right)\right]\frac{2V_0L}{n^2\pi^2c}\sin\left(\frac{n\pi}{L}x\right)\sin(\omega_n t)$

40. $\sum_{n=1}^{\infty} \frac{5 f_0 L^4}{192 E I n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi}{L}x\right) \cos(\omega_n t)$

41. $0.2605 \text{ m} < h < 1.905 \text{ m}$

42. $\sum_{n=1}^{\infty} \frac{2F_0 L}{\rho\left(n^2\pi^2c^2-\omega^2L^2\right)} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin(\omega t)$

43. $\sum_{n=1}^{\infty} \frac{2F_0 L}{\rho\left(n^2\pi^2c^2\right)} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right)\left[1-\cos\left(\frac{n\pi c}{L}t\right)\right]$

44. $\sum_{n=1}^{\infty} \frac{2 f_0 L^2}{\rho\left(n^3\pi^3c^2\right)}\left[1-(-1)^n\right] \sin\left(\frac{n\pi x}{L}\right)\left[1-\cos\left(\frac{n\pi c}{L}t\right)\right]$

45.
$$\sum_{n=1}^{\infty} \frac{2F_0 L}{\rho\left(n^2\pi^2c^2\right)} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right)\left[1-\cos\left(\frac{n\pi c}{L}t\right)\right]$$
$$-u(t-t_0)\sum_{n=1}^{\infty} \frac{2F_0 L}{\rho\left(n^2\pi^2c^2\right)} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right)\left[1-\cos\left\{\frac{n\pi c}{L}(t-t_0)\right\}\right]$$

46.

(a) $\sum_{n=1}^{\infty} \frac{2F_0}{\rho A L\left(\omega^2-\omega_n^2\right)} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right) \sin(\omega t);\ \omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$

(b) $\sum_{n=1}^{\infty} \frac{2F_0}{\rho A L\omega_n^2} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right)[1-\cos(\omega_n t)];\ \omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$

47. $\sum_{n=1}^{\infty} \frac{2F_0}{\rho A L\omega_n^3 t_0} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right)[\omega_n t_0(1-\cos\omega_n t)-\omega_n t+\sin\omega_n t]$

$-u(t-t_0)\sum_{n=1}^{\infty} \frac{2F_0}{\rho A L\omega_n^3 t_0} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right)[\omega_n(t-t_0)-\sin\omega_n(t-t_0)];$

$\omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$

48. $\sum_{n=1}^{\infty} \frac{2L}{\rho A n^3\pi^3c^2}\left[F_0(n\pi)\sin\left(\frac{n\pi}{2}\right)+f_0L(1-\cos n\pi)\right] \sin\left(\frac{n\pi x}{L}\right)\left[1-\cos\left(\frac{n\pi c}{L}t\right)\right]$

49.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{8F_0 L}{EA\left[(2n-1)^2\pi^2c^2-8\omega^2L\left(\frac{L}{2}+\frac{M}{\rho A}\right)\right]} \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\} \sin(\omega t)$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{8F_0 L}{EA(2n-1)^2\pi^2c^2} \sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}[1-\cos(\omega_n t)];\ \omega_n=\frac{(2n-1)\pi c}{2\sqrt{2L\left(\frac{L}{2}+\frac{M}{\rho A}\right)}}$

50.

(a) $\sum_{n=1}^{\infty} \frac{2T_0}{\rho J_{ps} L\left(\omega^2-\omega_n^2\right)} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right) \sin(\omega t);\ \omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$

(b) $\sum_{n=1}^{\infty} \frac{2T_0}{\rho J_{ps} L\omega_n^2} \sin\left(\frac{\omega_n}{c}L\right) \sin\left(\frac{\omega_n}{c}x\right)[1-\cos(\omega_n t)];\ \omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$

51. $$\sum_{n=1}^{\infty}\frac{2T_0L}{\rho J_{ps}L\omega_n^2}\sin\left\{(2n-1)\frac{\pi}{2}\right\}\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}\left[1-\cos\left\{\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}t\right\}\right]$$
$$-u(t-t_0)\sum_{n=1}^{\infty}\frac{2T_0L}{\rho J_{ps}L\omega_n^2}\sin\left\{(2n-1)\frac{\pi}{2}\right\}\sin\left\{\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right\}\left[1-\cos\left\{\left(\frac{2n-1}{2}\right)\frac{\pi c}{L}(t-t_0)\right\}\right]$$

52. $\sum\limits_{n=1}^{\infty}\frac{2F_0L^3}{\left(\rho AL^4\omega^2-\pi^4EIn^4\right)}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin(\omega t)$

53. $$\frac{F_0\sin(\omega t)}{2EI\beta^3[1+\cos(\beta L)\cosh(\beta L)]}\left[\{\cos(\beta L)+\cosh(\beta L)\}\{\sin(\beta x)-\sinh(\beta x)\}-\{\sin(\beta L)+\sinh(\beta L)\}\{\cos(\beta x)-\cosh(\beta x)\}\right]$$

54. $$\sum_{n=1}^{\infty}\frac{2F_0}{\rho AL\omega_n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)[1-\cos(\omega_n t)]-u(t-t_0)\sum_{n=1}^{\infty}\frac{2F_0}{\rho AL\omega_n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)[1-\cos\{\omega_n(t-t_0)\}];$$
$$\omega_n=\frac{\pi^2n^2}{L^2}\sqrt{\frac{EI}{\rho A}}$$

55. $\sum\limits_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{2f_0L^4}{n\pi\left(\rho AL^4\omega^2-\pi^4EIn^4\right)}\sin\left(\frac{n\pi x}{L}\right)\sin(\omega t)$

56. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2F_0L^4}{n\pi\left(\rho AL^4\omega^2-\pi^4EIn^4\right)}\sin\left(\frac{n\pi x}{L}\right)\sin(\omega t)$

57. $\frac{F_0\rho AL^4}{\left(\pi^4EI-\rho AL^4\omega^2\right)}\sin\left(\frac{\pi x}{L}\right)\sin(\omega t)$

58. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{88.3125-3367.2278n^4}\sin(2.0944nx)\sin(150t)$

59. $$\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{Lf_0}{\rho An\pi\omega_n^2}\sin\left(\frac{n\pi x}{L}\right)[1-\cos(\omega_n t)]-u(t-t_0)\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{Lf_0}{\rho An\pi\omega_n^2}\sin\left(\frac{n\pi x}{L}\right)[1-\cos\{\omega_n(t-t_0)\}];$$
$$\omega_n=\frac{\pi^2n^2}{L^2}\sqrt{\frac{EI}{\rho A}}$$

60. $\sum\limits_{n=1}^{\infty}\left[\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right]\frac{2F_0L^4}{n\pi\left(\rho AL^4\omega^2-\pi^4EIn^4\right)}\sin\left(\frac{n\pi x}{L}\right)\sin(\omega t)$

Chapter 8
Approximate Methods

8.1 Introduction

In previous chapters, different methods to determine exact solution for the response of different systems have been explained. Methods to determine the response of a single degree of freedom system have been explained in Chap. 4 and the methods to determine the response of a discrete system with a degree of freedom of two or higher have been explained in Chaps. 5 and 6. Similarly, the methods to determine the response of a continuous system have been explained in Chap. 7. These methods can be applied if the system under consideration is similar to the standard simplified model and their equations of motion appear in standard mathematical form.

However in real practice, all physical systems cannot be fitted into the standard simplified model and obviously their equations of motion will be different from the standard mathematical form. In such problems, we cannot determine the exact or closed form solutions and we have to use approximate methods.

As we know that any multi-degree of freedom system or continuous system may have large of natural frequencies and in some cases our objectives is fulfilled by determination of fundamental frequency or few natural frequencies. Hence for such problems, instead of solving the system for all natural frequencies and the mode shapes, we can use suitable approximate method to determine the fundamental frequency only.

Approximate methods when applied for a discrete multi-degree of freedom system give the approximate values of natural frequencies and the corresponding mode shapes.

Some approximate methods applicable to a continuous system gives the approximate value of natural frequency and the approximate expression for the corresponding mode shape, whereas other approximate methods can only give numerical results.

Common approximate methods used for vibration analysis are presented in this chapter.

M. C. Luintel, *Textbook of Mechanical Vibrations*,
https://doi.org/10.1007/978-981-99-3614-4_8

8.2 Rayleigh Method

Rayleigh method is based on energy conservation principle and hence can be applied for a conservative system to determine the fundamental natural frequency. Although it can be applied for both discrete and continuous system, it is commonly used to determine the natural frequency of a beam or a shaft carrying a number of concentrated inertia elements at different point along its span.

8.2.1 Rayleigh Method for a Single Degree of Freedom System

Consider a single degree of freedom system consisting of mass and a spring as shown in Fig. 8.1.

The potential and kinetic energies of the system are given as

$$V = \frac{1}{2}kx^2 \tag{8.1}$$

$$T = \frac{1}{2}m\dot{x}^2 \tag{8.2}$$

We can assume that the response of the conservative system will have a form of

$$x = A\sin\omega_n t \tag{8.3}$$

Substituting Eq. (8.3) into Eqs. (8.1) and (8.2), we can rewrite the expressions for the kinetic and potential energies of the system as

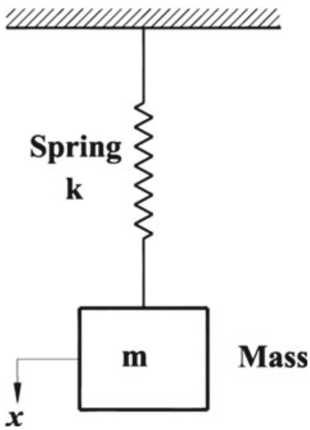


Fig. 8.1 Single degree of freedom system consisting of a spring and a mass

$$V = \frac{1}{2}k(A\sin\omega_n t)^2 \tag{8.4}$$

$$T = \frac{1}{2}m(-\omega_n A\cos\omega_n t)^2 \tag{8.5}$$

From Eqs. (8.4) and (8.5), we can write the expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}kA^2 \tag{8.6}$$

$$T_{\max} = \frac{1}{2}m\omega_n^2 A^2 \tag{8.7}$$

For the conservative system, the maximum potential energy should be equal to the maximum kinetic energy, i.e.

$$\frac{1}{2}kA^2 = \frac{1}{2}m\omega_n^2 A^2$$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{m}} \tag{8.8}$$

8.2.2 *Rayleigh Method for a Discrete Multi Degree of Freedom System*

Consider a system with n degree of freedom as shown in Fig. 8.2. The potential energy of the system can expressed as

$$\begin{aligned} V &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \cdots + \frac{1}{2}k_n(x_n - x_{n-1})^2 + \frac{1}{2}k_{n+1}x_n^2 \\ \therefore \quad V &= \frac{1}{2}(k_1 + k_2)x_1^2 - k_2x_1x_2 + \frac{1}{2}(k_2 + k_3)x_2^2 - k_3x_2x_3 + \cdots + \\ &\quad - k_nx_{n-1}x_n + \frac{1}{2}(k_n + k_{n+1})x_n^2 \end{aligned} \tag{8.9}$$

Similarly the kinetic energy of the system is given as

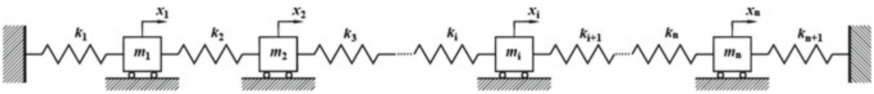


Fig. 8.2 Spring-mass system with n degree of freedom

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \cdots + \frac{1}{2}m_n\dot{x}_n^2 \tag{8.10}$$

Equations (8.9) and (8.10) can be expressed in matrix forms as

$$V = \frac{1}{2}\{x\}^T[K]\{x\} \tag{8.11}$$

$$T = \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\} \tag{8.12}$$

where $[K]$ is the stiffness matrix, $[M]$ is the mass matrix, $\{x\}$ is the displacement vector and $\{\dot{x}\}$ is the velocity vector.

The response of a multi degree of freedom system can be assumed as

$$\{x\} = \{X\}\sin\omega_n t \tag{8.13}$$

Substituting Eq. (8.13) into Eqs. (8.11) and (8.12), we can rewrite the expressions for the kinetic and potential energies of the system as

$$V = \frac{1}{2}\{X\}^T[K]\{X\}(\sin\omega_n t)^2 \tag{8.14}$$

$$T = \frac{1}{2}\omega_n^2\{X\}^T[M]\{X\}(\cos\omega_n t)^2 \tag{8.15}$$

From Eqs. (8.14) and (8.15), we can write the expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}\{X\}^T[K]\{X\} \tag{8.16}$$

$$T_{\max} = \frac{1}{2}\omega_n^2\{X\}^T[M]\{X\} \tag{8.17}$$

For the conservative system, the maximum potential energy should be equal to the maximum kinetic energy, i.e.

$$\frac{1}{2}\{X\}^T[K]\{X\} = \frac{1}{2}\omega_n^2\{X\}^T[M]\{X\}$$

$$\therefore \quad \omega_n^2 = \frac{\{X\}^T[K]\{X\}}{\{X\}^T[M]\{X\}} \tag{8.18}$$

Equation (8.18) is called a Rayleigh's quotient and gives the approximate fundamental frequency of the system for the assumed mode $\{X\}$.

The fundamental frequency obtained from this method is somehow insensitive to the assumed mode. To verify this, we can assume the displacement of the system as a linear combination of normal modes $\{X\}_i$ as

$$\{X\} = \{X\}_1 + C_2\{X\}_2 + \cdots + C_n\{X\}_n \tag{8.19}$$

then

$$\begin{aligned}\{X\}^T[K]\{X\} = \{X\}_1^T[K]\{X\}_1 + C_2^2\{X\}_2^T[K]\{X\}_2 \\ + \cdots + C_n^2\{X\}_n^T[K]\{X\}_n\end{aligned} \tag{8.20}$$

$$\begin{aligned}\{X\}^T[M]\{X\} = \{X\}_1^T[M]\{X\}_1 + C_2^2\{X\}_2^T[M]\{X\}_2 \\ + \cdots + C_n^2\{X\}_n^T[M]\{X\}_n\end{aligned} \tag{8.21}$$

Substituting $\{X\}_i^T[K]\{X\}_i = \omega_i^2\{X\}_i^T[M]\{X\}_i$ into Eq. (8.20), we get

$$\{X\}^T[K]\{X\} = \omega_1^2\{X\}_1^T[M]\{X\}_1 + \omega_2^2C_2^2\{X\}_2^T[M]\{X\}_2 + \cdots \tag{8.22}$$

Then substituting Eqs. (8.21) and (8.22) into Eq. (8.18), we get

$$\begin{aligned}
\omega_n^2 &= \frac{\omega_1^2\{X\}_1^T[M]\{X\}_1 + \omega_2^2C_2^2\{X\}_2^T[M]\{X\}_2 + \cdots}{\{X\}_1^T[M]\{X\}_1 + C_2^2\{X\}_2^T[M]\{X\}_2 + \cdots} \\
&= \omega_1^2\frac{\left[1 + C_2^2\frac{\omega_2^2}{\omega_1^2}\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]}{\left[1 + C_2^2\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]} \\
&= \omega_1^2\left[1 + C_2^2\frac{\omega_2^2}{\omega_1^2}\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]\left[1 + C_2^2\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]^{-1} \\
&= \omega_1^2\left[1 + C_2^2\frac{\omega_2^2}{\omega_1^2}\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]\left[1 - C_2^2\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right] \\
&= \omega_1^2\left[1 + C_2^2\frac{\omega_2^2}{\omega_1^2}\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} - C_2^2\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right]
\end{aligned}$$

$$\therefore\quad \omega_n^2 = \omega_1^2\left[1 + C_2^2\left(\frac{\omega_2^2}{\omega_1^2} - 1\right)\frac{\{X\}_2^T[M]\{X\}_2}{\{X\}_1^T[M]\{X\}_1} + \cdots\right] \tag{8.23}$$

If the eigenvectors are normalized, the Eq. (8.23) reduces to

$$\omega_n^2 = \omega_1^2\left[1 + C_2^2\left(\frac{\omega_2^2}{\omega_1^2} - 1\right) + \cdots\right] \tag{8.24}$$

It can be noticed from Eq. (8.24) that C_2 represents the deviation of assumed amplitudes (eigenvectors) from the exact amplitudes. Therefore Eq. (8.24) will give an exact value of the fundamental natural frequency if the assumed amplitudes are identical to the exact amplitudes because for this case $C_2, C_3, \ldots, C_n$ all are equal to zero.

For any other assumed amplitudes, the computed fundamental frequency will be slightly higher than the exact value.

8.2.3 Rayleigh Method for a Shaft or a Beam Carrying a Number of Lumped Inertia Elements

Consider a shaft or a beam carrying a number of lumped masses as shown in Fig. 8.3. Let $w_1, w_2, \ldots, w_n$ be the deflections of the shaft/beam at the points at which lumped masses are attached.

The total potential and kinetic energies of the system can be determined as

$$V = \frac{1}{2}F_1 w_1 + \frac{1}{2}F_2 w_2 + \cdots + \frac{1}{2}F_i w_i + \cdots + \frac{1}{2}F_n w_n \tag{8.25}$$

$$T = \frac{1}{2}M_1 \dot{w}_1^2 + \frac{1}{2}M_2 \dot{w}_2^2 + \cdots + \frac{1}{2}M_i \dot{w}_i^2 + \cdots + \frac{1}{2}M_n \dot{w}_n^2 \tag{8.26}$$

Substituting $w_i = W_i \sin \omega_n t$ and $F_i = M_i g$, we get the expressions for maximum potential and kinetic energies of the system as

$$V_{\max} = \frac{1}{2}g[M_1 W_1 + M_2 W_2 + \cdots + M_n W_n] = \frac{1}{2}g \sum_{i=1}^{n} M_i W_i \tag{8.27}$$

$$T_{\max} = \frac{1}{2}\omega_n^2\left[M_1 W_1^2 + M_2 W_2^2 + \cdots + M_n W_n^2\right] = \frac{1}{2}\omega_n^2 \sum_{i=1}^{n} M_i W_i^2 \tag{8.28}$$

Then equating the maximum potential energy and the maximum kinetic energy of the system, i.e.

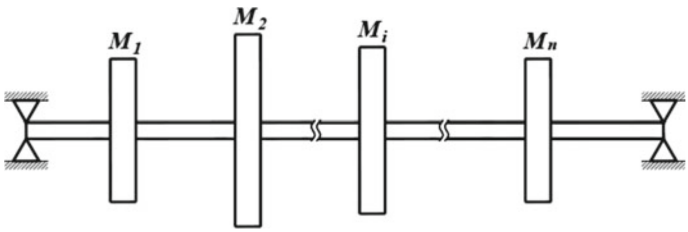


Fig. 8.3 A shaft/beam carrying a number of lumped masses

$$\frac{1}{2}g\sum_{i=1}^{n} M_i W_i = \frac{1}{2}\omega_n^2\sum_{i=1}^{n} M_i W_i^2$$

$$\therefore \quad \omega_n^2 = \frac{g\sum_{i=1}^{n} M_i W_i}{\sum_{i=1}^{n} M_i W_i^2} \tag{8.29}$$

where W_i is given by

$$W_i = a_{i1}F_1 + a_{i2}F_2 + \cdots + a_{in}F_n = \sum a_{ij}F_j = \sum a_{ij}M_j g \tag{8.30}$$

8.2.4 Rayleigh Method for a Continuous System

Rayleigh method for equating maximum kinetic energy and maximum potential energy can also be applied for different continuous system to determine the fundamental natural frequency of the system.

Maximum potential and maximum kinetic energies are determined with reference to an assumed mode shape as a continuous function and is generally taken as the static deflection of the system due to static load.

(a) **Rayleigh Method for a String**

Kinetic and potential energies of a string undergoing transverse vibration are given as

$$V = \frac{1}{2}\int_0^L T\left(w'\right)^2 \mathrm{d}x \tag{8.31}$$

$$T = \frac{1}{2}\int_0^L \rho(\dot{w})^2 \mathrm{d}x \tag{8.32}$$

where L is the length of the string, T is the tension in string, ρ is the mass per unit length of the string and $w(x,t)$ is the transverse deflection of the string.

If $W(x)$ is the assumed mode shape, then we can write an expression for the transverse displacement $w(x,t)$ as

$$w(x,t) = W(x)\sin\omega_n t \tag{8.33}$$

Substituting Eq. (8.33) into Eqs. (8.31) and (8.32), we get expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}\int_0^L T\left(W'\right)^2 \mathrm{d}x \tag{8.34}$$

$$T_{\max} = \frac{1}{2}\omega_n^2 \int_0^L \rho(W)^2 \mathrm{d}x \tag{8.35}$$

Then equating the maximum potential energy and the maximum kinetic energy of the system, i.e.

$$\frac{1}{2}\int_0^L T\left(W'\right)^2 \mathrm{d}x = \frac{1}{2}\omega_n^2 \int_0^L \rho(W)^2 \mathrm{d}x$$

$$\therefore \quad \omega_n^2 = \frac{\int_0^L T\left(W'\right)^2 \mathrm{d}x}{\int_0^L \rho(W)^2 \mathrm{d}x} \tag{8.36}$$

(b) **Rayleigh Method for a Bar**

Kinetic and potential energies of a bar undergoing longitudinal vibration are given as

$$V = \frac{1}{2}\int_0^L EA\left(u'\right)^2 \mathrm{d}x \tag{8.37}$$

$$T = \frac{1}{2}\int_0^L \rho A(\dot{u})^2 \mathrm{d}x \tag{8.38}$$

where L is the length of the bar, E is the modulus of elasticity of a bar material, ρ is the density of the bar material, A is the cross-section area of bar and $u(x,t)$ is the longitudinal deflection of the bar.

If $U(x)$ is the assumed mode shape, then we can write an expression for the longitudinal displacement $u(x,t)$ as

$$u(x,t) = U(x)\sin\omega_n t \tag{8.39}$$

Substituting Eq. (8.39) into Eqs. (8.37) and (8.38), we get expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}\int_0^L EA\left(U'\right)^2 \mathrm{d}x \tag{8.40}$$

$$T_{\max} = \frac{1}{2}\omega_n^2 \int_0^L \rho A(U)^2 \mathrm{d}x \tag{8.41}$$

Then equating the maximum potential energy and the maximum kinetic energy of the system, i.e.

$$\frac{1}{2}\int_0^L EA\left(U'\right)^2 \mathrm{d}x = \frac{1}{2}\omega_n^2 \int_0^L \rho A(U)^2 \mathrm{d}x$$

$$\therefore \quad \omega_n^2 = \frac{\int_0^L EA\left(U'\right)^2 \mathrm{d}x}{\int_0^L \rho A(U)^2 \mathrm{d}x} \tag{8.42}$$

(c) **Rayleigh Method for a Shaft**

Kinetic and potential energies of a shaft undergoing torsional vibration are given as

$$V = \frac{1}{2}\int_0^L GJ\left(\theta'\right)^2 \mathrm{d}x \tag{8.43}$$

$$T = \frac{1}{2}\int_0^L \rho J\left(\dot{\theta}\right)^2 \mathrm{d}x \tag{8.44}$$

where L is the length of the shaft, G is the shear modulus of elasticity of a shaft material, ρ is the density of the shaft material, A is the cross section area of shaft and $\theta(x, t)$ is the torsional deflection of the shaft.

If $\Theta(x)$ is the assumed mode shape, then we can write an expression for the longitudinal displacement $\theta(x, t)$ as

$$\theta(x, t) = \Theta(x) \sin \omega_n t \tag{8.45}$$

Substituting Eq. (8.45) into Eqs. (8.43) and (8.44), we get expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}\int_0^L GJ\left(\Theta'\right)^2 \mathrm{d}x \tag{8.46}$$

$$T_{\max} = \frac{1}{2}\omega_n^2 \int_0^L \rho J(\Theta)^2 \mathrm{d}x \tag{8.47}$$

Then equating the maximum potential energy and the maximum kinetic energy of the system, i.e.

$$\frac{1}{2}\int_0^L GJ\left(\Theta'\right)^2\mathrm{d}x = \frac{1}{2}\omega_n^2\int_0^L \rho J(\Theta)^2\mathrm{d}x$$

$$\therefore \quad \omega_n^2 = \frac{\int_0^L GJ\left(\Theta'\right)^2\mathrm{d}x}{\int_0^L \rho J(\Theta)^2\mathrm{d}x} \tag{8.48}$$

(d) **Rayleigh Method for a Beam**

Kinetic and potential energies of a beam undergoing transverse vibration are given as

$$V = \frac{1}{2}\int_0^L EI\left(w''\right)^2\mathrm{d}x \tag{8.49}$$

$$T = \frac{1}{2}\int_0^L \rho A(\dot{w})^2\mathrm{d}x \tag{8.50}$$

where L is the length of the beam, E is the modulus of elasticity of a beam material, ρ is the density of the beam material, A is the cross section area of beam, I is the moment of inertia of the beam section and $w(x, t)$ is the longitudinal deflection of the beam.

If $W(x)$ is the assumed mode shape, then we can write an expression for the longitudinal displacement $w(x, t)$ as

$$w(x, t) = W(x)\sin\omega_n t \tag{8.51}$$

Substituting Eq. (8.51) into Eqs. (8.49) and (8.50), we get expressions for the maximum potential and maximum kinetic energies as

$$V_{\max} = \frac{1}{2}\int_0^L EI\left(W''\right)^2\mathrm{d}x \tag{8.52}$$

$$T_{\max} = \frac{1}{2}\omega_n^2\int_0^L \rho A(W)^2\mathrm{d}x \tag{8.53}$$

Then equating the maximum potential energy and the maximum kinetic energy of the system, i.e.

$$\frac{1}{2}\int_0^L EI\left(W''\right)^2 \mathrm{d}x = \frac{1}{2}\omega_n^2 \int_0^L \rho A(W)^2 \mathrm{d}x$$

$$\therefore \quad \omega_n^2 = \frac{\int_0^L EI\left(W''\right)^2 \mathrm{d}x}{\int_0^L \rho A(W)^2 \mathrm{d}x} \tag{8.54}$$

8.3 Dunkerley's Method

Rayleigh method gives the upper bound to the fundamental natural frequency whereas Dunkerley's method gives the lower bound to the fundamental natural frequency. Hence these two methods provide a convenient way to check whether there occurs a resonance in the first mode for a moderate speed rotary machine.

To explain Dunkerley's method, let us start with the equation of motion of a system in terms of flexibility matrix as

$$[A][M]\{\ddot{x}\} + \{x\} = \{0\} \tag{8.55}$$

Substituting $\{x\} = \{X\}\sin\omega_n t$, Eq. (8.55) reduces to

$$-\omega_n^2[A][M]\{X\} + \{X\} = \{0\}$$

$$\text{or} \quad [A][M]\{X\} - \frac{1}{\omega_n^2}\{X\} = \{0\}$$

$$\therefore \quad \left[[A][M] - \frac{1}{\omega_n^2}[I]\right]\{X\} = \{0\} \tag{8.56}$$

Nontrivial solution of Eq. (8.56) can be determine as

$$\left|[A][M] - \frac{1}{\omega_n^2}[I]\right| = 0 \tag{8.57}$$

For simplicity, expanding Eq. (8.57) for a three degree of freedom system, we get

$$\begin{vmatrix} a_{11}m_1 - \frac{1}{\omega_n^2} & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 - \frac{1}{\omega_n^2} & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 - \frac{1}{\omega_n^2} \end{vmatrix} = 0$$

$$\therefore \quad \left(\frac{1}{\omega_n^2}\right)^3 - [a_{11}m_1 + a_{22}m_2 + a_{33}m_3]\left(\frac{1}{\omega_n^2}\right)^2 + \cdots = 0 \tag{8.58}$$

If ω_1, ω_2 and ω_3 are three natural frequencies of the system, then

$$\left(\frac{1}{\omega_n^2}-\frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega_n^2}-\frac{1}{\omega_2^2}\right)\left(\frac{1}{\omega_n^2}-\frac{1}{\omega_3^2}\right)=0$$

$$\therefore\quad \left(\frac{1}{\omega_n^2}\right)^3-\left[\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}\right]\left(\frac{1}{\omega_n^2}\right)^2+\cdots=0 \tag{8.59}$$

Comparing Eqs. (8.58) and (8.59), we get

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}=a_{11}m_1+a_{22}m_2+a_{33}m_3 \tag{8.60}$$

We know that $a_{ii}m_i = 1/\omega_{ii}^2$

$$a_{ii}m_i=\frac{1}{\omega_{ii}^2} \tag{8.61}$$

where ω_{ii} is the natural frequency of the system where the system consists of a rigid mass (a rigid disk) m_i only.

Substituting Eq. (8.61) into Eq. (8.60), we get

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}=\frac{1}{\omega_{11}^2}+\frac{1}{\omega_{22}^2}+\frac{1}{\omega_{33}^2} \tag{8.62}$$

Since $\omega_1 < \omega_2 < \omega_3$, Eq. (8.62) can be rewritten for an approximate value of ω_1 as

$$\frac{1}{\omega_1^2}=\frac{1}{\omega_{11}^2}+\frac{1}{\omega_{22}^2}+\frac{1}{\omega_{33}^2} \tag{8.63}$$

Equation (8.63) can be extended for a beam or shaft carrying a number of lumped masses (disks) as

$$\frac{1}{\omega_1^2}=\frac{1}{\omega_{11}^2}+\frac{1}{\omega_{22}^2}+\cdots+\frac{1}{\omega_{nn}^2}+\frac{1}{\omega_s^2} \tag{8.64}$$

where ω_s is the natural frequency of the beam or shaft due to its own weight.

8.4 Matrix Iteration Method

This method provides an approximate value of the natural frequencies and the corresponding mode shapes of a multi degree of freedom system.

8.4.1 Matrix Iteration Using Flexibility Matrix

Again, let us start with the equation of motion of a system in terms of flexibility matrix as

$$[A][M]\{\ddot{x}\} + \{x\} = \{0\} \tag{8.65}$$

Substituting $\{x\} = \{X\}\sin\omega_n t$, we get

$$\{X\} = \omega_n^2[A][M]\{X\} \tag{8.66}$$

The iterative process is started with any assumed mode shape $\{X\}_1$ and the refined mode shape is determined from Eq. (8.66) as

$$\{X\}_2 = \omega_n^2[A][M]\{X\}_1 \tag{8.67}$$

This iterative process is continued until we get two almost identical successive vectors for the mode shape, i.e.,

$$\{X\}_{i+1} \approx \{X\}_i \tag{8.68}$$

Under this condition,

$$\begin{aligned} &\{X\}_{i+1} \approx \omega_n^2[A][M]\{X\}_i \\ &\therefore \quad \omega_n^2[A][M] = 1 \end{aligned} \tag{8.69}$$

Equation (8.69) gives the natural frequency of the system and the vector $\{X\}_i$ is the mode shape corresponding to the natural frequency thus obtained.

The iteration process explained above converges to the lowest value of ω_n^2 such that we get the fundamental natural frequency and the corresponding fundamental mode shape.

8.4.2 Determination of Higher Order Modes

As explained above, the matrix iteration method converges to the fundamental mode. So, to determine the natural frequencies corresponding to higher modes, the orthogonality principle can be used to obtain a modified governing equation such that it does not contain the determined lower modes.

For a multi degree of freedom system, the general displacement of the system is given by

$$\{X\} = C_1\{X\}_1 + C_2\{X\}_2 + \cdots + C_n\{X\}_n \tag{8.70}$$

If we want to determine the second mode, we should eliminate the first mode by making $C_1 = 0$. For this pre-multiply Eq. (8.70) by $\{X\}_1^T[M]$, we get

$$\{X\}_1^T[M]\{X\} = \{0\} \tag{8.71}$$

For simplicity, expanding Eq. (8.71) for a three degree of freedom system, we get

$$\begin{Bmatrix} X_{11} & X_{21} & X_{31} \end{Bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or,}\quad X_{11}m_1X_1 + X_{21}m_2X_2 + X_{31}m_3X_3 = 0$$

$$\therefore\quad X_1 = -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1}X_2 - \frac{X_{31}}{X_{11}}\frac{m_3}{m_1}X_3 \tag{8.72}$$

Equation (8.72) with two identities $X_2 = X_2$ and $X_3 = X_3$ can also be expressed in matrix form as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0 & -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1} & -\frac{X_{31}}{X_{11}}\frac{m_3}{m_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$\therefore\quad \{X\} = [S_1]\{X\} \tag{8.73}$$

where

$$[S_1] = \begin{bmatrix} 0 & -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1} & -\frac{X_{31}}{X_{11}}\frac{m_3}{m_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting $\{X\} = [S_1]\{X\}$ into Eq. (8.66), we get

$$\{X\} = \omega_n^2[A][M][S_1]\{X\} \tag{8.74}$$

Since Eq. (8.74) does not contain the lowest mode, iteration applied to this equation converges to the second mode.

Similarly to eliminate both the first and second modes in Eq. (8.70), we should make $C_1 = 0$ and $C_2 = 0$. Then pre-multiplying Eq. (8.70) by $\{X\}_1^T[M]$ and $\{X\}_2^T[M]$, we respectively get

$$\{X\}_1^T[M]\{X\} = \{0\} \tag{8.75}$$

$$\{X\}_2^T[M]\{X\} = \{0\} \tag{8.76}$$

Expanding Eqs. (8.75) and (8.76), we get

$$X_1 = -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1}X_2 - \frac{X_{31}}{X_{11}}\frac{m_3}{m_1}X_3 \tag{8.77}$$

$$X_2 = -\frac{X_{12}}{X_{22}}\frac{m_1}{m_2}X_1 - \frac{X_{32}}{X_{22}}\frac{m_3}{m_2}X_3 \tag{8.78}$$

Equations (8.77) and (8.78) with an identity $X_3 = X_3$ can also be expressed in matrix form as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0 & -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1} & -\frac{X_{31}}{X_{11}}\frac{m_3}{m_1} \\ -\frac{X_{12}}{X_{22}}\frac{m_1}{m_2} & 0 & -\frac{X_{32}}{X_{22}}\frac{m_3}{m_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$\therefore \quad \{X\} = [S_2]\{X\} \tag{8.79}$$

where

$$[S_2] = \begin{bmatrix} 0 & -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1} & -\frac{X_{31}}{X_{11}}\frac{m_3}{m_1} \\ -\frac{X_{12}}{X_{22}}\frac{m_1}{m_2} & 0 & -\frac{X_{32}}{X_{22}}\frac{m_3}{m_2} \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting $\{X\} = [S_2]\{X\}$ into Eq. (8.66), we get

$$\{X\} = \omega_n^2[A][M][S_2]\{X\} \tag{8.80}$$

Since Eq. (8.80) does not contain the lowest mode, iteration applied to this equation converges to the third mode.

8.4.3 *Matrix Iteration Using Dynamic Matrix*

We can also use matrix iteration method by using equation of motion in terms of dynamic matrix $[D]$, i.e.,

$$\{\ddot{x}\} + [D]\{x\} = \{0\} \tag{8.81}$$

where $[D] = [M]^{-1}[K]$

Substituting $\{x\} = \{X\}\sin\omega_n t$, we get

$$-\omega_n^2\{X\} + [D]\{X\} = \{0\}$$

$$\therefore \quad \{X\} = \frac{1}{\omega_n^2}[D]\{X\} \tag{8.82}$$

Then again, we start any iterative process with any assumed mode shape $\{X\}_1$ and the refined mode shape is determined from Eq. (8.82) as

$$\{X\}_2 = \frac{1}{\omega_n^2}[D]\{X\}_1 \tag{8.83}$$

This iterative process is continued until we get two almost identical successive vectors for the mode shape. Under this condition,

$$\frac{1}{\omega_n^2}[D] = 1 \tag{8.84}$$

Equation (8.84) gives the natural frequency of the system and the vector $\{X\}_i$ is the mode shape corresponding to the natural frequency thus obtained.

The iteration process explained above converges to the lowest value of $1/\omega_n^2$ such that we get the approximate value of the highest natural frequency and the corresponding mode shape.

8.5 Stodola's Method

Stodola's method is an iterative method used to determine the fundamental natural frequency and the corresponding mode shape of an undamped multi-degree of freedom system.

To explain the method, consider a three degree of freedom system as shown in Fig. 8.4. Assume any suitable displacements of each mass as $\{X\}_1 = \left\{ X_1 \ X_2 \ X_3 \right\}^T$.

Then the inertia forces of each mass of the system are determined as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_1 = \omega^2 \begin{Bmatrix} m_1 X_1 \\ m_2 X_2 \\ m_3 X_3 \end{Bmatrix} \tag{8.85}$$

Then the refined deflections of each mass due inertia forces can be determined as

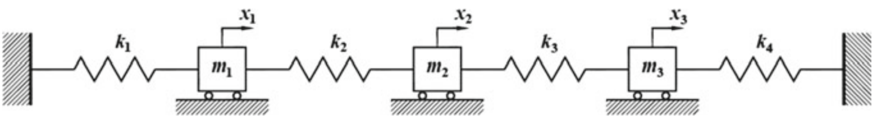


Fig. 8.4 A three degree of freedom system

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} a_{11}F_1 + a_{12}F_2 + a_{13}F_3 \\ a_{21}F_1 + a_{22}F_2 + a_{23}F_3 \\ a_{31}F_1 + a_{32}F_2 + a_{33}F_3 \end{Bmatrix} \tag{8.86}$$

where a_{ij} are the flexibility influence coefficients of the system.

This process is repeated for $\{F\}_i$ and $\{X\}_{i+1}$ using Eqs. (8.85) and (8.86) until we get two almost identical successive vectors for the mode shape, i.e.,

$$\{X\}_{i+1} \approx \{X\}_i \tag{8.87}$$

Under this condition, $\{X\}_i$ or $\{X\}_{i+1}$ gives the mode shape for the fundamental mode and its natural frequency can be determined from Eq. (8.85).

8.6 Holzer's Method

Holzer's method is a trial and error method which is applicable for both conservative and non-conservative systems. This method is based on successive assumptions of natural frequency of the system and determination of the resulting configuration according to the assumed frequency. It can be used to determine all natural frequencies of the system.

Although Holzer's method is mainly used for the analysis of torsional vibration of a shaft consisting of a number of disks, it can also be applied to any other systems. Here the steps of Holzer's method are explained for a spring-mass system from which, analogous equations for a torsional vibration can also be developed.

8.6.1 Holzer's Method for a System Without a Branch

Consider a mass at ith position of a spring-mass system and springs attached to this as shown in Fig. 8.5. With reference to the free-body diagram of the ith mass, its equation of motion can be written as

$$F_i^R - F_i^L = m_i \ddot{x}_i \tag{8.88}$$

where F_i^L and F_i^R are the forces acting at left at right and right side of the mass m_i.

For the sinusoidal oscillation, Eq. (8.88) can also be expressed as

$$F_i^R = F_i^L - m_i \omega^2 X_i \tag{8.89}$$

where X_i is the vibration amplitude of the mass m_i.

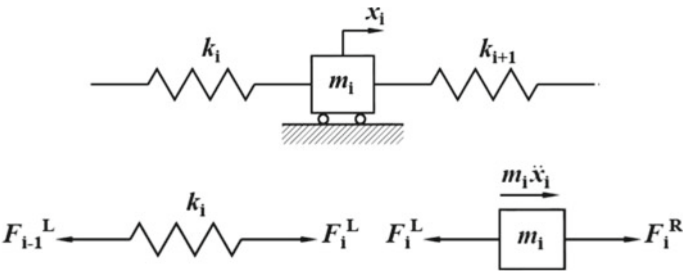


Fig. 8.5 Mass at ith position of a spring-mass system

The displacements at the left or right side of the mass m_i should be equal to X_i, i.e.,

$$X_i^R = X_i^L = X_i \tag{8.90}$$

Equations (8.89) and (8.90) can be combined in the form of a matrix equation as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_i \begin{Bmatrix} X \\ F \end{Bmatrix}_i^L \tag{8.91}$$

Equation (8.91) can be expressed in short form as

$$\{S\}_i^R = [P]_i \{S\}_i^L \tag{8.92}$$

where $\{S\}$ is called a state vector and $[P]$ is called a point matrix. A state vectors gives the displacement and force at a particular station (point) whereas a point matrix gives the state vector at the right of a station in terms of the state vector at left of the same station.

Again, with reference to the free-body diagram of the spring k_i, the force and displacement relations can be written as

$$F_i^L = F_{i-1}^R \tag{8.93}$$

$$X_i^L - X_{i-1}^R = \frac{F_{i-1}^R}{k_i} \tag{8.94}$$

Equations (8.93) and (8.94) can be combined in the form of a matrix equation as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_i^L = \begin{bmatrix} 1 & 1/k \\ 0 & 1 \end{bmatrix}_i \begin{Bmatrix} X \\ F \end{Bmatrix}_{i-1}^R \tag{8.95}$$

Equation (8.95) can be expressed in short form as

$$\{S\}_i^L = [F]_i \{S\}_{i-1}^R \tag{8.96}$$

where $[F]$ is called a field matrix and gives the transfer matrix for a field (spring).

Now substituting Eq. (8.96) into Eq. (8.92), we get

$$\{S\}_i^R = [P]_i [F]_i \{S\}_{i-1}^R = [T]_i \{S\}_{i-1}^R \tag{8.97}$$

where $[T]$ is the transfer matrix and can be determined as

$$[T] = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1 - \frac{m\omega^2}{k} \end{bmatrix} \tag{8.98}$$

(a) Holzer's Method for a System Fixed at Both Ends

Now consider a discrete system with n number of masses as shown in Fig. 8.6. Now taking the fixed left end as a station 0 and the right fixed as a station $n+1$. Then the relationship for successive stations can be expressed as

$$\begin{aligned}
&\{S\}_1^R = [T]_1 \{S\}_0 \\
&\{S\}_2^R = [T]_2 \{S\}_1^R = [T]_2 [T]_1 \{S\}_0 \\
&\ldots \\
&\{S\}_i^R = [T]_i [T]_{i-1} \ldots [T]_2 [T]_1 \{S\}_0 \\
&\{S\}_n^R = [T]_n [T]_{n-1} \ldots [T]_2 [T]_1 \{S\}_0 \\
&\{S\}_{n+1} = [F]_{n+1} [T]_n [T]_{n-1} \ldots [T]_2 [T]_1 \{S\}_0
\end{aligned} \tag{8.99}$$

Equation (8.99) can be expressed in short form to establish a relationship between state vectors of stations 0 and $n+1$ as

$$\{S\}_{n+1} = [U]\{S\}_0 \tag{8.100}$$

where

$$[U] = [F]_{n+1} [T]_n [T]_{n-1} \ldots [T]_2 [T]_1 \tag{8.101}$$

is called overall transfer matrix of the system.

Equation (8.100) can also be expressed as

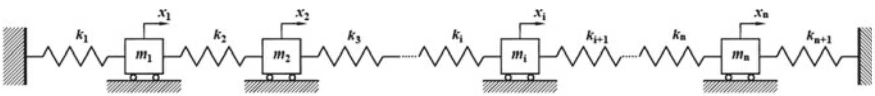


Fig. 8.6 A discrete n degree of freedom system fixed at both ends

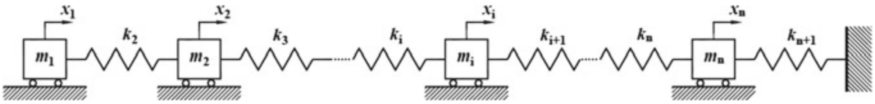


Fig. 8.7 A discrete n degree of freedom system free at left and fixed at right end

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_0 \tag{8.102}$$

For a system fixed at both ends, $X_0 = 0$ and $X_{n+1} = 0$, Eq. (8.102) reduces to

$$\begin{Bmatrix} 0 \\ F \end{Bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0$$

$$\therefore \quad U_{12} = 0 \tag{8.103}$$

In this method, a trail frequency ω is assumed and transfer matrix is determined by using Eq. (8.101). If the assumed trail frequency ω is not a natural frequency then Eq. (8.103) is not satisfied. Hence we repeat the process for some other ω such that Eq. (8.103) is satisfied.

For a system with relatively few degree of freedom, Eq. (8.103) can also be used directly as a frequency equation.

(b) Holzer's Method for a System Fixed at one End and Free at other End

Now consider a discrete system with n number of masses free at the left end and fixed at the right end as shown in Fig. 8.7. Since $k_1 = 0$, the overall transfer matrix for the system can be expressed as

$$[U] = [F]_{n+1}[T]_n[T]_{n-1} \ldots [T]_2[P]_1 \tag{8.104}$$

Then the relationship between state vectors of station 1 and station $n+1$ can be expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1 \tag{8.105}$$

For a free end at the left, $F_1 = 0$ and for the fixed right end, $X_{n+1} = 0$, Eq. (8.105) reduces to

$$\begin{Bmatrix} 0 \\ F \end{Bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} X \\ 0 \end{Bmatrix}_1$$

$$\therefore \quad U_{11} = 0 \tag{8.106}$$

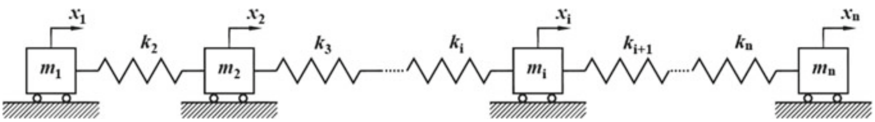


Fig. 8.8 A discrete n degree of freedom system free at both ends

(c) Holzer's Method for a System Free at Both Ends

Now consider a discrete system with n number of masses free at both ends as shown in Fig. 8.8. Since $k_1 = k_{n+1} = 0$, the overall transfer matrix for the system can be expressed as

$$[U] = [T]_n[T]_{n-1}\dots[T]_2[P]_1 \tag{8.107}$$

Then the relationship between state vectors of station 1 and station n can be expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_n = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1 \tag{8.108}$$

For a system free at both ends, $F_1 = 0$ and $F_n = 0$, Eq. (8.108) reduces to

$$\begin{Bmatrix} X \\ 0 \end{Bmatrix}_n = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{Bmatrix} X \\ 0 \end{Bmatrix}_1$$

$$\therefore \quad U_{21} = 0 \tag{8.109}$$

8.6.2 Holzer's Method for a Branched System

Holzer's method can also be used for a branched system. Consider a system consisting of three branches A, B and C as shown in Fig. 8.9.

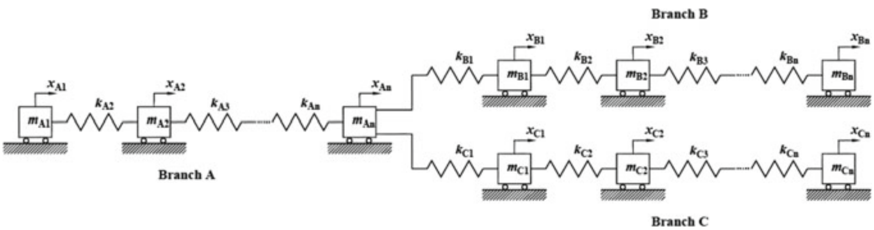


Fig. 8.9 Branched system

The overall transfer matrices for each branch can be expressed as

$$\{S\}^R_{An} = [U]_A\{S\}_{A1} \tag{8.110}$$

$$\{S\}^R_{Bn} = [U]_B\{S\}_{B0} \tag{8.111}$$

$$\{S\}^R_{Cn} = [U]_C\{S\}_{C0} \tag{8.112}$$

For a common point of the branches, displacements and forces can be related as

$$x_{An} = x_{B0} = x_{C0} \tag{8.113}$$

$$F_{An} = F_{B0} + F_{C0} \tag{8.114}$$

The left end of the branch A is free, i.e., $F_{A1} = 0$ and assuming $X_{A0} = 1$, Eq. (8.110) can be expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_{An} = \begin{bmatrix} U_{11A} & U_{12A} \\ U_{21A} & U_{22A} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} U_{11A} \\ U_{21A} \end{Bmatrix} \tag{8.115}$$

Similarly, the right end of the branch B is free, i.e., $F_{Bn} = 0$, Eq. (8.111) can be expressed as

$$\begin{Bmatrix} X \\ 0 \end{Bmatrix}^R_{Bn} = \begin{bmatrix} U_{11B} & U_{12B} \\ U_{21B} & U_{22B} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_{B0} \tag{8.116}$$

Substituting $X_{B0} = X_{An}$ from Eq. (8.113) and $X_{An} = U_{11A}$ from Eq. (8.115), we get

$$\begin{Bmatrix} X \\ 0 \end{Bmatrix}^R_{Bn} = \begin{bmatrix} U_{11B} & U_{12B} \\ U_{21B} & U_{22B} \end{bmatrix} \begin{Bmatrix} X_{An} \\ F_{B0} \end{Bmatrix} = \begin{bmatrix} U_{11B} & U_{12B} \\ U_{21B} & U_{22B} \end{bmatrix} \begin{Bmatrix} U_{11A} \\ F_{B0} \end{Bmatrix}$$

$$\text{or,} \quad U_{21B}U_{11A} + U_{22B}F_{B0} = 0$$

$$\therefore \quad F_{B0} = -\frac{U_{21B}U_{11A}}{U_{22B}} \tag{8.117}$$

Following the similar procedure for the branch C, we get

$$F_{C0} = -\frac{U_{21C}U_{11A}}{U_{22C}} \tag{8.118}$$

Now substituting Eqs. (8.117) and (8.118) into Eq. (8.110) and also using $F_{An} = U_{21A}$ from Eq. (8.114), we get frequency equation for the branched system as

$$U_{21A} = -\frac{U_{21B}U_{11A}}{U_{22B}} - \frac{U_{21C}U_{11A}}{U_{22C}}$$
$$U_{21A}U_{22B}U_{22C} + U_{11A}(U_{21B}U_{22C} + U_{21C}U_{22B}) = 0 \tag{8.119}$$

8.7 Myklestad-Prohl Method for Transverse Bending Vibration

Myklestad-Prohl method is an approximate method used to analyze vibration problem of a shaft or a beam carrying a number of lumped inertial elements.

Consider a beam carrying n number of lumped masses as shown in Fig. 8.10. Free-body diagram of ith element of the beam is shown in Fig. 8.11.

For this problem, a state vector consists of four variables: transverse deflection along z axis (w), slope or rotation about z axis (θ), shear force along z axis (V_z) and the bending moment about y axis (M_y), i.e., $\{S\} = \left\{ w \ \theta \ V_z \ M_y \right\}^T$.

With reference to the free-body diagram shown in Fig. 8.11, equilibrium equations for force and moment can be expressed as

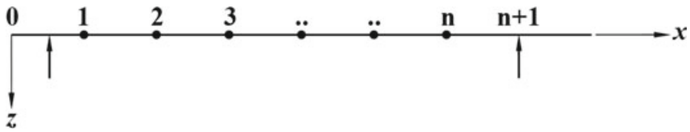


Fig. 8.10 A beam carrying n number of lumped masses

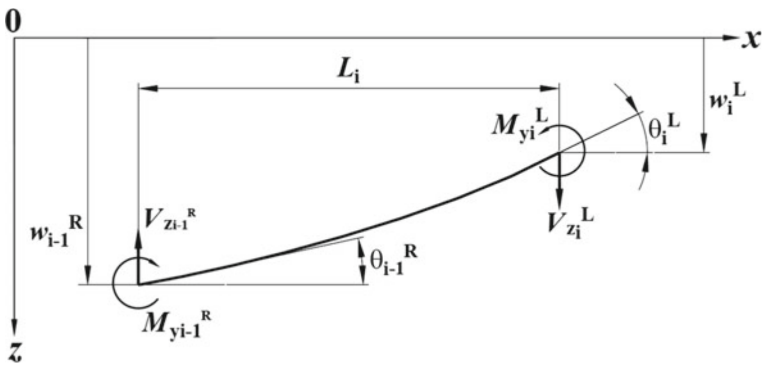


Fig. 8.11 Free-body diagram of ith element of the beam

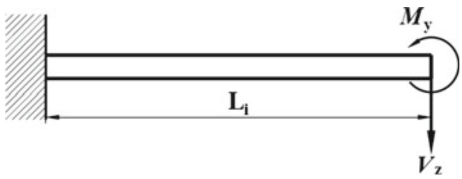


Fig. 8.12 A cantilever beam subjected to a transverse load V_z and an external moment M_y

$$V_{z.i}^{L} = V_{z.i-1}^{R} \tag{8.120}$$

$$M_{y.i}^{L} = M_{y.i-1}^{R} + V_{z.i}^{L} L_i \tag{8.121}$$

The deflection and slope at free end of a cantilever beam subjected to a transverse load V_z and an external moment M_y, as shown in Fig. 8.12, can be expressed as

$$w = -\frac{M_y L_i^2}{2EI} + \frac{V_z L_i^3}{3EI} \tag{8.122}$$

$$\theta = \frac{M_y L_i}{EI} - \frac{V_z L_i^2}{2EI} \tag{8.123}$$

Using Eqs. (8.122) and (8.123) again with reference to Fig. 8.11, we get

$$w_i^L = w_{i-1}^R - \theta_{i-1}^R L_i - \frac{M_{y,i}^L L_i^2}{2E_i I_i} + \frac{V_{z,i}^L L_i^3}{3E_i I_i} \tag{8.124}$$

$$\theta_i^L = \theta_{i-1}^R + \frac{M_{y,i}^L L_i}{E_i I_i} - \frac{V_{z,i}^L L_i^2}{2E_i I_i} \tag{8.125}$$

Using Eqs. (8.120) and (8.121) into Eqs. (8.124) and (8.125) and simplifying, we get

$$-w_i^L = -w_{i-1}^R + \theta_{i-1}^R L_i + \frac{M_{y,i-1}^R L_i^2}{2E_i I_i} + \frac{V_{z,i-1}^R L_i^3}{6E_i I_i} \tag{8.126}$$

$$\theta_i^L = \theta_{i-1}^R + \frac{M_{y,i-1}^R L_i}{E_i I_i} + \frac{V_{z,i-1}^R L_i^2}{2E_i I_i} \tag{8.127}$$

Equations (8.124), (8.125), (8.121) and (8.120) can be combined in a matrix form as

$$\begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_i^L = \begin{bmatrix} 1 & L & \frac{L^2}{2EI} & \frac{L^3}{6EI} \\ 0 & 1 & \frac{L}{EI} & \frac{L^2}{2EI} \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_{i-1}^R \tag{8.128}$$

Equation (8.128) can be expressed in short form as

$$\{S\}_i^L = [F]_i \{s\}_{i-1}^R \tag{8.129}$$

where $\{S\}$ is the state vector and $[F]_i$ is the field matrix for the ith segment of the beam.

Again with reference to the free-body diagram of lumped mass at ith position shown in Fig. 8.13 and using $V_{z.i}^R = V_{z.i}^L - m_i \ddot{w}_i = V_{z.i}^L + m_i \omega^2 w_i^L$, the state vectors at the left and right of the mass can be related as

$$\begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_i^L \tag{8.130}$$

Equation (8.130) can be expressed in short form as

$$\{S\}_i^R = [P]_i \{S\}_i^L \tag{8.131}$$

where $[P]_i$ is the point matrix for the ith mass.

Substituting $\{S\}_i^L$ from Eq. (8.129) into Eq. (8.131), we get

$$\{S\}_i^R = [P]_i [F]_i \{s\}_{i-1}^R = [T]_i \{s\}_{i-1}^R \tag{8.132}$$

where

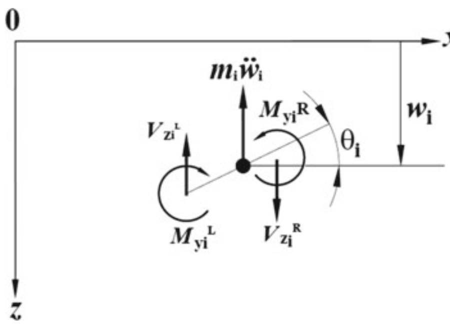


Fig. 8.13 Free-body diagram of a lumped mass at ith position

$$[T]_i = \begin{bmatrix} 1 & L & \frac{L^2}{2EI} & \frac{L^3}{6EI} \\ 0 & 1 & \frac{L}{EI} & \frac{L^2}{2EI} \\ 0 & 0 & 1 & L \\ m\omega^2 & m\omega^2 L & \frac{m\omega^2 L^2}{2EI} & 1 + \frac{m\omega^2 L^2}{6EI} \end{bmatrix}_i \tag{8.133}$$

is the transfer matrix for the ith element.

By following the procedure explained for the Holzer's method, the relationship between state vectors of stations 0 and $n+1$ as can be expressed as

$$\{S\}_{n+1} = [U]\{S\}_0 \tag{8.134}$$

where

$$[U] = [F]_{n+1}[T]_n[T]_{n-1}\dots[T]_2[T]_1 \tag{8.135}$$

is called overall transfer matrix of the system.

Equation (8.135) can also be expressed as

$$\begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{bmatrix} \begin{Bmatrix} -w \\ \theta \\ M_y \\ V_z \end{Bmatrix}_0 \tag{8.136}$$

(a) **Myklestad-Prohl Method for a Simply Supported Beam**

For a simply supported beam, the boundary conditions are $w_0 = w_{n+1} = 0$ and $M_{y,0} = M_{y,n+1} = 0$. Substituting these into Eq. (8.136), we get relations for nonzero variables θ and V_z as

$$\begin{bmatrix} U_{12} & U_{14} \\ U_{32} & U_{34} \end{bmatrix} \begin{Bmatrix} \theta \\ V_z \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.137}$$

From Eq. (8.137), the frequency equation for a simply supported beam can be expressed as

$$U_{12}U_{34} - U_{14}U_{32} = 0 \tag{8.138}$$

Assuming $\theta_0 = 1$, Eq. (8.137) gives

$$V_{z,0} = -\frac{U_{12}}{U_{14}} \tag{8.139}$$

Then the state vector for station 0 becomes $\left\{0\ 1\ 0\ -U_{12}/U_{14}\right\}^T$.Then, a trail frequency ω is assumed and transfer matrix is determined by using Eq. (8.135). If the assumed trail frequency ω is not a natural frequency then Eq. (8.138) is not satisfied. Hence we repeat the process for some other ω such that Eq. (8.138) is satisfied.

(b) Myklestad-Prohl Method for a Cantilever Beam

For a cantilever beam fixed at the left end and free at the right end, the boundary conditions are $w_0 = \theta_0 = 0$ and $V_{z,n} = M_{y,n} = 0$. Substituting these into Eq. (8.136), we get relations for nonzero variables θ and V_z as

$$\begin{bmatrix} U_{33} & U_{34} \\ U_{43} & U_{44} \end{bmatrix} \begin{Bmatrix} M_y \\ V_z \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.140}$$

From Eq. (8.140), the frequency equation for a cantilever beam can be expressed as

$$U_{33}U_{44} - U_{34}U_{43} = 0 \tag{8.141}$$

Assuming $M_{y,0} = 1$, Eq. (8.140) gives

$$V_{z,0} = -\frac{U_{33}}{U_{34}} \tag{8.142}$$

Then the state vector for station 0 becomes $\{0\ 0\ 1\ -U_{33}/U_{34}\}^T$.Then, a trail frequency ω is assumed and transfer matrix is determined by using Eq. (8.135). If the assumed trail frequency ω is not a natural frequency then Eq. (8.141) is not satisfied. Hence we repeat the process for some other ω such that Eq. (8.141) is satisfied.

(c) Myklestad-Prohl Method for a Fixed Beam

For a fixed beam, the boundary conditions are $w_0 = \theta_0 = 0$ and $w_{n+1} = \theta_{n+1} = 0$. Substituting these into Eq. (8.136), we get relations for nonzero variables θ and V_z as

$$\begin{bmatrix} U_{13} & U_{14} \\ U_{23} & U_{24} \end{bmatrix} \begin{Bmatrix} M_y \\ V_z \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.143}$$

From Eq. (8.140), the frequency equation for a fixed beam can be expressed as

$$U_{13}U_{24} - U_{14}U_{23} = 0 \tag{8.144}$$

Assuming $M_{y,0} = 1$, Eq. (8.143) gives

$$V_{z,0} = -\frac{U_{13}}{U_{14}} \tag{8.145}$$

Then the state vector for station 0 becomes $\left\{0\ 0\ 1\ -U_{13}/U_{14}\right\}^T$. Then, a trail frequency ω is assumed and transfer matrix is determined by using Eq. (8.135). If the assumed trail frequency ω is not a natural frequency then Eq. (8.144) is not satisfied. Hence we repeat the process for some other ω such that Eq. (8.144) is satisfied.

8.8 Rayleigh–Ritz Method

Rayleigh–Ritz method is an extension of Rayleigh's method and can be used to determine natural frequencies of fundamental as well as higher modes. To explain the method, recall Rayleigh's quotient for a beam as

$$\omega_n^2 = \frac{\int_0^L EI\left(W''\right)^2 \mathrm{d}x}{\int_0^L \rho A (W)^2 \mathrm{d}x} \tag{8.146}$$

Let $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, …, $\phi_n(x)$ be a set of linearly independent functions each of which satisfy the boundary conditions of the problem. Then, we assume an approximate solution of $W(x)$ as a linear combination of ϕ_i as

$$W(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots = \sum c_i\phi_i(x) \tag{8.147}$$

Substituting Eq. (8.147) into Eq. (8.146), we get

$$\omega_n^2(c_1, c_2, \ldots) = \frac{\int_0^L EI\left\{\sum c_i\phi_i''(x)\right\}^2 \mathrm{d}x}{\int_0^L \rho A\left\{\sum c_i\phi_i(x)\right\}^2 \mathrm{d}x} \tag{8.148}$$

In Rayleigh–Ritz method, we seek for the values of constants c_1, c_2, c_3, … such that ω_n^2 is minimized, that is

$$\frac{\partial\left(\omega_n^2\right)}{\partial c_i} = 0 \tag{8.149}$$

Equation (8.149) gives n simultaneous linear equations and ith equation can be expressed as

$$\begin{aligned} &\left[\int_0^L \rho A\left\{\sum c_i\phi_i(x)\right\}^2 \mathrm{d}x\right] \int_0^L EI\phi_i''(x)\left\{\sum c_i\phi_i''(x)\right\}\mathrm{d}x \\ &\quad - \left[\int_0^L EI\left\{\sum c_i\phi_i''(x)\right\}^2 \mathrm{d}x\right] \int_0^L \rho A\phi_i(x)\left\{\sum c_i\phi_i(x)\right\}\mathrm{d}x = 0 \end{aligned} \tag{8.150}$$

Now Using Eq. (8.148) into Eq. (8.150), we get

$$\int_0^L EI\phi_i''(x)\Big\{\sum c_i\phi_i''(x)\Big\}\mathrm{d}x - \omega_n^2\int_0^L \rho A\phi_i(x)\Big\{\sum c_i\phi_i(x)\Big\}\mathrm{d}x = 0 \tag{8.151}$$

We will get n number of such equations and they can be solved for c_i's. For nontrivial solution of $\{c_i\}$, Eq. (8.151) can be expressed as

$$\big|[K] - \omega_n^2[M]\big| = 0 \tag{8.152}$$

where the elements of matrices $[K]$ and $[M]$ are given respectively as

$$k_{ij} = \int_0^L EI\phi_i''(x)\phi_j''(x)\mathrm{d}x \tag{8.153}$$

$$m_{ij} = \int_0^L \rho A\phi_i(x)\phi_j(x)\mathrm{d}x \tag{8.154}$$

Similar analogous expressions for the string, bar and shaft can be directly expressed respectively as

(a) **For a string**

$$k_{ij} = \int_0^L T\phi_i'(x)\phi_j'(x)\mathrm{d}x \tag{8.155}$$

$$m_{ij} = \int_0^L \rho\phi_i(x)\phi_j(x)\mathrm{d}x \tag{8.156}$$

(b) **For a bar**

$$k_{ij} = \int_0^L EA\phi_i'(x)\phi_j'(x)\mathrm{d}x \tag{8.157}$$

$$m_{ij} = \int_0^L \rho A\phi_i(x)\phi_j(x)\mathrm{d}x \tag{8.158}$$

(c) **For a shaft**

$$k_{ij} = \int_0^L GJ\phi_i'(x)\phi_j'(x)\mathrm{d}x \tag{8.159}$$

$$m_{ij} = \int_0^L \rho J\phi_i(x)\phi_j(x)\mathrm{d}x \tag{8.160}$$

8.9 Assumed Mode Method

Assumed mode method can be used with Lagrange equation to convert governing equation of any continuous system in the form of partial differential equation into a system of ordinary differential equations with only time as an independent variable.

To explain the method, consider the Lagrangian functional for a beam as

$$L = \frac{1}{2}\int_0^L \rho A(\dot{w})^2\mathrm{d}x - \frac{1}{2}\int_0^L EI\left(w''\right)^2\mathrm{d}x \tag{8.161}$$

Then the approximate solution of $w(x, t)$ can be assumed as

$$w(x, t) = \sum \phi_i(x)q_i(t) \tag{8.162}$$

where ϕ_i's are the linearly independent functions which satisfy the boundary conditions of the problem.

Substituting Eq. (8.162) into Eq. (8.161), we get

$$L = \frac{1}{2}\int_0^L \rho A\left\{\sum \phi_i(x)\dot{q}_i(t)\right\}^2\mathrm{d}x - \frac{1}{2}\int_0^L EI\left\{\sum \phi_i''(x)q_i(t)\right\}^2\mathrm{d}x \tag{8.163}$$

Now applying Lagrange equation for a general coordinate q_i, we get

$$\int_0^L \rho A\phi_i(x)\left\{\sum \phi_i(x)\ddot{q}_i(t)\right\}\mathrm{d}x + \int_0^L EI\phi_i''(x)\left\{\sum \phi_i''(x)q_i(t)\right\}\mathrm{d}x = 0 \tag{8.164}$$

Combining such equations for all general coordinates q_i's in matrix form, we get

$$[M]\{\ddot{q}_i(t)\} + [K]\{q_i(t)\} = \{0\} \tag{8.165}$$

where the elements of matrices $[K]$ and $[M]$ are given respectively as

$$k_{ij} = \int_0^L EI\phi_i''(x)\phi_j''(x)\mathrm{d}x \tag{8.166}$$

$$m_{ij} = \int_0^L \rho A\phi_i(x)\phi_j(x)\mathrm{d}x \tag{8.167}$$

Equations (8.166) and (8.167) are identical to Eqs. (8.153) and (8.154). Hence Assumed mode method gives same result as the Rayleigh–Ritz method. Analogous expressions for string, bar and shaft will also be similar and therefore are not repeated here.

8.10 Weighted Residual Method

Weighted residual method is a general method used to determine an approximate analytical solution of any boundary value problem. Weighted residual statement which is also called weighted integral statement is stated as

$$\int_0^L w_i(x)\mathcal{R}\,\mathrm{d}x = 0 \tag{8.168}$$

where w_i is an arbitrary weighting function and $\mathcal{R}$ is the residual remaining in the governing equation when the assumed solution is substituted into it.

To explain the method, consider an equation of motion for a beam undergoing transverse vibration as

$$\frac{\mathrm{d}}{\mathrm{d}x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{8.169}$$

Then the approximate solution of Eq. (8.169) can be assumed as

$$w(x,t) = \sum \phi_i(x)q_i(t) \tag{8.170}$$

where ϕ_i's are the linearly independent functions which satisfy the boundary conditions of the problem.

Substituting Eq. (8.170) into Eq. (8.169), we get residual as

$$\mathcal{R} = \frac{\mathrm{d}}{\mathrm{d}x^2}\left[EI\frac{\partial^2}{\partial x^2}\left\{\sum \phi_i(x)\right\}\right]q_i(t) + \rho A\left\{\sum \phi_i(x)\right\}\ddot{q}_i(t) = 0 \tag{8.171}$$

Now substituting $\mathcal{R}$ from Eq. (8.171) into Eq. (8.168), we get a system of linear equation as

$$[M]\{\ddot{q}_i(t)\} + [K]\{q_i(t)\} = \{0\} \tag{8.172}$$

where the elements of matrices $[K]$ and $[M]$ are given respectively as

$$k_{ij} = \int_0^L w_i(x)\frac{\mathrm{d}}{\mathrm{d}x^2}\left[EI\frac{\partial^2}{\partial x^2}\phi_j(x)\right]\mathrm{d}x \tag{8.173}$$

$$m_{ij} = \int_0^L w_i(x)\left\{\rho A\phi_j(x)\right\}\mathrm{d}x \tag{8.174}$$

Similar analogous expressions for the string, bar and shaft can be directly expressed respectively as

(a) **For a string**

$$k_{ij} = \int_0^L w_i(x)\left\{T\phi_j''(x)\right\}\mathrm{d}x \tag{8.175}$$

$$m_{ij} = \int_0^L w_i(x)\left\{\rho\phi_j(x)\right\}\mathrm{d}x \tag{8.176}$$

(b) **For a bar**

$$k_{ij} = \int_0^L w_i(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[EA\frac{\partial}{\partial x}\phi_j(x)\right]\mathrm{d}x \tag{8.177}$$

$$m_{ij} = \int_0^L w_i(x)\left\{\rho A\phi_j(x)\right\}\mathrm{d}x \tag{8.178}$$

(c) **For a shaft**

$$k_{ij} = \int_0^L w_i(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[GJ\frac{\partial}{\partial x}\phi_j(x)\right]\mathrm{d}x \tag{8.179}$$

$$m_{ij} = \int_0^L w_i(x)\left\{\rho J \phi_j(x)\right\}\mathrm{d}x \tag{8.180}$$

Based on the choice of weight function $w_i(x)$, different methods of weighted residual are named. The common methods of weighted residuals are explained below.

(a) Petrov–Galerkin Method

In this method, weight function w_i's are chosen such that they are different from ϕ_i's, i.e.,

$$w_i(x) \neq \phi_i(x) \tag{8.181}$$

(b) Galerkin Method

In this method, weight function w_i's are taken same as ϕ_i's, i.e.,

$$w_i(x) = \phi_i(x) \tag{8.182}$$

(c) Point Collocation Method

In this method, we seek a combination of q_i's such that the residual or error at some selected points of the domain will be zero. Hence, for this method, the weight function is taken as a Dirac delta function, i.e.,

$$w_i(x) = \delta_d(x - x_i) \tag{8.183}$$

Substituting Eq. (8.183) into Eq. (8.168), we get

$$\mathcal{R}_i(x_i) = 0 \tag{8.184}$$

Solved Examples

Example 8.1

Determine the fundamental natural frequency of a three degree of freedom system shown in Figure E8.1.

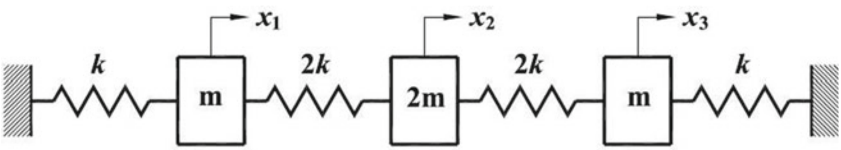


Figure E8.1

Solution

Mass and stiffness matrices of the system are given as

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{bmatrix}$$

Assuming mode shape as $\{X\} = \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix}^T$, we get Rayleigh's quotient as

$$\omega_n^2 = \frac{\{X\}^T[K]\{X\}}{\{X\}^T[M]\{X\}} \tag{a}$$

The numerator and denominator of the quotient can be obtained respectively as

$$\{X\}^T[K]\{X\} = \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix} \begin{bmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} k \\ 0 \\ k \end{Bmatrix} = 2k$$

$$\{X\}^T[M]\{X\} = \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} m \\ 2m \\ m \end{Bmatrix} = 4m$$

Substituting $\{X\}^T[K]\{X\}$ and $\{X\}^T[M]\{X\}$ into Eq. (**a**), we get

$$\omega_n^2 = \frac{2k}{4m} = \frac{0.5k}{m}$$

Then the fundamental natural frequency of the system is determined as

$$\omega_n = \sqrt{\frac{0.5k}{m}} = 0.7071\sqrt{\frac{k}{m}}$$

The exact value of fundamental natural frequency is $0.6622\sqrt{k/m}$. Hence this method gives the value of the fundamental natural frequency 6.78 % higher than the exact value.

Example 8.2

Determine the fundamental natural frequency of a beam carrying three lumped masses as shown in Figure E8.2 by using Rayleigh's method. Assume simply supported end conditions for the beam. Take $M_1 = M_3 = 50$ kg, $M_2 = 80$ kg, $E = 200$ GPa, $I = 2 \times 10^{-6}\,\text{m}^4$, $a = b = 0.4$ m.

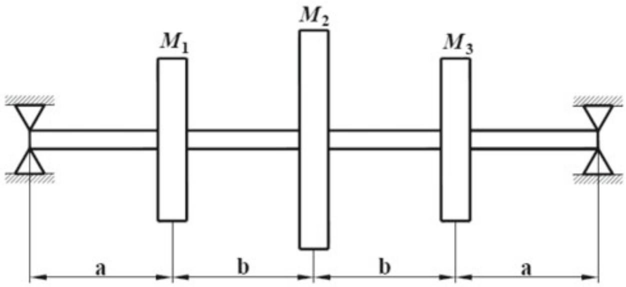


Figure E8.2

Solution

The deflection formula for a simply supported beam shown in **Figure E8.2(a)** is given by

$$w(x) = \begin{cases} \frac{P(L-a)x}{6EIL}\left[L^2 - (L-a)^2 - x^2\right] & x \le a \\ \frac{Pa(L-x)}{6EIL}\left(2Lx - x^2 - a^2\right) & x \ge a \end{cases}$$

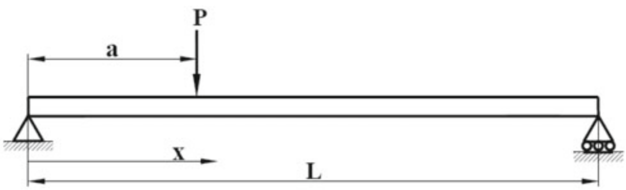


Figure E8.2(a)

Using the given deflection formula, the flexibility influence coefficients of the system are determined as

$$\begin{aligned} a_{11} &= \frac{1 \times (1.6-0.4) \times 0.4}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[(1.6)^2 - (1.2)^2 - (0.4)^2\right] \\ &= 12 \times 10^{-8}\text{m/N} \\ a_{21} &= \frac{1 \times 0.4 \times (1.6-0.8)}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[2 \times 1.6 \times 0.8 - (0.8)^2 - (0.4)^2\right] \\ &= 14.6667 \times 10^{-8}\text{m/N} \\ a_{31} &= \frac{1 \times 0.4 \times (1.6-1.2)}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[2 \times 1.6 \times 1.2 - (1.2)^2 - (0.4)^2\right] \\ &= 9.3333 \times 10^{-8}\text{m/N} \end{aligned}$$

$$
\begin{aligned}
a_{12} &= a_{21} = 14.6667 \times 10^{-8}\mathrm{m/N} \\
a_{22} &= \frac{1 \times (1.6 - 0.8) \times 0.8}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[(1.6)^2 - (0.8)^2 - (0.8)^2\right] \\
&= 21.3333 \times 10^{-8}\mathrm{m/N} \\
a_{32} &= \frac{1 \times 0.8 \times (1.6 - 1.2)}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[2 \times 1.6 \times 1.2 - (1.2)^2 - (0.8)^2\right] \\
&= 14.6667 \times 10^{-8}\mathrm{m/N} \\
a_{13} &= a_{31} = 9.3333 \times 10^{-8}\mathrm{m/N} \\
a_{23} &= a_{32} = 14.6667 \times 10^{-8}\mathrm{m/N} \\
a_{33} &= \frac{1 \times (1.6 - 0.4) \times 1.2}{6 \times 200 \times 10^9 \times 2 \times 10^{-6} \times 1.6}\left[(1.6)^2 - (1.2)^2 - (1.2)^2\right] \\
&= 12 \times 10^{-8}\mathrm{m/N}
\end{aligned}
$$

Then the deflection of the beam at the three given stations can be determined as

$$
\begin{aligned}
W_1 &= a_{11}F_1 + a_{12}F_2 + a_{13}F_3 \\
&= 12 \times 10^{-8} \times 50 \times g + 14.6667 \times 10^{-8} \times 80 \times g \\
&\quad + 9.3333 \times 10^{-8} \times 50 \times g = 22.4 \times 10^{-6} \times g \\
W_2 &= a_{21}F_1 + a_{22}F_2 + a_{23}F_3 \\
&= 14.6667 \times 10^{-8} \times 50 \times g + 21.3333 \times 10^{-8} \times 80 \times g \\
&\quad + 14.6667 \times 10^{-8} \times 50 \times g = 31.7333 \times 10^{-6} \times g \\
W_3 &= a_{31}F_1 + a_{32}F_2 + a_{33}F_3 \\
&= 9.3333 \times 10^{-8} \times 50 \times g + 14.6667 \times 10^{-8} \times 80 \times g \\
&\quad + 12 \times 10^{-8} \times 50 \times g = 22.4 \times 10^{-6} \times g
\end{aligned}
$$

Then we can determine the Rayleigh's quotient of the system as

$$
\begin{aligned}
\omega_n^2 &= \frac{g\sum_{i=1}^{n} M_i W_i}{\sum_{i=1}^{n} M_i W_i^2} \\
&= \frac{g(M_1W_1 + M_2W_2 + M_3W_3)}{\left(M_1W_1^2 + M_2W_2^2 + M_3W_3^2\right)} \\
&= \frac{g\left(50 \times 22.4 \times 10^{-6} \times g + 80 \times 31.7333 \times 10^{-6} \times g + 50 \times 22.4 \times 10^{-6} \times g\right)}{50 \times \left(22.4 \times 10^{-6} \times g\right)^2 + 80 \times \left(31.7333 \times 10^{-6} \times g\right)^2 + 50 \times \left(22.4 \times 10^{-6} \times g\right)^2} \\
&= 36{,}551.9342
\end{aligned}
$$

Then the fundamental natural frequency of the system is determined as

$$
\omega_n = \sqrt{36{,}551.9342} = 191.186\,\mathrm{rad/s}
$$

Example 8.3

Determine the fundamental natural frequency of a linearly tapered bar shown in Figure E8.3 by using Rayleigh's method. The cross sectional area of the bar decreases from A_1 at its left fixed end to A_2 at its right free end.

(a) **Take algebraic function as an assume mode function.**
(b) **Take trigonometric function as an assume mode function.**
(c) **Compare the results obtained in (a) and (b) when $A_1/A_2 = 2$.**

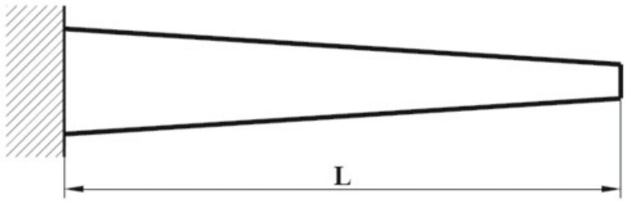


Figure E8.3

Solution

The cross sectional area of the bar at any intermediate distance x is given by

$$A(x) = A_1 - \left(\frac{A_1 - A_2}{L}\right)x$$

(a) An algebraic function which satisfies the boundary conditions for the given bar, i.e. $U(0) = 0$ and $U'(0) = 0$, can be taken as

$$U(x) = ax^2 + bx + c$$

Using the boundary condition $U(0) = 0$, we get $c = 0$. Again Using the boundary condition $U'(L) = 0$ and assuming $a = 1$, we get $b = -2L$. Hence, we assume the mode shape function of the system as $U(x) = x^2 - 2Lx$.

As we know that the Rayleigh's quotient for a bar is given by, i.e.

$$\omega_n^2 = \frac{\int_0^L EA\left(U'\right)^2 \mathrm{d}x}{\int_0^L \rho A(U)^2 \mathrm{d}x} \tag{a}$$

The numerator of the quotient can be determined as

$$\begin{aligned}
\int_0^L EA\left(U'\right)^2 \mathrm{d}x &= \int_0^L E\left[A_1 - \left(\frac{A_1 - A_2}{L}\right)x\right][2(x-L)]^2 \mathrm{d}x \\
&= 4EA_1 \int_0^L (x-L)^2 \mathrm{d}x - 4E\left(\frac{A_1 - A_2}{L}\right) \int_0^L x(x-L)^2 \mathrm{d}x \\
&= 4EA_1 \int_0^L \left(x^2 - 2Lx + L^2\right) \mathrm{d}x \\
&\quad - 4E\left(\frac{A_1 - A_2}{L}\right) \int_0^L \left(x^3 - 2Lx^2 + L^2 x\right) \mathrm{d}x \\
&= 4EA_1\left(\frac{1}{3} - 1 + 1\right)L^3 - 4E(A_1 - A_2)\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right)L^3 \\
&= \frac{4}{3}EA_1L^3 - \frac{1}{3}E(A_1 - A_2)L^3 \\
&= EA_1L^3 + \frac{1}{3}EA_2L^3 = \frac{1}{3}E(3A_1 + A_2)L^3
\end{aligned}$$

Similarly, the denominator of the quotient can be determined as

$$\begin{aligned}
\int_0^L \rho A(U)^2 \mathrm{d}x &= \int_0^L \rho\left[A_1 - \left(\frac{A_1 - A_2}{L}\right)x\right]\left(x^2 - 2Lx\right)^2 \mathrm{d}x \\
&= \rho A_1 \int_0^L \left(x^2 - 2Lx\right)^2 \mathrm{d}x - \rho\left(\frac{A_1 - A_2}{L}\right) \int_0^L x\left(x^2 - 2Lx\right)^2 \mathrm{d}x \\
&= \rho A_1 \int_0^L \left(x^4 - 4Lx^3 + 4L^2x^2\right) \mathrm{d}x \\
&\quad - \rho\left(\frac{A_1 - A_2}{L}\right) \int_0^L \left(x^5 - 4Lx^4 + 4L^2x^3\right) \mathrm{d}x \\
&= \rho A_1\left(\frac{1}{5} - 1 + \frac{4}{3}\right)L^5 - \rho(A_1 - A_2)\left(\frac{1}{6} - \frac{4}{5} + 1\right)L^5 \\
&= \frac{8}{15}\rho A_1L^5 - \frac{11}{30}\rho(A_1 - A_2)L^5 \\
&= \frac{1}{6}\rho A_1L^5 + \frac{11}{30}\rho A_2L^5 = \frac{1}{30}\rho(5A_1 + 11A_2)L^5
\end{aligned}$$

Then the fundamental natural frequency of the system is determined as

$$\omega_n = \sqrt{\frac{1}{3}E(3A_1 + A_2)L^3 \times \frac{30}{\rho(5A_1 + 11A_2)L^5}} = \frac{1}{L}\sqrt{\frac{10E(3A_1 + A_2)}{\rho(5A_1 + 11A_2)}}$$

(b) Again, a trigonometric function which satisfies the boundary conditions for the given bar, i.e. $U(0) = 0$ and $U'(0) = 0$, can be taken as

$$U(x) = \sin\left(\frac{\pi}{2L}x\right)$$

The numerator of the quotient with the assumed trigonometric function can be determined as

$$\begin{aligned}
\int_0^L EA\left(U'\right)^2 \mathrm{d}x &= \int_0^L E\left[A_1 - \left(\frac{A_1 - A_2}{L}\right)x\right]\left[\frac{\pi}{2L}\left\{\cos\left(\frac{\pi}{2L}x\right)\right\}\right]^2 \mathrm{d}x \\
&= \frac{\pi^2}{4L^2}EA_1 \int_0^L \left[\cos\left(\frac{\pi}{2L}x\right)\right]^2 \mathrm{d}x \\
&\quad - \frac{\pi^2}{4L^2}E\left(\frac{A_1 - A_2}{L}\right)\int_0^L x\left[\cos\left(\frac{\pi}{2L}x\right)\right]^2 \mathrm{d}x \\
&= \frac{\pi^2}{8L^2}EA_1 \int_0^L \left[1 + \cos\left(\frac{\pi}{L}x\right)\right]\mathrm{d}x \\
&\quad - \frac{\pi^2}{8L^2}\left(\frac{A_1 - A_2}{L}\right)\int_0^L \left[x + x\cos\left(\frac{\pi}{L}x\right)\right]\mathrm{d}x \\
&= \frac{\pi^2}{8L}EA_1 - \frac{\pi^2}{8L}E(A_1 - A_2)\left(\frac{1}{2} - \frac{2}{\pi^2}\right) \\
&= \frac{EA_1}{L}\left(\frac{\pi^2}{16} + \frac{1}{4}\right) + \frac{EA_2}{L}\left(\frac{\pi^2}{16} - \frac{1}{4}\right) \\
&= \frac{E}{16L}\left[A_1\left(4\pi^2 + 16\right) + A_2\left(4\pi^2 - 16\right)\right]
\end{aligned}$$

Similarly, the denominator of the quotient can be determined as

$$\int_0^L \rho A(U)^2 \mathrm{d}x = \int_0^L \rho\left[A_1 - \left(\frac{A_1 - A_2}{L}\right)x\right]\left\{\sin\left(\frac{\pi}{2L}x\right)\right\}^2 \mathrm{d}x$$

$$= \rho A_1 \int_0^L \left[\sin\left(\frac{\pi}{2L}x\right)\right]^2 \mathrm{d}x - \rho\left(\frac{A_1 - A_2}{L}\right)\int_0^L x\left\{\sin\left(\frac{\pi}{2L}x\right)\right\}^2 \mathrm{d}x$$

$$= \frac{1}{2}\rho A_1 \int_0^L \left[1 - \cos\left(\frac{\pi}{L}x\right)\right]\mathrm{d}x$$

$$- \frac{1}{2}\rho\left(\frac{A_1 - A_2}{L}\right)\int_0^L \left[x - x\cos\left(\frac{\pi}{L}x\right)\right]\mathrm{d}x$$

$$= \frac{1}{2}\rho A_1 L - \frac{1}{2}\rho(A_1 - A_2)\left(\frac{1}{2} + \frac{2}{\pi^2}\right)L$$

$$= \rho A_1 L\left(\frac{1}{4} - \frac{1}{\pi^2}\right) + \rho A_2 L\left(\frac{1}{4} + \frac{1}{\pi^2}\right)$$

$$= \frac{\rho L}{4\pi^2}\left[A_1\left(\pi^2 - 4\right) + A_2\left(\pi^2 + 4\right)\right]$$

Then the fundamental natural frequency of the system is determined as

$$\omega_n = \sqrt{\frac{E}{16L}\left[A_1\left(4\pi^2 + 16\right) + A_2\left(4\pi^2 - 16\right)\right] \times \frac{4\pi^2}{\rho L\left[A_1\left(\pi^2 - 4\right) + A_2\left(\pi^2 + 4\right)\right]}}$$

$$= \frac{\pi}{2L}\sqrt{\frac{E}{\rho}\frac{\left[A_1\left(4\pi^2 + 16\right) + A_2\left(4\pi^2 - 16\right)\right]}{\left[A_1\left(\pi^2 - 4\right) + A_2\left(\pi^2 + 4\right)\right]}}$$

(c) Using $A_1/A_2 = 2$, the fundamental natural frequencies obtained in (**a**) and (**b**) reduce to

$$(\omega_n)_{\text{algebraic}} = \frac{1}{L}\sqrt{\frac{70E}{21\rho}} = \frac{1.8257}{L}\sqrt{\frac{E}{\rho}}$$

$$(\omega_n)_{\text{trigonometric}} = \frac{\pi}{2L}\sqrt{\frac{E\left(\left(12\pi^2 + 16\right)\right)}{\rho\left(2\pi^2 - 4\right)}} = \frac{1.7995}{L}\sqrt{\frac{E}{\rho}}$$

Example 8.4

Determine the fundamental natural frequency of a simply supported beam carrying a concentrated mass *M* as shown in Figure E8.4 by using Rayleigh's method. Equate maximum kinetic energy and potential energy of the system.

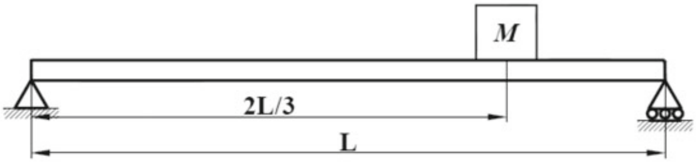


Figure E8.4

Solution

A function which satisfies the boundary conditions for a simply supported beam, i.e. $W(0) = W''(0) = 0$ and $W(L) = W''(L) = 0$, can be taken as

$$W(x) = x^4 + bx^3 + cx^2 + dx + e$$

Using the boundary conditions, we get $b = -2L$, $c = 0$, $d = L^3$ and $e = 0$. Hence, we assume the mode shape function of the system as $W(x) = x^4 - 2Lx^3 + L^3x$.

Then kinetic energy and potential energy of the system can be determined respectively as

$$T = \frac{1}{2}\int_0^L \rho A\dot{w}^2 \mathrm{d}x + \frac{1}{2}M\dot{w}^2\Big|_{x=2L/3}$$

$$V = \frac{1}{2}\int_0^L EI\left(w''\right)^2 \mathrm{d}x$$

Assuming harmonic oscillation, $w(x,t) = W(x)\sin(\omega_n t)$, expressions for kinetic and potential energies reduces to

$$T = \frac{1}{2}\omega_n^2\left[\int_0^L \rho AW^2 \mathrm{d}x + MW^2\big|_{x=2L/3}\right]\cos^2(\omega_n t)$$

$$V = \frac{1}{2}\left[\int_0^L EI\left(W''\right)^2 \mathrm{d}x\right]\sin^2(\omega_n t)$$

Then the maximum kinetic energy and potential energy of the system can be determined respectively as

$$T_{\max} = \frac{1}{2}\omega_n^2\left[\int_0^L \rho A W^2 \mathrm{d}x + M W^2\big|_{x=2L/3}\right]$$

$$V_{\max} = \frac{1}{2}\left[\int_0^L EI\left(W''\right)^2 \mathrm{d}x\right]$$

Equating maximum kinetic energy and potential energy and also substituting assumed deflection, we get

$$\omega_n^2\left[\int_0^L \rho A W^2 \mathrm{d}x + M W^2\big|_{x=2L/3}\right] = \left[\int_0^L EI\left(W''\right)^2 \mathrm{d}x\right]$$

Substituting the assumed deflection, we get

$$\omega_n^2\left[\int_0^L \rho A\left(x^4 - 2Lx^3 + L^3x\right)^2 \mathrm{d}x + M\left(x^4 - 2Lx^3 + L^3x\right)^2\Big|_{x=2L/3}\right]$$

$$= \left[\int_0^L EI\left(12x^2 - 12Lx\right)^2 \mathrm{d}x\right]$$

$$\text{or,}\quad \omega_n^2\left[\frac{31}{630}\rho A L + \frac{484}{6561}M\right]L^8 = \frac{24}{5}EIL^5$$

$$\text{or,}\quad \omega_n^2 = 2{,}204{,}496\left[\frac{EI}{22{,}599\rho A L^4 + 33{,}880 M L^3}\right]$$

$$= \frac{2{,}204{,}496}{22{,}599}\left[\frac{EI}{\rho A L^4 + 1.4992 M L^3}\right]$$

$$\therefore\quad \omega_n = 975.48\sqrt{\frac{EI}{\rho A L^4 + 1.4992 M L^3}}$$

Example 8.5

Determine the fundamental natural frequency of a beam carrying three lumped masses as shown in Figure E8.5 by using Dunkerley's method. Take $M_1 = 60$ kg, $M_2 = 50$ kg, $M_3 = 40$ kg, $E = 200$ GPa, $I = 2 \times 10^{-6}$m^4, $a = b = c = 0.4$ m.

Solution

Deflection at point 1 of the beam due to M_1 only can be determined as

$$\Delta_1 = \frac{M_1 g a^3}{3EI} = \frac{60 \times g \times (0.4)^3}{3 \times 200 \times 10^9 \times 2 \times 10^{-6}} = 3.2 \times 10^{-6}\,\text{g m}$$

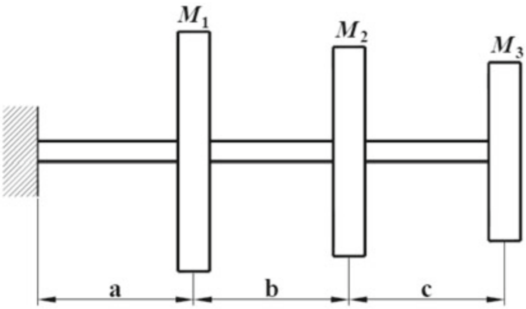


Figure E8.5

Similarly, deflection at point 2 of the beam due to M_2 only can be determined as

$$\Delta_2 = \frac{M_2 g (a+b)^3}{3EI} = \frac{50 \times g \times (0.8)^3}{3 \times 200 \times 10^9 \times 2 \times 10^{-6}} = 21.3333 \times 10^{-6}\,\text{g m}$$

Again, deflection at point 3 of the beam due to M_3 only can be determined as

$$\Delta_3 = \frac{M_3 g (a+b+c)^3}{3EI} = \frac{40 \times g \times (1.2)^3}{3 \times 200 \times 10^9 \times 2 \times 10^{-6}} = 57.6 \times 10^{-6}\,\text{g m}$$

Then the natural frequency of the system due to M_1 only can be determined as

$$\omega_{11}^2 = \frac{g}{\Delta_1} = \frac{1}{3.2 \times 10^{-6}} = 312{,}500$$

Similarly, the natural frequency of the system due to M_2 only can be determined as

$$\omega_{22}^2 = \frac{g}{\Delta_2} = \frac{1}{21.3333 \times 10^{-6}} = 46{,}875$$

Again, the natural frequency of the system due to M_3 only can be determined as

$$\omega_{33}^2 = \frac{g}{\Delta_3} = \frac{1}{57.6 \times 10^{-6}} = 17{,}361.11$$

Then using Dunkerley's formula, we get the fundamental natural frequency as

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}$$

$$\text{or,}\quad \frac{1}{\omega_1^2} = \frac{1}{312{,}500} + \frac{1}{46{,}875} + \frac{1}{17{,}361.11} = 82.13 \times 10^{-6}$$

$$\text{or,}\quad \omega_1^2 = \frac{1}{82.13 \times 10^{-6}} = 12{,}175.3247$$

$$\therefore\quad \omega_1 = \sqrt{12{,}175.3247} = 110.3419\,\text{rad/s}$$

Example 8.6

Determine the natural frequencies and the corresponding mode shapes of the three degree of freedom system shown in Figure E8.6 using matrix iteration method.

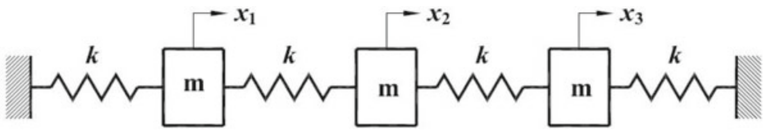


Figure E8.6

Solution

Mass and flexibility matrices of the system are given as (Refer **Example 6.1** for the flexibility matrix)

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad [A] = \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix}$$

For harmonic oscillation, equation of motion of a multi-degree of freedom system can be expressed

$$\{X\} = \omega_n^2 [A][M]\{X\} \tag{a}$$

Substituting $[A]$ and $[M]$ into Eq. (**a**), we get

$$\{X\} = \omega_n^2 \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \{X\} = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \{X\} \tag{b}$$

First Iteration

Assuming mode shape as $\{X\}_0 = \left\{ 1 \; 1 \; 1 \right\}^T$ and substituting it into Eq. (**b**), we get

$$\{X\}_1 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} \frac{3m}{2k} \\ \frac{2m}{k} \\ \frac{3m}{2k} \end{Bmatrix} = \omega_n^2 \frac{3m}{2k} \begin{Bmatrix} 1 \\ 1.3333 \\ 1 \end{Bmatrix}$$

Second Iteration

We get refined mode shape as $\{X\}_1 = \left\{ 1 \ 1.3333 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_2 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 4/3 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.6667\frac{m}{k} \\ 2.3333\frac{m}{k} \\ 1.6667\frac{m}{k} \end{Bmatrix} = 1.6667\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4 \\ 1 \end{Bmatrix}$$

Third Iteration

We get refined mode shape as $\{X\}_2 = \left\{ 1 \ 1.4 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_3 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.7\frac{m}{k} \\ 2.4\frac{m}{k} \\ 1.7\frac{m}{k} \end{Bmatrix} = 1.7\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4117 \\ 1 \end{Bmatrix}$$

Fourth Iteration

We get refined mode shape as $\{X\}_3 = \left\{ 1 \ 1.4117 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_4 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4117 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.7058\frac{m}{k} \\ 2.4117\frac{m}{k} \\ 1.7058\frac{m}{k} \end{Bmatrix} = 1.7058\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4137 \\ 1 \end{Bmatrix}$$

Fifth Iteration

We get refined mode shape as $\{X\}_4 = \left\{ 1 \ 1.4137 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_5 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4137 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.7068\frac{m}{k} \\ 2.4137\frac{m}{k} \\ 1.7068\frac{m}{k} \end{Bmatrix} = 1.7068\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4141 \\ 1 \end{Bmatrix}$$

Sixth Iteration

We get refined mode shape as $\{X\}_5 = \left\{ 1 \ 1.4141 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_6 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4141 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.7070\frac{m}{k} \\ 2.4141\frac{m}{k} \\ 1.7070\frac{m}{k} \end{Bmatrix} = 1.7070\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix}$$

Seventh Iteration

We get refined mode shape as $\{X\}_6 = \left\{ 1 \ 1.4142 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_7 = \omega_n^2 \begin{bmatrix} \frac{3m}{4k} & \frac{m}{2k} & \frac{m}{4k} \\ \frac{m}{2k} & \frac{m}{k} & \frac{m}{2k} \\ \frac{m}{4k} & \frac{m}{2k} & \frac{3m}{4k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 1.7071\frac{m}{k} \\ 2.4142\frac{m}{k} \\ 1.7071\frac{m}{k} \end{Bmatrix} = 1.7071\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix}$$

Since $\{X\}_7 \approx \{X\}_6$,

$$\begin{aligned} &1.7071\frac{m\omega_n^2}{k} = 1 \\ \text{or,} \quad &\omega_n^2 = \frac{k}{1.7071m} \\ \therefore \quad &\omega_n = \sqrt{\frac{k}{1.7071m}} = 0.7637\sqrt{\frac{k}{m}} \end{aligned}$$

Hence, the fundamental natural frequency is $0.7636\sqrt{k/m}$ and the corresponding mode shape is $\left\{ 1 \ 1.4142 \ 1 \right\}^T$.

Iteration for Second Mode

Using Eq. (8.74), we get

$$\{X\} = \omega_n^2[A][M][S_1]\{X\}$$

where

$$[S_1] = \begin{bmatrix} 0 & -\frac{X_{21}}{X_{11}}\frac{m_2}{m_1} & -\frac{X_{31}}{X_{11}}\frac{m_3}{m_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1.4142 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting $[A]$, $[M]$ and $[S_1]$, we get

$$\{X\} = \omega_n^2 \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 0 & -1.4142 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{X\}$$

$$\therefore \quad \{X\} = \omega_n^2 \begin{bmatrix} 0 & -0.5606\frac{m}{k} & -0.5\frac{m}{k} \\ 0 & 0.2929\frac{m}{k} & 0 \\ 0 & 0.1464\frac{m}{k} & 0.5\frac{m}{k} \end{bmatrix} \{X\} \tag{c}$$

First Iteration for Second Mode

Assuming mode shape as $\{X\}_0 = \left\{ 1 \ 1 \ 1 \right\}^T$ and substituting it into Eq. (**c**), we get

$$\{X\}_1 = \omega_n^2 \begin{bmatrix} 0 & -0.5606\frac{m}{k} & -0.5\frac{m}{k} \\ 0 & 0.2929\frac{m}{k} & 0 \\ 0 & 0.1464\frac{m}{k} & 0.5\frac{m}{k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} -1.0606\frac{m}{k} \\ 0.2929\frac{m}{k} \\ 0.6464\frac{m}{k} \end{Bmatrix}$$

$$= \omega_n^2 1.0606\frac{m}{k} \begin{Bmatrix} -1 \\ 0.2761 \\ 0.6094 \end{Bmatrix}$$

Second Iteration for Second Mode

We get refined mode shape as $\{X\}_1 = \left\{ -1 \ 0.2761 \ 0.6094 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_2 = \omega_n^2 \begin{bmatrix} 0 & -0.5606\frac{m}{k} & -0.5\frac{m}{k} \\ 0 & 0.2929\frac{m}{k} & 0 \\ 0 & 0.1464\frac{m}{k} & 0.5\frac{m}{k} \end{bmatrix} \begin{Bmatrix} -1 \\ 0.2761 \\ 0.6094 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} -0.4595\frac{m}{k} \\ 0.0808\frac{m}{k} \\ 0.3451\frac{m}{k} \end{Bmatrix}$$

$$= 0.4595\frac{m\omega_n^2}{k} \begin{Bmatrix} -1 \\ 0.1759 \\ 0.7511 \end{Bmatrix}$$

Third Iteration for Second Mode

We get refined mode shape as $\{X\}_2 = \left\{ -1 \ 0.1759 \ 0.7511 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_3 = \omega_n^2 \begin{bmatrix} 0 & -0.5606\frac{m}{k} & -0.5\frac{m}{k} \\ 0 & 0.2929\frac{m}{k} & 0 \\ 0 & 0.1464\frac{m}{k} & 0.5\frac{m}{k} \end{bmatrix} \begin{Bmatrix} -1 \\ 0.1759 \\ 0.7511 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} -0.4742\frac{m}{k} \\ 0.0515\frac{m}{k} \\ 0.4013\frac{m}{k} \end{Bmatrix}$$

$$= 0.4742\frac{m\omega_n^2}{k} \begin{Bmatrix} -1 \\ 0.1086 \\ 0.8462 \end{Bmatrix}$$

Going through similar iterations, we get second natural frequency as $1.4142\sqrt{k/m}$ and the corresponding mode shape as $\left\{ -1 \ 0 \ 1 \right\}^T$.

Iteration for Third Mode

Using Eqs. (8.75) and (8.76), we get

$$m_1 X_{11} X_1 + m_2 X_{21} X_2 + m_3 X_{31} X_3 = 0$$
$$m_1 X_{12} X_1 + m_2 X_{22} X_2 + m_3 X_{32} X_3 = 0$$

Substituting $m_1 = m_2 = m_3 = m$ and eigen vectors of first and second mode

$$X_1 = -1.4142 X_2 - X_3$$
$$X_1 = X_3$$

Substituting $X_3 = X_1$, first equation reduces to

$$X_2 = -1.4142 X_3$$

Combining with $X_1 = X_3$ and $X_2 = -1.4142 X_3$ with an identity $X_3 = X_3$, we get

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1.4142 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$
$$\therefore \quad \{X\} = [S_2]\{X\}$$

where

$$[S_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1.4142 \\ 0 & 0 & 1 \end{bmatrix}$$

Then equation for the third mode can be expressed as

$$\{X\} = \omega_n^2 [A][M][S_2]\{X\}$$

Substituting $[A]$, $[M]$ and $[S_2]$, we get

$$\{X\} = \omega_n^2 \begin{bmatrix} \frac{3}{4k} & \frac{1}{2k} & \frac{1}{4k} \\ \frac{1}{2k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{4k} & \frac{1}{2k} & \frac{3}{4k} \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1.4142 \\ 0 & 0 & 1 \end{bmatrix} \{X\}$$
$$\therefore \quad \{X\} = \omega_n^2 \begin{bmatrix} 0 & 0 & 0.2929\frac{m}{k} \\ 0 & 0 & -0.4142\frac{m}{k} \\ 0 & 0 & 0.2929\frac{m}{k} \end{bmatrix} \{X\} \qquad \text{(d)}$$

First Iteration for Third Mode

Assuming mode shape as $\{X\}_0 = \left\{ 1 \ 1 \ 1 \right\}^T$ and substituting it into Eq. (**c**), we get

$$\{X\}_1 = \omega_n^2 \begin{bmatrix} 0 & 0 & 0.2929\frac{m}{k} \\ 0 & 0 & -0.4142\frac{m}{k} \\ 0 & 0 & 0.2929\frac{m}{k} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 0.2929\frac{m}{k} \\ -0.4142\frac{m}{k} \\ 0.2929\frac{m}{k} \end{Bmatrix}$$

$$= 0.2929\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ -1.4141 \\ 1 \end{Bmatrix}$$

Second Iteration for Third Mode

We get refined mode shape as $\{X\}_1 = \left\{ 1 \ -1.4141 \ 1 \right\}^T$ and using it as the assumed mode, we get

$$\{X\}_2 = \omega_n^2 \begin{bmatrix} 0 & 0 & 0.2929\frac{m}{k} \\ 0 & 0 & -0.4142\frac{m}{k} \\ 0 & 0 & 0.2929\frac{m}{k} \end{bmatrix} \begin{Bmatrix} 1 \\ -1.4141 \\ 1 \end{Bmatrix} = \omega_n^2 \begin{Bmatrix} 0.2929\frac{m}{k} \\ -0.4142\frac{m}{k} \\ 0.2929\frac{m}{k} \end{Bmatrix}$$

$$= 0.2929\frac{m\omega_n^2}{k} \begin{Bmatrix} 1 \\ -1.4141 \\ 1 \end{Bmatrix}$$

Since $\{X\}_2 \approx \{X\}_1$,

$$0.2929\frac{m\omega_n^2}{k} = 1$$

$$\text{or,} \quad \omega_n^2 = \frac{k}{0.2929m}$$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{0.2929m}} = 1.8477\sqrt{\frac{k}{m}}$$

Hence, the fundamental natural frequency is $1.8477\sqrt{k/m}$ and the corresponding mode shape is $\left\{ 1 \ -1.4141 \ 1 \right\}^T$.

Example 8.7

Determine the fundamental natural frequency and the corresponding mode shape of a three degree of freedom system shown in Figure E8.7 using Stodola's method.

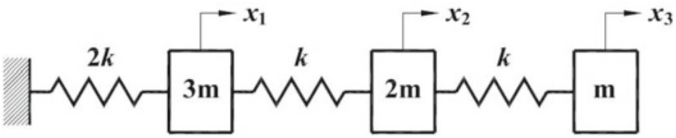


Figure E8.7

Solution

Mass and flexibility matrices of the system are given as

$$[M]=\begin{bmatrix}3m & 0 & 0\\ 0 & 2m & 0\\ 0 & 0 & m\end{bmatrix} \quad \text{and} \quad [A]=\begin{bmatrix}\frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k}\end{bmatrix}$$

First Iteration

Assuming mode shape as $\{X\}_0=\left\{1\ 1\ 1\right\}^T$, we get inertia forces as

$$\begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}_1=\omega_n^2[M]\{X\}_0=\omega_n^2\begin{bmatrix}3m & 0 & 0\\ 0 & 2m & 0\\ 0 & 0 & m\end{bmatrix}\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}=\begin{Bmatrix}3m\omega_n^2\\ 2m\omega_n^2\\ m\omega_n^2\end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix}X_1\\ X_2\\ X_3\end{Bmatrix}_1=[A]\{F\}_1=\begin{bmatrix}\frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k}\end{bmatrix}\begin{Bmatrix}3m\omega_n^2\\ 2m\omega_n^2\\ m\omega_n^2\end{Bmatrix}=\begin{Bmatrix}\frac{3m\omega_n^2}{k}\\ \frac{6m\omega_n^2}{k}\\ \frac{7m\omega_n^2}{k}\end{Bmatrix}=\frac{3m\omega_n^2}{k}\begin{Bmatrix}1\\ 2\\ 2.3333\end{Bmatrix}$$

Second Iteration

Assuming mode shape as $\{X\}_1=\left\{1\ 2\ 2.3333\right\}^T$, we get inertia forces as

$$\begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}_2=\omega_n^2[M]\{X\}_1=\omega_n^2\begin{bmatrix}3m & 0 & 0\\ 0 & 2m & 0\\ 0 & 0 & m\end{bmatrix}\begin{Bmatrix}1\\ 2\\ 2.3333\end{Bmatrix}=\begin{Bmatrix}3m\omega_n^2\\ 4m\omega_n^2\\ 2.3333m\omega_n^2\end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix}X_1\\ X_2\\ X_3\end{Bmatrix}_2=[A]\{F\}_2=\begin{bmatrix}\frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k}\\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k}\end{bmatrix}\begin{Bmatrix}3m\omega_n^2\\ 4m\omega_n^2\\ 2.3333m\omega_n^2\end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{4.6667m\omega_n^2}{k} \\ \frac{11m\omega_n^2}{k} \\ \frac{13.3333m\omega_n^2}{k} \end{Bmatrix} = \frac{4.6667m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.3571 \\ 2.8571 \end{Bmatrix}$$

Third Iteration

Assuming mode shape as $\{X\}_2 = \left\{ 1 \ 2.3571 \ 2.8571 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_3 = \omega_n^2[M]\{X\}_2 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.3571 \\ 2.8571 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.7142m\omega_n^2 \\ 2.8571m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_3 = [A]\{F\}_3 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.7142m\omega_n^2 \\ 2.8571m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.2857m\omega_n^2}{k} \\ \frac{12.8571m\omega_n^2}{k} \\ \frac{15.7142m\omega_n^2}{k} \end{Bmatrix} = \frac{5.2857m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4324 \\ 2.9729 \end{Bmatrix}$$

Fourth Iteration

Assuming mode shape as $\{X\}_3 = \left\{ 1 \ 2.4324 \ 2.9729 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_4 = \omega_n^2[M]\{X\}_3 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.4324 \\ 2.9729 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.8648m\omega_n^2 \\ 2.9729m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_4 = [A]\{F\}_4 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.8648m\omega_n^2 \\ 2.9729m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.4189m\omega_n^2}{k} \\ \frac{13.2567m\omega_n^2}{k} \\ \frac{16.2297m\omega_n^2}{k} \end{Bmatrix} = \frac{5.4189m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4463 \\ 2.9950 \end{Bmatrix}$$

Fifth Iteration

Assuming mode shape as $\{X\}_4 = \left\{ 1 \ 2.4463 \ 2.9950 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_5 = \omega_n^2[M]\{X\}_4 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.4463 \\ 2.9950 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.8927m\omega_n^2 \\ 2.9950m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_5 = [A]\{F\}_5 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.8927m\omega_n^2 \\ 2.9950m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.4438m\omega_n^2}{k} \\ \frac{13.3316m\omega_n^2}{k} \\ \frac{16.3266m\omega_n^2}{k} \end{Bmatrix} = \frac{5.4438m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4489 \\ 2.9990 \end{Bmatrix}$$

Sixth Iteration

Assuming mode shape as $\{X\}_5 = \left\{ 1 \ 2.4489 \ 2.9990 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_6 = \omega_n^2[M]\{X\}_5 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.4489 \\ 2.9990 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.8978m\omega_n^2 \\ 2.9990m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_6 = [A]\{F\}_6 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.8978m\omega_n^2 \\ 2.9990m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.4484m\omega_n^2}{k} \\ \frac{13.3453m\omega_n^2}{k} \\ \frac{16.3444m\omega_n^2}{k} \end{Bmatrix} = \frac{5.4484m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4493 \\ 2.9998 \end{Bmatrix}$$

Seventh Iteration

Assuming mode shape as $\{X\}_6 = \left\{ 1 \ 2.4493 \ 2.9998 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_7 = \omega_n^2[M]\{X\}_6 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.4493 \\ 2.9998 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.8987m\omega_n^2 \\ 2.9998m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_7 = [A]\{F\}_7 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.8987m\omega_n^2 \\ 2.9998m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.4493m\omega_n^2}{k} \\ \frac{13.3479m\omega_n^2}{k} \\ \frac{16.3477m\omega_n^2}{k} \end{Bmatrix} = \frac{5.4493m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4494 \\ 2.9999 \end{Bmatrix}$$

Eighth Iteration

Assuming mode shape as $\{X\}_7 = \left\{ 1 \ 2.4494 \ 2.9999 \right\}^T$, we get inertia forces as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_8 = \omega_n^2[M]\{X\}_7 = \omega_n^2 \begin{bmatrix} 3m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.4494 \\ 2.9999 \end{Bmatrix} = \begin{Bmatrix} 3m\omega_n^2 \\ 4.8989m\omega_n^2 \\ 2.9999m\omega_n^2 \end{Bmatrix}$$

Then deflection of each mass due to these forces can be determined as

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_8 = [A]\{F\}_8 = \begin{bmatrix} \frac{1}{2k} & \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{3}{2k} \\ \frac{1}{2k} & \frac{3}{2k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} 3m\omega_n^2 \\ 4.8989m\omega_n^2 \\ 2.9999m\omega_n^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{5.4494m\omega_n^2}{k} \\ \frac{13.3483m\omega_n^2}{k} \\ \frac{16.3483m\omega_n^2}{k} \end{Bmatrix} = \frac{5.4494m\omega_n^2}{k} \begin{Bmatrix} 1 \\ 2.4494 \\ 2.9999 \end{Bmatrix}$$

Since $\{X\}_8 \approx \{X\}_7$,

$$\frac{5.4494m\omega_n^2}{k} = 1$$

$$\text{or,} \quad \omega_n^2 = \frac{k}{5.4494m}$$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{5.4494m}} = 0.4283\sqrt{\frac{k}{m}}$$

Hence, the fundamental natural frequency is $0.4283\sqrt{k/m}$ and the corresponding mode shape is $\left\{ 1 \ 2.4494 \ 2.9999 \right\}^T$.

Example 8.8

Determine the natural frequencies and the corresponding mode shapes of a system shown in Figure E8.8 using Holzer's method. Take $m = 10$ kg and $k = 100$ N/m.

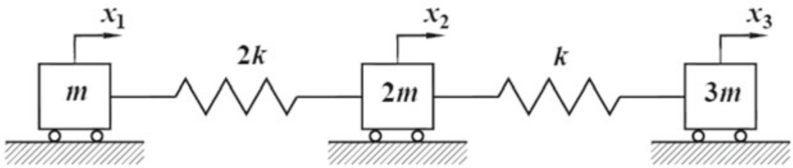


Figure E8.8

Solution

Point matrix for station 1 can be determined as

$$[P]_1 = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix}$$

Similarly, transfer matrices for stations $1-2$ and $2-3$ can be determined as

$$[T]_2 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{2k} \\ -2m\omega^2 & 1-\frac{2m\omega^2}{2k} \end{bmatrix} = \begin{bmatrix} 1 & 0.005 \\ -20\omega^2 & 1-0.1\omega^2 \end{bmatrix}$$
$$[T]_3 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k} \\ -3m\omega^2 & 1-\frac{3m\omega^2}{k} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -30\omega^2 & 1-0.3\omega^2 \end{bmatrix}$$

Then overall transfer matrix of the system can be determined as

$$[U] = [T]_3[T]_2[P]_1 = \begin{bmatrix} 1 & 0.01 \\ -30\omega^2 & 1-0.3\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0.005 \\ -20\omega^2 & 1-0.1\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.01\omega^4 - 0.35\omega^2 + 1 & -0.001\omega^2 + 0.015 \\ -\omega^2\left(0.3\omega^4 - 11.5\omega^2 + 60\right) & 0.03\omega^4 - 0.55\omega^2 + 1 \end{bmatrix}$$

Frequency equation for a system free at both ends is given by

$$\begin{aligned} &U_{21} = 0 \\ \text{or,}\quad &\omega^2\left(0.3\omega^4 - 11.5\omega^2 + 60\right) = 0 \\ \text{or,}\quad &\omega_1^2 = 0,\quad \omega_2^2 = 6.2298,\quad \omega_3^2 = 32.1034 \\ \therefore\quad &\omega_1 = 0,\quad \omega_2 = 2.4959\,\text{rad/s},\quad \omega_3 = 5.6659\,\text{rad/s} \end{aligned}$$

The state vectors between left and right side of the station 1 can be related as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}\begin{Bmatrix} X \\ F \end{Bmatrix}_1^L$$

Assuming $X_1^L = 1$ and substituting $F_1^L = 0$ for the free end, we get

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -10\omega^2 \end{Bmatrix}$$

Now the relationship between state vectors between stations 1 and 2 can expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{2k} \\ -2m\omega^2 & 1 - \frac{2m\omega^2}{2k} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R$$
$$= \begin{bmatrix} 1 & 0.005 \\ -20\omega^2 & 1 - 0.1\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ -10\omega^2 \end{Bmatrix} = \begin{Bmatrix} 1 - 0.05\omega^2 \\ -30\omega^2 + \omega^4 \end{Bmatrix} \quad \text{(a)}$$

Similarly, the relationship between state vectors between stations 2 and 3 can expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k} \\ -3m\omega^2 & 1 - \frac{3m\omega^2}{k} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R$$
$$= \begin{bmatrix} 1 & 0.01 \\ -30\omega^2 & 1 - 0.3\omega^2 \end{bmatrix} \begin{Bmatrix} 1 - 0.05\omega^2 \\ -30\omega^2 + \omega^4 \end{Bmatrix} = \begin{Bmatrix} 1 - 0.08\omega^2 + 0.01\omega^4 \\ -60\omega^2 + 11.5\omega^4 - 0.3\omega^6 \end{Bmatrix} \quad \text{(b)}$$

Substituting $\omega = \omega_1 = 0$ into Eqs. (**a**) and (**b**), we get $X_2 = 1$ and $X_3 = 1$. Hence the mode shape corresponding the first mode is $\{1 \; 1 \; 1\}^T$.

Similarly, substituting $\omega = \omega_2 = 2.4959$ rad/s into Eqs. (**a**) and (**b**), we get $X_2 = 0.6885$ and $X_3 = -0.7923$. Hence the mode shape corresponding the second mode is $\{1 \; 0.6885 \; -0.7923\}^T$.

Again, substituting $\omega = \omega_3 = 5.6659$ rad/s into Eqs. (**a**) and (**b**), we get $X_2 = -0.6051$ and $X_3 = 0.0701$. Hence the mode shape corresponding the third mode is $\{1 \; -0.6051 \; 0.0701\}^T$.

Alternative Method for Natural Frequencies (Trial and Error)

We start with an assumption $X_1^L = 1$, and successive amplitudes are determined by using Eqs. (8.92) and (8.97).

Trial 1

Assuming $\omega^2 = 0$, Eqs. (8.92) and (8.97) give

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1-\frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1-\frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\therefore \quad F_3\left(\omega^2 = 0\right) = 0$$

Trial 2

Assuming $\omega = 1\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -10 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1-\frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -20 & 0.9 \end{bmatrix} \begin{Bmatrix} 1 \\ -10 \end{Bmatrix} = \begin{Bmatrix} 0.95 \\ -29 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1-\frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ -30 & 0.7 \end{bmatrix} \begin{Bmatrix} 0.95 \\ -29 \end{Bmatrix} = \begin{Bmatrix} 0.66 \\ -48.8 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 1) = -48.8\,\text{N}$$

Trial 3

Assuming $\omega = 2\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -40 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -40 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1-\frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -80 & 0.6 \end{bmatrix} \begin{Bmatrix} 1 \\ -40 \end{Bmatrix} = \begin{Bmatrix} 0.8 \\ -104 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1-\frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ -120 & -0.2 \end{bmatrix} \begin{Bmatrix} 0.8 \\ -104 \end{Bmatrix} = \begin{Bmatrix} -0.24 \\ -75.2 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 2) = -75.2\,\text{N}$$

Trial 4

Assuming $\omega = 3\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -90 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -90 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1 - \frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -180 & 0.1 \end{bmatrix} \begin{Bmatrix} 1 \\ -90 \end{Bmatrix} = \begin{Bmatrix} 0.55 \\ -189 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1 - \frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ -270 & -1.7 \end{bmatrix} \begin{Bmatrix} 0.55 \\ -189 \end{Bmatrix} = \begin{Bmatrix} -1.34 \\ 172.8 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 3) = 172.8\,\text{N}$$

Trial 5

Assuming $\omega = 4\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -160 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -160 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1 - \frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -320 & -0.6 \end{bmatrix} \begin{Bmatrix} 1 \\ -160 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ -224 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1 - \frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ -480 & -3.8 \end{bmatrix} \begin{Bmatrix} 0.2 \\ -224 \end{Bmatrix} = \begin{Bmatrix} -2.04 \\ 755.2 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 4) = 755.2\,\text{N}$$

Trial 6

Assuming $\omega = 5\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -250 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -250 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1 - \frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -500 & -1.5 \end{bmatrix} \begin{Bmatrix} 1 \\ -250 \end{Bmatrix} = \begin{Bmatrix} -0.25 \\ -125 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1 - \frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & 0.01 \\ -750 & -6.5 \end{bmatrix} \begin{Bmatrix} 0.25 \\ -125 \end{Bmatrix} = \begin{Bmatrix} -1.5 \\ 1000 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 5) = 1000\,\text{N}$$

Trial 7

Assuming $\omega = 6\,\text{rad/s}$,

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^L = \begin{bmatrix} 1 & 0 \\ -360 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -360 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & 1 - \frac{m_2\omega^2}{k_2} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R$$

$$= \begin{bmatrix} 1 & 0.005 \\ -720 & -2.6 \end{bmatrix} \begin{Bmatrix} 1 \\ -360 \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ 216 \end{Bmatrix}$$

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & 1 - \frac{m_3\omega^2}{k_3} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R$$

$$= \begin{bmatrix} 1 & 0.01 \\ -1080 & -9.8 \end{bmatrix} \begin{Bmatrix} -0.8 \\ 216 \end{Bmatrix} = \begin{Bmatrix} 1.36 \\ -1252.8 \end{Bmatrix}$$

$$\therefore \quad F_3(\omega = 6) = -1252.8\,\text{N}$$

Magnitudes of force at station 3 (F_3) for different trial frequencies are listed in **Table E8.8**.

Table E8.8

ω (rad/s)	0	1	2	3	4	5	6
F_3(N)	0	−48.8	−75.2	172.8	755.2	1000	−1252.8

Variation of F_3 with ω is shown in **Figure E8.8(a)**.

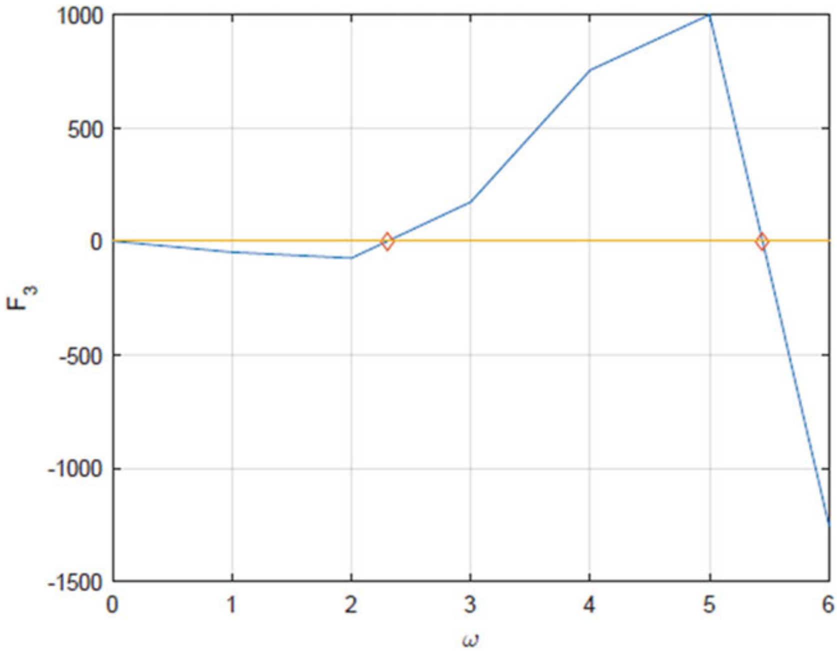


Figure E8.8(a): Variation of F_3 with ω for Example 8.8

Since the station 3 is free, F_3 should be 0 at this station. Therefore the values of ω which make $F_3 = 0$ are the natural frequencies of the system. It can be noticed from the plot that the natural frequencies of the system are 0, 2.4 and 5.5 rad/s.

Example 8.9

Determine the natural frequencies and the corresponding mode shapes of a system shown in Figure E8.9 using Holzer's method. Take $I = 1\,\text{kg m}^2$ and $k_t = 100\,\text{N m/rad}$.

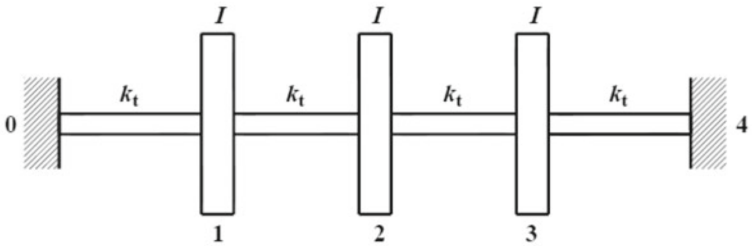


Figure E8.9

Solution

Using analogous parameters, transfer matrices for stations 0 − 1, 1 − 2 and 2 − 3 can be determined as

$$[T]_1 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix}_1 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1 - 0.01\omega^2 \end{bmatrix}$$

$$[T]_2 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1 - 0.01\omega^2 \end{bmatrix}$$

$$[T]_3 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1 - \frac{I\omega^2}{k_t} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1 - 0.01\omega^2 \end{bmatrix}$$

Field matrix for the shaft between stations 3 and 4 can be determined as

$$[F]_4 = \begin{bmatrix} 1 & \frac{1}{k_t} \\ 0 & 1 \end{bmatrix}_4 = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix}$$

Then overall transfer matrix of the system can be determined as

$$
\begin{aligned}
[U] &= [F]_4[T]_3[T]_2[T]_1 \\
&= \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \\
&= \begin{bmatrix} -1\times 10^{-6}\omega^6 + 5\times 10^{-4}\omega^4 - 0.06\omega^2 + 1 & -1\times 10^{-8}\omega^6 + 6\times 10^{-6}\omega^4 - 0.001\omega^2 + 0.04 \\ -0.0001\omega^6 + 0.04\omega^4 - 3\omega^2 & -1\times 10^{-6}\omega^6 + 5\times 10^{-4}\omega^4 - 0.06\omega^2 + 1 \end{bmatrix}
\end{aligned}
$$

Frequency equation for a system fixed at both ends is given by

$$
\begin{aligned}
&U_{12} = 0 \\
&\text{or,}\quad -1\times 10^{-8}\omega^6 + 6\times 10^{-6}\omega^4 - 0.001\omega^2 + 0.04 = 0 \\
&\text{or,}\quad \omega_1^2 = 58.5786,\quad \omega_2^2 = 200,\quad \omega_3^2 = 341.4213 \\
&\therefore\quad \omega_1 = 7.6536\,\text{rad/s},\quad \omega_2 = 14.1421\,\text{rad/s},\quad \omega_3 = 18.4775\,\text{rad/s}
\end{aligned}
$$

Now the relationship between state vectors between stations 0 and 1 can expressed as

$$
\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1-\frac{I\omega^2}{k_t} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_0
$$

Since $\theta_0 = 0$, we get

$$
\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \begin{Bmatrix} 0 \\ T_{r,0} \end{Bmatrix} = \begin{Bmatrix} 0.01T_{r,0} \\ \left(1-0.01\omega^2\right) \end{Bmatrix}
$$

For the mode shape, we need relative amplitudes of each disk. Dividing each element of the state vector of station 1 by $0.01T_{r,0}$, we get refined state vector for the station 1 as

$$
\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R = \begin{Bmatrix} 1 \\ \left(100-\omega^2\right)T_{r,0} \end{Bmatrix}
$$

Similarly, the relationship between state vectors between stations 1 and 2 can expressed as

$$
\begin{aligned}
\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2^R &= \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1-\frac{I\omega^2}{k_t} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1 \\
\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2^R &= \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ \left(100-\omega^2\right) \end{Bmatrix} = \begin{Bmatrix} \left(2-0.01\omega^2\right) \\ \left(100-3\omega^2+0.01\omega^4\right) \end{Bmatrix} \qquad \text{(a)}
\end{aligned}
$$

Again, the relationship between state vectors between stations 2 and 3 can expressed as

$$\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_t} \\ -I\omega^2 & 1-\frac{I\omega^2}{k_t} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2$$

$$= \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1-0.01\omega^2 \end{bmatrix} \begin{Bmatrix} \left(2-0.01\omega^2\right) \\ \left(100-3\omega^2+0.01\omega^4\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(3-0.04\omega^2+0.0001\omega^4\right) \\ \left(100-6\omega^2+0.05\omega^4-0.0001\omega^6\right) \end{Bmatrix} \tag{b}$$

Substituting $\omega = \omega_1 = 7.6536\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $\theta_2 = 1.4142$ and $\theta_3 = 1$. Hence the mode shape corresponding the first mode is $\left\{ 1 \; 1.4142 \; 1 \right\}^T$.

Similarly, substituting $\omega = \omega_2 = 14.1421\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $\theta_2 = 0$ and $\theta_3 = -1$. Hence the mode shape corresponding the second mode is $\left\{ 1 \; 0 \; -1 \right\}^T$.

Again, substituting $\omega = \omega_3 = 5.6659\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $\theta_2 = -1.4142$ and $\theta_3 = 1$. Hence the mode shape corresponding the third mode is $\left\{ 1 \; -1.4142 \; 1 \right\}^T$.

Example 8.10

Determine the natural frequencies and the corresponding mode shapes of a system shown in Figure E8.10 using Holzer's method. Take $m = 10\,\text{kg}$ and $k = 100\,\text{N/m}$.

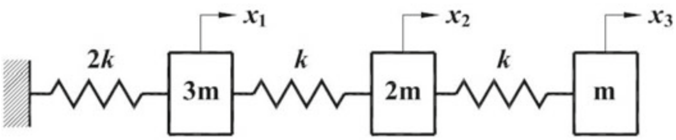


Figure E8.10

Solution

Transfer matrices for stations $0-1$, $1-2$ and $2-3$ can be determined as

$$[T]_1 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_1 = \begin{bmatrix} 1 & \frac{1}{2k} \\ -3m\omega^2 & 1-\frac{3m\omega^2}{2k} \end{bmatrix} = \begin{bmatrix} 1 & 0.005 \\ -30\omega^2 & 1-0.15\omega^2 \end{bmatrix}$$

$$[T]_2 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k} \\ -2m\omega^2 & 1-\frac{2m\omega^2}{k} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -20\omega^2 & 1-0.2\omega^2 \end{bmatrix}$$

$$[T]_3 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -10\omega^2 & 1-0.1\omega^2 \end{bmatrix}$$

Then overall transfer matrix of the system can be determined as

$$
\begin{aligned}
[U] &= [T]_3[T]_2[T]_1 \\
&= \begin{bmatrix} 1 & 0.01 \\ -10\omega^2 & 1-0.1\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0.01 \\ -20\omega^2 & 1-0.2\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0.005 \\ -30\omega^2 & 1-0.15\omega^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-0.8\omega^2+0.06\omega^4 & 0.025-0.06\omega^2+0.0003\omega^4 \\ -60\omega^2+14\omega^4-0.6\omega^6 & 1-0.7\omega^2+0.09\omega^4-0.003\omega^6 \end{bmatrix}
\end{aligned}
$$

Frequency equation for a system fixed at the left end and free at the right end is given by

$$
\begin{aligned}
& U_{22} = 0 \\
& \text{or,} \quad 1-0.7\omega^2+0.09\omega^4-0.003\omega^6 = 0 \\
& \text{or,} \quad \omega_1^2 = 1.8350, \quad \omega_2^2 = 10, \quad \omega_3^2 = 18.1649 \\
& \therefore \quad \omega_1 = 1.3546\,\text{rad/s}, \quad \omega_2 = 3.1622\,\text{rad/s}, \quad \omega_3 = 4.2620\,\text{rad/s}
\end{aligned}
$$

Now the relationship between state vectors between stations 1 and 2 can expressed as

$$
\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \frac{1}{2k} \\ -3m\omega^2 & 1-\frac{3m\omega^2}{2k} \end{bmatrix}\begin{Bmatrix} X \\ F \end{Bmatrix}_0 = \begin{bmatrix} 1 & 0.005 \\ -30\omega^2 & 1-0.15\omega^2 \end{bmatrix}\begin{Bmatrix} X \\ F \end{Bmatrix}_0
$$

Since $X_0 = 0$, we get

$$
\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0.005 \\ -30\omega^2 & 1-0.15\omega^2 \end{bmatrix}\begin{Bmatrix} 0 \\ F_{r,0} \end{Bmatrix} = \begin{Bmatrix} 0.005F_{r,0} \\ \left(1-0.15\omega^2\right)F_{r,0} \end{Bmatrix}
$$

For the mode shape, we need relative amplitudes of each disk. Dividing each element of the state vector of station 1 by $0.005F_{r,0}$, we get refined state vector for the station 1 as

$$
\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{Bmatrix} 1 \\ \left(200-30\omega^2\right)F_{r,0} \end{Bmatrix}
$$

Similarly, the relationship between state vectors between stations 1 and 2 can expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{k} \\ -2m\omega^2 & 1 - \frac{2m\omega^2}{k} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R$$

$$= \begin{bmatrix} 1 & 0.01 \\ -20\omega^2 & 1 - 0.2\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ \left(200 - 30\omega^2\right) F_{r,0} \end{Bmatrix} = \begin{Bmatrix} \left(3 - 0.3\omega^2\right) \\ \left(200 - 90\omega^2 + 6\omega^4\right) \end{Bmatrix} \tag{a}$$

Again, the relationship between state vectors between stations 2 and 3 can expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1 - \frac{m\omega^2}{k} \end{bmatrix} \begin{Bmatrix} X \\ F \end{Bmatrix}_2$$

$$= \begin{bmatrix} 1 & 0.01 \\ -\omega^2 & 1 - 0.01\omega^2 \end{bmatrix} \begin{Bmatrix} \left(2 - 0.3\omega^2\right) \\ \left(200 - 90\omega^2 + 6\omega^4\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(5 - 1.2\omega^2 + 0.06\omega^4\right) \\ \left(200 - 140\omega^2 + 18\omega^4 - 0.6\omega^6\right) \end{Bmatrix} \tag{b}$$

Substituting $\omega = \omega_1 = 1.3546\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $X_2 = 2.4495$ and $X_3 = 3$. Hence the mode shape corresponding the first mode is $\begin{Bmatrix} 1 & 2.4495 & 3 \end{Bmatrix}^T$.

Similarly, substituting $\omega = \omega_2 = 3.1622\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $X_2 = 0$ and $X_3 = -1$. Hence the mode shape corresponding the second mode is $\begin{Bmatrix} 1 & 0 & -1 \end{Bmatrix}^T$.

Again, substituting $\omega = \omega_3 = 4.2620\,\text{rad/s}$ into Eqs. (**a**) and (**b**), we get $X_2 = -2.4495$ and $X_3 = 3$. Hence the mode shape corresponding the third mode is $\begin{Bmatrix} 1 & -2.4495 & 3 \end{Bmatrix}^T$.

Example 8.11

Determine the natural frequencies and the corresponding mode shapes of a system shown in Figure E8.11 using Holzer's method. Take $I_1 = 10\,\text{kg m}^2$, $I_2 = 8\,\text{kg m}^2$, $I_3 = I_5 = 4\,\text{kg m}^2$, $I_4 = I_6 = 6\,\text{kg m}^2$, $k_{t1} = 120\,\text{N m/rad}$, $k_{t2} = k_{t4} = 100\,\text{N m/rad}$ and $k_{t3} = k_{t5} = 80\,\text{N m/rad}$.

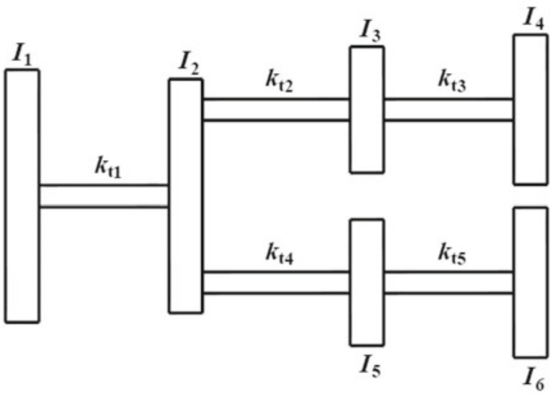


Figure E8.11

Solution

Point matrix for the station 1 and transfer matrices for stations $1-2$ of the main branch can be determined as

$$[P]_1 = \begin{bmatrix} 1 & 0 \\ -I\omega^2 & 1 \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ -I_1\omega^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix}$$

$$[T]_2 = \begin{bmatrix} 1 & \frac{1}{k} \\ -I\omega^2 & 1-\frac{I\omega^2}{k} \end{bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_{t1}} \\ -I_2\omega^2 & 1-\frac{I_2\omega^2}{k_{t1}} \end{bmatrix} = \begin{bmatrix} 1 & 0.00833 \\ -8\omega^2 & 1-0.0667\omega^2 \end{bmatrix}$$

Similarly, transfer matrices for stations $2-3$ and $3-4$ for the upper right branch can be determined as

$$[T]_3 = \begin{bmatrix} 1 & \frac{1}{k} \\ -I\omega^2 & 1-\frac{I\omega^2}{k} \end{bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k_{t2}} \\ -I_3\omega^2 & 1-\frac{I_3\omega^2}{k_{t2}} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1-0.04\omega^2 \end{bmatrix}$$

$$[T]_4 = \begin{bmatrix} 1 & \frac{1}{k} \\ -I\omega^2 & 1-\frac{I\omega^2}{k} \end{bmatrix}_4 = \begin{bmatrix} 1 & \frac{1}{k_{t3}} \\ -I_4\omega^2 & 1-\frac{I_4\omega^2}{k_{t3}} \end{bmatrix} = \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1-0.075\omega^2 \end{bmatrix}$$

Similarly, transfer matrices for stations $2-5$ and $5-6$ for the lower right branch can be determined as

$$[T]_5 = \begin{bmatrix} 1 & \frac{1}{k} \\ -I\omega^2 & 1-\frac{I\omega^2}{k} \end{bmatrix}_5 = \begin{bmatrix} 1 & \frac{1}{k_{t4}} \\ -I_5\omega^2 & 1-\frac{I_5\omega^2}{k_{t4}} \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1-0.04\omega^2 \end{bmatrix}$$

$$[T]_6 = \begin{bmatrix} 1 & \frac{1}{k} \\ -I\omega^2 & 1-\frac{I\omega^2}{k} \end{bmatrix}_6 = \begin{bmatrix} 1 & \frac{1}{k_{t5}} \\ -I_6\omega^2 & 1-\frac{I_6\omega^2}{k_{t5}} \end{bmatrix} = \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1-0.075\omega^2 \end{bmatrix}$$

Then overall transfer matrix of the main branch of the system can be determined as

$$\begin{aligned}[U]_A &= [T]_2[P]_1 \\ &= \begin{bmatrix} 1 & 0.00833 \\ -8\omega^2 & 1-0.06667\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-0.0833\omega^2 & 0.00833 \\ -18\omega^2+0.6667\omega^4 & 1-0.06667\omega^2 \end{bmatrix}\end{aligned}$$

Similarly, overall transfer matrix of the right upper branch of the system can be determined as

$$\begin{aligned}[U]_B &= [T]_4[T]_3 \\ &= \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1-0.075\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1-0.04\omega^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-0.05\omega^2 & 0.0225-0.0005\omega^2 \\ -10\omega^2+0.3\omega^4 & 1-0.175\omega^2+0.003\omega^4 \end{bmatrix}\end{aligned}$$

Again, overall transfer matrix of the right lower branch of the system can be determined as

$$\begin{aligned}[U]_C &= [T]_6[T]_5 \\ &= \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1-0.075\omega^2 \end{bmatrix}\begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1-0.04\omega^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-0.05\omega^2 & 0.0225-0.0005\omega^2 \\ -10\omega^2+0.3\omega^4 & 1-0.175\omega^2+0.003\omega^4 \end{bmatrix}\end{aligned}$$

Then frequency equation of the system [Eq. (8.119)] can be expressed as

$$\begin{aligned}&U_{21A}U_{22B}U_{22C} + U_{11A}(U_{21B}U_{22C} + U_{21C}U_{22B}) = 0 \\ &\text{or,}\quad \left(-18\omega^2+0.6667\omega^4\right)\left(1-0.175\omega^2+0.003\omega^4\right)\left(1-0.175\omega^2+0.003\omega^4\right) \\ &\quad + \left(1-0.0833\omega^2\right)\left[\left(-10\omega^2+0.3\omega^4\right)\left(1-0.175\omega^2+0.003\omega^4\right)\right. \\ &\quad \left. + \left(-10\omega^2+0.3\omega^4\right)\left(1-0.175\omega^2+0.003\omega^4\right)\right] = 0 \\ &\text{or,}\quad \omega^2(6\times10^{-6}\omega^{10} - 0.001012\omega^8 + 0.058867\omega^6 \\ &\quad - 1.399225\omega^4 + 12.7333\omega^2 - 38) = 0 \\ &\text{or,}\quad \omega_1^2 = 0;\ \omega_2^2 = 6.421;\ \omega_3^2 = 8.873;\ \omega_4^2 = 29.943;\ \omega_5^2 = 51.912;\ \omega_6^2 = 71.518 \\ &\therefore\quad \omega_1 = 0, \omega_1 = 2.53\,\text{rad/s}, \omega_2 = 2.978\,\text{rad/s}, \omega_4 = 5.472\,\text{rad/s}, \\ &\omega_5 = 7.205\,\text{rad/s}, \omega_6 = 2.534\,\text{rad/s}\end{aligned}$$

The state vectors between left and right side of the station 1 can be related as

$$\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -I_1\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^L$$

Assuming $\theta_1^L = 1$ and substituting $T_r,_1^L = 0$ for the free end, we get

$$\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -10\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -10\omega^2 \end{Bmatrix}$$

Now the relationship between state vectors between stations 1 and 2 can expressed as

$$\begin{aligned} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2^R &= \begin{bmatrix} 1 & \frac{1}{k_{t1}} \\ -I_2\omega^2 & 1 - \frac{I_2\omega^2}{k_{t1}} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_1^R \\ &= \begin{bmatrix} 1 & 0.00833 \\ -8\omega^2 & 1 - 0.06667\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ -10\omega^2 \end{Bmatrix} = \begin{Bmatrix} 1 - 0.0833\omega^2 \\ -18\omega^2 + 0.6667\omega^4 \end{Bmatrix} \end{aligned} \quad \text{(a)}$$

Similarly, the relationship between state vectors between stations $2-3$, $3-4$, $2-5$ and $2-6$ can expressed as

$$\begin{aligned} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_3^R &= \begin{bmatrix} 1 & \frac{1}{k_{t2}} \\ -I_3\omega^2 & 1 - \frac{I_3\omega^2}{k_{t2}} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2^R \\ &= \begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1 - 0.04\omega^2 \end{bmatrix} \begin{Bmatrix} 1 - 0.0833\omega^2 \\ -18\omega^2 + 0.6667\omega^4 \end{Bmatrix} \\ &= \begin{Bmatrix} 1 - 0.2633\omega^2 + 0.00667\omega^4 \\ -22\omega^2 + 1.72\omega^4 - 0.02667\omega^6 \end{Bmatrix} \end{aligned} \quad \text{(b)}$$

$$\begin{aligned} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_4^R &= \begin{bmatrix} 1 & \frac{1}{k_{t3}} \\ -I_4\omega^2 & 1 - \frac{I_4\omega^2}{k_{t3}} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_3^R \\ &= \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1 - 0.075\omega^2 \end{bmatrix} \begin{Bmatrix} 1 - 0.2633\omega^2 + 0.00667\omega^4 \\ -22\omega^2 + 1.72\omega^4 - 0.02667\omega^6 \end{Bmatrix} \\ &= \begin{Bmatrix} 1 - 0.5383\omega^2 + 0.02817\omega^4 - 0.00033\omega^4 \\ -28\omega^2 + 4.95\omega^4 - 0.19567\omega^6 + 0.002\omega^8 \end{Bmatrix} \end{aligned} \quad \text{(c)}$$

$$\begin{aligned} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_5^R &= \begin{bmatrix} 1 & \frac{1}{k_{t2}} \\ -I_3\omega^2 & 1 - \frac{I_3\omega^2}{k_{t2}} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_2^R \\ &= \begin{bmatrix} 1 & 0.01 \\ -4\omega^2 & 1 - 0.04\omega^2 \end{bmatrix} \begin{Bmatrix} 1 - 0.0833\omega^2 \\ -18\omega^2 + 0.6667\omega^4 \end{Bmatrix} \end{aligned}$$

$$= \begin{Bmatrix} 1 - 0.2633\omega^2 + 0.00667\omega^4 \\ -22\omega^2 + 1.72\omega^4 - 0.02667\omega^6 \end{Bmatrix} \tag{d}$$

$$\begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_6^R = \begin{bmatrix} 1 & \frac{1}{k_{t3}} \\ -I_4\omega^2 & 1 - \frac{I_4\omega^2}{k_{t3}} \end{bmatrix} \begin{Bmatrix} \theta \\ T_r \end{Bmatrix}_5^R$$

$$= \begin{bmatrix} 1 & 0.0125 \\ -6\omega^2 & 1 - 0.075\omega^2 \end{bmatrix} \begin{Bmatrix} 1 - 0.2633\omega^2 + 0.00667\omega^4 \\ -22\omega^2 + 1.72\omega^4 - 0.02667\omega^6 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 - 0.5383\omega^2 + 0.02817\omega^4 - 0.00033\omega^4 \\ -28\omega^2 + 4.95\omega^4 - 0.19567\omega^6 + 0.002\omega^8 \end{Bmatrix} \tag{e}$$

Substituting $\omega_1^2 = 0$, $\omega_2^2 = 6.421$, ..., $\omega_6^2 = 71.518$ into Eqs. (**a**) to (**e**), we get corresponding mode shapes corresponding to first to sixth mode are respectively obtained as

$$\{X\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \{X\}_2 = \begin{Bmatrix} 1 \\ 0.4649 \\ -0.4160 \\ -1.3836 \\ -0.4160 \\ -1.3836 \end{Bmatrix}, \{X\}_3 = \begin{Bmatrix} 1 \\ 0.2606 \\ -0.8166 \\ -1.7918 \\ -0.8166 \\ -1.7918 \end{Bmatrix}, \{X\}_4 = \begin{Bmatrix} 1 \\ -1.4952 \\ -0.9077 \\ 1.1856 \\ -0.9077 \\ 1.1856 \end{Bmatrix},$$

$$\{X\}_5 = \begin{Bmatrix} 1 \\ -3.3260 \\ 5.2956 \\ 2.3272 \\ 5.2956 \\ 2.3272 \end{Bmatrix}, \{X\}_6 = \begin{Bmatrix} 1 \\ -4.9598 \\ 16.2656 \\ -15.3667 \\ 16.2656 \\ -15.3667 \end{Bmatrix}$$

Example 8.12

Determine the natural frequencies and corresponding mode shapes of a beam carrying two lumped masses shown in Figure E8.12 using Myklestad-Prohl method. Take $M_1 = 50\,\text{kg}$, $M_2 = 60\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\text{m}^4$, $a = b = 0.4\,\text{m}$.

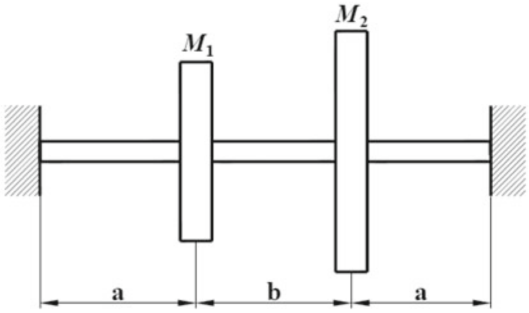


Figure E8.12

Solution

Taking the left fixed end and right fixed end as stations 0 and 3, we get transfer matrices for stations $0-1$ and $1-2$ can be determined as

$$[T]_1 = \begin{bmatrix} 1 & L_1 & \frac{L_1^2}{2EI_1} & \frac{L_1^3}{6EI_1} \\ 0 & 1 & \frac{L_1}{EI_1} & \frac{L_1^2}{2EI_1} \\ 0 & 0 & 1 & L_1 \\ M_1\omega^2 & M_1\omega^2 L_1 & \frac{M_1\omega^2 L_1^2}{2EI_1} & 1+\frac{M_1\omega^2 L_1^2}{6EI_1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 50\omega^2 & 20\omega^2 & 0.00001\omega^2 & 1+3.3333\times10^{-6}\omega^2 \end{bmatrix}$$

$$[T]_2 = \begin{bmatrix} 1 & L_2 & \frac{L_2^2}{2EI_2} & \frac{L_2^3}{6EI_2} \\ 0 & 1 & \frac{L_2}{EI_2} & \frac{L_2^2}{2EI_2} \\ 0 & 0 & 1 & L_1 \\ M_2\omega^2 & M_2\omega^2 L_2 & \frac{M_2\omega^2 L_2^2}{2EI_2} & 1+\frac{M_2\omega^2 L_2^2}{6EI_2} \end{bmatrix}_2$$

$$= \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 60\omega^2 & 24\omega^2 & 0.000012\omega^2 & 1+1.6\times10^{-6}\omega^2 \end{bmatrix}$$

Field matrix for the shaft between stations 2 and 3 can be determined as

$$[F]_3 = \begin{bmatrix} 1 & L_3 & \frac{L_3^2}{2EI_3} & \frac{L_3^3}{6EI_3} \\ 0 & 1 & \frac{L_3}{EI_3} & \frac{L_3^2}{2EI_3} \\ 0 & 0 & 1 & L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then overall transfer matrix of the system can be determined as

$$[U] = [F]_3[T]_2[T]_1$$

$$= \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 60\omega^2 & 24\omega^2 & 0.000012\omega^2 & 1+1.6\times10^{-6}\omega^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 50\omega^2 & 20\omega^2 & 0.00001\omega^2 & 1+3.3333\times10^{-6}\omega^2 \end{bmatrix}$$

$$= \left[\begin{matrix} 1+12.2667\times10^{-6}\omega^2+2.1333\times10^{-12}\omega^4 & 1.2+5.5467\times10^{-6}\omega^2+8.5333\times10^{-12}\omega^4 \\ 52\times10^{-6}\omega^2+1.6\times10^{-11}\omega^4 & 1+25.6\times10^{-6}\omega^2+6.4\times10^{-12}\omega^4 \\ 64\omega^2+32\times10^{-6}\omega^4 & 35.2\omega^2+12.8\times10^{-6}\omega^4 \\ 110\omega^2+80\times10^{-6}\omega^4 & 68\omega^2+32\times10^{-6}\omega^4 \end{matrix}\right.$$

$$\left.\begin{matrix} 1.8\times10^{-6}+3.4133\times10^{-12}\omega^2+4.2667\times10^{-18}\omega^4 & 0.72\times10^{-6}+6.2578\times10^{-13}\omega^2+5.6889\times10^{-20}\omega^4 \\ 3\times10^{-6}+17.6\times10^{-12}\omega^2+3.2\times10^{-18}\omega^4 & 1.8\times10^{-6}+3.6267\times10^{-12}\omega^2+4.2667\times10^{-19}\omega^4 \\ 1+27.2\times10^{-6}\omega^2+6.4\times10^{-12}\omega^4 & 1.2+6.1867\times10^{-6}\omega^2+8.5333\times10^{-13}\omega^4 \\ 58\times10^{-6}\omega^2+16\times10^{-12}\omega^4 & 1+14.1333\times10^{-6}\omega^2+2.1333\times10^{-12}\omega^4 \end{matrix}\right]$$

Then frequency equation for a fixed beam is given by

$$U_{13}U_{24} - U_{14}U_{23} = 0$$

$$\text{or,}\quad \left(1.8\times10^{-6}+3.4133\times10^{-12}\omega^2+4.2667\times10^{-18}\omega^4\right)$$
$$\left(1.8\times10^{-6}+3.6267\times10^{-12}\omega^2+4.2667\times10^{-19}\omega^4\right)$$
$$-\left(0.72\times10^{-6}+6.2578\times10^{-13}\omega^2+5.6889\times10^{-20}\omega^4\right)$$
$$\left(3\times10^{-6}+17.6\times10^{-12}\omega^2+3.2\times10^{-18}\omega^4\right) = 0$$

$$\text{or,}\quad 10^{-46}\omega^8+4.2667\times10^{-25}\omega^4-1.8773\times10^{-18}\omega^2+1.08\times10^{-12}=0$$

Real roots of the above equation for ω^2 can be determined as

$$\omega_1^2 = 680{,}542.8631;\quad \omega_2^2 = 3.7195\times10^6$$

Hence, natural frequencies of the system are found to be

$$\omega_1 = 824.9502\,\text{rad/s},\quad \omega_2 = 1928.5894\,\text{rad/s}$$

For the mode shape corresponding to the first natural frequency $\left(\omega^2 = \omega_1^2 = 680{,}542.8631\right)$, the state vector of the station 0 can be assumed as

$$\{S\}_0 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -U_{13}/U_{14} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -3.6858 \end{Bmatrix}$$

Then the state vector for the station 1 can be determined as

$$\{S\}_1 = [T]_1\{S\}_0 = \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 3.4027\times10^{7} & 1.3611\times10^{7} & 6.8054 & 1.9074 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -3.6858 \end{Bmatrix}$$
$$= \begin{Bmatrix} 1.0171\times10^{-7} \\ 2.6284\times10^{-7} \\ -0.4743 \\ -0.2247 \end{Bmatrix}$$

Then, the state vector of the station 1 can be refined as

$$\{S\}_1 = \begin{Bmatrix} 1 \\ 2.5842 \\ -4.6632\times10^{6} \\ -2.2099\times10^{6} \end{Bmatrix}$$

Similarly, the state vector for the station 2 can be determined as

$$\{S\}_2$$
$$= [T]_2\{S\}_1$$
$$= \begin{bmatrix} 1 & 0.4 & 2\times10^{-7} & 2.6667\times10^{-8} \\ 0 & 1 & 1\times10^{-6} & 2\times10^{-7} \\ 0 & 0 & 1 & 0.4 \\ 4.0832\times10^{7} & 1.6333\times10^{7} & 8.1665 & 2.0888 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.5842 \\ -4.6632\times10^{6} \\ -2.2099\times10^{6} \end{Bmatrix}$$
$$= \begin{Bmatrix} 1.0421 \\ -2.5210 \\ -5.5472\times10^{6} \\ 4.0341\times10^{7} \end{Bmatrix}$$

From these the mode shape corresponding to the first natural frequency for the transverse deflection of the system can be found to be $\{0\ 1\ 1.0421\ 0\}^T$.

Following the similar procedure for the second natural frequency, we get the mode shape corresponding to the second natural frequency as $\{0\ 1\ -0.7996\ 0\}^T$.

Example 8.13

Determine the natural frequency of a bar shown in Figure E8.13 using Rayleigh–Ritz method. The bar material has a density of ρ, modulus of elasticity of E and its cross-sectional area is A. Take two approximate (admissible) functions.

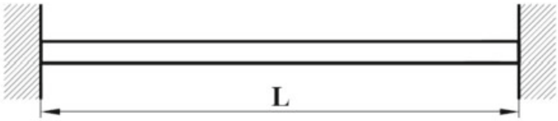


Figure E8.13

Solution

The admissible functions which satisfies the boundary conditions for the given bar, i.e. $U(0) = 0$ and $U(L) = 0$, can be taken as $\phi_1(x) = x(L - x)$ and $\phi_2(x) = x^2(L - x)$.

Then the elements of stiffness and mass matrices are given as

$$k_{11} = \int_0^L EA\phi_1'(x)\phi_1'(x)\mathrm{d}x = EA\int_0^L \left(L^2 - 4Lx + 4x^2\right)\mathrm{d}x = \frac{EAL^3}{3}$$

$$k_{12} = \int_0^L EA\phi_1'(x)\phi_2'(x)\mathrm{d}x = EA\int_0^L \left(2L^2x - 7Lx^2 + 6x^3\right)\mathrm{d}x = \frac{EAL^4}{6}$$

$$k_{21} = \int_0^L EA\phi_2'(x)\phi_2'(x)\mathrm{d}x = EA\int_0^L \left(2L^2x - 7Lx^2 + 6x^3\right)\mathrm{d}x = \frac{EAL^4}{6}$$

$$k_{22} = \int_0^L EA\phi_2'(x)\phi_2'(x)\mathrm{d}x = EA\int_0^L \left(4L^2x^2 - 12Lx^3 + 9x^4\right)\mathrm{d}x = \frac{2EAL^5}{15}$$

$$m_{11} = \int_0^L \rho A\phi_1(x)\phi_1(x)\mathrm{d}x = \rho A\int_0^L \left(L^2x^2 - 2Lx^3 + x^4\right)\mathrm{d}x = \frac{\rho AL^5}{30}$$

$$m_{12} = \int_0^L \rho A\phi_1(x)\phi_2(x)\mathrm{d}x = \rho A\int_0^L \left(L^2x^3 - 2Lx^4 + x^5\right)\mathrm{d}x = \frac{\rho AL^6}{60}$$

$$m_{21} = \int_0^L \rho A\phi_2(x)\phi_1(x)\mathrm{d}x = \rho A\int_0^L \left(L^2x^3 - 2Lx^4 + x^5\right)\mathrm{d}x = \frac{\rho AL^6}{60}$$

$$m_{22} = \int_0^L \rho A \phi_2(x)\phi_2(x)\mathrm{d}x = \rho A \int_0^L \left(L^2x^4 - 2Lx^5 + x^6\right)\mathrm{d}x = \frac{\rho A L^7}{105}$$

Then, frequency equation of the system is given by

$$\left|[K] - \omega_n^2[M]\right| = 0$$

$$\text{or,} \quad \begin{vmatrix} \frac{EAL^3}{3} - \omega_n^2 \frac{\rho A L^5}{30} & \frac{EAL^4}{6} - \omega_n^2 \frac{\rho A L^6}{60} \\ \frac{EAL^4}{6} - \omega_n^2 \frac{\rho A L^6}{60} & \frac{2EAL^3}{15} - \omega_n^2 \frac{\rho A L^5}{105} \end{vmatrix} = 0$$

$$\text{or,} \quad \frac{\rho^2 A^2 L^{12}}{25{,}200}\omega_n^4 - \frac{13\rho E A^2 L^{10}}{6300}\omega_n^2 + \frac{E^2A^2L^8}{60} = 0$$

$$\text{or,} \quad \rho^2 L^4 \omega_n^4 - 52\rho E L^2 \omega_n^2 + 420E^2 = 0$$

$$\therefore \quad \omega_1^2 = \frac{10E}{L^2\rho} \quad \text{and} \quad \omega_2^2 = \frac{42E}{L^2\rho}$$

Hence, natural frequencies of the system are determined as

$$\omega_1 = \frac{3.1623}{L}\sqrt{\frac{E}{\rho}} \quad \text{and} \quad \omega_2 = \frac{6.5807}{L}\sqrt{\frac{E}{\rho}}$$

These are closed to exact values

$$\omega_1 = \frac{3.1416}{L}\sqrt{\frac{E}{\rho}} \quad \text{and} \quad \omega_2 = \frac{6.2832}{L}\sqrt{\frac{E}{\rho}} \quad \text{respectively.}$$

Example 8.14

Determine the natural frequency of a tapered cantilever beam shown in Figure E8.14 using Rayleigh–Ritz method. The beam material has a density of ρ, modulus of elasticity of E. Its cross-sectional area is circular with radius R_0 at the fixed end. Take single approximate (admissible) function.

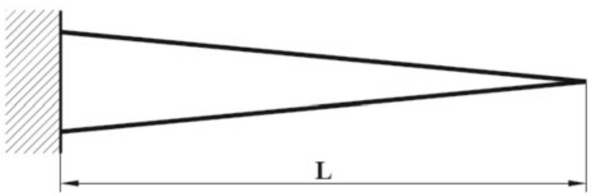


Figure E8.14

Solution

A function which satisfies the boundary conditions for a simply supported beam, i.e. $W(0) = W'(0) = 0$ and $W''(L) = W'''(L) = 0$, can be taken as

$$\phi(x) = x^4 + bx^3 + cx^2 + dx + e$$

Using the boundary conditions, we get $b = -4L$, $c = 6L^2$, $d = 0$ and $e = 0$. Hence, we assume an admissible function of the system as $\phi(x) = x^4 - 4Lx^3 + 6L^2x^2$.

Radius of the beam at any intermediate distance x is given by

$$R(x) = R_0\left(1 - \frac{x}{L}\right)$$

The area and moment of inertia of the section of the beam can determined as

$$A(x) = \pi R^2 = \pi R_0^2\left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right)$$
$$I(x) = \frac{\pi}{4}R^4 = \frac{\pi}{4}R_0^4\left(1 - \frac{4x}{L} + \frac{6x^2}{L^2} - \frac{4x^3}{L^3} + \frac{x^4}{L^4}\right)$$

Then the elements of stiffness and mass matrices are given as

$$\begin{aligned}
k_{11} &= \int_0^L EI(x)\phi_1''(x)\phi_1''(x)\mathrm{d}x \\
&= \frac{\pi E}{4}R_0^4\int_0^L\left(1 - \frac{4x}{L} + \frac{6x^2}{L^2} - \frac{4x^3}{L^3} + \frac{x^4}{L^4}\right)\left(12x^2 - 24Lx + 12L^2\right)^2\mathrm{d}x \\
&= 4\pi ER_0^4L^5 \\
m_{11} &= \int_0^L \rho A(x)\phi_1(x)\phi_1(x)\mathrm{d}x \\
&= \pi\rho R_0^2\int_0^L\left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right)\left(x^4 - 4Lx^3 + 6L^2x^2\right)^2\mathrm{d}x \\
&= \frac{57\pi\rho R_0^2L^9}{385}
\end{aligned}$$

Then, natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k_{11}}{m_{11}}} = 5.1978\frac{R_0}{L^2}\sqrt{\frac{E}{\rho}}$$

Example 8.15

A concentrated mass M is attached at the free end of a bar of length L undergoing longitudinal vibration as shown in Figure E8.15. The bar material has a density of ρ, modulus of elasticity of E and its cross-sectional area is A. Determine the natural frequencies of the system using assumed mode method if $M = 10\,\text{kg}$, $L = 1\,\text{m}$, $\rho = 7860\,\text{kg/m}^3$, $E = 200\,\text{GPa}$ and $A = 0.005\,\text{m}^2$. Take two assumed mode functions.

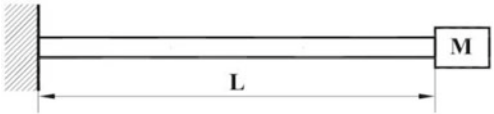


Figure E8.15

Solution

Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A(\dot{u})^2 \mathrm{d}x + \frac{1}{2}M(\dot{u})^2\Big|_{x=L} - \frac{1}{2}\int_0^L EA\left(u'\right)^2 \mathrm{d}x$$

The assumed mode shapes which satisfy the boundary conditions for the given bar, i.e. $U(0) = 0$ and $U'(L) = 0$, can be taken as $\phi_1(x) = x^2 - 2Lx$ and $\phi_2(x) = x^3 - 3L^2x$.

Then, we can assume the solution of the form

$$\begin{aligned} u(x,t) &= \phi_1(x)q_1(t) + \phi_2(x)q_2(t) \\ &= \left(x^2 - 2Lx\right)q_1(t) + \left(x^3 - 3L^2x\right)q_2(t) \end{aligned}$$

Substituting the assumed solution into the Lagrangian functional and simplifying, we get

$$\begin{aligned} L = T - V = \frac{1}{2}\int_0^L &\rho A\left[\left(x^2 - 2Lx\right)\dot{q}_1(t) + \left(x^3 - 3L^2x\right)\dot{q}_2(t)\right]^2 \mathrm{d}x \\ &+ \frac{1}{2}M\left[\left(-L^2\right)\dot{q}_1(t) + \left(-2L^3\right)\dot{q}_2(t)\right]^2 \\ &- \frac{1}{2}\int_0^L EA\left[(2x - 2L)q_1(t) + \left(3x^2 - 3L^2\right)q_2(t)\right]^2 \mathrm{d}x \end{aligned}$$

Now using Lagrange equation for $q_1(t)$, we get

$$\frac{\partial L}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 0$$

$$\text{or,} \quad -EA\int_0^L (2x-2L)\left[(2x-2L)q_1(t) + \left(3x^2-3L^2\right)q_2(t)\right]\mathrm{d}x$$

$$-\rho A\int_0^L \left(x^2-2Lx\right)\left[\left(x^2-2Lx\right)\ddot{q}_1(t) + \left(x^3-3L^2x\right)\ddot{q}_2(t)\right]\mathrm{d}x$$

$$-M\left(-L^2\right)\left[\left(-L^2\right)\ddot{q}_1(t) + \left(-2L^3\right)\ddot{q}_2(t)\right] = 0$$

$$\therefore \quad \left[ML^4 + \frac{8}{15}\rho AL^5\right]\ddot{q}_1(t) + \left[2ML^5 + \frac{61}{60}\rho AL^6\right]\ddot{q}_2(t)$$

$$+\frac{4}{3}EAL^3q_1(t) + \frac{5}{2}EAL^4q_2(t) = 0 \tag{a}$$

Similarly, using Lagrange equation for $q_2(t)$, we get

$$\frac{\partial L}{\partial q_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 0$$

$$\text{or,} \quad -EA\int_0^L \left(3x^2-3L^2\right)\left[(2x-2L)q_1(t) + \left(3x^2-3L^2\right)q_2(t)\right]\mathrm{d}x$$

$$-\rho A\int_0^L \left(x^3-3L^2x\right)\left[\left(x^2-2Lx\right)\ddot{q}_1(t) + \left(x^3-3L^2x\right)\ddot{q}_2(t)\right]\mathrm{d}x$$

$$-M\left(-2L^3\right)\left[\left(-L^2\right)\ddot{q}_1(t) + \left(-2L^3\right)\ddot{q}_2(t)\right] = 0$$

$$\therefore \quad \left[2ML^5 + \frac{61}{60}\rho AL^6\right]\ddot{q}_1(t) + \left[4ML^6 + \frac{68}{35}\rho AL^7\right]\ddot{q}_2(t)$$

$$+\frac{5}{2}EAL^4q_1(t) + \frac{24}{5}EAL^5q_2(t) = 0 \tag{b}$$

Substituting $M = 10\,\mathrm{kg}$, $L = 1\,\mathrm{m}$, $\rho = 7860\,\mathrm{kg/m^3}$, $E = 200\,\mathrm{GPa}$ and $A = 0.005\,\mathrm{m}^2$ into Eqs. (**a**) and (**b**), we get

$$30.96\ddot{q}_1(t) + 59.955\ddot{q}_2(t) + 1.3333\times 10^9 q_1(t) + 2.5\times 10^9 q_2(t) = 0 \tag{c}$$

$$59.955\ddot{q}_1(t) + 116.3543\ddot{q}_2(t) + 2.5\times 10^9 q_1(t) + 4.8\times 10^9 q_2(t) = 0 \tag{d}$$

From which mass and stiffness matrices of the system can be written as

$$M = \begin{bmatrix} 30.96 & 59.955 \\ 59.955 & 116.3543 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1.3333 \times 10^9 & 2.5 \times 10^9 \\ 2.5 \times 10^9 & 4.8 \times 10^9 \end{bmatrix}$$

Then, frequency equation of the system is given by

$$\left|[K] - \omega_n^2[M]\right| = 0$$

$$\text{or,} \quad \begin{vmatrix} 1.3333 \times 10^9 - 30.96\omega_n^2 & 2.5 \times 10^9 - 59.955\omega_n^2 \\ 2.5 \times 10^9 - 59.955\omega_n^2 & 4.8 \times 10^9 - 116.3543\omega_n^2 \end{vmatrix} = 0$$

$$\text{or,} \quad 7.7267\omega_n^4 - 3.9720 \times 10^9\omega_n^2 + 1.5 \times 10^{17} = 0$$

$$\therefore \quad \omega_1^2 = 4.1040 \times 10^7 \quad \text{and} \quad \omega_2^2 = 4.7303 \times 10^8$$

Hence, natural frequencies of the system are determined as

$$\omega_1 = 6406.2712\,\text{rad/s} \quad \text{and} \quad \omega_2 = 21{,}749.2561\,\text{rad/s}$$

Example 8.16

Three lumped masses are attached to a beam of length L shown in Figure E8.16. The beam material has a density of ρ and modulus of elasticity of E. Its area and moment of inertia of section are A and I. Determine natural frequencies of the system by using assumed mode method if $M_1 = M_3 = 10\,\text{kg}$, $M_2 = 15\,\text{kg}$, $L = 1.6\,\text{m}$, $\rho = 7860\,\text{kg/m}^3$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2.4 \times 10^{-6}\text{m}^4$. Take two assumed mode trigonometric functions.

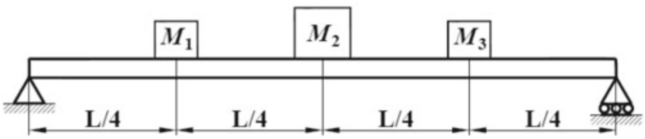


Figure E8.16

Solution

Lagrangian functional for the system can be determined as

$$L = T - V = \frac{1}{2}\int_0^L \rho A(\dot{w})^2 \mathrm{d}x + \frac{1}{2}M_1(\dot{w})^2\Big|_{x=L/4} + \frac{1}{2}M_2(\dot{w})^2\Big|_{x=L/2}$$

$$+ \frac{1}{2}M_1(\dot{w})^2\Big|_{x=3L/4} - \frac{1}{2}\int_0^L EI\left(w''\right)^2 \mathrm{d}x$$

The assumed mode shapes which satisfy the boundary conditions for the given beam, i.e. $\phi_i(0) = \phi_i''(0) = 0$ and $\phi_i(L) = \phi_i''(L) = 0$, can be taken as $\phi_1(x) = \sin(\pi x/L)$ and $\phi_2(x) = \sin(2\pi x/L)$.

Then, we can assume the solution of the form

$$w(x,t) = \phi_1(x)q_1(t) + \phi_2(x)q_2(t) = \sin\left(\frac{\pi x}{L}\right)q_1(t) + \sin\left(\frac{2\pi x}{L}\right)q_2(t)$$

Substituting the assumed solution into the Lagrangian functional and simplifying, we get

$$\begin{aligned} L = T - V = {} & \frac{1}{2}\int_0^L \rho A\left[\sin\left(\frac{\pi x}{L}\right)\dot{q}_1(t) + \sin\left(\frac{2\pi x}{L}\right)\dot{q}_2(t)\right]^2 \mathrm{d}x \\ & + \frac{1}{2}M_1\left[\left(\frac{1}{\sqrt{2}}\right)\dot{q}_1(t) + \dot{q}_2(t)\right]^2 + \frac{1}{2}M_2[\dot{q}_1(t)]^2 \\ & + \frac{1}{2}M_3\left[\left(\frac{1}{\sqrt{2}}\right)\dot{q}_1(t) + (-1)\dot{q}_2(t)\right]^2 \\ & - \frac{1}{2}\int_0^L EA\frac{\pi}{L}\left[\cos\left(\frac{\pi x}{L}\right)q_1(t) + \frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right)q_2(t)\right]^2 \mathrm{d}x \end{aligned}$$

Now using Lagrange equation for $q_1(t)$, we get

$$\begin{aligned} & \frac{\partial L}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 0 \\ \text{or,} \quad & -EA\int_0^L \cos\left(\frac{\pi x}{L}\right)\left[\cos\left(\frac{\pi x}{L}\right)q_1(t) + \frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right)q_2(t)\right]\mathrm{d}x \\ & - \rho A\int_0^L \sin\left(\frac{\pi x}{L}\right)\left[\sin\left(\frac{\pi x}{L}\right)\ddot{q}_1(t) + \sin\left(\frac{2\pi x}{L}\right)\ddot{q}_2(t)\right]\mathrm{d}x \\ & - M_1\left(\frac{1}{\sqrt{2}}\right)\left[\left(\frac{1}{\sqrt{2}}\right)\ddot{q}_1(t) + \ddot{q}_2(t)\right] \\ & - M_2\ddot{q}_1(t) - M_3\left(\frac{1}{\sqrt{2}}\right)\left[\left(\frac{1}{\sqrt{2}}\right)\ddot{q}_1(t) - \ddot{q}_2(t)\right] = 0 \\ \therefore \quad & \left[\frac{1}{2}\rho AL + \frac{1}{2}M_1 + M_2 + \frac{1}{2}M_3\right]\ddot{q}_1(t) + \left[\frac{1}{\sqrt{2}}M_1 - \frac{1}{\sqrt{2}}M_3\right]\ddot{q}_2(t) \\ & + \frac{\pi^4 EI_s}{2L^3}q_1(t) = 0 \end{aligned} \tag{a}$$

Similarly, using Lagrange equation for $q_2(t)$, we get

$$\frac{\partial L}{\partial q_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 0$$

$$\text{or,}\quad -EA\int_0^L \cos\left(\frac{2\pi x}{L}\right)\left[\cos\left(\frac{\pi x}{L}\right)q_1(t) + \frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right)q_2(t)\right]\mathrm{d}x$$

$$-\rho A\int_0^L \sin\left(\frac{2\pi x}{L}\right)\left[\sin\left(\frac{\pi x}{L}\right)\ddot{q}_1(t) + \sin\left(\frac{2\pi x}{L}\right)\ddot{q}_2(t)\right]\mathrm{d}x$$

$$-M_1\left[\left(\frac{1}{\sqrt{2}}\right)\ddot{q}_1(t) + \ddot{q}_2(t)\right] + M_3\left[\left(\frac{1}{\sqrt{2}}\right)\ddot{q}_1(t) - \ddot{q}_2(t)\right] = 0$$

$$\therefore\quad \left[\frac{1}{\sqrt{2}}M_1 - \frac{1}{\sqrt{2}}M_3\right]\ddot{q}_1(t) + \left[\frac{1}{2}\rho AL + \frac{1}{2}M_1 + M_2 + \frac{1}{2}M_3\right]\ddot{q}_2(t)$$

$$+\frac{8\pi^4 EI_s}{L^3}q_2(t) = 0 \tag{b}$$

Substituting $M_1 = M_3 = 10\,\mathrm{kg}$, $M_2 = 15\,\mathrm{kg}$, $L = 1.6\,\mathrm{m}$, $\rho = 7860\,\mathrm{kg/m^3}$, $E = 200\,\mathrm{GPa}$, $A = 0.005\,\mathrm{m^2}$ and $I = 2.4\times10^{-6}\mathrm{m^4}$ into Eqs. (**a**) and (**b**), we get

$$56.44\ddot{q}_1(t) + 5.7076\times10^6 q_1(t) = 0 \tag{c}$$

$$51.44\ddot{q}_2(t) + 91.3210\times10^6 q_2(t) = 0 \tag{d}$$

Equations (**c**) and (**d**) are decoupled equations. Hence natural frequencies of the system can be directly determined as

$$\omega_1 = \sqrt{\frac{5.7076\times10^6}{56.44}} = 318.0035\,\mathrm{rad/s}$$

and

$$\omega_2 = \sqrt{\frac{91.3210\times10^6}{51.44}} = 1332.4009\,\mathrm{rad/s}$$

Example 8.17

A beam fixed at both ends of length L has area and moment of inertia of section A and I, respectively. The beam material has a density of ρ and modulus of elasticity of E. Determine its natural frequencies Take two approximate (admissible) functions. Use

(a) **Galerkin method,**

(b) **Petrov–Galerkin method and**
(c) **Point collocation method.**

Solution

Equation of motion for the given system can be expressed as

$$\rho A\ddot{w} + EIw^{iv} = 0 \tag{a}$$

The functions which satisfy the boundary conditions for the fixed beam, i.e. $\phi_i(0) = \phi_i'(0) = 0$ and $\phi_i(L) = \phi_i'(L) = 0$, can be taken as $\phi_1(x) = x^4 - 2Lx^3 + L^2x^2$ and $\phi_2(x) = x^5 - 3L^2x^3 + 2L^3x^2$.

Then, we can assume the solution of the form

$$\begin{aligned} w_a(x,t) &= \phi_1(x)q_1(t) + \phi_2(x)q_2(t) \\ &= \left(x^4 - 2Lx^3 + L^2x^2\right)q_1(t) + \left(x^5 - 3L^2x^3 + 2L^3x^2\right)q_2(t) \end{aligned}$$

Substituting the assumed solution into the equation of motion, we get the residual of the approximation as

$$\begin{aligned} \mathcal{R} &= \rho A\ddot{w}_a + EIw_a^{iv} \\ &= \left[\rho A\left(x^4 - 2Lx^3 + L^2x^2\right)\right]\ddot{q}_1(t) + \left[\rho A\left(x^5 - 3L^2x^3 + 2L^3x^2\right)\right]\ddot{q}_2(t) \\ &\quad + 24EIq_1(t) + 120EIxq_2(t) \end{aligned}$$

(a) **The Galerkin method**

For Galerkin method, we choose weight function as $w_i = \phi_i$, i.e.,

$$w_1 = x^4 - 2Lx^3 + L^2x^2 \quad \text{and} \quad w_2 = x^5 - 3L^2x^3 + 2L^3x^2$$

Now applying the weighted residual statement with the weight function w_1,

$$\int_0^L w_1\mathcal{R}\mathrm{d}x = 0$$

$$\text{or,} \quad \int_0^L \left(x^4 - 2Lx^3 + L^2x^2\right)\left[\left\{\rho A\left(x^4 - 2Lx^3 + L^2x^2\right)\right\}\ddot{q}_1(t)\right.$$

$$\left. + \left\{\rho A\left(x^5 - 3L^2x^3 + 2L^3x^2\right)\right\}\ddot{q}_2(t) + 24EIq_1(t) + 120EIxq_2(t)\right]\mathrm{d}x = 0$$

$$\therefore \quad \frac{1}{630}\rho AL^9\ddot{q}_1(t) + \frac{1}{252}\rho AL^{10}\ddot{q}_2(t) + \frac{4}{5}EIL^5q_1(t) + 2EIL^6q_2(t) = 0 \tag{b}$$

Again, applying the weighted residual statement with the weight function w_2,

$$\int_0^1 w_2 \mathcal{R} \mathrm{d}x = 0$$

$$\text{or,} \quad \int_0^L \left(x^5 - 3L^2x^3 + 2L^3x^2\right)\left[\left\{\rho A\left(x^4 - 2Lx^3 + L^2x^2\right)\right\}\ddot{q}_1(t)\right.$$
$$\left. + \left\{\rho A\left(x^5 - 3L^2x^3 + 2L^3x^2\right)\right\}\ddot{q}_2(t) + 24EIq_1(t) + 120EIxq_2(t)\right]\mathrm{d}x = 0$$
$$\therefore \quad \frac{1}{252}\rho AL^{10}\ddot{q}_1(t) + \frac{23}{2310}\rho AL^{11}\ddot{q}_2(t) + 2EIL^6q_1(t) + \frac{36}{7}EIL^7q_2(t) = 0 \tag{c}$$

With reference to Eqs. (**b**) and (**c**), we can write mass and stiffness matrices of the system as

$$[M] = \begin{bmatrix} \frac{1}{630}\rho AL^9 & \frac{1}{252}\rho AL^{10} \\ \frac{1}{252}\rho AL^{10} & \frac{23}{2310}\rho AL^{11} \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} \frac{4}{5}EIL^5 & 2EIL^6 \\ 2EIL^6 & \frac{36}{7}EIL^7 \end{bmatrix}$$

Then, dynamic matrix can be determined as

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{173880}{\rho AL^9} & -\frac{69300}{\rho AL^{10}} \\ -\frac{69300}{\rho AL^{10}} & \frac{27720}{\rho AL^{11}} \end{bmatrix} \begin{bmatrix} \frac{4}{5}EIL^5 & 2EIL^6 \\ 2EIL^6 & \frac{36}{7}EIL^7 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{504EI}{\rho AL^4} & -\frac{8640EI}{\rho AL^3} \\ 0 & \frac{3960EI}{\rho AL^4} \end{bmatrix}$$

Then characteristic equation of the system can be determined as

$$\begin{vmatrix} \frac{504EI}{\rho AL^4} - \lambda & -\frac{8640EI}{\rho AL^3} \\ 0 & \frac{3960EI}{\rho AL^4} - \lambda \end{vmatrix} = 0$$
$$\text{or,} \quad \left(\frac{504EI}{\rho AL^4} - \lambda\right)\left(\frac{3960EI}{\rho AL^4} - \lambda\right) = 0$$
$$\therefore \quad \lambda_1 = \frac{504EI}{\rho AL^4} \quad \text{and} \quad \lambda_2 = \frac{3960EI}{\rho AL^4}$$

Hence the first two natural frequencies are found to be

$$\omega_1 = \sqrt{\lambda_1} = 22.45\sqrt{\frac{EI}{\rho AL^4}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = 62.93\sqrt{\frac{EI}{\rho AL^4}}$$

(b) **The Petrov–Galerkin method**

For Petrov–Galerkin method, we choose weight function as

$$w_1 = 1 \quad \text{and} \quad w_2 = x$$

Now applying the weighted residual statement with the weight function w_1,

$$\int_0^L w_1 \mathcal{R} \mathrm{d}x = 0$$

$$\text{or,} \quad \int_0^L \left[\left\{\rho A\left(x^4 - 2Lx^3 + L^2x^2\right)\right\}\ddot{q}_1(t) + \left\{\rho A\left(x^5 - 3L^2x^3 + 2L^3x^2\right)\right\}\ddot{q}_2(t)\right.$$

$$\left.+24EIq_1(t) + 120EIxq_2(t)\right]\mathrm{d}x = 0$$

$$\therefore \quad \frac{1}{30}\rho AL^5\ddot{q}_1(t) + \frac{1}{12}\rho AL^6\ddot{q}_2(t) + 24EILq_1(t) + 60EIL^2q_2(t) = 0 \qquad \text{(d)}$$

Again, applying the weighted residual statement with the weight function w_2,

$$\int_0^1 w_2 \mathcal{R} \mathrm{d}x = 0$$

$$\text{or,} \quad \int_0^L x\left[\left\{\rho A\left(x^4 - 2Lx^3 + L^2x^2\right)\right\}\ddot{q}_1(t) + \left\{\rho A\left(x^5 - 3L^2x^3 + 2L^3x^2\right)\right\}\ddot{q}_2(t)\right.$$

$$\left.+24EIq_1(t) + 120EIxq_2(t)\mathrm{d}x = 0\right]$$

$$\therefore \quad \frac{1}{60}\rho AL^6\ddot{q}_1(t) + \frac{3}{70}\rho AL^7\ddot{q}_2(t) + 12EIL^2q_1(t) + 40EIL^3q_2(t) = 0 \qquad \text{(e)}$$

With reference to Eqs. (**d**) and (**e**), we can write mass and stiffness matrices of the system as

$$[M] = \begin{bmatrix} \frac{1}{30}\rho AL^5 & \frac{1}{12}\rho AL^6 \\ \frac{1}{60}\rho AL^6 & \frac{3}{70}\rho AL^7 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 24EIL & 60EIL^2 \\ 12EIL^2 & 40EIL^3 \end{bmatrix}$$

Then, dynamic matrix can be determined as

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1080}{\rho AL^5} & -\frac{2100}{\rho AL^6} \\ -\frac{420}{\rho AL^6} & \frac{840}{\rho AL^7} \end{bmatrix}\begin{bmatrix} 24EIL & 60EIL^2 \\ 12EIL^2 & 40EIL^3 \end{bmatrix} = \begin{bmatrix} \frac{720EI}{\rho AL^4} & -\frac{19200EI}{\rho AL^3} \\ 0 & \frac{8400EI}{\rho AL^4} \end{bmatrix}$$

Then characteristic equation of the system can be determined as

$$\begin{vmatrix} \frac{720EI}{\rho AL^4} - \lambda & -\frac{19200EI}{\rho AL^3} \\ 0 & \frac{8400EI}{\rho AL^4} - \lambda \end{vmatrix} = 0$$

$$\text{or,}\quad \left(\frac{720EI}{\rho AL^4}-\lambda\right)\left(\frac{8400EI}{\rho AL^4}-\lambda\right)=0$$

$$\therefore\quad \lambda_1=\frac{720EI}{\rho AL^4}\quad\text{and}\quad \lambda_2=\frac{8400EI}{\rho AL^4}$$

Hence the first two natural frequencies are found to be

$$\omega_1=\sqrt{\lambda_1}=26.83\sqrt{\frac{EI}{\rho AL^4}}\quad\text{and}\quad \omega_2=\sqrt{\lambda_2}=91.65\sqrt{\frac{EI}{\rho AL^4}}$$

(c) **Point Collocation method**

For point collocation method, we take collocation points as $x_1 = L/3$ and $x_2 = 2L/3$. Then the weighing functions are defined as

$$w_1=\delta_d\left(x-\frac{L}{3}\right)\quad\text{and}\quad w_2=\delta_d\left(x-\frac{2L}{3}\right)$$

Now applying the weighted residual statement with the weight function w_1,

$$\mathcal{R}\left(x=\frac{L}{3}\right)=0$$

$$\therefore\quad \frac{4}{81}\rho AL^4\ddot{q}_1(t)+\frac{28}{243}\rho AL^5\ddot{q}_2(t)$$
$$+24EIq_1(t)+40EILq_2(t)=0 \tag{f}$$

Again, applying the weighted residual statement with the weight function w_2,

$$\mathcal{R}\left(x=\frac{2L}{3}\right)=0$$

$$\therefore\quad \frac{4}{81}\rho AL^4\ddot{q}_1(t)+\frac{32}{243}\rho AL^5\ddot{q}_2(t)$$
$$+24EIq_1(t)+80EILq_2(t)=0 \tag{g}$$

With reference to Eqs. (**f**) and (**g**), we can write mass and stiffness matrices of the system as

$$[M]=\begin{bmatrix}\frac{4}{81}\rho AL^4 & \frac{28}{243}\rho AL^5\\ \frac{4}{81}\rho AL^4 & \frac{32}{243}\rho AL^5\end{bmatrix}\quad\text{and}\quad [K]=\begin{bmatrix}24EI & 40EIL\\ 24EI & 80EIL\end{bmatrix}$$

Then, dynamic matrix can be determined as

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{162}{\rho AL^4} & -\frac{567}{4\rho AL^4} \\ -\frac{243}{4\rho AL^5} & \frac{243}{4\rho AL^5} \end{bmatrix} \begin{bmatrix} 24EIL & 60EIL^2 \\ 12EIL^2 & 40EIL^3 \end{bmatrix} = \begin{bmatrix} \frac{486EI}{\rho AL^4} & -\frac{4860EI}{\rho AL^3} \\ 0 & \frac{2430EI}{\rho AL^4} \end{bmatrix}$$

Then characteristic equation of the system can be determined as

$$\begin{vmatrix} \frac{486EI}{\rho AL^4} - \lambda & -\frac{4860EI}{\rho AL^3} \\ 0 & \frac{2430EI}{\rho AL^4} - \lambda \end{vmatrix} = 0$$

$$\text{or,} \quad \left(\frac{486EI}{\rho AL^4} - \lambda\right)\left(\frac{2430EI}{\rho AL^4} - \lambda\right) = 0$$

$$\therefore \quad \lambda_1 = \frac{486EI}{\rho AL^4} \quad \text{and} \quad \lambda_2 = \frac{2430EI}{\rho AL^4}$$

Hence the first two natural frequencies are found to be

$$\omega_1 = \sqrt{\lambda_1} = 22.05\sqrt{\frac{EI}{\rho AL^4}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = 49.3\sqrt{\frac{EI}{\rho AL^4}}$$

Example 8.18

A rigid mass M is attached at the free end of a cantilever beam of length L shown in Figure E8.18. The beam material has a density of ρ and modulus of elasticity of E. Its area and moment of inertia of section are A and I. Determine natural frequencies of the system if $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2.4 \times 10^{-6}\text{m}^4$. Take two approximate (admissible) functions. Use Galerkin method.

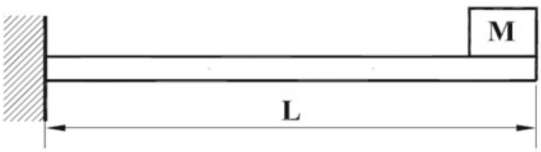


Figure E8.18

Solution

Equation of motion for the given system can be expressed as

$$\rho A\ddot{w} + M\delta_d(x - L)\ddot{w} + EIw^{iv} = 0 \tag{a}$$

The functions which satisfy the boundary conditions for the given beam, i.e. $W(0) = W'(0) = 0$ and $W''(L) = W'''(L) = 0$, can be taken as $\phi_1(x) = x^4 - 4Lx^3 + 6L^2x^2$ and $\phi_2(x) = x^5 - 10L^2x^3 + 20L^3x^2$.

Then, we can assume the solution of the form

$$\begin{aligned} w(x,t) &= \phi_1(x)q_1(t) + \phi_2(x)q_2(t) \\ &= \left(x^4 - 4Lx^3 + 6L^2x^2\right)q_1(t) + \left(x^5 - 10L^2x^3 + 20L^3x^2\right)q_2(t) \end{aligned}$$

Substituting the assumed solution into the equation of motion, we get the residual of the approximation as

$$\begin{aligned} \mathcal{R} &= \rho A\ddot{w} + M\delta_d(x-L)\ddot{w} + EIw^{iv} \\ &= \left[\rho A\left(x^4 - 4Lx^3 + 6L^2x^2\right) + M\delta_d(x-L)\left(x^4 - 4Lx^3 + 6L^2x^2\right)\right]\ddot{q}_1(t) \\ &\quad + \left[\rho A\left(x^5 - 10L^2x^3 + 20L^3x^2\right) + M\delta_d(x-L)\right]\ddot{q}_2(t) + 24EIq_1(t) \\ &\quad + 120EIxq_2(t) \end{aligned}$$

For Galerkin method, we choose weight function as $w_i = \phi_i$, i.e.,

$$w_1 = x^4 - 4Lx^3 + 6L^2x^2 \quad \text{and} \quad w_2 = x^5 - 10L^2x^3 + 20L^3x^2$$

Now applying the weighted residual statement with the weight function w_1,

$$\begin{aligned} &\int_0^L w_1\mathcal{R}\mathrm{d}x = 0 \\ &\therefore \quad \left[\frac{104}{45}\rho AL^9 + 9ML^8\right]\ddot{q}_1(t) + \left[\frac{2644}{315}\rho AL^{10} + 33ML^9\right]\ddot{q}_2(t) \\ &\quad + \frac{144}{5}EIL^5q_1(t) + 104EIL^6q_2(t) = 0 \end{aligned} \tag{b}$$

Again, applying the weighted residual statement with the weight function w_2,

$$\begin{aligned} &\int_0^1 w_2\mathcal{R}\,\mathrm{d}x = 0 \\ &\therefore \quad \left[\frac{2644}{315}\rho AL^{10} + 33ML^9\right]\ddot{q}_1(t) + \left[\frac{21128}{693}\rho AL^{11} + 121ML^{10}\right]\ddot{q}_2(t) \\ &\quad + 104EIL^6q_1(t) + \frac{2640}{7}EIL^7q_2(t) = 0 \end{aligned} \tag{c}$$

With reference to Eqs. (**b**) and (**c**), we can write mass and stiffness matrices of the system as

$$[M] = \begin{bmatrix} \frac{104}{45}\rho AL^9 + 9ML^8 & \frac{2644}{315}\rho AL^{10} + 33ML^9 \\ \frac{2644}{315}\rho AL^{10} + 33ML^9 & \frac{21128}{693}\rho AL^{11} + 121ML^{10} \end{bmatrix}$$

and

$$[K] = \begin{bmatrix} \frac{144}{5}EIL^5 & 104EIL^6 \\ 104EIL^6 & \frac{2640}{7}EIL^7 \end{bmatrix}$$

Substituting given parameters $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2.4 \times 10^{-6}\text{m}^4$, we get

$$[M] = \begin{bmatrix} 270.8267 & 989.8705 \\ 989.8705 & 3618.1680 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 1.3824 \times 10^7 & 4.992 \times 10^7 \\ 4.992 \times 10^7 & 1.8103 \times 10^8 \end{bmatrix}$$

Then, dynamic matrix can be determined as

$$\begin{aligned}[D] = [M]^{-1}[K] &= \begin{bmatrix} 68.5131 & -18.7440 \\ -18.7440 & 5.1283 \end{bmatrix} \begin{bmatrix} 1.3824 \times 10^7 & 4.992 \times 10^7 \\ 4.992 \times 10^7 & 1.8103 \times 10^8 \end{bmatrix} \\ &= \begin{bmatrix} 1.1422 \times 10^7 & 2.6967 \times 10^7 \\ -3.1112 \times 10^6 & -7.3276 \times 10^6 \end{bmatrix}\end{aligned}$$

Then characteristic equation of the system can be determined as

$$\begin{aligned}&\begin{vmatrix} 1.1422 \times 10^7 - \lambda & 2.6967 \times 10^7 \\ -3.1112 \times 10^6 & -7.3276 \times 10^6 - \lambda \end{vmatrix} = 0 \\ \text{or,} \quad & \lambda^2 - 4.0949 \times 10^6\lambda + 1.9944 \times 10^{11} = 0 \\ \therefore \quad & \lambda_1 = 49299.0112 \quad \text{and} \quad \lambda_2 = 4.0456 \times 10^6\end{aligned}$$

Hence the first two natural frequencies are found to be

$$\omega_1 = \sqrt{\lambda_1} = 222.0338\,\text{rad/s} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = 2011.3634\,\text{rad/s}$$

Review Questions

1. Why are approximate methods used for vibrational analysis?
2. List the common approximate methods used for vibration analysis.
3. What is a Rayleigh's quotient?
4. How can Rayleigh method be used to determine the fundamental natural frequency of a single degree of freedom system and a multi degree of freedom system?
5. How can Rayleigh method be used to determine the fundamental natural frequency of a shaft or beam carrying a number of lumped inertia elements?
6. How can Rayleigh method be used to determine the fundamental natural frequency of a string, a bar, a shaft and a beam?

7. How can Dunkerley's method be used to determine the fundamental natural frequency of a system?
8. How can Matrix iteration method be used determine the natural frequencies of a system using flexibility matrix?
9. How can matrix iteration method be used to determine the natural frequencies of a system using dynamic matrix?
10. How can Stodala's method be used to determine the fundamental natural frequency of a system?
11. How can Holzer method be used to determine the fundamental natural frequency of a system without a branch and with branches?
12. What are state vector, point matrix, field matrix, transfer matrix and overall transfer matrix in the context of Holzer method?
13. How can Myklestad-Prohl method be used for bending vibration analysis of a shaft or beam carrying a number of lumped inertia elements?
14. What are state vector, point matrix, field matrix, transfer matrix and overall transfer matrix in the context of Myklestad-Prohl method?
15. How can Rayleigh–Ritz method be used to determine the natural frequencies of a string, a bar, a shaft and a beam?
16. How can assume mode method be used to determine the natural frequencies of a continuous system?
17. How can weighted residual method be used to determine the natural frequencies of a continuous system?
18. List common weighted residual methods used for vibration analysis of continuous systems.

Exercise

1. Determine the fundamental natural frequency of discrete systems shown in **Figure P8.1** by using Rayleigh's method. Take unit displacements of each mass as the assumed mode shape.

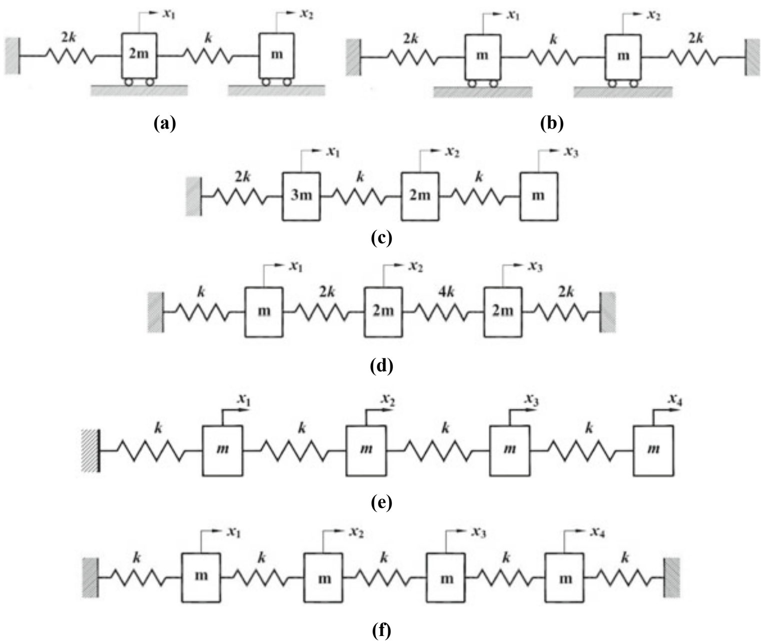


Figure P8.1

2. Determine the fundamental natural frequency of beams carrying lumped masses as shown in **Figure P8.2** by using Rayleigh's method. Assume fixed end conditions for the beam. Take $M_1 = M_2 = M_3 = 50\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\text{m}^4$, $a = b = 0.4\,\text{m}$.

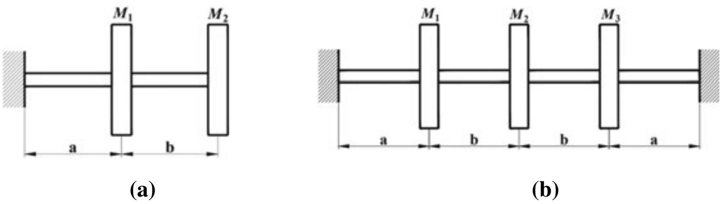
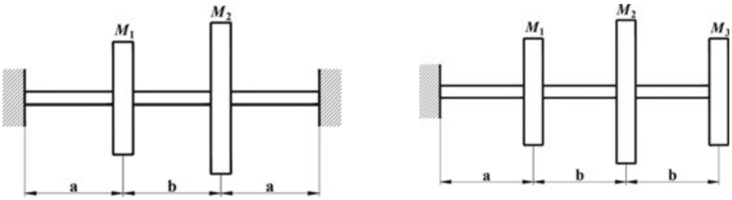


Figure P8.2

3. Determine the fundamental natural frequency of beams carrying lumped masses as shown in **Figure P8.3** by using Rayleigh's method. Assume fixed end conditions for the beam. Take $M_1 = M_3 = 50\,\text{kg}$, $M_2 = 80\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\,\text{m}^4$, $a = b = 0.4\,\text{m}$.



(a) **(b)**

Figure P8.3

4. Determine the fundamental natural frequency of a beam carrying lumped masses as shown in **Figure P8.4** by using Rayleigh's method. Assume simply supported end conditions for the beam. Take $M_1 = 50\,\text{kg}$, $M_2 = 80\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\,\text{m}^4$, $a = b = 0.4\,\text{m}$.
5. Determine the fundamental natural frequency of a beam carrying lumped masses as shown in **Figure P8.5** by using Rayleigh's method. Assume simply supported end conditions for the beam. Take $M_1 = M_2 = M_3 = 50\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\,\text{m}^4$, $a = b = 0.4\,\text{m}$.

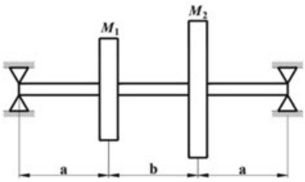


Figure P8.4

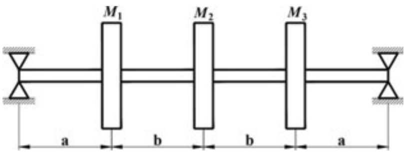
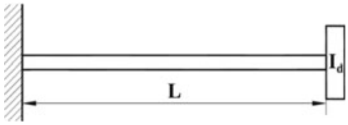


Figure P8.5

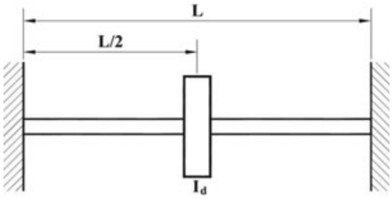
6. A string of length L and mass per unit length ρ is stretched under a tension T. Determine the fundamental natural frequency of the string by using Rayleigh's method when it is fixed at both ends.

   (a) Take $\sin(\pi x/L)$ as an admissible function.
   (b) Take $x^2 - Lx$ as an admissible function.

7. Use Rayleigh's method to determine the fundamental natural frequency of shaft-disk system shown in **Figure P8.7**. The mass moment of inertia of the rigid disk is I_D, length of the shaft is L and shaft diameter is d_s. The shaft material has a density of ρ, modulus of elasticity of E and shear modulus of elasticity of G. Take $I_D = 15\,\text{kg m}^2$, $L = 1.2\,\text{m}$, $d_s = 8\,\text{cm}$, $\rho = 7850\,\text{kg/m}^3$ and $G = 84\,\text{GPa}$. Take $x^2 - 2Lx$ and $x^2 - Lx$ as admissible functions for (**a**) and (**b**) respectively.



(a)



(b)

Figure P8.7

8. A bar of length L is fixed at one end and free at the other end as shown in **Figure P8.8**. The bar material has a density of ρ, modulus of elasticity of E and its cross-sectional area is A. Determine the fundamental natural frequency of the system by using Rayleigh's method.
   (a) Take $\cos(\pi x/2L)$ as an admissible function.
   (b) Take $x^2 - 2Lx$ as an admissible function.

9. A stepped bar, having length of each portion $L/2$, is fixed at both ends as shown in **Figure P8.9**. Each portion of the bar has the same material with a density of ρ and the modulus of elasticity of E. The left part of the bar has a cross-sectional area of $2A$ and that of the right part of the has a cross-sectional area of A. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $\sin(\pi x/L)$ as an admissible function.

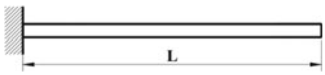


Figure P8.8

Figure P8.9

10. A concentrated mass M is attached at the free end of a tapered bar of length L as shown in **Figure P8.10**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $\rho = 7850\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1.6\,\text{m}$, $E = 210\,\text{GPa}$ and $M = 20\,\text{kg}$. Take $x^2 - 2Lx$ as an admissible function.
11. A spring of stiffness k is attached to the left end of a tapered bar of length L as shown in **Figure P8.11**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $\rho = 7850\,\text{kg/m}^3$, $A_1 = 1.2 \times 10^{-4}\text{m}^2$, $A_2 = 2.4 \times 10^{-4}\text{m}^2$, $L = 1\,\text{m}$, $E = 210\,\text{GPa}$ and $k = 250\,\text{kN/m}$. Take $x^2 - L^2$ as an admissible function.

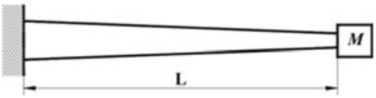


Figure P8.10

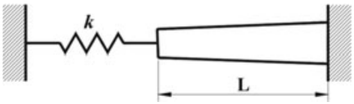


Figure P8.11

12. A concentrated mass M is attached at the mid span of a simply supported beam of length L as shown in **Figure P8.12**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $\sin(\pi x/L)$ as an admissible function.
13. Three lumped masses are attached to a simply supported beam shown in **Figure P8.12**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $\sin(\pi x/L)$ as an admissible function.

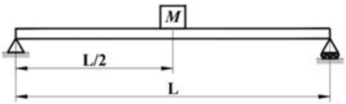


Figure P8.12

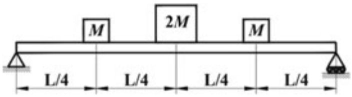


Figure P8.13

14. A tapered cantilever beam shown in **Figure P8.14** has a square cross section of $b \times b$ at its left fixed end and has a length of L. The beam material has a density of ρ and modulus of elasticity of E. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $x^4 - 4Lx^3 + 6L^2x^2$ as an admissible function.
15. Two lumped masses are attached to a cantilever beam shown in **Figure P8.15**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $x^4 - 4Lx^3 + 6L^2x^2$ as an admissible function.

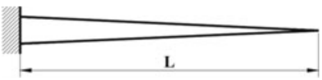


Figure P8.14

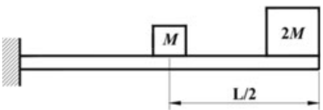


Figure P8.15

16. A stepped beam, having length of each portion $L/2$, is fixed at both ends as shown in **Figure P8.16**. Each portion of the beam has the same material with a density of ρ and the modulus of elasticity of E. The left part of the beam

section has a width of b and a depth of h and those of the right part are b and $0.8h$ respectively. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $x^4 - 2Lx^3 + L^2x^2$ as an admissible function.

17. A concentrated mass M is attached at the mid span of a beam of length L as shown in **Figure P8.17**. The beam is fixed at the left end and simply supported at the right end. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $2x^4 - 5Lx^3 + 3L^2x^2$ as an admissible function.

Figure P8.16

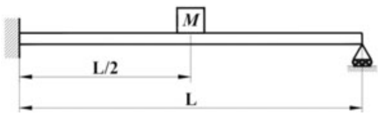


Figure P8.17

18. Three lumped masses are attached to a beam shown in **Figure P8.18**. The beam is fixed at the left end and simply supported at the right end. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using Rayleigh's method. Take $2x^4 - 5Lx^3 + 3L^2x^2$ as an admissible function.

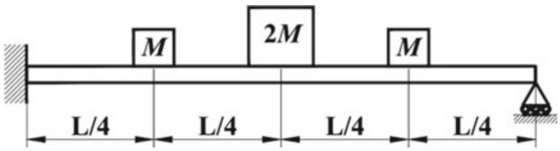
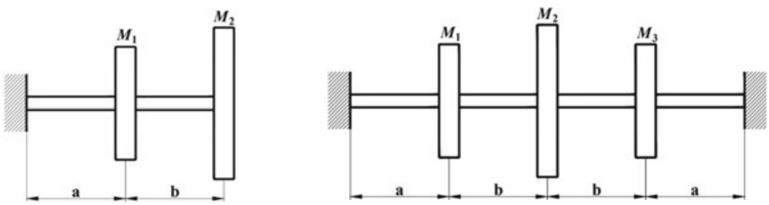


Figure P8.18

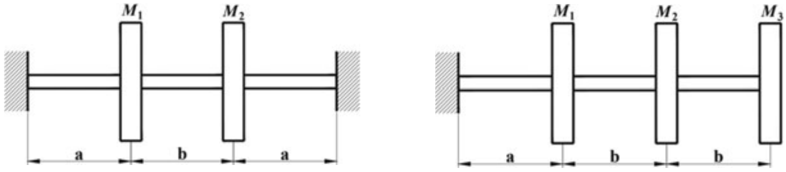
19. Determine the fundamental natural frequency of a beam carrying lumped masses as shown in **Figure P8.19** by using Dunkerley's method. Take $M_1 = 40\,\text{kg}$, $M_2 = 50\,\text{kg}$, $M_3 = 40\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\text{m}^4$, $a = b = 0.4\,\text{m}$.



(a) **(b)**

Figure P8.19

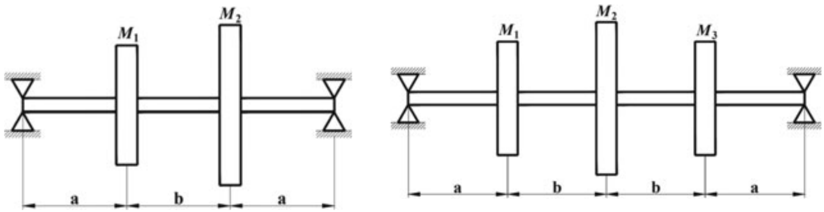
20. Determine the fundamental natural frequency of a beam carrying lumped masses as shown in **Figure P8.20** by using Dunkerley's method. Take $M_1 = M_2 = M_3 = 50\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\text{m}^4$, $a = b = 0.4\,\text{m}$.



(a) **(b)**

Figure P8.20

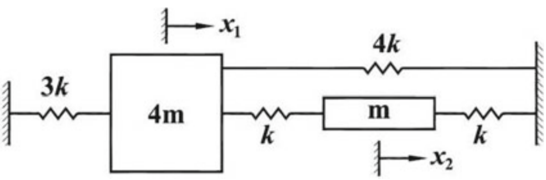
21. Determine the fundamental natural frequency of a beam carrying lumped masses as shown in **Figure P8.21** by using Dunkerley's method. Assume simply supported end conditions for the beam. Take $M_1 = 40\,\text{kg}$, $M_2 = 50\,\text{kg}$, $M_3 = 40\,\text{kg}$, $E = 200\,\text{GPa}$, $I = 2 \times 10^{-6}\text{m}^4$, $a = b = 0.4\,\text{m}$.



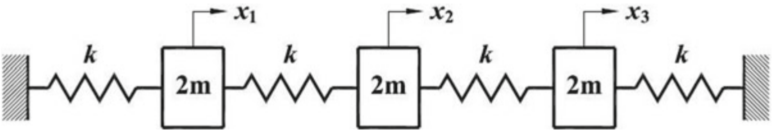
(a) **(b)**

Figure P8.21

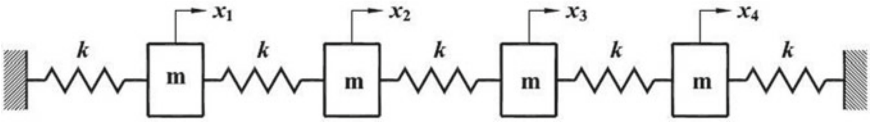
22. Determine the fundamental natural frequency and the corresponding mode shape of the systems shown in **Figure P8.22** using matrix iteration method.
23. Determine the natural frequencies and the corresponding mode shapes of the systems shown in **Figure P8.23** using matrix iteration method.



(a)

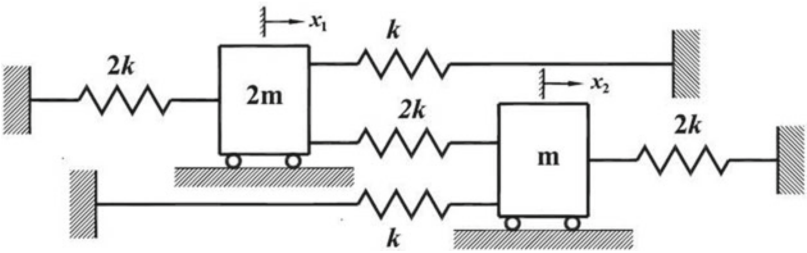


(b)

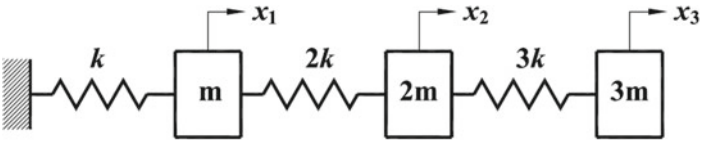


(c)

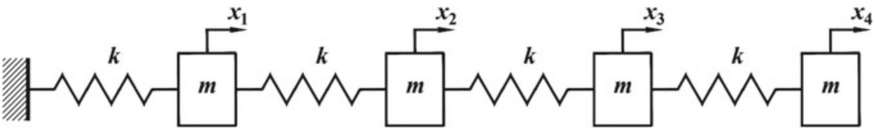
Figure P8.22



(a)



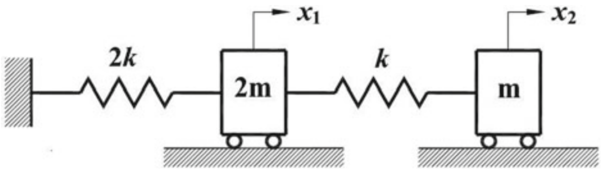
(b)



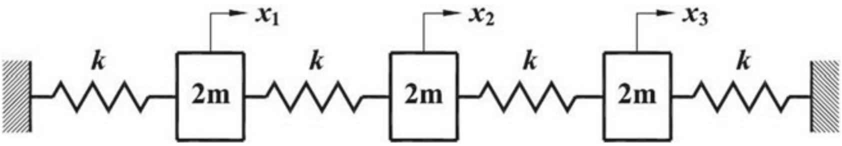
(c)

Figure P8.23

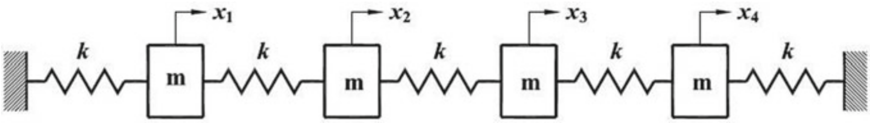
24. Determine the fundamental natural frequency and the corresponding mode shape of the systems shown in **Figure P8.24** using Stodola's method.



(a)



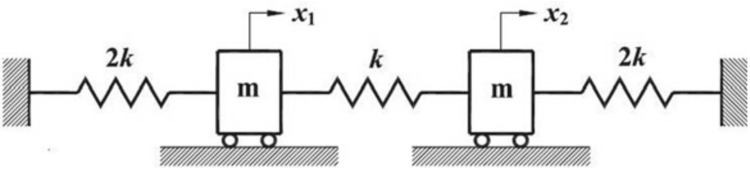
(b)



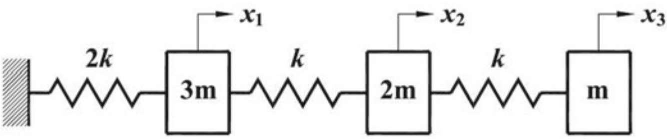
(c)

Figure P8.24

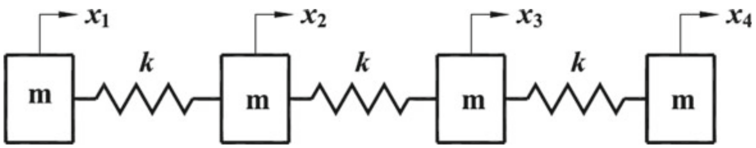
25. Determine the natural frequencies and the corresponding mode shapes of the systems shown in **Figure P8.25** using Holzer's method. Take $m = 10\,\text{kg}$ and $k = 100\,\text{N/m}$.



(a)



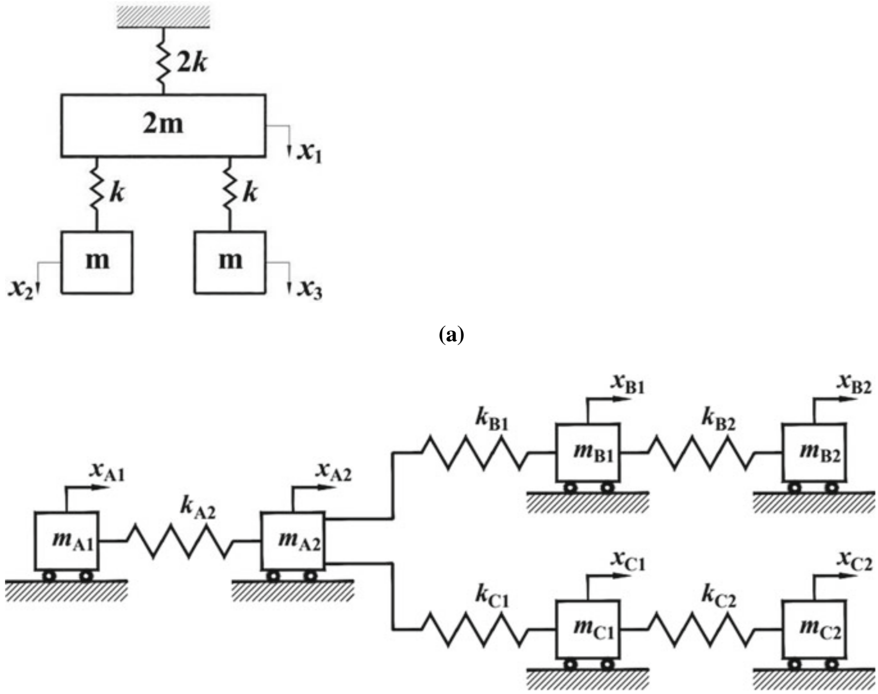
(b)



(c)

Figure P8.25

26. Determine the natural frequencies and the corresponding mode shapes of the systems shown in **Figure P8.26** using Holzer's method. Take $m = 10\,\text{kg}$ and $k = 100\,\text{N/m}$ for **(a)** and $m_{A1} = m_{A2} = 15\,\text{kg}$, $m_{B1} = m_{B2} = m_{C1} = m_{C2} = 10\,\text{kg}$, $k_{A2} = 100\,\text{N/m}$ and $k_{B1} = k_{B2} = k_{C1} = k_{C2} = 120\,\text{N/m}$ for **(b)**.

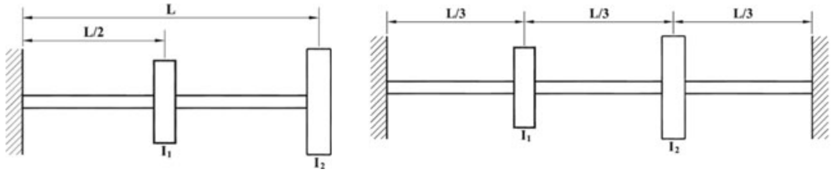


(b)

Figure P8.26

27. Determine the natural frequencies and the corresponding mode shapes for the torsional vibration of the shaft-disk systems shown in **Figure P8.27** using Holzer's method. All shaft segments have a diameter of 200 mm. Total length

of the shaft is 1.8 m and mass moment of inertia of disks are $I_1 = 80\,\text{kgm}^2$ and $I_2 = 100\,\text{kgm}^2$. Take $G = 84\,\text{GPa}$.



(a) **(b)**

Figure P8.27

28. **Figure P8.28** shows a three-cylinder engine with a flywheel. Determine the natural frequencies and the corresponding mode shapes of system using Holzer's method. The mass moment of inertia of each engine is $0.04\,\text{kg m}^2$ and that of the flywheel is $1.5\,\text{kg m}^2$ and torsional stiffness of each shaft is $1.6\,\text{MN m/rad}$.

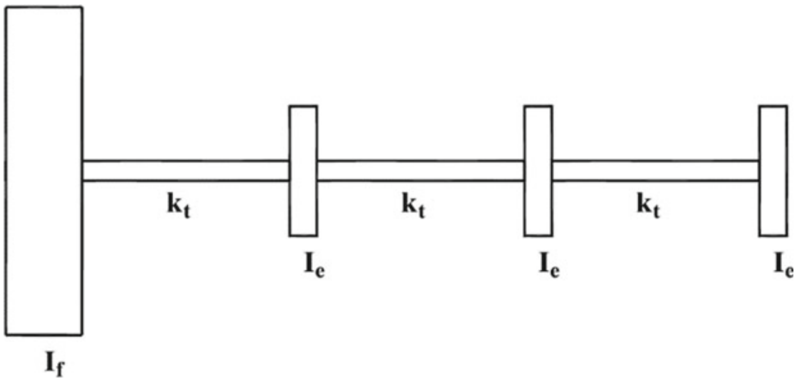
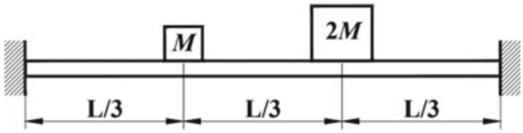


Figure P8.28

29. Determine the natural frequencies and corresponding mode shapes of a beam carrying lumped masses shown in **Figure P8.29** using Myklestad-Prohl method. Take $M = 20\,\text{kg}$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $I = 2 \times 10^{-6}\text{m}^4$.

(a) **(b)**



(c)

Figure P8.29

30. A stepped bar, having length of each portion $L/2$, is fixed at one end and free at the other end as shown in **Figure P8.30**. Each portion of the bar has the same material with a density of ρ and the modulus of elasticity of E. The left part of the bar has a cross-sectional area of $2A$ and the right part has a cross-sectional area of A. Determine natural frequencies of the system by using Rayleigh–Ritz method. Use two approximate (admissible) functions as $\sin(\pi x/2L)$ and $\sin(3\pi x/2L)$.
31. A tapered bar shown in **Figure P8.31** has a length of L. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine natural frequencies of the system by using Rayleigh–Ritz method. Use two approximate (admissible) functions as $x^2 - 2Lx$ and $x^3 - 3L^2x$. Take $\rho = 7850\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1.6\,\text{m}$ and $E = 200\,\text{GPa}$.

Figure P8.30

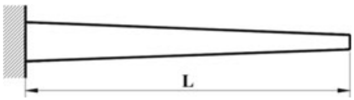


Figure P8.31

32. A spring of stiffness k is attached to the right end of a tapered bar of length L as shown in **Figure P8.32**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine natural frequency of the system by using Rayleigh–Ritz method. Use a single approximate (admissible) function as $\sin(\pi x/2L)$. Take $\rho = 7850\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1\,\text{m}$, $E = 200\,\text{GPa}$ and $k = 250\,\text{kN/m}$.
33. Determine the natural frequencies of a simply supported beam shown in **Figure P8.33** using Rayleigh–Ritz method. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Use two approximate (admissible) functions as $\sin(\pi x/L)$ and $\sin(2\pi x/L)$. Take $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $I = 2 \times 10^{-6}\text{m}^4$.

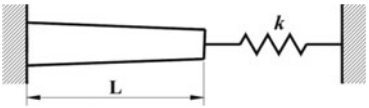


Figure P8.32

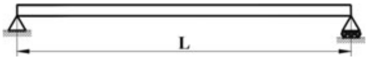


Figure P8.33

34. A concentrated mass M is attached to a simply supported beam of length L as shown in **Figure P8.34**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system if **(a)** $x = L/4$, **(b)** $x = L/2$ and **(c)** $x = 3L/4$. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$. Use a single approximate (admissible) function as $\sin(\pi x/L)$.
35. A tapered cantilever beam shown in **Figure P8.35** has a length of L. The beam material has a density of ρ, modulus of elasticity of E. The beam has a uniform width of b and the depth of the beam varies linearly from H_1 at the left end to H_2 at the right end. Determine natural frequencies of the system by using Rayleigh–Ritz method. Use two approximate (admissible) functions as $x^4 - 4Lx^3 + 6L^2x^2$ and $x^5 - 10L^2x^3 + 20L^3x^2$. Take $\rho = 7850\,\text{kg/m}^3$, $H_1 = 60\,\text{mm}$, $H_2 = 30\,\text{mm}$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $b = 40\,\text{mm}$.

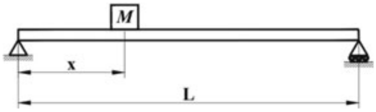


Figure P8.34

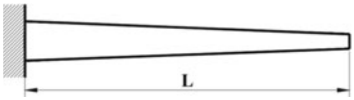


Figure P8.35

36. A tapered fixed beam shown in **Figure P8.36** has a length of L. The beam material has a density of ρ, modulus of elasticity of E. The beam has a uniform width of b and the depth of the beam varies linearly from H_1 at the left end to H_2 at the right end. Determine natural frequencies of the system by using Rayleigh–Ritz method. Use two approximate (admissible) functions as $x^4 - 2Lx^3 + L^2x^2$ and $x^5 - 3L^2x^3 + 2L^3x^2$. Take $\rho = 7860\,\text{kg/m}^3$, $H_1 = 60\,\text{mm}$, $H_2 = 30\,\text{mm}$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $b = 40\,\text{mm}$.
37. Two lumped masses are attached to a fixed beam shown in **Figure P8.37**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequency of the system by using Rayleigh–Ritz method. Use a single approximate (admissible) function as $x^4 - 2Lx^3 + L^2x^2$. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.

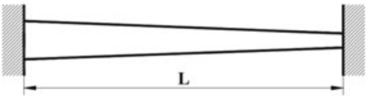


Figure P8.36

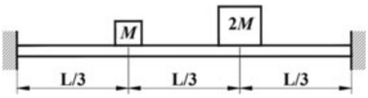


Figure P8.37

38. Two lumped masses are attached to a beam shown in **Figure P8.38**. The beam is fixed at the left end and simply supported at the right end. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequency of the system by using Rayleigh–Ritz method. Use a single approximate (admissible) function as $2x^4 - 5Lx^3 + 3L^2x^2$. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.
39. Three lumped masses are attached to a cantilever beam shown in **Figure P8.39**. The beam is fixed at the left end and simply supported at the right end. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequency of the system by using Rayleigh–Ritz method. Use a single approximate (admissible) function as $x^4 - 4Lx^3 + 6L^2x^2$. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.

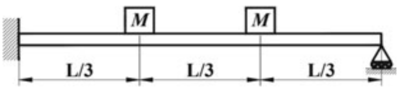


Figure P8.38

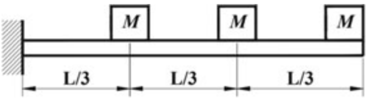


Figure P8.39

40. A tapered bar shown in **Figure P8.40** has a square cross section of $b \times b$ at its left fixed end and has a length of L. The bar material has a density of ρ and modulus of elasticity of E. Determine the fundamental natural frequency of the system by using assumed mode method. Consider first two mode shapes as $\sin(\pi x/2L)$ and $\sin(3\pi x/2L)$. Take $b = 50\,\text{mm}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$ and $E = 200\,\text{GPa}$.
41. A tapered bar shown in **Figure P8.41** has a length of L. The bar material has a density of ρ and modulus of elasticity of E. The beam has a uniform width of b and the depth of the beam varies linearly from H_1 at the left end to H_2 at the right end. Determine the fundamental natural frequency of the system by using assumed mode method. Consider first two mode shapes as $x^2 - Lx$ and $x^3 - Lx^2$. Take $\rho = 7860\,\text{kg/m}^3$, $H_1 = 60\,\text{mm}$, $H_2 = 30\,\text{mm}$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $b = 40\,\text{mm}$.

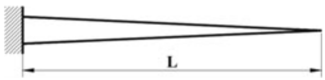


Figure P8.40

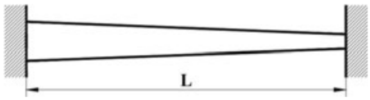


Figure P8.41

42. A concentrated mass M is attached at the free end of a tapered bar of length L as shown in **Figure P8.42**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine the fundamental natural frequency of the system by using assumed mode method. Consider an assumed mode as $x^2 - 2Lx$. Take $\rho = 7860\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $M = 10\,\text{kg}$.
43. A spring of stiffness k is attached to the right end of a bar of length L as shown in **Figure P8.43**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar is A. Determine the fundamental natural frequency of the system by using assumed mode method. Consider an assumed mode as $\sin(\pi x/2L)$. Take $\rho = 7850\,\text{kg/m}^3$, $A = 2 \times 10^{-4}\text{m}^2$, $L = 1\,\text{m}$, $E = 200\,\text{GPa}$ and $k = 250\,\text{kN/m}$.

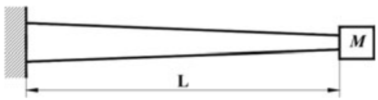


Figure P8.42

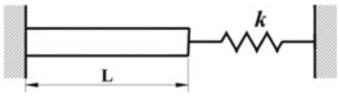


Figure P8.43

44. Determine the natural frequencies of a cantilever beam shown in **Figure P8.44** using assumed mode method. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Consider first two mode shapes as $x^4 - 4Lx^3 + 6L^2x^2$ and $x^5 - 10L^2x^3 + 20L^3x^2$. Take $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.
45. A tapered beam fixed at the left and simply supported at the right end shown in **Figure P8.45** has a length of L. The beam material has a density of ρ, modulus of elasticity of E. The beam has a uniform width of b and the depth of the beam varies linearly from H_1 at the left end to H_2 at the right end. Determine natural frequencies of the system by using assumed mode method. Consider first two mode shapes as $2x^4 - 5Lx^3 + 3L^2x^2$ and $2x^5 - 9L^2x^3 + 7L^3x^2$. Take $\rho = 7850\,\text{kg/m}^3$, $H_1 = 60\,\text{mm}$, $H_2 = 30\,\text{mm}$, $L = 1.4\,\text{m}$, $E = 200\,\text{GPa}$ and $b = 40\,\text{mm}$.

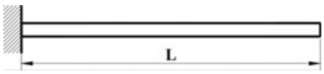


Figure P8.44

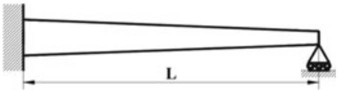


Figure P8.45

46. A stepped beam, having length of each portion $L/2$, is fixed at both ends as shown in **Figure P8.46**. Each portion of the beam has the same material with a density of ρ and the modulus of elasticity of E. The left part of the beam section has a width of b and a depth of h and those of the right part are b and $0.8h$ respectively. Determine natural frequencies of the system by using assumed mode method. Consider first two mode shapes as $x^4 - 2Lx^3 + L^2x^2$ and $x^5 - 3L^2x^3 + 2L^3x^2$. Take $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $b = 40\,\text{mm}$, $h = 50\,\text{mm}$ and $E = 200\,\text{GPa}$.
47. A concentrated mass M is attached at the free end of a cantilever beam of length L as shown in **Figure P8.47**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using assumed mode method. Consider an assumed mode as $x^4 - 4Lx^3 + 6L^2x^2$. Take $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$, $M = 20\,\text{kg}$ and $I = 2 \times 10^{-6}\text{m}^4$.

Figure P8.46

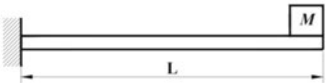


Figure P8.47

48. A concentrated mass M is attached at the mid span of a fixed beam of length L as shown in **Figure P8.48**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine the fundamental natural frequency of the system by using assumed mode method. Consider an assumed mode as $x^4 - 2Lx^3 + L^2x^2$. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.

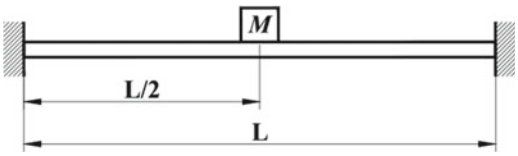


Figure P8.48

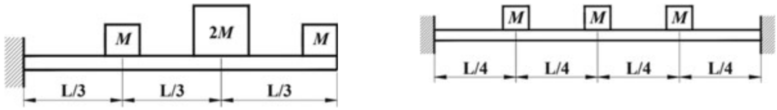
49. Two lumped masses are attached to the beams shown in **Figure P8.49**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia

of section of I. Determine the fundamental natural frequency of the system by using assumed mode method. Consider an assumed mode as $x^4 - 2Lx^3 + L^3x$ for (**a**) and $2x^4 - 5Lx^3 + 3L^2x^2$ for (**b**). Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.

(a) (b)

Figure P8.49

50. Three lumped masses are attached to the beams shown in **Figure P8.50**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequencies of the system by using assumed mode method. Consider an assumed mode as $x^4 - 4Lx^3 + 6L^2x^2$ for (**a**) and $x^4 - 2Lx^3 + L^2x^2$ for (**b**). Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$, $A = 0.005\,\text{m}^2$ and $I = 2 \times 10^{-6}\text{m}^4$.



(a) (b)

Figure P8.50

51. A tapered bar shown in **Figure P8.51** has a length of L. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine natural frequencies of the system by using two terms Petrov–Galerkin method. Use admissible functions as $x^2 - 2Lx$ and $x^3 - 3L^2x$ and weight functions as 1 and x. Take $\rho = 7850\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1.2\,\text{m}$ and $E = 200\,\text{GPa}$.
52. A tapered bar shown in **Figure P8.52** has a length of L. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar varies linearly from A_1 at the left end to A_2 at the right end. Determine natural frequencies of the system by using two terms Galerkin method. Use admissible functions as $\sin(\pi x/L)$ and $\sin(2\pi x/L)$. Take $\rho = 7860\,\text{kg/m}^3$, $A_1 = 2.4 \times 10^{-4}\text{m}^2$, $A_2 = 1.2 \times 10^{-4}\text{m}^2$, $L = 1.2\,\text{m}$ and $E = 200\,\text{GPa}$.

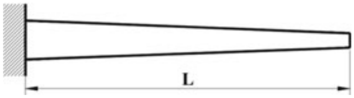


Figure P8.51

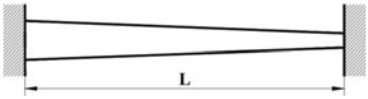


Figure P8.52

53. A concentrated mass M is attached at the free end of a bar of length L as shown in **Figure P8.53**. The bar material has a density of ρ and its cross-sectional area is A. Determine natural frequencies of the system by using one term Galerkin method. Use $\sin(\pi x/2L)$ as an admissible function. Take $\rho = 7860\,\text{kg/m}^3$, $A = 2 \times 10^{-4}\text{m}^2$, $L = 1.2\,\text{m}$, $M = 10\,\text{kg}$ and $E = 200\,\text{GPa}$.
54. A spring of stiffness k is attached to the left end of a bar of length L as shown in **Figure P8.54**. The bar material has a density of ρ, modulus of elasticity of E. The cross-sectional area of the bar is A. Determine natural frequencies of the system by using one term Galerkin method. Use $\cos(\pi x/2L)$ as an admissible function. Take $\rho = 7850\,\text{kg/m}^3$, $A = 2 \times 10^{-4}\text{m}^2$, $L = 1\,\text{m}$, $E = 200\,\text{GPa}$ and $k = 250\,\text{kN/m}$.

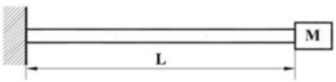


Figure P8.53

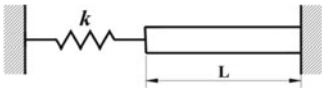


Figure P8.54

55. A tapered simply supported beam shown in **Figure P8.55** has a length of L. The beam material has a density of ρ, modulus of elasticity of E. The beam has a uniform width of b and the depth of the beam varies linearly from H_1 at the left end to H_2 at the right end. Determine natural frequencies of the system by using two terms point collocation method. Use admissible functions as $\sin(\pi x/L)$ and $\sin(2\pi x/L)$. Take $\rho = 7860\,\text{kg/m}^3$, $H_1 = 60\,\text{mm}$, $H_2 = 30\,\text{mm}$, $L = 1.4\,\text{m}$, $E = 200\,\text{GPa}$ and $b = 40\,\text{mm}$.
56. A stepped beam has two portions of equal length $L/2$ as shown in **Figure P8.56**. Each portion of the beam has the same material with a density of ρ and the modulus of elasticity of E. The left part of the beam section has a width of b and a depth of h and those of the right part are b and $0.8h$ respectively. Determine natural frequencies of the system by using two terms Petrov–Galerkin method. Use admissible functions as $2x^4 - 5Lx^3 + 3L^2x^2$ and $2x^5 - 9L^2x^3 + 7L^3x^2$ and weight functions as 1 and x. Take $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $b = 40\,\text{mm}$, $h = 50\,\text{mm}$ and $E = 200\,\text{GPa}$.

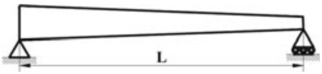


Figure P8.55

Figure P8.56

57. Two lumped masses are attached to the beams shown in **Figure P8.57**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequencies of the system by using one term point collocation method. Use $x^4 - 4Lx^3 + 6L^2x^2$ for (**a**) and $x^4 - 2Lx^3 + L^2x^2$ for (**b**) as admissible functions. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $I = 2 \times 10^{-6}\text{m}^4$.

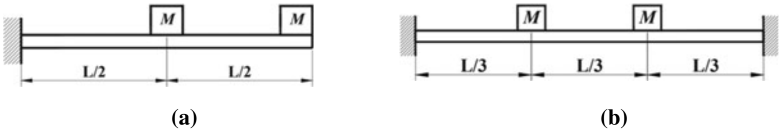


Figure P8.57

58. Three lumped masses are attached to the beams shown in **Figure P8.58**. The beam material has a density of ρ, modulus of elasticity of E and moment of inertia of section of I. Determine natural frequencies of the system by using one term Galerkin method. Use $x^4 - 2Lx^3 + L^3x$ for (**a**), $x^4 - 4Lx^3 + 6L^2x^2$ for (**b**), $x^4 - 2Lx^3 + L^2x^2$ for (**c**) and $2x^4 - 5Lx^3 + 3L^2x^2$ for (**d**) as admissible functions. Take $M = 20\,\text{kg}$, $\rho = 7860\,\text{kg/m}^3$, $L = 1.2\,\text{m}$, $E = 200\,\text{GPa}$ and $I = 2 \times 10^{-6}\text{m}^4$.

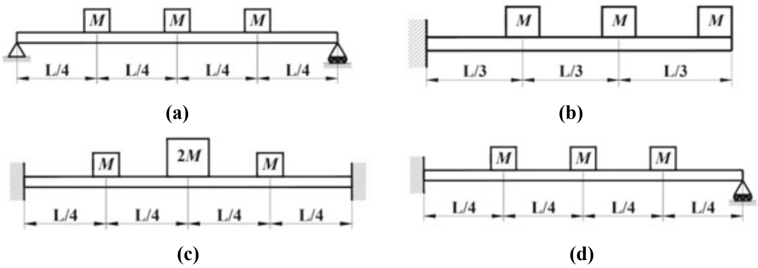


Figure P8.58

Answers

1.
   (a) $\sqrt{\frac{2k}{3m}}$
   (b) $\sqrt{\frac{2k}{m}}$
   (c) $\sqrt{\frac{k}{3m}}$
   (d) $\sqrt{\frac{3k}{5m}}$
   (e) $\frac{1}{2}\sqrt{\frac{k}{m}}$
   (f) $\sqrt{\frac{k}{2m}}$
2.
   (a) 66.0963 rad/s
   (b) 150.24468 rad/s
3.
   (a) 241.1158 rad/s
   (b) 31.1464 rad/s
4. 108.2484 rad/s
5. 69.6236 rad/s
6.
   (a) $\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$
   (b) $\frac{1}{L}\sqrt{\frac{10T}{\rho}}$
7.
   (a) 158.0740 rad/s
   (b) 316.1480 rad/s
8.
   (a) $\frac{\pi}{2L}\sqrt{\frac{E}{\rho}}$
   (b) $\frac{1}{L}\sqrt{\frac{5E}{2\rho}}$
9. $\frac{\pi}{L}\sqrt{\frac{E}{\rho}}$
10. 5411.2785 rad/s
11. 9443.2921 rad/s
12. $\pi^2\sqrt{\frac{EI}{L^3(\rho AL+2M)}}$
13. $\pi^2\sqrt{\frac{EI}{L^3(\rho AL+6M)}}$
14. $\frac{3b}{L^2}\sqrt{\frac{E}{\rho}}$

15. $\sqrt{\frac{EI}{L^3(0.0803\rho AL+0.6642M)}}$
16. $5.9397\frac{h}{L^2}\sqrt{\frac{E}{\rho}}$
17. $\sqrt{\frac{EI}{L^3(0.0042\rho AL+0.0087M)}}$
18. $\sqrt{\frac{EI}{L^3(0.0042\rho AL+0.0254M)}}$
19.
    (a) 206.4307 rad/s
    (b) 473.1602 rad/s
20.
    (a) 795.4951 rad/s
    (b) 102.0621 rad/s
21.
    (a) 395.2847 rad/s
    (b) 221.1308 rad/s
22.
    (a) $1.2250\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$
    (b) $0.5412\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix}$
    (c) $0.6180\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.6180 \\ 1.6180 \\ 1 \end{Bmatrix}$
23.
    (a) $1.3647\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 0.6375 \end{Bmatrix}; 2.3743\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -3.1375 \end{Bmatrix}$
    (b) $0.3376\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.4430 \\ 1.6286 \end{Bmatrix}; 1.4142\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \end{Bmatrix};$
    $2.0943\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -0.6930 \\ 0.2046 \end{Bmatrix}$

(c) $0.3473\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.8794 \\ 2.5321 \\ 2.8794 \end{Bmatrix}; \sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{Bmatrix}; 1.5321\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -0.3473 \\ -0.8794 \\ 0.6527 \end{Bmatrix};$

$1.8794\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -1.5321 \\ 1.3473 \\ -0.5321 \end{Bmatrix}$

24.

(a) $0.7071\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

(b) $0.5412\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.4142 \\ 1 \end{Bmatrix}$

(c) $0.6180\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1.6180 \\ 1.6180 \\ 1 \end{Bmatrix}$

25.

(a) $4.4721\,\text{rad/s}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}; 6.3246\,\text{rad/s}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(b) $1.3546\,\text{rad/s}, \begin{Bmatrix} 1 \\ 2.4495 \\ 3 \end{Bmatrix}; 3.1623\,\text{rad/s}, \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix};$

$4.2620\,\text{rad/s}, \begin{Bmatrix} 1 \\ -2.4495 \\ 3 \end{Bmatrix}$

(c) $0, \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}; 2.4203\,\text{rad/s}, \begin{Bmatrix} 1 \\ 0.4142 \\ -0.4142 \\ 1 \end{Bmatrix}; 4.4721\,\text{rad/s}, \begin{Bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{Bmatrix};$

$5.8431\,\text{rad/s}, \begin{Bmatrix} 1 \\ -2.4142 \\ 2.4142 \\ -1 \end{Bmatrix}$

26.

(a) $1.9544\,\text{rad/s}, \begin{Bmatrix} 1 \\ 1.6180 \\ 1.6180 \end{Bmatrix}$; $3.1623\,\text{rad/s}, \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$;

$5.1167\,\text{rad/s}, \begin{Bmatrix} 1 \\ -0.6180 \\ -0.6180 \end{Bmatrix}$

(b) $0, \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$; $2.1409\,\text{rad/s}, \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 1.6180 \\ -1 \\ -1.6180 \end{Bmatrix}$; $2.4163\,\text{rad/s}, \begin{Bmatrix} 1 \\ 0.1242 \\ -0.2860 \\ -0.5571 \\ -0.2860 \\ -0.5571 \end{Bmatrix}$;

$4.3477\,\text{rad/s}, \begin{Bmatrix} 1 \\ -1.8354 \\ -0.8484 \\ 1.4750 \\ -0.8484 \\ 1.4750 \end{Bmatrix}$; $5.6050\,\text{rad/s}, \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -0.6180 \\ -1 \\ 0.6180 \end{Bmatrix}$;

$6.3712\,\text{rad/s}, \begin{Bmatrix} 1 \\ -5.0888 \\ 5.2845 \\ -2.2179 \\ 5.2845 \\ -2.2179 \end{Bmatrix}$

27.

(a) $243.2834\,\text{rad/s}, \begin{Bmatrix} 1 \\ 1.6770 \end{Bmatrix}$; $673.7507\,\text{rad/s}, \begin{Bmatrix} 1 \\ -0.4770 \end{Bmatrix}$

(b) $492.8089\,\text{rad/s}, \begin{Bmatrix} 1 \\ 1.1165 \end{Bmatrix}$; $864.1418\,\text{rad/s}, \begin{Bmatrix} 1 \\ -0.7165 \end{Bmatrix}$

28. $0, \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$; $2915.0132\,\text{rad/s}, \begin{Bmatrix} 1 \\ -6.9662 \\ -13.4226 \\ -17.0812 \end{Bmatrix}$; $7910.3886\,\text{rad/s},$

$\begin{Bmatrix} 1 \\ -57.6634 \\ -26.1207 \\ 46.2840 \end{Bmatrix}$; $11401.5401\,\text{rad/s}, \begin{Bmatrix} 1 \\ -120.8704 \\ 150.0733 \\ -66.7029 \end{Bmatrix}$

29.

(a) 612.3724 rad/s, $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$; 2371.7082 rad/s, $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

(b) 128.6184 rad/s, $\begin{Bmatrix} 1 \\ 3.1583 \end{Bmatrix}$; 1154.4105 rad/s, $\begin{Bmatrix} 1 \\ -0.1583 \end{Bmatrix}$

(c) 1104.5766 rad/s, $\begin{Bmatrix} 1 \\ 1.1588 \end{Bmatrix}$; 2789.2491 rad/s, $\begin{Bmatrix} 1 \\ -0.4315 \end{Bmatrix}$

30. $\frac{1.9367}{L^2}\sqrt{\frac{E}{\rho}}$; $\frac{4.4480}{L^2}\sqrt{\frac{E}{\rho}}$
31. 5659.6369 rad/s, 15878.5635 rad/s
32. 6063.7526 rad/s
33. $\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\frac{4\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}$
34. 579.4321 rad/s, 508.6268 rad/s, 579.4321 rad/s
35. 232.8559 rad/s, 1307.1409 rad/s
36. 994.6823 rad/s, 3103.9045 rad/s
37. 915.0268 rad/s
38. 717.3208 rad/s
39. 137.8144 rad/s
40. 13227.5490 rad/s, 26575.5848 rad/s
41. 13184.3307 rad/s, 27274.3688 rad/s
42. 2079.4411 rad/s
43. 7943.6647 rad/s
44. 246.3340 rad/s, 1591.2402 rad/s
45. 550.1624 rad/s, 1914.2538 rad/s
46. 1014.4359 rad/s, 2916.1880 rad/s
47. 151.8838 rad/s
48. 1100.2281 rad/s
49.

(a) 459.1115 rad/s
(b) 611.4127 rad/s

50.

(a) 127.9229 rad/s
(b) 1036.4071 rad/s

51. 7545.6458 rad/s, 25511.7192 rad/s
52. 13133.6391 rad/s, 26448.1371 rad/s
53. 1938.5280 rad/s
54. 7948.7227 rad/s
55. 328.1532 rad/s, 1247.6967 rad/s

56. 759.8453 rad/s, 3373.0105 rad/s
57.
   (a) 194.0035 rad/s
   (b) 1022.1035 rad/s

58.
   (a) 431.8430 rad/s
   (b) 160.4257 rad/s
   (c) 895.1513 rad/s
   (d) 689.7173 rad/s