Student Names and IDs:

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Homework 6

Part 1: Newton's Method

Problem 1.1

Write a function with header

```
def Newton(f, g, H, z0, maxK=1000, delta=hw6.small, , normalize=False):
```

that takes three functions `f`, `g`, `H` that compute value, gradient, and Hessian of some function f and a starting point `z0` (a one-dimensional `numpy` array) and finds a minimum of f by Newton's method. The optional parameter `maxK` is the maximum number of iterations, and `delta` is a threshold on the Euclidean norm of $\mathbf{z}_{k+1} - \mathbf{z}_k$. The algorithm stops when that norm falls below `delta`. See the programming notes for the meaning of `normalize`.

Show your code and the value `zNewton[-1]` found by running `Newton` on the Rosenbrock function, starting at $\mathbf{z}_0 = (-1.2, 1)$. Also print out the number of points in `zNewton`.

Answer

In [79]:
```
import hw6
import numpy as np
import math
```

In [80]:
```
def Newton(f, g, H, z0, maxK=1000, delta=hw6.small, normalize=False):
    '''
    implementation of Newton's method. returns the input of f that gives
 the minimum.
    '''

    count = 0
    answer = []
    answer.append(z0)
    dz_norm = 100000 # intialize the norm to a very high number

    while count < maxK:

        eig_vals = np.linalg.eigvals(H(z0)) #finds eigenvalues

        if np.all(eig_vals >= 0): # checks if it is positive semi-defini
te (evals >= 0)

            dz = np.dot(np.linalg.pinv(H(z0)), -g(z0))
            z1 = dz + z0
            dz_norm = np.linalg.norm(dz)
            if dz_norm < hw6.small:
                break

        if not np.all(eig_vals >= 0): # if even one is negative, use lin
eSearch()
            z1 = hw6.lineSearch(f, g, z0, delta=hw6.small)
            if normalize:
                if not np.all(z1 == 0):
                    z1 /= np.linalg.norm(z1)
            dz = z1 - z0
            dz_norm = np.linalg.norm(dz)
            if dz_norm < hw6.small:
                break


        if normalize:
            if not np.all(z1 == 0):
                z1 /= np.linalg.norm(z1)

        answer.append(z1)
        z0 = z1
        count += 1

    return answer
```

In [81]:
```
z0 = np.array([-1.2, 1])
zNewton = Newton(hw6.Rosenbrock, hw6.RosenGrad, hw6.RosenHess, z0, maxK=
1000, delta=hw6.small, normalize=False) # list of steps
minNewton = zNewton[-1] # minimum found by Newton
stepsNewton = len(zNewton) # number of steps taken in Newton
#print("Values in zNewton", zNewton)
print("The minimum found by Newton's method is", minNewton)
print("Number of points in zNewton:", stepsNewton)
```

```
The minimum found by Newton's method is [1. 1.]
Number of points in zNewton: 7
```

Problem 1.2

How does the number of iterations from `Newton` compare with that you found for steepest descent in homework 5 on the same optimization problem? You need not give exact numbers.

Answer

There are orders of magnitude less steps in Newton's method. We take 6 steps in Newton's method to reach the global minimum (first value of zNewton is the starting point). Compare this to the 1000 steps we took in steepest descent using line search and we still did not find the minimum. Trying to use line search with steepest descent in the above function for a 1000 iterations only brings us to [0.91289501 0.83317358], which is still quite a ways from the minimum. Newton's method dramatically trumps steepest descent with line search.

Part 2: The Geometry of Newton's Method

Problem 2.1

Write a function with header

```
def ellipse(z0, z1, Hessian, color='red'):
```

that takes two consecutive points `z0` and `z1` on a Newton optimization path and a function `Hessian` that computes the Hessian of some function from $\mathbb{R}^2$ to $\mathbb{R}$ and returns the ellipse E associated with that step.

Show your code and display the ellipse for the first Newton step of your optimization in the previous part.

Answer

In [82]:
```
import matplotlib
import matplotlib.pyplot as plt
%precision %g
%config InlineBackend.figure_format = 'retina'
matplotlib.rcParams['savefig.dpi'] = 120
matplotlib.rcParams['figure.dpi'] = 120
%matplotlib inline
```

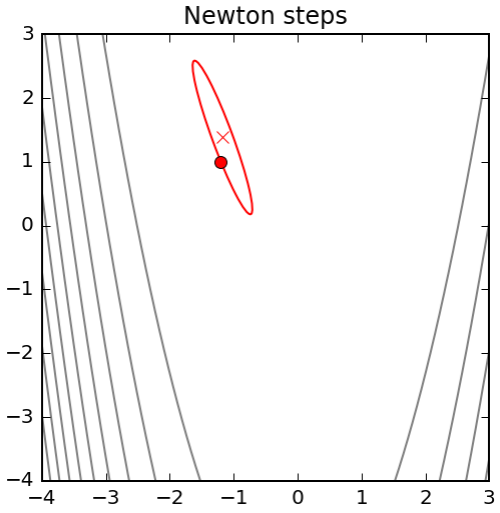
In [83]:
```
def ellipse(z0, z1, Hessian, color='red'):
    '''
    returns the components needed to describe an ellipse
    -center of ellipse (tuple in x,y)
    -angle between first axis and reference axis
    -full lengths
    '''

    # determine center
    center = z1

    # determine angle
    H0 = Hessian(z0)
    evals, V = np.linalg.eigh(H0)
    theta_rad = np.arctan2(V[1][0], V[0][0]) # in radians
    theta = theta_rad * (180/np.pi)          # convert radians to degree
s

    # determine width and height
    dz = z1 - z0
    dzH0 = np.dot(dz, H0)
    a = np.dot(dzH0, dz)                     # scaling factor
    D = np.diag(evals)                       # square matrix with eigenv
alues of H0 on the diagonal
    Linv = np.sqrt(D/a)
    L = np.linalg.inv(np.sqrt(D / a))
    width = 2*L[0][0]
    height = 2*L[1][1]

    ellipse = matplotlib.patches.Ellipse(center, width, height, theta, e
dgecolor = color, facecolor = 'none')

    return ellipse
```

In [84]:
```
z0 = zNewton[0]
z1 = zNewton[1]
x0, y0 = z0[0], z0[1]
x1, y1 = z1[0], z1[1]
```

In [85]:
```
fig, ax = hw6.RosenContours()
ell = ellipse(z0, z1, hw6.RosenHess, color = 'red')
ax.add_artist(ell)
plt.plot(x0, y0, 'ro')
plt.plot(x1, y1, 'rx')
plt.title("Newton steps")
```

```
/anaconda3/lib/python3.6/site-packages/numpy/ma/core.py:6434: MaskedArr
ayFutureWarning: In the future the default for ma.minimum.reduce will b
e axis=0, not the current None, to match np.minimum.reduce. Explicitly
pass 0 or None to silence this warning.
  return self.reduce(a)
/anaconda3/lib/python3.6/site-packages/numpy/ma/core.py:6434: MaskedArr
ayFutureWarning: In the future the default for ma.maximum.reduce will b
e axis=0, not the current None, to match np.maximum.reduce. Explicitly
pass 0 or None to silence this warning.
  return self.reduce(a)
```

Out[85]: <matplotlib.text.Text at 0x1a0b1f3320>



Problem 2.2

Make a new plot that displays the ellipses for the first four steps of your Newton optimization path you found in the previous part of this assignment.

Show your code and the resulting plot.

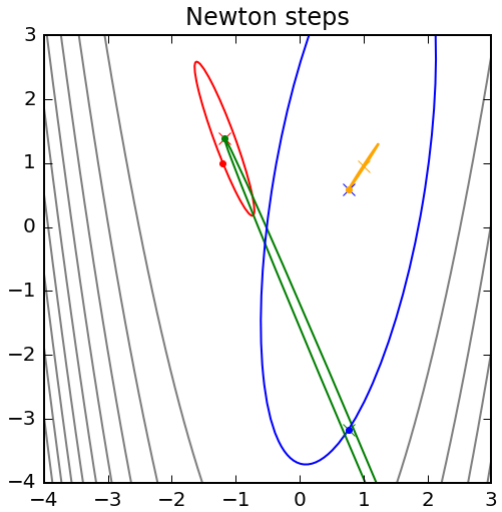
Answer

In [86]:
```
# initial and first 4 steps
z0 = zNewton[0]
z1 = zNewton[1]
z2 = zNewton[2]
z3 = zNewton[3]
z4 = zNewton[4]
x0, y0 = z0[0], z0[1]
x1, y1 = z1[0], z1[1]
x2, y2 = z2[0], z2[1]
x3, y3 = z3[0], z3[1]
x4, y4 = z4[0], z4[1]
```

In [87]:
```
fig, ax = hw6.RosenContours()
ellipse1 = ellipse(z0, z1, hw6.RosenHess, color = 'red')
ellipse2 = ellipse(z1, z2, hw6.RosenHess, color = 'green')
ellipse3 = ellipse(z2, z3, hw6.RosenHess, color = 'blue')
ellipse4 = ellipse(z3, z4, hw6.RosenHess, color = 'orange')
ax.add_artist(ellipse1)
ax.add_artist(ellipse2)
ax.add_artist(ellipse3)
ax.add_artist(ellipse4)
plt.plot(x0, y0, 'r.')
plt.plot(x1, y1, 'rx')
plt.plot(x1, y1, 'g.')
plt.plot(x2, y2, 'gx')
plt.plot(x2, y2, 'b.')
plt.plot(x3, y3, 'bx')
plt.plot(x3, y3, '.', color = 'orange')
plt.plot(x4, y4, 'x', color = 'orange')
plt.title("Newton steps")
```

```
/anaconda3/lib/python3.6/site-packages/numpy/ma/core.py:6434: MaskedArr
ayFutureWarning: In the future the default for ma.minimum.reduce will b
e axis=0, not the current None, to match np.minimum.reduce. Explicitly
pass 0 or None to silence this warning.
  return self.reduce(a)
/anaconda3/lib/python3.6/site-packages/numpy/ma/core.py:6434: MaskedArr
ayFutureWarning: In the future the default for ma.maximum.reduce will b
e axis=0, not the current None, to match np.maximum.reduce. Explicitly
pass 0 or None to silence this warning.
  return self.reduce(a)
```

Out[87]: <matplotlib.text.Text at 0x1a0c0767f0>



Part 3: Binary Linear Classifiers

Problem 3.1

Define functions with headers

```
def lossXE(y, p):
def score(x, v):
```

that implement the cross-entropy loss and score function for a logistic-regression classifier.

Show your code, and a single diagram that shows the two plots of `lossXE(0, p)` and `lossXE(1, p)` for p between 10^{-3} and $1 - 10^{-3}$.

Answer

In [88]:
```
T, S = hw6.makeData()
```

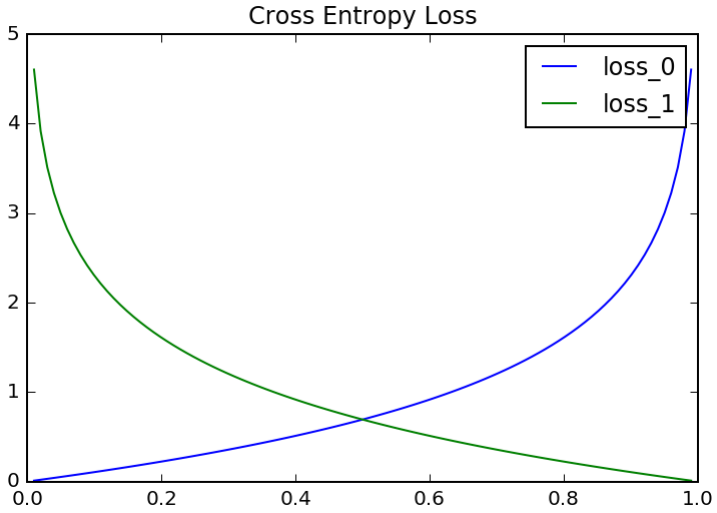
In [89]:
```
def lossXE(y,p):
    '''
    computes the cross entropy loss
    '''
    loss = (-y*np.log(p)) - ((1-y)*np.log(1-p))
    return loss
```

In [90]:
```
def score(x,v):
    '''
    assigns a p value using the logistic function
    '''
    a = v[0] + np.dot(v[1:], x)
    a = max(-10, min(10, a))
    score = 1 / (1 + np.exp(-a))
    return score
```

In [91]:
```
p_range = np.linspace(10e-3, 1-10e-3, 100) # set up the range of p

# initialize lists holding losses of y = 0 and y = 1
loss_0 = []
loss_1 = []

for p in p_range:

    loss = lossXE(0, p)
    loss_0.append(loss)
    loss = lossXE(1, p)
    loss_1.append(loss)
```

In [92]:
```
plt.plot(p_range, loss_0, label = 'loss_0')
plt.plot(p_range, loss_1, label = 'loss_1')
plt.title("Cross Entropy Loss")
plt.legend(loc = "upper right")
```

Out[92]: <matplotlib.legend.Legend at 0x1a0d191278>



Problem 3.2

Define functions with headers

```
def riskXE(v):
def riskXEGrad(v):
def riskXEHess(v):
```

that implement the cross-entropy risk on set `T` (loaded by `hw6.makeData()`), its gradient, and Hessian, following the formulas in the notes.

Then run your own `Newton` to train the optimal parameter vector $\mathbf{v}$ on `T`, starting with a vector $\mathbf{v}_0$ of all zeros.

Show your code and give the number of iterations to convergence.

Answer

In [93]:
```
def riskXE(v):
    '''
    calculates risk of the cross entropy loss
    '''

    risk = 0         # initialize risk to 0
    N = len(T['x'])  # number of training examples in T
    x_data = T['x']
    y_data = T['y']

    for i in range(N):
        p = score(x_data[i], v)
        loss = lossXE(y_data[i], p)
        risk += loss
    risk = (1/N) * risk

    return risk
```

In [94]:
```
def riskXEGrad(v):
    '''
    calculates the risk of the gradient of the cross entropy loss
    '''
    risk = 0         # initialize risk to 0
    N = len(T['x'])  # number of training examples in T
    x_data = T['x']
    y_data = T['y']

    for i in range(N):
        p = score(x_data[i], v)
        constant = p - y_data[i]      # constant to multiply vector by
        b = np.insert(x_data[i], 0, 1)
        loss = constant*b
        risk += loss

    risk = (1/N) * risk

    return risk
```

In [95]:
```
def riskXEHess(v):
    '''
    calculates the risk of the Hessian of the cross entropy loss
    '''

    risk = 0
    N = len(T['x'])  # number of training examples in T
    x_data = T['x']
    y_data = T['y']

    for i in range(N):
        p = score(x_data[i], v)
        constant = p * (1-p)           # constant to multiply matrix by
        #a = np.array(x_data[i])
        a = x_data[i]
        b = np.insert(a, 0, 1)
        d = np.reshape(b, (len(b), 1))
        loss = constant * d * b
        risk += loss

    risk = (1/N) * risk

    return risk
```

In [96]:
```
# initialize
m = len(T['x'][0]) + 1 # m = d + 1
v0 = np.zeros(m)       # starting weights are all 0
```

In [97]:
```
zNewton = Newton(riskXE, riskXEGrad, riskXEHess, v0, maxK=30, delta=hw6.
small, normalize=True)
print("Length of zNewton:", len(zNewton))
```

```
Length of zNewton: 31
```

Problem 3.3

Why do we normalize the value of $\mathbf{v}$ at every iteration? Answer this question by explaining why the norm of $\mathbf{v}$ is meaningless for this problem.

Answer

We want to assign a label $y = 0$ or $y = 1$. The norm of $\mathbf{v}$ is meaningless because logistic-regression classification does not care about constants. If they were important, we would need to introduce a scaling factor f and then find a way to determine this value. We're better off without it.

Problem 3.4

Write functions with headers

```
def h(x, v):
def risk01(v, S):
```

that implement the logistic-regression classifier h with parameter $\mathbf{v}$ and the zero-one risk on set S.

Show your code and the values of *zero-one* training and test risk for the optimal parameter vector $\mathbf{v}^*$ you found in the previous problem. Express these risks as *percentage* values.

Answer

In [98]:
```
def h(x, v):
    '''

    '''
    a = v[0] + np.dot(v[1:], x)
    a = max(-10, min(10, a))
    score = 1 / (1 + np.exp(-a))

    return score
```

In [99]:
```
def risk01(v, S):
    '''

    '''
    risk = 0
    x_data = S['x']
    y_data = S['y']
    N = len(S['x'])

    for i in range(N):
        p = h(x_data[i], v_star)
        loss = -y_data[i]*np.log(p) - (1 - y_data[i])*np.log(1-p)
        risk += loss

    answer = (1/N) * risk

    return answer
```

In [100]:
```
v_star = zNewton[-1]                # most optimal found above
test_risk = risk01(v_star, S)       # test set risk
training_risk = riskXE(v_star)      # training set risk
```

In [101]:
```
print("Training risk:", training_risk)
print("Test risk", test_risk)
print("Percentage training to test risk:", 100 * training_risk / test_ri
sk, "%")
```

```
Training risk: 0.38022897757800966
Test risk 0.3949603662545597
Percentage training to test risk: 96.27016026538335 %
```

The training risk and test risk value are similar; the closer the training and tests risks are to each other, the closer to 100 the percentage will be. From this, I can confidently that this set of parameters will do well on unseen data.