Pure Book 2, Exercise 6C

Question 1 Rewrite the following as powers of $\sec\theta$, $\csc\theta$, or $\cot\theta$.

$$\frac{1}{\sin^3\theta} = \csc^3\theta$$
$$\frac{4}{\tan^6\theta} = 4\cot^6\theta$$
$$\frac{1}{2\cos^2\theta} = 2\sec^2\theta$$
$$\frac{1-\sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$$
$$\frac{\sec\theta}{\cos^4\theta} = \sec^5\theta$$
$$\sqrt{\csc^3\theta\cot\theta\sec\theta} = \sqrt{\frac{\cos\theta}{\sin^4\theta\cos\theta}} = \frac{1}{\sqrt{\sin^4\theta}} = \frac{1}{\sin^2\theta} = \csc^2\theta$$
$$\frac{2}{\sqrt{\tan\theta}} = 2\cot^{\frac{1}{2}}\theta$$
$$\frac{\csc^2\theta\tan^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\sin^2\theta\cos^3\theta} = \frac{1}{\cos^3\theta} = \sec^3\theta$$

Question 2 Write down the value(s) of $\cot x$ in each of the following equations.

$$\begin{aligned} 5\sin x &= 4\cos x \\ \frac{\cos x}{\sin x} &= \frac{5}{4} \\ \therefore \cot x &= \frac{5}{4} \end{aligned} \tag{a}$$

$$\begin{aligned} \tan x &= -2 \\ \therefore \cot x &= -\frac{1}{2} \end{aligned} \tag{b}$$

$$\begin{aligned} 3\frac{\sin x}{\cos x} &= \frac{\cos x}{\sin x} \\ 3\sin^2 x &= \cos^2 x \\ \frac{\cos^2 x}{\sin^2 x} &= 3 \\ \cot^2 x &= 3 \\ \cot x &= \pm\sqrt{3} \end{aligned} \tag{c}$$

Question 3 Using the definitions of sec, csc, cot, and tan, simplify the following expressions.

$$\sin\theta\cot\theta = \frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \frac{\sin\theta\cos\theta}{\sin\theta} = \cos\theta$$
$$\tan\theta\cot\theta = \frac{\tan\theta}{\tan\theta} = 1$$
$$\tan 2\theta\csc 2\theta = \frac{\tan 2\theta}{\sin 2\theta} = \frac{\frac{\sin 2\theta}{\cos 2\theta}}{\sin 2\theta} = \frac{\sin 2\theta}{\sin 2\theta\cos 2\theta} = \frac{1}{\cos 2\theta} = \sec 2\theta$$

$$\cos\theta\sin\theta(\cot\theta+\tan\theta) = \cos\theta\sin\theta\cot\theta + \cos\theta\sin\theta\tan\theta = \frac{\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin^2\theta\cos\theta}{\cos\theta}$$
$$= \frac{\sin\theta\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta\cos\theta}{\cos\theta} = \sin^2\theta + \cos^2\theta = 1$$

$$\sin^3 x\csc x + \cos^3 x\sec x = \frac{\sin^3 x}{\sin x} + \frac{\cos^3 x}{\cos x} = \sin^2 x + \cos^2 x = 1$$
$$\sec A - \sec A\sin^2 A = \sec A\cdot(1-\sin^2 A) = \frac{\cos^2 A}{\cos A} = \cos A$$

$$\sec^2 x\cos^5 x + \cot x\csc x\sin^4 x = \frac{\cos^5 x}{\cos^2 x} + \frac{\sin^4 x}{\frac{\sin^2 x}{\cos x}} = \cos^3 x + \sin^2 x\cos x$$
$$= \cos x\cdot(\sin^2 x + \cos^2) = \cos x$$

Question 4 Prove that:

$$\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$$
$$\cos\theta + \frac{\sin^2\theta}{\cos\theta} \equiv \frac{\sin^2\theta+\cos^2\theta}{\cos\theta} \equiv \frac{1}{\cos\theta} \equiv \sec\theta \tag{a}$$

$$\cot\theta + \tan\theta \equiv \csc\theta\sec\theta$$
$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \equiv \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta} \equiv \frac{1}{\sin\theta\cos\theta} \equiv \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} \equiv \csc\theta\sec\theta \tag{b}$$

$$\csc\theta - \sin\theta \equiv \cos\theta\cot\theta$$
$$\frac{1}{\sin\theta} - \sin\theta \equiv \frac{1-\sin^2\theta}{\sin\theta} \equiv \frac{\cos^2\theta}{\sin\theta} \equiv \cos\theta\cdot\frac{\cos\theta}{\sin\theta} \equiv \cos\theta\cot\theta \tag{c}$$

$$
\begin{gathered}
(1-\cos x)(1+\sec x) \equiv \sin x \tan x \\
1+\sec x-\cos x-\cos x \sec x \equiv \sec x-\cos x \equiv \tfrac{1}{\cos x}-\cos x \\
\equiv \tfrac{1-\cos^2 x}{\cos x} \equiv \tfrac{\sin^2 x}{\cos x} \equiv \sin x \cdot \tfrac{\sin x}{\cos x} \equiv \sin x \tan x
\end{gathered} \tag{d}
$$

$$
\begin{gathered}
\tfrac{\cos x}{1-\sin x}+\tfrac{1-\sin x}{\cos x} \equiv 2 \sec x \\
\tfrac{\cos^2 x+(1-\sin x)^2}{\cos x \cdot(1-\sin x)} \equiv \tfrac{cos^2 x+1-2 \sin x+\sin^2 x}{\cos x \cdot(1-\sin x)} \equiv \tfrac{2-2 \sin x}{\cos x \cdot(1-\sin x)} \equiv \tfrac{2(1-\sin x)}{\cos x \cdot(1-\sin x)} \\
\equiv \tfrac{2}{\cos x} \equiv 2 \sec x
\end{gathered} \tag{e}
$$

$$
\begin{gathered}
\tfrac{\cos \theta}{1+\cot \theta} \equiv \tfrac{\sin \theta}{1+\tan \theta} \\
\tfrac{\cos \theta}{1+\frac{1}{\tan \theta}} \equiv \tfrac{\cos \theta}{\frac{1+\tan \theta}{\tan \theta}} \equiv \tfrac{\cos \theta \tan \theta}{1+\tan \theta} \equiv \tfrac{\sin \theta}{1+\tan \theta}
\end{gathered} \tag{f}
$$

Question 5 Solve, for values of θ in the interval $0 \leq \theta \leq 360°$, the following equations. Give your answers to 3 significant figures where necessary.

$$
\begin{gathered}
\sec \theta=\sqrt{2} \\
\Rightarrow \tfrac{1}{\cos \theta}=\sqrt{2} \\
\therefore \cos \theta=\tfrac{1}{\sqrt{2}} \quad \therefore \theta=45^{\circ} \\
\text{Solutions in the interval are } 45^{\circ} \text{ and } 315^{\circ}
\end{gathered} \tag{a}
$$

$$
\begin{gathered}
\csc \theta=-3 \\
\Rightarrow \tfrac{1}{\sin \theta}=-3 \\
\therefore \sin \theta=-\tfrac{1}{3} \quad \therefore \theta=-19.5^{\circ} \text{ (3 s.f.)} \\
\text{Solutions in the interval are } 199^{\circ} \text{ and } 341^{\circ} \text{ (3 s.f.)}
\end{gathered} \tag{b}
$$

$$
\begin{gathered}
5 \cot \theta=-2 \\
\therefore \tan \theta=-2.5 \quad \theta=-68.2 \text{ (3 s.f.)} \\
\text{Solutions in the interval are } 112^{\circ} \text{ and } 292^{\circ} \text{ (3 s.f.)}
\end{gathered} \tag{c}
$$

$$\begin{gathered}\csc\theta = 2\\ \therefore \sin\theta = \tfrac{1}{2} \quad \theta = 30^\circ\\ \text{Solutions in the interval are } 30^\circ \text{ and } 150^\circ\end{gathered} \tag{d}$$

$$\begin{gathered}3\sec^2\theta - 4 = 0\\ \therefore \cos\theta = \pm\tfrac{\sqrt{3}}{2} \quad \theta = \pm 30^\circ\\ \text{Solutions in the interval are } 30^\circ,\ 150^\circ,\ 210^\circ, \text{ and } 330^\circ\end{gathered} \tag{e}$$

$$\begin{gathered}5\cos\theta = 3\cot\theta\\ \Rightarrow 5\cos\theta\sin\theta = 3\cos\theta \quad \therefore \sin\theta = \tfrac{3}{5} \quad \therefore \theta = 36.9^\circ \text{ (3 s.f.)}\\ \Rightarrow \cos\theta\cdot(5\sin\theta - 3\cos\theta) = 0 \quad \therefore \cos\theta = 0 \quad \therefore \theta = 90^\circ\\ \text{Solutions in the interval are } 36.9^\circ \text{ (3 s.f.)},\ 90^\circ,\ 143^\circ \text{ (3 s.f.), and } 270^\circ\end{gathered} \tag{f}$$

$$\begin{gathered}\cot^2\theta - 8\tan\theta = 0\\ \therefore \tan\theta = \tfrac{1}{2} \quad \therefore \theta = 26.6^\circ \text{ (3 s.f.)}\\ \text{Solutions in the interval are } 26.6^\circ \text{ and } 207^\circ \text{ (3 s.f.)}\end{gathered} \tag{g}$$

$$\begin{gathered}2\sin\theta = \csc\theta\\ \therefore \sin\theta = \pm\tfrac{1}{\sqrt{2}} \quad \therefore \theta = \pm 45^\circ\\ \text{Solutions in the interval are } 45^\circ,\ 135^\circ,\ 225^\circ, \text{ and } 315^\circ\end{gathered} \tag{h}$$

Question 6 Solve, for values of θ in the interval $-180^\circ \le \theta \le 180^\circ$, the following equations:

$$\begin{gathered}\csc\theta = 1\\ \therefore \sin\theta = 1 \qquad \therefore \theta = 90^\circ\\ \text{The only solution in the interval is } 90^\circ\end{gathered} \tag{a}$$

$$\begin{gathered}\sec\theta = -3\\ \therefore \cos\theta = -\tfrac{1}{3} \qquad \therefore \theta = 109^\circ \text{ (3 s.f.)}\\ \text{The solutions in the interval are } -109^\circ \text{ and } 109^\circ \text{ (3 s.f.)}\end{gathered} \tag{b}$$

$$\begin{gathered}\cot\theta = 3.45\\ \therefore \tan\theta = \tfrac{1}{3.45} \qquad \therefore \theta = 16.2 \text{ (3 s.f.)}\\ \text{The solutions in the interval are } -164^\circ \text{ and } 16.2^\circ \text{ (3 s.f.)}\end{gathered} \tag{c}$$

$$\begin{gathered}2\csc^2\theta - 3\csc\theta = 0\\ \therefore \sin\theta = \tfrac{2}{3} \qquad \therefore \theta = 41.8 \text{ (3 s.f.)}\\ \text{The solutions in the interval are } 41.8^\circ\text{, and } 138^\circ \text{ (3 s.f.)}\end{gathered} \tag{d}$$

$$\begin{gathered}\sec\theta = 2\cos\theta\\ \therefore \cos\theta = \pm\tfrac{1}{\sqrt{2}} \qquad \therefore \theta = 45^\circ\\ \text{The solutions in the interval are } \pm 45^\circ \text{ and } \pm 135^\circ\end{gathered} \tag{e}$$

$$\begin{aligned} 3\cot\theta &= 2\sin\theta \\ \Rightarrow 3\cos\theta &= 2\sin^2\theta \\ \Rightarrow 2\cos^2\theta + 3\cos\theta - 2 &= 0 \\ \Rightarrow \cos\theta &= \tfrac{-3\pm 5}{4} \\ \therefore \cos\theta = \tfrac{1}{2} \quad & \therefore \theta = 60^\circ \end{aligned} \tag{f}$$

The solutions in the interval are $\pm 60^\circ$

$$\csc 2\theta = 4$$

$$\therefore \sin 2\theta = \tfrac{1}{4} \quad \therefore 2\theta = 14.48^\circ \text{ (3 s.f.)}$$

The solutions in the interval are -173°, -97.2°, 7.24°, and 82.8° (3 s.f.)

(g)

$$2\cot^2\theta - \cot\theta - 5 = 0$$

$$\Rightarrow \cot\theta = \tfrac{1\pm\sqrt{41}}{4}$$

$$\therefore \tan\theta = \tfrac{-1+\sqrt{41}}{10}, -\tfrac{1+\sqrt{41}}{10} \quad \therefore \theta = 28.4^\circ, -36.5^\circ \text{ (3 s.f.)}$$

The solutions in the interval are -152°, -36.5°, 28.4°, and 143° (3 s.f.)

(h)

Question 7 Solve the following equations for values of θ in the interval $0 \le \theta \le 2\pi$. Give your answers in terms of π.

$$\begin{gathered} \sec\theta = -1 \\ \therefore \cos\theta = -1 \quad \therefore \theta = \pi \text{ rad} \end{gathered} \tag{a}$$

The only solution in the interval is π rad

$$\cot\theta = -\sqrt{3}$$
$$\therefore \tan\theta = -\tfrac{1}{\sqrt{3}} \qquad \therefore \theta = -\tfrac{1}{6}\pi \text{ rad} \tag{b}$$
$$\text{The solutions in the interval are } \frac{5}{6}\pi \text{ rad and } \frac{11}{6}\pi \text{ rad}$$

$$\csc(\tfrac{1}{2}\theta) = \tfrac{2\sqrt{3}}{3}$$
$$\therefore \sin(\tfrac{1}{2}\theta) = \tfrac{\sqrt{3}}{2} \qquad \therefore \tfrac{1}{2}\theta = \tfrac{1}{3}\pi \text{ rad} \tag{c}$$
$$\text{The solutions in the interval are } \frac{2}{3}\pi \text{ rad and } \frac{4}{3}\pi \text{ rad}$$

$$\sec\theta = \sqrt{2}\tan\theta \quad (\theta \neq \tfrac{1}{2}\pi, \theta \neq \tfrac{3}{2}\pi)$$
$$\therefore \sin\theta = \tfrac{1}{\sqrt{2}} \qquad \therefore \theta = \tfrac{1}{4}\pi \text{ rad} \tag{d}$$
$$\text{The solutions in the interval are } \frac{1}{4}\pi \text{ rad and } \frac{3}{4}\pi \text{ rad}$$

Question 8 In the diagram $AB = 6$cm is the diameter of the circle and BT is the tangent to the circle at B. The chord AC is extended to meet this tangent at D and $\angle DAB = \theta$.

$$\textit{Show that } CD = 6(\sec\theta - \cos\theta)\,cm$$

$$\begin{gathered} \text{In } \triangle ABD,\ 6\sec\theta = AD \\ \text{In } \triangle ABC,\ 6\cos\theta = AC \\ CD = AD - AC = 6\sec\theta - 6\cos\theta \\ \therefore CD = 6(\sec\theta - \cos\theta) \end{gathered} \tag{a}$$

Given that $CD = 16cm$, calculate the length of the chord AC.

$$6(\sec\theta - \cos\theta) = 16$$
$$3(\tfrac{1}{\cos\theta} - \cos\theta) = 8$$
$$8\cos\theta = 3 - 3\cos^2\theta$$
$$3\cos^2\theta + 8\cos\theta - 3 = 0$$
$$\cos\theta = \tfrac{-8\pm10}{6} = \tfrac{1}{3}, -3 \text{ (-3 is not possible)}$$

(b)

$$AC = 6\cos\theta = 6\cdot\tfrac{1}{3} = 2\text{cm}$$

Question 9

$$\textit{Prove that } \frac{\csc x - \cot x}{1-\cos x} \equiv \csc x$$

(a)

$$\frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{1-\cos x} \equiv \frac{\frac{1-\cos x}{\sin x}}{1-\cos x} = \frac{(1-\cos x)}{\sin x\cdot(1-\cos x)} \equiv \frac{1}{\sin x} = \csc x$$

$$\textit{Hence solve, in the interval } -\pi \leq x \leq \pi \textit{ the equation } \frac{\csc x - \cot x}{1-\cos x} = \mathit{2}$$

$$\csc x = 2$$
$$\therefore \sin x = \tfrac{1}{2} \quad \therefore x = \tfrac{1}{6}\pi \text{ rad}$$
$$\text{The solutions in the interval are } \frac{1}{6}\pi \text{ rad and } \frac{5}{6}\pi \text{ rad}$$

(b)

Question 10

$$\textit{Prove that } \frac{\sin x \tan x}{1-\cos x} - 1 \equiv \sec x$$

$$\frac{\frac{\sin^2 x}{\cos x}}{1-\cos x} - 1 \equiv \frac{\sin^2 x}{\cos x \cdot (1-\cos x)} - 1 \equiv \frac{\sin^2 + \cos^2 x - \cos x}{\cos x \cdot (1-\cos x)} \equiv \frac{1-\cos x}{\cos x \cdot (1-\cos x)} \equiv \frac{1}{\cos x} \equiv \sec x$$

(a)

$$\textit{Hence explain why the equation } \frac{\sin x \tan x}{1-\cos x} - 1 = -\frac{1}{2} \textit{ has no solutions.}$$

$$\sec x = -\tfrac{1}{2}$$

$$\Rightarrow \cos x = -2$$

$\cos x$'s range is between -1 and 1, therefore $\cos(-2)$ has no solutions.

(b)

Question 11 Solve, in the interval $0 \leq x \leq 360°$, the equation $\frac{1+\cot x}{1+\tan x} = 5$.

$$\tfrac{1+\cot x}{1+\tan x} = 5$$

$$\Rightarrow \tfrac{\sin x + \cos x}{\sin x} \div \tfrac{\sin x + \cos x}{\cos x} = 5$$

$$\Rightarrow \tfrac{\cos x \cdot (\sin x + \cos x)}{\sin x \cdot (\sin x + \cos x)} = 5$$

$$\cot x = 5$$

$$\therefore \tan x = \tfrac{1}{5} \quad \therefore x = 11.3° \text{ (3 s.f.)}$$

The solutions in the interval are $11.3°$ and $191.3°$ (3 s.f.)