

Suppose we have an n th degree polynomial, such that

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \dots + c_0 = \sum_{i=0}^n c_{n-i} x^{n-i}$$

By the power rule of differentiation, we can conclude that the first derivative of $f(x)$ is as follows

$$\begin{aligned} f'(x) &= n c_n x^{n-1} + (n-1) c_{n-1} x^{n-2} + (n-2) c_{n-2} x^{n-3} + \dots + c_1 \\ &= \sum_{i=0}^{n-1} (n-i) c_{n-i} x^{n-i-1} \end{aligned}$$

And the second derivative is

$$\begin{aligned} f''(x) &= n(n-1) c_n x^{n-2} + (n-1)(n-2) c_{n-1} x^{n-3} + (n-2)(n-3) c_{n-2} x^{n-4} + \dots + c_2 \\ &= \sum_{i=0}^{n-2} (i-n)(i-n+1) a_{n-i} x^{n-i-2} \end{aligned}$$

After starting with an initial guess, the next iteration of Halley's method is given by

$$x_k - \frac{2f(x_k)f'(x_k)}{2[f'(x_k)]^2 - f(x_k)f''(x_k)}.$$

This means that we must first find $f(x)f'(x)$, $[f'(x)]^2$, and $f(x)f''(x)$. These all consist of the multiplication of two series - there is a nice general form to this problem stated below

$$\left(\sum_{i=0}^n x_i \right) \left(\sum_{j=0}^m y_j \right) = \sum_{i=0}^n \sum_{j=0}^m x_i y_j$$

From this, we can conclude that:

$$\begin{aligned} f(x)f'(x) &= \left(\sum_{i=0}^n c_{n-i}x^{n-i}\right)\left(\sum_{i=0}^{n-1}(n-i)c_{n-i}x^{n-i-1}\right) \\ &= \sum_{i=0}^n \sum_{j=0}^{n-1} (n-j)c_{n-i}c_{n-j}x^{2n-i-j-1} \end{aligned}$$

$$\begin{aligned} [f'(x)]^2 &= \left(\sum_{i=0}^{n-1}(n-i)c_{n-i}x^{n-i-1}\right)\left(\sum_{i=0}^{n-1}(n-i)c_{n-i}x^{n-i-1}\right) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n-i)(n-j)c_{n-i}c_{n-j}x^{2(n-1)-i-j} \end{aligned}$$

$$\begin{aligned} f(x)f''(x) &= \left(\sum_{i=0}^n c_{n-i}x^{n-i}\right)\left(\sum_{i=0}^{n-2}(i-n)(i-n+1)c_{n-i}x^{n-i-2}\right) \\ &= \sum_{i=0}^n \sum_{j=0}^{n-2} (j-n)(j-n+1)c_{n-i}c_{n-j}x^{2(n-1)-i-j} \end{aligned}$$

And all that is left to do is to plug it into the formula for Halley's method, leaving us with the following:

$$x_{k+1} = x_k - \frac{2 \sum_{i=0}^n \sum_{j=0}^{n-1} (n-j)c_{n-i}c_{n-j}x^{2n-i-j-1}}{2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n-i)(n-j)c_{n-i}c_{n-j}x^{2(n-1)-i-j} - \sum_{i=0}^n \sum_{j=0}^{n-2} (j-n)(j-n+1)c_{n-i}c_{n-j}x^{2(n-1)-i-j}}$$