

Pure Book 2, Exercise 6C

Question 1 Rewrite the following as powers of $\sec \theta$, $\csc \theta$, or $\cot \theta$.

$$\begin{aligned}\frac{1}{\sin^3 \theta} &= \csc^3 \theta \\ \frac{4}{\tan^6 \theta} &= 4 \cot^6 \theta \\ \frac{1}{2 \cos^2 \theta} &= 2 \sec^2 \theta \\ \frac{1 - \sin^2 \theta}{\sin^2 \theta} &= \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta \\ \frac{\sec \theta}{\cos^4 \theta} &= \sec^5 \theta \\ \sqrt{\csc^3 \theta \cot \theta \sec \theta} &= \sqrt{\frac{\cos \theta}{\sin^4 \theta \cos \theta}} = \frac{1}{\sqrt{\sin^4 \theta}} = \frac{1}{\sin^2 \theta} = \csc^2 \theta \\ \frac{2}{\sqrt{\tan \theta}} &= 2 \cot^{\frac{1}{2}} \theta \\ \frac{\csc^2 \theta \tan^2 \theta}{\cos \theta} &= \frac{\sin^2 \theta}{\sin^2 \theta \cos^3 \theta} = \frac{1}{\cos^3 \theta} = \sec^3 \theta\end{aligned}$$

Question 2 Write down the value(s) of $\cot x$ in each of the following equations.

$$\begin{aligned}5 \sin x &= 4 \cos x \\ \frac{\cos x}{\sin x} &= \frac{5}{4} \\ \therefore \cot x &= \frac{5}{4}\end{aligned}\tag{a}$$

$$\begin{aligned}\tan x &= -2 \\ \therefore \cot x &= -\frac{1}{2}\end{aligned}\tag{b}$$

$$\begin{aligned}3 \frac{\sin x}{\cos x} &= \frac{\cos x}{\sin x} \\ 3 \sin^2 x &= \cos^2 x \\ \frac{\cos^2 x}{\sin^2 x} &= 3 \\ \cot^2 x &= 3 \\ \cot x &= \pm \sqrt{3}\end{aligned}\tag{c}$$

Question 3 Using the definitions of sec, csc, cot, and tan, simplify the following expressions.

$$\sin \theta \cot \theta = \frac{\sin \theta}{\tan \theta} = \frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}} = \frac{\sin \theta \cos \theta}{\sin \theta} = \cos \theta$$

$$\tan \theta \cot \theta = \frac{\tan \theta}{\tan \theta} = 1$$

$$\tan 2\theta \csc 2\theta = \frac{\tan 2\theta}{\sin 2\theta} = \frac{\frac{\sin 2\theta}{\cos 2\theta}}{\sin 2\theta} = \frac{\sin 2\theta}{\sin 2\theta \cos 2\theta} = \frac{1}{\cos 2\theta} = \sec 2\theta$$

$$\begin{aligned} \cos \theta \sin \theta (\cot \theta + \tan \theta) &= \cos \theta \sin \theta \cot \theta + \cos \theta \sin \theta \tan \theta = \frac{\sin \theta \cos \theta}{\frac{\sin \theta}{\cos \theta}} + \frac{\sin^2 \theta \cos \theta}{\cos \theta} \\ &= \frac{\sin \theta \cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta \cos \theta}{\cos \theta} = \sin^2 \theta + \cos^2 \theta = 1 \end{aligned}$$

$$\sin^3 x \csc x + \cos^3 x \sec x = \frac{\sin^3 x}{\sin x} + \frac{\cos^3 x}{\cos x} = \sin^2 x + \cos^2 x = 1$$

$$\sec A - \sec A \sin^2 A = \sec A \cdot (1 - \sin^2 A) = \frac{\cos^2 A}{\cos A} = \cos A$$

$$\begin{aligned} \sec^2 x \cos^5 x + \cot x \csc x \sin^4 x &= \frac{\cos^5 x}{\cos^2 x} + \frac{\sin^4 x}{\frac{\sin^2 x}{\cos x}} = \cos^3 x + \sin^2 x \cos x \\ &= \cos x \cdot (\sin^2 x + \cos^2 x) = \cos x \end{aligned}$$

Question 4 Prove that:

$$\cos \theta + \sin \theta \tan \theta \equiv \sec \theta$$

$$\cos \theta + \frac{\sin^2 \theta}{\cos \theta} \equiv \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} \equiv \frac{1}{\cos \theta} \equiv \sec \theta \quad (a)$$

$$\cot \theta + \tan \theta \equiv \csc \theta \sec \theta$$

$$\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \equiv \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \equiv \frac{1}{\sin \theta \cos \theta} \equiv \frac{1}{\sin \theta} \cdot \frac{1}{\cos \theta} \equiv \csc \theta \sec \theta \quad (b)$$

$$\csc \theta - \sin \theta \equiv \cos \theta \cot \theta$$

$$\frac{1}{\sin \theta} - \sin \theta \equiv \frac{1 - \sin^2 \theta}{\sin \theta} \equiv \frac{\cos^2 \theta}{\sin \theta} \equiv \cos \theta \cdot \frac{\cos \theta}{\sin \theta} \equiv \cos \theta \cot \theta \quad (c)$$

$$(1 - \cos x)(1 + \sec x) \equiv \sin x \tan x$$

$$\begin{aligned} 1 + \sec x - \cos x - \cos x \sec x &\equiv \sec x - \cos x \equiv \frac{1}{\cos x} - \cos x \\ &\equiv \frac{1 - \cos^2 x}{\cos x} \equiv \frac{\sin^2 x}{\cos x} \equiv \sin x \cdot \frac{\sin x}{\cos x} \equiv \sin x \tan x \end{aligned} \quad (\text{d})$$

$$\begin{aligned} \frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} &\equiv 2 \sec x \\ \frac{\cos^2 x + (1 - \sin x)^2}{\cos x \cdot (1 - \sin x)} &\equiv \frac{\cos^2 x + 1 - 2 \sin x + \sin^2 x}{\cos x \cdot (1 - \sin x)} \equiv \frac{2 - 2 \sin x}{\cos x \cdot (1 - \sin x)} \equiv \frac{2(1 - \sin x)}{\cos x \cdot (1 - \sin x)} \\ &\equiv \frac{2}{\cos x} \equiv 2 \sec x \end{aligned} \quad (\text{e})$$

$$\begin{aligned} \frac{\cos \theta}{1 + \cot \theta} &\equiv \frac{\sin \theta}{1 + \tan \theta} \\ \frac{\cos \theta}{1 + \frac{1}{\tan \theta}} &\equiv \frac{\cos \theta}{\frac{1 + \tan \theta}{\tan \theta}} \equiv \frac{\cos \theta \tan \theta}{1 + \tan \theta} \equiv \frac{\sin \theta}{1 + \tan \theta} \end{aligned} \quad (\text{f})$$

Question 5 Solve, for values of θ in the interval $0 \leq \theta \leq 360^\circ$, the following equations. Give your answers to 3 significant figures where necessary.

$$\begin{aligned} \sec \theta &= \sqrt{2} \\ \Rightarrow \frac{1}{\cos \theta} &= \sqrt{2} \\ \therefore \cos \theta &= \frac{1}{\sqrt{2}} \quad \therefore \theta = 45^\circ \end{aligned} \quad (\text{a})$$

Solutions in the interval are 45° and 315°

$$\begin{aligned} \csc \theta &= -3 \\ \Rightarrow \frac{1}{\sin \theta} &= -3 \\ \therefore \sin \theta &= -\frac{1}{3} \quad \therefore \theta = -19.5^\circ \text{ (3 s.f.)} \end{aligned} \quad (\text{b})$$

Solutions in the interval are 199° and 341° (3 s.f.)

$$\begin{aligned} 5 \cot \theta &= -2 \\ \therefore \tan \theta &= -2.5 \quad \theta = -68.2^\circ \text{ (3 s.f.)} \end{aligned} \quad (\text{c})$$

Solutions in the interval are 112° and 292° (3 s.f.)

$$\csc \theta = 2$$

$$\therefore \sin \theta = \frac{1}{2} \quad \theta = 30^\circ \quad (d)$$

Solutions in the interval are 30° and 150°

$$3 \sec^2 \theta - 4 = 0$$

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2} \quad \theta = \pm 30^\circ \quad (e)$$

Solutions in the interval are 30° , 150° , 210° , and 330°

$$5 \cos \theta = 3 \cot \theta$$

$$\Rightarrow 5 \cos \theta \sin \theta = 3 \cos \theta \quad \therefore \sin \theta = \frac{3}{5} \quad \therefore \theta = 36.9^\circ \text{ (3 s.f.)} \quad (f)$$

$$\Rightarrow \cos \theta \cdot (5 \sin \theta - 3 \cos \theta) = 0 \quad \therefore \cos \theta = 0 \quad \therefore \theta = 90^\circ$$

Solutions in the interval are 36.9° (3 s.f.), 90° , 143° (3 s.f.), and 270°

$$\cot^2 \theta - 8 \tan \theta = 0$$

$$\therefore \tan \theta = \frac{1}{2} \quad \therefore \theta = 26.6^\circ \text{ (3 s.f.)} \quad (g)$$

Solutions in the interval are 26.6° and 207° (3 s.f.)

$$2 \sin \theta = \csc \theta$$

$$\therefore \sin \theta = \pm \frac{1}{\sqrt{2}} \quad \therefore \theta = \pm 45^\circ \quad (h)$$

Solutions in the interval are 45° , 135° , 225° , and 315°

Question 6 Solve, for values of θ in the interval $-180^\circ \leq \theta \leq 180^\circ$, the following equations:

$$\csc \theta = 1$$

$$\therefore \sin \theta = 1 \quad \therefore \theta = 90^\circ \quad (a)$$

The only solution in the interval is 90°

$$\sec \theta = -3$$

$$\therefore \cos \theta = -\frac{1}{3} \quad \therefore \theta = 109^\circ \text{ (3 s.f.)} \quad (b)$$

The solutions in the interval are -109° and 109° (3 s.f.)

$$\cot \theta = 3.45$$

$$\therefore \tan \theta = \frac{1}{3.45} \quad \therefore \theta = 16.2^\circ \text{ (3 s.f.)} \quad (c)$$

The solutions in the interval are -164° and 16.2° (3 s.f.)

$$2 \csc^2 \theta - 3 \csc \theta = 0$$

$$\therefore \sin \theta = \frac{2}{3} \quad \therefore \theta = 41.8^\circ \text{ (3 s.f.)} \quad (d)$$

The solutions in the interval are 41.8° , and 138° (3 s.f.)

$$\sec \theta = 2 \cos \theta$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{2}} \quad \therefore \theta = 45^\circ \quad (e)$$

The solutions in the interval are $\pm 45^\circ$ and $\pm 135^\circ$

$$\begin{aligned}
3 \cot \theta &= 2 \sin \theta \\
\Rightarrow 3 \cos \theta &= 2 \sin^2 \theta \\
\Rightarrow 2 \cos^2 \theta + 3 \cos \theta - 2 &= 0 \\
\Rightarrow \cos \theta &= \frac{-3 \pm 5}{4} \\
\therefore \cos \theta = \frac{1}{2} \quad \therefore \theta &= 60^\circ
\end{aligned} \tag{f}$$

The solutions in the interval are $\pm 60^\circ$

$$\begin{aligned}
\csc 2\theta &= 4 \\
\therefore \sin 2\theta &= \frac{1}{4} \quad \therefore 2\theta = 14.48^\circ \text{ (3 s.f.)}
\end{aligned}$$

The solutions in the interval are -173° , -97.2° , 7.24° , and 82.8° (3 s.f.)

(g)

$$\begin{aligned}
2 \cot^2 \theta - \cot \theta - 5 &= 0 \\
\Rightarrow \cot \theta &= \frac{1 \pm \sqrt{41}}{4} \\
\therefore \tan \theta &= \frac{-1 + \sqrt{41}}{10}, -\frac{1 + \sqrt{41}}{10} \quad \therefore \theta = 28.4^\circ, -36.5^\circ \text{ (3 s.f.)}
\end{aligned}$$

The solutions in the interval are -152° , -36.5° , 28.4° , and 143° (3 s.f.)

(h)

Question 7 Solve the following equations for values of θ in the interval $0 \leq \theta \leq 2\pi$. Give your answers in terms of π .

$$\begin{aligned}
\sec \theta &= -1 \\
\therefore \cos \theta &= -1 \quad \therefore \theta = \pi \text{ rad}
\end{aligned} \tag{a}$$

The only solution in the interval is π rad

$$\cot \theta = -\sqrt{3}$$

$$\therefore \tan \theta = -\frac{1}{\sqrt{3}} \quad \therefore \theta = -\frac{1}{6}\pi \text{ rad} \quad (\text{b})$$

The solutions in the interval are $\frac{5}{6}\pi$ rad and $\frac{11}{6}\pi$ rad

$$\csc(\frac{1}{2}\theta) = \frac{2\sqrt{3}}{3}$$

$$\therefore \sin(\frac{1}{2}\theta) = \frac{\sqrt{3}}{2} \quad \therefore \frac{1}{2}\theta = \frac{1}{3}\pi \text{ rad} \quad (\text{c})$$

The solutions in the interval are $\frac{2}{3}\pi$ rad and $\frac{4}{3}\pi$ rad

$$\sec \theta = \sqrt{2} \tan \theta \quad (\theta \neq \frac{1}{2}\pi, \theta \neq \frac{3}{2}\pi)$$

$$\therefore \sin \theta = \frac{1}{\sqrt{2}} \quad \therefore \theta = \frac{1}{4}\pi \text{ rad} \quad (\text{d})$$

The solutions in the interval are $\frac{1}{4}\pi$ rad and $\frac{3}{4}\pi$ rad

Question 8 In the diagram $AB = 6\text{cm}$ is the diameter of the circle and BT is the tangent to the circle at B . The chord AC is extended to meet this tangent at D and $\angle DAB = \theta$.

Show that $CD = 6(\sec \theta - \cos \theta)\text{cm}$

$$\text{In } \triangle ABD, 6 \sec \theta = AD \quad (\text{a})$$

$$\text{In } \triangle ABC, 6 \cos \theta = AC$$

$$CD = AD - AC = 6 \sec \theta - 6 \cos \theta$$

$$\therefore CD = 6(\sec \theta - \cos \theta)$$

Given that $CD = 16\text{cm}$, calculate the length of the chord AC .

$$\begin{aligned}
 6(\sec \theta - \cos \theta) &= 16 \\
 3\left(\frac{1}{\cos \theta} - \cos \theta\right) &= 8 \\
 8 \cos \theta &= 3 - 3 \cos^2 \theta \\
 3 \cos^2 \theta + 8 \cos \theta - 3 &= 0 \\
 \cos \theta &= \frac{-8 \pm 10}{6} = \frac{1}{3}, -3 \quad (-3 \text{ is not possible})
 \end{aligned}
 \tag{b}$$

$$AC = 6 \cos \theta = 6 \cdot \frac{1}{3} = 2\text{cm}$$

Question 9

$$\text{Prove that } \frac{\csc x - \cot x}{1 - \cos x} \equiv \csc x
 \tag{a}$$

$$\frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{1 - \cos x} \equiv \frac{\frac{1 - \cos x}{\sin x}}{1 - \cos x} = \frac{(1 - \cos x)}{\sin x \cdot (1 - \cos x)} \equiv \frac{1}{\sin x} = \csc x$$

Hence solve, in the interval $-\pi \leq x \leq \pi$ the equation $\frac{\csc x - \cot x}{1 - \cos x} = 2$

$$\csc x = 2$$

$$\therefore \sin x = \frac{1}{2} \quad \therefore x = \frac{1}{6}\pi \text{ rad}$$

$$\text{The solutions in the interval are } \frac{1}{6}\pi \text{ rad and } \frac{5}{6}\pi \text{ rad}
 \tag{b}$$

Question 10

Prove that $\frac{\sin x \tan x}{1 - \cos x} - 1 \equiv \sec x$

$$\frac{\frac{\sin^2 x}{\cos x}}{1 - \cos x} - 1 \equiv \frac{\sin^2 x}{\cos x \cdot (1 - \cos x)} - 1 \equiv \frac{\sin^2 + \cos^2 x - \cos x}{\cos x \cdot (1 - \cos x)} \equiv \frac{1 - \cos x}{\cos x \cdot (1 - \cos x)} \equiv \frac{1}{\cos x} \equiv \sec x \quad (\text{a})$$

Hence explain why the equation $\frac{\sin x \tan x}{1 - \cos x} - 1 = -\frac{1}{2}$ has no solutions.

$$\begin{aligned} \sec x &= -\frac{1}{2} \\ \Rightarrow \cos x &= -2 \end{aligned}$$

$\cos x$'s range is between -1 and 1 , therefore $\cos(-2)$ has no solutions. (b)

Question 11 Solve, in the interval $0 \leq x \leq 360^\circ$, the equation $\frac{1 + \cot x}{1 + \tan x} = 5$.

$$\begin{aligned} \frac{1 + \cot x}{1 + \tan x} &= 5 \\ \Rightarrow \frac{\sin x + \cos x}{\sin x} \div \frac{\sin x + \cos x}{\cos x} &= 5 \\ \Rightarrow \frac{\cos x \cdot (\sin x + \cos x)}{\sin x \cdot (\sin x + \cos x)} &= 5 \\ \cot x &= 5 \end{aligned}$$

$$\therefore \tan x = \frac{1}{5} \quad \therefore x = 11.3^\circ \text{ (3 s.f.)}$$

The solutions in the interval are 11.3° and 191.3° (3 s.f.)