A split-step Padé solution for the parabolic equation method

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A split-step Padé solution is derived for the parabolic equation (PE) method. Higher-order Padé approximations are used to reduce both numerical errors and asymptotic errors (e.g., phase errors due to wide-angle propagation). This approach is approximately two orders of magnitude faster than solutions based on Padé approximations that account for asymptotic errors but not numerical errors. In contrast to the split-step Fourier solution, which achieves similar efficiency for some problems, the split-step Padé solution is valid for problems involving very wide propagation angles, large depth variations in the properties of the waveguide, and elastic ocean bottoms. The split-step Padé solution is practical for global-scale problems.

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INTRODUCTION

The parabolic equation (PE) method[1,2] is useful for modeling range-dependent wave propagation, including sound propagation in the ocean.[3] Since the ocean environments that are typically of interest are very large compared to a wavelength, efficient solution techniques are important for this application of the PE method. The split-step Fourier solution,[4,5] which is based on the fast Fourier transform, allows large range and depth grid spacings. The initial success of the PE method in ocean acoustics is largely due to the efficiency of this algorithm. The PE method is based on the asymptotic assumptions that energy propagates nearly horizontally and the speed of sound varies weakly. With the split-step Fourier solution, it is possible to relax one, but not both, of these assumptions.[6,7]

Numerical solutions may also be obtained using finite differences.[8] Although finite-difference solutions tend to require smaller grid spacings than the split-step Fourier solution, they are not as limited asymptotically. This was first demonstrated[9,10] using the wide-angle PE, which is based on a Padé approximation[11] and handles wide propagation angles and sound-speed variations more accurately than the split-step Fourier solution. It was subsequently demonstrated[12] that the higher-order PE, which is based on a higher-order Padé approximation,[13] suppresses asymptotic errors for problems involving very wide propagation angles and large sound-speed variations. Since these finite-difference solutions are relatively inefficient, the split-step Fourier solution has remained in widespread use despite the accuracy improvements. In this paper, we derive the split-step Padé solution, which offers the accuracy of the higher-order PE and the efficiency of the split-step Fourier solution.

The PE method is based on Padé approximations of nonrational functions such as the square-root function, which arises when the wave equation is factored into incoming and outgoing solutions. Other radical functions have been approximated for applications such as the self-starter.[14] The split-step Padé solution is based on higher-order Padé approximations of a composition of the exponential and square-root functions. To achieve a combination of accuracy and efficiency, the Padé approximations are used to reduce both asymptotic and numerical errors. The split-step Padé solution is valid for problems involving very wide propagation angles, large depth variations in the properties of the waveguide, and elastic ocean bottoms.

The split-step Padé solution is approximately two orders of magnitude faster than the Crank–Nicolson solution of the wide-angle PE. This gain in efficiency is achieved by using a Padé approximation that consists of a sum of rational-linear terms. A related Padé approximation involving a single rational-quadratic term achieves a gain in asymptotic accuracy but does not achieve a significant gain in efficiency.[15] Other related Padé approximations have been used to efficiently solve three-dimensional wave propagation problems.[16] With the additional efficiency achieved through algorithm improvements,[17] the present implementation of the split-step Padé solution is more than two orders of magnitude faster than the original implementation of the wide-angle PE.[9] The split-step Padé solution has been applied to model global-scale sound propagation in the ocean.[18] To fully realize the improved efficiency, the solution should be implemented on a computer with modest parallel processing capability. We have implemented the solution on both four-processor and eight-processor SiliconGraphics Iris computers.

I. THE HIGHER-ORDER PARABOLIC EQUATION

In this section, we derive the higher-order acoustic PE and describe its numerical solution and discuss the higher-order elastic PE. We work in cylindrical coordinates, where z is the depth below the ocean surface, r is the range from a point source at $z=z_0$, and θ is the azimuth. The term in the wave equation that accounts for energy coupling between azimuths may be neglected for a large class of problems,[19] and the wave equation may be solved for a particular value of θ. Range dependence is handled by dividing the environment into a sequence of range-independent regions. If a sufficient number of regions is used, an arbitrary level of accuracy may be achieved with this approach, which is commonly used in propagation modeling. In each of the range-independent regions, the wave equation may be solved accurately using the PE method.

The spreading factor $r^{-1/2}$ is removed from the complex acoustic pressure p. In each range-independent region, p is assumed to satisfy the far-field equation,

$$\frac{\partial^2 p}{\partial r^2}+\rho\frac{\partial}{\partial z}\rho^{-1}\frac{\partial p}{\partial z}+k^2p=0, \tag{1}$$

where ρ is the density and k is the complex wave number. Factoring Eq. (1) and keeping the outgoing factor, we obtain

$$\frac{\partial p}{\partial r}=ik_0\sqrt{1+X}p, \tag{2}$$

where k_0 is a representative wave number and

$$k_0^2X=\rho\frac{\partial}{\partial z}\rho^{-1}\frac{\partial}{\partial z}+k^2-k_0^2. \tag{3}$$

Removing the factor $\exp(ik_0r)$ from p, we obtain

$$\frac{\partial p}{\partial r}=ik_0(-1+\sqrt{1+X})p. \tag{4}$$

One approach for solving Eq. (4), which is valid for problems involving very wide propagation angles and large depth variations in the acoustic properties, is to use a Padé approximation of the form,

$$-1+\sqrt{1+X}\cong\sum_{j=1}^{n}\frac{\alpha_{j,n}X}{1+\beta_{j,n}X}. \tag{5}$$

There are various choices for the coefficients. To guarantee a high level of accuracy, the first $2n$ derivatives of both sides of Eq. (5) are required to agree at $X=0$, where X is treated as a complex variable, and the coefficients of Ref. 13 are obtained. It might be possible to achieve accuracy more efficiently, especially if n is relatively large, by using the minimax approach used in Refs. 10 and 20. Stability must also be considered when designing Padé approximations for the elastic PE.[21] This was accomplished in Ref. 22 by modifying one of the accuracy constraints. The Padé approximations of Ref. 22 have an additional advantage: they annihilate the evanescent spectrum.[23] An approach based on Newton's method for computing the coefficients is described in Ref. 24.

Substituting Eq. (5) into Eq. (4), we obtain

$$\frac{\partial p}{\partial r}=ik_0\sum_{j=1}^{n}\alpha_{j,n}(1+\beta_{j,n}X)^{-1}Xp. \tag{6}$$

The splitting solution of Eq. (6), which is described in Ref. 12, requires the numerical solution of the following equation for each term of the Padé approximation:

$$\frac{\partial p}{\partial r}=ik_0\alpha_{j,n}(1+\beta_{j,n}X)^{-1}Xp. \tag{7}$$

The Crank–Nicolson solution for Eq. (7) is given by

$$p(r+\Delta r)-p(r)=\tfrac{1}{2}ik_0\,\Delta r\,\alpha_{j,n}(1+\beta_{j,n}X)^{-1}\times X[p(r+\Delta r)+p(r)], \tag{8}$$

which reduces to

$$p(r+\Delta r)=[1+(\beta_{j,n}-\tfrac{1}{2}ik_0\,\Delta r\,\alpha_{j,n})X]^{-1}\times[1+(\beta_{j,n}+\tfrac{1}{2}ik_0\,\Delta r\,\alpha_{j,n})X]p(r). \tag{9}$$

The depth operator in Eq. (9) is discretized using finite differences as described in Ref. 12. The resulting matrices are tridiagonal.

The derivation of the higher-order elastic PE is analogous to the derivation of the higher-order acoustic PE and involves factoring a vector wave equation.[25] The solution of the higher-order elastic PE involves the splitting method used to solve the higher-order acoustic PE and heptadiagonal matrices.

II. THE SPLIT-STEP PADÉ SOLUTION

To obtain the split-step Padé solution, we solve Eq. (4) analytically before applying a Padé approximation. Given the field over all depth at the arbitrary range r, the solution of Eq. (4) at the range $r+\Delta r$ is

$$p(r+\Delta r)=\exp[i\sigma(-1+\sqrt{1+X})]p(r), \tag{10}$$

where $\sigma=k_0\Delta r$. To implement Eq. (10), we apply the Padé approximation,

$$\exp[i\sigma(-1+\sqrt{1+X})]\cong1+\sum_{j=1}^{n}\frac{a_{j,n}X}{1+b_{j,n}X}=\prod_{j=1}^{n}\frac{1+\lambda_{j,n}X}{1+\mu_{j,n}X}. \tag{11}$$

Coefficients for approximations analogous to the approximations of Refs. 13 and 22 are determined numerically using the approach of Ref. 24. Although this requires a relatively small effort, it is useful to tabulate the coefficients for various values of n and σ. Since each value of σ corresponds to a particular number of grid points per wavelength, a table would be useful for a wide range of frequencies. Substituting Eq. (11) into Eq. (10), we obtain the split-step Padé solution,

$$p(r+\Delta r)=p(r)+\sum_{j=1}^{n}a_{j,n}(1+b_{j,n}X)^{-1}Xp(r). \tag{12}$$

An analogous solution applies for the elastic case.

To solve Eq. (12), the depth operator is discretized using the finite-difference schemes used to solve Eq. (9). For the case $n=1$, Eq. (12) reduces to Eq. (9). For $n>1$, each term is handled with the numerical approach used to solve Eq. (9). The product representation,

$$p(r+\Delta r)=\prod_{j=1}^{n}(1+\mu_{j,n}X)^{-1}(1+\lambda_{j,n}X)p(r), \tag{13}$$

is not as useful for computation as the equivalent sum representation because it does not permit parallel processing. Since $\lambda_{j,n}$ and $\mu_{j,n}$ are complex conjugates (for the approximations analogous to the approximations of Ref.

13), we conclude that the split-step Padé solution is stable from Eq. (13).

The Padé approximation is used to achieve a high level of accuracy for both the asymptotics and the range integration. For other finite-difference PE solutions, Padé approximations are used to minimize the asymptotic error (where X is treated as a small parameter) but not the numerical error. Since the split-step Padé solution is correct to higher order in σ than other finite-difference PE solutions, it allows much larger range steps. The split-step Padé solution is suitable for parallel processing because each of the terms in the sum may be evaluated separately. In contrast, the terms in the splitting solution described in Sec. I may not be solved in parallel. On a computer with n or more processors, the split-step Padé solution is approximately two orders of magnitude faster than other finite-difference PE solutions. The split-step Padé solution becomes more accurate as either X or Δr decreases. For some problems involving strong range dependence, it is not possible to take full advantage of the efficiency of the split-step Padé solution because a relatively small value is required for Δr to adequately sample the environment. This restriction also applies to the split-step Fourier solution.

III. THE DEPTH GRID

The second-order difference schemes that are commonly used to discretize depth operators can lead to large phase errors in PE solutions, especially for long-range and wide-angle propagation. Fourth-order difference schemes have been implemented to achieve improved accuracy.[10,26] In this section, we generalize the Padé approximation to account for the depth grid spacing Δz for the case of a homogeneous medium. It should be possible to generalize this higher-order approach to depth-dependent problems.

In a homogeneous waveguide, the depth operator in the PE reduces to $X=k_0^{-2}\,\partial^2/\partial z^2$. The standard difference operator δ_z^2 that is used to approximate $\partial^2/\partial z^2$ is defined by

$$\delta_z^2 p=\frac{p(z+\Delta z)-2p(z)+p(z-\Delta z)}{(\Delta z)^2}=\frac{\partial^2 p}{\partial z^2}+O(\Delta z)^2. \tag{14}$$

From the Taylor series,

$$p(z\pm\Delta z)=\exp(\pm\tau X^{1/2})p(z), \tag{15}$$

we obtain

$$\tau^2\bar{X}=2\cosh(\tau X^{1/2})-2, \tag{16}$$

where $\bar{X}=k_0^{-2}\delta_z^2$ and $\tau=k_0\Delta z$. Solving Eq. (16) for X, we obtain

$$X=\Gamma(\bar{X})=\tau^{-2}\log^2\left[1+\tfrac{1}{2}\tau^2\bar{X}+\sqrt{(1+\tfrac{1}{2}\tau^2\bar{X})^2-1}\right]. \tag{17}$$

Substituting Eq. (17) into Eq. (10), we obtain

$$p(r+\Delta r)=\exp\{i\sigma[-1+\sqrt{1+\Gamma(\bar{X})}]\}p(r). \tag{18}$$

To implement Eq. (18), we apply the Padé approximation,

TABLE I. The first 16 coefficients of the Taylor approximation for $\Gamma(\bar{X})$.

$A_1=1$	$A_9=1969110$
$A_2=-12$	$A_{10}=-9237800$
$A_3=90$	$A_{11}=42678636$
$A_4=-560$	$A_{12}=-194699232$
$A_5=3150$	$A_{13}=878850700$
$A_6=-16632$	$A_{14}=-3931426800$
$A_7=84084$	$A_{15}=17450721000$
$A_8=-411840$	$A_{16}=-76938289920$

$$\exp\{i\sigma[-1+\sqrt{1+\Gamma(\bar{X})}]\}\cong 1+\sum_{j=1}^{n}\frac{\bar{a}_{j,n}\bar{X}}{1+\bar{b}_{j,n}\bar{X}}. \tag{19}$$

The Padé coefficients may be determined using the approach described in Ref. 24. To evaluate the first $2n$ derivatives of Γ, we use the Taylor series,

$$\Gamma(\bar{X})=\sum_{j=1}^{\infty}A_j^{-1}\tau^{2j-2}\bar{X}^j. \tag{20}$$

The coefficients A_j, which are tedious to evaluate by hand, were obtained using a symbolic calculus software package[27] and are given in Table I for $j\leqslant 16$. Substituting Eq. (19) into Eq. (18), we obtain

$$p(r+\Delta r)=p(r)+\sum_{j=1}^{n}\bar{a}_{j,n}(1+\bar{b}_{j,n}\bar{X})^{-1}\bar{X}p(r). \tag{21}$$

For finite τ, this solution is more accurate than the Crank–Nicolson solution of Sec. I and the split-step Padé solution of Sec. II, which have $O(\tau^2)$ truncation errors.

IV. EXAMPLES

In this section, we illustrate the performance of the split-step Padé solutions with examples. For each of the examples, we place a 25-Hz source at $z=100$ m, take $c=c_0=1500$ m/s in the water column, and generate a reference solution using the approach of Sec. I. The Padé coefficients appearing in Table II were generated using the accuracy and stability constraints used in Ref. 22. The stability constraint, which is important for both the elastic PE and the energy-conserving PE, forces the evanescent modes to decay with range. This is accomplished by requiring that the right side of Eq. (5) map the point X_0 on the negative real axis to the point Y_0 in the upper half of the complex plane (see Ref. 22 for details). For the split-step Padé solution, we require that X_0 is mapped to $\exp(i\sigma Y_0)$.

Example A involves an ocean of depth 200 m overlying a fluid sediment in which $c=1700$ m/s, $\rho=1.5$ g/cm^3, and the attenuation is 0.5 dB/λ. Example B is identical to example A with the exception that the sediment is elastic; the compressional and shear wave speeds are 1700 and 800 m/s, and both of the attenuations are 0.5 dB/λ. Examples A and B consist of case 1 ($n=4$, $\Delta r=200$ m) and case 2 ($n=8$, $\Delta r=800$ m) with the self-starter applied to construct an initial condition at $r=100$ m. Split-step Padé solutions and the Crank–Nicolson solution (i.e., the solution described in Sec. I) appear in Figs. 1 and 2.

TABLE II. The coefficients of the Padé approximations for examples A, B, and D. The coefficients correspond to a 25-Hz source with $c_0=1500$ m/s, $\Delta r=100$ m ($n=3$), $\Delta r=200$ m ($n=4$), and $\Delta r=800$ m ($n=8$).

	$a_{j,n}$	$b_{j,n}$
$n=3$	$(-2.659750,-0.6787369)$	$(-0.7691565,-0.8772290)$
	$(0.2835737,6.602732)$	$(0.2693234,-1.461761)$
	$(2.376176,-0.6880072)$	$(1.309326,-0.8455583)$
$n=4$	$(-21.54856,11.00260)$	$(-0.4017556,-1.993056)$
	$(-1.364474,-4.719127)$	$(1.827321,-1.006914)$
	$(1.373595,-5.162516)$	$(-1.301058,-1.033010)$
	$(21.53944,9.351020)$	$(0.9323825,-1.981306)$
$n=8$	$(-5183.040,826.1732)$	$(-0.3982593,-4.019302)$
	$(-2246.202,-902.1266)$	$(2.112127,-3.480347)$
	$(-393.3105,276.3862)$	$(-2.466623,-2.501479)$
	$(-19.39545,-13.68648)$	$(3.313007,-1.173942)$
	$(20.30401,-14.56783)$	$(-2.804012,-1.187054)$
	$(385.1797,256.4947)$	$(2.978688,-2.489223)$
	$(2265.491,-1002.982)$	$(-1.597574,-3.488805)$
	$(5170.974,616.1964)$	$(0.9142057,-4.016299)$

For example A, both of the split-step Padé solutions are in agreement with the reference solution. For $n=1$, the Crank–Nicolson solutions generated using $\Delta r=200$ m and $\Delta r=800$ m have large errors. To estimate the efficiency gain, we assume the Crank–Nicolson solution requires $\Delta r=10$ m and $n=1$. This value for Δr is twice the value used in Ref. 28 for a problem that is similar to example A but has a sloping bottom. With these reasonable values for n and Δr, the split-step Padé solution is faster than the Crank–Nicolson solution by a factor of 20 for case 1 and by a factor of 80 for case 2. On a computer with a single processor, these factors would be reduced to 5 for case 1 and 10 for case 2. For example B, both of the split-step Padé solutions are in agreement with the reference solution. The estimated efficiency gains are similar to the gains for example A.

Example C involves a lossless homogeneous waveguide of depth 400 m with pressure-release top and bottom boundaries. The initial field consists of the first six modes, which propagate up to 27 deg from the horizontal. The Padé approximations for example C were generated using the accuracy constraints described in Sec. I and the correction for the depth truncation error described in Sec. III. Padé coefficients appear in Table III for case 1 ($n=4$, $\Delta r=100$ m, $\Delta z=10$ m) and case 2 ($n=8$, $\Delta r=400$ m, $\Delta z=10$ m). For $n=4$, the coefficients of Sec. III differ slightly from the coefficients of Sec. II. Split-step Padé solutions appear in Fig. 3. The split-step Padé solutions of Sec. III are in agreement with the reference solution. The split-step Padé solution of Sec. II, which was generated

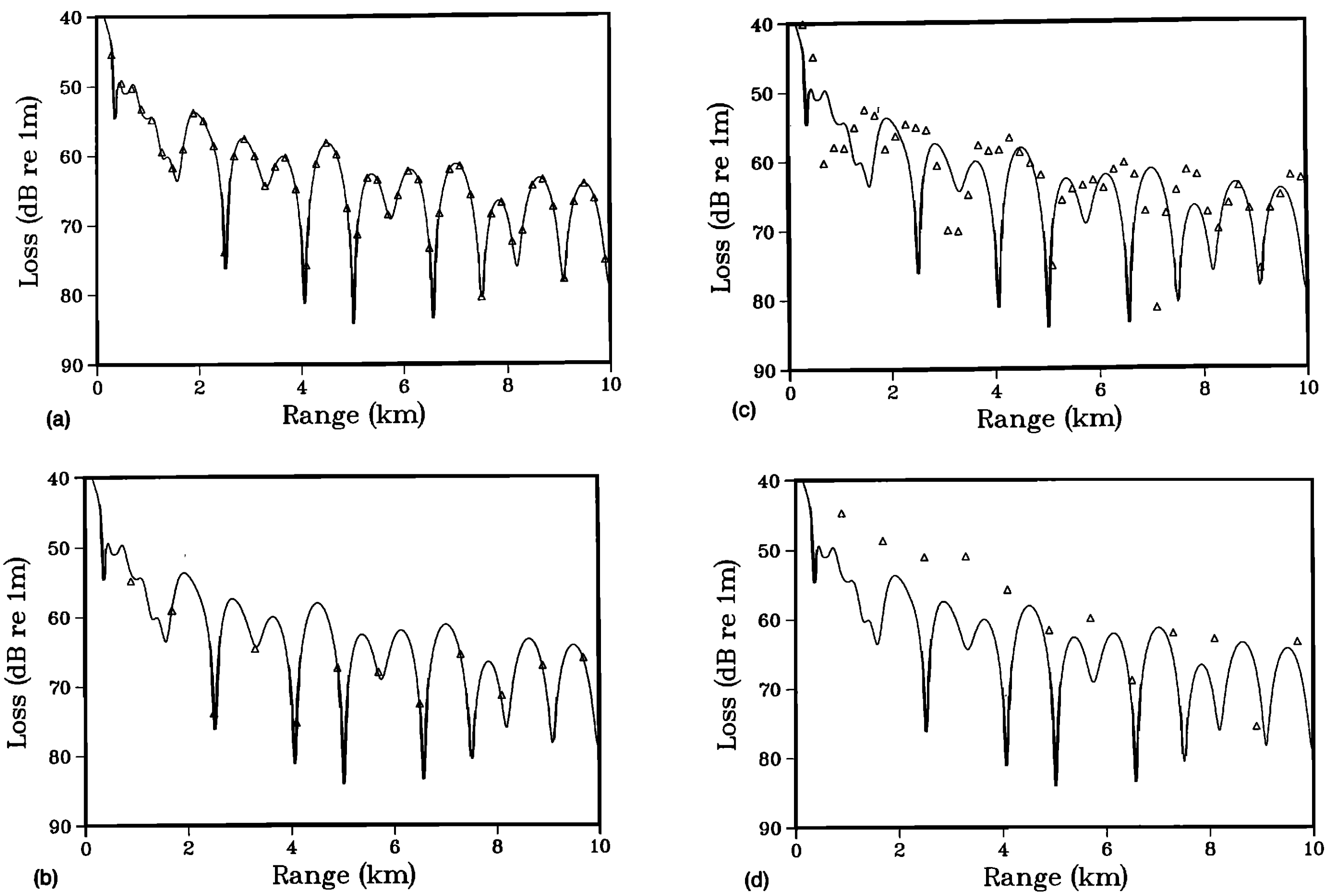

FIG. 1. Results at $z=30$ m for example A, which involves an ocean of depth 200 m overlying a fluid bottom. The densely sampled reference solution appears as a solid curve and is compared with sparsely sampled solutions represented by triangles. The split-step Padé solutions are accurate for (a) the case ($n=4$, $\Delta r=200$ m) and (b) the case ($n=8$, $\Delta r=800$ m). The Crank–Nicolson solution breaks down for (c) the case ($n=1$, $\Delta r=200$ m) and (d) the case ($n=1$, $\Delta r=800$ m).

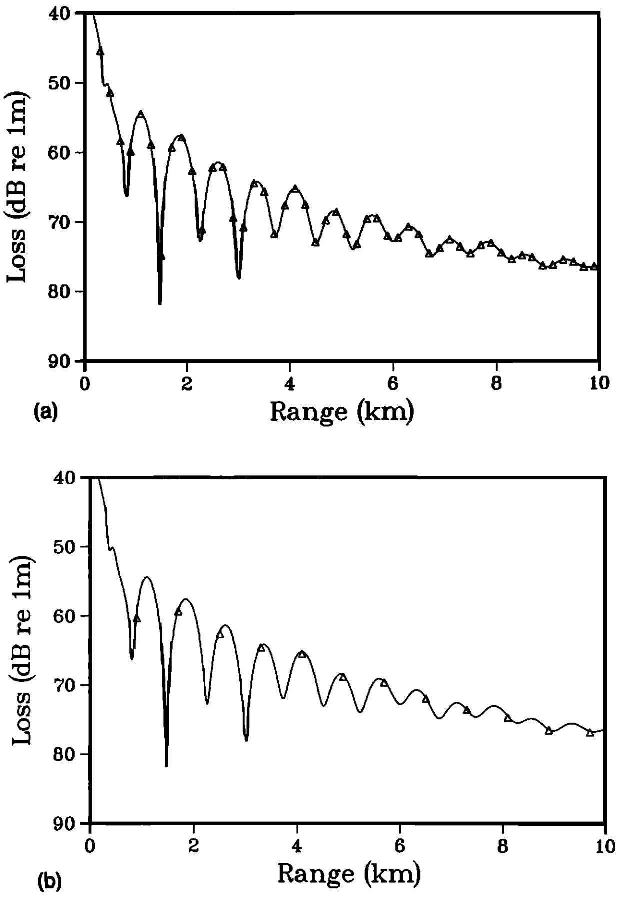

FIG. 2. Results at $z=30$ m for example B, which involves an ocean of depth 200 m overlying an elastic bottom. The densely sampled reference solution appears as a solid curve and is compared with sparsely sampled solutions represented by triangles. The split-step Padé solutions are accurate for (a) the case ($n=4$, $\Delta r=200$ m) and (b) the case ($n=8$, $\Delta r=800$ m).

TABLE III. The coefficients of the Padé approximations for example C for a 25-Hz source with $c_0=1500$ m/s. The coefficients designed for $\Delta z=0$ correspond to $\Delta r=100$ m. The coefficients designed for $\Delta z=10$ m correspond to $\Delta r=100$ m ($n=4$) and $\Delta r=400$ m ($n=8$).

	$a_{j,n}$	$b_{j,n}$
$n=4$	$(-18.48514, 7.638847)$	$(-3.1638578E\text{-}02, -0.8486589)$
	$(-2.285711, -2.865772)$	$(0.9970016, -0.4476350)$
	$(2.834464, -3.768919)$	$(-0.4463122, -0.4906366)$
	$(17.93638, 4.231832)$	$(0.5463279, -0.8310634)$
	$\bar{a}_{j,n}$	$\bar{b}_{j,n}$
$n=4$	$(-18.05664, 7.470541)$	$(5.6471091E\text{-}02, -0.8482449)$
	$(-2.209945, -2.898923)$	$(1.094351, -0.4488921)$
	$(2.685907, -3.728057)$	$(-0.3646472, -0.4887160)$
	$(17.58068, 4.392426)$	$(0.6402838, -0.8321408)$
$n=8$	$(-5332.384, 481.3416)$	$(4.6549164E\text{-}02, -1.836432)$
	$(-2424.226, -408.0659)$	$(1.185077, -1.605197)$
	$(-490.2043, 115.6771)$	$(-0.9069701, -1.188776)$
	$(-25.15266, -2.012668)$	$(1.795754, -0.6035424)$
	$(27.02235, -1.086116)$	$(-1.095259, -0.6130431)$
	$(463.0874, 105.1988)$	$(1.599184, -1.178962)$
	$(2501.379, -508.1337)$	$(-0.4979587, -1.611978)$
	$(5280.478, 238.0249)$	$(0.6381223, -1.834044)$

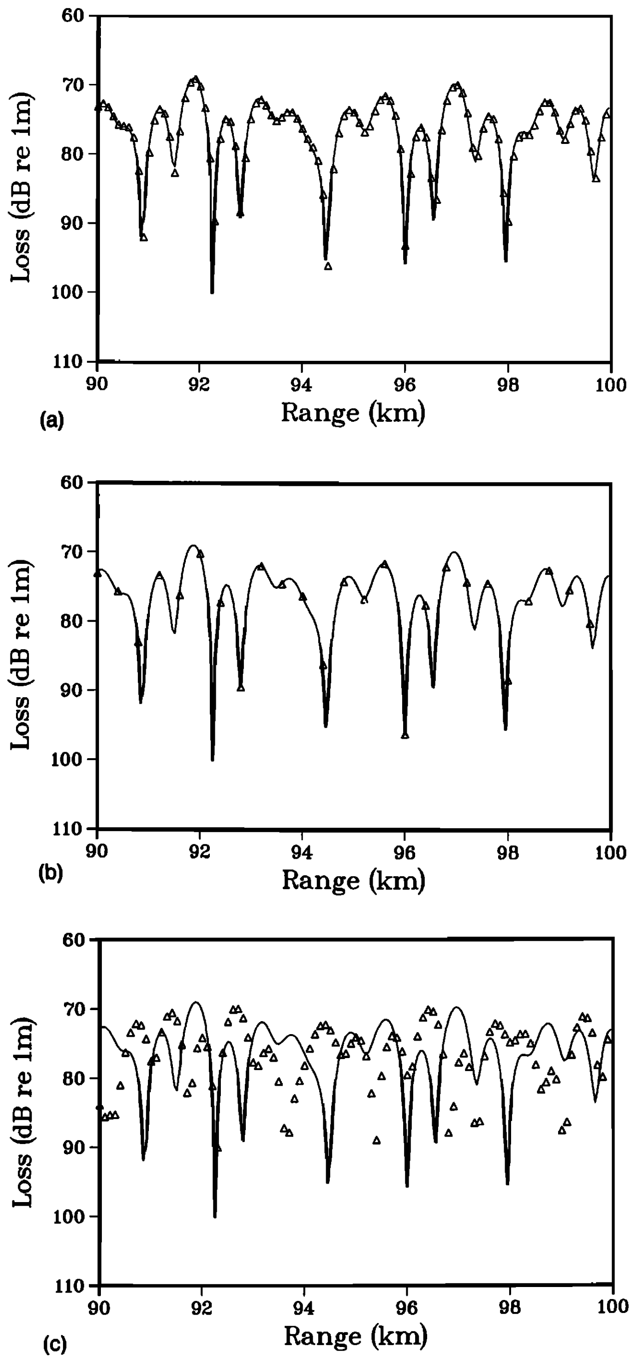

FIG. 3. Results at $z=30$ m for example C, which involves a homogeneous waveguide of thickness 400 m. The densely sampled reference solution appears as a solid curve and is compared with sparsely sampled solutions represented by triangles. The split-step Padé solution of Sec. III is accurate for (a) the case ($n=4$, $\Delta r=100$ m, $\Delta z=10$ m) and (b) the case ($n=8$, $\Delta r=400$ m, $\Delta z=10$ m). (c) The split-step Padé solution of Sec. II is completely out of phase with the reference solution for the case ($n=4$, $\Delta r=100$ m, $\Delta z=5$ m).

using ($n=4$, $\Delta r=100$ m, $\Delta z=5$ m), is completely out of phase with the reference solution. For example C, the parameter values ($n=2$, $\Delta r=5$ m, $\Delta z=1$ m) are reasonable for the Crank–Nicolson solution. For case 2, the split-step Padé solution of Sec. III is 1600 times faster.

To demonstrate that the split-step Padé solution is efficient for range-dependent problems, we consider one of the range-dependent benchmark problems of Refs. 17 and 28. Example D is similar to example A with the exception

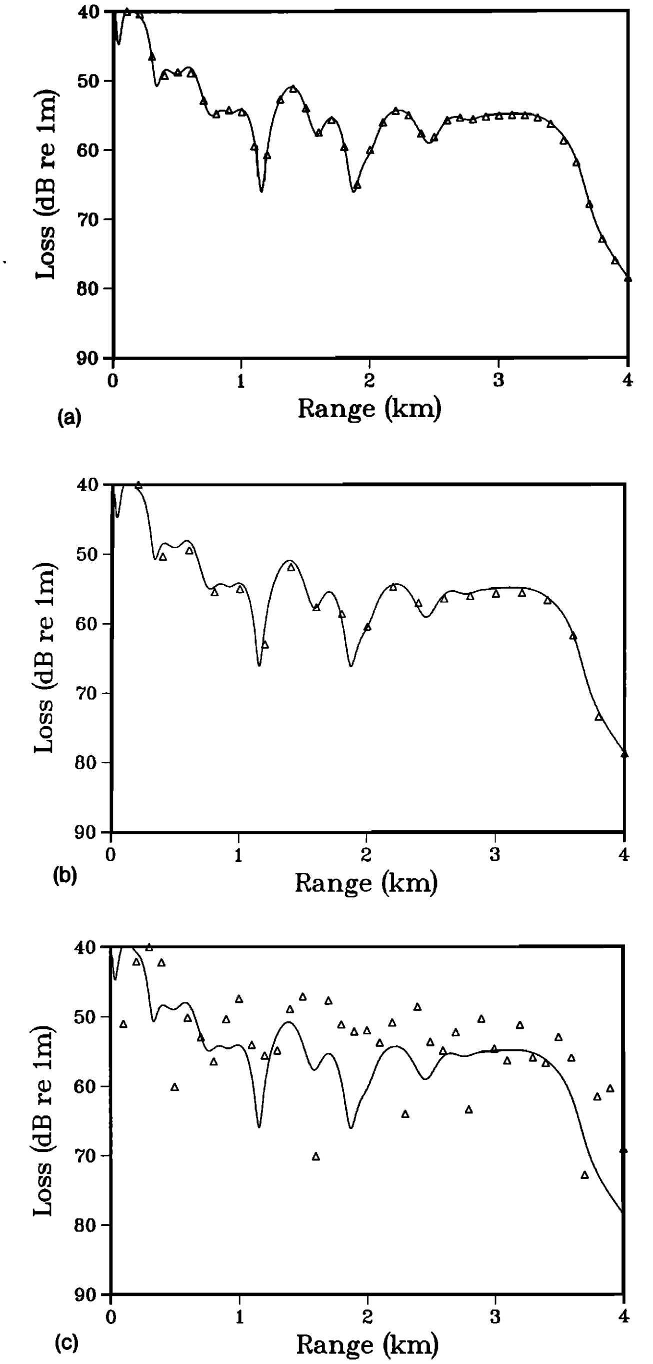

FIG. 4. Results at $z=30$ m for example D, which involves a sloping ocean bottom. The densely sampled reference solution appears as a solid curve and is compared with sparsely sampled solutions represented by triangles. The split-step Padé solution is accurate for (a) the case ($n=3$, $\Delta r=100$ m) and (b) the case ($n=4$, $\Delta r=200$ m). (c) The Crank–Nicolson solution breaks down for $\Delta r=100$ m.

that the ocean depth decreases linearly from 200 m at $r=0$ to zero at $r=4$ km. We use the Padé coefficients appearing in Table II for case 1 ($n=3$, $\Delta r=100$ m) and case 2 ($n=4$, $\Delta r=200$ m). To conserve energy as the environment varies with range, we use the formulation described in Ref. 23. The field was initialized with the starter of Ref. 10. Solutions for example D appear in Fig. 4. The agreement between the split-step Padé solution and the reference solution is excellent for case 1 and good for case 2. The Crank–Nicolson solution, which was generated using $\Delta r=100$ m, has large errors. The split-step Padé solution for case 1 was obtained 20 times faster than the Crank–Nicolson solution generated using $n=1$ and $\Delta r=5$ m (the values used in Ref. 28). With the additional efficiency gain due to the algorithm improvements described in Ref. 17, the overall efficiency gain for example D is more than two orders of magnitude.

V. CONCLUSION

The split-step Padé solution is as accurate as higher-order PE solutions and approximately two orders of magnitude faster than other finite-difference solutions of the PE. The improved efficiency is achieved by using Padé approximations to account for both asymptotic and numerical accuracy. Large range steps are possible for most problems (the exception being problems involving relatively rapid range dependence). A coarse depth grid is also possible for some problems. Although the full realization of the efficiency gain requires the use of a computer with a small number of processors, a significant improvement may be achieved with a single-processor computer. The split-step Padé solution is valid for problems involving very wide propagation angles, large depth variations in the waveguide properties, and elastic layers. In contrast, the split-step Fourier solution, which is also efficient for many problems, is not applicable to these types of problems. It should be possible to achieve efficiency gains well beyond the level suggested by the examples by designing improved Padé approximations using the minimax approach of Refs. 10 and 20.

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