Differentiable Probabilistic Models

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Abstract

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4.1 Transform (Base Class)

4.2 Inverse Transform

4.3 Affine

- **Parameters**
  - Location $\mu \in \mathbb{R}^n$
  - Scale $\sigma > 0$
- **Forward**

$$f(x) = \mu + \sigma \cdot x \tag{1}$$

- **Inverse**

$$f^{-1}(y) = \frac{y - \mu}{\sigma} \tag{2}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = \log |\sigma| \tag{3}$$

4.4 Exp

- **Parameters**
  - None
- **Forward**

$$f(x) = e^x \tag{4}$$

- **Inverse**

$$f^{-1}(y) = \log y \tag{5}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = x \tag{6}$$

4.5 Expm1

- **Parameters**
  - None
- **Forward**

$$f(x) = e^x - 1 \tag{7}$$

- **Inverse**

$$f^{-1}(y) = \log(1 + y) \tag{8}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = x \tag{9}$$

4.6 Gumbel

- **Parameters**
  - Location $\mu \in \mathbb{R}^n$
  - Scale $\sigma > 0$
- **Forward**

$$f(x) = \exp\left(-\exp\left(-\frac{x - \mu}{\sigma}\right)\right) \tag{10}$$

- **Inverse**

$$f^{-1}(y) = \mu - \sigma \cdot \log\left(-\log\left(y\right)\right) \tag{11}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = -\log\left(\frac{\sigma}{-\log(y) \cdot y}\right) \tag{12}$$

4.7 Identity

4.8 Log

- **Parameters**
  - None
- **Forward**

$$f(x) = \log x \tag{13}$$

- **Inverse**

$$f^{-1}(y) = \exp y \tag{14}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = -y \tag{15}$$

4.9 Logit

- **Parameters**
  - None
- **Forward**

$$f(x) = \log\left(\frac{x}{1-x}\right) \tag{16}$$

- **Inverse**

$$f^{-1}(y) = \frac{1}{1+e^{-y}} \tag{17}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = \log\left(1+e^{-y}\right) + \log\left(1+e^{y}\right) \tag{18}$$

4.10 Planar

4.11 Power

- **Parameters**
  - Power p
- **Forward**

$$f(x) = \begin{cases} e^{x} & p = 0 \\ (1 + x \cdot p)^{1/p} & \text{otherwise} \end{cases} \tag{19}$$

- **Inverse**

$$f^{-1}(y) = \begin{cases} \log y & p = 0 \\ y^{p-1}/p & \text{otherwise} \end{cases} \tag{20}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = \begin{cases} x & p = 0 \\ \left(\frac{1}{p} - 1\right) \cdot \log\left(x \cdot p + 1\right) & \text{otherwise} \end{cases} \tag{21}$$

4.12 Radial

4.13 Reciprocal

- **Parameters**
  - None
- **Forward**

$$f(x) = 1/x \tag{22}$$

- **Inverse**

$$f^{-1}(y) = 1/y \tag{23}$$

- **Log Absolute Determinant Jacobian**

$$\log |\det \mathbf{J}|(x, y) = -2 \cdot \log |x| \tag{24}$$

4.14 Sigmoid

- **Parameters**
  - None
- **Forward**

$$f(x) = \frac{1}{1 + e^{-x}} \tag{25}$$

- **Inverse**

$$f^{-1}(y) = \log\left(\frac{y}{1 - y}\right) \tag{26}$$

- **Log Absolute Determinant Jacobian**

$$\log|\det \mathbf{J}|(x, y) = -\log\left(1 + e^{-x}\right) - \log\left(1 + e^{x}\right) \tag{27}$$

4.15 SinhArcsinh

4.16 Softplus

4.17 Softsign

4.18 Square

4.19 Tanh

5 Criterion

The criterion and divergences listed here can be used to quantify the "distance" between two distributions. Hence, in conjunction with torch optimizers, one can minimize said difference to learn the paramters of a distribution. For sake of notation clarity, p is the true distribution and q is the learned distribution. Hence we "fit" q to match p. In addition, we provide the Monte Carlo approximation.

		P		Q	
	Criterion	$\log p(x)$	$x \sim P$	$\log q(x)$	$x \sim Q$
Divergence	Cross-Entropy		✓	✓	
	Perplexity		✓	✓	
	Forward KL	✓	✓	✓	
	Reverse KL	✓		✓	✓
	JS Divergence	✓	✓	✓	✓
Adversarial	GAN		✓		✓
	MMGAN		✓		✓
	WGAN		✓		✓
	LSGAN		✓		✓

5.1 Divergences

5.1.1 Cross-Entropy

$$\begin{aligned} H(p, q) &= -\int p(x) \log q(x) dx \\ &= -\frac{1}{n} \sum_{x \sim p} \log q(x) \end{aligned} \tag{28}$$

5.1.2 Perplexity

$$
\begin{aligned}
H(p,q) &= \exp\left(-\int p(x)\log q(x)dx\right) \\
&= \exp\left(-\frac{1}{n}\sum_{x\sim p}\log q(x)\right)
\end{aligned}
\tag{29}
$$

5.1.3 Forward KL Divergence

$$
\begin{aligned}
H(p,q) &= \int p(x)\log\frac{p(x)}{q(x)}dx \\
&= \frac{1}{n}\sum_{x\sim p}\log\frac{p(x)}{q(x)}
\end{aligned}
\tag{30}
$$

5.1.4 Reverse KL Divergence

$$
\begin{aligned}
H(p,q) &= \int q(x)\log\frac{q(x)}{p(x)}dx \\
&= \frac{1}{n}\sum_{x\sim q}\log\frac{q(x)}{p(x)}
\end{aligned}
\tag{31}
$$

5.1.5 Jensen-Shannon Divergence

5.1.6 Earth Mover's Distance

5.2 ELBO

5.3 Adversarial Loss

Adversarial Losses are criterion functions that allow for sample-sample based training between models p and q. More formally, it hides a Discriminator model that attempts to discriminate between the real data from p and fake data generated from q.

5.3.1 Adversarial Loss

5.3.2 GAN Loss

5.3.3 MMGAN Loss

5.3.4 WGAN Loss

5.3.5 LSGAN Loss

5.3.6 Gradient Penalty

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6 Models

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7 Monte Carlo

7.1 Monte Carlo Integration