0.1 Problem Description

We model a one-vehicle car following system using the kinematic single-track model:

$$\begin{aligned}\dot{v} &= a_{long} \\ \dot{s}_x &= v\end{aligned} \tag{1}$$

And we provide the ego vehicle with a closed-loop controller that keeps the distance from a front vehicle. The controller is expressed in an analytical form w.r.t. the state variables in the system and is differentiable.

The front vehicle is modeled by the same dynamics but with a different acceleration/deceleration capability.

The question we seek to answer is: given a set of initial conditions of the one-vehicle system state $x_0 \in S_0$ (which is part of our ODD definition), for all the provided reference trajectories of the front car, will the following vehicle $\dot{x} = f(x)$ remain as collision-free within a predefined time window $[0, T]$?

0.2 Mathematical Formulation

The full state of the two-vehicle system is $\mathbf{x} = [v_1, s_{x1}, v_2, s_{x2}]^{\mathrm{T}}$, and the state-space equations are:

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{v}_1 \\ \dot{s}_{x1} \\ \dot{v}_2 \\ \dot{s}_{x2}\end{bmatrix} = \begin{bmatrix}a_1 \\ v_1 \\ a_2 \\ v_2\end{bmatrix} \tag{2}$$

This is a first-order nonlinear ODE with 2 control inputs $[a_1, a_2]$. The ego vehicle's control inputs a_1 is based on a closed-loop control (IDM Model):

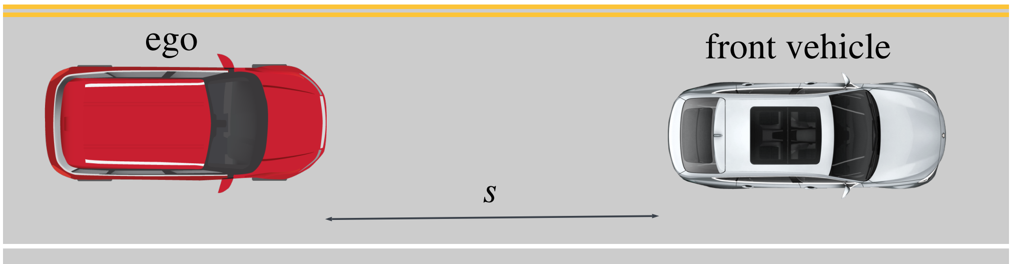
$$a_1 = a_0 \left[1 - \left(\frac{v_1}{v_0}\right)^{\delta} - \left(\frac{s^*}{s}\right)^2\right] \tag{3}$$

where a_0 is a nominal value for acceleration to mimic everyday traffic, v_0 is a desired driving speed on a free road, δ is a pre-defined constant (typically $\delta = 4$), s is the distance from the front car, and s^* stands for the desired dynamical distance to the front car, which is calculated as:

$$s^*(v_1, \Delta v) = s_0 + max\left[0, v_1 T_h + \frac{v_1 \Delta v}{2\sqrt{a_0 b_0}}\right] \tag{4}$$

where s_0 is a minimum bumper-to-bumper distance, T_h is a time headway, $\Delta v = v_1 - v_l$ is the speed difference between the ego car and the lead car, b_0 is a reference deceleration value for stop-and-go motion.

For simplicity, we do not define any steering behavior for the ego vehicle, therefore $\delta_1 = 0$.



Figur 1: The car-following scenario: the ego vehicle is following a front vehicle with a bumper-to-bumper distance of s.

This closed-loop control reduces the two-vehicle system to only one control input a_2. The state-space equation can be re-written as:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{s}_{x1} \\ \dot{v}_2 \\ \dot{s}_{x2} \end{bmatrix} = \begin{bmatrix} a_0 \left[1 - \left(\frac{v_1}{v_0}\right)^{\delta} - \left(\frac{s^*}{s}\right)^2\right] \\ v_1 \\ 0 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \tag{5}$$

Given a set of initial conditions $\mathbf{x}_0 \in S_0$, the reachable states within the time interval $[0,T]$ can be numerically approximated using a selected reachability computation tool:

$$R_{[0,T]}(S_0) = \{\mathbf{x}(t) \in \mathbb{R}^{10} | \mathbf{x} = \int_0^t f(\mathbf{x},\mathbf{u}), \forall t \in [0,T], \forall \mathbf{u}(\cdot) = [a_2(\cdot), v_{\delta 2}(\cdot)]\} \tag{6}$$

We define a simple point-to-point version of collision freedom as:

$$\mathbf{x} \in \mathcal{X}_{cf} := \{\mathbf{x} | \sqrt{(s_{x1} - s_{x2})^2 + (s_{y1} - s_{y2})^2} > d_{min}\} \tag{7}$$

where d_{min} is a pre-defined minimum collision distance.

Initial States

If the state of the system is initially fixed, i.e.,

$$\mathbf{x}(0) = [v_1(0), s_{x1}(0), v_2(0), s_{x2}(0)]^{\mathrm{T}}$$

is assigned a single vector, then the system reachability within $[0, T]$ can only represent a particular traffic scene, and the analysis would have very limited use.

Instead, if the initial state of the system can take a range of values (ideally this range would match the ODD), then the system reachability can provide safety analysis for the ego vehicle controller across a range of scenes. For simplicity, we limit the diversity of the initial state to the range of values for $v_2(0)$, while keeping all other sub-states fixed.

An initial set of values are suggested in Table 1

Parameters

The parameters used in this formulation are listed in Table 2.

Tabel 1: Suggested initial state sets of the car-following scenario

State	I.C	State	I.C
$v_1(0)$	[14,16]	s_{x1}	[0,2]
$v_2(0)$	[10,20]	s_{x2}	[19,21]

Tabel 2: Model Parameters

Parameter	Symbol	Value
Speed-following weight	δ	4
Minimal bumper-to-bumper distance	s_0	1 m
Time headway	Th	0.5 s
acceleration (for nominal traffic)	a_0	2 m/s^2
deceleration (for nominal stop-n-go traffic)	b_0	3 m
Vehicle length (for both vehicles)	L	2 m

0.3 Proposed Workflow

The proposed workflow consists of the following steps:

1. We apply the selected reachability tool to the system in (5), and obtain the over-approximated reachable states $R_{[0,T]}(S_0)$ within time $[0, T]$ and a set of initial states $\mathbf{x} \in S_0$.;

2. We use the collision definition to post-process $R_{[0,T]}(S_0)$, and back-track what values of $v_2(0)$ lead to collisions.

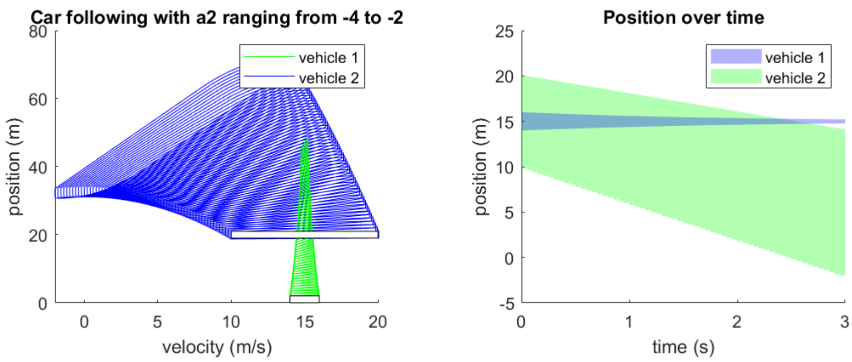
0.4 Implementation in CORA

CORA is a tool for reachability analysis. We use CORA first to build the dynamics model, then calculate the reachability of the system in $[0, T]$ with the initial condition specified in Section 0.2.

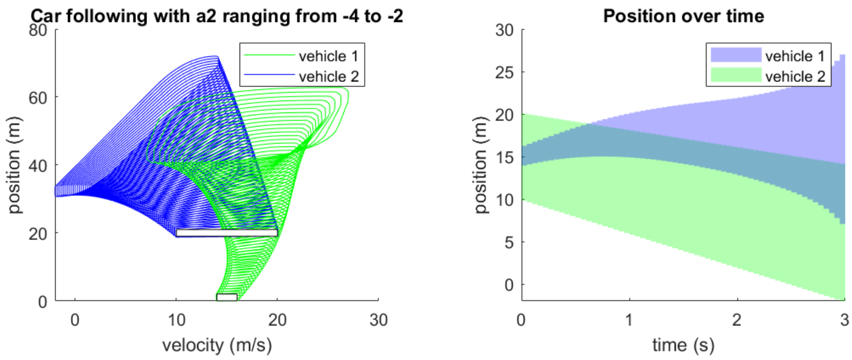
1. We use the template of `example_nonlinear_reach_02_vanDerPol_polyZonotope.m` as a template to build the system above;

2. We provide the initial state sets suggested in Table 1 and run the reachability calculation;

3. We visualize the reachable set results and develop additional visualizations.



Figur 2: constant speed cruising (without distance-keeping)



Figur 3: crusing with a somewhat distance-keeping term

0.5 Results

First, we show results of the ego car only doing constant speed cruising (without distance-keeping to the front vehicle) in Figure 2.

Next, we add a term $+a_0(\frac{s}{10})^2$ that will use the bumper-to-bumper distance s to affect the acceleration a_2 of vehicle 2. This term is not exactly a distance-keeping term, but at least it works with the CORA solver. The result is shown in Figure 3.

We have yet to implement the IDM car-following model. Currently there are errors implementing it.