0.1 Problem Description

We model a two-vehicle (the ego vehicle and the cut-in vehicle) system using the kinematic single-track model:

$$\begin{aligned} \dot{\delta} &= v_\delta \\ \dot{\psi} &= \frac{v}{l_{wb}} \tan(\delta) \\ \dot{v} &= a_{1ong} \\ \dot{s}_x &= v\cos(\psi) \\ \dot{s}_y &= v\sin(\psi) \end{aligned} \tag{1}$$

And we provide the ego vehicle with a closed-loop controller that keeps the distance from a front vehicle or a cut-in vehicle. The controller is expressed in an analytical form w.r.t. the state variables in the system and is differentiable.

The cut-in vehicle is assigned the same kinematic single-track model with full control authority to perform any control sequences.

The question we seek to answer is: given a set of initial conditions of the two-vehicle system state $x_0 \in S_0$ (which is part of our ODD definition), for all the possible controls of the cut-in vehicle, will the two-vehicle system $\dot{x} = f(x)$ remain as collision-free within a predefined time window $[0,T]$?

0.2 Mathematical Formulation

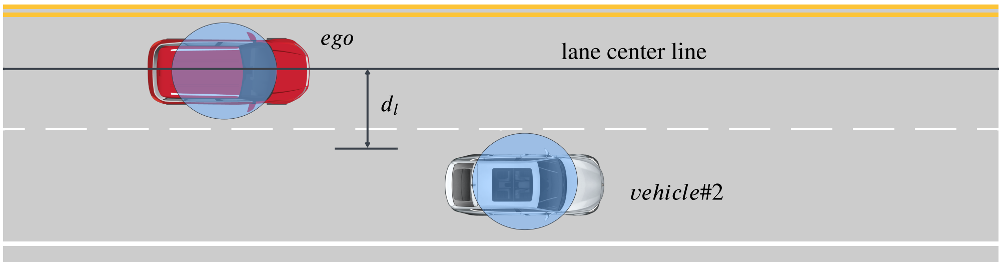
The full state of the two-vehicle system is $\mathbf{x} = [v_1,s_{x1},s_{y1},\psi_1,\delta_1,v_2,s_{x2},s_{y2},\psi_2,\delta_2]^{\mathrm{T}}$, and the state-space equations are:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{s}_{x1} \\ \dot{s}_{y1} \\ \dot{\psi}_1 \\ \dot{v}_1 \\ \dot{\delta}_1 \\ \dot{s}_{x2} \\ \dot{s}_{y2} \\ \dot{\psi}_2 \\ \dot{v}_2 \\ \dot{\delta}_2 \end{bmatrix} = \begin{bmatrix} v_1\cos(\psi_1) \\ v_1\sin(\psi_1) \\ \frac{v_1}{l_{wb1}}\tan(\delta_1) \\ a_1 \\ v_{\delta 1} \\ v_2\cos(\psi_2) \\ v_2\sin(\psi_2) \\ \frac{v_2}{l_{wb2}}\tan(\delta_2) \\ a_2 \\ v_{\delta 2} \end{bmatrix} \tag{2}$$

This is a first-order nonlinear ODE with 4 control inputs $[a_1,v_{\delta 1},a_2,v_{\delta 2}]$. The ego vehicle's control inputs $[a_1,v_{\delta 1}]$ is based on a closed-loop control (IDM Model):

$$a_1 = a_0\left[1 - \left(\frac{v_1}{v_0}\right)^{\delta} - \left(\frac{s^*}{s}\right)^2\right] \tag{3}$$

where a_0 is a nominal value for acceleration to mimic everyday traffic, v_0 is a desired driving speed on a free road, δ is a pre-defined constant (typically $\delta = 4$), s is the distance from the front car, and s^* stands for the desired dynamical distance to the



Figur 1: The cut-in status determination: if vehicle #2's mass center is closer than d_l to the lane center line of the ego vehicle's lane, then the ego vehicle considers vehicle #2 as entering the cut-in status.

front car, which is calculated as:

$$s^*(v_1,\Delta v) = s_0 + max\left[0, v_1 T_h + \frac{v_1 \Delta v}{2\sqrt{a_0 b_0}}\right] \tag{4}$$

where s_0 is a minimum bumper-to-bumper distance, T_h is a time headway, $\Delta v = v_1 - v_l$ is the speed difference between the ego car and the lead car, b_0 is a reference deceleration value for stop-and-go motion.

An important detail is the determination of a cut-in action. A cut-in action is defined as vehicle #2's mass center getting closer than d_l to the ego vehicle's lane center, as depicted in Figure 1. Therefore, the distance s in the IDM model can be expressed as a piecewise function:

$$s = \begin{cases} s_{x2} - s_{x1} & \text{if } dist((s_{x2},s_{y2}),C) < d_l \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

For simplicity, we do not define any steering behavior for the ego vehicle, therefore $\delta_1 = 0$.

With this closed-loop control, the two-vehicle system is reduced to 2 control inputs $[a_2, v_{\delta 2}]$. The state-space equation can be re-written as:

$$\begin{bmatrix} \dot{s}_{x1} \\ \dot{s}_{y1} \\ \dot{\psi}_1 \\ \dot{v}_1 \\ \dot{\delta}_1 \\ \dot{s}_{x2} \\ \dot{s}_{y2} \\ \dot{\psi}_2 \\ \dot{v}_2 \\ \dot{\delta}_2 \end{bmatrix} = \begin{bmatrix} v_1 \cos(\psi_1) \\ v_1 \sin(\psi_1) \\ \frac{v_1}{l_{wb1}} \tan(\delta_1) \\ a_0 \left[1 - \left(\frac{v_1}{v_0}\right)^{\delta} - \left(\frac{s^*}{s}\right)^2\right] \\ 0 \\ v_2 \cos(\psi_2) \\ v_2 \sin(\psi_2) \\ \frac{v_2}{l_{wb2}} \tan(\delta_2) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \\ v_{\delta 2} \end{bmatrix} \tag{6}$$

Given a set of initial conditions $\mathbf{x}_0 \in S_0$, the reachable states within the time interval $[0,T]$ can be numerically approximated using a selected reachability computation tool:

$$R_{[0,T]}(S_0) = \{\mathbf{x}(t) \in \mathbb{R}^{10} | \mathbf{x} = \int_0^t f(\mathbf{x},\mathbf{u}), \forall t \in [0,T], \forall \mathbf{u}(\cdot) = [a_2(\cdot), v_{\delta 2}(\cdot)]\} \tag{7}$$

We define a simple point-to-point version of collision freedom as:

$$\mathbf{x} \in \mathcal{X}_{cf} := \{\mathbf{x} | \sqrt{(s_{x1} - s_{x2})^2 + (s_{y1} - s_{y2})^2} > d_{min}\} \tag{8}$$

where d_{min} is a pre-defined minimum collision distance.

Initial States

If the state of the system is initially fixed, i.e.,

$$\mathbf{x}(0) = [v_1(0), s_{x1}(0), s_{y1}(0), \psi_1(0), \delta_1(0), v_2(0), s_{x2}(0), s_{y2}(0), \psi_2(0), \delta_2(0)]^{\mathrm{T}}$$

is assigned a single vector, then the system reachability within $[0, T]$ can only represent a very specific traffic scene, and the analysis would have very limited use.

Instead, if the initial state of the system can take a range of values (ideally this range would match the ODD), then the system reachability can provide safety analysis for the ego vehicle controller across a range of scenes. For simplicity, we limit the diversity of the initial state to the range of values for $v_2(0)$, while keeping all other sub-states fixed.

0.3 Proposed Workflow

The proposed workflow consists of the following steps:

1. We apply the selected reachability tool to the system in (6), and obtain the over-approximated reachable states $R_{[0,T]}(S_0)$ within time $[0, T]$ and a set of initial states $\mathbf{x} \in S_0$.;

2. We use the collision definition to post-process $R_{[0,T]}(S_0)$, and back-track what values of $v_2(0)$ lead to collisions.