Linear Stability Analysis of the Gray-Scott Model: Finding Turing Pattern Parameter Ranges

The Non-Dimensional Gray-Scott Model

The non-dimensional Gray-Scott equations are:

$$\frac{\partial u}{\partial t} = \nabla^2 u - uv^2 + f(1-u) \tag{1}$$
$$\frac{\partial v}{\partial t} = \delta \nabla^2 v + \lambda uv^2 - (f+\kappa)v \tag{2}$$

where:

- u: dimensionless reactant concentration
- v: dimensionless autocatalyst concentration
- f: dimensionless feed rate
- δ: diffusion ratio (D_v/D_u)
- λ: reaction rate parameter
- κ: dimensionless kill rate

Finding Homogeneous Steady States

Trivial Steady State

Set $\nabla^2 u = \nabla^2 v = 0$ and $\partial u/\partial t = \partial v/\partial t = 0$.

From the equations:

$$0 = -uv^2 + f(1-u)$$
$$0 = \lambda uv^2 - (f+\kappa)v$$

The trivial steady state is:

$$(u_0, v_0) = (1, 0)$$

Non-Trivial Steady States

Assuming $v \neq 0$, solve:

$$\begin{aligned} uv^2 &= f(1-u) \\ \lambda uv^2 &= (f+\kappa)v \end{aligned}$$

From these:

$$\lambda f(1-u) = (f+\kappa)v$$

These steady states exist only for specific parameter ranges.

Linearization Around the Trivial Steady State

Consider small perturbations:

$$\begin{aligned} u(x,y,t) &= 1 + \tilde{u}(x,y,t) \\ v(x,y,t) &= 0 + \tilde{v}(x,y,t) \end{aligned}$$

Define $F(u,v) = -uv^2 + f(1-u)$ and $G(u,v) = \lambda uv^2 - (f+\kappa)v$.
The Jacobian matrix at $(1,0)$:

$$J = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix}_{(1,0)} = \begin{pmatrix} -f & 0 \\ 0 & -(f+\kappa) \end{pmatrix}$$

So without diffusion we can have two conditions by finding the real part of eigen value and that it should be less than 0

$$-(2f+\kappa) < 0 \quad \text{and} \quad f^2 + f\kappa > 0$$

Now for with diffusion part -

Fourier Mode Analysis

Assume perturbations of the form:

$$\begin{aligned} \tilde{u}(x,y,t) &= \hat{u}e^{\lambda t}e^{i(k_x x + k_y y)} \\ \tilde{v}(x,y,t) &= \hat{v}e^{\lambda t}e^{i(k_x x + k_y y)} \end{aligned}$$

where $k^2 = k_x^2 + k_y^2$.

Dispersion Relation

Substituting into the linearized equations:

$$\begin{pmatrix} -f-k^2 & 0 \\ 0 & -(f+\kappa)-\delta k^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \lambda \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$$

Turing Instability Conditions

Stability Without Diffusion

$$-(2f+\kappa) < 0 \quad \text{and} \quad f^2 + f\kappa > 0$$

These are always satisfied for $f, \kappa > 0$.

Instability With Diffusion

Turing instability occurs if $\mathrm{Re}(\lambda_1, \lambda_2) > 0$. i.e. while denying the trace and matrix conditions for stability, the condition becomes:

$$\mathrm{Re}(\lambda) < 0 \quad \text{for stable condition,} \qquad \mathrm{Re}(\lambda) > 0 \quad \text{for instability}$$

$$trace(J) = -f - k^2 - (f+\kappa) - \delta k^2 \quad \Rightarrow \quad \text{only } \det(J) > 0 \text{ for instability (as trace always } < 0)$$

$$\delta k^4 + k^2(f + \kappa + f + \delta) + f^2 + f\kappa < 0$$

This cannot be satisfied since $k^2 \geq 0$ and $f, \kappa, \delta > 0$.

so that's why i rejected the trivial state steady state condition and for specified parameters calculated eigenvalue through the code and plotted it against the values of k but it is inconsistent with our values of parameters where we are getting patterns. It is confusing as for our patterns we took the initial state to be the trivial steady state ?

Linearization around Non-Trivial Steady State

The steady states satisfy

$$\frac{\partial u}{\partial t} = 0, \quad \frac{\partial v}{\partial t} = 0,$$

leading to:

$$\begin{cases} -uv^2 + F(1-u) = 0 \\ uv^2 - (F+\kappa)v = 0 \end{cases}$$

Solving the second equation for u:

$$u^* = \frac{F+\kappa}{\lambda v^*}$$

Substituting into the first equation yields:

$$\left(\frac{F+\kappa}{\lambda v^*}\right) v^{*2} - F\left(1 - \frac{F+\kappa}{\lambda v^*}\right) = 0$$

This nonlinear equation is solved numerically for v^* using `fsolve`, with u^* derived from v^*.

Jacobian Matrix Construction

The Jacobian matrix J at steady state (u^*, v^*) is:

$$J = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} -v^{*2} - F & -2u^* v^* \\ \lambda v^{*2} & 2\lambda u^* v^* - (F+\kappa) \end{pmatrix}$$

where

$$f = -uv^2 + F(1-u), \quad g = \lambda uv^2 - (F+\kappa)v.$$

Dispersion Relation with Diffusion

For spatial perturbations with wavenumber k, the eigenvalues λ of the matrix:

$$J_{\text{full}} = \begin{pmatrix} a_{11} - k^2 & a_{12} \\ a_{21} & a_{22} - \delta k^2 \end{pmatrix}$$

determine stability, where $\delta = \frac{D_v}{D_u}$. The eigenvalue with the largest real part ($\lambda_{\max}$) defines the dispersion relation.

Code Implementation Summary

Steady State Solver:

```
u_star = (f + kappa) / (lam * v_star)  # From steady-state equations
```

Jacobian Elements:

```
a11 = -v_star**2 - f
a12 = -2 * u_star * v_star
a21 = lam * v_star**2
a22 = 2 * lam * u_star * v_star - (f + kappa)
```

Dispersion Relation Calculation:

For each wavenumber k, construct J_{full} and compute its eigenvalues:

```
J = np.array([[a11 - k**2, a12], [a21, a22 - delta * k**2]])
eigs = np.linalg.eigvals(J)
lam_max[i] = eigs.real.max()  # Maximum real part
```

Plotting:

The output visualizes $\mathrm{Re}(\lambda_{\max})$ vs. k, where positive values indicate instability (pattern formation).

The dispersion relation reveals Turing instability when

$$\mathrm{Re}(\lambda_{\max}) > 0 \quad \text{for some } k > 0,$$

indicating spontaneous pattern formation. This occurs when diffusion destabilizes a stable homogeneous steady state.