1 Exercise 1

Write the Euler Lagrange Equation to find the optimizer for the inequality

$$\int_{-\pi}^{\pi} dx\,[f(x)]^4 \leq c \int_{-\pi}^{\pi} dx\,[f'(x)]^4$$

Let use define the set

$$X = \left\{ f \in C^2_{2\pi\text{ periodic}}([-\pi,\pi]) \mid \int_{-\pi}^{\pi} f = 0,\ \int_{-\pi}^{\pi} f^4 = 1 \right\}$$

We also define the function J as

$$J(f) = \int_{-\pi}^{\pi} [f'(x)]^4\,dx$$

X and J are defined such that we want to minimize J on X in order to find the most optimal c value. The right hand side is assumed to take a constant value (in this case 1) and thus we want to minimize the integral on the right hand side (in order to maximize c). Suppose that a minimizer of J exists, and let us call it $u(x)$. Let us assume that $v(x,t)$ is defined such that $u(x) + v(x,t) \in X$ for all t and $v(x,0) = 0$. Then, we have that $J(u + v(t))$ is minimized at $t = 0$ by assumption. It follows that the derivative is zero at that point as well. Thus, we compute

$$\begin{aligned}
\frac{dJ}{dt}(u + v(t)) &= \frac{d}{dt}\int_{-\pi}^{\pi} (u'(x) + \partial_x v(x,t))^4\,dx \\
&= \frac{d}{dt}\int_{-\pi}^{\pi} (u')^4 + 4(u')^3\partial_x v + 6(u')^2(\partial_x v)^2 + 4u'(\partial_x v)^3 + (\partial_x v)^4\,dx \\
&= \int_{-\pi}^{\pi} 4(u')^3\partial_x\partial_t v + 12(u')^2(\partial_x v)\partial_x\partial_t v + 12u'(\partial_x v)^2\partial_x\partial_t v + 4(\partial_x v)^3\partial_x\partial_t v\,dx
\end{aligned}$$

Evaluating at $t = 0$, we get that each of the terms containing $\partial_x v$ go to zero since $v(0) = 0$ for all x. Then, we get that

$$\frac{dJ}{dt}(u(x) + v(x,t))\Big|_{t=0} = \int_{-\pi}^{\pi} 4(u'(x))^3\partial_x\partial_t v(x,t) = 0$$

We wish to calculate $\partial_t(v(x,0))$, and we note that since $u + v(t) \in X$, we have

$$0 = \int_{-\pi}^{\pi} dx\,u(x) + v(x,t) = \int_{-\pi}^{\pi} dx\,v(x,t)$$

and taking derivatives with respect to t and evaluating at 0 gives

$$\int_{-\pi}^{\pi} dx\,\partial_t v(x,0) = 0$$

Similarly, we get that

$$1 = \int_{-\pi}^{\pi} dx\,u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4$$

and thus as the integral of u^4 is assumed to be 1,

$$\int_{-\pi}^{\pi} dx\,4u^3v + 6u^2v^2 + 4uv^3 + v^4 = 0$$

once again we take derivatives with respect to t of both sides and evaluate at $t = 0$. We get that

$$\frac{d}{dt}\int_{-\pi}^{\pi} dx\, 4u^3v + 6u^2v^2 + 4uv^3 + v^4\Big|_{t=0} = \int_{-\pi}^{\pi} dx\, 4u^3\partial_t v + 12u^2v\partial_t v + 12uv^2\partial_t v + 4v^3\partial_t v\Big|_{t=0}$$
$$= \int_{-\pi}^{\pi} dx\, 4[u(x)]^3\partial_t v(x,0) = 0$$

Then, let us set $w = \partial_t v(x,0)$, where w are the *admissible variations*. Then, we get that

$$\int_{-\pi}^{\pi} dx\, 2u'(x)w'(x) = 0$$

for all w satisfying $\int_{-\pi}^{\pi} dx\, w = \int_{-\pi}^{\pi} dx\, u^3 w = 0$. These two conditions come from the above results regarding $\partial_t v(x,0)$. Now, we can remove the constant and then integrate by parts to get that

$$\int_{-\pi}^{\pi} dx\, u''(x)w(x) = 0$$

There are no boundary terms since the functions we are considering are 2π periodic. If w were any function, we could simply conclude that $u''(x) = 0$. However, we still have conditions on w that need to be satisfied. Thus, the next step is to remove these conditions on w. For any f, we can write

$$f = \frac{1}{2\pi}\int_{-\pi}^{\pi} dy\, f(y) + f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} dy\, f(y)$$

Let us call the first term, a constant, c, and the latter term $\tilde{f}$. We see by construction that

$$\int_{-\pi}^{\pi} dx\, \tilde{f} = \int_{-\pi}^{\pi} dx\, f(x) - c = \int_{-\pi}^{\pi} dx\, f(x) - 2\pi c = 0$$

If our condition holds for the functions with mean of 0 (they integrate to 0), then we get that for any arbitrary f, we have

$$\int_{-\pi}^{\pi} dx\, u''(x)f(x) = \int_{-\pi}^{\pi} dx\, u''(x)\tilde{f}(x) + cu''(x) = 0 + c\int_{-\pi}^{\pi} dx\, u''(x) = c[u'(x)]_{-\pi}^{\pi} = 0$$

since u is again 2π periodic and thus its derivatives at the boundaries are the same. We note that the $u''\tilde{f}$ term disappears since $\tilde{f}$ satisfies the first condition and thus when integrated against u'', we get 0. Thus, if the boundary condition holds for functions with mean 0, then any function also satisfies the condition, and we can remove the first condition that $\int_{-\pi}^{\pi} dx\, w = 0$.

For the second condition, we note that for any arbitrary f, we can write $f = ku(x) + f(x) - ku(x)$ for some $k \in \mathbb{R}$ such that $\int_{-\pi}^{\pi} dx\, (f(x) - ku(x))[u(x)]^3 = 0$. Let us denote the latter two terms, $f(x) - ku(x)$, as $f^*(x)$. We can solve for k directly and see that

$$\int_{-\pi}^{\pi} dx\, (f - ku)(u^3) = 0 \implies \int_{-\pi}^{\pi} dx\, fu^3 = k\int_{-\pi}^{\pi} dx\, u^4 = k$$

Then, if the our condition holds for functions that satisfy $\int_{-\pi}^{\pi} dx\, u^3 w = 0$, we see that for any arbitrary f

$$\int_{-\pi}^{\pi} dx\, u''(x)f(x) = \int_{-\pi}^{\pi} dx\, u''(ku + f^*) = \int_{-\pi}^{\pi} dx\, u''ku = k\int_{-\pi}^{\pi} dx\, u''u = \int_{-\pi}^{\pi} dx\, u''u\int_{-\pi}^{\pi} dx\, fu^3$$

Here, just as above, the $u''f^*$ term disappears since f^* satisfies the second condition. We are left with two integrals, one which is always constant (as u is fixed), and one that is varying. We note that we denoted the latter integral as k (a constant) for a fixed f, but f can now be any function (as we have lifted all the constraints). Thus, we denote $\lambda = \int_{-\pi}^{\pi} dx\, u''u$ and get that

$$\int_{-\pi}^{\pi} dx\, u''f = \lambda \int_{-\pi}^{\pi} dx\, u^3 f$$

for any f, which implies that

$$\int_{-\pi}^{\pi} dx\, (u'' - \lambda u^3)f = 0 \qquad \forall\, f \in C^2([-\pi, \pi])$$

implying that $u'' - \lambda u^3 = 0$. This last equation is our Euler Lagrange Analog for this given J.