An Assessment of Maximum Flows

Introduction

For this problem, we have a map of a pipeline network, where each connection between nodes (pipes) have a certain maximum capacity. There are eight nodes and eleven connection between them (Figure 1a). It is important to note the pipes are unidirectional (from V_o to V_t). The Part1 problem asks us to find the maximum flow which starts from V_0 and sink to V_t. When one does (Dawkins, 2018), they are trying to find the solution which outputs the maximum/minimum value of all the feasible solutions. Since in this problem we are asked to find the maximum flow through the pipes, we are faced with an optimization problem.

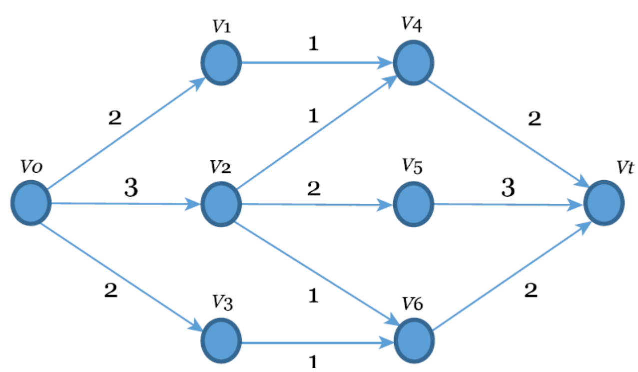

Figure 1a (Taken from assignment 9)

In Part 2 of the problem, a set of new restrictions are added to the initial state. Every day there is a chance that pipes require maintenance, which causes reduction in their maximum capacity of flow. In other words, some of the pipes on this network are not 100% reliable. Maintenances are either major or minor. Major maintenance causes a greater reduction in capacity than minor maintenance. The pipes which might need maintenance are pipes: V_0-V_3, V_0-V_1, and V_5-V_t. As a result, there are 3 out of 11 connections affected. With these new restrictions, the maximum flow differs every time that one of these pipes need maintenance. It is important to note that pipes are independent of each other, and they can be under maintenance at the same time. The problem asks us to find the expected value of maximum flow for any given day. To model this problem correctly, we must find every variation of maintenance, and calculate the maximum flow and chance of occurrence for each. In summary, Part 1 has a constant maximum flow on any given day, while in Part 2, we can only predict the probability of different maximum flows for a given day.

Methods

The best strategy to solve a problem is to figure out which fields of mathematics it falls in to. Part 1 is a weighted graph problem (Weisstein, 2000), where edges are the "pipes". Part 2 adds the probability for different weights; this can be considered as a network reliability (Harris, 2010) problem. For both parts, we are trying to find the maximum flow (weight) of the pipes' connection from V_0 to V_t.

Part 1

The methodology for this part is rather simple. It can be either done by hand, or Maple. The pen and paper method are shown in Figure 1. This method is Minimum Cut/Maximum Flow:

a- Draw straight lines from the direction of flow (top to bottom) and cut as much as edges as possible.

b- Now for every straight line, add the weights of pipes that it cuts. List the values and the smallest number would be the maximum flow.

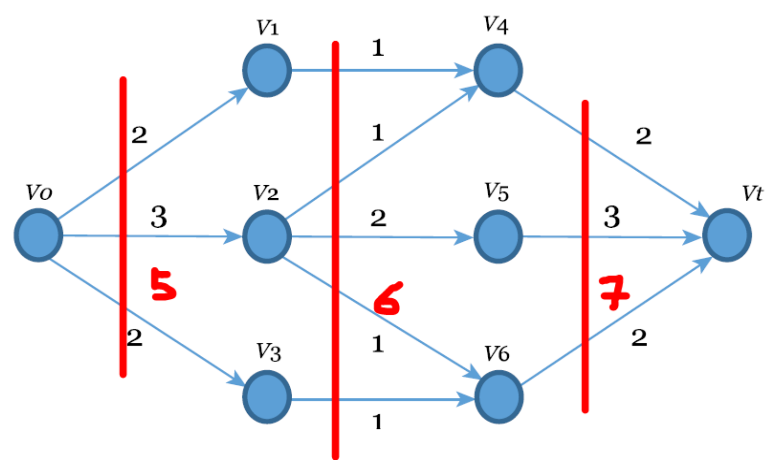

Figure 1(Taken from assignment 9)

Since we need to find several cases of max flow for Part 2, to minimize errors, we construct a Maple (MapleSoft, 2015)code which prints the maximum flow (Figure2):

> *with*(*GraphTheory*) :

$A := Matrix([[0, 2, 3, 2, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 2, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 2], [0, 0, 0, 0, 0, 0, 0, 3], [0, 0, 0, 0, 0, 0, 0, 2], [0, 0, 0, 0, 0, 0, 0, 0]]) :$

- The code above asks Maple to use graph theory commands. Then we set an 8x8 matrix denotes as A. Every row represents the weight of connection between a certain node (station) and other nodes. The matrix above represents the pipe system given in Part1. For the sake of simplicity, let V_0=1, V_1=2, …. V_t=8

$N := Digraph(A, weighted) :$

Graph 3: a directed weighted graph with 8 vertices and 11 arc(s)

- Above code introduces N which is a directed weighted graph with relations defined on matrix A

$IsNetwork(N) : MaxFlow(N, 1, 8); DrawNetwork(N);$

- This code asks Maple to output maximum flow network N where it starts from V0=1 and sinks at Vt=8. Then it asks Maple to print the network.

$$5, \begin{bmatrix} 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

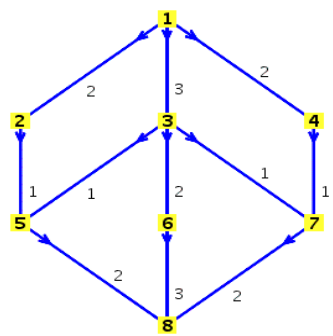

Figure 2

Part 2

Preceding to part 2, we should start off to find all the possible cases of maintenances. First, we know that the best-case scenario is when all the pipes are fully functional, and worst-case scenario is when all the three pipes are affected by major maintenance. For each pipe, there are three cases: no maintenance, minor maintenance, and major maintenance. So to count all the possible cases, we have to choose one case for each pipe: ${}^{1}_{3}C \text{ and } {}^{1}_{3}C \text{ and } {}^{1}_{3}C = 3 \times 3 \times 3 = 27$ cases. Initially, to get a rough idea, a sketch of the cases was done by pen and paper (Figure 3). This is a simple tree diagram, which pipe V_0-V_1 is denoted as "a", V_0-V_3 denoted as 'b', and V_5-V_t is denoted as "c".

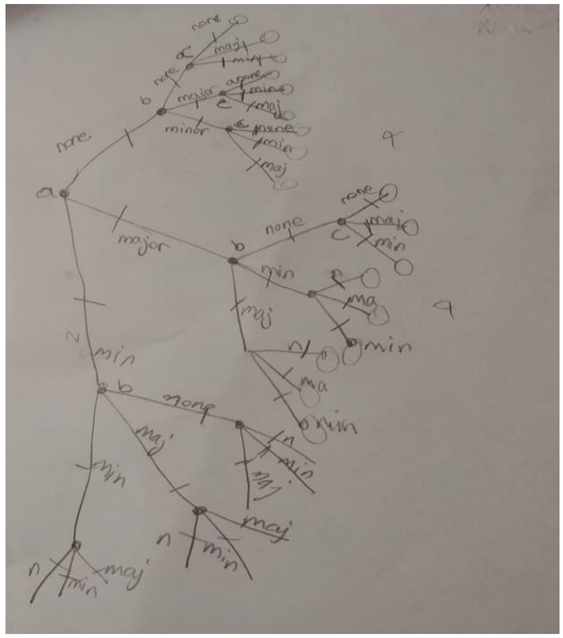

Figure 2

Now, all the possible outcomes are written for each step. After producing a draft of the cases, they were written on an Excel sheet as follows (Table 1): Let us assume that no maintenance=1, minor maintenance=2, and major maintenance=3. There are 27 possible cases of these listed on column A. For every row, columns B-D are states of pipes V_0-V_1, V_0-V_3, and V_5-V_t respectively. We know every pipe has a probability for each maintenance situation. Since there are 81 probabilities, we ask Excel to do that for us. Table 2 is a snapshot of the simulation process for Part2. First, we construct a table to include all the probabilities for the states. On columns E-G, we introduce a double if loop for every single pipe state. The loop asks if the cell has value 1, print the corresponding probability, if not, go to the inner loop. Now, if the value is 2 print the probability, if not (it must be 3), print the probability for state 3. Here is an example for pipe V_0-V_1, using the probabilities on Table 2: IF (B2=1,0.95,IF(B2=2,0.8*0.05,0.2*0.05)). Now, we have all the possible combinations of probabilities. On column H, we order Excel to find the product of probabilities for every case. To make sure we are right, we add all the probabilities on column H and get 1 on row 29, which confirms our probabilities for each case is right.

Now, we must find the maximum flow for each case. A combination of sieving/elimination techniques can be used to find the maximum flow for each case. From inspection based on Figure 1, we can see that the maintenance state of pipe V_5-V_t does not make any change on maximum flow. As a result, max flows should be filtered based on the two remaining pipes. Starting from the worst-case scenario , where both V_0-V_1 and V_0-V_3 need major maintenance: we create an IF loop telling Excel that if both columns B and C have value of 3 on any row, then print the appropriate max flow of 3, otherwise print "Next" on column I. Here is the example Excel code: =IF(AND(B2=3,C2=3),"maxflow=3","Next"). It is important to eliminate any scenario which has a maxflow=3 for the next step. For the next step, consider all the scenarios where both V_0-V_1 and V_0-V_3 need no maintenance. As it was previously mentioned, V_5-V_t has no effect on maxflow whether it is in state 1, 2 or 3. Thus, max flow for this scenario is 5. Using the same IF loop, we ask Excel to print the outcomes on column J. Again, we eliminate any row which has a result other than "Next". For columns K-L, we follow the same scheme for the scenarios where one of V_0-V_1 or V_0-V_3 have no maintenance and the other has minor maintenance. Using the Maple code introduced, we can check that maxflow=5. Note that we only need to change the first row of matrix A, where the two values are bolded. The valid entries are 2 (no maintenance capacity), 1 (minor maintenance capacity) and 2(major maintenance capacity): $A := Matrix([[0, \underline{\mathbf{2}}, 3, \underline{\mathbf{2}}, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 2, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 2], [0, 0, 0, 0, 0, 0, 0, 3], [0, 0, 0, 0, 0, 0, 0, 2], [0, 0, 0, 0, 0, 0, 0, 0]]);$

For columns M-O, we use the same loop. By now, we can see that every step eliminates 3 cases. Columns M and O are cases where one of V_0-V_1 or V_0-V_3 requires major maintenance and the other pipe requires no maintenance. By the Maple code, we find the maxflow=4 for all such cases. Column N is all the scenarios where both pipes have minor maintenance. The maxflow=5 for all those scenarios. Proceeding to column P, we can see all the remaining rows have either of V_0-V_1 or V_0-V_3 has major maintenance and the other has a minor one. Without using the IF loop, we can confirm with the Maple code that all such scenarios have max flow of 4. Now, we have identified the maximum flow and probability of each case. The cases are color-coded by maximum flow on Table 1: red is maxflow of 3, yellow is maxflow=4 and green is max flow=5.

The next step is to find xp(x) value of each case and add all of them up to find the expected maximum flow for any given day. This can be done using the formula $\sum xp(x)$where x is the maximum flow of each case and p(x) is its probability. This can be easily done using the Sum function in Excel. The results are presented in the next section.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Case	v0-v1	v0-v3	v5-vt	P(v0-v1)	P(v0-v3)	P(v5-vt)	P(v0-v1)andP(v0-v3)andP(v5-vt)								
2	1	1	1	1	0.95	0.95	0.92	0.8303	Next	maxflow=5						
3	2	1	1	3	0.95	0.95	0.04	0.0361	Next	maxflow=5						
4	3	1	1	2	0.95	0.95	0.04	0.0361	Next	maxflow=5						
5	4	1	3	1	0.95	0.01	0.92	0.00874	Next	Next	Next	Next	Next	Next	maxflow=4	
6	5	1	3	2	0.95	0.01	0.04	0.00038	Next	Next	Next	Next	Next	Next	maxflow=4	
7	6	1	3	3	0.95	0.01	0.04	0.00038	Next	Next	Next	Next	Next	Next	maxflow=4	
8	7	1	2	1	0.95	0.04	0.92	0.03496	Next	Next	Next	maxflow=5				
9	8	1	2	2	0.95	0.04	0.04	0.00152	Next	Next	Next	maxflow=5				
10	9	1	2	3	0.95	0.04	0.04	0.00152	Next	Next	Next	maxflow=5				
11	10	3	1	1	0.01	0.95	0.92	0.00874	Next	Next	Next	Next	maxflow=4			
12	11	3	1	3	0.01	0.95	0.04	0.00038	Next	Next	Next	Next	maxflow=4			
13	12	3	1	2	0.01	0.95	0.04	0.00038	Next	Next	Next	Next	maxflow=4			
14	13	3	2	1	0.01	0.04	0.92	0.000368	Next	Next	Next	Next	Next	Next	Next	maxflow=4
15	14	3	2	3	0.01	0.04	0.04	0.000016	Next	Next	Next	Next	Next	Next	Next	maxflow=4
16	15	3	2	2	0.01	0.04	0.04	0.000016	Next	Next	Next	Next	Next	Next	Next	maxflow=4
17	16	3	3	1	0.01	0.01	0.92	0.000092	maxflow=3							
18	17	3	3	2	0.01	0.01	0.04	0.000004	maxflow=3							
19	18	3	3	3	0.01	0.01	0.04	0.000004	maxflow=3							
20	19	2	1	1	0.04	0.95	0.92	0.03496	Next	Next	maxflow=5					
21	20	2	1	2	0.04	0.95	0.04	0.00152	Next	Next	maxflow=5					
22	21	2	1	3	0.04	0.95	0.04	0.00152	Next	Next	maxflow=5					
23	22	2	3	1	0.04	0.01	0.92	0.000368	Next	Next	Next	Next	Next	Next	Next	maxflow=4
24	23	2	3	2	0.04	0.01	0.04	0.000016	Next	Next	Next	Next	Next	Next	Next	maxflow=4
25	24	2	3	3	0.04	0.01	0.04	0.000016	Next	Next	Next	Next	Next	Next	Next	maxflow=4
26	25	2	2	1	0.04	0.04	0.92	0.001472	Next	Next	Next	Next	Next	maxflow=5		
27	26	2	2	2	0.04	0.04	0.04	0.000064	Next	Next	Next	Next	Next	maxflow=5		
28	27	2	2	3	0.04	0.04	0.04	0.000064	Next	Next	Next	Next	Next	maxflow=5		
29								1								
30																

Table 1

Pipe	State Probability 1	2	3
v0-v1	P(not broken)=0.95	p(broken and minor)=0.05x0.8=0.04	p(broken and major)=0.05x0.8=0.05
v0-v3	0.92	0.04	0.04
v5-vt	0.95	0.01	0.04

Table 2

Results

Part 1

The result for Part 1 is the maximum flow of 5. This means pipes network have the maximum capacity if it starts from V_0 and sinks at V_t. Since the reliability of every edge is 100%, we will always have maximum flow of 5. The results were obtained from minimum cut/max flow technique (Figure 1) and, was confirmed through the Maple code introduced in the Methods section.

Part 2

On Table 1, different cases of maintenance scenarios are listed. By a quick inspection, we can see that only 3 pipes are affected: V_0-V_1 , V0-V3, and V_5-V_t.There are 27 combinations of maintenance states, and pipes' maintenance state are not dependent on each other. Following values were assigned to each maintenance state: no maintenance=1, minor maintenance=2, and major maintenance=3. On column H of Table 1, we have the probability of each case, where individual probability of each pipe's maintenance state is taken from Table 2. To confirm that all probabilities are correct, we checked to see if sum of all probabilities equals 1, which indeed was (cell H29). Now, we must find maximum flow for each case. Using a quick inspection on Figure 1, we can see that if all the three pipes are under major maintenance, our maximum flow is 3. This can be confirmed by the Maple code. Also, we already know that maximum possible flow for this network is 5. Thus, we have a maximum flow range of 3-5 for Part 2. To find the maximum flow for each case, we have used the sieving/elimination scheme described in the Methods section (columns I-P). The calculated xp(x) for each case is listed in Table 3. The probabilities are taken from Table 2 and, multiplied to the corresponding maximum flow. The expected value of maximum flow for any random day is 4.98.

Case	Probability of the cases	Maxflow	expected Value xp(x)
1	0.8303	5	4.1515
2	0.0361	5	0.1805
3	0.0361	5	0.1805
4	0.00874	4	0.03496
5	0.00038	4	0.00152
6	0.00038	4	0.00152
7	0.03496	5	0.1748
8	0.00152	5	0.0076
9	0.00152	5	0.0076
10	0.00874	4	0.03496
11	0.00038	4	0.00152
12	0.00038	4	0.00152
13	0.000368	4	0.001472
14	0.000016	4	0.000064
15	0.000016	4	0.000064
16	0.000092	3	0.000276
17	0.000004	3	0.000012
18	0.000004	3	0.000012
19	0.03496	5	0.1748
20	0.00152	5	0.0076
21	0.00152	5	0.0076
22	0.000368	4	0.001472
23	0.000016	4	0.000064
24	0.000016	4	0.000064
25	0.001472	5	0.00736
26	0.000064	5	0.00032
27	0.000064	5	0.00032
Expected value			4.98

Table 3

The expected maximum flow is very intuitive. Table 4 is a summation of probabilities of different max flows taken from Table3. we can see the probability of max flow = 5 is 0.98, which means almost every time we get a max flow of 5. Since maximum flows are whole numbers, we can say that expected value of the maxflow for any given day is 5, but this can be considered careless assumption because the expected value is 0.02 less than 5. Therefore, we must round down and state that expected value of maximum flow for any given day is 4.

Flows	Probabilities
max flow 4	0.0198
max flow 3	0.0001
max flow 5	0.9801

Table 4

Conclusions

Based on the Maple calculation and min-cut/max flow techniques on the Part1, the pipe network always has a maximum flow of 5. The pipes (edges) are 100% reliable on Part1. Thus, the expected value of flow in such a pipe system is 5. A full analysis of this part is explained in the Methods section.

For Part2 of the problem, we are faced with 3 pipes which are not always 100% reliable. Based on the Maple codes explained in the Methods section, we have a range of maximum flows where they range from 3 to 5. Maximum flow of 3 happens when V_0-V_1 and V_0-V_3 are under major maintenance. Since this only happens 0.01% of the time, it is negligible. Max flow of 4 happens ≈ 2% of the time, which means 2 days out of 100 days. The maximum flow of 4 should not be ignored, because it can happen around 6-7 times per year. Maxflow of 5 happens 98% of the time, which is indeed closest to the expected value of 4.98. However, to be cautious one would round down the results to expected maximum flow of 4, which happens 99.99% of the time. Table 4 provides an exact chance of every maximum flow, on a random day of the year.

References

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