Project Thesis/Report

Contents

Chapter 1

Introduction

Experimenting on biological, physical, and artificial systems in order to generate a more informative dynamical model is a well-established practice in modern science. Traditional methods for modelling physical systems are based on laws of physics that are based either on empirical relationships or on intuition. For systems that evolve with time, physical laws yield mathematical equations that govern how the quantities evolve with time. However our world around is, for the most part, much more complex than that which can be distilled into elegant equations. We do not fully understand many of the more complex systems, nor do they even provide us with a good physical intuition of the underlying principles governing the dynamics. To complicate this - the underlying systems we observe often display sensitive dependence on initial conditions; despite having highly similar initial conditions, differing orbits diverge quickly and to such an extent that it becomes seemingly impossible to retrace their steps back to the original conditions. Even the presence of usually 'negligible' computational noise or measurement error renders long-term point-wise prediction infeasible. There are many difficulties that one encounters while modelling such systems:

i. One may not have access to the complete states of the systems
ii. The actual system may be described by functions that behave wildly in the sense their graphs have a wild oscillatory behaviour, i.e., they have a large functional complexity.

Models derived primarily from data can be classified into three categories:

i. Interpretable models (i.e., they establish relationships between internal physical mechanisms),
ii. Partially interpretable models capturing some modes of the dynamics,
iii. Non-Interpretable models, defined as such mainly due to the fact that they are defined on a different phase space that is usually high-dimensional.

Examples in the literature attempting to forecast data from such systems have tried a number of different approaches with differing degrees of success. When we have complete access to states of the system, an ordinary differential equation can be obtained from data (Citations to Sindy and [**?**, **?**]) wherein one could approximate the vector field by a library of functions to obtain interpretable models.

Recent partially interpretable models available in the literature have been based on the Koopman operator (e.g.,[**?**, **?**]) to employ observables mapping the data onto a higher-dimensional space. This then makes the dynamics in the higher-dimensional space more amenable for approximation by a linear transformation. Such methods, obviously, do not guarantee an exact reconstructions for nonlinear models, and in practice provide poor long-term consistency for a large class of systems. (Add citation)

The non-interpretable models include the delay embedding and the machine learning algorithms. One could learn a system conjugate to the underlying system by applying the Takens delay embedding [**?**] when one has "good" observations from the system; this learnt system could then be used for forecast the observed data. Takens delay embedding theorem [**?**] and its various generalisations (e.g., [**?**, **?**, **?**, **?**]) establishes the learnability of a system constructed by concatenating sufficiently large previous time-series observations of a dynamical system into a vector (called delay coordinates). This then establishes the existence of a map on the space of delay coordinates equivalent (or topologically conjugate) to the underlying map from which the observed time-series was first obtained. Although topological conjugacy guarantees an alternate representation of the underlying system, the quality of this representation still depends on numerous parameters, making the comprehension of the dynamics rather unreliable (e.g., [**?**]). One reason for this fragility is that the embedded attractor in the reconstruction space not always an attractor of the map that is learnt in the reconstruction space, despite unquestionably being an invariant set. When the embedded object is not **(definitely/always?)** an attractor (as explained in Chapter 3) it can cause predictions to fail.

Practically, the application of Takens embedding involves learning a map through some technique and consequently one wishes that these would have low functional complexity[**?**], i.e., functions with fewer oscillatory graphs. Pure machine learning methodology that processes temporal information (like the echo state networks (Citations to echo state networks)) by mapping data onto a higher dimensional space for further processing. Although they perform well on forecasting some dynamical systems, they fail completely on others (Citations here) as there is often no guarantee that the right function was learnt during training.

This thesis deals with the implementation and analysis of non-interpretable models that can guarantee **exact/accurate** reconstruction. The project work concerns the study and implementation of a method (Citation to paper with Adriaan) that incorporates learning a function by mapping the data on to a higher dimensional space using what is called a driven dynamical system (See Chapter 4). With a clear understanding of how the data is mapped onto the higher dimensional space, the method then permits the learning of a dynamical system topologically conjugate to that of the underlying system. Instead of linear regression as in the training of echo state networks, deep learning methods are employed to learn the correct function. With slight modifications to the implementation in the paper (with Adriaan), we show that one can construct accurate non-interpretable models with the ability to reconstruct attractors from more hard-to-forecast systems like the double pendulum. (The forecasting of the time-series from a double pendulum has not been reported before.) Moreover, we also demonstrate that long-term statistical consistency is preserved.

By solidifying the mathematical underpinnings of our theory, we hope to guarantee the ability to construct models with predictive power ranging from molecular biology to neuroscience. in the near future. (We can modify this sentence when the report is completed.)

The report is organised as follows:
In Chapter 2, we recall the definition of a discrete-time dynamical system, how a discrete-system arises from a flow of an ODE and then proceed to define the inverse limit space and topological conjugacy of autonomous systems.
In Chapter 3, we introduce the problem of forecasting dynamical systems, state the Takens delay embedding theorem and discuss various issues faced while forecasting.
In Section 4, a driven dynamical system is defined and discuss the properties of a specific class of driven dynamical systems that we make use of in this thesis/report/paper.
Finally in Section 5 we show the implementation of these forecasting methods, and conclusions are provided in Section 6.

Chapter 2

Discrete-time Dynamical Systems

In this chapter, we provide a brief description of what a discrete-time dynamical system is and what it means for it to exhibit chaos. We refer to [**?**, **?**] for more details.

At its most elementary level, a dynamical system is just something that evolves deterministically through time. In the context of this thesis, deterministic refers to the fact that a system evolves according to specified rules rather than based on random events. Dynamical systems arise in a variety of situations. A continuous-time dynamical system describes the states for all values of the time. Specifically, if the motion of a pendulum in which the quantities such as the angular position and angular momentum are known at all times, then it is a continuous-time dynamical system. The equations of the dynamical system can take the form of one or more ordinary differential equations that determine the relevant quantities at any future time if we know the initial location and momentum. In ecology, discrete-time dynamical systems are widely used to model population growth. The model in this case is a function calculating the following generation's population given the population of the previous generation. If we know the starting population, we may once again calculate the population at any time in the future.

Formally, a function $T : U \to U$, where U is some set is a *discrete-time dynamical system* and its iterates $\{u, Tu, T^2u, \ldots\}$, where T^n denotes the n-fold composition of T with itself, describe the evolution of an initial condition $u \in U$ (Note that we frequently drop the brackets and denote $T(u)$ by Tu so as to simplify notations).

Continuous-time dynamical systems modelled using ordinary differential equations can give rise to discrete-time dynamical systems. To see this, consider a differential equation $\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$ given to have a unique solution passing through each point $x \in \mathbb{R}^n$, call it $x(t)$ where $x(0) = x_0$.

Definition 1 (**Flow of an Equation**). *The flow of the equation $\dot{x} = f(x)$ is defined to be a mapping $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ where $\varphi(x_0, t) = x(t)$ for the solution $x(t)$ with $x(0) = x_0$*

By fixing $t = K \in (0, \infty)$, we can define the *time-K map* as $T(x) := \varphi(x_0, K)$, and it is easily verified that $T \circ T(x_0) = \varphi(x_0, 2K)$. In general the m^{th} iterate of x_0 under T would be the value of the solution of the ODE evaluated at time mK with the initial condition x_0, i.e. $\varphi(x_0, mK)$. Thus ordinary differential equations give rise to a discrete-time dynamical system by sampling the value of the solution $x(t)$ at time intervals K units apart.

A numerical discretization of a differential equation can also give rise to a discrete-time dynamical system. For instance, Euler's method approximates $\dot{x}(t)$ by $(x(t+h) - x(t))/h$; if h is fixed throughout, the solution of a differential equation $\dot{x} = f(x)$ can be approximated at the time instant $t + (m+1)h$ by iterating the equation

$$x(t + (m+1)h) = x(t + mh) + hf(x(t + mh)) \tag{2.1}$$

Adopting more succinct notation by replacing $x(t + mh)$ with u_m, we rewrite the above equation as

$$u_{m+1} = u_m + hf(u_m) \tag{2.2}$$

or in even simpler terms as the discrete-time dynamical system with map $T(u) = u + hf(u)$, where T, u and f are understood to be as above.

Of course, discrete-time dynamical systems need not always arise through a differential equation, and once again we may consider the field of ecology where discrete dynamical systems are often directly derived or assumed. **Cite.** There is a school of thought that advocates discrete-time dynamical systems to be more natural for modelling real-world observations than differential equations and we refer the interested reader to [**?**].

2.1 Invariant Sets

A core concept in the study of dynamical systems is that of invariance.

Definition 2 (**Invariant Set**). *Given a discrete-time dynamical system $T : U \to U$, a subset $A \subset U$ is said to be an* invariant set *if $T(A) = A$.*

We also define the *orbit of T* to be a sequence $\bar{u} = \{u_n\}_{n \in \mathbb{Z}}$ obeying the update equation, $u_{n+1} = Tu_n$, $n \in \mathbb{Z}$.

Two examples of invariant sets include a fixed point where $Tu = u$ for $u \in U$, and a periodic orbit, i.e., a set of iterates $\{u, Tu, T^2u, \ldots, T^pu\}$, where $T^{p+1}u = u$ for some $p \in \mathbb{Z}$. The entire space U could be also be invariant. Consider for example the space $U = [0, 1]$, where $Tu = 4u(1-u)$ and then U is invariant (as every $u \in U$ can be written as $u = 4x(1-x)$ for some $x \in U$).

We may learn a great deal about the iterates of a dynamical system by considering the types of invariant sets of a discrete-time dynamical system. For example, if $U = [0, 1]$ has map $Tu = u/2$, then the only invariant set is $\{0\}$, and all orbits approach this invariant set as time flows in the forward direction. Indeed, if a some non-zero invariant set (call it B) exists, then there is some $r \in (0, 1] \cap B$. But $r \notin T(B)$ since any orbit with initial value r will be a decreasing sequence. Moreover, every orbit of T will be decreasing and therefore approach the value 0 as $n \to \infty$.

One may ask if every orbit approaches an invariant set? In general the answer is no, since for the dynamical system $T : \mathbb{R} \to \mathbb{R}$ defined by $Tu = 2u$, any orbit that does not intersect the invariant set $\{0\}$, will not approach any invariant set.

However, when the space U is compact, all orbits approach an invariant set. This is since the set of limit points of the orbit can be shown to be invariant [**?**]. So, when U is compact, the ω-limit set $\omega(u; T)$ of a point u defined to the collection of limit points of the sequence $\{x, Tu, T^2u, \ldots\}$ is nonempty, and $\omega(u; T)$ is invariant.

Example 1. *For the map $Tu = u^2$ defined on $[0, 1]$ all orbits lie in the invariant set $\{1\}$ or else would would approach the invariant set $\{0\}$.*

Invariant sets have various properties. Invariant sets can be attracting or repelling depending on how orbits in its vicinity behave. Recall the examples $Tu = u/2$ and $Tu = 2u$ defined on $\mathbb{R}$ (where $\{0\}$ "attracts" orbits) or the example, $Tu = u^2$ defined on $[0, 1]$ (where $\{1\}$ "repels" orbits). We are interested in attractive invariant sets since they capture the long-term dynamics as time increases. In particular we are interested in those invariant sets named attractors.

Definition 3 (**Attractor**)**.** *Let $T : U \to U$, where U is a metric space with metric d. A compact subset $A \subset U$ is said to be an attractor if it satisfies the three conditions:*

1. *A is invariant.*
2. *A is asymptotically stable, i.e., for every $\epsilon > 0$ and for all u so that $d(u, A) < \epsilon$, we have $d(T^n u, A) \to 0$ as $n \to \infty$.*
3. *A has Lyapunov stability, i.e., for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ so that $d(u, A) < \delta$ implies $d(T^n u, A) < \epsilon$ for all $n \geq 0$.*

Indeed in our previous examples, it can be verified that the system $Tu = u/2$ defined on $\mathbb{R}$ and $Tu = u^2$ defined on $[0, 1]$ the singleton set $\{0\}$ is an attractor. For the system, $Tu = 1 - |2u - 1|$ on $[0, 1]$, one may easily verify that the only attractor is the entire space $[0, 1]$. This is since between any two points $u < v$ in $[0, 1]$, and for any a, b so that $u \leq a < b \leq v$, we can find an n so that $T^n(a, b) = [0, 1]$ (as would be explained later in this chapter).

[1] EdN:1

[1]EdNote: For let B be an attractor of U. From above, we know that B is itself invariant, is asymptotically stable and has Lyapunov stability. Now suppose there is some attractor $C \subsetneq U = B$, so there is a $b \in U$ which is not in C. ....
M: Have changed the example

A dynamical system can have several attractors, and may also be contained in another attractor. For the example, $Tu = u^2$ on $[0, 1]$, both $\{0\}$ and $[0, 1]$ are attractors. It is a known result that for dynamical systems on a compact space there always exists an attractor.

The dynamics restricted to an invariant set can be complicated. For instance an invariant set could be just a single point or it could have an infinite set. If the invariant set is infinite then there is a possibility of complicated dynamics, and a particular well-studied phenomenon of such complexity gives rise to what is called chaotic behaviour, and has been studied in detail over the last fifty years.

2.2 Chaos

In the 1960s, a number of mathematicians and mathematically interested scientists independently discovered chaos in the mathematical sense. The meteorologist Edward Lorenz may have been the first to explain this phenomenon in his 1963 paper [**?**]. The notions of invariance, attractivity and chaos may also be described for continuous systems, and Lorenz's system comprised of a system of differential equations. The narrative of Lorenz's discovery of chaos, as well as the history of other forerunners in this subject, are fascinating. For those who are interested, we highly recommend James Gleick's book Chaos: The Making of a New Science [**?**]. It clearly illustrates these experiences, explains why chaos was such a startling and crucial mathematical and scientific discovery, and describes the underlying mathematical notions for non-specialists.

There are several intricately different definitions of chaos available in the literature **(Cite)**, with EdN:2 each of them indicating some aspect of complexity. [2]n practice, it is not possible to verify which definition or notion does the underlying dynamical system satisfies when only data is observed from it. We do, however, specifically recall the definition of chaos in the sense of Devaney [**?**, **?**] so as to understand some nuances behind the complexity. Devaney's definition of chaos has three requirements, with the first being the notion of sensitive dependence on initial conditions.

Definition 4 (**Sensitive Dependence on Initial Conditions).** *A dynamical system $T : U \to U$ is said to have sensitive dependence on initial conditions (SDIC) if there exists a $\delta > 0$ such that for every $u \in U$ and in every neighborhood of $u \in U$ there exists a $v \in U$ and an integer $N := N(u, v) \in \mathbb{Z}$ such that $d(T^N u, T^N v) > \delta$.*

It is important to acquire a sense of this definition as it can easily be misunderstood. It is common misconception to interpret SDIC as two close points (u and v) that eventually become separated by a distance δ under iteration by T. But this is not true. To fully comprehend all the subtleties of the concept, one needs to discuss it more thoroughly by considering each sentence with care and

[2] EdNote: |

attention, whereafter one may examine how they fit together to convey the concept of sensitive dependence.

To this end, we make 3 remarks:

1. First, the $\delta > 0$ in the definition of SDIC is independent of u.

2. Second, in every neighborhood of u, we may not necessarily find all points v in the neighborhood distinct from u that would separate from the forward iterates of u.

3. Finally, N depends upon u and v chosen, and their iterates may not separate for ever (i.e., for all $n > N$) and we allow their iterates to get arbitrarily close in the future.

To illustrate the concept of SDIC, we present two examples - one from Mathematics and the other from the field of Physics.

Example 2. *The logistic map(LM), a recursion relation of the form* $x_{n+1} = rx_n(1 - x_n)$ *where* $x_n \in [0, 1]$ *and* $r < 4$ *is of particular interest to us as it is classically used to illustrate the influence of chaos on a system* ***(Cite)****. Below are plotted the first 75 iterates of the LM for* $r = 3.7$ *and* $x_0 = 0.2$ *with an adjustment of 0.001 in* x_0 *and then* r*.*

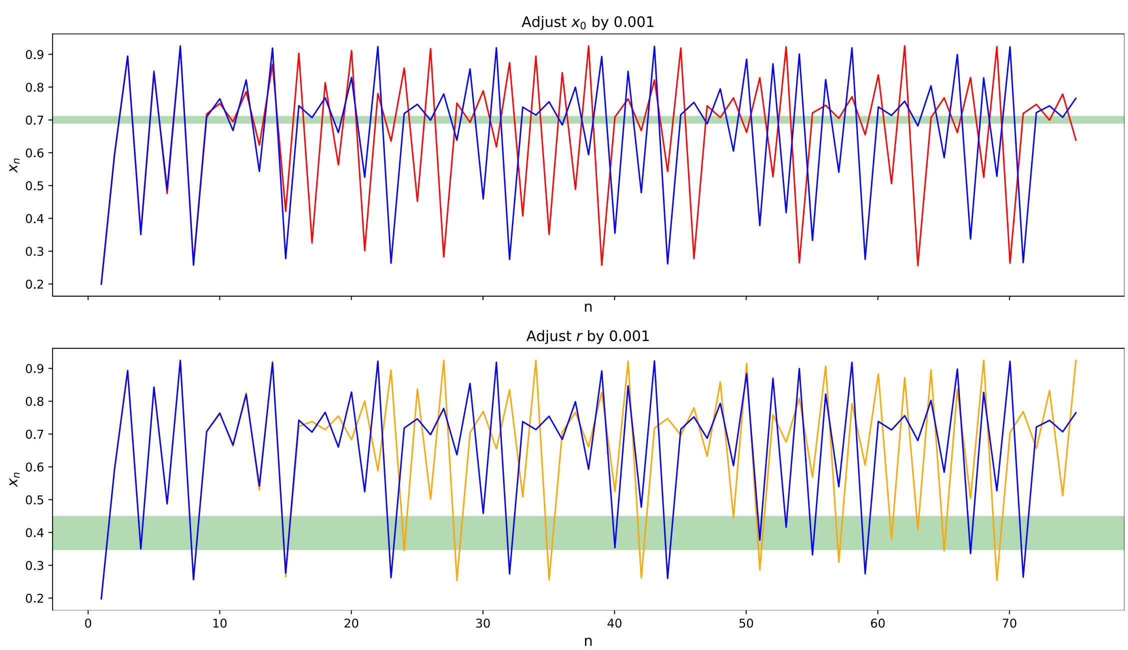

Fig. 1. Logistic Map for $r = 3.7$, $x_0 = 0.2$

For an intuitive sense of the fact that the LM exhibits SDIC, we draw two green neighbourhoods of diameters $\delta_1 = 0.1, \delta_2 = 0.02$ *around the points* $u_1 = 0.7, u_2 = 0.4$ *respectively. It is easy to see that every neighbourhood around these two points will contain some* $v_1, v_2 \in [0, 1]$ *and natural number* $N_1, N_2 \in \mathbb{N}$ *respectively s.t. the distance* $d(T^N u_1, T^N v) > \delta_1 = 0.02$ *and* $d(T^N u_2, T^N v) > \delta_2 = 0.1$.

EdN:3 *In a more rigourous fashion ....* [3]

Example 3. *A second example pertains to a simple physical system called the double pendulum - a pendulum with another pendulum attached to its head given an initial position and angular velociy.*

*Below are two figures denoting the trajectories of a double pendulum's second head after some elapsed time. Depicted are three pendulum heads with equal angular velocity, but differing ever so slightly in initial position . As can be seen below, the trajectories diverge very quickly to become completely different. The double pendulum will be discussed in much greater detail in a later chapter (**??**), but we mention the example here to provide a numerical example of a practical system exhibiting SDIC.*

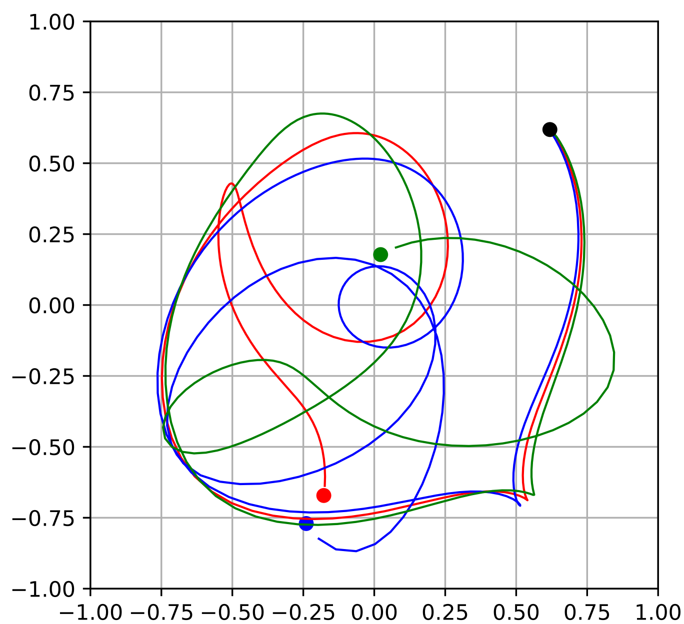

Fig. 2. Three double pendulum heads with initial angle differing by 0,025 radians

The second part in Devaney's definition of chaos concerns topological transitivity.

Definition 5 (Topologically Transitive). *A dynamical system* $F : U \to U$ *is topologically transitive if for any pair of nonempty open sets* E_1 *and* E_2 *there exists a* $n \in \mathbb{N}$ *such that* $T^n(E_1) \cap E_2 \neq \emptyset$.

[3] EdNote: B:Should I be more rigourous here?

Topologically transitivity implies that the iterates of an open set of initial conditions gets mixed up with other open sets. On a compact metric space, it can be shown that topological transitivity also implies the existence of a point whose forward iterates are dense [**?**]; or in other words, the orbit going through this point will be dense in the compact metric space. In fact, it is topologically more likely that the choice of an arbitrary point will be one whose iterates are almost dense.

Definition 6 (**Meager Set**). *A subset of a topological space U is said to be a meager set if it can be written as a countable union of sets of with empty interior. The complement of the a meager set is set to be a residual set.*

To be more precise (see [**?**]), for a discrete-time dynamical system on a compact space, the set of points with dense iterates are residual, and they are typical or likely to be observed in practice. In this thesis, we assume that when data is observed from a topologically transitive system, we assume that the data arises from a dense orbit.

We may now define the notion of Chaos as formulated by Devaney[**?**].

Definition 7 (**Devaney's Chaos**). *A dynamical system $T : U \to U$ is said to exhibit chaos in the sense of Devaney if it satisfies the three properties:*

1. *T has SDIC.*
2. *T is topologically transitive.*
3. *The set of periodic points of T are dense in U.*

Example 4. *The standard tent map $Tu = 1 - |2u - 1|$ defined on $[0, 1]$ is a well-known example of a dynamical system satisfying these three properties.*

We now reason as to why this is true. The graph of the map T is piecewise linear with two straight lines, one connecting the points $(0, 0)$ and $(\frac{1}{2}, 1)$ and the other connecting $(\frac{1}{2}, 1)$ with $(1, 0)$. They form a so-called tent with base centered at $u = 1/2$. The graph of the map T^2 comprises two symmetric tents with their base centered at $1/4$ and $3/4$. See Figure 3

In general, the graph of T^k contains 2^{k-1} tents. Given any point u and an open neighborhood $W \subset [0, 1]$ of u, we can find $k \in \mathbb{Z}$ large enough so that $T^k(W) = [0, 1]$ because we can accommodate a tent whose base is contained in W. This implies the existence of two points w_1 and w_2 in W so that $d(T^k w_1, T^k w_2) = 1$, and by the triangle inequality at least one of the inequalities $d(T^k u, T^k w_1) > \delta$ or $d(T^k u, T^k w_2) > \delta$ hold when $\delta \in (0, \frac{1}{2})$. So T has sensitive dependence on initial conditions.

Next, let W_1 and W_2 be two nonempty open sets. Given any open set W_1, we can find a $k \in \mathbb{N}$ large enough so that $T^k(W_1) = [0, 1]$ because we can accommodate a tent with base contained in W. So $T^k(W_1) \cap W_2 \neq \emptyset$, and thus T has topological transitivity.

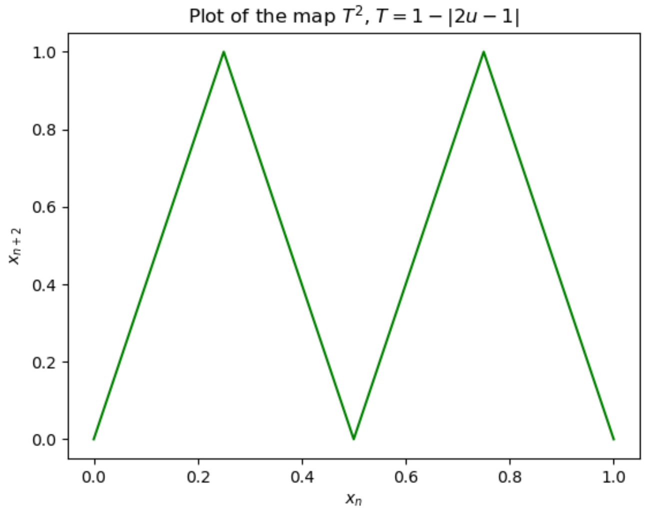

Fig. 3. Three double pendulum heads with initial angle differing by 0,025 radians

Finally, for every open interval (a, b), the graph of T^k intersects the graph of the identity map on $[0, 1]$. This follows from the fact established above that there exists a $k \in \mathbb{N}$ such that $T^k(a, b) = [0, 1]$.If the intersection point has coordinates of the form (p, p), the point p is then a fixed point of T^k and therefore also a periodic point of T. Since we have found a periodic point in the interval W, the set of periodic points are dense in T. This follows because any fixed point of T^k is also a periodic point, and we have found this in an arbitrary interval.

2.3 Conjugacy

We now turn our attention to the subject of conjugacy that describes when two dynamical systems are equivalent dynamically.

To show topological similarity or sameness between two metric or topological spaces, one needs to establish a homeomorphism between the two spaces. However, in the study of dynamical systems defined on two spaces, establishing a homeomorphism does not indicate that the systems are dynamically related in any way. For instance, the maps $Tu = u^2$ and $Tu = 1 - |2u - 1|$ defined on $[0, 1]$ have totally different behaviour. To find systems that are dynamically similar, one rather needs to establish a kind of dynamical equivalence which we illustrate in a simple commutativity diagram below.

$$\begin{array}{cc} U & U \\ & \\ V & V. \end{array} \tag{2.3}$$

To understand the diagram we note that if we travel “right and then down,” the diagram instructs us to use T (top arrow right) first, followed by ϕ. (right arrow downwards). Consequently the pathway amounts to finding $\phi(T(u))$. If we proceed ”down and then right”, the diagram instructs us to apply ϕ (left arrow downwards) first, and then apply S (bottom arrow right). This then amounts to finding $S(\phi(u))$. When the relationships in this diagram hold, we say $\phi(T(u)) = S(\phi(u))$, and we formally denote it as $\phi \circ T = S \circ \phi$.

Definition 8 (Conjugacy & Semi-Conjugacy). *Consider the dynamical systems $T : U \to U$, $S : V \to V$ and suppose the relationship $\phi \circ T = S \circ \phi$ holds. If $\phi : U \to V$ is a homeomorphism, then S is said to be conjugate to T and ϕ is called a conjugacy. If we relax the criterion on ϕ and merely require ϕ to be continuous and surjective where $\phi : U \to V$, then ϕ is a semi-conjugacy between T and S where T has domain U and S has domain V; S is said to be semi-conjugate to T.*

[4] When S is conjugate to T, the dynamics of the two systems are in some way ”dynamically equivalent”. Specifically, they are in one-to-one correspondence with one another. However when S is semi-conjugate to T with $\phi : U \to V$ a many-to-one mapping, the dynamics on V provide merely a coarse-grained description of the dynamics on U [**?**]. When S is semi-conjugate to T, it is also common to call S a *factor of* T, or conversely that T is an *extension of* S. In essence, an extension (e.g., [**?**]) is a larger system capturing all of the important dynamics of its factor. EdN:4

It is a very hard or nearly an impossible task to establish the existence of a conjugacy or a semi-conjugacy ϕ between two systems [**?**]. However, one can verify that a function ϕ satisfies the commutativity diagram 2.3.

NB NB FINISH

Example 5. *For example $\phi(x) = \sin(\frac{\pi}{2}x)^2$ is a conjugacy between two systems $Tu = 1 - |2u - 1|$ and $Sv = 4v(1 - v)$ defined on $U = [0, 1]$.*

To see this, we consider $\phi \circ T = \sin(\frac{\pi}{2}(1 - |2u - 1|))^2$

and $S \circ \phi = 4\sin(\frac{\pi}{2}x)^2(1 - \sin(\frac{\pi}{2}x)^2)$

In establishing that two systems are conjugate (semi-conjugate) to one another, we have also shown that one may choose to work with one system as opposed to the second and still be guaranteed to obtain information on the latter. This is especially useful while forecasting the future evolution of dynamical systems.

[4] EdNote: M: Saying ϕ is then a semi-conjugacy between T and S could be ambiguous as it does not specify which system is on the top in the commutativity. So please edit accordingly. B: Does defining explicitly the domain and codomain of phi make it clear enough?

Chapter 3

The Learning Problem

In this chapter we acquaint ourselves with the question of forecasting dynamical systems with unknown underlying structures, state and discuss the shortcomings of the Takens delay embedding theorem and discuss various issues faced while forecasting.

Consider a relatively simple learning problem: Given the sequence $(u_0, u_1, \ldots, u_m)$ for $m \in \mathbb{Z}$, a finite segment of an orbit of the map T where T is defined by the update equation $u_{n+1} = Tu_n$, forecast the values u_{m+1}, u_{m+1} where the map T is unknown, given that u_m.

A practical example of this would be the subsequent scenario: given the time-sequential coordinates of an object moving in space, predict the future positions of that object. However we are very rarely, if at all, presented with a problem where the entire state information is available to us in the form u_n at some time-step $n \in \mathbb{N}$. Consider then a more involved learning problem:

Suppose we only have the observations $\theta(u_0), \theta(u_1), \ldots, \theta(u_m)$ of the true system states u in an unknown dynamical system (U, T) and we wish to predict the values $\theta(u_{m+1}), \theta(u_{m+2})$ and so forth.

First we establish the method of Takens delay embedding. In this method, one considers a discrete-time dynamical system defined on a 'nice' space (a smooth manifold)[5] that can be obtained EdN:5 as time-K map of a continuous time-dynamical system [6]. We make observations from such a system, EdN:6 i.e., we observe the evolution of $\theta(w_0)$ (where w_0 represents the initial state) i.e., we observe a finite set of values of $u_n := \theta(w_n) = \theta(Tw_{n-1})$. The sequence $\{u_n\}$ represents a scalar time-series and intuitively θ may be thought to represent a probe inserted into a larger system which is itself only measuring/extracting a small part of the greater system state at time $t = n$. Consider for example a thermometer erected to measure the ambient temperature in a local village. This measurement

[5]EdNote: Should this be further explained?
[6]EdNote: B: Should one add here: "These notions will be made formal below"?

function, the thermometer, is capturing only a single aspect, the temperature, of a much grander dynamical system entailing the present weather of the surrounding area. Even more than that though- it is measuring a miniscule part of the global weather system.

However, the true state u of a system is seldom, if ever, fully known. In almost all cases we can at most insert probes into a system to obtain partial information by means of the measurements taken. Moreover, the process of taking a measurement itself introduces 2 additional aspects which complicate the problem: [7] EdN:7

i. A series of measurements over a specific time-interval is inherently a discretisation procedure of the underlying continuous-time dynamical system. (Hence why we restricted our attention to discrete-time systems in the preceding section)
ii. A probe will never be fully accurate, and so the act of measurement introduces a certain measure of numerical noise/inaccuracy.

From here we construct a multidimensional observable using the method of stacking previous observations, i.e., we create delay-coordinate map defined by $\Phi_{k,\theta}(w) := (\theta(T^{-k}w)\ldots,\theta(T^{-1}w),\theta(w))$. The essence of Takens theorem relates to the fact that when k is sufficiently large, we can define a dynamical system on the space $\mathbb{R}^{k+1}$ whose states are $\Phi_{k,\theta}(w)$, $\Phi_{k,\theta}(Tw)$, $\Phi_{k,\theta}(T^2w)$, etc. and this dynamical system is topologically conjugate to the unknown underlying system (W,T). We recall the Takens delay embedding theorem next.

3.1 Taken Embedding Theorem

We define first the concepts of homeomorphism and embedding:

Definition 9 (**Homeomorphism**)**.** *A homeomorphism is a function $f : Z \to Y$ between two topological spaces Z and Y that is continuous, bijective and has a continuous inverse.*

Definition 10 (**Embedding**)**.** *Consider a homeomorphism $f : Z \to Y$ for $Y \subset X$. Z is said to be embedded in X by f.*

Takens Theorem states a result establishing a relationship between the observed and underlying dynamical systems by showing that the concatenation of a sufficiently large number of previous observations into a vector will, under certain conditions, generate a map between the vectors from the respective systems. We formulate the theorem from [**?**].

Theorem 1 (**Takens Embedding Theorem (adopted from [?])**)**.** *Let W be a compact manifold of dimension m, and $d \geq m$ so that $2d$ is an integer. It is a generic property for the pair (T,θ),*

[7] EdNote: M: Maybe replace difficulty by saying that we need to consider two aspects. B: Perhaps this edit makes more sense?

where $T : W \to W$ is a smooth diffeomorphism, and $\theta : W \to \mathbb{R}$ a smooth function, the map $\Phi_{2d,\theta} : W \to \mathbb{R}^{2d+1}$ defined on W by $\Phi_{2d,\theta}(w) := (\theta(T^{-2d}w) \ldots, \theta(T^{-1}w), \theta(w))$ is a diffeomorphic embedding; by 'smooth' we mean at least C^2. Consequently, there exists a map $F_\theta : \Phi_{2d,\theta}(W) \to \Phi_{2d,\theta}(W)$ defined by

$$F_\theta : (\theta(T^{-2d}w), \ldots, \theta(T^{-1}w), \theta(w)) \mapsto (\theta(T^{-2d+1}w), \ldots, \theta(w), \theta(Tw))$$

so that (W, T) is topologically conjugate to $(\Phi_{2d,\theta}(W), F_\theta)$.

By generic we mean a residual set on a certain topology on appropriate spaces of functions (that we do not describe here) **(Cite)**. By C^2, we make reference to a twice-differentiable function with a continuous second derivative. In our scenario (and this remains important throughout) the input space U is considered the attractor.

Below we reproduce Figure 1 from [**?**].

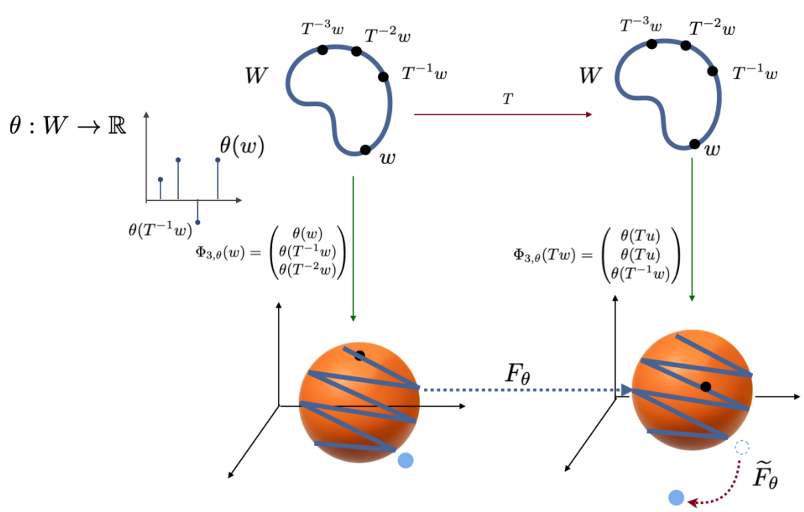

Fig. 1. Schematic of Takens Embedding Theorem's usage and consequence of how iterates ...

8 9 EdN:8
EdN:9

If we can learn the map F_θ, from sufficient number of data points of $\{\phi_{2d,\theta}(w_n)\}$, then we also know how to forecast how $\theta(w_n)$.

[8] EDNOTE: M: Here, and throughout if you reproduce a figure, it must be stated where it was reproduced from. B: How does one write that a figure is reproduced? Also, should $\widetilde{F}$ be left out for now?

[9] EDNOTE: B: Could I perhaps have the original figure? This is only a screenshot used as a temporary placeholder.

3.2 Practical Issues of Takens theorem

It is a fact that Takens' Embedding Theorem is both a powerful result and that it provides compelling reason to believe that one could conceivably reconstruct accurately a system conjugate to the underlying system. Nonetheless, it does present some serious practical limitations:

1. Even supposing that we could indeed find F_θ , our *approximation* of F_θ is a map from a larger set $\mathbb{R}^{2d+1}$ containing the embedded attractor. There are, however, no theoretical guarantees that F_θ will retain $V = \Phi_{2d,\theta}(W)$ as an attractor although W could itself be an attractor.

2. Takens theorem is stated for noiseless observations. Due to noise ϵ_n the delay vector $\Phi_{2d,\theta}(w_n)+\epsilon_n$ may lie outside V.Furthermore, due to the chaotic nature of the underlying system (i.e. the fact that it has SIDC), the evolution of $\Phi_{2d,\theta}(w_n) + \epsilon_n$ under the map F_θ could move out of V completely. This problem can be overcome by using a driven dynamical system with some properties, and we discuss this in Chapter *.

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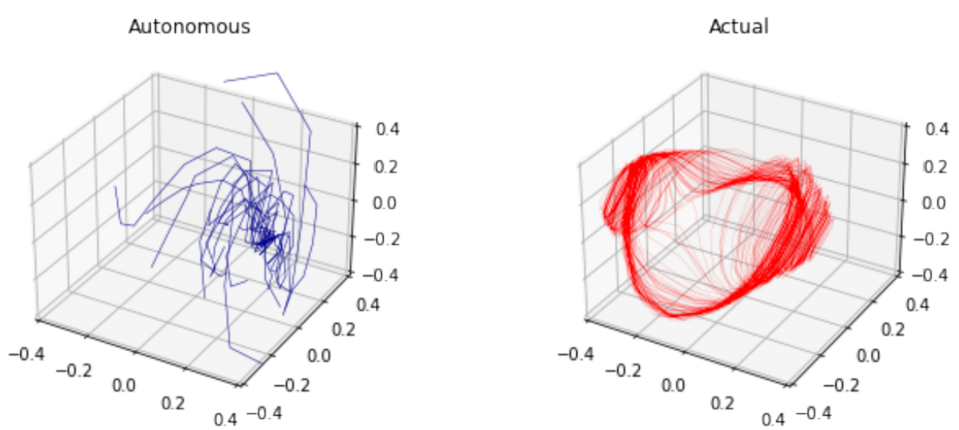

Fig. 2. Illustration of a system having failed to learn the map

Explanation continued here

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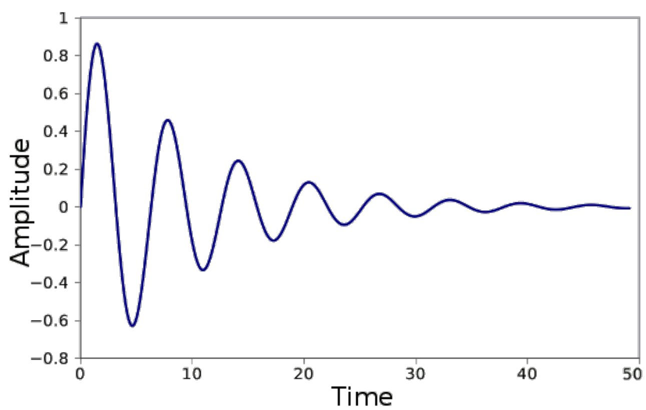


Fig. 3. A damped pendulum

Chapter 4

Driven Dynamical Systems to Forecast Problems

In this chapter, we discuss results pertaining to the mapping of temporal data obtained from a discrete-dynamical system onto a different space through the notion of a driven dynamical system. We also consider the conditions a driven system should have to avoid adding distortion to its state space representation, as well as then ensuring that the single-delay dynamics (SDD) are conjugate (or at least semi-conjugate to the system). The SDD can then be used to forecast and reconstruct the underlying attractor. [10] EdN:10

4.1 Nonautonomous and Driven Dynamical Systems

A nonautonomous dynamical system is simply a dynamical system (as defined before in **??**) where the input u from the input space U, a topological space, is time-dependent.

We immediately recall the figure first defined in 2.3 and remind ourselves of our ultimate goal to forecast the evolution of an unknown nonautonomous dynamical system (U, T) by constructing a conjugate system such as (V, S) below.

$$\begin{array}{cc} \mathrm{U} & \mathrm{U} \\ & \\ \mathrm{V} & \mathrm{V}. \end{array}$$

[10] EdNote: Is this attractor assumed to exist?

To do this, the need arises to consider a driven dynamical system where input is taken from both an input space U, and an underlying state space X. This is done to account, in a manner more similar to the true scenario, for the influence of both the input and the actual state of the system at a specific timestep.

Definition 11 (**Driven and Compactly Driven Dynamical System).** *A driven dynamical system comprises two topological metric spaces U, X and a continuous function $g : U \times X \to X$ where $g(u_n, x_n) = x_{n+1}$. The dynamics on X are generated by the update equation $x_{n+1} = g(u_n, x_n)$ where $n \in \mathbb{Z}$, input u_n from U and state x_n belonging to X, where X is compact. If the input space U is compact, we refer to the system as compactly driven. Abbreviated, we shall refer only to the* driven system *g, with all other entities being understood implicitly.*

In particular, a nonautonomous dynamical system may be generated from U; any input $\overline{u}$, a bi-infinite sequence from U, gives rise to the sequence of self-maps $\{g(u_n, \cdot)\}_{n\in\mathbb{Z}}$ contained in X. Physically, one may think of this bi-infinite sequence as referring to a system that has been running for an incredibly long period at the time of the first measurement taken from the system (alternatively, the first time a probe is inserted into the system to take an observation).

Definition 12 (**Entire Solution).** *A sequence $\{x_n\}_{n\in\mathbb{Z}} \subseteq U$ is called an entire solution(or simply a solution) to the driven system g with input $\overline{u}$ when it satisfies*

$$g(u_{n-1}, x_{n-1}) = x_n$$

for all $n \in \mathbb{Z}$

It is important to emphasise that a sequence satisfying the update equation above can only be a solution if $x_n \in U$ for all $n \in \mathbb{Z}$. Consider the example below.

Example 6. *The only solution $\{x_n\}_{n\in\mathbb{Z}}$ to the driven system $g(u, x) = \frac{ux}{2}$, where $X = [0, 1]$, $U = [0, 1]$, is the zero solution $x_n \equiv 0$. To see this, consider any $x_n = a \in [0, 1]$ where $a \neq 0$. Let $\overline{u} \in U$ be an non-zero constant sequence , say $u_n = 0.5$. The driven system may be rewritten as $x_{n-1} = \frac{2x_n}{u_{n-1}} = 4x_n$ and the iterates of x_n in backward time will increase by a factor of 4 at each timestep. Thus for some $m \leq n$, $1 < x_m$ i.e. $x_m \notin X$. So $\{x_n\}_{n\in\mathbb{Z}}$ is not a solution and it follows that the only possible solution is the zero solution.*

A system may also have multiple solutions as is evidenced in the example below.

Example 7. *Consider the driven system $g(u_n, x_n) = x^n$ for $X = [0, 1]$, $U = \mathbb{R}$. The system has an uncountable number of solutions, as there exists a solution for every $x \in X$ which also passes through the point x and $lim_{n\to\infty} x_n = 0$, $lim_{n\to-\infty} x_n = 1$. The proof is deferred to immediately after the next paragraph.*

As the solutions to a driven system are often considered, we next identify a subspace X_U of X that contains all possible solutions. To realize such a subspace of a driven system g, the concept of a reachable set is defined.

Definition 13 (**Reachable Set**)**.** *The reachable set of a driven system g is exactly the union of all the elements of all the solutions, i.e.,*

$$X_U := \Big\{x \in X : x = x_k \text{ where } \{x_n\} \text{ is a solution for some } \bar{u}\Big\}.$$

The set of all reachable states at a specific time n for input $\overline{u}$ is denoted by $X_n(\overline{u})$

We note that $x \in X_n(\overline{u})$ if and only g has a solution $\{x_k\}$ for $x_n = x$ and input $\overline{u}$. **Cite**. This will lead to a result established later, but which is worth taking note of now: *g being a topological contraction is equivalent to the existence of a unique entire solution.* To this end we define the notion of a topological contraction.

Definition 14 (**Topological Contraction**)**.** *A function $g : U \times X \to X$ is a topological contraction if for all $n \in \mathbb{Z}$ and all $\overline{u} \subseteq U$, $X_n(\overline{u})$ is a singleton subset of X.*

Proof of Example7. To see this, we show that for every input $\overline{u} = \{u_n\}_{n\in\mathbb{Z}}$, $g(u_n, \cdot)$ is a contraction map on X for every $k \in \mathbb{Z}$. Indeed $g(u_n, x_n) = x^n$ depends only on the state at time n and so for each input $\overline{u}$, $X_n(\overline{u}) = \{x^n\}$ is a singleton subset. We then easily conclude that $\{x_n\} \equiv 0$ is the only solution. □

Thus far it has been demonstrated that a system may have one or more solutions; one may ask if a driven system always has a solution and, if so, whether it satisfies certain properties such as uniqueness. Should the driven system be compact, existence follows immediately as shown in the following result.

Theorem 2. *If X is compact then for each input $\overline{u}$, there exists at least one solution to the driven system $g(.\,, x)$*

Proof. Consider an input $\overline{u} = \{u_n\}_{n\in\mathbb{Z}} \subseteq U$ and driven system $g : U \times X \to X$ generating a sequence $\{x_n\}_{n\in\mathbb{Z}}$ in the compact space X. Since X is a metric space, it follows immediately from a well-known result in Analysis (**Cite**) that $\{x_n\}$ has a convergent subsequence which then is a solution. □

We may easily construct many systems with trivial solution-sets, such as $g(u, x) = x$ which has only the constant solution x and so for $U = [-1, 1]$, the system would have no solution if $|x| > 1$. To refine the scenario, we consider only systems with unique solutions.

4.2 Unique Solution Property

Definition 15 (Unique Solution Property). *A driven system g is said to have the Unique Solution Property (USP) if for each input $\overline{u}$ there exists exactly one solution. Alternatively we may formulate the USP as follows: g has the Unique Solution Property if there exists a well-defined map $\Psi : U \to X$ with $\Psi(\overline{u})$ denoting the unique solution.*

One of the first result obtained upon defining the USP, is the fact that every solution will attract any different initial conditions towards the component parts of the solution, i.e. that the solution
EdN:11 Ψ is a non-autonomous uniform attractor. [11]

Theorem 3. *If g has USP, then any solution to g is a non-autonomous uniform attractor*

Proof. [?] □

EdN:12 [12] Having already defined the reachable set X_U, we pause for a moment to fix additional notation. Letting $\overleftarrow{u}^n := (\dots, u_{n-2}, u_{n-1})$ be the left-infinite subsequence of an input, $\overleftarrow{U}$ then denotes all these left-infinite sequences in U. Moreover, $\overleftarrow{u}^n v := (\dots, u_{n-2}, u_{n-1}, v)$ is to symbolise the input up to time n with $v \in U$ being the specific input value at time n. The introduction of a new input at time n can be described by a mapping $\sigma_v : \overleftarrow{u}^n \mapsto \overleftarrow{u}^n v$.

The question now becomes whether we may establish a conjugacy as presented below for g driven

$$\begin{array}{cc} \overleftarrow{U} & \overleftarrow{U} \\ & \\ X_U & X_U . \end{array} \tag{4.1}$$

Restricting our attention more and more, we now consider a specific subclass of conjugacies.

Definition 16 (Universal Semi-Conjugacy). *Given a driven system g, we call a continuous and surjective map $h : \overleftarrow{U} \to X_U$ a universal semi-conjugacy if diagram 4.1 commutes for all $v \in U$.*

If the universal semi-conjugacy h exists (i.e. the diagram in 4.1 commutes) then the solution Ψ will be at least a coarse-grained representation of the input u.

[11] EdNote: Define Uniform Attractor before stating theorem.
[12] EdNote: Additional discussion on USP?

Does such a function h for the driven system defined above exist? If g has the USP and $\Psi(u) = \{x_n\}_{n\in\mathbb{Z}}$ then h, defined by $h(\overleftarrow{u}_n) := g(u_n, x_{n-1}) = x_n$, will satisfy the semi-conjugacy in the graph above 4.1. Regrettably, the mapping h is not guaranteed to exist in general.

Re-sketching the graph4.1 above by fixing the input v in g and replacing X_U by its left-infinite sequence space $\overleftarrow{U}$, we obtain the graph below. In this case, the function $H : \overleftarrow{U} \to \overleftarrow{X}_U$, a map that is both continuous and surjective, is called a *causal mapping.* make special note of a

Definition 17 (**Causal Mapping).** *A continuous, surjective map $H : \overleftarrow{U} \to \overleftarrow{X}_U$ such that*

$$H \circ \tilde{g}_v = \sigma \circ H$$

where $\tilde{g}_v$ maps $(\ldots, u_{-2}, u_{-1})$ to $(\ldots, u_{-2}, u_{-1}, g(v, u_{-1}))$.

$$\begin{array}{cc} \overleftarrow{U} & \overleftarrow{U} \\ \overleftarrow{X}_U & \overleftarrow{X}_U. \end{array} \tag{4.2}$$

Theorem 4. *For a compactly driven system, a causal mapping H exists if and only if g has the USP.*

*Proof. See proof from p.15 of [**?**]* **Expand.** □

Note that even when h does exist, we are not guaranteed its injectivity. Considering again example 6, we see that even if h were to exist, it could not be injective as $X_U = \{0\}$.

An embedding of the space $\overleftarrow{U}$ would allow one to establish topological conjugacy, which in turn provides stronger results than merely obtaining a coarse-grained representation via a causal mapping.

Before formalising this, we make mention of the concept of the inverse-limit system of a dynamical system (U, T). In broad sweeps, the inverse-limit system of a dynamical system is a self-map on a subset of an infinite-dimensional space **(Expand)** where each point in the inverse-limit space corresponds to a backward orbit of the map T. In literature, the inverse-limit space is denoted by $\hat{U}_T \subseteq \overleftarrow{U}$ [**?**]. The map T, then, induces a self-map $\hat{T} : \widehat{T} : (\ldots, u_{-2}, u_{-1}) \mapsto (\ldots, u_{-2}, u_{-1}, T(u_{-1}))$ on $\widehat{U}_T$

Definition 18 (**Causal Embedding).** *A driven system g is said to causally embed the dynamical system (U, T) if*

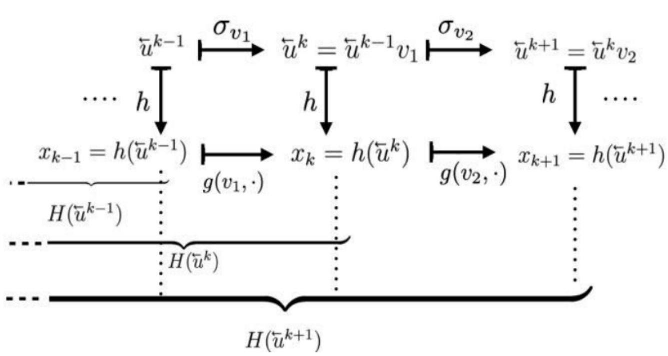
i. The diagram in 4.2 commutes (i.e. g is a universal semi-conjugacy), *ii.* $H_2(\overline{u}) := (h(r\overline{u}, h(\overline{u})))$ *embeds the inverse-limit space* $(\hat{U}, \hat{T})$ *of* (U, T) *in* $X \times X$.

The driven system g can induce an embedding of $\overleftarrow{U}$ in $\overleftarrow{X}$ as follows: If $H : \overleftarrow{U} \to \overleftarrow{X}_U$ is also injective (in addition to being surjective, and hence bijective), it becomes the embedding of $\overleftarrow{U}$ in $\overleftarrow{X}$ induced by the driven system g and we refer to H as a *causal embedding.*

Theorem 5. *If* $g(.,x)$ *is invertible and has the USP then* H *is a causal embedding.*

Proof. *See proof from [**?**]* **Expand.** □

When g has the USP, the diagram below illustrates the operation of the the mappings h and H. The mapping $h : \overleftarrow{U} \to X_U$ is an observable as discussed in an above section (**??**).



The next result is of relative import but to fully comprehend its influence, some discussion still remains. We state it in part and postpone the proof.

Theorem 6. *The following statements are equivalent:*

i. g *has the USP.*
ii. g *is a topological contraction.*

Proof. (postponed) □

4.3 Choosing the driven system g

One must be careful to avoid a choice of g which would add complexity to the obtained solution.

When a causal embedding H exists for the driven system g, one can map an arbitrary input u onto the solution space X without additional distortion or information-loss. [13] When an embedding is established, the question of possible additional complexity in the solution is removed by guaranteeing that, since the systems are conjugate (semi-conjugate, (refer)), g does not add any (some) complexity to the system. It is, however, a balancing act as it also undesirable to choose a function g that quenches the temporal structure in u by contracting to such a degree that the ability to recover information from the original system is lost completely. To obtain a suitably complex function g, it is desired that the the reachable set of a driven system be large enough to relate to the input. [14] EdN:13 EdN:14

In the example above (6), the input's temporal variation cannot be related to the reachable set as X_U consists of a single element and so little, if not no, information is encoded. Rather, the reachable set of a driven system must therefore be such that the inverse-limit space of U_T can be embedded in some finite self-product of the reachable set of T [15]. To this end, consider the notion of State-Input (SI) Invertibility. EdN:15

Definition 19 (**SI-Invertibility**). *A map g is said to be SI-Invertible if g is invertible for all $x \in X$. Alternatively it may be said that if, given x_n and x_{n-1}, u_n can be uniquely determined from $x_n = g(u_n, x_{n-1})$, then g is said to be SI-invertible.*

SI-invertibility promises that 'enough' information is retained when g is chosen without introducing additional unwanted complexity. [16] EdN:16

It is worth taking note of a specific driven system. The function

$$g(u, x) = (1 - a)x + a{\cdot}tanh(Au + \alpha aBx) \tag{4.3}$$

is both SI-invertible and possesses the USP. The proof of these two facts is rather involved and hence the proof, instead of being replicated here, may be found in **(cite relevant article)** This specific driving function g is used in our implementation and is more completely discussed in chapter **??**

Despite the ease that one may work with a left-infinite history in the realm of theory, it is impossible to obtain such a sequence in any real-life application. One does not in practice, fortunately, need the entire left-infinite history of the input thanks to the Uniform Attraction Property (UAP).

Definition 20 (**Uniform Attraction Property**). *A driven system g has the Uniform Attraction Property (UAP) if, regardless of starting position, all trajectories **(of what?)** converge to a single trajectory as time flows forward. This trajectory is also the unique solution sequence x to the input sequence u as mentioned above* **??**.

This definition is stated in a not completely rigourous manner as the more formal definition makes

[13] EdNote: (Cite.)
[14] EdNote: (Expand here. It stops a bit abruptly.)
[15] EdNote: (refer or cite)
[16] EdNote: Expand why this is true.

use of processes, a concept which would take some time to establish and detracts from the principal
EdN:17 thrust of this paper/project/thesis. [17]

As the Unique Attraction Property guarantees that all trajectories *(which ones?)* will converge to the same trajectory as time moves forward. Incredibly, this permits one to initialise a driven system g with an altogether arbitrary initial value $y_m \in X$ where $m \in \mathbb{Z}$ and the UAP then guarantees that the sequence $\{y_{m+1}, y_{m+2}, \ldots\}$ which satisfies the relation $y_n = g(u_n, y_{n-1})$ for $k \geq m$ will uniformly approach the elements $\{x_n\}$ of the actual solution. (see [41, Theorem 1] or [11, Eqn. (18)])

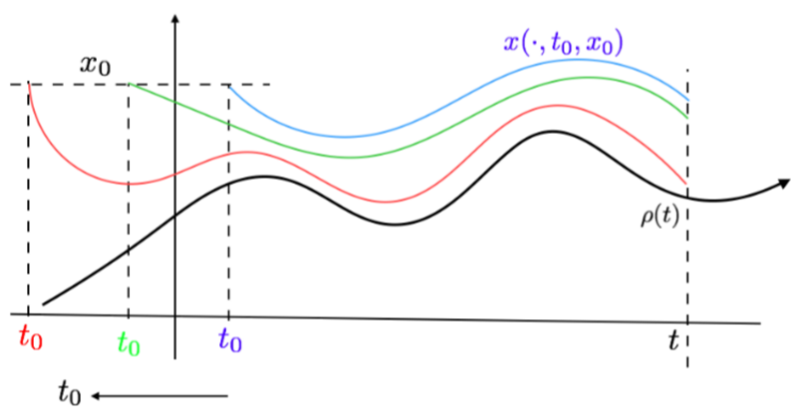


Fig. 1. Approaching the true solution if g has the UAP

One may now appreciate an even more astounding result: g having the USP is equivalent to the UAP. We restate the above theorem.

Theorem 7. *The following statements are equivalent:*

i. g has the USP.
ii. g is a topological contraction.
iii. g has the UAP.

Proof. (We'll show this) [11] eqn.6 □

One need therefore not establish any additional results to ensure that the sequence uniformly approaches the unique solution Ψ. This vastly simplifies the effort necessary to set up a problem in order to guarantee that the underlying system will be accurately represented by the conjugate system.

This also solves the problem of perturbation/noise introduced by the observable or measurement function. Recall from before that an observable is inherently a map that discretises the underlying continuous-time system. Moreover, measurement and device error introduce mistakes. Since we

[17] EdNote: Is it VERY obvious to the reader at this point what the general thrust of the paper is?

are working with a chaotic system displaying sensitive dependence on initial conditions, these small errors could potentially send the trajectory into a completely different attractor. In the presence of the UAP, however, small measurement errors introduced into the system do not pose the same danger as before. [18] EdN:18

4.4 The next step in Dynamics

Next we define the relation Y_T induced by (U, T) on $X_U \times X_U$ for a driven system g possessing SI-invertibility. To describe the single-delay lag dynamics formally, we consider a dynamical system $T : U \to U$ and we define a relation on the reachable set X_U, i.e., a subset defined on $X_U \times X_U$ by

$$Y_T := \{(x_{n-1}, x_n) : \{x_k\}_{k\in\mathbb{Z}} \text{ is a solution for some orbit of } T \text{ and } n \in \mathbb{Z}\}.$$

The following theorem establishes the existence of a well-defined map G_T describing the single-delay dynamics(SDD) of the system above.

Theorem 8. *If we let $G_T : Y_T \to Y_T$ be a map defined by the relation $(x_{n-1}, x_n) \mapsto (x_n, x_{n-1})$, then G_T is well-defined (and this results holds even in the absence of g possessing over the USP)*

We're now getting quite close to where we want to be and our results carry more and more weight. Recall that g is only being given inputs from the orbits of T.

Definition 21 (Inverse-Limit Space). *The inverse-limit space $\widehat{U}_T$, a subspace of $\overleftarrow{U}$, is defined by*

$$\widehat{U}_T := \{(\dots, u_{-2}, u_{-1}) : T(u_n) = u_{n+1}\}$$

The inverse-limit space is well-defined since $T : U \to U$ is surjective by assumption. [19] EdN:19

Theorem 9. *Graph 4.1 is exactly the inverse-limit system $(\hat{U}, \hat{T})$.*

Note that H_2 maps an entire left-infinite solution sequence from Ψ to an element in $X \times X$. We now have the following (compare with 4.2 above):

$$\begin{array}{cc} \widehat{U}_T & \widehat{U}_T \\ \\ Y_T & Y_T \end{array} \tag{4.4}$$

[18] EdNote: (Rephrase and/or expand)

[19] EdNote: Should I explain this in more details or am I then over-simplifiying?

Theorem 10. *(Y_T, G_T) is semi-conjugate to $(\widehat{U}, \widehat{T})$.*

Proof. **Show** □

Summarising the discussing thus far:

It is easy to lose the birds-eye view, so we take a moment to review our progress up until this point.

1. We are interested in a some dynamical system (U, T) with unknown dynamics.
2. To determine properties about this system (U, T) and predict its future evolution, we determine the dynamics of the inverse-limit system $(\widehat{U}, \widehat{T})$. Given certain assumptions, we can guarantee that $(\widehat{U}, \widehat{T})$ is at least semi-conjugate to (U, T).
3. If the driven system g is SI-invertible (and $\{u_n\} \in U$ is an orbit of T), the map G_T exists.
4. If, furthermore, g has the USP, (Y_T, G_T) is semi-conjugate to $(\widehat{U}, \widehat{T})$.
5. If we can assume that T is a homeomorphism, (Y_T, G_T) is topologically conjugate to $(\widehat{U}, \widehat{T})$, an extension space of (U, T)

One can therefore learn the SDD of the driven system states via G_T with enough data thanks to the USP/UAP. This enables us to do at least 2 things:

- Forecast future values of x_n via iterates of G_T (as G_T can be determined.
- Forecast future values of u_n.

4.5 A discussion of G_T

In the above sections, we established the map G_T describing the SDD of a driven system. By
EdN:20 establishing the existence of this map, we've essentially embedded the attractor U [20] into the higher
EdN:21 dimensional space $X \times X$. [21] In layman's terms, this ensures that there is more "dimensional room" for the underlying system's underlying dynamics to "move". As the dynamics aren't as "squashed", we might therefore hope that the dynamics of G_T are in some sense simpler than that of T. (Taken note of the fact that G_T is a homeomorphism even when T is just continuous)

[20] EdNote: (So far we haven't really spoken about an attractor)
[21] EdNote: (Why is it true that it's higher-dimensional?)

In [?] it is illustrated in an empirical fashion in that the map G_T describes dynamics which are less functionally complex than that of T or of the map $\Phi_{2d,\theta}$. This is done by implementing a Recurrent Neural Network (RNN), but not discussed as of yet. [22] EdN:22

We opt to learn G_T in an indirect manner by defining a new map $\Gamma : (x_{n-1}, x_n) \mapsto u_n$. It will follow immediately from G_T's existence that Γ also exists.

The reasons for taking this roundabout approach remain to be discussed in 4.6, but a pat answer may immediately be given: When Γ has been learnt, the system can be driven autonomously and then G_T is known anyway. We formalise this with a theorem.

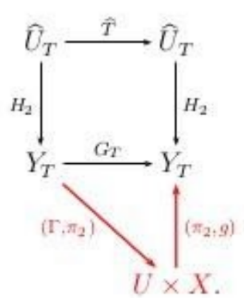
Theorem 11. *When x_n and x_{n-1} are successive points on a solution obtained for input-orbit of T $\{u_n\}$, then the map $\Gamma : X \times X \to U$ defined by $(x_{n-1}, x_n) \mapsto u_n$ exists whenever G_T exists*

Proof. From [Koopman, Theorem 3c] □

Projection mappings π_i are defined in the traditional meaning where a k-dimensionalvector is projected to it's i^{th} component such that

$$\pi_i : (a_1, a_2, ., a_{k-1}, a_k) \to a_i$$

The graph in equation 4.4 is then extended as below:



The problem finally simplifies to the issue of learning the map Γ and combining this with the projection mapping π_2 and the function g, which will be known. A final set of equations is obtained - equations that have been entirely constructed from data.

$$u_{k+1} = \pi_1 \circ (\Gamma, \pi_2) \circ (\pi_2, g)(u_k, x_k) \tag{4.5}$$
$$x_{k+1} = \pi_2 \circ (\Gamma, \pi_2) \circ (\pi_2, g)(u_k, x_k). \tag{4.6}$$

[22] EdNote: (Add in that we'll be using Pearson coefficient?)

4.6 Advantages of learning Γ

One may immediately ask why we opt to take such a roundabout route; why not just learn the map G_T from the get-go? On the surface, this seems to be an arbitrary decision path with no real reasoning, so we take a pause again and discuss the motivation for learning Gamma. There are a number of distinct advantages.

Learning Γ saves computational resources. This is due to the fact that the input u_n lies in a lower-dimensional subspace of the high-dimensional space $X \times X$. In practice, if the input is of a lower dimension, one may easily embed it into the space $X \times X$ by padding the vector u_n with zeroes.

Can we discuss how to put this across well? I would like to add it in as I think it beneficial to the discussion.

Moreover, two types of stability are achieved when g has the USP.

Input- and parameter-related stability is achieved if g has the USP
Define/Explain the two types of stabilities
Sentence on u, v, tails, USP and product topology
If we opt to work with Y_T, errors can occur. Expand
Γ is more stable. Explain why.

EdN:23 Thirdly, Γ is globally dissipative whereas G_T is not guaranteed to be(*Cite*). [23] Global dissipativity prevents large numerical errors due to input data.

The ω-limit set $\omega(u;T)$ of a point u is defined to be the collection of limit points of the sequence $\{x, Tu, T^2u, \ldots\}$. This set is nonempty, and $\omega(u;T)$ is invariant. **(Change)**

Definition 22 (**Global Dissipativity**). *We say a dynamical system (Z, f) is globally dissipative if there exists a nonempty proper closed subset B of Z so that for all $x \in Z \setminus B$, (i). $\omega(x; f) \subset B$ and (ii). B is positively invariant that is $f(B) \subset B$. We call any such closed subset B a trapping set of f.*

If a system is not globally dissipative, neglibly small errors result in major errors which thus induce predictions to fail utterly. If one learns G_T directly, we are in danger of replicating the problems arising for Takens.

[23] EdNote: Add linking sentence.