CONSTRUCTING PROPER AFFINE ACTIONS

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Graduate student Topology & Geometry Conference

4/11/2021

AFFINE MANIFOLDS

Def: A manifold M^n is an affine manifold if its charts lie in $\mathbb{R}^n$ and the transition maps are affine.

$\hookrightarrow \quad x \longmapsto Ax + u, \quad A \in GL_n(\mathbb{R}), \; u \in \mathbb{R}^n$

An affine manifold M^n is complete if we can realize it as a quotient $\mathbb{R}^n/\Gamma$, where $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ is a discrete subgroup acting properly discontinuously on $\mathbb{R}^n$.

Example: $T^n = \mathbb{R}^n / \mathbb{Z}^n$

Question: Which groups in $\mathrm{Aff}(\mathbb{R}^n)$ can act properly discontinuously on $\mathbb{R}^n$?

- Auslander conjecture: If $\mathbb{R}^n/\Gamma$ is compact, Γ is virtually solvable.

- Question by Milnor: What if we drop the cocompactness condition?

  I.e. can a free group act properly discontinuously on $\mathbb{R}^n$?

Yes!

- Margulis
- Mess
- Drumm
- Choi-Goldman
- Danciger-Guéritaud-Kassel
- Charette-Drumm-Goldman

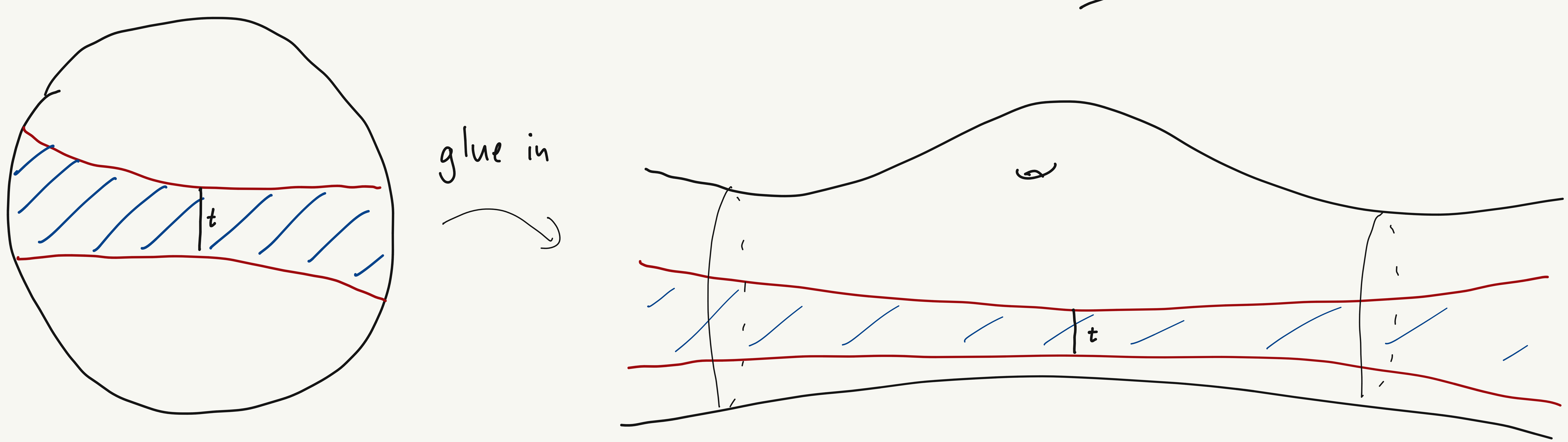
mostly for actions on $\mathbb{R}^3$

- Smilga
- Burelle-Treib

work in higher dimensions

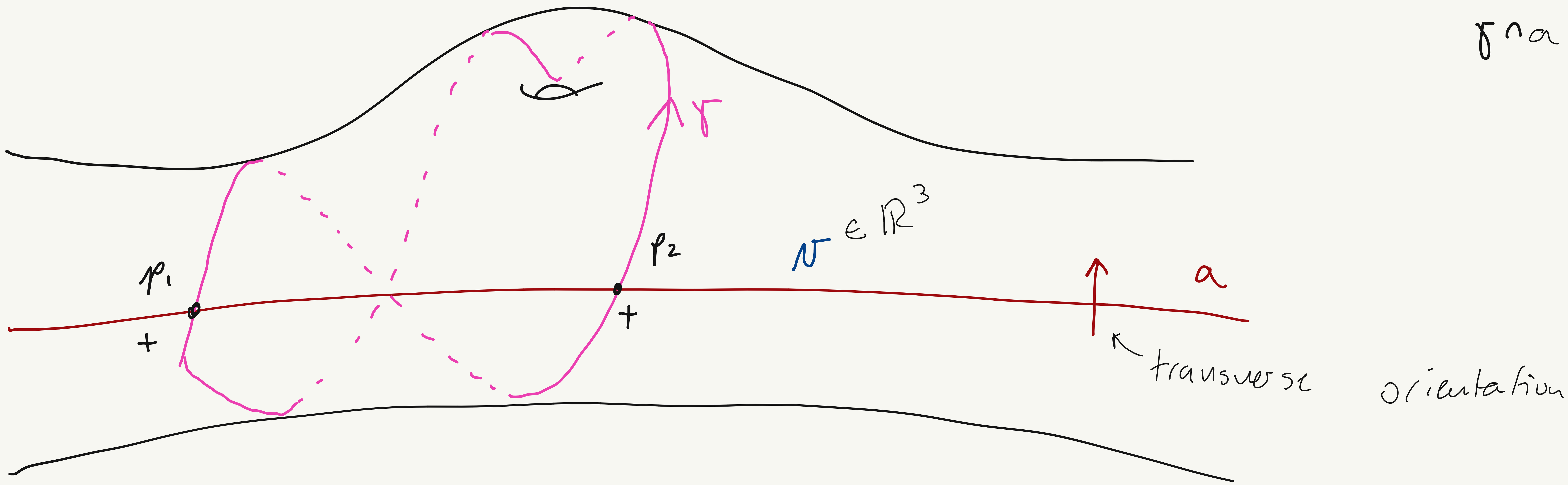
STRIP DEFORMATIONS (Danciger–Guéritaud–Kassel)

For $\Gamma = \pi_1(S) \leq PSL_2\mathbb{R}$, construct a path of hyperbolic structures on S, giving rise to a cocycle $u : \Gamma \to \mathbb{R}^3$.



Infinitesimally :

$$\mu(\gamma) = \sum_{\gamma \cap a} \varepsilon_p \, v$$



$$\gamma \circ x = \gamma x + 2v$$

THE MARGULIS INVARIANT

$$\begin{tikzcd}
\Gamma = \pi_1 S \leq PSL_2 \mathbb{R} \arrow[rr, hook, "\text{irreducible}"] \arrow[dr, hook] & & PSL(4n-1) \\
 & SO(2n,2n-1) \arrow[ur, hook] \arrow[d, hook] & \\
 & SO(2n,2n-1) \ltimes \mathbb{R}^{2n,2n-1} &
\end{tikzcd}$$

$$\Gamma \xrightarrow{\rho_0} SO(2n,2n-1), \qquad \Gamma \xrightarrow{\rho_u} SO(2n,2n-1) \ltimes \mathbb{R}^{2n,2n-1}$$

$$\rho_u(\gamma)\cdot x = \rho_0(\gamma)x + u(\gamma)$$

Assume all elements in Γ are hyperbolic. Then $\rho_0(\gamma)$ has eigenvalues

$\lambda^{4n-2}, \lambda^{4n-4}, \dots, \lambda^{2}, 1, \lambda^{-2}, \dots, \lambda^{-(4n-2)}$, where λ is an eigenvalue of γ.

γ acts on $\mathbb{R}^{2n,2n-1}$ by expanding in $2n-1$ directions, contracting in $2n-1$ directions, and translating along one affine line.

The Margulis invariant is a conjugation-invariant function $\alpha_u : \Gamma \to \mathbb{R}$,

$$\alpha_u(\gamma) = \frac{1}{\ell(\gamma)} \langle x^0(\gamma), u(\gamma) \rangle_{2n,2n-1}$$

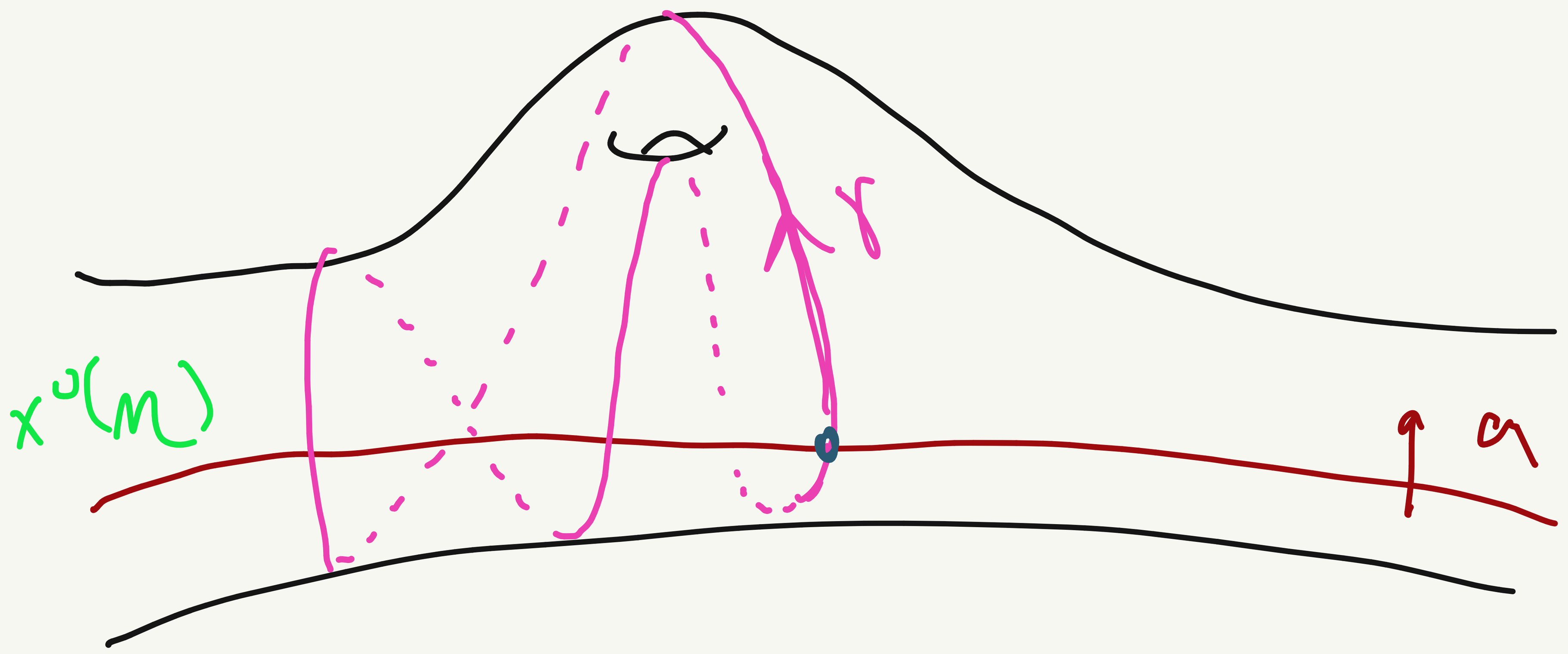
the eigenvalue 1 eigenvector of $\rho_0(\gamma)$, consistently normed and oriented

the translational part of γ

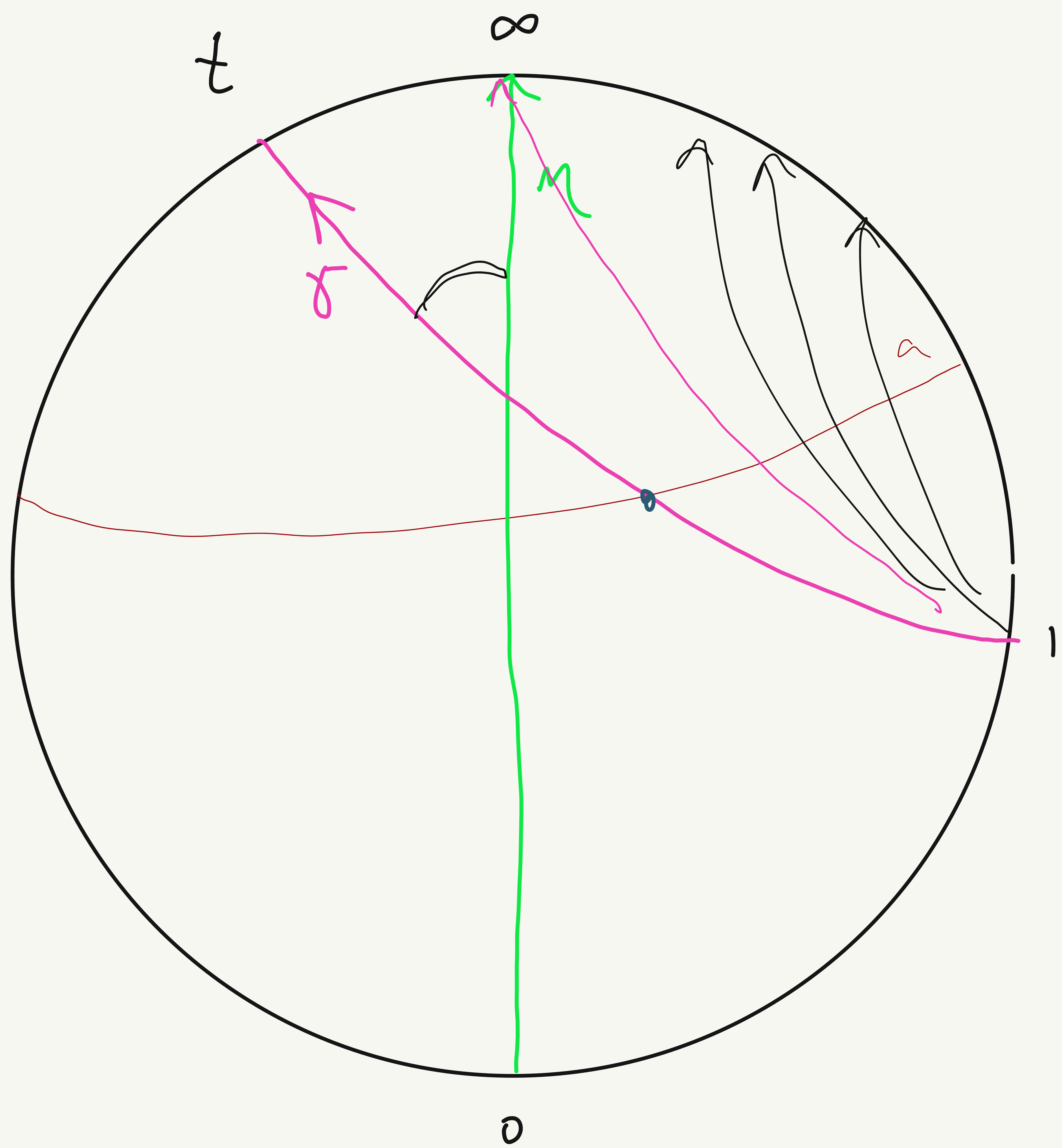
Thm (Margulis' opposite-sign lemma): If there exist $\gamma, \gamma' \in \Gamma$ such that $\alpha_u(\gamma) > 0 > \alpha_u(\gamma')$, ρ_u does not give rise to a proper action.

Thm (Goldman-Labourie-Margulis): ρ_u gives rise to a proper action if and only if the (normed) Margulis invariant is bounded away from 0.

HIGHER STRIP DEFORMATIONS

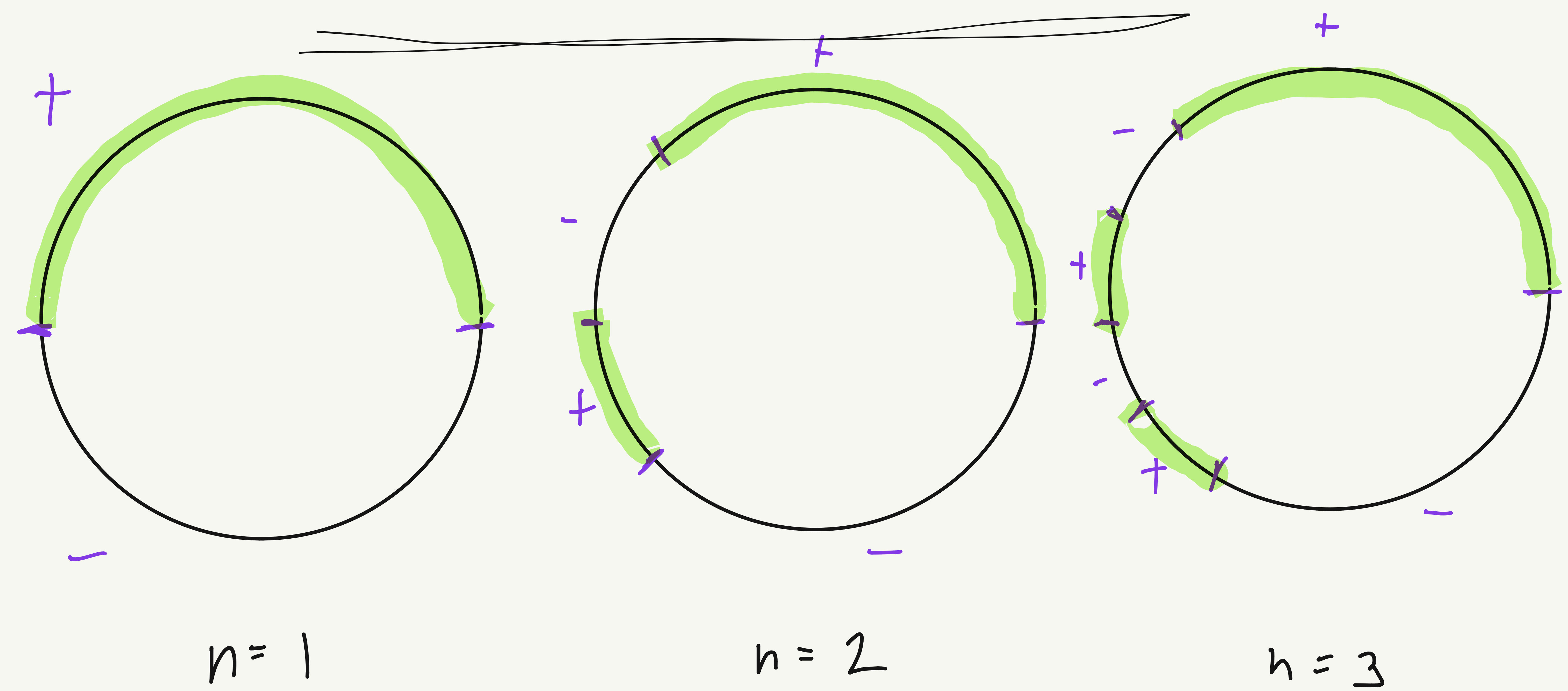


We can compute the contribution to the Margulis invariant for each time γ crosses α: $\langle x^0(\gamma), x^0(\eta) \rangle$ in terms of t:



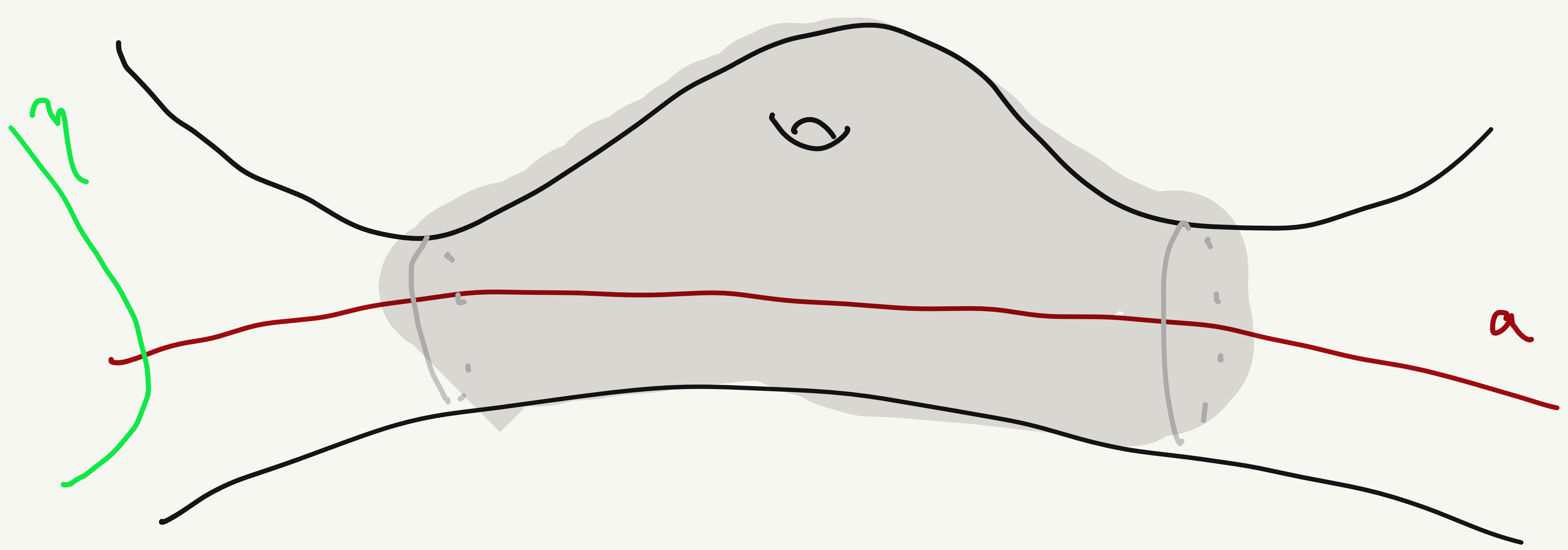
Lemma: This contribution is

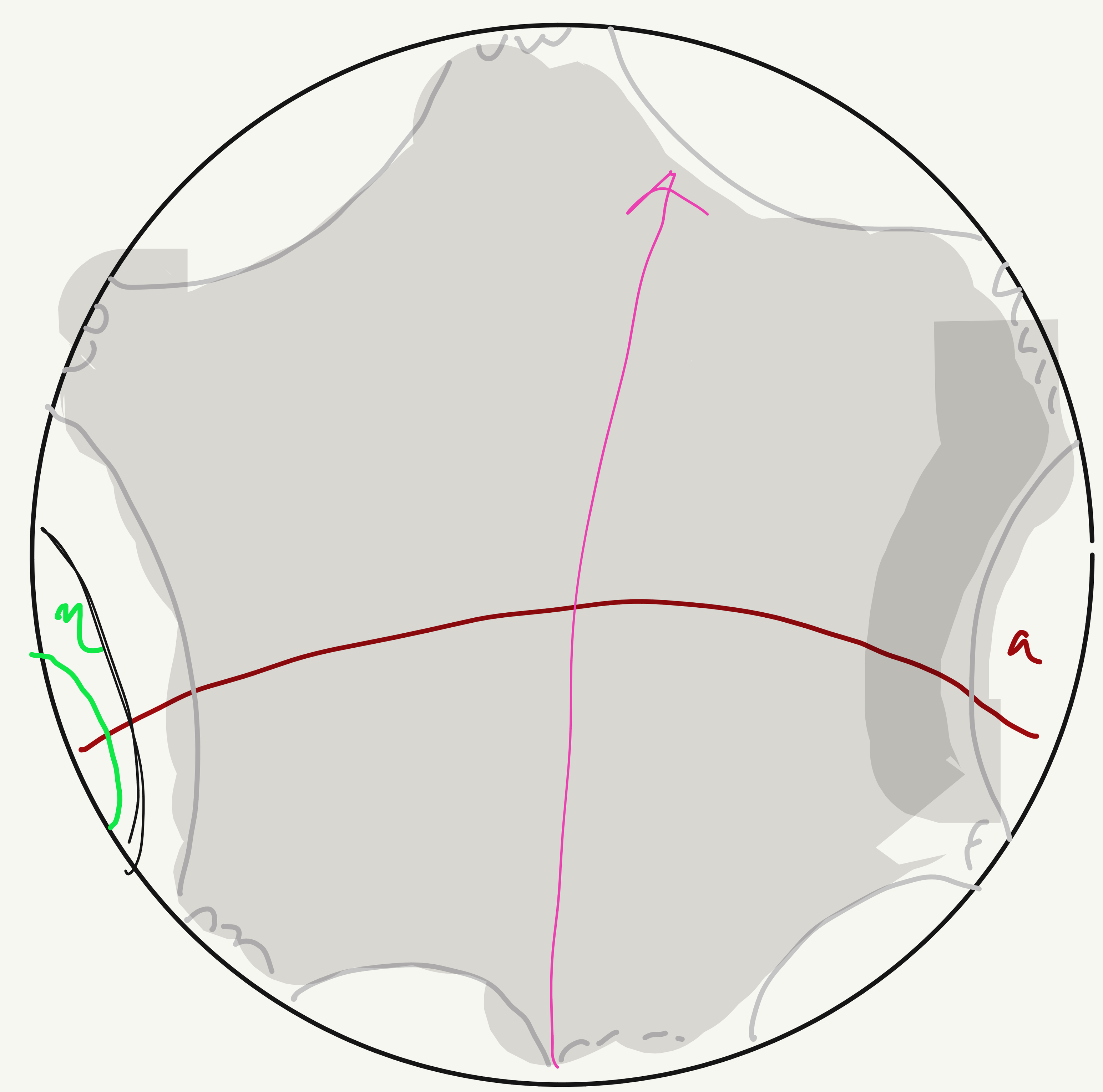
$$\frac{1}{(t-1)^{2n-1}} \sum_{k=0}^{2n-1} \binom{2n-1}{k}^2 t^k$$



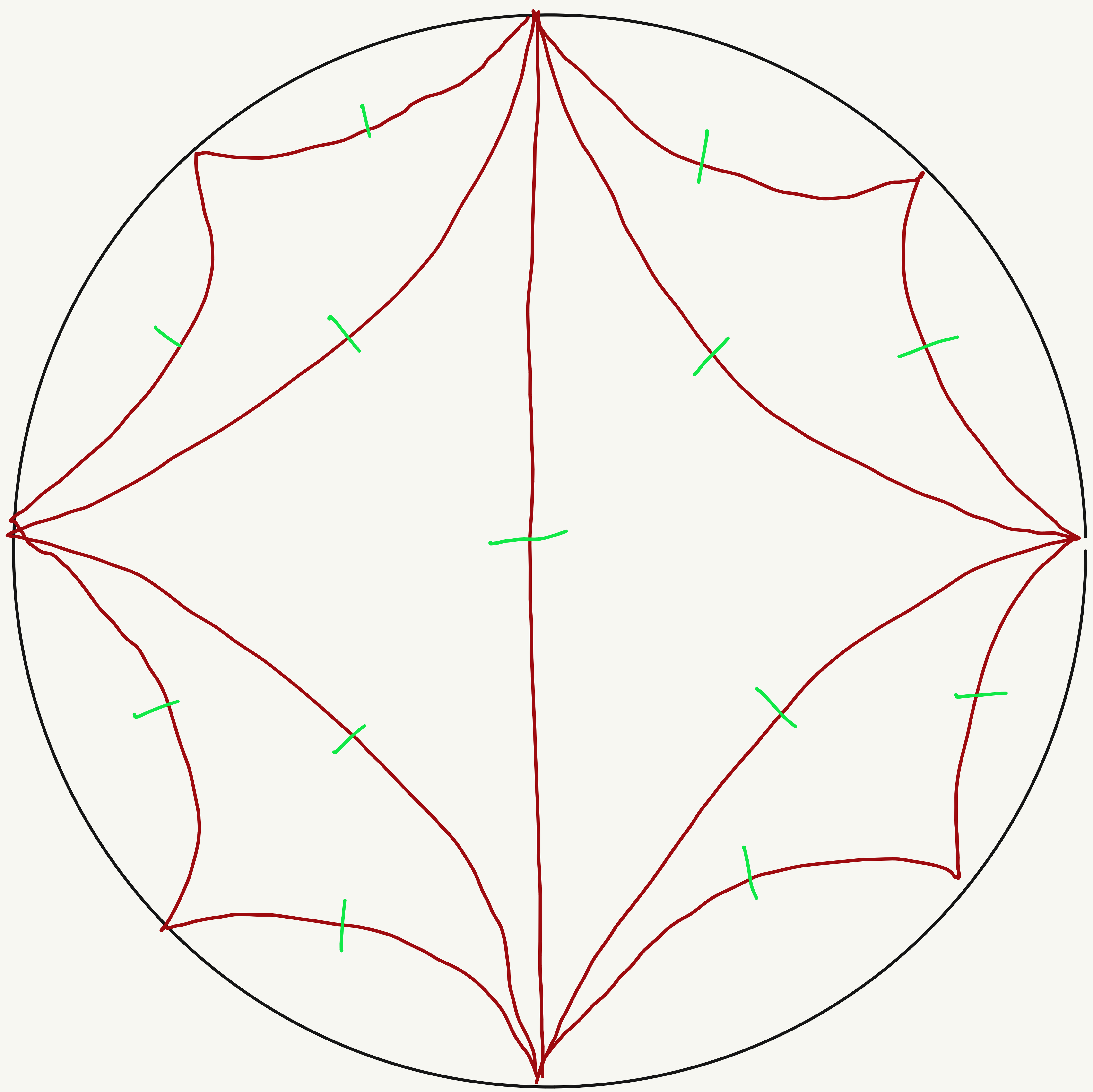
Thm: Every convex cocompact Fuchsian representation into $SO(2n,2n-1)$ admits a deformation acting properly affinely on $\mathbb{R}^{2n,2n-1}$.

Proof: Let $\underset{\sim}{a}$ be a filling system of arcs on S. If we associate to each arc an η whose axis lies outside the convex core, all the Margulis invariants are positive.





We can also study groups with finite-index free subgroups, for example $SL_2\mathbb{Z}$



Crossing two consecutive edges in $\mathbb{R}^{4,3}$ gives a positive Margulis invariant.

Here, we need to be careful about dealing with parabolics, but the Margulis invariant can be defined in this case too (e.g. by Choi-Drumm-Goldman) and a similar properness criterion should hold (upcoming work by Danciger-Guéritand-Kassel)