Computational Geosciences

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Preface

Welcome to the Computational Geosciences resource at the University of Stavanger (UiS). Computational Geosciences is not a new subject. A course in Computational Geology was offered for the first time at the University of South Florida in 1996 (Vacher, 2000). Computational Geosciences makes connections between mathematics, computation and geology. It promotes a mathematical problem-solving disposition (Vacher, 2000).

This resource is designed based on the same principles and with emphasis on problem-solving. However, the access to data and the tools used to visualize data have improved tremendously over the last 20 years. Today, a geologist carrying a mobile device in the field has access to a collection of sensors collecting data in real time (magnetometer, accelerometers, gravity, GPS, etc.), and accurate databases of topography, aerial photos, and satellite imagery. Such information not only supports the geologist in the field, but also allows her to test hypotheses and take decisions. Computation is greatly facilitated by high-level programming languages (e.g. Matlab and Python) that focus on visualizing and solving problems, rather than on coding details. The digital era is here, and to analyze the large number of data associated with it, we need math and computing.

To develop this resource, we put together an interesting group of faculty from the Departments of Energy Resources (IER) and Mechanical and Structural Engineering (IMBM) with expertise in Geographic Information Science (GIS, Lisa Watson, IER), Geophysics (Wiktor Weibull, IER), Structural Geology (Nestor Cardozo, IER), and Fluid Mechanics (Knut Giljarhus, IMBM). Master students from Computational Engineering (Angela Hoch), Offshore Engineering (Adham Amer), and Geosciences (Vania Mansoor) were instrumental. They wrote our scattered code into functions and notebooks and help editing the resource in Overleaf. They also tested the resource.

Python is the programming language of choice. The resource consists of ten chapters covering an introduction to computation in Geosciences and Python (chapter 1), understanding location (chapter 2), orientation and display of geologic features (chapter 3), coordinate systems and vectors (chapter 4), transformations (chapter 5), tensors (chapter 6), stress (chapter 7), strain (chapter 8), elasticity (chapter 9), and the inverse problem (chapter 10). Each chapter describes shortly the basic theory before going directly into applications and problems. Exercises at the end of each chapter are essential to master the material.

Much of the material is based on the book Structural Geology Algorithms: Vectors and Tensors (Allmendinger et al., 2012), and the lab manual: Modern Structural Practice (Allmendinger, 2019). However, we have also included additional GIS and Geophysics topics. The resource is focused on our areas of expertise, but we hope you can use and further develop the material for other geoscience areas.

You can download or subscribe to changes in the resource using our git repository . We hope you enjoy this resource and learn from it, as well as use it for teaching. This will be the measure of our success. We also hope to spark enough interest for users to contribute to the resource with additional material and more chapters in the future. Finally, we are grateful to the Faculty of Science and Technology at the University of Stavanger for sponsoring this project.

References

Allmendinger, R.W., Cardozo, N. and Fisher, D.M. 2012. Structural Geology Algorithms: Vectors and Tensors. Cambridge University Press, 302 p.

Allmendinger, R.W. 2019. Modern Structural Practice: A structural geology laboratory manual for the 21st century. Available at this site.

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Contents

Chapter 1

Computation in Geosciences

1.1 Solving problems by computation

Geology is an interpretive and historical science (Frodeman, 1995). We observe, collect, analyze, and interpret data (what), to tell a story (why). To collect data, we need to take measurements. All measurements have some uncertainty, and therefore uncertainty and error propagation are very important in geosciences, and they are a recurring topic in this resource.

For the last 50 years or more, the methods geoscientists have used to visualize, analyze and interpret data are mostly graphical. For example, in structural geology, students typically learn two types of graphical constructions: orthographic and spherical projections (stereonets) (Ragan, 2009). Although these methods are great to visualize and solve geometrical problems in three-dimensions, they are not amenable to computation, and therefore applying these methods to large datasets with thousands of entries is impractical. Plane and spherical trigonometry allow deriving formulas (e.g. apparent dip formula) for computation (Ragan, 2009). However, these formulas give little insight about the problems. They are just formulas associated with complex geometric constructions, which bear no relation to each other, and which are difficult to combine to solve more complicated problems.

It turns out that many of the most interesting problems in geosciences can be described and solved using linear algebra, and more specifically vectors and tensors (Allmendinger et al., 2012). Linear algebra also happens to be the language of data and computation. The main purpose of this resource is to show how to solve problems in geosciences using computation. There are several advantages of following this approach. It will enhance your mathematical and computational skills, as well as promote your geological-mathematical problem solving disposition. In today's digital age, these skills are very useful.

1.2 Why Python?

The choice of programming language is important. While computer languages such as C or C++ are ideal to work with very large datasets and computer-intensive operations, they involve a steep learning curve associated with their syntax, compilation, and execution (Jacobs et al., 2016). These coding details have little to do with the problem-solving approach of this resource. Interpretive languages such as Python, R or Matlab are a better choice because of their simpler syntax, and the interpretation and execution of commands as they are called (no need for compilation). In addition, these languages have access to an integrated development environment (IDE) that facilitates writing and debugging programs, and to many standard libraries that perform advanced tasks such as matrix operations and data visualization. Thus, Python, R or Matlab are "scientific packages" rather than just programming languages.

In this resource, the language of choice is Python. Besides the reasons above, Python has the following advantages:

- Python can be learned quickly. It typically involves less code than other languages and its syntax is easier to read.
- Python comes with robust standard libraries for arrays and mathematical functions (NumPy), visualization (PyPlot), and scientific computing (SciPy).
- Python is one of the most popular programming languages, with a large base of developers and users. It is used by every major technology company and it is almost a skill you must have in your CV to land a job as a geoscientist.
- Because of its large developers base, Python has access to a large amount of additional libraries, including several libraries for geosciences. We make use of some of these libraries in this resource.

- Python can be installed easily through a single distribution that includes all the standard libraries and provides access to additional libraries (see next section).

- Last but not least, Python is free and open source. This is probably why Python is more popular than its commercial counterpart Matlab.

1.3 Installing Python

We recommend installing Python using the free Anaconda distribution. This distribution includes Python as well as many other useful libraries, including Jupyter, which is the system we use to write the notebooks in this resource. Anaconda can be easily installed on any major operating system, including Windows, macOS, or Linux.

The installation process is quite straightforward. From the Anaconda distribution page, go to the Download section. The website will recognize your operating system and present you with two possible installers, one for Python 3 and another for Python 2. We recommend you install the Python 3 version. Download the installer. Windows and macOS users just need to run the installer and follow the steps to install Anaconda. Linux users need to type a set of commands in a terminal window. Further instructions can be found in the online Anaconda documentation, installation section.

1.4 A first introduction to Python

In this section, we use our first Jupyter notebook to learn the basics of Python. Download the notebook ch1.ipynb from the resource git repository. Open Anaconda and then launch Jupyter notebook. This will open a browser with a list of files and folders in your home directory. Navigate to the notebook ch1.ipynb and open it. Alternatively, follow the notebook in the sections below. Surprisingly, few lines of code are required to introduce key topics such as conditionals, loops, functions, array mathematics, and plotting. This shows the power of Python.

1.4.1 Basics

A notebook is divided into computational units called *cells*. Cells can contain text such as this one or Python code. Below is a cell with some typical Python statements. Try changing the variables and re-run the cell. To run a cell, either click the *Run* button, or type *Ctrl+Enter*.

```python
a = 2
b = 9.0
c = a + b
print('The sum is: ', c)

# This is just a comment

name = 'Donald'
print('Hello, my name is', name)
```

Output:
The sum is: 11.0
Hello, my name is Donald

There are some other useful shortcuts you should know. To run a cell and move to the next cell, type *Shift+Enter*. To run a cell and insert a new cell below, type *Alt+Enter*. You can use the arrow keys to move quickly between cells. To run all the cells of a notebook, choose the *Cell* → *Run all* menu.

1.4.2 Conditionals

A conditional is used to perform different operations depending on a conditional statement. In Python, this is expressed in the following way:

```python
a = 3
b = 5
if a > b:
    print('a is bigger than b')
elif a < b:
    print('a is smaller than b')
else:
    print('a is equal to b')
```

Output:
a is smaller than b

Try changing the values of *a* and *b* to see how the output changes. Also, note that Python cares about white spaces, so there must be a tab indent or 4 spaces for each operation in the if statement. You can also use the boolean operators *and*, *or*, and *not* in the conditional statement:

```
age = 30
if age > 18 and age < 34:
    print('You are a young adult')

if age < 18 or age > 80:
    print('You are not allowed to drive a car')
```

Output:
You are a young adult

1.4.3 Loops

A loop is used to execute a group of statements multiple times. For instance, to print all numbers from 1 to 10 divisible by 3, we can use a *for* loop together with an *if* statement, and the modulus operator %:

```
print('Number divisible by three:')
for i in range(1, 11):
    if i % 3 == 0:
        print(i)
```

Output:
Numbers divisible by three:
3
6
9

range is a Python function that iterates from the given first number up to the second number (but not including it). If we only give one number, the iteration will go from zero up to (but not including) the given number. We will give more examples of *for* loops later in this notebook.

1.4.4 Functions and modules

If we have written a useful piece of code, we often want to use it again without copying and pasting the code multiple times. To do this, we use functions and modules. For instance, if we want to convert an angle from degrees to radians, we can use the following formula:

$$\alpha_{\text{radians}} = \alpha_{\text{degrees}} \frac{\pi}{180} \tag{1.1}$$

To put this into a callable function, we use the *def* keyword:

```
def deg_to_rad(angle_degrees):
    pi = 3.141592
    return angle_degrees*pi/180.0

angle_degrees = 45.0
print('Radians', deg_to_rad(angle_degrees))
```

Output:
Radians 0.785398

We can also include code from other places. This is useful to make your own library of functions that you can then use in many different notebooks. This is basically the modus operandi of this resource. We will write functions that solve common problems in geosciences. Using a text editor, create a file called *mylib.py* and put it in the same folder as your notebook. In the file, write a function to convert from radians to degrees:

```
def rad_to_deg(angle_radians):
    pi = 3.141592
    return angle_radians*180/pi
```

We can then import in the notebook the code from the file and use it like this:

```
try:
    import mylib
    angle_radians = 0.785398
```

```
    print('Degrees', mylib.rad_to_deg(angle_radians))

except ModuleNotFoundError:
    print('Create a file called mylib.py')
```

Output:
Degrees 45.0

Note: If you make a change in *mylib.py*, the changes will not be immediately available in the notebook and it needs to be restarted . To circumvent this, we can use the following commands to always reload imported modules:

```
%load_ext autoreload
%autoreload
```

1.4.5 Mathematics

To use Python as an environment for numerical mathematics, it is useful to use the NumPy library for arrays and matrices, and the Matplotlib for plotting. See the links in the *Help* menu for more information on these libraries. The following two lines import these libraries. The third line makes sure the plots are rendered in the notebook:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

To define an array, we use the NumPy *array* function:

```
a = np.array( [1, 2, 3, 4] )
print(a)
```

Output:
[1 2 3 4]

To access an array element, we use brackets with the index of the element. A very important difference compared to Matlab is that in Python the first element has index zero (like most other programming languages). We can also use negative indices to access values starting from the end of the array.

```
print(a[0], a[2])
print(a[-1])
```

Output:
1 3
4

Slicing is a very useful feature to extract subarrays. For instance:

```
print(a[2:])
print(a[1:3])
```

Output:
[3 4]
[2 3]

Matrices are defined as multi-dimensional arrays:

```
a_matrix = np.array( [[1, 2, 3],
                      [4, 5, 6],
                      [7, 8, 9]] )
b_matrix = np.array( [[2, 4],
                      [3, 5],
                      [5, 7]] )
print(a_matrix)
print(b_matrix)
```

Output:
[[1 2 3]
[4 5 6]
[7 8 9]]
[[2 4]
[3 5]
[5 7]]

We can get the number of rows and columns of the matrix from the *shape* variable:

```
nrow, ncol = b_matrix.shape
print('b has {} rows and {} columns'.format(nrow, ncol))
```

Output:
b has 3 rows and 2 columns.

Let us make a function to multiply two matrices. Consider a $n \times m$ matrix $\mathbf{A}$ and a $m \times p$ matrix $\mathbf{B}$. The formula to multiply these matrices can be written as:

$$\mathbf{C} = \mathbf{AB} = \sum_{k=1}^{m} A_{ik}B_{kj} \tag{1.2}$$

for $i = 1,\ldots,n$ and $j = 1,\ldots,p$. Here $\mathbf{C}$ will be a $n \times p$ matrix. To implement this formula, we need to use a triple-nested loop, as shown in the function below:

```
def matrix_multiply(A,B):
   n, m = A.shape
   nrow_B, p = B.shape

   # Check that matrices are conformable
   if not nrow_B == m:
         print('Error, the number of columns in A must be equal to the number of rows in B!')
         return -1
    # Initialize C using the numpy zeros function
    C = np.zeros((n,p))
    for i in range(n):
        for j in range (p):
            for k in range (m):
                C [i,j] = C[i,j] + A[i,k]*B[k,j]
    return C

print(matrix_multiply(a_matrix, b_matrix))
```

Output:
[[23. 35.]
[53. 83.]
[83. 131.]]

Verify by hand calculation that the above result is correct. Remember, the element in the first row and first column of **C** is equal to the sum of the product of the elements in the first row of **A** times the elements in the first column of **B**, and so on. What happens if you try the multiplication **BA**? Try it.

Note that although the above function is elegant, it is not very efficient. The NumPy library contains super-optimized code for common operations such as matrix multiplication. The NumPy *dot* function can be used for matrix multiplication. Let's repeat the matrix multiplication above using the *dot* function:

```
C = np.dot(a_matrix , b_matrix)
print(C)
```

Output:
[[23 35]
[53 83]
[83 131]]

When working with large matrices, there is a significant impact on the runtime. To illustrate this, let's generate two 100 x 100 matrices and time how long it takes to multiply them. The *%%timeit* command will run the cell a number of times and output the average time spent per run. The NumPy *random.rand* function generates the arrays and fill them with random numbers.

```
%%timeit
N = 100
A = np.random.rand(N,N)
B = np.random.rand(N,N)
C = matrix_multiply(A,B)
```

Output:
428 ms ± 3.02 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)

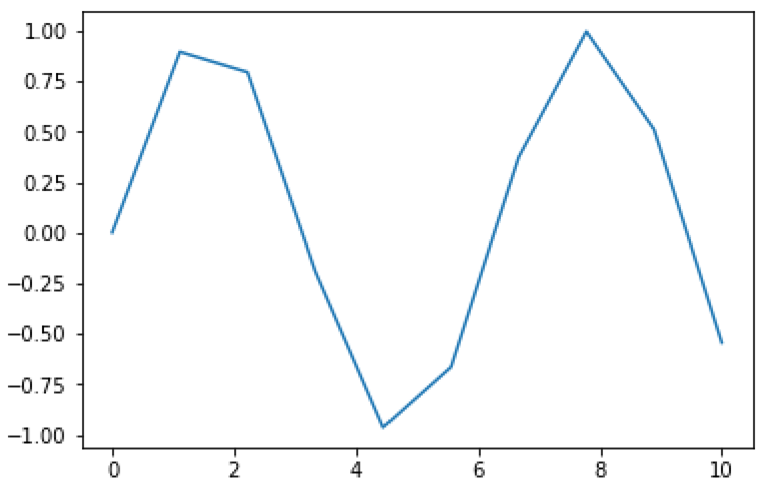
Try changing the multiplication function from *matrix_multiply* to *np.dot* and note the difference in runtime. On a standard computer, our *matrix_multiply* function uses ≈ 500 milliseconds, while NumPy *dot* function uses ≈ 200 microseconds. The NumPy *dot* function is a staggering 2500 times faster!

1.4.6 Plotting

Arrays can be easily plotted using the Matplotlib *plot* command. Below we plot the sinusoidal function. We use the NumPy *linspace* function to generate an array with equally spaced values between the start and end point, and the NumPy *sin* function to take the sine of the array. With a low number of points, the curve is actually jagged. Increase the number of points n in the *linspace* command to get a smoother curve. Try values of $n = 100$, 1000, and 10000.

```
# The linspace command gives us an equally spaced array
# The syntax is:
# linspace(start_point, end_point, number_of_points)
n = 10
x = np.linspace(0, 10, n)
y = np.sin(x)
plt.plot(x, y)
```

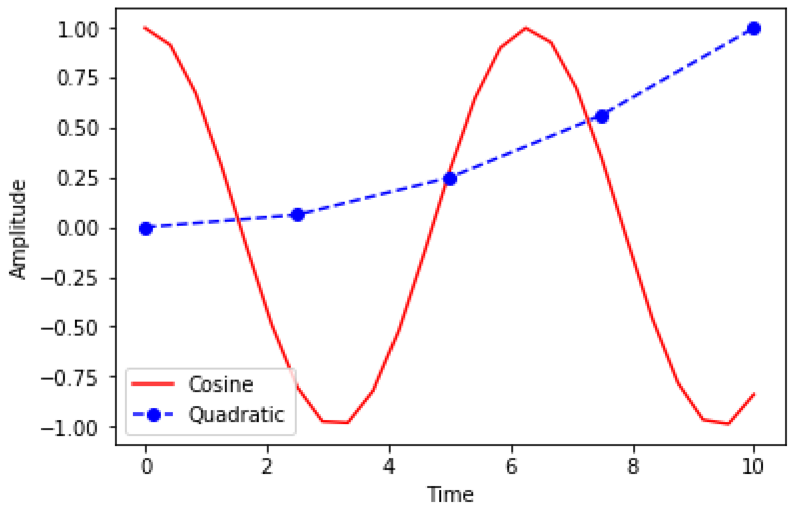
Output:



We end with a slightly more advanced plot, showing how to change line style and markers, and add axes labels and a legend. The NumPy *cos* function takes the cosine of the array, and *xlabel*, *ylabel* and *legend* are all Matplotlib commands to add labels to the axes and a legend to the graph.

```
n = 25
x = np.linspace(0, 10,25)
y = np.cos(x)
plt.plot(x, y, 'r')
x = np.linspace(0,10,5)
y = 0.01*x**2
plt.plot(x,y,'bo--')
plt.xlabel('Time')
plt.ylabel('Amplitude')
plt.legend(['Cosine', 'Quadratic'])
```

Output:



1.5 Exercises

1. Write a program that prints each number from 1 to 20 on a new line. For each multiple of 3, print "Fizz" instead of the number. For each multiple of 5, print "Buzz" instead of the number. For numbers which are multiples of both 3 and 5, print "FizzBuzz" instead of the number. The correct answer is: 1 2 Fizz 4 Buzz Fizz 7 8 Fizz Buzz 11 Fizz 13 14 FizzBuzz 16 17 Fizz 19 Buzz.

2. Write a more complete function to convert an angle from degrees to radians or from radians to degrees. The function should accept two inputs: the angle, and a flag to tell the function whether the angle should be converted from degrees to radians (flag $= 1$) or from radians

to degrees (flag = 2).

3. Given two 3×3 matrices $\mathbf{A}$ = [[1, 2, 3], [4, 5, 6], [7, 8, 9]] and $\mathbf{B}$ = [[5, 7, 2], [3, 5, 1], [2, 4, 3]], compute:

   (a) the sum of the matrices ($\mathbf{A} + \mathbf{B}$),

   (b) The difference of the matrices ($\mathbf{A}$ - $\mathbf{B}$),

   (c) The product of the matrices ($\mathbf{AB}$),

   (d) The square root of matrix $\mathbf{A}$,

   (e) The sum of all elements of matrix $\mathbf{B}$,

   (f) The column sum of matrix $\mathbf{A}$,

   (g) The row sum of matrix $\mathbf{A}$,

   (h) The transpose of matrix $\mathbf{A}$ ($\mathbf{A}^T$),

   (i) The product $\mathbf{AA}^T$. What is this product equal to?

   Hint: Look at the functions *add, subtract, dot, sqrt, sum* and *transpose* in the NumPy library.

4. The apparent dip α of a plane is given by the following equation:

$$\tan\alpha = \tan\delta\sin\beta \tag{1.3}$$

   where δ is the true dip of the plane, and β is the structural bearing (Fig. 3.1b). We will discuss this equation in chapter 3.

   (a) Make a function to compute the apparent dip α from the true dip δ and structural bearing β.

   (b) Use this function in a notebook to make a graph of apparent dip α (0 to 90°, vertical axis) versus the structural bearing β (0 to 90°, horizontal axis), for values of true dip δ of 10, 20, 30, 40, 50, 60, 70, and 80°.

   The graph should look like the figure below:

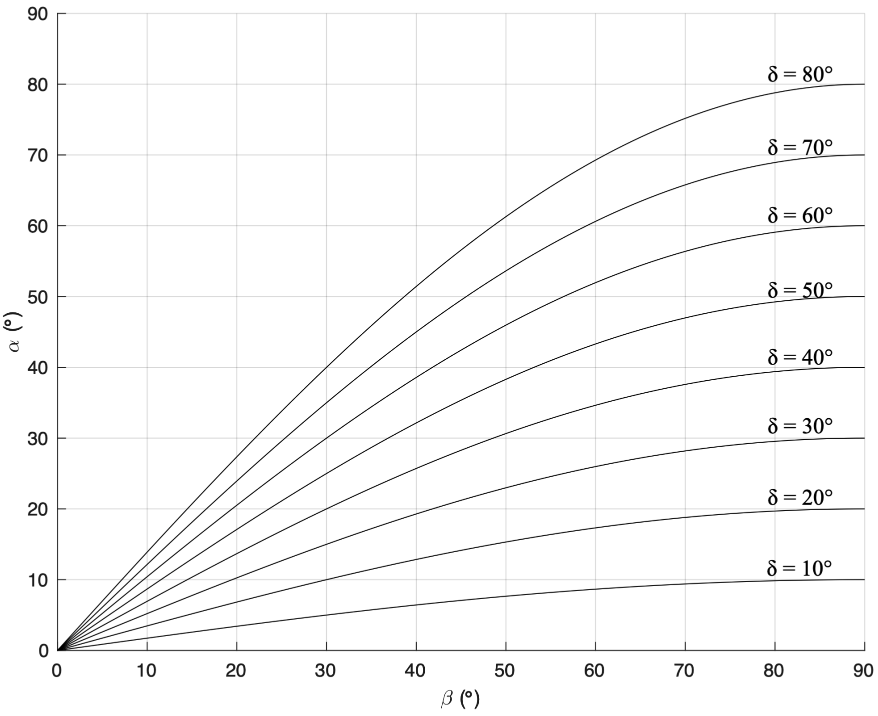


Figure 1.1: Apparent dip α as function of section bearing β and true dip δ

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Chapter 2

Understanding location

2.1 Locations

2.2 Geodesy basics

2.3 Projections

2.4 Reference systems and datums

2.5 Coordinate conversion and transformation

2.6 Exercises

Chapter 3

Geologic features

3.1 Primitive objects: Lines and planes

The fundamental geometric features of geology are lines (e.g. a lineation or a fold axis) and planes (e.g. bedding or a foliation). A *line* is the element generated by a moving point. It can be straight or curved. We will treat straight lines here. A *plane* is a flat surface; a line joining two points on the plane lies wholly on its surface, and two intersecting lines on the plane define the plane. This is equivalent to say that three non-collinear points on the plane define the plane (this is the principle of the well-known three-point problem). Obviously, linear features can be curved (e.g. the intersection of bedding with irregular topography), and surfaces can be non-planar (e.g. bedding on a fold). However, even these more complex cases can be expressed as a collection of lines and planes.

3.2 Lines and planes orientations

Two important properties of lines and planes are location (chapter 2) and orientation (this chapter). Lines and planes orientations are measured with respect to the geographic north and the angle downward or upward from the horizontal. We refer to this coordinate system as the spherical coordinate system, and the measurements defining the lines or planes orientations as the spherical coordinates.

3.2.1 Planes: Strike and dip

A plane orientation can be defined by the angle a horizontal line on the plane makes with the geographic north, known as the *strike* and the maximum angle measured downward from the horizontal to the plane, known as the *dip* (Fig 3.1a). The strike is measured as an azimuth, an angle between 0 and 360°(0 = north, 90 = east, 180 = south, 270 = west). The dip is an angle between 0 (horizontal plane) and 90°(vertical plane). The projection of the dip onto the horizontal is known as the *dip direction* and is always 90°from the strike. However, is the dip direction plus or minus 90°the given strike? Which end of the strike line should we use? To avoid ambiguities, we will use a format known as the *right hand rule* (RHR). In the RHR format, one gives the strike such that the dip direction is always the strike plus 90°, i.e. the dip direction is to the right of the strike (Fig. 3.1a).

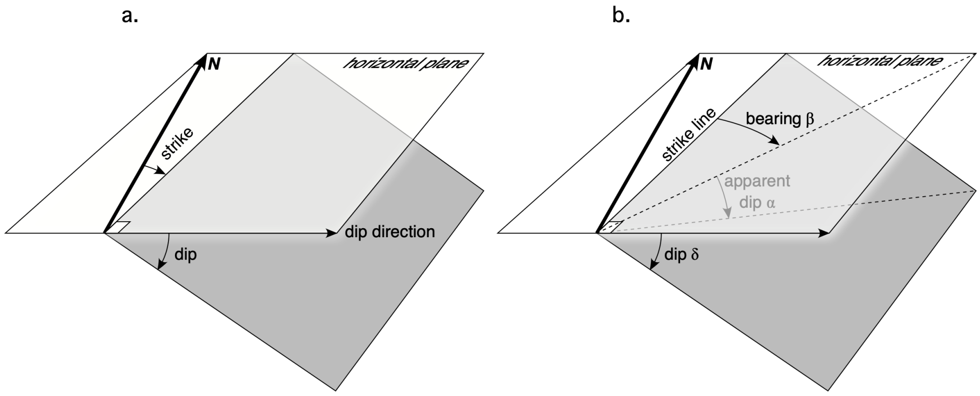


Figure 3.1: **a.** Strike and dip of a plane, **b.** Apparent dip of a plane. Modified from Allmendinger et al. (2012)

It is only along the dip direction that the true dip can be determined, any other direction will give a lower apparent dip (Fig. 3.1b). The relation between the dip (δ) and the apparent dip (α) is given by the equation:

$$\tan\alpha = \tan\delta \sin\beta \tag{3.1}$$

where β is the angle between the strike (horizontal) line on the plane and the vertical section on which the apparent dip is measured (Fig. 3.1b). This is also Eq. 1.3, which we plotted in problem 4 of chapter 1 (Fig. 1.1). You can quickly verify that it works by setting $\beta = 0$ (a cross section parallel to strike) which gives $\alpha = 0$ (since sin(0) is 0), and $\beta = 90°$(a cross section perpendicular to strike) which gives $\alpha = \delta$ (since sin(90) is 1). This leads to a very important observation: *Any plane on a vertical section parallel to strike looks horizontal (even if it's dipping), and the true dip of the plane can only be observed on a vertical section perpendicular to strike.* This is why we should always visualize planes (bedding, faults, etc.) on cross sections perpendicular to strike.

3.2.2 Lines: Trend and plunge or rake

The orientation of a line is specified by the azimuth of the horizontal projection of the line, or *trend*, and the vertical angle measured downward from the horizontal to the line, or *plunge* (Fig. 3.2a). The plunge has a range between -90 and 90°. Positive plunge indicates lines pointing downwards, and negative plunge lines pointing upwards. To measure the trend and plunge one must determine the vertical plane containing the line. This is quite difficult and often results in errors (section 3.2.5). For this reason, and if the line is on a plane, it is more accurate (and convenient) to measure the angle on the plane between the strike line and the line. This angle is known as the rake or pitch (Fig. 3.2b). To avoid any confusion, the rake should be always measured from the given strike and thus it varies between 0 and 180°.

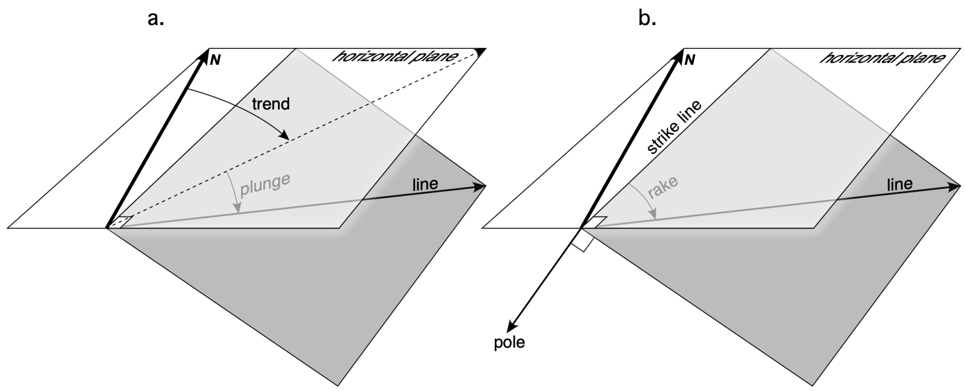


Figure 3.2: **a.** Trend and plunge of a line, **b**. Rake of a line and pole to a plane. Modified from Allmendinger et al. (2012).

3.2.3 The pole to the plane

Any plane can be uniquely represented by its downward normal. This line is known as the pole to the plane (Fig. 3.2b). If we use the RHR format, the orientation of the pole is given by:

$$\begin{aligned} trend\,of\,pole &= strike\,of\,plane - 90^\circ \\ plunge\,of\,pole &= 90^\circ - dip\,of\,plane \end{aligned} \tag{3.2}$$

The pole facilitates analyzing planes graphically and by computation. Our first Python function *Pole* below computes the pole of a plane ($k = 1$) or the plane from its pole ($k = 0$). It is followed by the helper function *ZeroTwoPi* which makes sure azimuths are always between 0 and 360°. Notice that angles (*trd* and *plg*) should be entered in radians, and the plane must follow the RHR format.

```python
import math
from ZeroTwoPi import ZeroTwoPi as ZeroTwoPi

def Pole(trd, plg, k):
    '''
    Pole returns the pole to a plane or the plane from a pole

    If k = 0, Pole returns the strike (trd1) and dip (plg1)
    of a plane, given the trend (trd) and plunge (plg)
    of its pole.

    If k = 1, Pole returns the trend (trd1) and plunge (plg1)
    of a pole, given the strike (trd) and dip (plg)
    of its plane.

    NOTE: Input/Output angles are in radians.
    Input/Output strike and dip follow the RHR format

    Pole uses function ZeroTwoPi
    '''
    # Some constants
    east = math.pi/2

    # Eq. 3.2
    # Calculate plane given its pole
    if k == 0:
        if plg >= 0:
```

```
            plg1 = east - plg
            trd1 = ZeroTwoPi(trd + east)
        else: # Unusual case of pole pointing upwards
            plg1 = east + plg
            trd1 = ZeroTwoPi(trd - east)
    # Else calculate pole given its plane
    elif k == 1:
      plg1 = east - plg;
      trd1 = ZeroTwoPi (trd - east)

    return trd1, plg1
```

```
import math

def ZeroTwoPi(a):
    '''
    This function makes sure input azimuth (a)
    is within 0  to 2*pi (b)

    NOTE: Azimuths a and b are input/output in radians

    Python function translated from the Matlab function
    ZeroTwoPi in Allmendinger et al. (2012)
    '''
    b=a
    twopi = 2*math.pi
    if b < 0:
        b += twopi
    elif b >= twopi:
        b -= twopi

    return b
```

3.2.4 Instruments used in the field

Traditionally, geologists use a geological compass/clinometer to measure the orientation of planes and lines in the field. Figure 3.3 shows four of the most common compasses used in geology: the Silva compass (Fig. 3.3a), the Brunton compass (Fig. 3.3b), the Krantz compass (Fig. 3.3c, a less expensive variant of the Freiberg compass), and the Brunton Geo compass (Fig. 3.3d). All these compasses have a magnetic needle that points to the magnetic north (N or white end of the needle), a horizontal level, and a clinometer (an instrument to measure vertical angles). The Silva compass has an azimuth scale that can be rotated to follow the magnetic needle, while in the other three compasses the azimuth scale is fixed. This is why east-west (E-W) are in the right place in the Silva compass, while they are flipped in the other three compasses (Fig. 3.3a-d).

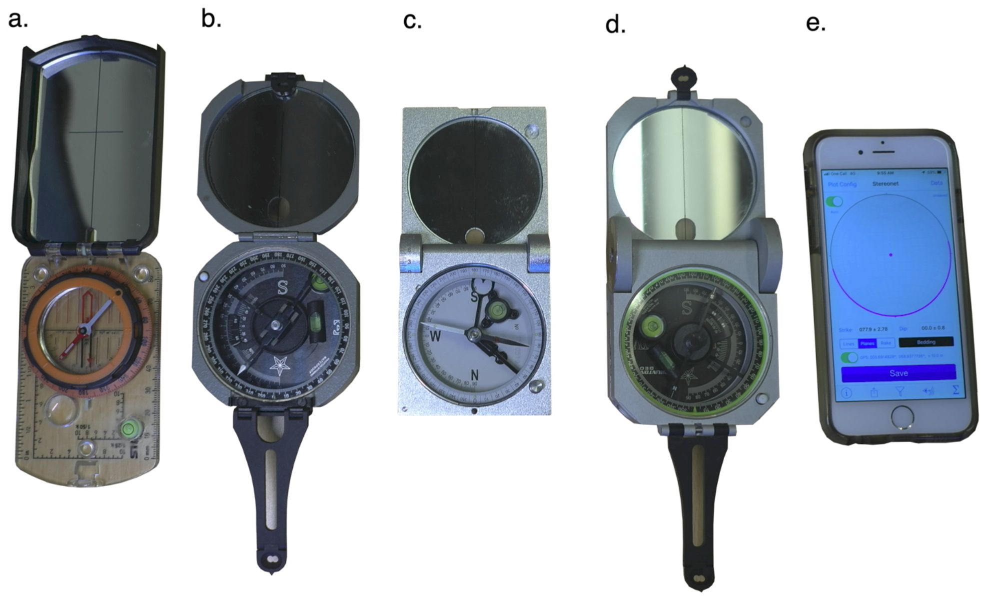


Figure 3.3: **a.** Silva, **b.** Brunton, **c.** Krantz, **d.** Brunton Geo, and **e.** Smartphone with Stereonet Mobile.

The Silva and Brunton compasses are designed to measure strike and dip through two measurements, while the Krantz compass measures dip direction and dip at once. The Brunton Geo compass can work either as a Brunton or Krantz compass, and it has higher precision than the other three compasses (it is also the most expensive). The use of these compasses is well explained in field geology books such as Compton (1985) and Coe (2010). For illustration, Figure 3.4 shows how strike and dip are measured with the Brunton compass (the same principles apply to the Silva compass). Notice that in this measurement, it is crucial to determine when the compass is horizontal (Fig. 3.4a). We will see that this can be a source of error (section 3.2.5).

These days, digital devices in the form of smartphone programs or apps (Fig. 3.3e) are slowly replacing the analog compasses. Smartphones contain instruments such as accelerometers, gyroscopes, and magnetometers, which enable apps such as Stereonet Mobile (Richard Allmendinger) or Fieldmove Clino (Petroleum Experts) to determine the exact orientation of the device in space. Measuring a plane or a line just requires placing the phone on the

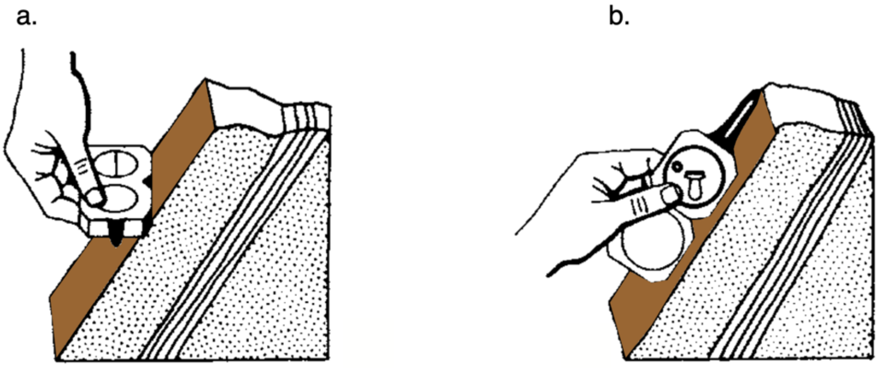


Figure 3.4: Measuring the **a.** strike and **b.** dip of a plane. Modified from Compton (1985).

plane or along the line. Thus, one can capture a large number of measurements quickly. However, smartphones are very sensitive to nearby magnetic fields and one can easily get spurious results (Novakova and Plavlis, 2017). Smartphones also have access to accurate geographic location (GPS, cell and wireless networks) as well as satellite imagery and raster data such as elevation. They can greatly facilitate mapping in the field.

3.2.5 Uncertainties in orientations

Geological planes and lines are irregular and therefore it is difficult to take exact measurements of them. Every plane or line measurement has an uncertainty (an error). There are different ways to try to reduce this error, either by placing a smooth planar object (e.g. a field notebook) on the plane or along the line, or by sighting the plane or line from the distance (Compton, 1985). Figure 3.5 illustrates the error associated to the strike and dip measurement of a plane. If the compass is not exactly horizontal then a direction other than the strike will be measured. The departure of the compass from the horizontal or operator error (ε_o) will give a strike error (ε_s).

From the three right-triangles and their corresponding equations in Figure 3.5, and by substituting the first two equations for w and l into the third equation for ε_s, one gets the following relation (Woodcock, 1976):

$$\sin \varepsilon_s = \frac{\tan \varepsilon_o}{\tan \delta} \tag{3.3}$$

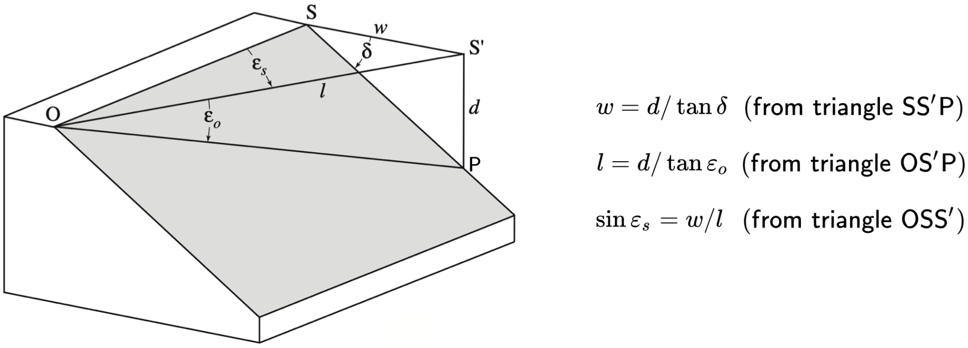


Figure 3.5: Geometrical relations for estimating the strike error ε_s from the operator error (or departure of the compass from the horizontal) ε_o. Modified from Ragan (2009).

where δ is the dip angle of the plane. This equation is plotted in Figure 3.6 for dip angles δ of 0 to 40°and operator errors ε_o of 1 to 5°. It is clear that the strike error ε_s increases with decreasing dip. For a gentle 5°dipping plane, an operator error ε_o of 2°(a compass just 2°off the horizontal) results in a strike error ε_s of about 24°! Thus, one should always be suspicious about the accuracy of strike and dip measurements, particularly if they are from gently dipping planes.

For line measurements, this situation is not better. When measuring the orientation of a line, it is common practice to align the compass in the direction of the horizontal projection of the line, which, as anyone who has tried this in the field knows, it is quite difficult. There will be an operator error and the measured trend β' will differ from the true trend β (Fig. 3.7a). The trend error ε_t ($|\beta' - \beta|$)in terms of the angle on the plane ε_o which the measured line makes with the true line, is given by the following equations (Woodcock, 1976):

$$\begin{aligned}\tan\varepsilon_t &= \frac{[\tan(r+\varepsilon_o) - \tan(r)]\cos\delta}{1 + [\tan(r+\varepsilon_o)\tan(r)]\cos^2\delta} \quad if\beta' > \beta \\ \tan\varepsilon_t &= \frac{[\tan(r) - \tan(r-\varepsilon_o)]\cos\delta}{1 + [\tan(r)\tan(r-\varepsilon_o)]\cos^2\delta} \quad if\beta' < \beta\end{aligned} \tag{3.4}$$

where r is the rake of the line, and δ is the dip of the plane (Fig. 3.7a).

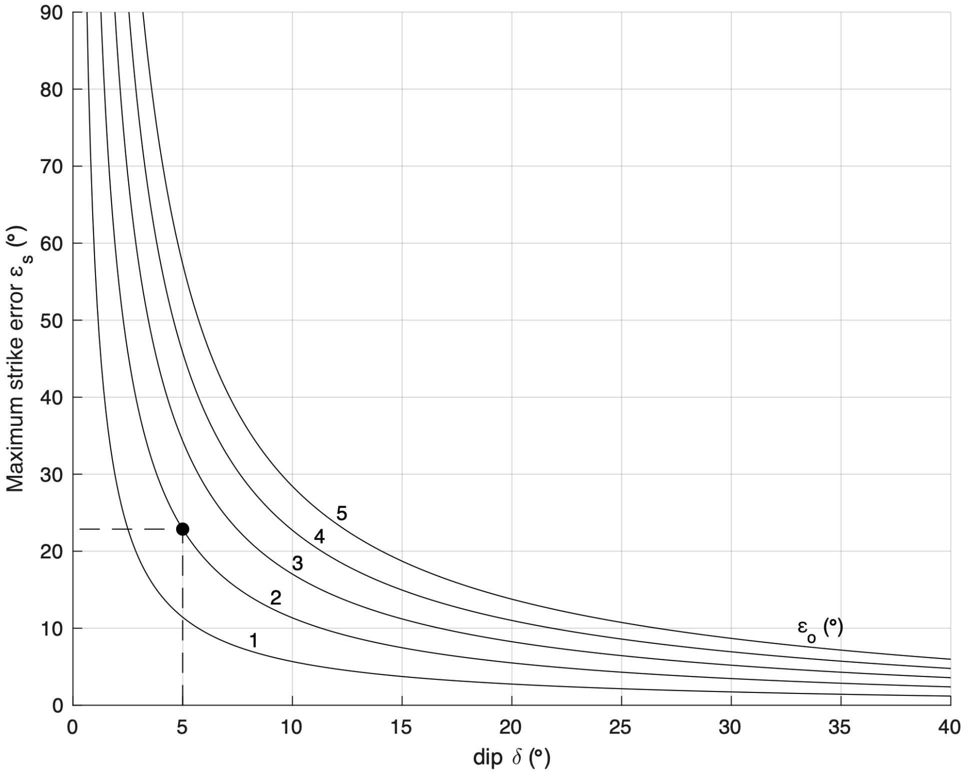


Figure 3.6: Strike error ε_s as a function of dip δ for values of operator error ε_o of 1-5°. The notebook that produced this graph is available in our git repository.

These equations are plotted in Figure 3.7b-c for an ε_o of 3°. The trend error is greater for a measured line on the down-dip ($\beta' > \beta$) side of the line (Fig. 3.7b), than for a measured line on the up-dip ($\beta' < \beta$) side of the line (Fig. 3.7c). This means that repeated measurements will not be symmetrically distributed around the true trend β. Also for a given ε_o, the trend error ε_t increases with the dip δ of the plane and the rake r of the line, i.e. a combination of a steep plane and a large rake may result in a large trend error.

Equations 3.3 and 3.4 allow determining the uncertainties associated to the measurement of planes and lines. As we will see in section 4.5, these errors propagate in any computation making use of these orientations.

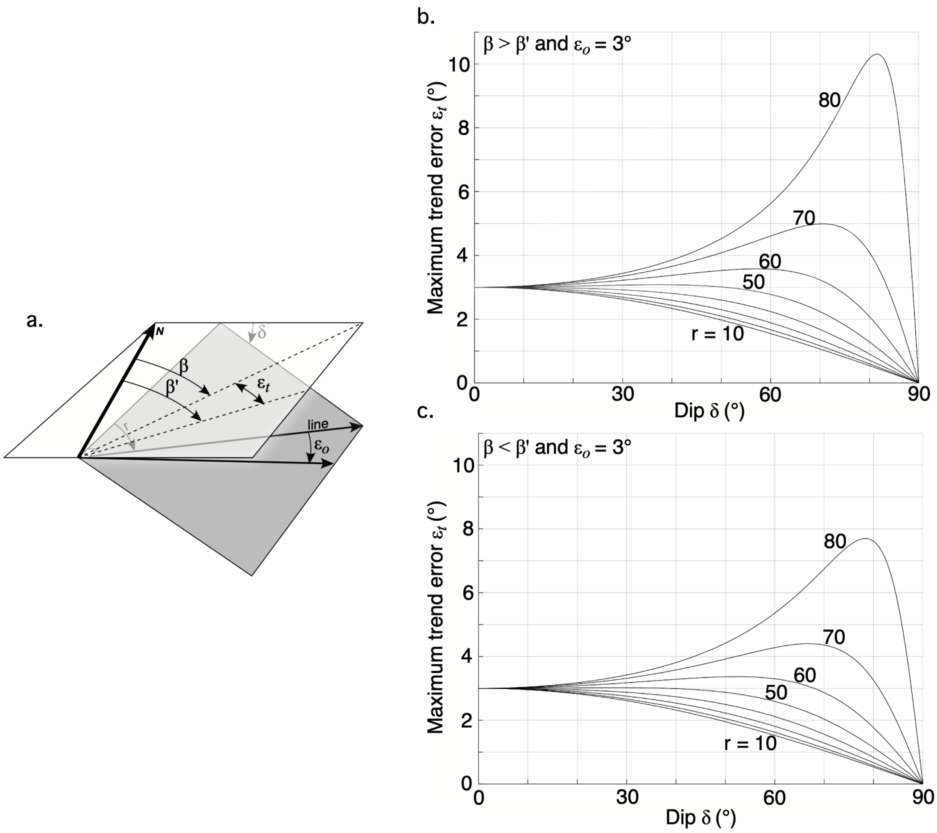


Figure 3.7: **a.** Geometrical relations for estimating the trend error ε_t from the rake r of the line, the dip δ of the plane, and the angle on the plane ε_o between the measured and the true lines. Trend error as function of dip for **b.** a measured line on the down-dip side of the line, and **c.** a measured line on the up-dip side of the line. In b and c, ε_o is 3°. The notebooks that produced graphs b and c are available in our git repository.

3.3 Displaying geologic features

There are two fundamental ways geologists display geologic features: maps and stereonets. In maps, we are concerned about the location and orientation of the features, and the spatial relation of one feature to another. In stereonets, we are just concerned with the orientation of the features.

3.3.1 Maps

All maps are a projection of surface or subsurface geologic features onto a horizontal plane. In section 2.3, we looked at the different methods used to project data from the approximately spherical Earth to a flat surface, and the distortions associated to these methods. Geologic features (bedding, faults, the ground surface) are rarely flat, and therefore to display the spatial variation of their elevation (or depth) on maps, we use contours. A contour line is a line joining the points in the map area of equal value for a specific parameter. On a topographic map, for example, contour lines join points of equal elevation on the ground surface. Contour lines should not cross (unless very unusual circumstances) or disappear in the middle of the map (unless the contoured feature is intersected by another). If the difference in value between adjacent contours or contour interval is held constant throughout the map, the gradient (rate of change) of the parameter in a given direction is proportional to the spacing of the contour lines: high gradient is represented by closely spaced contours, and low gradient by widely spaced contours. This is expressed by the following relation:

$$gradient = \arctan \frac{parameter\ change\ between\ contours}{map\ distance\ between\ contours} \tag{3.5a}$$

For a topographic map, this relation becomes:

$$slope\ angle = \arctan \frac{elevation\ change\ between\ contours}{map\ distance\ between\ contours} \tag{3.5b}$$

which is why when choosing the walking path to a high ground area, you should look for the widely spaced contours (unless you are a climber or a goat).

Geologic features are rarely isolated, and they usually have different orientations, so we should expect them to intersect. The intersection of two non-parallel planes (e.g. bedding contacts) is a straight line. In chapter 4, we will see how to determine this type of intersection using vector operations. If one of the surfaces is not planar but is irregular, the intersection is a curved line which is more difficult to determine. One of the most fundamental mapping problem geology students are early confronted with is the intersection of a planar feature (e.g. bedding or a fault) with the irregular land surface across a valley. This is elegantly summarized by the *Rule of V's* (Fig. 3.8).

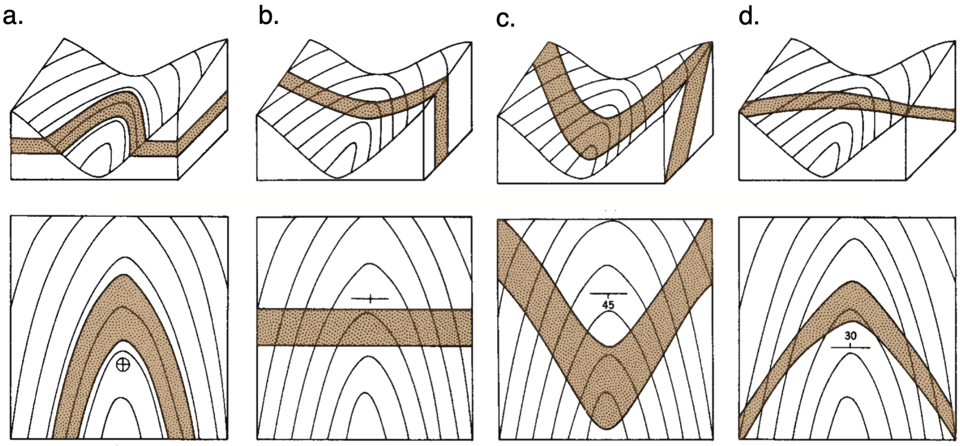


Figure 3.8: Outcrop pattern across a valley of **a.** Horizontal bed, **b.** Vertical bed, **c.** Bed dipping downstream, and **d.** Bed dipping upstream. Modified from Ragan (2009).

The Rule of V's says that when a planar contact crosses a valley, its outcrop pattern will V or curve in the direction that the contact is dipping, but only if the contact if steeper than the slope of the valley, which is normally the case (Fig. 3.8c-d). There are two exceptions: 1. If the contact is horizontal, its outcrop pattern will follow the topographic contours, which makes sense since the contours are the intersection of horizontal planes of different elevation with the ground (Fig. 3.8a), and 2. If the contact is vertical, its outcrop pattern across the valley is a straight line. Vertical planes "ignore" topography.

Determining the outcrop trace of a planar contact on irregular topography is not straightforward. Graphically, this problem involves making elevation contours on the planar contact. These are called structure contours. Then one should look at the locations where the structure contours of the contact have the same elevation than the topographic contours of the land surface. On these locations, the contact outcrops. Finally, one should join these locations with a line, to make the outcrop trace of the contact. Figure 3.9 illustrates this procedure for a plane dipping north and intersecting irregular topography. Notice how in the stream valleys, the outcrop trace of the plane curves to the north, clearly following the Rule of V's.

This graphical approach requires a great deal of patience and drawing skills. In chapter 4, we sill see that if we have the plane's orientation and one

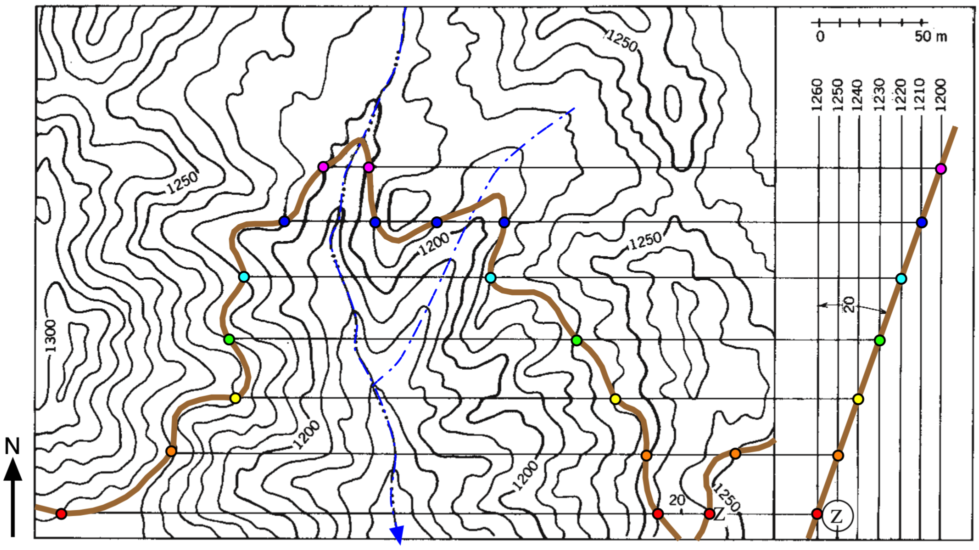


Figure 3.9: Outcrop trace of a plane dipping 20°N. The left figure is the map, and the right figure is a N-S cross section. Color points are the locations where the plane's structure contours have the same elevation than the topographic contours. The line joining these points is the outcrop trace of the plane. Modified from Ragan (2009).

outcrop location, it is possible to project the plane throughout the terrain using computation, provided we have a digital elevation model (DEM) of the terrain. This saves a lot of time and it's a great way to quality control mapping, test different hypotheses, and take better decisions in the field.

3.3.2 Stereonets

Spherical projections can be used to represent the orientation of a plane or a line if the plane or line is positioned so that it passes through the center of the sphere. A plane will intersect the sphere along a great circle, and a line will pierce the sphere at a point (Fig. 3.10a). Obviously, it would be inconvenient to carry a sphere everywhere. Fortunately, it is possible to project the sphere onto a plane using, for example, an azimuthal projection (section 2.3).

A *stereonet* or stereographic projection is a special kind of azimuthal projection, where the point source or viewpoint lies on the surface of the sphere,

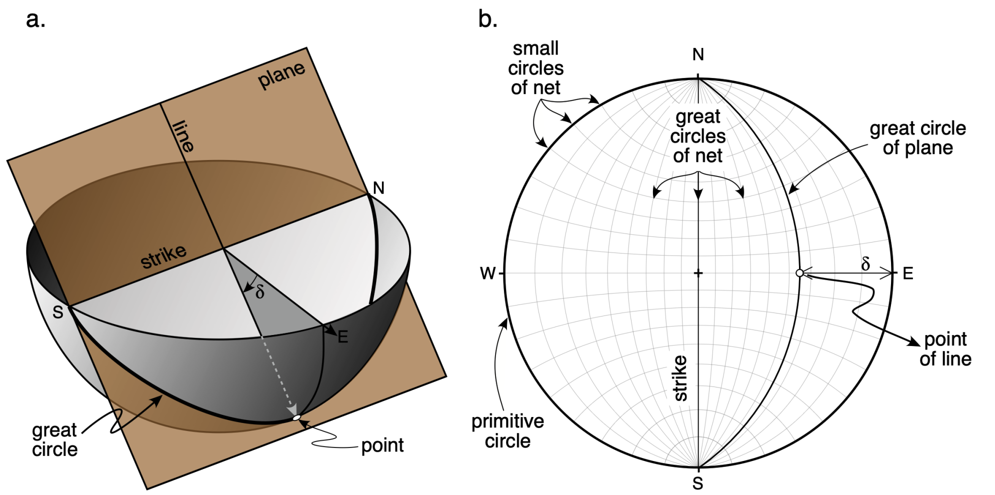


Figure 3.10: **a.** Plane and line intersecting the lower half of a sphere. The rake of the line is 90°and therefore its plunge is equal to the plane's dip δ. **b.** Lower hemisphere stereographic projection of plane and line. Modified from Allmendiger et al. (2012) and Allmendinger (2019).

and the projection plane passes through the center of the sphere. In a stereonet, the viewpoint is at the top of the sphere or zenith, the view direction is downwards, the projection plane is the equatorial plane dividing the sphere into lower and upper hemispheres, and the lower hemisphere (bowl in Fig. 3.10a) is projected. In the stereonet, the rim of the bowl is called the primitive circle and it represents a horizontal plane (Fig. 3.10b). A net consisting of great circles representing N-S striking, 0-90°E and W dipping planes, and small circles representing cones of N-S horizontal axis and 0-90°apical radius opening to the S and N, helps drawing any plane or line on the stereonet (Fig. 3.10b). Several books explain how to do this and solve orientation problems (including rotations) using the stereonet (e.g. Marshak and Mitra, 1988).

For our purpose, it is more important to know how this projection actually works. Figure 3.11a illustrates this on a vertical section passing through the center of the sphere. Any line from the zenith (the top of the sphere) pinches the equatorial plane at one point, and this is the location where the point plots in the stereonet. This is defined by the following equation:

$$x = R \tan\left(45^\circ - \frac{\phi}{2}\right) \tag{3.6}$$

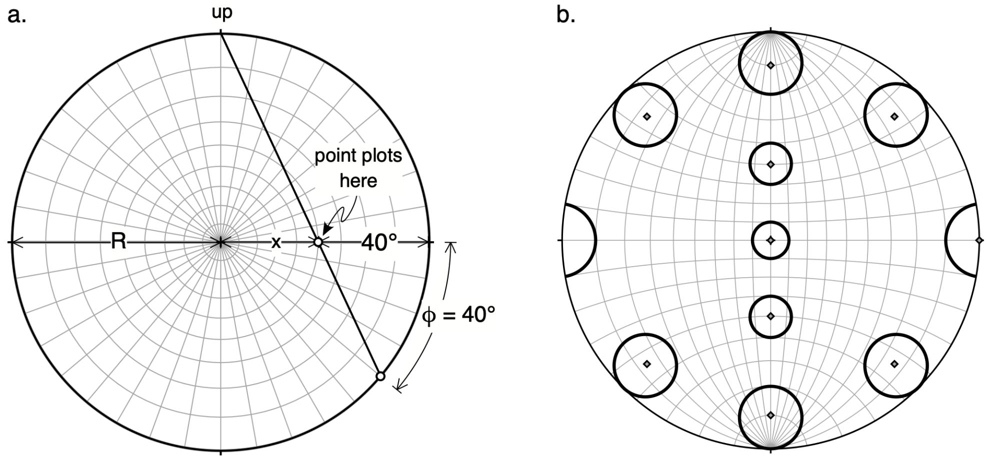


Figure 3.11: **a.** The equal angle stereonet illustrated on a vertical plane passing through the center of the sphere. **b.** Lower hemisphere equal angle projection of small circles of 10°radius but different axis orientations. Modified from Allmendinger et al. (2012).

where x is the is the distance of the point from the center of the net, R is the radius of the net, and ϕ is the plunge of the line. This method preserves angles perfectly and thus, on the primitive circle, degrees are equally spaced, and a small circle will be a circle anywhere on the net (Fig. 3.11b). This is why this projection is called the equal angle or Wulff stereonet. However, the preservation of angles has a disadvantage: areas are distorted. Thus, for example, a 10°radius small circle will look smaller near the center of the net but larger near the edges (Fig. 3.11b). This poses a problem when trying to assess visually or graphically the density of points plotted on the net.

The equal area or Schmidt net (Fig. 3.12) overcomes this problem. Strictly speaking, this projection is not a stereographic projection because the projection plane is at the bottom of the sphere. The point of intersection of the line and the surface of the lower hemisphere, is projected to the horizontal plane at the lowest point of the sphere, along a circular arc centered at the bottom of the sphere. The x distance of the point is then scaled by a factor of $\sqrt{2}$ to fit the radius R of the net (Fig. 3.12a). This is expressed by the following equation:

$$x = R\sqrt{2}\sin\left(45^\circ - \frac{\phi}{2}\right) \tag{3.7}$$

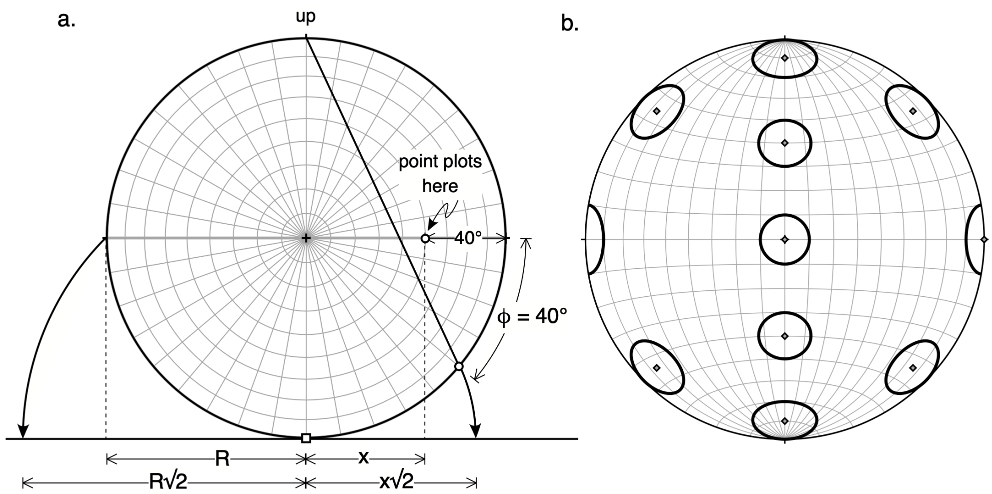


Figure 3.12: **a.** The equal area stereonet illustrated on a vertical plane passing through the center of the sphere. **b.** Lower hemisphere equal area projection of small circles of 10°radius but different axis orientations. Modified from Allmendinger et al. (2012).

The tradeoff is that angles are no longer preserved, and small circles are no longer true circles (Fig. 3.12b). The equal angle or Wulff net is used in problems where visualizing correctly angles on the net is important such as in crystallography and geography, while the equal area or Schmidt net is used in cases where analyzing the concentration of points on the net is important such as in structural analysis.

The Python function *StCoordLine* below computes the coordinates of a line in an equal angle or an equal area net (equations 3.7 and 3.8). Notice that angles (*trd* and *plg*) should be entered in radians.

```
import math
from ZeroTwoPi import ZeroTwoPi as ZeroTwoPi

def StCoordLine(trd, plg, sttype):
    '''
    StCoordLine computes the coordinates of a line
    in an equal angle or equal area stereonet of unit radius

    trd    = trend of line
    plg    = plunge of line
    sttype  = Stereonet type: 0 = equal angle, 1 = equal area
    xp and yp = Coordinates of the line in the stereonet

```

```
    NOTE: trend and plunge should be entered in radians

    StCoordLine uses function ZeroTwoPi

    Python function translated from the Matlab function
    StCoordLine in Allmendinger et al. (2012)
    '''

    # Take care of negative plunges
    if plg < 0:
        trd = ZeroTwoPi(trd+math.pi)
        plg = -plg

    # Some constants
    piS4 = math.pi/4
    s2 = math.sqrt(2)
    plgS2 = plg/2

    # Equal angle stereonet, Eq. 3.6
    if sttype == 0:
        xp = math.tan(piS4 - plgS2)*math.sin(trd)
        yp = math.tan(piS4 - plgS2)*math.cos(trd)
    # Equal area stereonet, Eq. 3.7
    elif sttype == 1:
        xp = s2*math.sin(piS4 - plgS2)*math.sin(trd)
        yp = s2*math.sin(piS4 - plgS2)*math.cos(trd)

    return xp, yp
```

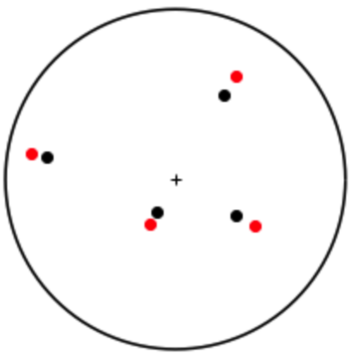
3.3.3 Plotting lines and poles in the stereonet

The notebook ch3.ipynb below illustrates the use of the *StCoordLine* and *Pole* functions to plot lines and poles to planes on an equal angle or an equal area stereonet. You will get the chance to practice more with these functions in the Exercises section.

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

from Pole import Pole as Pole
from StCoordLine import StCoordLine as StCoordLine

# Plot the following four lines (trend and plunge)
# on an equal angle or equal area stereonet
```

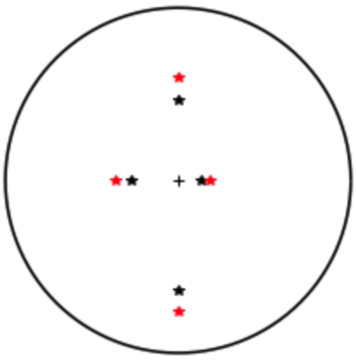
```
lines = np.array([[30, 30],[120, 45],[210, 65],[280, 15]])
pi = np.pi
linesr = lines * pi/180 # lines in radians

# Plot the primitive of the stereonet
r = 1; # unit radius
TH = np.arange(0,360,1)*pi/180
x = r * np.cos(TH)
y = r * np.sin(TH)
plt.plot(x,y,'k')
# Plot center of circle
plt.plot(0,0,'k+')
# Make axes equal and remove them
plt.axis('scaled')
plt.axis('off')

# Find the coordinates of the lines in the
# equal angle or equal area stereonet
nrow, ncol = lines.shape
eqAngle = np.zeros((nrow, ncol))
eqArea = np.zeros((nrow, ncol))

for i in range(nrow):
    # Equal angle coordinates
    eqAngle[i,0], eqAngle[i,1] = StCoordLine(linesr[i,0],
    linesr[i,1],0)
    # Equal area coordinates
    eqArea[i,0], eqArea[i,1] = StCoordLine(linesr[i,0],linesr
    [i,1],1)

# Plot the lines
# Equal angle as black dots
plt.plot(eqAngle[:,0],eqAngle[:,1],'ko')
# Equal area as red dots
plt.plot(eqArea[:,0],eqArea[:,1],'ro')
```

Output:



```
# Plot the following four planes (strike and dip, RHR)
# as poles on an equal angle or equal area stereonet
planes = np.array([[0, 30], [90, 50], [180, 15], [270, 65]])
planesr = planes * pi/180 # planes in radians

# Plot the primitive of the stereonet
plt.plot(x,y,'k')
# Plot center of circle
plt.plot(0,0,'k+')
# Make axes equal and remove them
plt.axis('scaled')
plt.axis('off')

# Find the coordinates of the poles to the planes in the
# equal angle or equal area stereonet
for i in range(nrow):
    # Compute pole of plane
    trend, plunge = Pole(planesr[i,0], planesr[i,1], 1)
    # Equal angle coordinates
    eqAngle[i,0], eqAngle[i,1] = StCoordLine(trend,plunge,0)
    # Equal area coordinates
    eqArea[i,0], eqArea[i,1] = StCoordLine(trend,plunge,1)

# Plot the poles
# Equal angle as black asterisks
plt.plot(eqAngle[:,0],eqAngle[:,1],'k*')
# Equal area as red asterisks
plt.plot(eqArea[:,0],eqArea[:,1],'r*')
```

Output:



3.4 Exercises

1. Modify the notebook that makes Fig. 3.6 to extend the range of dip δ angles from 0 to 90°and the operator error ε_o from 1 to 10°.

2. Modify the notebooks that make Fig. 3.7 b and c for an ε_o of 5°.

3. You can draw a great circle on a stereonet by plotting closely spaced points along the great circle. These are lines on the plane. The following arrays contain the trend and plunge of lines on a plane of orientation 030/40 (strike and dip, RHR format):

   trend = [30, 34, 38, 42, 46, 50, 54, 58, 63, 67, 72, 78, 83, 89, 95, 101, 107, 113, 120, 127, 133, 139, 145, 151, 157, 162, 168, 173, 177, 182, 186, 190, 194, 198, 202, 206, 210]

   plunge = [0, 3, 6, 10, 13, 16, 19, 22, 24, 27, 29, 32, 34, 36, 37, 38, 39, 40, 40, 40, 39, 38, 37, 36, 34, 32, 29, 27, 24, 22, 19, 16, 13, 10, 6, 3, 0]

   Plot these lines on an equal angle and an equal area stereonet. From the resulting great circle, can you guess how a plane of orientation 050/60 (RHR) would look like on the stereonet?

4. You can also draw a small circle on a stereonet by plotting closely spaced points along the small circle. These are lines on the conical surface. The following arrays contain the trend and plunge of lines on a small circle of axis 050/30 (trend and plunge) and radius 20°:

   trend = [50, 53, 57, 60, 63, 66, 68, 70, 72, 73, 73, 73, 72, 70, 68, 64, 60, 55, 50, 45, 40, 36, 32, 30, 28, 27, 27, 27, 28, 30, 32, 34 37, 40, 43, 47, 50]

   plunge = [10, 10, 11, 12, 14, 16, 19, 22, 25, 28, 31, 35, 38, 41, 44, 47, 48, 50, 50, 50, 48, 47, 44, 41, 38, 35, 31, 28, 25, 22, 19, 16 14, 12, 11, 10, 10]

   Plot these lines on an equal angle and an equal area stereonet. What is the difference between the small circle in the equal angle and equal area stereonets?

5. The strike and dip arrays below contain the strike and dip (RHR format) of 50 bedding surfaces in a fold:

   strike = [8, 22, 19, 33, 27, 37, 41, 47, 55, 40, 32, 55, 65, 68, 89, 79, 102, 105, 108, 122, 132, 136, 145, 159, 156, 164, 176, 169, 179, 173, 167, 160, 145, 148, 141, 125, 108, 92, 75, 57, 50, 39, 22, 10, 1, 9, 15, 16, 114, 78]

dip = [75, 79, 68, 72, 61, 46, 50, 67, 51, 66, 55, 42, 49, 58, 54, 45, 35, 49, 63, 45, 52, 66, 52, 59, 76, 64, 72, 83, 78, 88, 72, 81, 73, 62, 50, 63, 42, 48, 56, 62, 50, 65, 76, 87, 81, 68, 74, 83, 56, 37]

Plot these planes as poles in an equal area stereonet. The resultant diagram is called a *point-*, *scatter-* or $\pi-$ diagram. You can approximate a great circle through the poles. What is the approximate orientation of this great circle? What does the pole to this great circle represent?

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Chapter 4

Coordinate systems and vectors

Strike and dip, and trend and plunge, are a convenient way to represent the orientation of planes and lines. However, it is difficult to handle these angles using computation. In this chapter, we will see how to convert linear features (lines and poles to planes) from spherical (trend and plunge) to Cartesian (direction cosines) coordinates, thus representing these features as vectors. This facilitates the analysis of planes and lines using linear algebra and computation, and it will allow us to solve a range of interesting problems using vector operations.

4.1 Coordinate systems

Any point or location in space can be represented by the coordinates of the point with respect to the three orthogonal axes of a Cartesian coordinate system. We will call the three axes of this coordinate system $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ (Fig. 4.1). In addition, we will follow a right-handed naming convention: If you hold your right hand so that your thumb points in the positive direction of the first axis $\mathbf{X_1}$, your other fingers should curl from the positive direction of the second axis $\mathbf{X_2}$ toward the positive direction of the third axis $\mathbf{X_3}$ (Fig. 4.1). Such a coordinate system is called a right-handed coordinate system.

In geosciences, we use mainly two types of right-handed coordinate systems: An east (**E**), north (**N**), up (**U**) coordinate system (Fig. 4.1a), and a north (**N**), east (**E**), down (**D**) coordinate system (Fig. 4.1b). The **ENU** coordinate system is used in GIS and Geophysics when dealing with elevations (e.g. topography), while the **NED** coordinate system is used in Structural Geology where, by convention, angles measured downwards from the horizontal (e.g. plunge of a downward pointing line) are considered positive. In this chapter, we will use the **NED** coordinate system, but when dealing with topography and elevations, we will use the **ENU** coordinate system.

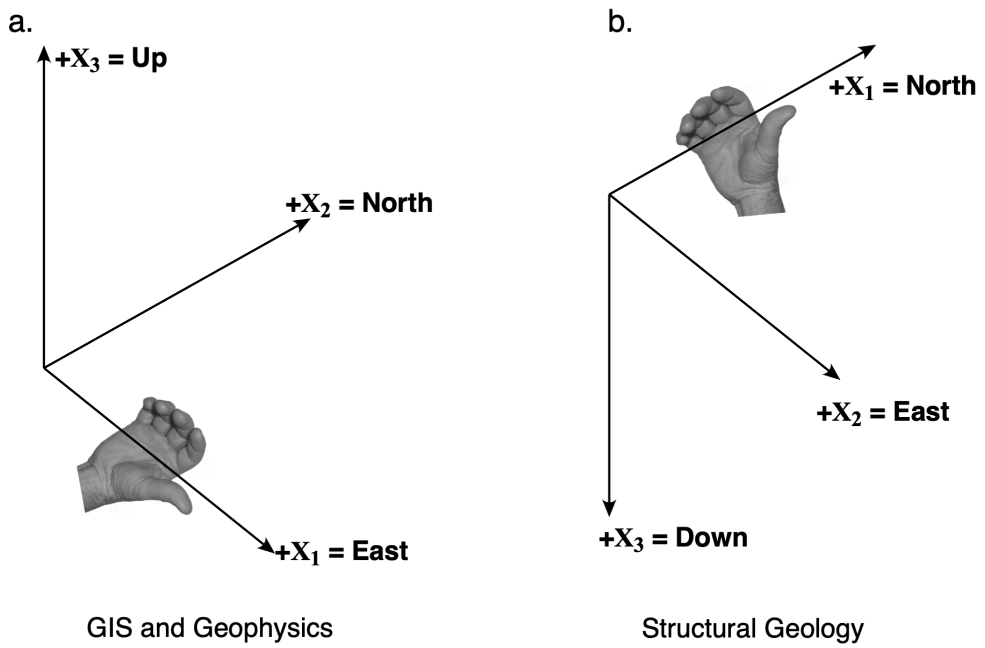


Figure 4.1: Right-handed Cartesian coordinate systems. **a.** The **ENU** coordinate system used when dealing with topography, and **b.** The **NED** coordinate system used in Structural Geology. Modified from Allmendinger et al. (2012).

4.2 Vectors

4.2.1 Vector components, magnitude, and unit vectors

A line from the origin of the Cartesian coordinate system to a point in space is the position *vector* of the point. A *vector* is an object that has both a magnitude and a direction. Displacement, velocity, force, acceleration, and poles to planes, are all vectors. A vector is defined by its three components with respect to the axes of the Cartesian coordinate system; these are the projections of the vector onto the axes $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ (Fig. 4.2a).

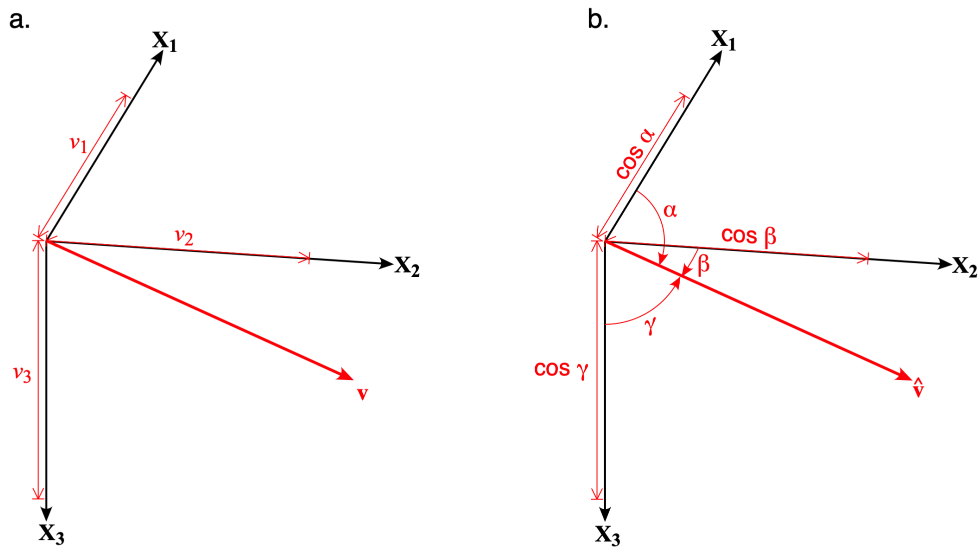


Figure 4.2: **a.** Components of a vector. **b.** Direction cosines of a unit vector. Modified from Allmendinger et al. (2012)

This is expressed by the following equation:

$$\mathbf{v} = [v_1, v_2, v_3] \tag{4.1}$$

We use lower capital letters to denote vectors. The magnitude (length) of a vector can be computed using Pythagoras' theorem:

$$v = (v_1^2 + v_2^2 + v_3^2)^{1/2} \tag{4.2}$$

The result is just a number, a scalar. We use regular, non-capital letters to denote scalars. If we divide each of the components of a vector by its magnitude, the result is a unit vector, a vector with the same orientation but with a magnitude (length) of one (Fig. 4.2b):

$$\hat{\mathbf{v}} = [v_1/v, v_2/v, v_3/v] \tag{4.3}$$

We use a hat to indicate unit vectors. There is a very interesting property of unit vectors; the components of a unit vector are the cosines of the angles the vector makes with the axes of the coordinate system (Fig. 4.2b):

$$\hat{\mathbf{v}} = [\cos\alpha, \cos\beta, \cos\gamma] \tag{4.4}$$

these are called the *direction cosines* of the vector. By convention, $\cos\alpha$ is the direction cosine of the vector with respect to $\mathbf{X_1}$, $\cos\beta$ is the direction cosine of the vector with respect to $\mathbf{X_2}$, and $\cos\gamma$ is the direction cosine of the vector with respect to $\mathbf{X_3}$ (Fig. 4.2b).

In Python, we can use the NumPy *linalg.norm* function to compute the magnitude of a vector and convert it to a unit vector as illustrated in the following notebook ch4-1.ipynb:

```
# Import numpy
import numpy as np
# Import linear algebra functions
from numpy import linalg as la
# Make vector
v = np.array([1,2,3])
print('Vector: ', v)
# Magnitude of the vector
length = la.norm(v)
print('Magnitude of the vector: ', length)
# Unit vector
v_hat = v / length
print('Unit Vector: ', v_hat)
# Magnitude of unit vector
length = la.norm(v_hat)
print('Magnitude of the unit vector: ', length)
```

Output:
Vector: [1 2 3]
Magnitude of the vector: 3.7416573867739413
Unit Vector: [0.26726124 0.53452248 0.80178373]
Magnitude of the unit vector: 1.0

4.2.2 Vector operations

To multiply a scalar times a vector, just multiply each component of the vector by the scalar:

$$x\mathbf{v} = [xv_1, xv_2, xv_3] \tag{4.5}$$

This operation is useful, for example, to reverse the direction of the vector; just multiply the vector by -1. To add two vectors, just sum their components:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = [u_1 + v_1, u_2 + v_2, u_3 + v_3] \tag{4.6}$$

Vector addition is commutative. However, vector subtraction is not. Graphically, vector addition obeys the parallelogram rule, whereby the resulting vector bisects the two vectors to be added (Fig. 4.3a).

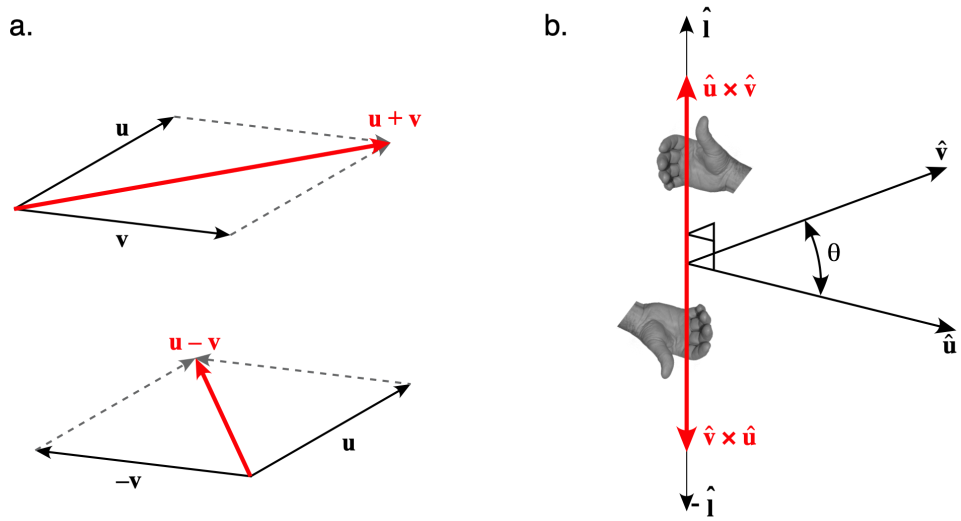


Figure 4.3: **a.** Vector addition and subtraction. **b.** Cross product of two unit vectors. Modified from Allmendinger et al. (2012).

There are two operations that are unique to vectors: the *dot product* and the *cross product*. The result of the dot product is a scalar and is equal to the magnitude of the first vector times the magnitude of the second vector times the cosine of the angle between the vectors:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = uv\cos\theta = u_1v_1 + u_2v_2 + u_3v_3 = u_iv_i \tag{4.7}$$

The dot product is commutative. If the two vectors are unit vectors, you can easily see that the dot product is:

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \cos\theta = u_1v_1 + u_2v_2 + u_3v_3 \tag{4.8}$$

or in terms of the direction cosines of the vectors:

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \cos\theta = cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 \tag{4.9}$$

which as we will see later, it is a great way to find the angle between two unit vectors.

The result of the cross product is another vector. This vector is perpendicular to the other two vectors, and has a magnitude that is equal to the product of the magnitude of each vector times the sine of the angle between the vectors:

$$\mathbf{u} \times \mathbf{v} = uv\sin\theta\hat{\mathbf{l}} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1] \tag{4.10}$$

The cross product is not commutative. If the vectors are unit vectors, the length of the resulting vector is equal to the sine of the angle between the vectors (Fig. 4.3b). The new vector obeys a right-hand rule; for $\mathbf{u} \times \mathbf{v}$, the fingers curl from $\mathbf{u}$ towards $\mathbf{v}$ and the thumb points in the direction of the resulting vector, and vice versa (Fig. 4.3b).

In Python, these operations are easy to perform using the NumPy library as shown in the following notebook ch4-2.ipynb:

```
# Import numpy
import numpy as np
# Make vectors
u = np.array([1,2,3])
v = np.array([3,2,1])
print('u = ', u)
print('v = ', v)
# Scalar multiplication of vector
sv = 3 * u
print('3 * u = ', sv)
# Sum of vectors
vsum = u + v
print('u + v = ', vsum)
#  Dot product of vectors
dotp = np.dot(u,v)
print('u . v = ', dotp)
# Cross product of vectors
crossp = np.cross(u,v)
print('u x v = ', crossp)
```

```
Output:
u = [1 2 3]
v = [3 2 1]
3 * u = [3 6 9]
u + v = [4 4 4]
u . v = 10
u x v = [-4 8 -4]
```

4.3 Geologic features as vectors

We have now all the mathematical tools to represent geologic features as vectors. Since we are only interested in the orientation of these features, we will treat lines and poles to planes as unit vectors. We will also use the Structural Geology **NED** coordinate system.

4.3.1 From spherical to Cartesian coordinates

Figure 4.4 shows a line as a unit vector $\hat{\mathbf{v}}$ in the **NED** coordinate system. Clearly, the angle that the line makes with the **D** axis is 90°- *plunge*, therefore:

$$\cos\gamma = \cos(90^\circ - plunge) = \sin(plunge) \tag{4.11a}$$

The horizontal projection of the line is cos(*plunge*) (Fig. 4.4). $\cos\alpha$ and $\cos\beta$ are just the **N** and **E** components of this horizontal line (Fig. 4.4):

$$\cos\alpha = \cos(trend)\cos(plunge) \tag{4.11b}$$

$$\cos\beta = \cos(90^\circ - trend)\cos(plunge) = \sin(trend)\cos(plunge) \tag{4.11c}$$

The magnitude and sign of the direction cosines tell us a lot about the orientation of the line (Fig. 4.5). A horizontal line (plunge = 0) has $\cos\gamma = 0$, a downward pointing line (plunge > 0) has $+\cos\gamma$, and if the line is vertical

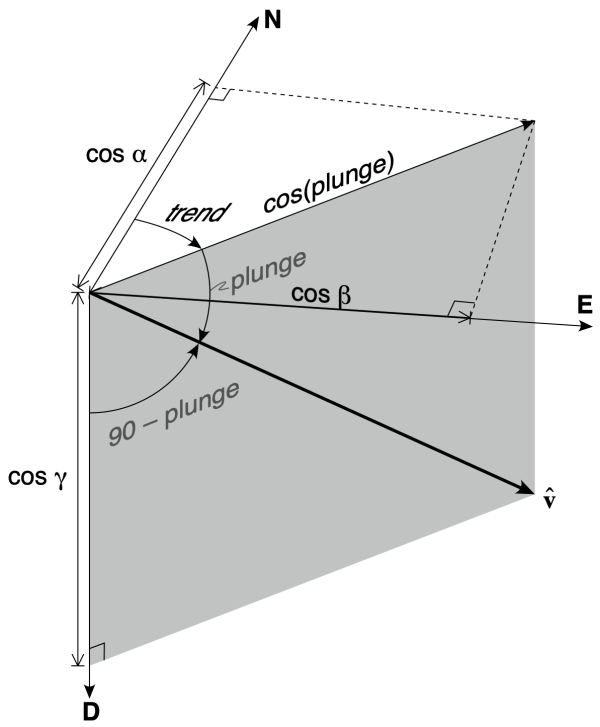


Figure 4.4: Diagram showing the relations between the trend and plunge and the direction cosines in the **NED** coordinate system. Gray plane is the vertical plane in which the plunge is measured. Modified from Allmendinger et al. (2012).

(plunge = 90°) $\cos\gamma = 1$ and the other two direction cosines are 0. A horizontal or downward pointing line (plunge ≥ 0) has $+\cos\alpha$ if it trends NE or NW (first or fourth quadrants), and $+\cos\beta$ if it trends NE or SE (first or second quadrants). If the line trends N or S, $\cos\beta = 0$; and if the line trends E or W, $\cos\alpha = 0$.

To determine the direction cosines of a pole to a plane, we just need to express the trend and plunge of the pole in terms of the strike and dip of the plane assuming a RHR format (Eq. 3.2), and use these in Eq. 4.11. The direction cosines of the pole to the plane are then:

$$\cos\alpha = \cos(strike - 90°)\cos(90° - dip) = \sin(strike)\sin(dip) \tag{4.12a}$$

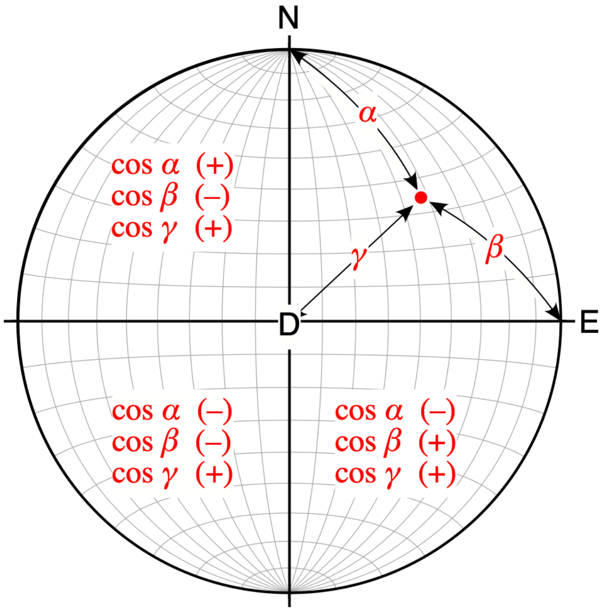


Figure 4.5: Lower hemisphere stereonet showing the sign of the direction cosines in each quadrant. In the NE quadrant, all three direction cosines are positive. Modified from Allmendinger et al. (2012).

$$\cos\beta = \sin(strike - 90^\circ)\cos(90^\circ - dip) = -\cos(strike)\sin(dip) \quad (4.12b)$$

$$\cos\gamma = \sin(90^\circ - dip) = \cos(dip) \quad (4.12c)$$

These equations are summarized in Table 4.1.

Axis	**Direction cosines**	**Lines**	**Poles to planes** (RHR format)
N	$\cos\alpha$	$\cos(trend)\cos(plunge)$	$\sin(strike)\sin(dip)$
E	$\cos\beta$	$\sin(trend)\cos(plunge)$	-$\cos(strike)\sin(dip)$
D	$\cos\gamma$	$\sin(plunge)$	$\cos(dip)$

Table 4.1: Conversion from spherical to Cartesian coordinates

The following Python function *SphToCart* converts a line ($k = 0$) or a plane ($k = 1$) from spherical to Cartesian coordinates. Notice that the angles (*trd* and *plg*) should be entered in radians:

```
import math

def SphToCart(trd,plg,k):
    '''
    SphToCart converts from spherical to
    Cartesian coordinates

    SphToCart(trd,plg,k) returns the north (cn),
    east (ce), and down (cd) direction cosines of a line.

    k: integer to tell whether the trend and plunge of a line
    (k = 0) or strike and dip of a plane in right hand rule
    (k = 1) are being sent in the trd and plg slots. In this
    last case, the direction cosines of the pole to the plane
    are returned

    NOTE: Angles should be entered in radians

    Python function translated from the Matlab function
    SphTpCart in Allmendinger et al. (2012)
    '''
    # If line  (see Table 4.1)
    if k == 0:
        cn = math.cos(trd) * math.cos(plg)
        ce = math.sin(trd) * math.cos(plg)
        cd = math.sin(plg)
    # Else pole to plane (see Table 4.1)
    elif k == 1:
        cn = math.sin(trd) * math.sin(plg)
        ce = -math.cos(trd) * math.sin(plg)
        cd = math.cos(plg)

    return cn, ce, cd
```

4.3.2 From Cartesian to spherical coordinates

Converting from direction cosines (Cartesian coordinates) to trend and plunge (spherical coordinates) is a little less straightforward. The plunge is easy:

$$plunge = \sin^{-1}(\cos\gamma) \tag{4.13a}$$

The trend can be determined as follows:

$$\frac{\cos\beta}{\cos\alpha} = \frac{\sin(trend)\cos(plunge)}{\cos(trend)\cos(plunge)} = \tan(trend)$$

or:

$$trend = \tan^{-1}\left(\frac{\cos\beta}{\cos\alpha}\right) \tag{4.13b}$$

The problem is that the trend varies from 0 and 360°. For the $\tan^{-1}$ function, there are two possible angles between 0 and 360°. Which one should we use? The answer is to use the signs of the direction cosines to determine in which quadrant the trend lies within. By inspection of Figure 4.5, one can see that:

$$trend = \tan^{-1}\left(\frac{\cos\beta}{\cos\alpha}\right) \;\; if \;\; \cos\alpha > 0 \tag{4.14a}$$

$$trend = 180^\circ + \tan^{-1}\left(\frac{\cos\beta}{\cos\alpha}\right) \;\; if \;\; \cos\alpha < 0 \tag{4.14b}$$

One should also check for the special case of $\cos\alpha = 0$:

$$trend = 90^\circ \; if \; (\cos\alpha = 0 \; and \; \cos(\beta) \geq 0) \tag{4.14c}$$

$$trend = 270^\circ \; if \; (\cos\alpha = 0 \; and \; \cos(\beta) < 0) \tag{4.14d}$$

The following Python function *CartToSph* converts a line from Cartesian to spherical coordinates. Notice that the trend and plunge are returned in radians:

```python
import math
from ZeroTwoPi import ZeroTwoPi as ZeroTwoPi

def CartToSph(cn,ce,cd):
    '''
```

```python
    CartToSph converts from Cartesian to spherical coordinates

    CartToSph(cn,ce,cd) returns the trend (trd)
    and plunge (plg) of a line for input north (cn),
    east (ce), and down (cd) direction cosines

    NOTE: Trend and plunge are returned in radians

    CartToSph uses function ZeroTwoPi

    Python function translated from the Matlab function
    CartToSph in Allmendinger et al. (2012)
    '''
    pi = math.pi
    # Plunge
    plg = math.asin(cd) # Eq. 4.13a

    #Trend
    #If north direction cosine is zero, trend is east or west
    #Choose which one by the sign of the east direction cosine
    if cn == 0.0:
        if ce < 0.0:
            trd = 3.0/2.0*pi # Eq. 4.14d, trend is west
        else:
            trd = pi/2.0 # Eq. 4.14c, trend is east
    # Else
    else:
        trd = math.atan(ce/cn) # Eq. 4.14a
        if cn < 0.0:
            #Add pi
            trd = trd+pi # Eq. 4.14b
        # Make sure trd is between 0 and 2*pi
        trd = ZeroTwoPi(trd)

    return trd, plg
```

4.4 Applications

4.4.1 Mean vector

An important problem in geosciences is to determine the average or mean vector that represents a group of lines. These lines can be for example poles to bedding, paleocurrent directions, paleomagnetic poles, or slip vectors on a fault surface. This problem can be solved using vector addition. The resultant vector **r** of the sum of the N unit vectors representing the lines has components:

$$r_1 = \sum_{i=1}^{N} \alpha_i \qquad r_2 = \sum_{i=1}^{N} \beta_i \qquad r_3 = \sum_{i=1}^{N} \gamma_i \tag{4.15a}$$

where α, β and γ are the direction cosines of the unit vectors. The length of the resultant vector **r** is:

$$r = \sqrt{r_1^2 + r_2^2 + r_3^2} \tag{4.15b}$$

and the direction cosines of the unit vector that is parallel to the mean of the individual vectors are:

$$\overline{\alpha} = \frac{r_1}{r} \qquad \overline{\beta} = \frac{r_2}{r} \qquad \overline{\gamma} = \frac{r_3}{r} \tag{4.15c}$$

These direction cosines define the orientation of the mean vector. A measure of how concentrated the individual vectors are or how representative the mean vector is, is given by the *mean resultant length*:

$$\overline{r} = \frac{r}{N} \qquad where \qquad 0 \leq \overline{r} \leq 1 \tag{4.15d}$$

The closer this value is to 1, the better the concentration. The Python function *CalcMV* below calculates the mean vector for a series of lines. It also calculates the Fisher statistics for the mean vector (Fisher et al., 1987), which is the standard way to represent uncertainties in this analysis. Notice that *CalcMV* uses our two previous functions *SphToCart* and *CartToSph* to convert from spherical to Cartesian coordinates, and vice versa.

```
import math
from SphToCart import SphToCart as SphToCart
from CartToSph import CartToSph as CartToSph

def CalcMV(T, P):
    '''
    CalcMV calculates the mean vector for a group of lines

    CalcMV(T,P) calculates the trend (trd) and plunge (plg)
    of the mean vector, its mean resultant length (Rave), and
    Fisher statistics (concentration factor (conc), 99 (d99)
    and 95 (d95) % uncertainty cones); for a series of lines
    whose trends and plunges are stored in the arrays T and P

    NOTE: Input/Output trends and plunges, as well as
    uncertainty cones are in radians

    CalcMV uses functions SphToCart and CartToSph

    Python function translated from the Matlab function
    CalcMV in Allmendinger et al. (2012)
    '''
    # Number of lines
    nlines = len(T)

    # Initialize the 3 direction cosines which contain the
    # sums of the individual vectors
    CNsum = 0.0
    CEsum = 0.0
    CDsum = 0.0

    #Now add up all the individual vectors. Eq. 4.15a
    for i in range(nlines):
        cn,ce,cd = SphToCart(T[i],P[i],0)
        CNsum += cn
        CEsum += ce
        CDsum += cd

    # R is the length of the resultant vector and
    # Rave is the mean resultant length. Eqs. 4.15b and d
    R = math.sqrt(CNsum*CNsum + CEsum*CEsum + CDsum*CDsum)
    Rave = R/nlines
    # If Rave is lower than 0.1, the mean vector is
    # insignificant, return error
    if Rave < 0.1:
        raise ValueError('Mean vector is insignificant')
```

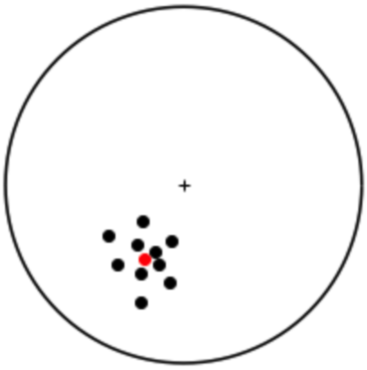
```
    #Else
    else:
        # Divide the resultant vector by its length to get
        # the direction cosines of the unit vector
        CNsum = CNsum/R
        CEsum = CEsum/R
        CDsum = CDsum/R
        # Convert the mean vector to the lower hemisphere
        if CDsum < 0.0:
            CNsum = -CNsum
            CEsum = -CEsum
            CDsum = -CDsum
        # Convert the mean vector to trend and plunge
        trd, plg = CartToSph(CNsum,CEsum,CDsum)
        # If there are enough measurements calculate the
        # Fisher statistics (Fisher et al., 1987)
        if R < nlines:
            if nlines < 16:
                afact = 1.0-(1.0/nlines)
                conc = (nlines/(nlines-R))*afact**2
            else:
                conc = (nlines-1.0)/(nlines-R)
        if Rave >= 0.65 and Rave < 1.0:
            afact = 1.0/0.01
            bfact = 1.0/(nlines-1.0)
            print(R, afact, bfact)
            d99 = math.acos(1.0-((nlines-R)/R)*(afact**bfact-1.0))
            afact = 1.0/0.05
            d95 = math.acos(1.0-((nlines-R)/R)*(afact**bfact-1.0))

    return trd, plg, Rave, conc, d99, d95
```

The notebook ch4-3.ipynb below shows the solution of the mean vector problem on pages 147-148 of Ragan (2009), which consists of finding the mean orientation for 10 poles of bedding:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

from StCoordLine import StCoordLine as StCoordLine
from CalcMV import CalcMV as CalcMV

# Arrays T and P contain the trend (T)
```

```
# and plunge (P) of the 10 poles
T = np.array ([206, 220, 204, 198, 200, 188, 192, 228, 236,
    218])
P = np.array ([32, 30, 46, 40, 20, 32, 54, 56, 36, 44])

# Convert T and P from degrees to radians
pi = np.pi
TR = T * pi/180
PR = P * pi/180

# Compute the mean vector and print orientation
# and mean resultant length
trd, plg, Rave, conc, d99, d95 = CalcMV(TR,PR)
print('Mean vector trend = {:.1f}, plunge {:.1f}'.format(trd
    *180/pi,plg*180/pi))
print('Mean resultant length = {:.3f}'.format(Rave))

# Plot the primitive of the stereonet
r = 1; # unit radius
TH = np.arange(0,360,1)*pi/180
x = r * np.cos(TH)
y = r * np.sin(TH)
plt.plot(x,y,'k')
# Plot center of circle
plt.plot(0,0,'k+')
# Make axes equal and remove them
plt.axis('scaled')
plt.axis('off')

# Plot the poles as black points
# on an equal angle stereonet
npoles = len(T)
eqAngle = np.zeros((npoles, 2))
for i in range(npoles):
    # Equal angle coordinates
    eqAngle[i,0], eqAngle[i,1] = StCoordLine(TR[i],PR[i],0)
plt.plot(eqAngle[:,0],eqAngle[:,1],'ko')

# Plot the mean vector as a red point
mvx, mvy = StCoordLine(trd,plg,0)
plt.plot(mvx,mvy,'ro')
```

Output:
Mean vector trend = 208.6, plunge 40.0
Mean resultant length = 0.963



Notice that the mean resultant length is 0.963, so that the mean vector (red dot) is a representative orientation of the individual vectors (black dots).

4.4.2 Angles, intersections, and poles

Many interesting problems in geology can be solved using the dot and cross product operations. The dot product can be used to find the angle between two lines or planes, while the cross product can be used to find a plane from two lines or the intersection of two planes. Table 2 lists some problems that can be solved using these operations.

Problem	Solution
Angle between two lines	arccos of dot product between lines
Angle between two planes	supplement of arccos of dot product between poles to planes
Intersection of two planes	Cross product of poles to planes
Plane containing two lines	Pole to plane is cross product of lines
Apparent dip of plane	Intersection of plane and vertical section of a given orientation
Plane from two apparent dips	Plane containing the two apparent dips (lines)

Table 4.2: Some problems that can be solved using the dot and cross product operations.

The Python function *Angles* below computes the angle between two lines (*ans0* = *'l'*), the angle between two planes (*ans0* = *'p'*), the plane from two lines (*ans0* = *'a'*), or the intersection of two planes (*ans0* = *'i'*).

```
import math
from SphToCart import SphToCart as SphToCart
from CartToSph import CartToSph as CartToSph
from Pole import Pole as Pole

def Angles(trd1,plg1,trd2,plg2,ans0):
    '''
    Angles calculates the angles between two lines,
    between two planes, the line which is the intersection
    of two planes, or the plane containing two apparent dips

    Angles(trd1,plg1,trd2,plg2,ans0) operates on
    two lines or planes with trend/plunge or
    strike/dip equal to trd1/plg1 and trd2/plg2

    ans0 is a character that tells the function what
    to calculate:

      ans0 = 'a' -> plane from two apparent dips
      ans0 = 'l' -> the angle between two lines

      In the above two cases, the user sends the trend
      and plunge of two lines

      ans0 = 'i' -> the intersection of two planes
      ans0 = 'p' -> the angle between two planes

      In the above two cases the user sends the strike
      and dip of two planes in RHR format

    NOTE: Input/Output angles are in radians

    Angles uses functions SphToCart, CartToSph and Pole

    Python function translated from the Matlab function
    Angles in Allmendinger et al. (2012)
    '''
    # If planes have been entered
    if ans0 == 'i' or ans0 == 'p':
        k = 1
    # Else if lines have been entered
    elif ans0 == 'a' or ans0 == 'l':
        k = 0

```

```
    # Calculate the direction cosines of the lines
    # or poles to planes
    cn1, ce1, cd1 = SphToCart(trd1,plg1,k)
    cn2, ce2, cd2 = SphToCart(trd2,plg2,k)

    # If angle between 2 lines or between
    # the poles to 2 planes
    if ans0 == 'l' or ans0 == 'p':
        # Use dot product
        ans1 = math.acos(cn1*cn2 + ce1*ce2 + cd1*cd2)
        ans2 = math.pi - ans1

    # If intersection of two planes or pole to
    # a plane containing two apparent dips
    if ans0 == 'a' or ans0 == 'i':
        # If the 2 planes or lines are parallel
        # return an error
        if trd1 == trd2 and plg1 == plg2:
            raise ValueError('Error: lines or planes are
   parallel')
        # Else use cross product
        else:
            cn = ce1*cd2 - cd1*ce2
            ce = cd1*cn2 - cn1*cd2
            cd = cn1*ce2 - ce1*cn2
            #Make sure the vector points downe
            if cd < 0.0:
                cn = -cn
                ce = -ce
                cd = -cd
            # Convert vector to unit vector
            r = math.sqrt(cn*cn+ce*ce+cd*cd)
            # Calculate line of intersection or pole to plane
            trd, plg = CartToSph(cn/r,ce/r,cd/r)
            # If intersection of two planes
            if ans0 == 'i':
                ans1 = trd
                ans2 = plg
            # Else if plane containing two dips,
            # calculate plane from its pole
            elif ans0 == 'a':
                ans1, ans2 = Pole(trd,plg,0)

    return ans1, ans2
```

The notebook ch4-4.ipynb below illustrates the use of the function *Angles* to solve several interesting problems. Let's start with the following problem from Leyshon and Lisle (1996): Two limbs of a chevron fold (A and B) have orientations (RHR) as follows: Limb A = 120/40, Limb B = 250/60. Determine: (a) the trend and plunge of the hinge line of the fold, (b) the rake of the hinge line in limb A, (c) the rake of the hinge line in limb B.

```python
import math
pi = math.pi

from Angles import Angles as Angles

# Strike and dip of the limbs in radians
str1 = 120 * pi/180 # SW dipping limb
dip1 = 40 * pi/180
str2 = 250 * pi/180 # NE dipping limb
dip2 = 60 * pi/180

# (a) Chevron folds have planar limbs. The hinge
# of the fold is the intersection of the limbs
htrd, hplg = Angles(str1,dip1,str2,dip2,'i')
print('Hinge trend = {:.1f}, plunge {:.1f}'.format(htrd*180/
    pi,hplg*180/pi))

# The rake of the hinge on either limb is the angle
# between the hinge and the strike line on the limb.
# This line is horizontal and has plunge = 0
plg = 0

# (b) For the SW dipping limb
ang1, ang2 = Angles(str1,plg,htrd,hplg,'l')
print('Rake of hinge in SW dipping limb = {:.1f}'.format(ang1
    *180/pi))

# (c) And for the NE dipping limb
ang1, ang2 = Angles(str2,plg,htrd,hplg,'l')
print('Rake of hinge in NE dipping limb = {:.1f}'.format(ang1
    *180/pi))
```

Output:
Hinge trend = 265.8, plunge 25.3
Rake of hinge in SW dipping limb = 138.4
Rake of hinge in NE dipping limb = 29.5

Let's do another problem from the same book: A quarry has two walls, one trending 002°and the other 135°. The apparent dip of bedding on the faces are 40°N and 30°SE respectively. Calculate the strike and dip of bedding.

```
# The apparent dips are just two lines on bedding
# These lines have orientations:
trd1 = 2 * pi/180
plg1 = 40 * pi/180
trd2 = 135 * pi/180
plg2 = 30 * pi/180

# Calculate bedding from these two apparent dips
strike, dip = Angles(trd1,plg1,trd2,plg2,'a')
print('Bedding strike = {:.1f}, dip {:.1f}'.format(strike*180/pi,dip*180/pi))
```

Output:
Bedding strike = 333.9, dip 60.7

And the final problem from Leyshon and Lisle (1996): Slickenside lineations trending 074°occur on a fault with orientation 300/50 (RHR). Determine the plunge of these lineations and their rake in the plane of the fault.

```
# The lineation on the fault is just the intersection
# of a vertical plane with a strike equal to
# the trend of the lineation, and the fault
str1 = 74 * pi/180
dip1 = 90 * pi/180
str2 = 300 * pi/180
dip2 = 50 * pi/180

# Find the intersection of these two planes which is
# the lineation on the fault
ltrd, lplg = Angles(str1,dip1,str2,dip2,'i')
print('Slickensides trend = {:.1f}, plunge {:.1f}'.format(ltrd*180/pi,lplg*180/pi))

# And the rake of this lineation is the angle
# between the lineation and the strike line on the fault
plg = 0
ang1, ang2 = Angles(str2,plg,ltrd,lplg,'l')
print('Rake of slickensides = {:.1f}'.format(ang1*180/pi))
```

Output:
Slickensides trend = 74.0, plunge 40.6
Rake of slickensides = 121.8

There are many interesting problems you can solve using the function *Angles*. You will find more problems in the Exercises section.

4.4.3 Three point problem

The three point problem is a fundamental problem in geology. It derives from the fact that three non-collinear points on a plane define the orientation of the plane. The graphical solution to this problem is introduced early in the Geosciences Bachelor. It involves finding the strike line (a line connecting two points of equal elevation) on the plane, and the dip from two strike lines (two structure contours) on the plane.

There is however an easier, faster, and perhaps more accurate solution to this problem using linear algebra and computation: The three points on the plane define two lines, and the downward unit vector parallel to the cross product between these two lines, is the pole to the plane, from which the orientation of the plane can be estimated. The Python function *ThreePoint* computes the strike and dip of plane from the east (**E**), north (**N**), and up (**U**) coordinates of three points on the plane:

```
import numpy as np
from numpy import linalg as la
from CartToSph import CartToSph as CartToSph
from Pole import Pole as Pole

def ThreePoint(p1,p2,p3):
    '''
    ThreePoint calculates the strike (strike) and dip (dip)
    of a plane given the east (E), north (N), and up (U)
    coordinates of three non-collinear points on the plane

    p1, p2 and p3 are 1 x 3 vectors defining the location
    of the points in an ENU coordinate system. For each one
    of these vectors the first entry is the E coordinate,
    the second entry the N coordinate, and the third entry
    the U coordinate

    NOTE: strike and dip are returned in radians and they
    follow the right-hand rule format

    ThreePoint uses functions CartToSph and Pole
    '''
    # if points are given as lists,
    # they must be converted to np.arrays
```

```
    p1 = np.array(p1)
    p2 = np.array(p2)
    p3 = np.array(p3)
    # make vectors v (p1 - p3) and u (p2 - p3)
    v = p1 - p2
    u = p2 - p3

    # take the cross product of v and u
    vcu = np.cross(v,u)

    # make this vector a unit vector
    mvcu = la.norm(vcu) # magnitude of the vector
    uvcu = vcu/mvcu # unit vector

    # make the pole vector in NED coordinates
    p = [uvcu[1], uvcu[0], -uvcu[2]]

    # Make pole point downwards
    if p[2] < 0:
        p = [-elem for elem in p]

    # find the trend and plunge of the pole
    trd, plg = CartToSph(p[0],p[1],p[2])

    # find strike and dip of plane
    strike, dip = Pole(trd, plg, 0)

    return strike , dip
```

Let's use this function in the following example. The geologic map in Fig. 4.6 shows a sequence of sedimentary units dipping south (you can figure out this by looking at the outcrop V in the valley). Three points 1, 2 and 3 are located on the base of unit S. From the **EN** origin at the lower left corner of the map, and the topographic contours, the **ENU** coordinates of the points are:

point1 = [509, 2041, 400]

point2 = [1323, 2362, 500]

point3 = [2003, 2913, 700]

Calculate the strike and dip of the plane. The Python notebook ch4-5.ipynb below shows the solution of this problem:

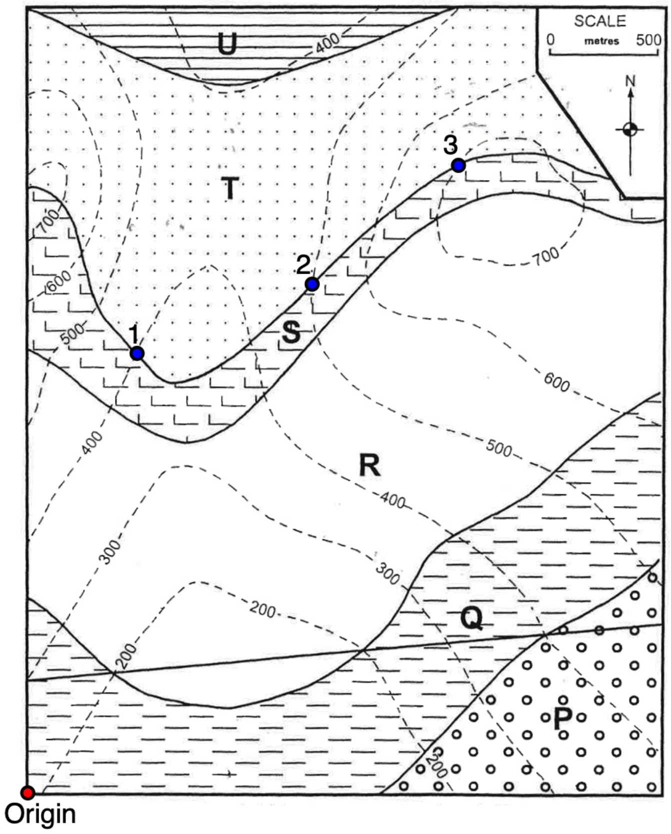


Figure 4.6: Geologic map of sedimentary units dipping south (Bennison et al., 2011). Points 1 to 3 on the base of unit S are used to estimate the orientation of bedding.

```
import numpy as np
pi = np.pi
from ThreePoint import ThreePoint as ThreePoint

# ENU coordinates of the three points
p1 = [509, 2041, 400]
p2 = [1323, 2362, 500]
p3 = [2003, 2913, 700]

# Compute the orientation of the plane
strike, dip = ThreePoint(p1,p2,p3)
print('Plane strike = {:.1f}, dip = {:.1f}'.format(strike
    *180/pi,dip*180/pi))
```

Output:
Plane strike = 84.5, dip = 22.5

4.5 Uncertainties

As we saw in section 3.2.5, strike and dip or trend and plunge measurements have errors (Figs. 3.6 and 3.7), and these errors will propagate in any computation making use of these angles. Also, as accurate as GPS and elevation measurements are today, they will also have errors. Thus, functions *Angles* and *ThreePoint* lack this important element of uncertainty.

Suppose that $x, \ldots, z$ are measured with uncertainties (or errors) $\delta x, \ldots, \delta z$ and the measured values are used to compute the function $q(x, \ldots, z)$. If the uncertainties in $x, \ldots, z$ are independent and random, then the uncertainty (or error) in q is (Taylor, 1997):

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \ldots + \left(\frac{\partial q}{\partial z}\delta z\right)^2} \tag{4.16}$$

This formula is easy to calculate for a few operations, but it can become quite complex for a long chain of operations. Fortunately, there is a Python package that handles calculations with numbers with uncertainties. This package is called *uncertainties* and it was developed by Eric Lebigot. You can find details about the package as well as instructions for installation in the uncertainties website. If you have the Python *pip* package-management system, you can install the uncertainties package by entering in a terminal:

```
pip install --upgrade uncertainties
```

After this, you can use the uncertainties package in your functions. The Python functions *AnglesU* and *ThreePointU* in our git repository, are the corresponding *Angles* and *ThreePoint* functions with uncertainties. We don't list these functions here (you can check them from our git repository), but rather illustrate their use in the following notebook ch4-6.ipynb.

In the first problem on page 64, the uncertainty in strike is 4°and in dip is 2°. What is the uncertainty in the hinge orientation and its rake on the limbs? This problem can be solved as follows:

```
import math
pi = math.pi
import uncertainties as unc
from AnglesU import AnglesU as AnglesU

# Strike and dip of the limbs in radians
str1 = 120 * pi/180 # SW dipping limb
dip1 = 40 * pi/180
str2 = 250 * pi/180 # NE dipping limb
dip2 = 60 * pi/180

# Errors in radians
ustr = 4 * pi/180 # Error in strike
udip = 2 * pi/180 # Error in dip

# Create the input values with uncertainties
str1 = unc.ufloat(str1, ustr)  # str1 = str1 +/-ustr
dip1 = unc.ufloat(dip1, udip)  # dip1 = dip1 +/-udip
str2 = unc.ufloat(str2, ustr)  # str2 = str2 +/-ustr
dip2 = unc.ufloat(dip2, udip)  # dip2 = dip2 +/-udip

# (a) Chevron folds have planar limbs. The hinge
# of the fold is the intersection of the limbs
htrd, hplg = AnglesU(str1,dip1,str2,dip2,'i')
print('Hinge trend = {:.1f}, plunge {:.1f}'.format(htrd*180/pi,hplg*180/pi))

# The rake of the hinge on either limb is the angle
# between the hinge and the strike line on the limb.
# This line is horizontal and has plunge = 0
plg = unc.ufloat(0, udip)  # plg = 0 +/-udip

# (b) For the SW dipping limb
ang1, ang2 = AnglesU(str1,plg,htrd,hplg,'l')
print('Rake of hinge in SW dipping limb = {:.1f}'.format(ang1*180/pi))

# (c) And for the NE dipping limb
ang1, ang2 = AnglesU(str2,plg,htrd,hplg,'l')
print('Rake of hinge in NE dipping limb = {:.1f}'.format(ang1*180/pi))
```

Output:
Hinge trend = 265.8+/-3.3, plunge 25.3+/-2.6
Rake of hinge in SW dipping limb = 138.4+/-4.6
Rake of hinge in NE dipping limb = 29.5+/-3.5

In the map of Fig. 4.6, the error in East and North coordinates is 10 m, and in elevation is 5 m. What is uncertainty in the strike and dip of the T-S contact?

```
import numpy as np
pi = np.pi
from ThreePointU import ThreePointU as ThreePointU

# ENU coordinates of the three points
# with uncertainties in E-N = 10, and U = 5
p1 = [unc.ufloat(509, 10), unc.ufloat(2041, 10), unc.ufloat(400, 5)]
p2 = [unc.ufloat(1323, 10), unc.ufloat(2362, 10), unc.ufloat(500, 5)]
p3 = [unc.ufloat(2003, 10), unc.ufloat(2913, 10), unc.ufloat(700, 5)]

# Compute the orientation of the plane
strike, dip = ThreePointU(p1,p2,p3)
print('Plane strike = {:.1f}, dip = {:.1f}'.format(strike*180/pi,dip*180/pi))
```

Output:
Plane strike = 84.5+/-3.5, dip = 22.5+/-2.7

4.6 Exercises

Problems 1-3 are from Marshak and Mitra (1988). Solve these three problems using the function *Angles*.

1. A fault surface has an orientation (RHR) 190/80. Slickenlines on the fault trend 300°.
   (a) What is the plunge of the lineation?
   (b) What is the rake of the lineation on the fault?
2. A shale bed has an orientation (RHR) 115/42. What is the apparent dip of the bed in the direction 265°?
3. A sandstone bed strikes 140°across a stream. The stream flows down a narrow gorge with vertical walls. The apparent dip of the bed on the walls of the gorge is 095/25. What is the true dip of the bed?

4. In the geologic map of Fig. 4.7, points 1 to 9 have the following **ENU** coordinates:

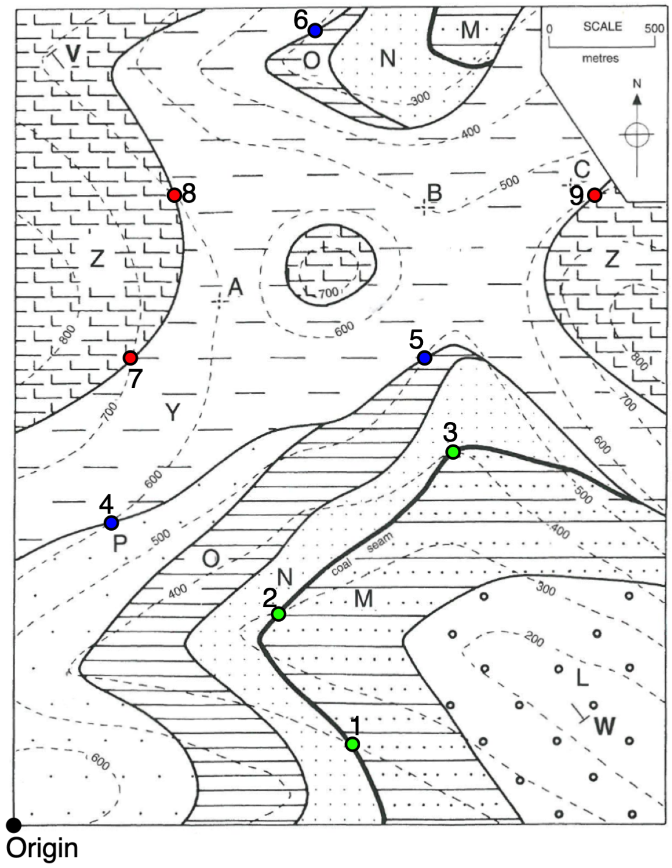


Figure 4.7: Map for exercise 2. This is map 10 of Bennison et al. (2011).

point1 = [1580, 379, 400]

point2 = [1234, 992, 300]

point3 = [2054, 1753, 400]

point4 = [448, 1424, 600]

point5 = [1921, 2195, 500]

point6 = [1408, 3737, 300]

point7 = [536, 2196, 700]

point8 = [743, 2963, 600]

point9 = [2720, 2963, 600]

(a) Compute the strike and dip of the coal seam (points 1-3).

(b) Compute the strike and dip of the contact where the blue points 4-6 are located. What kind of contact is this? Is the coal seam below or above this contact?

(c) Compute the strike and dip of the contact between units Y and Z (points 7-9). Is this contact below or above the contact in b?

(d) The line of section V-W has a trend of 142°. What is the apparent dip of the three contacts above along the section?

(e) Draw a schematic cross section along line V-W

Hint: Use function *ThreePoint* to solve a, b and c. Use function *Angles* to solve d.

5. The map of Fig. 4.8 shows an area of a reconnaissance survey in the Appalachian Valley and Ridge Province of western Maryland, USA. On the western half of the map, the contact between a shale horizon (B) and a sandstone unit (C) has been located in two areas. Three points on this contact have the following **ENU** coordinates:

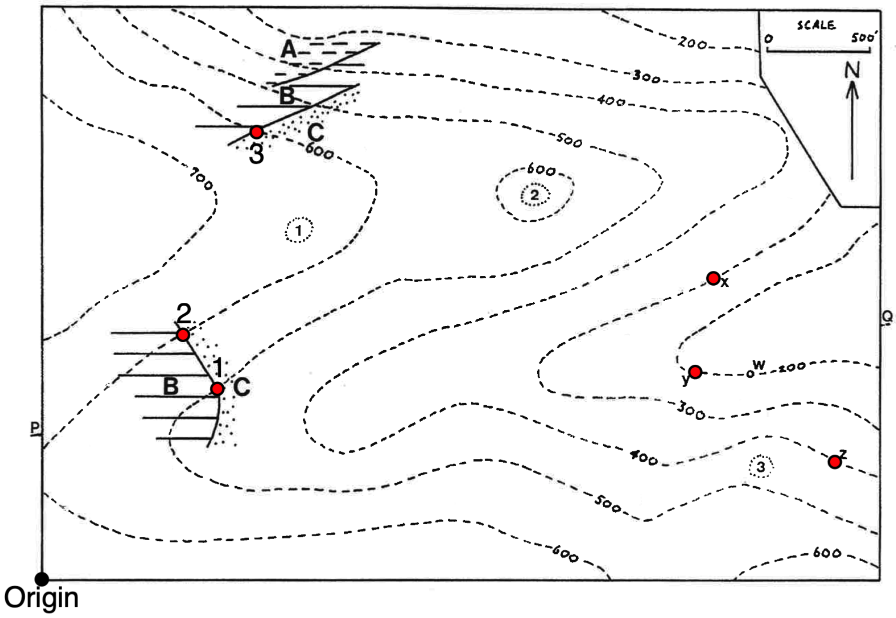


Figure 4.8: Map for exercise 3

point1 = [862, 943, 500]

point2 = [692, 1212, 600]

point3 = [1050, 2205, 600]

In the eastern half of the map, the contact between B and C was found exposed at three locations labelled x, y, and z. The **ENU** coordinates of these points are:

pointx = [3298, 1487, 300]

pointy = [3203, 1031, 200]

pointz = [3894, 590, 400]

(a) Compute the strike and dip of the contact on the western half of the map.

(b) Compute the strike and dip of the contact in the eastern half of the map.

(c) What kind of structure are the western and eastern contacts part of?

(d) Compute the intersection of the western and eastern contacts. What does this line represent?

Hint: Use function *ThreePoint* to solve a and b. Use function *Angles* to solve d.

6. In problem 3, the error in azimuth is 5°and in apparent dip is 3°. What is the uncertainty in the true dip of the bed? *Hint*: Use function *AnglesU*.

7. In the map of Fig. 4.7, the east and north coordinates of the points have 15 m error, and the elevations have 5 m error.

   (a) What is the uncertainty in the strike and dip of the coal seam?

   (b) What is the uncertainty in the strike and dip of the unconformity?

   (c) What is the angle between the unconformity and the coal seam and what is the uncertainty in this angle?

   Hint: Use functions *ThreePointU* and *AnglesU*.

References

Allmendinger, R.W., Cardozo, N. and Fisher, D.W. 2012. Structural Geology Algorithms: Vectors and Tensors. Cambridge University Press, 302 p.

Bennison, G.M., Olver, P.A. and Moseley, K.A. 2011. An Introduction to Geological Structures and Maps, 8th edition. Hodder Education, 168 p.

Fisher, N.I., Lewis, T. and Embleton, B.J.J. 1987. Statistical analysis of spherical data. Cambridge University Press, 329 p.

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Marshak, S. and Mitra, G. 1988. Basic Methods of Structural Geology. Prentice Hall, 446 p.

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Chapter 5

Transformations

Many problems in geology, as in life, become simpler when viewed from another perspective. For example, when studying the movement of continents through time because of plate tectonics (Fig. 5.1a), two coordinate systems are required, one in a present-day geographic frame, and another one attached to the continent. Or to analyze a fault (Fig. 5.1b), we need one coordinate system attached to the fault (with one axis parallel to the pole and another to the slickensides), which we want to relate to the more familiar **NED** system. A change in coordinate system is called a coordinate transformation and this is an operation that happens everytime and everywhere. Computer games, flight simulators, and 3D interpretation programs rely heavily on coordinate transformations.

5.1 Transforming coordinates and vectors

5.1.1 Coordinate transformations

A transformation involves a change in the origin and orientation of the coordinate system. We will refer to the new axes as the primed coordinate system, $\mathbf{X}'$, and the old coordinate system as the unprimed system, $\mathbf{X}$ (Fig. 5.2a). Let's assume the origin of the old and new coordinate systems is the same. The change in orientation of the new coordinate system is defined by the angles between the new coordinate axes and the old axes. These angles are marked systematically, the first subscript refers to the new coordinate

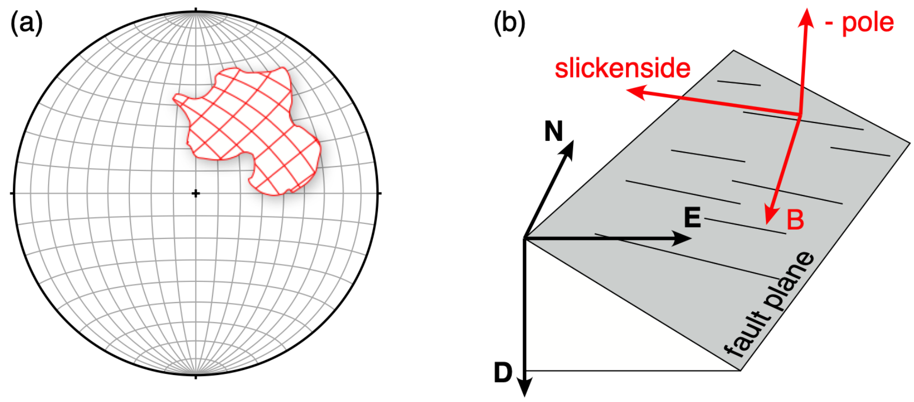

Figure 5.1: Examples of coordinate transformations in geology. **a.** Continental drift, **b.** A fault plane. Red is the coordinate system for analysis, and gray is the geographic coordinate system. Modified from Allmendinger et al. (2012).

axis and the second subscript to the old coordinate axis. For example, θ_{23} is the angle between the $\mathbf{X_2}'$ axis and the $\mathbf{X_3}$ axis (Fig. 5.2a).

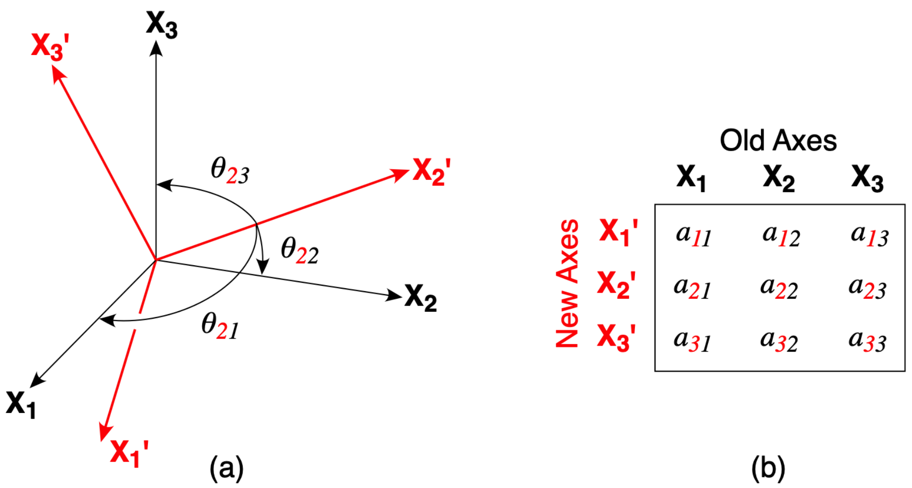

Figure 5.2: **a.** A rotation of the coordinate system. The new axes are primed and red. Only three of the nine possible angles are shown. **b.** Graphic device for remembering how the subscript of the direction cosines relate to the new and the old axes. Modified from Allmendinger et al. (2012).

To define the transformation, we use the cosines of these angles rather than the angles themselves (Fig. 5.2b). These are the direction cosines of the new axes with respect to the old axes. The subscript convention is exactly the same. For example, a_{23} is the direction cosine of the $\mathbf{X_2}'$ axis with respect to the $\mathbf{X_3}$ axis. There are nine direction cosines that form a 3×3 array, where each row refers to a new axis and each column to an old axis (Fig. 5.2b). This matrix **a** of direction cosines is known as the *transformation matrix*, and it is the key element to determine for a transformation.

Fortunately, not all the nine direction cosines in the transformation matrix are independent. Since the base vectors of the new coordinate system are unit vectors, their magnitude is 1:

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1 \end{aligned} \tag{5.1}$$

and because the base vectors of the new coordinate system are perpendicular to each other, the dot product of two of these axes is zero:

$$\begin{aligned} a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0 \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} &= 0 \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0 \end{aligned} \tag{5.2}$$

Equations 5.1 and 5.2 are known as the *orthogonality relations*. Since we have nine unknowns (i.e. the nine direction cosines) and six equations, there are only three independent direction cosines in the transformation matrix. If we know three of the direction cosines defining the transformation, we can calculate the other six.

5.1.2 Transformation of vectors

Once we know the transformation matrix **a** and the components of a vector in the old coordinate system, we can calculate the components of the vector in the new coordinate system. The equations are fairly simple:

$$\begin{aligned} {v_1}' &= a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ {v_2}' &= a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ {v_3}' &= a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{aligned} \tag{5.3}$$

or using the Einstein summation notation:

$${v_i}' = a_{ij}v_j \tag{5.4}$$

where i is the free suffix, and j is the dummy suffix. Assuming that $(i,\ j = 1,2,3)$, Equation 5.4 represents three separate equations (the three indexes of i), each one with three terms (the three indexes of j). These equations are easy to implement in Python code:

```
for i in range(0,3,1):
    v_new[i] = 0
    for j in range(0,3,1):
        v_new[i] = a[i][j]*v_old[j] + v_new[i]
```

You will see this code snippet repeatedly in the functions of this chapter. Linear algebra is very elegant. To convert the vector from the new coordinate system back to the old coordinate system, you just need to do:

$$v_i = a_{ji}{v_j}' \tag{5.5}$$

or in Python code:

```
for i in range(0,3,1):
    v_old[i] = 0
    for j in range(0,3,1):
        v_old[i] = a[j][i]*v_new[j] + v_old[i]
```

5.1.3 A simple transformation: From ENU to NED

There is a simple coordinate transformation that nicely illustrates the theory above: the transformation from an **ENU** to a **NED** coordinate system (Fig.

4.1). It is simple because the angles involved are either 0, 90, or 180°. The direction cosines of the new axes (**NED**) with respect to the old axes (**ENU**), and the transformation matrix **a** is:

$$\mathbf{a} = \begin{bmatrix} \cos 90 & \cos 0 & \cos 90 \\ \cos 0 & \cos 90 & \cos 90 \\ \cos 90 & \cos 90 & \cos 180 \end{bmatrix} \tag{5.6}$$

which simplifies to:

$$\mathbf{a} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{5.7}$$

When we use this matrix in Eq. 5.4, we get:

$$v_1{}' = v_2; \quad v_2{}' = v_1; \quad v_3{}' = -v_3; \tag{5.8}$$

Notice that in this special case **a** is symmetric ($a_{ij} = a_{ji}$), so the elements of **a** are also the direction cosines of the axes **ENU** with respect to the axes **NED**. In the following section, we will look at more complicated coordinate transformations, but the principles will still be the same.

5.2 Applications

5.2.1 Stratigraphic thickness

The thickness of a stratigraphic unit is the perpendicular distance between the parallel planes bounding the unit. This is also called the true or stratigraphic thickness (Ragan, 2009). A general problem though is that points on the planes bounding the unit, are commonly given in a geographic (e.g. **ENU**) coordinate system. One therefore must use ortographic projections and trigonometry to determine the stratigraphic thickness of the unit from the points (Ragan, 2009).

An easier approach is to use a transformation. Figure 5.3 illustrates the situation. Points on the top and base of the unit are given in an **ENU** coordinate system. We can transform these points into a coordinate system attached to the bounding planes, where the strike (RHR) of the planes is the first axis, the dip the second axis, and the pole to the planes the third axis. We will call this coordinate system the **SDP** system. The thickness of the unit is just the difference between the **P** coordinate of a point on the top of the unit and the **P** coordinate of any point on the base.

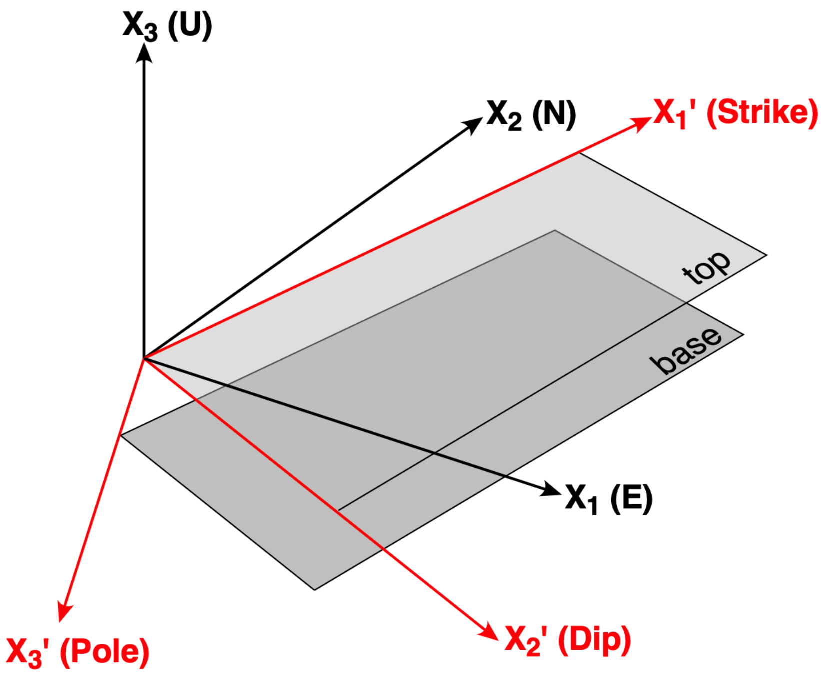

Figure 5.3: Coordinate transformation from an **ENU** to a Strike-Dip-Pole (**SDP**) coordinate system. The thickness of the unit can be calculated by subtracting the **P** coordinates of any point on the top and any point on the base. Modified from Allmendinger (2019).

We can find the elements of the matrix **a** for this transformation using trigonometry. However, we are going to follow a more didactic approach. We will reference the two **ENU** and **SDP** coordinate systems with respect to the Structural Geology **NED** coordinate system, and use the dot product to determine the direction cosines of **SDP** into **ENU**.

The direction cosines of the **ENU** coordinate system with respect to the **NED** coordinate system are given by Eq. 5.7. The direction cosines of the

SDP coordinate system with respect to the **NED** coordinate system, can be found using Table 4.1:

$$
\begin{gathered}
\mathbf{S} = [\cos(strike), \sin(strike), 0] \\
\mathbf{D} = [\cos(strike + 90)\cos(dip), \sin(strike + 90)\cos(dip), \sin(dip)] \\
\mathbf{P} = [\cos(strike - 90)\cos(90 - dip), \sin(strike - 90)\cos(90 - dip), \sin(90 - dip)]
\end{gathered}
$$

which simplifies to:

$$
\begin{aligned}
\mathbf{S} &= [\cos(strike), \sin(strike), 0] \\
\mathbf{D} &= [-\sin(strike)\cos(dip), \cos(strike)\cos(dip), \sin(dip)] \\
\mathbf{P} &= [\sin(strike)\sin(dip), -\cos(strike)\sin(dip), \cos(dip)]
\end{aligned} \tag{5.9}
$$

Now, the transformation matrix **a** from the **ENU** to the **SDP** coordinate system has as elements the direction cosines of the new **SDP** axes into the old **ENU** axes. From Eq. 4.9, one can see that these are just the dot product between the new and old axes:

$$
\mathbf{a} = \begin{bmatrix} \mathbf{S}\cdot\mathbf{E} & \mathbf{S}\cdot\mathbf{N} & \mathbf{S}\cdot\mathbf{U} \\ \mathbf{D}\cdot\mathbf{E} & \mathbf{D}\cdot\mathbf{N} & \mathbf{D}\cdot\mathbf{U} \\ \mathbf{P}\cdot\mathbf{E} & \mathbf{P}\cdot\mathbf{N} & \mathbf{P}\cdot\mathbf{U} \end{bmatrix}
$$

which is equal to:

$$
\mathbf{a} = \begin{bmatrix} \sin(strike) & \cos(strike & 0 \\ \cos(strike)\cos(dip) & -\sin(strike)\cos(dip) & -\sin(dip) \\ -\cos(strike)\sin(dip) & \sin(strike)\sin(dip) & -\cos(dip) \end{bmatrix} \tag{5.10}
$$

So if point 1 is at the top of the unit and has coordinates $[E_1, N_1, U_1]$ and point 2 is at the base of the unit and has coordinates $[E_2, N_2, U_2]$, the **P** coordinates of these points are:

$$
\begin{aligned}
P_1 &= -\cos(strike)\sin(dip)E_1 + \sin(strike)\sin(dip)N_1 - \cos(dip)U_1 \\
P_2 &= -\cos(strike)\sin(dip)E_2 + \sin(strike)\sin(dip)N_2 - \cos(dip)U_2
\end{aligned} \tag{5.11}
$$

and the thickness of the unit is:

$$t = P_2 - P_1 \tag{5.12}$$

The Python function *TrueThickness* calculates the thickness of a unit given the strike and dip of the unit, and the **ENU** coordinates of two, top and base, points.

```python
import numpy as np

def TrueThickness(strike, dip, top, base):
    '''
    TrueThickness calculates the thickness (t) of a unit
    given the strike (strike) and dip (dip) of the unit,
    and points at its top (top) and base (base)

    top and base are 1 x 3 vectors defining the location
    of top and base points in an ENU coordinate system.
    For each of these vectors, the first entry is the E
    coordinate, the second entry the N coordinate,
    and the third entry the U coordinate

    NOTE: strike and dip should be input in radians
    '''
    # make the transformation matrix from ENU coordinates
    # to SDP coordinates. Eq. 5.10
    a = [[np.sin(strike), np.cos(strike), 0],
    [np.cos(strike)*np.cos(dip), -np.sin(strike)*np.cos(dip),
    -np.sin(dip)],
    [-np.cos(strike)*np.sin(dip), np.sin(strike)*np.sin(dip),
    -np.cos(dip)]]

    # transform the top and base points
    # from ENU to SDP coordinates. Eq. 5.4
    topn = np.zeros(3)
    basen = np.zeros(3)
    for i in range(0,3,1):
        for j in range(0,3,1):
            topn[i] = a[i][j]*top[j] + topn[i]
            basen[i] = a[i][j]*base[j] + basen[i]

    # compute the thickness of the unit. Eq. 5.12
    t = np.abs(basen[2] - topn[2])

    return t
```

Let's use this function to determine the thickness of the sedimentary units T to Q in the geologic map of Fig. 4.6. This time we have points at the top and base of these units (Fig. 5.4), and their **ENU** coordinates are:

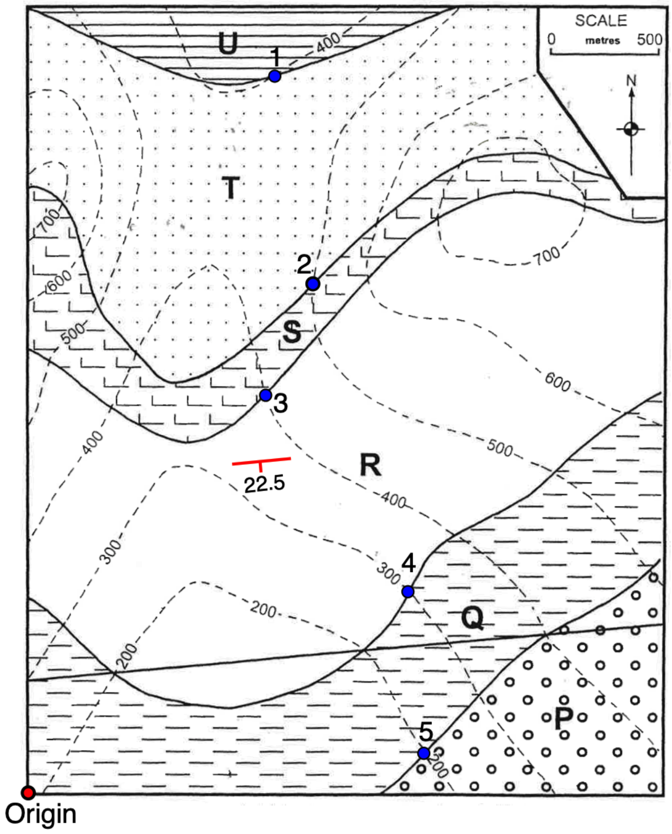

Figure 5.4: Geologic map of sedimentary units with an orientation 84.5/22.5 (RHR) (Bennison etal., 2011). Points at the top and base of units T to Q are used to determine the thickness of these units.

point1 = [1147, 3329, 400]

point2 = [1323, 2362, 500]

point3 = [1105, 1850, 400]

point4 = [1768, 940, 300]

point5 = [1842, 191, 200]

The Python notebook ch5-1.ipynb shows the solution of this problem:

5.2.2 Outcrop trace of a plane

5.2.3 Downplunge projection

5.2.4 Rotations

5.2.5 Plotting great and small circles on stereonets

5.3 Exercises

References