The card pairs problem

Ernest A. C.

July 13, 2018

The problem was posted on Reddit[1] in 2017 and is stated as follows:

> You have one hundred cards numbered $1, \ldots, 100$. You randomly pair all of the cards. Whichever of the pair is a higher number is considered to be a 'winner'. On average, what percentage of cards from the upper half $(51, \ldots, 100)$ will be considered to be 'winners'?

Let A and B be the sets of cards in the lower and upper halves of the deck respectively, and let W be the number of winners in B after randomly pairing all the cards in the deck with one another. We are asked to find the expected value of $W/50$. Since any card in B wins when paired with any card in A, the largest amount of upper half winners occurs when every card in B is paired with some card in A, in which case $w = 50$. Conversely, the least number of upper half winners occurs when every card in B is paired with another card in B, in which case $w = 25$. It follows that that $\Pr(W = w) = 0$ for any $w > 50$ and for any $w < 25$. Therefore we can write the expected value of $W/50$ as

$$\mathrm{E}(W/50) = \frac{1}{50} \sum_{w=25}^{50} w \Pr(W = w). \tag{1}$$

Our task now is to find $\Pr(W = w)$ for all $w \in \{25, \ldots, 50\}$. Let M be the number of cards from the upper half of the deck that are paired with cards from the lower half, and let R be the number of cards from the upper half that are paired with other cards from the upper half. The number of winners from the upper half of the deck is

$$w = M + \frac{R}{2}.$$

Since $M + R = 50$, it is clear that M and R are uniquely determined by the number of winners from the upper half, w. For example, if $w = 34$, then $M = 18$ and $R = 32$.

There are $\binom{50}{M}$ different M-element subsets of B. The first card in the subset can be paired with any of the 50 cards in A. The second card can be paired with any of the 49 remaining cards in A, and so forth. The R remaining cards in B are paired with one another. The first card can be paired with $R - 1$ cards, the second with $R - 3$ cards, and so forth. The same thing applies to the R remaining cards in

[1] `https://www.reddit.com/r/statistics/comments/7eginx/`

A. Thus, by the multiplication principle, the number of ways the cards in the deck can be paired with one another resulting in a particular number of winners from the upper half of the deck is

$$\binom{50}{M} \prod_{k=1}^{M} (50 - k + 1) \left(\prod_{k=1}^{R/2} (R - 2k + 1) \right)^2,$$

where $R = 100 - 2w$ and $M = 50 - R$. Dividing by the size of the sample space

$$\prod_{k=1}^{50} (100 - 2k + 1)$$

we get the probability $\Pr(W = w)$, and substituting in (1) we find that

$$\mathrm{E}(W/50) = \frac{37.6262}{50} = 0.7525.$$

Thus, on average, 75.25% of the cards from the upper half of the deck will be considered to be winners.