Technical details of equilibrium of my model

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May 2, 2018

1 Preliminaries

Final goods production technology:

$$Y = L_F^{\beta}\left(\left(\int_0^1 q_j^{\beta} x_j^{1-\beta} dj\right)^{1/(1-\beta)}\right)^{1-\beta} \tag{1}$$

Final goods optimization:

$$\max_{\{x_j\}_{j\in[0,1]}} \int_0^1 q_j^{\beta} x_j^{1-\beta}$$

Subject to constraint:

$$\int_0^1 p_j x_j dj \leq E$$

FOCs from Lagrangean: for each $j \in [0,1]$,

$$\begin{aligned}(1-\beta)q_j^{\beta}x_j^{-\beta} &= \lambda p_j \\ q_j^{\beta} &= \lambda p_j (1-\beta)^{-1} x_j^{\beta}\end{aligned}$$

where λ is a Lagrange multiplier.

For all i, j, we have

$$x_i = x_j \frac{q_i}{q_j}\left(\frac{p_j}{p_i}\right)^{1/\beta}$$

Multiplying both sides of the above by p_i and integrating yields

$$E = \int_0^1 p_i x_i di = \int_0^1 p_i x_j \frac{q_i}{q_j} \left(\frac{p_j}{p_i}\right)^{1/\beta} dj \tag{2}$$

Denote the elasticity of substitution by σ; have $\sigma = \frac{1}{\beta}$. Define the price index

$$P = \left(\int_0^1 q_i p_i^{1-\sigma} di \right)^{1/(1-\sigma)} \tag{3}$$

Substituting (3) into (2) yields the final demand equation:

$$\begin{align} \frac{x_j}{q_j} &= \frac{E}{P}\left(\frac{p_j}{P}\right)^{-1/\beta} \tag{4} \\ &= \frac{E}{P}\left(\frac{p_j}{P}\right)^{-\sigma} \tag{5} \end{align}$$

Define effective aggregate capital input:

$$\begin{align} X &\equiv \left(\int_0^1 q_j^{\beta} x_j^{1-\beta} dj \right)^{1/(1-\beta)} \tag{6} \\ &= \left(\int_0^1 q_j^{1/\sigma} x_j^{(\sigma-1)/\sigma} dj \right)^{\sigma/(\sigma-1)} \tag{7} \end{align}$$

Equations (3), (5), and (7) imply that the price of obtaining one unit of X is the price index P:

$$\begin{align} X &= \left(\int_0^1 q_j^{1/\sigma} x_j^{(\sigma-1)/\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \left(\int_0^1 q_j^{1/\sigma} q_j^{(\sigma-1)/\sigma} \frac{E^{(\sigma-1)/\sigma}}{P^{((\sigma-1)/\sigma)(1-\sigma)}} p_j^{1-\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \frac{E}{P^{1-\sigma}} \left(\int_0^1 q_j p_j^{1-\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \frac{E}{P^{1-\sigma}} P^{-\sigma} \\ &= \frac{E}{P} \tag{8} \end{align}$$

Plugging into (1) yields

$$Y = L_F^{\beta} X^{1-\beta}$$

Profit maximization by the final goods firms over L_F, X implies

$$\beta L_F^{\beta-1} X^{1-\beta} = w \tag{9}$$

$$(1-\beta) L_F^{\beta} X^{-\beta} = P \tag{10}$$

The inverse demand of the intermediate goods producers can be derived by using rearranging (10) to obtain an expression for X in terms of L_F, P and parameters; and then substituting this and (8) into (5) to obtain an expression relating x_j, p_j, L_F.

First, rearranging (10) we get

$$X = \left(\frac{1-\beta}{P}\right)^{1/\beta} L_F \tag{11}$$

Substituting (8) into (4) yields:

$$\frac{x_j}{q_j} = X\left(\frac{p_j}{P}\right)^{-1/\beta} \tag{12}$$

Now substitute (11) into (12) to obtain

$$\frac{x_j}{q_j} = \left(\frac{p_j}{1-\beta}\right)^{-1/\beta} L_F$$

$$p_j = (1-\beta) L_F^{\beta} q_j^{\beta} x_j^{-\beta} \tag{13}$$

Since the producers face the same demand curve for their goods as in AK 2017 (conditional on L_F), prices and quantities are the same in equilibrium (except for extra constant $(1-\beta)$ in x_j expression):

$$x_j = \left[\frac{(1-\beta)^2 \overline{q}}{w}\right]^{1/\beta} L_F q_j \tag{14}$$

$$p_j = \frac{w}{(1-\beta)\overline{q}} \tag{15}$$

Now we plug back into the profit equation to obtain equilibrium profits (as a function of

$L_F, q_j, w)$.[1]

$$\begin{aligned}
\pi_j &= (p_j - c_j)x_j \\
&= \left(\frac{1}{1-\beta} - 1\right)\frac{w}{\bar{q}}x_j \\
\pi_j &= \beta(1-\beta)^{\frac{2-\beta}{\beta}}\left(\frac{\bar{q}}{w}\right)^{\frac{1-\beta}{\beta}} L_F q_j \qquad (16)
\end{aligned}$$

Next, since $p_j \equiv \bar{p}$ in equilibrium, we can derive an expression for P in terms of $\beta, w, \bar{q}$,

$$\begin{aligned}
P &= \left(\int_0^1 q_j p_j^{\frac{\beta-1}{\beta}} dj\right)^{\frac{\beta}{\beta-1}} \\
P &= \bar{p}\bar{q}^{\frac{\beta}{\beta-1}} \\
&= \frac{w}{(1-\beta)\bar{q}}\bar{q}^{\frac{\beta}{\beta-1}} \\
P(w) &= \frac{w}{1-\beta}\bar{q}^{\frac{1}{\beta-1}} \qquad (17)
\end{aligned}$$

Using (9), (10) and (17), we obtain a system of two equations in (L/X) and the wage w:

$$\begin{aligned}
\beta(\frac{L}{X})^{\beta-1} &= w \\
(1-\beta)(\frac{L}{X})^{\beta} &= P(w) \\
&= \frac{w}{1-\beta}\bar{q}^{\frac{1}{\beta-1}}
\end{aligned}$$

Solving this system for w yields[2]

$$\begin{aligned}
w &= \tilde{\beta}\bar{q} \qquad (18) \\
\tilde{\beta} &= \beta^{\beta}(1-\beta)^{2-2\beta} \qquad (19)
\end{aligned}$$

[1] If, as in AK 2017, I had multiplied the final goods technology by a factor $(1-\beta)^{-1}$, I would get $\pi = \beta(1-\beta)^{\frac{1-\beta}{\beta}}\left(\frac{\bar{q}}{w}\right)^{\frac{1-\beta}{\beta}}$, as they obtain.

[2] If I scale the final goods production function by $(1-\beta)^{-1}$ I would get $\tilde{\beta} = \beta^{\beta}(1-\beta)^{1-2\beta}$, as in AK 2017

2 Lab equipment model

If R&D were done using final goods, we can write E as a function of L_F using the equation:

$$\begin{aligned} L_F &= 1 - \int_0^1 l_j dj \\ &= 1 - \frac{E}{p} \end{aligned}$$

Further, we can substitute to obtain an expression for production in terms of L_F, E, assuming expenditures on capital goods are optimal given the quality distribution. First, do some algebra to get an expression for the optimal CES aggregator given price p, qualities $\{q_j\}_{j\in[0,1]}$ and spending E:

$$\begin{aligned} \Big(\Big(\int_0^1 q_j^\beta x_j^{1-\beta} dj\Big)^{1/(1-\beta)}\Big)^{1-\beta} &= \Big(\Big(\int_0^1 q_j^\beta \Big(\frac{q_j}{\overline{q}}\frac{E}{p}\Big)^{1-\beta} dj\Big)^{1/(1-\beta)}\Big)^{1-\beta} \\ &= \Big(\frac{1}{\overline{q}}\frac{E}{p}\Big)^{1-\beta}\Big(\Big(\int_0^1 q_j dj\Big)^{1/(1-\beta)}\Big)^{1-\beta} \\ &= \Big(\frac{1}{\overline{q}p}\Big)^{1-\beta}\overline{q}E^{1-\beta} \\ &= \overline{q}^\beta p^{\beta-1}E^{1-\beta} \end{aligned}$$

Substitute this into the final goods production function:

$$Y(L_F, E; \overline{q}) = \overline{q}^\beta p^{\beta-1} L_F^\beta E^{1-\beta}$$

This yields FOCs for L_F and E:

$$\begin{aligned} \beta\overline{q}^\beta p^{\beta-1} L_F^{\beta-1} E^{1-\beta} &= w \\ (1-\beta)\overline{q}^\beta p^{\beta-1} L_F^{\beta} E^{-\beta} &= 1 \end{aligned}$$

because the price of one unit of E is, by definition, equal to 1.

Finally recall our equation for p:

$$p = \frac{w}{\overline{q}(1-\beta)}$$

Hence we have four equations in four unknowns $\{L_F, E, w, p\}$ and parameters:

$$L_F = 1 - \frac{E}{p} \tag{20}$$

$$\beta \overline{q}^{\beta} p^{\beta-1} L_F^{\beta-1} E^{1-\beta} = w \tag{21}$$

$$(1-\beta)\overline{q}^{\beta} p^{\beta-1} L_F^{\beta} E^{-\beta} = 1 \tag{22}$$

$$p = \frac{w}{\overline{q}(1-\beta)} \tag{23}$$

This part of the model is therefore determined separately from the R&D side of the model. Intuitively, I haven't proven that there exists a closed-form solution – this is shown by Akcigit & Kerr 2017, which is exactly the same framework. To check these conditions we could substitute that solution and check there is no contradiction.

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My model In my model, R&D is done using labor drawn from the same pool as intermediate and final goods production. Now we cannot derive (20) because

$$L_F = 1 - \int_0^1 l_j^I dj - \int_0^1 l_j^{RD} dj$$

Hence, we cannot derive a formula relating E and L without appealing to $z(m), \hat{z}(m)$ in order to compute the last term in the equation above. But those require solving the HJBs, etc. The static and dynamic aspects of the model now interact.

Possible solutions The only way to eliminate this feature is to entirely decouple the production and R&D labor markets. In addition, we must assume elastic labor supply in the R&D market in order to make the model an endogenous growth model. Also note that we can't endogenize the elasticity of R&D labor supply by using some kind of decision to specialize in final goods production or R&D with some initial heterogeneity in relative productivities in each form of employment, because this couples the labor markets, eliminating the tractability. Hence, the only way to have a tractable, non-trivial model is to assume a separate population of potential R&D workers with some aggregate labor supply elasticity.

New algorithm In light of this, we need a new algorithm.

1. Guess L^{RD}, the BGP labor supply to R&D

2. Now we know the labor supply available to production, hence can solve for all static production variables L^F, L^I, w, p, π in closed form
3. Given these, solve HJBs numerically using iterative procedure described above
4. Next, solve KF equation to compute stationary distribution $\mu(m)$
5. Using $\mu(m)$ and policy functions from previous step, integrate to compute aggregate labor demand
6. Check market clearing in R&D market $L^{RD} = \int l(m) + \hat{l}(m) d\mu(m)$. If market does not clear, update guess L^{RD} and go back to Step 1

My original algorithm was needlessly complex.