

# Equilibrium production wage determination

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## 1 Preliminaries

**Final goods production technology:**

$$Y = L_F^\beta \left( \left( \int_0^1 q_j^\beta x_j^{1-\beta} dj \right)^{1/(1-\beta)} \right)^{1-\beta} \quad (1)$$

**Final goods optimization:**

$$\max_{\{x_j\}_{j \in [0,1]}} \int_0^1 q_j^\beta x_j^{1-\beta}$$

Subject to constraint:

$$\int_0^1 p_j x_j dj \leq E$$

FOCs from Lagrangean: for each  $j \in [0, 1]$ ,

$$\begin{aligned} (1 - \beta) q_j^\beta x_j^{-\beta} &= \lambda p_j \\ q_j^\beta &= \lambda p_j (1 - \beta)^{-1} x_j^\beta \end{aligned}$$

where  $\lambda$  is a Lagrange multiplier.

For all  $i, j$ , we have

$$x_i = x_j \frac{q_i}{q_j} \left( \frac{p_j}{p_i} \right)^{1/\beta}$$

Multiplying both sides of the above by  $p_i$  and integrating yields

$$E = \int_0^1 p_i x_i di = \int_0^1 p_i x_j \frac{q_i}{q_j} \left( \frac{p_j}{p_i} \right)^{1/\beta} dj \quad (2)$$

Denote the elasticity of substitution by  $\sigma$ ; have  $\sigma = \frac{1}{\beta}$ . Define the price index

$$P = \left( \int_0^1 q_i p_i^{1-\sigma} di \right)^{1/(1-\sigma)} \quad (3)$$

Substituting (3) into (2) yields the final demand equation:

$$\frac{x_j}{q_j} = \frac{E}{P} \left( \frac{p_j}{P} \right)^{-1/\beta} \quad (4)$$

$$= \frac{E}{P} \left( \frac{p_j}{P} \right)^{-\sigma} \quad (5)$$

Define effective aggregate capital input:

$$X \equiv \left( \int_0^1 q_j^\beta x_j^{1-\beta} dj \right)^{1/(1-\beta)} \quad (6)$$

$$= \left( \int_0^1 q_j^{1/\sigma} x_j^{(\sigma-1)/\sigma} dj \right)^{\sigma/(\sigma-1)} \quad (7)$$

Equations (3), (5), and (7) imply that the price of obtaining one unit of  $X$  is the price index  $P$ :

$$\begin{aligned} X &= \left( \int_0^1 q_j^{1/\sigma} x_j^{(\sigma-1)/\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \left( \int_0^1 q_j^{1/\sigma} q_j^{(\sigma-1)/\sigma} \frac{E^{(\sigma-1)/\sigma}}{P^{((\sigma-1)/\sigma)(1-\sigma)}} p_j^{1-\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \frac{E}{P^{1-\sigma}} \left( \int_0^1 q_j p_j^{1-\sigma} dj \right)^{\sigma/(\sigma-1)} \\ &= \frac{E}{P^{1-\sigma}} P^{-\sigma} \\ &= \frac{E}{P} \end{aligned} \quad (8)$$

Plugging into (1) yields

$$Y = L_F^\beta X^{1-\beta}$$

Profit maximization by the final goods firms over  $L_F, X$  implies

$$\beta L_F^{\beta-1} X^{1-\beta} = w \quad (9)$$

$$(1 - \beta) L_F^\beta X^{-\beta} = P \quad (10)$$

The inverse demand of the intermediate goods producers can be derived by using rearranging (10) to obtain an expression for  $X$  in terms of  $L_F, P$  and parameters; and then substituting this and (8) into (5) to obtain an expression relating  $x_j, p_j, L_F$  and hopefully  $P$  cancels out...(it should):

First, rearranging (10) we get

$$X = \left( \frac{1 - \beta}{P} \right)^{1/\beta} L_F \quad (11)$$

Substituting (8) into (4) yields:

$$\frac{x_j}{q_j} = X \left( \frac{p_j}{P} \right)^{-1/\beta} \quad (12)$$

Now substitute (11) into (12) to obtain

$$\begin{aligned} \frac{x_j}{q_j} &= \left( \frac{p_j}{1 - \beta} \right)^{-1/\beta} L_F \\ p_j &= (1 - \beta) L_F^\beta q_j^\beta x_j^{-\beta} \end{aligned} \quad (13)$$

Since the producers face the same demand curve for their goods as in AK 2017 (conditional on  $L_F$ ), prices and quantities are the same in equilibrium:

$$x_j = \left[ \frac{(1 - \beta)}{P} \right]$$

## 2 Lab equipment model

If R&D were done using final goods, we can write  $E$  as a function of  $L_F$  using the equation:

$$\begin{aligned} L_F &= 1 - \int_0^1 l_j dj \\ &= 1 - \frac{E}{p} \end{aligned}$$

Further, we can substitute to obtain an expression for production in terms of  $L_F, E$ , assuming expenditures on capital goods are optimal given the quality distribution. First, do some algebra to get an expression for the optimal CES aggregator given price  $p$ , qualities  $\{q_j\}_{j \in [0,1]}$  and spending  $E$ :

$$\begin{aligned} \left( \left( \int_0^1 q_j^\beta x_j^{1-\beta} dj \right)^{1/(1-\beta)} \right)^{1-\beta} &= \left( \left( \int_0^1 q_j^\beta \left( \frac{q_j}{\bar{q}} \frac{E}{p} \right)^{1-\beta} dj \right)^{1/(1-\beta)} \right)^{1-\beta} \\ &= \left( \frac{1}{\bar{q}} \frac{E}{p} \right)^{1-\beta} \left( \left( \int_0^1 q_j dj \right)^{1/(1-\beta)} \right)^{1-\beta} \\ &= \left( \frac{1}{\bar{q}p} \right)^{1-\beta} \bar{q} E^{1-\beta} \\ &= \bar{q}^\beta p^{\beta-1} E^{1-\beta} \end{aligned}$$

Substitute this into the final goods production function:

$$Y(L_F, E; \bar{q}) = \bar{q}^\beta p^{\beta-1} L_F^\beta E^{1-\beta}$$

This yields FOCs for  $L_F$  and  $E$ :

$$\begin{aligned} \beta \bar{q}^\beta p^{\beta-1} L_F^{\beta-1} E^{1-\beta} &= w \\ (1-\beta) \bar{q}^\beta p^{\beta-1} L_F^\beta E^{-\beta} &= 1 \end{aligned}$$

because the price of one unit of  $E$  is, by definition, equal to 1.

Finally recall our equation for  $p$ :

$$p = \frac{w}{\bar{q}(1-\beta)}$$

Hence we have four equations in four unknowns  $\{L_F, E, w, p\}$  and parameters:

$$L_F = 1 - \frac{E}{p} \quad (14)$$

$$\beta \bar{q}^\beta p^{\beta-1} L_F^{\beta-1} E^{1-\beta} = w \quad (15)$$

$$(1 - \beta) \bar{q}^\beta p^{\beta-1} L_F^\beta E^{-\beta} = 1 \quad (16)$$

$$p = \frac{w}{\bar{q}(1 - \beta)} \quad (17)$$

This part of the model is therefore determined separately from the R&D side of the model. Intuitively, I haven't proven that there exists a closed-form solution – this is shown by Akcigit & Kerr 2017, which is exactly the same framework. To check these conditions we could substitute that solution and check there is no contradiction.

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**My model** In my model, R&D is done using labor drawn from the same pool as intermediate and final goods production. Now we cannot derive (14) because

$$L_F = 1 - \int_0^1 l_j^I dj - \int_0^1 l_j^{RD} dj$$

Hence, we cannot derive a formula relating  $E$  and  $L$  without appealing to  $z(m), \hat{z}(m)$  in order to compute the last term in the equation above. But those require solving the HJBs, etc. The static and dynamic aspects of the model now interact.

**Possible solutions** The only way to eliminate this feature is to entirely decouple the production and R&D labor markets. In addition, we must assume elastic labor supply in the R&D market in order to make the model an endogenous growth model. Also note that we can't endogenize the elasticity of R&D labor supply by using some kind of decision to specialize in final goods production or R&D with some initial heterogeneity in relative productivities in each form of employment, because this couples the labor markets, eliminating the tractability. Hence, the only way to have a tractable, non-trivial model is to assume a separate population of potential R&D workers with some aggregate labor supply elasticity.

**New algorithm** In light of this, we need a new algorithm.

1. Guess  $L^{RD}$ , the BGP labor supply to R&D

2. Now we know the labor supply available to production, hence can solve for all static production variables  $L^F, L^I, w, p, \pi$  in closed form
3. Given these, solve HJBs numerically using iterative procedure described above
4. Next, solve KF equation to compute stationary distribution  $\mu(m)$
5. Using  $\mu(m)$  and policy functions from previous step, integrate to compute aggregate labor demand
6. Check market clearing in R&D market  $L^{RD} = \int l(m) + \hat{l}(m) d\mu(m)$ . If market does not clear, update guess  $L^{RD}$  and go back to Step 1

My original algorithm was needlessly complex.