Want to solve the HJBs

$$\begin{aligned}
(r+\theta-g)A(q,m,n) &= \max_z \pi(q)+\theta B(q) \\
&\quad +\chi_I z\phi(z+z_E(q,m))A(q,m,n) \\
&\quad +\nu(z_E(q,m)+z)A_m(q,m,n) \\
&\quad -gqA_q(q,m,n) \\
(r+\theta-g)B(q) &= \max_z \pi(q)+\chi_I z\phi(z+\xi F(q))(A((1+\lambda)q,0,0)-B(q)) \\
&\quad -\chi_E \xi F(q)\phi(z+\xi F(q))B(q)-gqB'(q)
\end{aligned}$$

First, we augment the first equation so that everything is a function of (q,m,n), possibly constant if necessary:

$$\begin{aligned}
(r+\theta-g)A(q,m,n) &= \max_z \pi(q,m,n)+\theta B(q,m,n) \\
&\quad +\chi_I z\phi(z+z_E(q,m,n))A(q,m,n) \\
&\quad +\nu(z_E(q,m,n)+z)A_m(q,m,n) \\
&\quad -gqA_q(q,m,n) \\
(r+\theta-g)B(q) &= \max_z \pi(q)+\chi_I z\phi(z+\xi F(q))(A((1+\lambda)q,0,0)-B(q)) \\
&\quad -\chi_E \xi F(q)\phi(z+\xi F(q))B(q)-gqB'(q)
\end{aligned}$$

We have our initial guesses, $A_i(q,m,n), B_i(q,m,n)$ and we are given $z_E(q,m,n)$. The following is the algorithm for updating to $A_{i+1}(q,m,n), B_{i+1}(q,m,n)$. We augment the HJBs with a time dependent term (we want time to go backwards, though, so we flip the sign):

$$\begin{aligned}
(r+\theta-g)A(q,m,n,t) &= \max_z \pi(q,m,n)+\theta B(q,m,n,t) \\
&\quad +\chi_I z\phi(z+z_E(q,m,n))A(q,m,n,t) \\
&\quad +\nu(z_E(q,m,n)+z)A_m(q,m,n,t) \\
&\quad -gqA_q(q,m,n,t)-A_t(q,m,n,t) \\
(r+\theta-g)B(q,t) &= \max_z \pi(q)+\chi_I z\phi(z+\xi F(q))(A((1+\lambda)q,0,0,t)-B(q,t)) \\
&\quad -\chi_E \xi F(q)\phi(z+\xi F(q))B(q,t)-gqB_q(q,t)-B_t(q,t)
\end{aligned}$$

Next, rearrange to obtain

$$\begin{aligned}
A_t(q,m,n,t) &= -(r+\theta-g)A(q,m,n,t) \\
&\quad +\max_z \pi(q,m,n)+\theta B(q,m,n,t) \\
&\quad +\chi_I z\phi(z+z_E(q,m,n))A(q,m,n,t) \\
&\quad +\nu(z_E(q,m,n)+z)A_m(q,m,n,t) \\
&\quad -gqA_q(q,m,n,t)+A_t(q,m,n,t) \\
B_t(q,t) &= -(r+\theta-g)B(q,t) \\
&\quad +\max_z \pi(q)+\chi_I z\phi(z+\xi F(q))(A((1+\lambda)q,0,0,t)-B(q,t)) \\
&\quad -\chi_E \xi F(q)\phi(z+\xi F(q))B(q,t)-gqB_q(q,t)+B_t(q,t)
\end{aligned}$$

First, I will work out how to do this with an explicit scheme. Once I have this ironed out, I can think about how to make it faster using an implicit scheme. Discretize the derivatives as follows (i.e. "upwind" scheme):

$$A_t(q,m,n,t_i) = \frac{A_{i+1}(q,m,n)-A_i(q,m,n)}{\Delta_t}$$

$$
\begin{aligned}
A_m(q,m,n,t_i) &= \frac{A_i(q,m_{j+1},n) - A_i(q,m_j,n)}{\Delta_m} \\
A_q(q,m,n,t_i) &= -\frac{A_i(q_{j-1},m,n) - A_i(q_j,m,n)}{\Delta_q} \\
A_n(q,m,n,t_i) &= \frac{A_i(q,m,n_{j+1}) - A_i(q,m,n_j)}{\Delta_n}
\end{aligned}
$$

NOTE: Interpolate before computation of derivatives. Will have to anyway, because need to interpolate to be able to compute the function off the grid, which will be necessary at each update step.

NOTE: Can we use the envelope theorem anywhere, as in Moll's HACT appendix?