I want to solve the ODE

$$\frac{d\pi_t}{dt} = \rho\pi(t) - \kappa x(t) \tag{1}$$

where x_t is some given function. The author claims that you can "solve forward" the ODE to obtain

$$\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t+s) ds$$

He does not derive it, and I'd like to be able to show this myself. It may involve assuming something about π_t not "exploding" as $t \to \infty$ (it definitely requires assuming some initial / terminal data based on economic theory, otherwise there clearly wouldn't be one solution)

I know that, in a difference equation, solving forward is just substituting in the difference equation iteratively forward. So I figured that by analogy solving forward a difference equation would begin by writing down the expression

$$\pi(t) = \lim_{l \to \infty} \pi(l) + \int_0^\infty \pi'(t+s) ds$$

assume the limit above is 0 (which is an unreasonable assumptionto make, honestly), and substitute in (1) to get

$$\pi(t) = \int_0^\infty \rho\pi(t+s) - \kappa x(t+s) ds$$

But this doesn't get me anywhere (and like I said above it requires me making a bizarre assumption, so the second step isn't even really justified).

I can however get the result in the text by solving a difference equation and taking a limit (heuristically at least). Just rewrite the ODE in finite difference (true in the limit as $\Delta \to 0$)

$$\begin{aligned}
\pi(t+\Delta) &= \pi(t) + \Delta(\rho\pi(t) - \kappa x(t)) \\
\Rightarrow \pi(t) &= \pi(t+\Delta) - \Delta(\rho\pi(t) - \kappa x(t)) \\
\Rightarrow (1+\Delta\rho)\pi(t) &= \pi(t+\Delta) + \Delta\kappa x(t) \\
\Rightarrow \pi(t) &= (1+\Delta\rho)^{-1}\Big(\pi(t+\Delta) + \Delta\kappa x(t)\Big)
\end{aligned}$$

Then recursively substitute this same equation starting at $\pi(t+\Delta)$, etc. to obtain

$$\pi(t) = \lim_{k\to\infty} (1+\Delta\rho)^{-k}\pi(t+k\Delta) + \sum_{j=1}^{\infty}(1+\Delta\rho)^{-j}\Delta\kappa x\big(t+(j-1)\Delta\big)$$

Taking the limit as $\Delta \to 0$, this (heuristically, i'm not being rigorous) becomes

$$\pi(t) = \lim_{T\to\infty} e^{-\rho T}\pi(T) + \kappa\int_0^\infty e^{-\rho s}x(t+s)ds$$

Then I just need to assume that $\pi(T)$ does not explode faster than exponential rate ρ as $T \to \infty$, which is an economically reasonable assumption, and I get the result in the text.

I basically want to understand how to do this rigorously using just the algebra of differentiation and integration...I know there's a way, I just don't know shit about differential equations.