Equilibrium Imitation and Growth

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The least productive agents in an economy can be vital in generating growth by spurring technology diffusion. We develop an analytically tractable model in which growth is created as a positive externality from risk taking by firms at the bottom of the productivity distribution imitating more productive firms. Heterogeneous firms choose to produce or pay a cost and search within the economy to upgrade their technology. Sustained growth comes from the feedback between the endogenously determined distribution of productivity, as evolved from past search decisions, and an optimal, forward-looking search policy. The growth rate depends on characteristics of the productivity distribution, with a thicker-tailed distribution leading to more growth.

I. Introduction

Productivity growth is a key mechanism for modeling limitless production and technology growth in a resource-constrained economy. Many models capture the economy's technology level as the frontier productivity. However, in any empirical distribution of productivity, there is a large mass of low-productivity firms, and there are very few at the frontier

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of the distribution.[1] Hence, there are potentially enormous gains in aggregate output from even marginally increasing the productivity of less productive agents. There are many established models of innovation that investigate the role of a researcher inventing a new frontier technology and mechanisms for the near-frontier firms to adopt it; instead, we develop a model of equilibrium imitation to analyze endogenous changes to the large mass at the lower tail of the productivity distribution.

This paper contributes an analytically tractable endogenous growth mechanism—with an emphasis on the evolution of the productivity distribution—driven by the decisions of the least productive. In the model, heterogeneous firms choose to produce or pay a cost and search within the economy to upgrade their technology. While they may get lucky and draw a high productivity, most will make only modest improvements. The expected benefit from risk taking—that is, the option value of search—will depend on characteristics of the productivity distribution. Search is risky since, ex post, a firm may regret having searched, depending on the draw, as the value of the improved productivity may be less than the cost of search.

With heterogeneous productivities in a search-theoretic growth model, a key question is which distribution to use for new productivity draws. One standard approach is to use an "external" exogenous distribution or stochastic process for idiosyncratic productivity. However, a natural candidate for the distribution of new productivity draws is to sample from the existing distribution of productivities itself. Featuring this "internal" productivity distribution in our model, when a firm searches, it will copy the ideas of an existing firm in the economy. The result of a meeting is that both firms receive the maximum productivity of the pair.[2] So, not only do the unproductive drive growth, but everyone in the economy does, through the diffusion of technology from the more to the less productive. This leads to the key feature of the model: The distribution of productivity evolves endogenously from firms' past, optimal, forward-looking decisions. Aggregate growth is generated as an externality from risk taking by firms at the bottom of the productivity distribution. Not only is growth endogenous, but risk is endogenous, as risk taking through search is an optimal choice.

Section II characterizes the dynamic equilibrium of the model, defines a balanced growth path (BGP) equilibrium, solves for the BGP in closed form, and explores off-BGP dynamics and asymptotic growth rates.

[1] In Gabaix (2009), firm size is shown to empirically fit a Pareto distribution, except for a large mass of small firms. A theoretical relationship between the firm size and productivity distributions is also established. See Aw, Chen, and Roberts (2001) for an estimation of productivity distributions across various industries and over time.

[2] Our modeling of imitation as a meet-and-copy process is similar to that in Jovanovic and Rob (1989) but does not include the potential for additional spillovers beyond the maximum productivity in the pairwise meeting technology.

Because the evolving distribution is endogenously generated by firms' choices, we can analyze the feedback relationship between characteristics of the productivity distribution and growth. We interpret a thicker-tailed productivity distribution with a higher Gini coefficient as having more dispersed opportunities. To understand why the dispersion of opportunities in the productivity distribution changes the growth rate of the economy, consider a firm sampling from a left-truncated distribution. With a fatter right tail of the distribution, firms face higher returns to search—similar to the effect of a mean-preserving spread of the wage offer distribution in a standard McCall search model. Tempering this mechanism, firms have an incentive to wait for others to push the productivity distribution forward before searching since imitation is costly and the value of search is increasing over time. Growth is moderated because infinitesimal firms do not internalize how their technology adoption policies affect the evolution of the distribution.

Solving a planner's problem enables us to investigate this externality, compare the socially efficient economy to the competitive equilibrium, and determine if the forces of the model affect firms and the planner in similar ways. Section III develops this problem, where the planner chooses which firms search. In contrast to the competitive equilibrium, the planner is able to internalize the effect that the search decision has on the evolution of the productivity distribution. Hence, the planner's economy always features higher growth than in the competitive equilibrium. Furthermore, the stronger the degree of inequality in the productivity distribution, the larger the wedge between the planner and the competitive equilibrium's growth rates.

A natural extension is to investigate whether a constrained planner can subsidize search and indirectly adjust the growth rate. In Section IV, we solve the problem of a constrained planner that subsidizes search and satisfies the balanced-budget constraint with linear taxes on productive firms. In the Markov perfect equilibrium, the firms operate in a competitive equilibrium in which the planner recursively chooses optimal taxes and subsidies, subject to a budget constraint. Unlike many tax distortions that affect an agent's elastic labor supply decision, in this economy, proportional taxes impede growth by decreasing the future value of a high productivity draw. However, we find that the constrained planner can overcome this distortion and still achieve the first-best outcome.

Relation to the literature.—Much of the endogenous growth literature, such as Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), captures the technology level of the economy either as the total number of differentiated products or as a frontier productivity/quality for each good in the economy. This literature then investigates how research expands the technology frontier, how this new technology is adopted across the economy, and the related intertemporal returns to

R&D. Since we do not have a theory of how the frontier expands, our model complements this strain of the growth literature by providing a theory of the evolution of the entire productivity distribution driven by technology diffusion.

In Romer (1990) and many other endogenous growth models, the returns to research are proportional to the current stock of research and, hence, generate geometric growth with a constant research investment. For search-theoretic models interested in the expansion of the technology frontier, the returns to productivity-enhancing investments depend crucially on the sampling distribution. In an early such example, Evenson and Kislev (1976) model applied technological advancement as obtaining, at some cost, higher productivity draws from a distribution over time. However, growth is sustainable only in the long run as a result of exogenous basic research that increases the mean of the distribution.[3] Kortum (1997) incorporates a mechanism by which firms draw from an exogenous distribution of productivities, adjusted for spillovers from the aggregate stock of research.[4]

Building on Kortum (1997) and Eaton and Kortum (1999), Alvarez, Buera, and Lucas (2008) and Lucas (2009) replace the exogenous productivity distribution with the existing cross-sectional distribution of productivity across agents, capturing the idea that each agent learns from surrounding agents. The economy grows by pulling itself up by its bootstraps, as better ideas diffuse across agents through an exogenous process of meeting and learning.

Whether it is exogenous basic research, an aggregate spillover function, or exogenous random meetings, these models depend on the distribution evolving over time and provide an exogenous or semi-endogenous mechanism that improves the distribution. Our paper provides a tractable framework that delivers a shift in the distribution much like that of Evenson and Kislev (1976), Kortum (1997), or Alvarez et al. (2008). Additionally, in our model, both the evolution of the productivity distribution and the technology adoption decision are jointly endogenously determined in equilibrium, as the least productive firms choose to adopt better technologies. Thus, our model is well suited to analyzing the effect that the productivity distribution has on adoption incentives, the effect of adoption behavior in generating the productivity distribution, and the corresponding growth implications of this link.

[3] Similarly, Bental and Peled (1996) have searchers drawing new technologies from a distribution in levels rather than making proportional improvements. In this general equilibrium setting, the frontier grows if an agent gets a lucky draw above the current frontier, and the economy grows as other firms can copy this technology next period.

[4] As in our model, Kortum (1997) and Jones (2005) investigate how a Pareto distribution of productivity/idea draws can be consistent with both a stationary distribution of firm characteristics and constant aggregate productivity growth. Similarly, Eaton and Kortum (1999) derive a BGP with a Pareto distribution for the quality of new ideas.

Our endogenous evolution of the distribution is most similar to that of Lucas and Moll (2014, in this issue), in which heterogeneous agents invest in studying to adopt new ideas and growth is generated as idea adoption evolves the productivity distribution. In contrast to our paper, Lucas and Moll emphasize the intensive margin of time dedicated to learning and develop continuous-time computational solution techniques.

Another complementary approach in the literature, as in the balanced growth models of Luttmer (2007, 2011), is to emphasize the role of selection and entry rather than the adoption of technology by incumbents. Luttmer (2007) studies a model in which both stochastic changes to incumbents' productivity and increased competition due to entering firms that have better technology cause the endogenous destruction of incumbent firms. Entrants draw a productivity and internalize the value of the growing economy, while incumbent firms choose when to exit, given that they must pay a fixed cost to operate. In contrast, in our model, a producing firm's period profits are constant, and the increasing value of technology adoption due to growth leads forward-looking incumbents, who internalize this positive value, to choose to search. While in both papers the existing distribution of technology used in production affects the returns to technology adoption, our paper complements Luttmer's by providing a different perspective on which firms benefit from aggregate growth of the economy.

Finally, this paper emphasizes how the degree of inequality in the economy determines the rate of growth and the strength of the free-riding incentive. An alternative approach, as in Eeckhout and Jovanovic (2002), is to investigate how imperfect technology spillovers can change the degree of inequality in an economy. There, agents do not copy a technology directly; instead their production function includes intratemporal spillovers from the current distribution of technologies. Eeckhout and Jovanovic find that the larger the free-riding incentive, the greater the inequality. Our paper finds that along a BGP, higher inequality increases both the growth rate and the free-riding incentive, while the degree of inequality remains constant over time.

II. The Model Economy

Time is discrete and infinite. There are two types of agents in the economy: consumers and firms. The model admits a representative consumer who simply consumes aggregate production each period. There is a fixed measure of firms with heterogeneous productivity levels, z. The function $F_t(z)$ is the cross-sectional productivity cumulative density function (cdf) in the economy, which will be the aggregate object affecting firms' decisions and will evolve over time endogenously in response to firms' actions. Given their idiosyncratic productivity, firms have a choice to either produce with their existing productivity or search and upgrade their productivity. If a firm chooses to search, it forgoes production and randomly imitates the technology of some producing firm in the existing economy. This new productivity level will be its idiosyncratic state in the period after search. When a firm searches, it samples from the existing productivity distribution. Because in equilibrium the unproductive will choose to search, the distribution of productivities evolves by shifting mass from lower to higher productivity levels. Productivity increases over time, despite the lack of any exogenous forcing process. The model is parsimoniously parameterized by a time discount factor, β, the initial productivity distribution, F_0, and the utility function of the consumer, $u(x)$. See figure 1 for intuition on the evolution of the productivity probability density function (pdf), where $m_t \equiv \min \text{support}\{F_t\}$.

A. Consumers

The lifetime utility of the representative consumer who consumes aggregate output, Y_t, each period is given by

$$\sum_{t=0}^{\infty} \beta^t u(Y_t).$$

The implied intertemporal optimization condition yields

$$\frac{1}{1 + r_t} = \beta \frac{u'(Y_{t+1})}{u'(Y_t)}, \tag{1}$$

where r_t is the equilibrium interest rate that ensures that there is no active market for claims to aggregate production. Since there is no aggregate uncertainty, the representative consumer will have a deterministic sequence of consumption.

B. Firms

While consumers are kept trivial and simply eat aggregate output, firms make the important decisions in the economy. There exists a continuum of mass one of risk-neutral, infinitely lived, heterogeneous firms indexed by productivity levels, z. Firms have access to costless linear production technology such that output equals productivity, that is, $y = z$ (maximum scale is one). Since consumers own the firms, a firm's objective is to maximize its discounted stream of output, discounting the future using the interest rate, r_t, determined from the consumer problem.

Agents forecast an equilibrium sequence of $\{r_t, F_t\}_{t\geq 0}$. Given this forecast and idiosyncratic state z, firms choose whether to search and upgrade their technology or to produce with their existing technology. The cost of

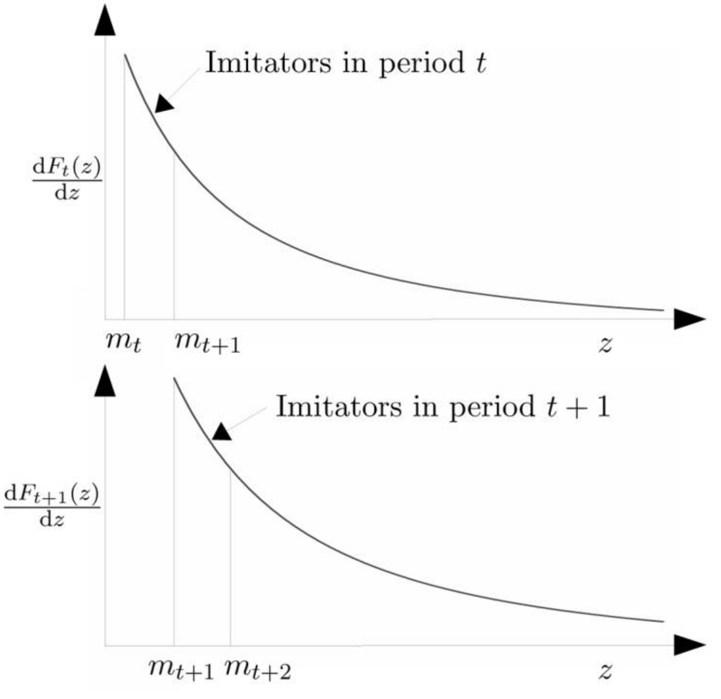

FIG. 1.—Evolution of the productivity pdf

search is forgone production. If the firm chooses to search, it enters the next period with a new productivity value drawn from the existing productivity distribution conditional on being a producer. Even though in equilibrium a firm receives a draw higher than its current productivity with certainty, search is still risky, since the firm may regret the search decision ex post, as the value of improved productivity may be less than the cost of search.[5] The firm weighs the value of producing versus the value of paying the opportunity cost of not producing to upgrade by randomly drawing a new productivity level. The firm solves

$$V_t(z) = \max\left\{ z + \frac{1}{1+r_t} V_{t+1}(z),\ \frac{1}{1+r_t} \int V_{t+1}(z')\, dF_t(z'|z' \geq m_{t+1}) \right\}. \quad (2)$$

[5] A variation of the model in which firms draw unconditionally from F_t and may reject lower draws is discussed in online App. G. Qualitative features of the model remain the same.

The distribution F_t is endogenous and depends on the decisions of all firms in the economy, requiring the firms to forecast the evolution of the distribution. Given a perceived sequence of distributions, it can be shown—following the arguments of the standard McCall search model—that the solution to this search problem is a reservation value of productivity for each t, $\hat{z}_t$, that solves equation (2) at the indifference point. Firms with productivity z will choose to search in period t if and only if $z \leq \hat{z}_t$.[6] We will restrict focus to a rational expectations equilibrium, in which the infinitesimal agents' forecasts match the evolution of the productivity distribution induced by their actions, that is, $\hat{z}_t = m_{t+1}$. The evolution of F_t is deterministic and perfectly foreseen—made possible by the absence of aggregate risk. Thus, if a firm chooses to search, it expects to and does receive a draw from the conditional distribution $z_{t+1} \sim F_t(z|z \geq m_{t+1})$.

C. *Evolution of the Productivity Distribution*

Given a productivity pdf, $f_t(z)$, the distribution evolves as a mass $F_t(m_{t+1})$ of searching firms draw new productivities from the conditional density of producers:

$$\begin{aligned} f_{t+1}(z) &= f_t(z) + f_t(z|z \geq m_{t+1})F_t(m_{t+1}) \\ &= \frac{f_t(z)}{1 - F_t(m_{t+1})}. \end{aligned} \tag{3}$$

That is, it is a truncation of the distribution F_t at m_{t+1}.[7] Thus, min support$\{F_{t+1}\} = m_{t+1}$ and max support$\{F_{t+1}\}$ = max support$\{F_t\}$. Iterating forward again, we can show that

$$f_{t+2}(z) = \frac{f_t(z)}{1 - F_t(m_{t+2})},$$

demonstrating that only the initial distribution and the last truncation point are necessary to characterize the distribution at any point in time.

[6] Note that the agent forecasts a deterministic evolution of the productivity distribution. Given $\{m_t\}$, the proof that optimal policy is a sequence of reservation productivities follows from the standard solution techniques of search in a nonstationary economy, as in Lippman and McCall (1976).

[7] The assumption that upgrading firms meet only producing firms yields the clean truncation of the distribution, with no mass of firms perpetually left behind. While this simplifying assumption is added for tractability and exposition, economically, it represents directed search toward only the productive firms. The primary mechanism for growth in this paper is from the endogenous selection of who searches rather than from the "selective sampling" of the right-hand tail, although directed search does increase the growth rate. Appendix G solves and analyzes a version of the model with unconditional draws, and Sec. II.G conducts numerical experiments with this model variation.

So, given the initial condition F_0 and a sequence $\{m_{t+1}\}$, the density generated by this law of motion at t is

$$f_t(z) = \frac{f_0(z)}{1 - F_0(m_t)}. \quad (4)$$

With this law of motion, the firm problem can be simplified to

$$V_t(z) = \max\left\{z + \frac{1}{1+r_t} V_{t+1}(z),\ \frac{1}{1+r_t}\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{f_0(z')}{1-F_0(m_{t+1})}dz'\right\}. \quad (5)$$

D. *Equilibrium Concepts*

1. Competitive Equilibrium

Definition 1 (Competitive equilibrium). A *competitive equilibrium* consists of an initial distribution F_0 and sequences of reservation productivity levels, value functions, and interests rates, $\{m_t, V_t(\cdot), r_t\}_{t\geq 0}$, such that given $\{r_t\}$, $\{m_{t+1}\}$ are the reservation productivity levels that solve the firm problem, with $\{V_t(\cdot)\}$ the associated value functions and given $\{m_t\}$, $\{r_t\}$ are consistent with the consumer's intertemporal marginal rate of substitution.

A sequence of reservation productivity levels $\{m_t\}$ fully characterizes the equilibrium. Given the law of motion in equation (4), from m_0, the sequence of distributions is characterized by the initial distribution and the sequence of reservation productivities,

$$F_t(z) = \frac{F_0(z) - F_0(m_t)}{1 - F_0(m_t)}.$$

Aggregate production is defined as $Y_t \equiv \int_{m_{t+1}}^{\infty} z dF_t(z)$.[8] The growth factor, $g_t \equiv Y_{t+1}/Y_t$, may diverge or converge depending on the characteristics of F_0. In general, numerical methods are required to solve for $\{m_t\}$. See online Appendix H for details.

As the goal is to fully analyze the growth mechanism and its dependence on model parameters, in this paper we focus on a BGP equilibrium that allows for analytical solutions.

[8] As in Lagos (2006), aggregate production is calculated as an integral of the output of producing firms and, hence, is dependent on properties of the current distribution and searching decisions. Also as in Lagos's paper, more searchers would result in higher mean productivity at the macro level within a given period, while the period's aggregate production would be lower.

2. BGP Distribution Evolution

A definition of a balanced growth path for scalar variables simply requires geometric growth at a constant rate. Defining a BGP for an evolving distribution requires additional restrictions. As the distribution evolves according to sequential truncation, the support of the distribution is certainly changing, and the shape of the distribution could potentially change in an unrestricted dynamic system. To introduce the idea of scale-invariant distributions, we want a concept that removes the scale and maintains the shape of the distribution.[9]

Definition 2 (Scale invariant). A sequence of distributions, $\{F_t\}$, and scales, $\{m_t\}$, are *scale invariant* if

$$F_t(\tilde{z}m_t) \text{ are identical for all } t \geq 0, \tilde{z} \in [1, \infty). \tag{6}$$

Intuitively, scale invariance requires that all quantiles of the initial distribution expand at the same rate each period as the distribution evolves.[10]

3. Balanced Growth Path

Definition 3 (BGP competitive equilibrium). A *BGP competitive equilibrium* is a competitive equilibrium with a constant $g > 1$ such that support $\{F_0\} = [m_0, \infty)$, $\{F_t, m_t\}$ is scale invariant, and $Y_{t+1} = gY_t$ for all $t \geq 0$.

The initial distribution must have infinite right-tailed support or the economy would not be able to grow indefinitely.[11] Requiring production to grow by the constant factor g and requiring scale invariance restricts the BGP equilibrium to be balanced. Restricting $g > 1$ ensures that the BGP equilibrium has growth.

As is well known, not every utility function is compatible with balanced growth. Restricting the lifetime utility function to being homothetic, increasing, and quasi-concave implies that period utility must be of the constant elasticity of substitution form, representing constant relative risk aversion preferences. Thus, for the remainder of the paper, we use a power utility function, $u(Y) = Y^{1-\gamma}/(1-\gamma)$, ensuring a constant intertemporal marginal rate of substitution (constant r) and consistency with balanced growth.

[9] An alternative approach is to require that the Lorenz curves of the distribution are identical for all t.

[10] Scale invariance is related to the normalization discussed in online App. F.

[11] A similar assumption for the support of potential improvements is used in papers such as Bental and Peled (1996), Kortum (1997), and Eaton and Kortum (1999). This paper does not model or emphasize the technology frontier, and the majority of searchers end up with only minor improvements. These papers, and other research that emphasizes R&D, provide a better description of the limitless growth of the technology frontier.

E. *Solution and Analysis*

Existence of an equilibrium is proved by construction via a guess and check strategy. It is straightforward to show that a Pareto distribution as the initial condition, F_0, will fulfill the evolution equilibrium requirements.[12] Thus, let F_0 be Pareto: $F_0(z; m_0, \alpha) = 1 - (m_0/z)^{\alpha}$ and $f_0(z; m_0, \alpha) = \alpha m_0^{\alpha} z^{-\alpha-1}$.

First, guess that the minimum of support will grow geometrically at the aggregate growth rate, so $m_t = m_0 g^t$. Second, guess that the value of imitation will grow geometrically at the aggregate growth rate, so, for some constant W, $V_t(z) = m_t W$ for $z \leq m_{t+1}$. These guesses will be verified during the solution process.

The solution strategy is to use these guesses to solve for the constants (W and g) as functions of parameters (m_0, α, β, and γ) and then to verify the guesses and that the solution fulfills all BGP equilibrium requirements. The key to the solution strategy is to use the indifference equations obtained by evaluating the value functions at reservation productivities. The problem is analytically tractable because the expectation of a piecewise-linear function of a Pareto random variable can be calculated explicitly.

Given parameter restrictions that guarantee positive and finite growth, the problem is solved analytically with the guesses and BGP requirements verified.

PROPOSITION 1. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$ and $\alpha > 1$ and

$$\frac{\alpha - 1}{\alpha} < \beta < \min\left\{\left(\frac{\alpha}{\alpha - 1}\right)^{(\gamma-1)/\alpha}, 1\right\},$$

a BGP competitive equilibrium exists with the following properties:

1. The growth factor is

$$g = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{1/(\gamma - 1 + \alpha)} > 1.$$

2. The reservation productivity level grows geometrically: $m_{t+1} = g m_t$.
3. The interest rate is $r = (g^{\gamma}/\beta) - 1$.
4. The mass of imitating firms is constant: $S_t = 1 - g^{-\alpha}$.
5. Output grows geometrically: $Y_t = [\alpha/(\alpha - 1)] g^{1-\alpha} m_t$.

[12] See App. C for a proof that the Pareto distribution is the unique distribution that can satisfy the BGP equilibrium requirements.

6. The value function is piecewise linear, with kinks at $\{m_{t+1}\}$. That is, for all $s \in \mathbb{N}$, for $z \in [m_0 g^{t+s}, m_0 g^{t+s+1}]$,

$$V_t(z) = \frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^s\right] z + \left(\frac{1}{1+r}\right)^s m_0 g^{t+s} W,$$

and for $z \leq m_0 g^t$,

$$V_t(z) = m_0 g^t W.$$

Proof. Given the guess that $m_t = m_0 g^t$, the Pareto distribution can be shown to be scale invariant and to generate constant geometric growth of aggregate production and a constant fraction of searching firms. The remaining task is to solve for constants W and g as a function of parameters, such that the reservation productivity, $m_{t+1} = gm_t$, is optimal for firms.

Using equation (5), plug in f_0 and the guess that $m_{t+1} = gm_t$:

$$V_t(z) = \max\left\{z + \frac{1}{1+r} V_{t+1}(z),\ \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^\infty V_{t+1}(z')z'^{-\alpha-1}dz'\right\}. \quad (7)$$

Given that the indifference level of productivity is gm_t, using the guess that the value of imitation grows geometrically, $V_t(gm_t) = m_t W$, gives two equalities:

$$m_t W = gm_t + \frac{1}{1+r} gm_t W \quad (8)$$

$$= \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^\infty V_{t+1}(z')z'^{-\alpha-1}dz'. \quad (9)$$

Equating the first of the two equalities in equation (8) provides an equation in W and g,

$$W = \frac{g}{1 - g/(1+r)}. \quad (10)$$

Equating the second equality between equation (8) and equation (9) and splitting the integral at the reservation productivity yields

$$gm_t + \frac{1}{1+r} gm_t W = \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} dz' + \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz'. \tag{11}$$

By the decision rule, firms will search at $t + 1$ if $z \le g^2 m_t$ with value $gm_t W$,

$$\int_{gm_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} dz' = gm_t W \int_{gm_t}^{g^2 m_t} z'^{-\alpha-1} dz' \tag{12}$$

$$= \frac{gm_t W}{\alpha}(gm_t)^{-\alpha}(1 - g^{-\alpha}). \tag{13}$$

By the decision rule, firms will produce at $t + 1$ if $z > g^2 m_t$:

$$\int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz' = \int_{g^2 m_t}^{\infty} \left[z' + \frac{1}{1+r} V_{t+2}(z') \right] z'^{-\alpha-1} dz' \tag{14}$$

$$= \frac{1}{\alpha-1}(g^2 m_t)^{1-\alpha} + \frac{1}{1+r}\int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{15}$$

Using the indifference equation at $t + 1$, where the reservation productivity is $g^2 m_t$,

$$V_{t+1}(g^2 m_t) = g^2 m_t + \frac{1}{1+r} g^2 m_t W = \frac{1}{1+r}\alpha(g^2 m_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{16}$$

Combining equations (13), (15), and (16) with equation (11) and simplifying yields

$$(1 + r) g^\alpha = -W + \frac{\alpha}{\alpha-1} g + g\left(1 + \frac{1}{1+r} W\right). \tag{17}$$

As m_t has dropped out of equations (17) and (10), W and g are not functions of time. Hence, the guess of the functional form $V_t(gm_t) = m_t W$ and the equilibrium requirement that g is constant are confirmed.

Combining equation (17) with (10) and solving for g shows that

$$g = \left(\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)^{1/(\alpha-1)}. \tag{18}$$

Given this fixed g, substituting $1/(1+r) = \beta g^{-\gamma}$ into equation (18) solves for g as a function of parameters

$$g = \left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma+\alpha-1)}. \tag{19}$$

The restrictions on parameters come from the requirements that both $g > 1$ and $W > 0$. For $W > 0$, it is necessary and sufficient that $g/(1+r) < 1$. The closed-form equation for the value function is derived in Appendix E by using this solution to the firm problem and working with the value function in sequence space. See Section C in Appendix H for a numerical analysis of the magnitude of g in relation to parameter values. QED

The constraint $\beta > (\alpha - 1)/\alpha$ guarantees that the discount factor is high enough, compared to the imitation opportunities, to ensure that firms want to search. If $\gamma > 1$, consumer period utility has an Inada condition, and the upper constraint of $\beta < 1$ restrains growth. Otherwise, the constraint $\beta < [\alpha/(\alpha-1)]^{(\gamma-1)/\alpha}$ ensures that patience or inequality is low enough—compared to the curvature of the utility function—to avoid infinite growth.

Model behavior can be summarized by the derivatives of the growth factor with respect to the parameters. As analytical derivatives are attainable, it is easy to show that $dg/d\beta > 0$, $dg/d\alpha < 0$, and $dg/d\gamma < 0$. The growth factor provided in proposition 1 and the above derivatives succinctly capture the connection between all model parameters and the endogenously determined growth rate. This allows a transparent analysis of the relationship of consumers' preferences and the shape of the productivity distribution to growth and the evolution of the productivity distribution.

As β increases and agents become more patient, growth increases because more value is put on higher future consumption, which yields more technology adoption. As γ increases, $1/\gamma$ decreases, and the firms wish to reflect the consumer's lower intertemporal elasticity of substitution by reducing search to produce more earlier.

A decrease in α corresponds to an increase in productivity inequality. As α decreases, the expected value of a draw increases, as the Pareto distribution has more weight in the tail and less weight around the minimum of the support. Thus, a searching firm is more likely to obtain a higher productivity in an economy with more inequality. Fatter tails incentivize more firms to search and generate higher growth.[13]

F. *Asymptotic Growth Rates*

As was just established, the initial distribution essentially determines the BGP growth rate. When the analysis is expanded outside of BGP equilibria, the initial distribution remains essential in determining asymptotic growth rates.

DEFINITION 4 (Power law). A function $L(z)$ is called slowly varying if $\lim_{m\to\infty} L(mz)/L(m) = 1$ for all $z \geq 0$. A distribution F is called a *power law* if, for $\alpha > 0$, the pdf and cdf have the form $f(z) \propto L(z) z^{-\alpha-1}$ and $F(z) \propto 1 - L(z) z^{-\alpha}$.

Section E in Appendix F shows that for all power laws with $\alpha > 1$,

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = \frac{\alpha}{\alpha - 1} > 1.$$

Alternatively, if a distribution has

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = 1,$$

then it is not a power law and intuitively can be thought of as having an infinite tail parameter (and an infinite number of moments).

PROPOSITION 2. Let the expectation of the truncated distribution exist for all $m < \max \operatorname{support}\{F_0\}$. If, for all t and some $\underline{g} > 1$, $g_t \geq \underline{g}$, then F_0 is a power law and an equilibrium exists where $\lim_{t\to\infty} g_t = g(\overline{\alpha})$ as determined by proposition 1. Alternatively, if for F_0

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = 1,$$

then $\lim_{t\to\infty} g_t = 1$.

Proof. See Appendix F, Section E.

From proposition 2, if the initial distribution is a power law, the growth rate is ultimately determined by the tail parameter. The condition that $g_t \geq \underline{g} > 1$ for all t is important because there are power law distributions with no asymptotic growth. For non–power law initial dis-

[13] An informal way to see this is to note that the mean of the Pareto distribution, $m_t\alpha/(\alpha - 1)$, is decreasing in α, while last period's truncation point, m_t, would also be increasing with more search. Also note, if the proportion of firms searching either grew or shrank monotonically with no limit in (0, 1), then there would be no growth in the limit.

tributions, growth can stop as the expectation of the truncated distribution gets close to its minimum of support, lowering the returns to search below the cost of forgone production. Initial distributions that lead growth rates to converge to zero and the economy to reach a finite asymptotic size include all with finite support, as well as the standard non–power law distributions, such as the normal, lognormal, and the exponential.

G. *Off-BGP Dynamics Example*

Off-BGP dynamics can highlight the impact of the shape of the productivity distribution on the growth of output. Fixing the tail parameter α, figure 2 shows the evolution of growth for both a Fréchet initial distribution and a right-truncated Pareto initial distribution with bounded support.[14]

For the Fréchet distribution, growth is initially high, as many firms are searching. Over time, the growth rate converges from above to the constant growth rate of a BGP equilibrium with a Pareto initial condition with the same tail index. Intuitively, the Fréchet distribution is like a lognormal distribution with a Pareto tail. Initially, for the same minimum of support as a Pareto, the Fréchet distribution has less mass close to the minimum of support and, thus, a higher expectation leading to a higher expected return to search. After repeated truncations, the Fréchet distribution looks more like the Pareto distribution, with more mass close to the minimum of support, leading to less search and slower growth. This confirms that the same forces that lead fatter tails to generate more growth on the BGP are present in the dynamic equilibrium.

For the right-truncated Pareto distribution, initial output is 5 percent lower than the unbounded Pareto since the right truncation of the initial Pareto distribution removes high–productivity tail firms. Growth rates are consequently lower, starting about 0.37 percent below the unbounded case, as the expectation of the truncated distribution is lower than the expectation of the unbounded distribution. However, even though the distribution has finite upper support and growth will eventually stop, as shown in proposition 2, growth rates drop very slowly over time. While the growth rate is asymptotically zero, starting from growth rates of 2.9 percent, after 50 years, growth has dropped by only 0.47 percentage points and, after 100 years, by 1.1 percentage points.

[14] These examples are roughly calibrated to target a 3.28 percent annual growth rate with $\gamma = 1$ and a Pareto tail parameter of $\alpha = 1.5$. In order to better calibrate growth and interest rates, these dynamic examples are calculated allowing for taxation and search costs similar to those in eq. (23); they are also conducted with firms drawing unconditionally from the entire distribution. Search costs and corporate taxes are calibrated such that an average firm takes 5 years to recoup the cost of upgrading its technology. See App. H for details on the bounded Pareto and Fréchet examples, as well as the numerical solution algorithm.

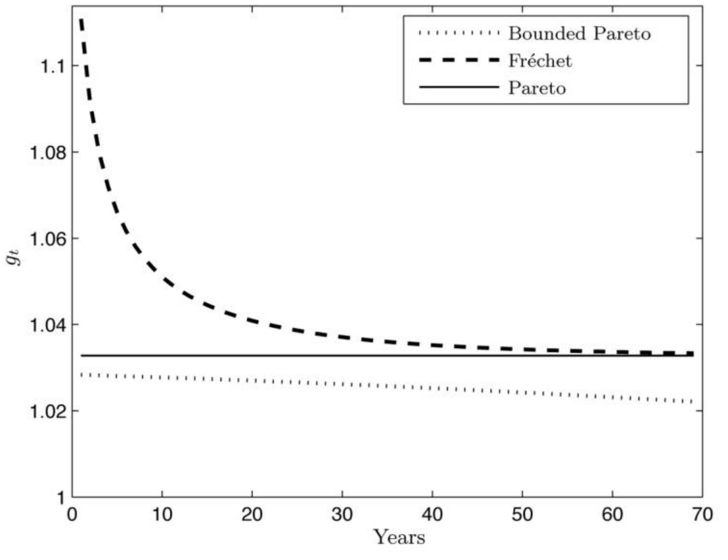

FIG. 2.—Growth transition from Pareto, bounded Pareto, and Fréchet initial conditions

This example shows that with bounded support and reasonably calibrated growth rates, a far-off finite upper bound on technology can have little impact on short- and medium-term growth dynamics. Economically, at any point in time, a far-off upper bound on technology has very little impact on a firm's choice, as its expected draw is far below the frontier. Thus, for the purposes of studying technology adoption by low-productivity firms, an unbounded Pareto distribution can be a useful approximation of the reality of a finite and growing upper bound on technology.

III. The Planner Problem

As in many other endogenous growth models, in this economy, growth is moderated since agents do not internalize the full social benefit of their private decision to improve. The magnitude of the externality is highlighted by solving a planner problem, where the planner internalizes the social benefit that adoption has on improving the productivity distribution for future adopters.

The planner chooses which firms produce and which upgrade in the economy. For clarity of exposition, the problem is defined recursively, with the distribution $F(z)$ as the aggregate state. It chooses a reservation value, $m'(F)$, below which firms will search. Equivalently, the planner chooses the growth factor $g(F) \geq 1$ such that $m'(F) = g(F)m(F)$, where

$m(F) \equiv \min \text{support}\,\{F\}$. Aggregate production is $Y(F) = \int_{gm(f)}^{\infty} z dF(z)$. The planner maximizes the lifetime utility of the representative agent, who derives utility from consuming aggregate production:

$$U(F) = \max_{g \geq 1} \left\{ \frac{\left[\int_{gm(f)}^{\infty} z dF(z)\right]^{1-\gamma}}{1-\gamma} + \beta U(F') \right\} \tag{20}$$

subject to

$$F'(z) = \frac{F(z) - F(m(F))}{1 - F(gm(F))}.$$

With a particular F_0, the problem can be solved in closed form by guessing and verifying that $U(f) = -A(m(F))^{1-\gamma}$ for some $A > 0$. For a Pareto $F_0(z)$, the optimal choice of g is shown to be independent of the aggregate state, F, and the first-order condition can be used to solve for A and g.

PROPOSITION 3. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$, $(\alpha - 1)/\alpha < \beta < 1$, $\gamma > 1$, and $\alpha > \gamma/(\gamma - 1)$, then the planner chooses a constant $g > 1$ each period such that

$$g = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{1/(\gamma-1)}. \tag{21}$$

Proof. See Appendix A.

We can summarize the sensitivity of growth to parameter values with the following derivatives: $dg/d\beta > 0$, $dg/d\alpha < 0$, and $dg/d\gamma < 0$. The explanation for $dg/d\beta$ and $dg/d\gamma$ is identical to that of the competitive equilibrium. Analysis of α is given below.

Since a lower α provides a stronger search incentive, the parameter requirement that $\alpha > \gamma/(\gamma - 1)$ ensures that the value of search is not sufficiently high to dominate the curvature of the utility function.[15]

Comparing the planner to the competitive equilibrium.—The growth factors in equations (19) and (21) are similar, but the competitive equilibrium growth factor contains an additional α in the exponent. This α in equation (19) decreases the growth rate and can be seen as capturing the growth penalty from adoption externalities. Firms do not internalize the

[15] Note that the constraints on the planner's problem are more restrictive than those of the competitive equilibrium to prevent infinite growth. For example, the planner's problem solution does not exist for $\gamma \leq 1$.

value of improving future search opportunities when making their search decisions.

The ratio of the planner's first-best g to the competitive equilibrium growth factor is

$$\frac{g_{fb}}{g_{ce}} = \frac{\left(\beta\dfrac{\alpha}{\alpha-1}\right)^{1/(\gamma-1)}}{\left(\beta\dfrac{\alpha}{\alpha-1}\right)^{1/(\gamma-1+\alpha)}} = \left(\beta\frac{\alpha}{\alpha-1}\right)^{\alpha/[(\gamma-1)(\gamma-1+\alpha)]}. \tag{22}$$

Note that $g_{fb}/g_{ce} > 1$ and that $d(g_{fb}/g_{ce})/d\alpha < 0$. Hence, the planner always chooses higher growth, as it internalizes the effect of search on the evolution of the productivity distribution. The efficiency wedge between the planner's growth factor and that of the competitive equilibrium is higher when productivity inequality is greater. Since the inequality parameter α also summarizes the strength of the externality, the larger the free-riding incentive, the larger the wedge.

IV. The Constrained Planner: Subsidies and Taxes

A natural alternative to the unconstrained planner's problem is to constrain the planner to operate within a competitive equilibrium and to use taxes and subsidies to indirectly influence the search decision. A trade-off exists for the constrained planner: Subsidies encourage search through a decrease in the search cost, but taxes discourage search by decreasing the future value of higher production. Hence, there exists a potential distortion of growth through the extensive margin.

A. *Firm Problem with Linear Taxes*

A firm solves its problem given its forecast of a constant taxation and subsidy policy of the planner (τ and ς, respectively):

$$\begin{aligned} V_t(z;\tau,\varsigma) = \max\Bigg\{&(1-\tau)z + \frac{1}{1+r_t}V_{t+1}(z), \\ &\varsigma z + \frac{1}{1+r_t}\int V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}dz'\Bigg\}. \end{aligned} \tag{23}$$

The resulting growth factor is

$$g(\tau, \varsigma) = \left\{ \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(\frac{r}{1+r} + \frac{1}{1+r} \frac{\alpha}{\alpha - 1} \right)} \right] \frac{1}{1+r} \frac{\alpha}{\alpha - 1} \right\}^{1/(\alpha-1)} > 1. \tag{24}$$

The proof closely follows that of the competitive equilibrium without taxation presented in Section II.E and is omitted for brevity (see App. D for a complete derivation). The analysis of this competitive equilibrium is similar to that in Section II.E. As expected, $dg/d\tau < 0$ and $dg/d\varsigma > 0$, capturing the meaningful trade-off between taxes and subsidies.

B. *Constrained Planner Problem with Optimal Linear Taxes*

The planner chooses ς and τ to maximize the utility of the representative consumer subject to a balanced-budget constraint. Additionally, the planner is constrained to work within the existing structure of the economy, and, thus, the growth rate generated must be consistent with the solution to the competitive equilibrium of the firms given in Section IV.A. The solution requires verifying that τ and ς are constant in order to validate the forecasts of the firms. The constrained planner solves

$$U(F) = \max_{0 \le \tau < 1,\, 0 \le \varsigma \le 1 - \tau} \left\{ \frac{\left[\int_{gm(f)}^{\infty} z dF(z) \right]^{1-\gamma}}{1 - \gamma} + \beta U(F') \right\} \tag{25}$$

subject to

$$g = \left\{ \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(\frac{r}{1+r} + \frac{1}{1+r} \frac{\alpha}{\alpha - 1} \right)} \right] \frac{1}{1+r} \frac{\alpha}{\alpha - 1} \right\}^{1/(\alpha-1)}, \tag{26}$$

$$F'(z) = \frac{F(z) - F(m(F))}{1 - F(gm(F))}, \tag{27}$$

$$\frac{1}{1+r} = \beta g^{-\gamma}, \tag{28}$$

$$\tau \int_{gm(F)}^{\infty} z dF(z) = \varsigma \int_{m(F)}^{gm(F)} z dF(z), \tag{29}$$

$$\tau, \varsigma \text{ confirm agent forecasts.} \tag{30}$$

PROPOSITION 4. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$, $(\alpha - 1)/\alpha < \beta < 1$, $\gamma > 1$, and $\alpha > \gamma/(\gamma - 1)$, then a feasible solution exists such that the constrained planner uses τ and ς to achieve the first-best solution of the unconstrained planner: $g = \{\beta[\alpha/(\alpha - 1)]\}^{1/(\gamma-1)}$.

Proof. See Appendix B.

Thus, subsidizing low-productivity firms to upgrade their productivity increases growth.

V. Conclusion

This paper contributes an analytically tractable mechanism for analyzing growth and the evolution of the productivity distribution, with both the evolution of the productivity distribution and the technology adoption decision jointly endogenously determined in equilibrium. Thus, we can analyze the effect the productivity distribution has on adoption incentives, the effect of adoption behavior in generating the productivity distribution, and the corresponding growth implications of this feedback loop. We develop a solution technique that obtains closed-form expressions for all equilibrium objects—including the growth factor—as a function of intrinsic parameters.

The closed-form solutions for both the distribution and the growth rate clearly illuminate the forces at work in the model that govern the strength of the growth mechanism. In particular, higher productivity inequality leads to more growth by fueling risk taking. The greater the dispersion of opportunities, the more firms are willing to take a risk, even if they might fail and need to continue costly searching.

The competitive equilibrium is inefficient, as firms do not internalize how their search decisions affect the evolution of the distribution. A planner that considers this externality, and chooses which firms search, achieves higher growth. This result suggests a government policy of using a tax-funded program to subsidize the upgrading of unproductive enterprises. Although a trade-off exists for the constrained planner, with subsidies encouraging search and taxation discouraging search, the first best can be achieved.

Appendix A

Solution to Planner's Problem: Proof

To prove proposition 3, guess and verify a functional form for the value function and use the first-order conditions to determine coefficients. To ensure that the growth rate is constant, it must be shown that with the guess, the planner's choice of g is independent of the state F. Finally, given the coefficient values, we need to show that the objective function in the maximization problem is globally concave to confirm that our choice of g is the global maximum.

The current minimum of support, m, summarizes the aggregate state. Substitute the Pareto F_0 into equation (20) and evaluate the integral

$$U(m) = \max_{g\geq 1}\left\{\frac{\left(\frac{\alpha}{\alpha-1}g^{1-\alpha}m\right)^{1-\gamma}}{1-\gamma} + \beta U(gm)\right\}. \tag{A1}$$

Guess that the form of the planner value function is $U(m) = -Am^{1-\gamma}$, where $A > 0$.

Substitute the guess into equation (A1) and define constant $Q \equiv [\alpha/(\alpha-1)]^{1-\gamma}$:

$$-Am^{1-\gamma} = \max_{g\geq 1}\left\{Q\frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma}m^{1-\gamma} - \beta Ag^{1-\gamma}m^{1-\gamma}\right\}. \tag{A2}$$

Using $m > 0$, divide by $m^{1-\gamma}$:

$$-A = \max_{g\geq 1}\left\{Q\frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma} - \beta Ag^{1-\gamma}\right\}. \tag{A3}$$

Since m has dropped out of the expression, A and g will not be functions of m, confirming our guess on the functional form of $U(m)$ and ensuring that g is constant.

Assume, for now, that g is interior and take the first-order condition

$$0 = g^{\alpha(\gamma-1)}(\alpha-1)Q - \beta A(\gamma-1). \tag{A4}$$

Solving for A,

$$A = \frac{Q(\alpha-1)g^{\alpha(\gamma-1)}}{\beta(\gamma-1)}. \tag{A5}$$

Substitute equation (A5) into equation (A3), dropping the max since this g is the argmax. Solving for g,

$$g = \left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma-1)}. \tag{A6}$$

Note that $\beta > (\alpha-1)/\alpha$ is necessary and sufficient for $g > 1$.

Define the maximization problem's objective, $\Omega(g)$, from equation (A3) to get

$$\Omega(g) \equiv Q\frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma} - \beta Ag^{1-\gamma}. \tag{A7}$$

Given the parameter restrictions that ensure that g is interior, to show that the optimal g found in equation (A6) is the global maximum, it is sufficient to show global strict concavity of $\Omega(g)$ for any fixed $A > 0$ and variable $g > 1$. Since $\alpha > 1$, $\gamma > 1$, and $A > 0$, a necessary and sufficient condition for $d^2\Omega(g)/dg^2 < 0$ for all $A > 0$ and $g > 1$ is $\alpha > \gamma/(\gamma - 1)$. QED

Appendix B

Constrained Planner Problem: Proof

This proof of proposition 4 shows that the constrained planner can achieve the first-best solution of the unconstrained planner with constant τ, ς. From equation (26),

$$g = \left\{ \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(\frac{r}{1+r} + \frac{1}{1+r}\frac{\alpha}{\alpha - 1}\right)} \right] \frac{1}{1+r}\frac{\alpha}{\alpha - 1} \right\}^{1/(\alpha-1)}. \tag{B1}$$

Substituting in $1/(1 + r) = \beta g^{-\gamma}$ and $r/(1 + r) = 1 - \beta g^{-\gamma}$ from the consumer problem,

$$g = \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(1 + \frac{1}{\alpha - 1}\beta g^{-\gamma}\right)} \beta \frac{\alpha}{\alpha - 1} \right]^{1/(\gamma - 1 + \alpha)}. \tag{B2}$$

The first-best solution, achieved by the planner in equation (21), is

$$g_{fb} = \left(\beta \frac{\alpha}{\alpha - 1} \right)^{1/(\gamma - 1)}. \tag{B3}$$

Assume that the decentralized planner is able to achieve the first best, $g = g_{fb}$, and substitute

$$g = \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(1 + \frac{1}{\alpha} g^{\gamma - 1} g^{-\gamma}\right)} g^{\gamma - 1} \right]^{1/(\gamma - 1 + \alpha)}. \tag{B4}$$

Solving for $\varsigma(\tau)$,

$$\varsigma = \frac{g\alpha(g^{\alpha} - 1)}{g^{\alpha}(1 + g\alpha) - g\alpha}(1 - \tau). \tag{B5}$$

Substituting for $F_t(z)$, the budget constraint of the constrained planner is

$$\tau = \varsigma(g^{\alpha - 1} - 1). \tag{B6}$$

Equations (B5) and (B6) are a system in (τ, ς). Solving yields equations for ς and τ in terms of model intrinsics that satisfy the budget constraint by construction:

$$\varsigma = \frac{g\alpha(1 - g^{-\alpha})}{1 + \alpha(g^{\alpha} - 1)}, \tag{B7}$$

$$\tau = \frac{g\alpha(1 - g^{-\alpha})(g^{\alpha-1} - 1)}{1 + \alpha(g^{\alpha} - 1)}. \tag{B8}$$

Both $\varsigma > 0$ and $\tau > 0$ follow from $g^{\alpha} > 1$. To check the final requirement that $\varsigma + \tau < 1$, add equations (B7) and (B8):

$$\varsigma + \tau = 1 - \frac{1}{\alpha(g^{\alpha} - 1) + 1}, \tag{B9}$$

$g^{\alpha} > 1$ and, hence, both $\varsigma + \tau < 1$ and $\varsigma + \tau > 0$ hold, proving that an interior solution for the subsidies and taxes can achieve the first-best solution. QED

Appendix C

Uniqueness of the Pareto Distribution

The following proposition proves that under the scale invariance requirement, the Pareto distribution is the unique initial condition that can fulfill the BGP equilibrium requirements.

PROPOSITION 5. Given the maintained assumptions, the Pareto distribution is the unique distribution that can satisfy the BGP equilibrium requirements.

Proof. Assume for clarity of exposition that $m_0 = 1 = \min \text{support}\,\{F_0\}$ (without loss of generality, or use the normalized version of the evolution developed in App. F). Combining the law of motion for the density and the scale invariance equation (in density form), (4) and (6), and rearranging,

$$f_0(\tilde{z}m_t) = \frac{1 - F_0(m_t)}{m_t} f_0(\tilde{z}) \propto f_0(\tilde{z}), \tag{C1}$$

a power law. Differentiate both sides of equation (C1) with respect to m_t:

$$\frac{df_0(\tilde{z}m_t)}{d\tilde{z}}\tilde{z} = \frac{F_0(m_t) - m_t F_0'(m_t) - 1}{m_t^2} f_0(\tilde{z}). \tag{C2}$$

Evaluate at $m_t = 1$ and use that for any initial condition with a minimum of support 1, $F_0'(1) \equiv f_0(1)$ and $F_0(1) = 0$:

$$\frac{df_0(\tilde{z})}{d\tilde{z}}\tilde{z} = [-f_0(1) - 1] f_0(\tilde{z}). \tag{C3}$$

This is an ordinary differential equation in $\tilde{z}$. If we arbitrarily choose the initial condition to be $f_0(1) = \alpha$, then the particular solution of the differential equation matches the parameterization for a Pareto$(1, \alpha)$:

$$f_0(\tilde{z}) = \alpha\tilde{z}^{-\alpha-1}. \tag{C4}$$

Hence, for any initial condition with support $[1, \infty)$, the only solution is the Pareto distribution. This can be renormalized for an arbitrary m_0 to give the Pareto density as the unique initial condition: $f_0(z) = \alpha m_0^{\alpha} z^{-\alpha-1}$. QED

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