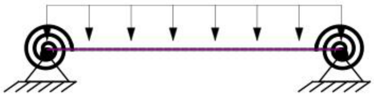
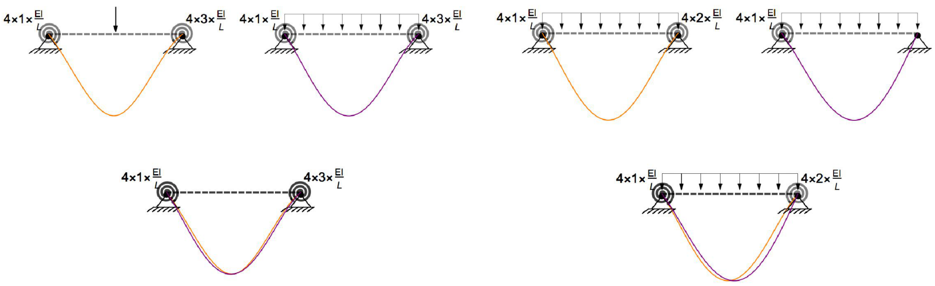
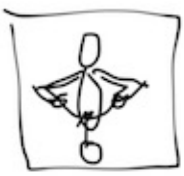
Scripts



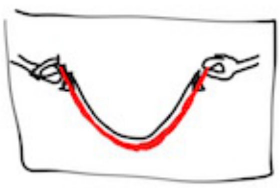
Having considered special cases, we look at a uniformly loaded beam with the ends connected to passive structures and that have no side-sway. We model the effect of the passive structures attached to each end using rotary springs with rotary stiffness factors denoted by 'k_L' (k sub-L) for the left support and 'k_R' (k sub-R) for the right support and these may generally be different. This case includes the cases discussed previously because a hinge would be equivalent to using a rotary stiffness factor of zero and a fixed end would be equivalent to using a very large rotary stiffness factor; by very large we mean taking the limiting value to infinity in the formulas, but practically in this case, using a value of 100 in the formulas would suffice as being very large. Again, we apply a uniform load over the whole beam.



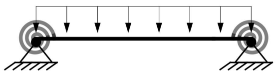
Now the deformed shape of a beam loaded with a central point force and a beam loaded with a uniform load are again practically visually indistinguishable if our aim is to sketch the shapes as we can see in the overlay comparison figure ... which we can see more clearly if we zoom in ... .



Therefore, I will demonstrate the behavior of a uniformly loaded straw with the visually near equivalent, namely the centrally loaded beam. Notice that I start with the simply supported case and apply some moment resistance at each end. I choose these moments so that they straighten the beam at the ends but not enough to make them tangent to the horizontal. Notice that the shape has a positive curvature in the middle while it has negative curvatures near the ends. The negative curvature at the ends occur because I am applying resisting moments. How much resisting moment depends on the rotary stiffness factor that I want to simulate at each end.



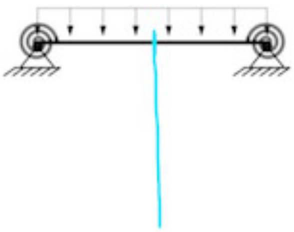
Again, if we compare the theoretical deformation with the deformation of the straw, we get a good match. In this case, the good match was obtained by taking ' k_L ≈ ?? ' and ' k_R ≈ ?? '.



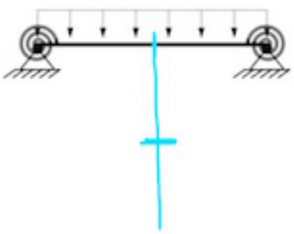
Now we consider how to sketch this case by hand.

Location of maximum displacement as % distance of length from center

k_L ↓ k_R →	0	1	1.5	2	3	∞
0	0.0%	-3.3%	-4.1%	-4.7%	-5.4%	-7.8%
1	3.3%	0.0%	-0.8%	-1.4%	-2.2%	-4.9%
1.5	4.1%	0.8%	0.0%	-0.6%	-1.4%	-4.1%
2	4.7%	1.4%	0.6%	0.0%	-0.8%	-3.3%
3	5.4%	2.2%	1.4%	0.8%	0.0%	-2.8%
∞	7.8%	4.9%	4.1%	3.3%	2.8%	0.0%

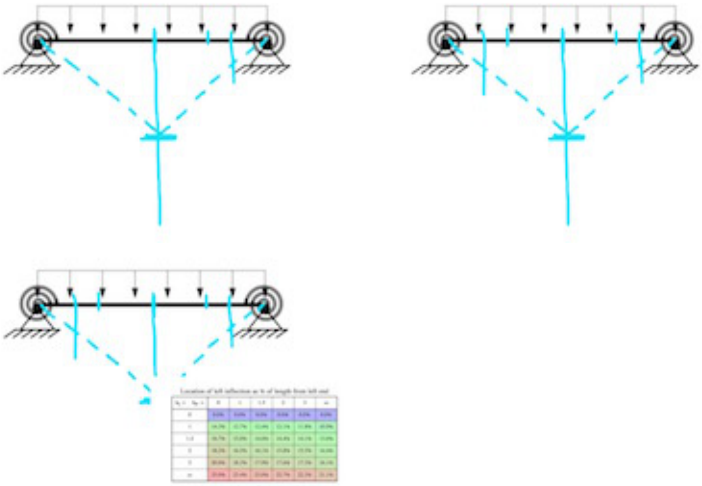


We start by locating the midpoint and then we may move a bit to the side where the support is less stiff to determine the x-coordinate of the where the displacement is maximum. Actually, when the rotary stiffness factors are both between 1 and 3 then the shift is less than 2.5% which for a sketch is practically insignificant. In the worst case, when one side is hinged and the other fixed, we have to move by 7.8% the length of the beam which is about a quarter of a quarter visually; again, very small and can be neglected in a sketch. The table shown presents the actual range of values. So we locate the midpoint and we draw a vertical guide there and the maximum displacement will occur close to this line.

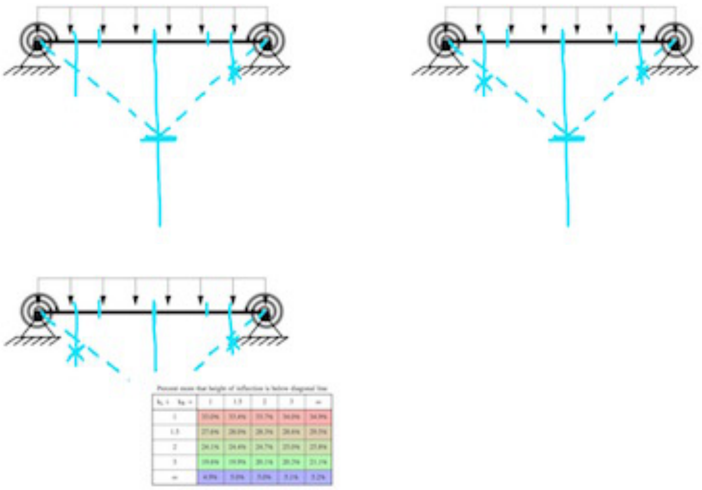


So along this vertical guide line, we choose a level for the maximum displacement and draw a short horizontal guide line.

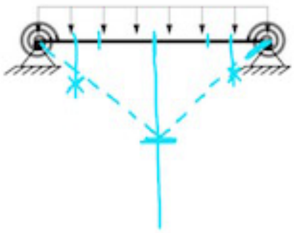
We next draw diagonal guide-lines that join the point of maximum displacement with each end.

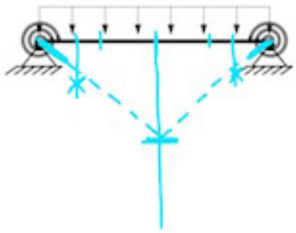


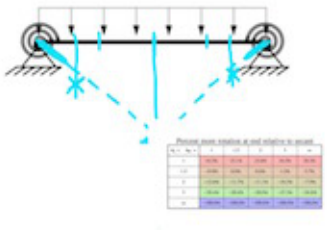
Now we locate the inflection points at each side. When the rotary stiffness factors are both between 1 and 3 then the inflection points are between 11 to 18% the length from each end. For the purpose of sketching, we simply use half the quarter length from each side (that's 12.5% if done to perfection) and draw a vertical guide line down at each side to identify the x-coordinates of the inflection points. The table shown presents the actual range of values for the left inflection point which would be analogous for the right inflection point.



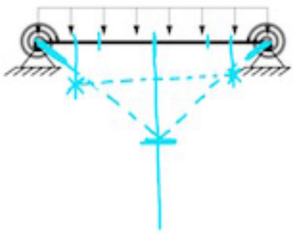
Along those vertical guides for the inflection points we mark points at about 25% below the diagonal guide lines. The actual range of values are shown in the lower table and these have values between about 20 to 35% when the rotary stiffness at the corresponding end is between 1 and 3. So 25% is a good choice.



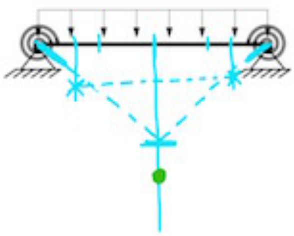


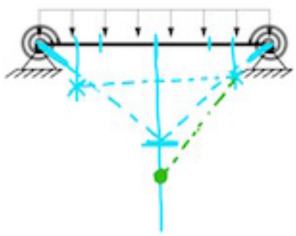


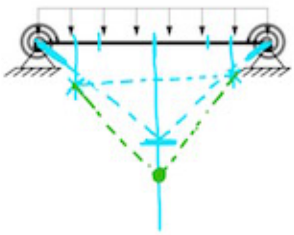
Next, draw two stubs at each end support with slopes along the diagonal. The deformed beam will be drawn tangent to these stubs at each end. The actual slopes will actually vary but not enough to be visually too significant. A table of values for the percent change in slope is shown in the table. For example, when the rotary stiffness factor at the left is 1 then the slope of the beam will be between about 14 to 18% steeper than the diagonal guide line. We will simply neglect this difference in our sketches.

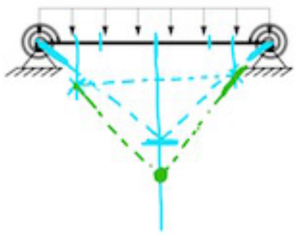


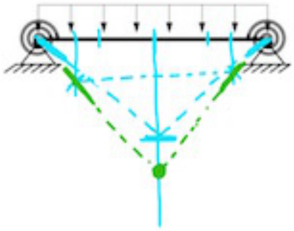
Now, draw a datum line joining the two inflection points.

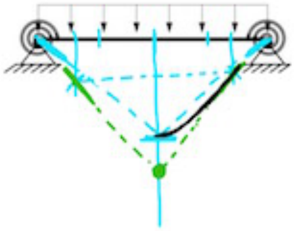


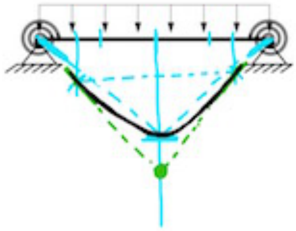




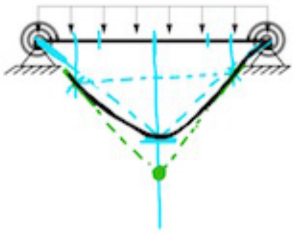


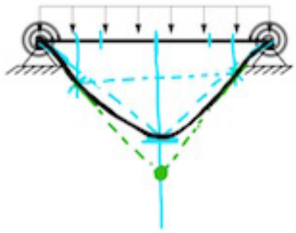




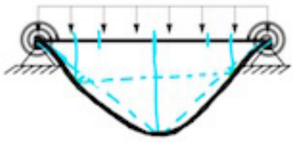


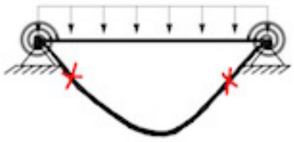
Relative to this datum line joining the two inflection points, we have exactly the shape of a simply supported beam because the bending moment is zero at the inflection points.
Using the directions described for the simply supported case, draw the deformed shape of the beam between the two inflection points. One small detail is to extend the slopes at the inflection points because these will guide us on how to sketch the end pieces. These slope extensions are shown in green in the figure.



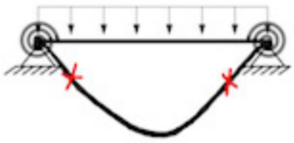


At each support, draw a smooth curve tangent to the guide stubs at the ends and tangent to the slope identified at the inflection points while sketching the middle portion.

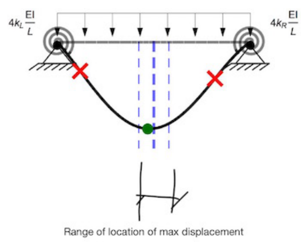




Finally, we remove or erase the extraneous lines, and further smooth out our curve to get our deformed shape.
Some or all of these steps may be omitted once we get a sufficient familiarity with sketching this shape. But this step-by-step approach makes us aware of all the salient details of the beam’s deformation. One important insight that we will use is that the portion between the inflection point and the hinge support is exactly like a simply supported beam in regards to its relative deformation and thus also its bending moment diagram.



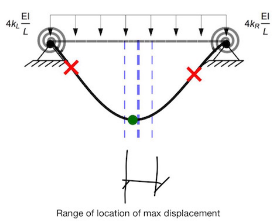
And now we compare with the theoretical shape, and, we again get a good match, this time between theory and sketch. The values used for the left and right rotary stiffness factors to achieve the match are ??? and ??? respectively.



Location of maximum displacement as % distance of length from center

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	0.0%	-3.3%	-4.1%	-4.7%	-5.4%	-7.8%
1	3.3%	0.0%	-0.8%	-1.4%	-2.2%	-4.9%
1.5	4.1%	0.8%	0.0%	-0.6%	-1.4%	-4.1%
2	4.7%	1.4%	0.6%	0.0%	-0.8%	-3.5%
3	5.4%	2.2%	1.4%	0.8%	0.0%	-2.8%
∞	7.8%	4.9%	4.1%	3.5%	2.8%	0.0%

Now let's go back to the beam and look at some quantitative results. In general, we have different rotary stiffness factors on the left and right. The location of the maximum displacement depends on those values. When they are equal, we have symmetry and the maximum displacement occurs in the middle. Otherwise, the maximum displacement is shifted towards the more flexible side or lower stiffness. That shift is, however, rather small. The largest shift occurs when one side is fixed and the other hinged. In that case, as we saw in before, the shift is 7.8% of the length from the center towards the hinge. When both rotary stiffness factors are between 1 and 3, the largest shift of the location of the maximum displacement is less than about 2.2% the length of the beam. So, the maximum displacement stays rather close to the center as long as the beam is connected to passive supports with no side-sway.



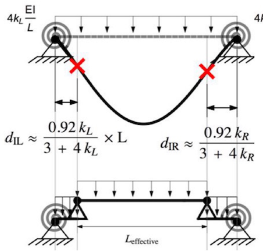
Location of maximum displacement as % distance of length from center

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	0.0%	-3.3%	-4.1%	-4.7%	-5.4%	-7.8%
1	3.3%	0.0%	-0.8%	-1.4%	-2.2%	-4.9%
1.5	4.1%	0.8%	0.0%	-0.6%	-1.4%	-4.1%
2	4.7%	1.4%	0.6%	0.0%	-0.8%	-3.5%
3	5.4%	2.2%	1.4%	0.8%	0.0%	-2.8%
∞	7.8%	4.9%	4.1%	3.5%	2.8%	0.0%

Maximum displacement divided by $w L^4 / (384 EI)$

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	5.000	3.305	3.027	2.852	2.644	2.080
1	3.305	2.333	2.159	2.046	1.909	1.525
1.5	3.027	2.159	2.000	1.897	1.771	1.416
2	2.852	2.046	1.897	1.800	1.682	1.345
3	2.644	1.909	1.771	1.682	1.571	1.257
∞	2.080	1.525	1.416	1.345	1.257	1.000

If we now look at the corresponding value of the maximum displacement, we find that it is a multiple of (w L^4 / (384 EI)) that varies between 1 and 5. It is 1 for the fixed-fixed case and 5 for the simply supported case. When both rotary stiffness factors are larger than 1, which is the usual case when there are connected members at each end, then the maximum displacement is between 1 and 2.33. From this observation and other values shown in the table in the lower right corner, we see that the difference between a free-to-rotate end and a modest resistance corresponding to a rotary stiffness factor of 1 is much more significant than any further increase in the rotary stiffness factor.



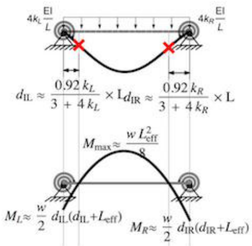
Location of left inflection as % of length from left end

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
1	14.3%	12.7%	12.4%	12.1%	11.8%	10.9%
1.5	16.7%	15.0%	14.6%	14.4%	14.1%	13.0%
2	18.2%	16.5%	16.1%	15.8%	15.5%	14.4%
3	20.0%	18.3%	17.9%	17.6%	17.3%	16.1%
∞	25.0%	23.4%	23.0%	22.7%	22.3%	21.1%

Next, we look at the location of inflection points. Whenever both rotary stiffness factors are nonzero, there are two inflection points, one near each end. Of course, when a rotary stiffness factor is zero at a support, we have zero moment at that support and there is no inflection point and no negative bending moments near that support.

When the rotary stiffness factor at an end is between 1 and 3 then the inflection point near that end is between about 11 and 20% the length of the beam from that side. The location of an inflection point near a support is mostly affected by the rotary stiffness at that support and very weakly by the rotary stiffness at the other end. An approximate formula for the location of an inflection point that uses that observation is shown in the figure. Specifically, the distance of an inflection point from a support is approximately given by the length of the beam times 0.92 times the rotary stiffness factor at that support divided by (3 + 4 times the rotary stiffness factor at that support).

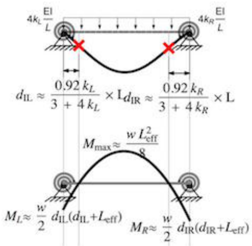
Once we calculate the location of inflection point, we can calculate the length between the two inflection points and this gives an effective length of a mid-section beam that acts as if it is simply supported with possibly some rigid rotation and translation.



The moment diagram for this basic case is again a parabola because the load is uniform. It is a parabola that has a maximum positive moment at the midpoint between the inflection points. This maximum has a value of $\frac{w\,L_{\text{effective}}^2}{8}$ which can be deduced from the observation that, in terms of forces and bending moments, the section of the beam between the inflection points is like a simply supported beam. The negative moments at the ends can be calculated using statics using the free body diagrams with cuts at the inflection points and using the fact that the bending moments there are zero. The resulting formula is rather simple. The bending moment at a support is half the distributed load times the distance of the support to the nearest inflection point times the distance of the support to farthest inflection point. Now, let’s look at some numerical ...

Maximum positive moment divided by $\frac{w L^2}{8}$

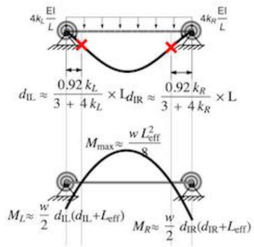
$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	1.00	0.73	0.69	0.67	0.64	0.56
1	0.73	0.56	0.53	0.51	0.49	0.43
1.5	0.69	0.53	0.50	0.48	0.46	0.41
2	0.67	0.51	0.48	0.47	0.45	0.40
3	0.64	0.49	0.46	0.45	0.43	0.38
∞	0.56	0.43	0.41	0.40	0.38	0.33



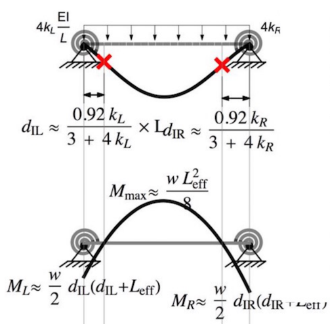
... results concerning the maximum positive bending moment. The largest possible positive moment occurs when the effective length is as large as possible which is the distance between the supports. Of course, this occurs with the simply supported case. By contrast, the smallest positive moment is when both sides are fixed ends. This gives one-third the maximum. In cases when both ends have rotary stiffness factors between 1 to 3, which is common, then the positive moment is between 43 and 56% the maximum case of w L^2/ 8. That is a relatively small variation and guessing a middle value of 50% for such cases gives an error less than 17%. Next, let's look at some numerical ...

Bending moment at left support divided by $\frac{w L^2}{8}$

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	0.00	0.00	0.00	0.00	0.00	0.00
1	0.57	0.44	0.42	0.41	0.39	0.33
1.5	0.67	0.53	0.50	0.48	0.46	0.40
2	0.73	0.58	0.55	0.53	0.51	0.44
3	0.80	0.65	0.62	0.60	0.57	0.50
∞	1.00	0.83	0.80	0.78	0.75	0.67



... results concerning the negative bending moments at support. The largest possible negative moment occurs when a suport is fixed and the other side is hinged. This largest value equals the largest possible positive moment of w L^2 / 8. By contrast, the smallest negative moment occurs whenever an end has a freely rotating condition such as at a hinge support. In cases when both ends have rotary stiffness factors between 1 to 3, which is common, then the negative moment at each support between 39 and 65% the maximum case of w L^2/ 8. With this variation, if we always guess a middle value of 50% for such cases we will always get an error less than 28%.



Maximum positive moment divided by $\frac{w\,L^2}{8}$

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	1.00	0.73	0.69	0.67	0.64	0.56
1	0.73	0.56	0.53	0.51	0.49	0.43
1.5	0.69	0.53	0.50	0.48	0.46	0.41
2	0.67	0.51	0.48	0.47	0.45	0.40
3	0.64	0.49	0.46	0.45	0.43	0.38
∞	0.56	0.43	0.41	0.40	0.38	0.33

Bending moment at left support divided by $\frac{w\,L^2}{8}$

$k_L \downarrow$ $k_R \rightarrow$	0	1	1.5	2	3	∞
0	0.00	0.00	0.00	0.00	0.00	0.00
1	0.57	0.44	0.42	0.41	0.39	0.33
1.5	0.67	0.53	0.50	0.48	0.46	0.40
2	0.73	0.58	0.55	0.53	0.51	0.44
3	0.80	0.65	0.62	0.60	0.57	0.50
∞	1.00	0.83	0.80	0.78	0.75	0.67

In any case, the process of calculating the approximate bending moment diagram involves the following steps. First, identify the values of the left and right rotary stiffness factors as discussed in the lecture on chapter 1 part 1. With those values, estimate the location of the inflection points on both sides and calculate the effective length. With those values, use the formulas shown to obtain the maximum positive moment and the negative moments at the ends. To draw the parabola for the bending moment diagram, draw a parabola starting and ending at zero at the inflection points with a maximum occurring in the middle between them. Then, extend both sides of that parabola in a continuation of the parabolic shape to meet the supports at each end. Mark the values obtained for the maximum positive moment and the negative moments at each side.