$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

While calculating the derivatives of $\cos(x)$ and $\sin(x)$, Professor Jerison said that $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$. This is true, but in order to be certain that our derivative formulas are correct we should understand *why* it's true.

As in the discussion of $\sin(\theta)/\theta$, our explanation involves looking at a diagram of the unit circle and comparing an arc with length $\theta$ to a straight line segment. (Remember that $\theta$ is measured in radians!) As shown in Figure 1, the vertical distance between the endpoints of the arc is $\cos\theta$, and the horizontal distance between the ends of the arc is $1 - \cos\theta$.

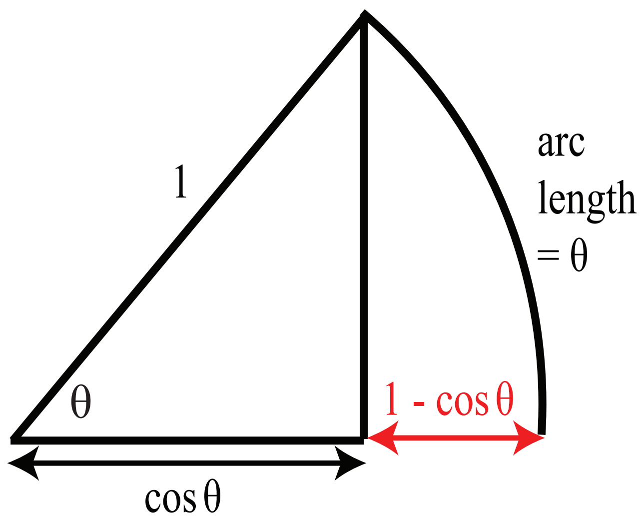


Figure 1: Same figure as for $\frac{\sin x}{x}$ except that the horizontal distance between the edge of the triangle and the perimeter of the circle is marked

From Fig. 2 we can see that as $\theta \to 0$, the horizontal distance $1 - \cos\theta$ between endpoints of the arc (what Professor Jerison calls "the gap") gets much smaller than the length $\theta$ of the arc. Hence, $\frac{1 - \cos\theta}{\theta} \to 0$.

If you find this hard to believe it may be helpful to use a calculator to verify that if $x$ is small, $1 - \cos x$ is much smaller. You might also study the graph of $y = 1 - \cos x$ near $x = 0$ or use a web application to compare the distance $1 - \cos\theta$ to the arc length $\theta$ for very small angles $\theta$.

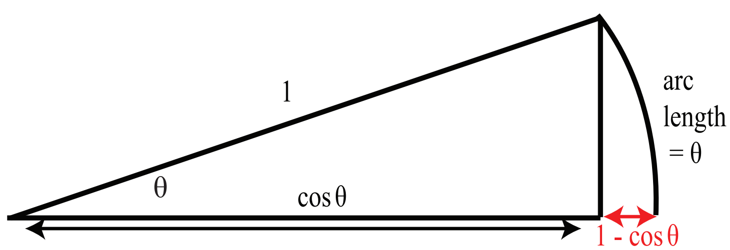


Figure 2: The sector in Fig. 1 as $\theta$ becomes very small