

Quotient Rule

Now that we know the product rule we can find the derivatives of many more functions than we used to be able to. Our next step toward “differentiating everything” will be to learn a formula for differentiating quotients (fractions). The rule is:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Why is this true? The definition of the derivative tells us that:

$$\left(\frac{u}{v}\right)' = \lim_{\Delta x \rightarrow 0} \frac{\frac{u(x + \Delta x)}{v(x + \Delta x)} - \frac{u(x)}{v(x)}}{\Delta x}$$

This is an unwieldy expression. We’ll start to make sense of it by simplifying the numerator and by creating two new variables $\Delta u = u(x + \Delta x) - u(x)$ and $\Delta v = v(x + \Delta x) - v(x)$.

$$\begin{aligned} \frac{u(x + \Delta x)}{v(x + \Delta x)} - \frac{u(x)}{v(x)} &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \quad (\text{since } u(x + \Delta x) = u(x) + \Delta u) \\ &= \frac{(u + \Delta u)v - u(v + \Delta v)}{(v + \Delta v)v} \quad (\text{common denominator}) \\ &= \frac{uv + (\Delta u)v - uv - u(\Delta v)}{(v + \Delta v)v} \quad (\text{distribute } u \text{ and } v) \\ &= \frac{(\Delta u)v - u(\Delta v)}{(v + \Delta v)v} \quad (\text{because } uv - uv = 0) \end{aligned}$$

Now that we’ve simplified the numerator, we can use it to simplify the difference quotient:

$$\begin{aligned} \frac{\frac{u(x + \Delta x)}{v(x + \Delta x)} - \frac{u(x)}{v(x)}}{\Delta x} &= \frac{\frac{(\Delta u)v - u(\Delta v)}{(v + \Delta v)v}}{\Delta x} \\ &= \frac{1}{\Delta x} \frac{(\Delta u)v - u(\Delta v)}{(v + \Delta v)v} \\ &= \frac{\left(\frac{\Delta u}{\Delta x}\right)v - u\left(\frac{\Delta v}{\Delta x}\right)}{(v + \Delta v)v} \end{aligned}$$

we’re assuming that v is differentiable and therefore continuous, so $\lim_{x \rightarrow 0} v(x + \Delta x) = v(x)$. Hence, by the definition of the derivative,

$$\frac{\left(\frac{\Delta u}{\Delta x}\right)v - u\left(\frac{\Delta v}{\Delta x}\right)}{(v + \Delta v)v} \rightarrow \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2} \quad \text{as } \Delta x \rightarrow 0.$$

We conclude that:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$