FIND SOLUTIONS ON NEXT PAGE

CHAPTER THREE

Solutions for Section 3.1

Exercises

1. The annual growth factor is $1+$ the growth rate, so we have 1.03.

2. The decennial growth factor (growth factor per 10 years) is $1+$ the growth per decade: 1.28.

3. The daily growth factor is $1+$ the daily growth. Since the mine's resources are shrinking, the growth is -0.01, giving 0.99.

4. The growth factor per century is $1+$ the growth per century. Since the forest is shrinking, the growth is negative, so we subtract 0.80, giving 0.20.

5. For a 10% increase, we multiply by 1.10 to obtain $500 \cdot 1.10 = 550$.

6. For a 100% increase, we multiply by $1 + 1.00 = 2$ to obtain $500 \cdot 2 = 1000$.

7. For a 1% decrease, we multiply by $1 - 0.01 = 0.99$ to obtain $500 \cdot 0.99 = 495$.

8. For a 42% decrease, we multiply by $1 - 0.42 = 0.58$ to obtain $500 \cdot 0.58 = 290$.

9. For a 42% increase, we multiply by 1.42 to obtain $500 \cdot 1.42 = 710$. For a 42% decrease, we multiply by $1 - 0.42 = 0.58$ to obtain $710 \cdot 0.58 = 411.8$.

10. For a 42% decrease, we multiply by $1 - 0.42 = 0.58$ to obtain $500 \cdot 0.58 = 290$. For a 42% increase, we multiply by 1.42 to obtain $290 \cdot 1.42 = 411.8$.

11. We have $a = 1750$, $b = 1.593$, and $r = b - 1 = 0.593 = 59.3\%$.

12. We have $a = 34.3$, $b = 0.788$, and $r = b - 1 = 0.788 - 1 = -0.212 = -21.2\%$.

13. Since $Q = 79.2(1.002)^t$, we have $a = 79.2$, $b = 1.002$, and $r = b - 1 = 0.002 = 0.2\%$.

14. We can rewrite this as

$$\begin{aligned} Q &= 0.0022(2.31^{-3})^t \\ &= 0.0022(0.0811)^t, \end{aligned}$$

so $a = 0.0022$, $b = 0.0811$, and $r = b - 1 = -0.9189 = -91.89\%$.

15. In the exponential formula $f(t) = ab^t$, the parameter a represents the vertical intercept. All the formulas given have graphs that intersect the vertical axis at 10, 20, or 30. We see that formulas (a) and (b) intersect the vertical axis at 10 and correspond (in some order) to graphs I and III.

Formula (c) intersects the vertical axis at 20 and must be graph II. Formulas (d), (e), and (f) intersect the vertical axis at 30, and correspond (in some order) to graphs IV, V, and VI. The parameter b in the exponential formula $f(t) = ab^t$ gives the growth factor. Since the growth factor for formula (b), 1.5, is greater than the growth factor for formula (a), 1.2, formula (a) must correspond to graph III while formula (b) corresponds to graph I.

Formula (f) represents exponential growth (and thus must correspond to graph IV), while formulas (e) and (f) represent exponential decay. Since formula (d) decays at a more rapid rate (15% per unit time compared to 5% per unit time) than formula (e), formula (d) corresponds to graph VI and formula (e) corresponds to graph V. We have:

(a) III
(b) I
(c) II
(d) VI
(e) V
(f) IV

16. **(a)** The formula $f(t) = ab^t$ represents exponential growth if the base $b > 1$ and exponential decay if $0 < b < 1$. Towns (i), (ii), and (iv) are growing and towns (iii), (v), and (vi) are shrinking.

(b) Town (iv) is growing the fastest since its growth factor of 1.185 is the largest. Since $1.185 = 1 + 0.185$, it is growing at a rate of 18.5% per year.

(c) Town (v) is shrinking the fastest since its growth factor of 0.78 is the smallest. Since $0.78 = 1 - 0.22$, it is shrinking at a rate of 22% per year.

(d) In the exponential function $f(t) = ab^t$, the parameter a gives the value of the function when $t = 0$. We see that town (iii) has the largest initial population (2500) and town (ii) has the smallest initial population (600).

17. The percent of change is given by

$$\text{Percent of change} = \frac{\text{Amount of change}}{\text{Old amount}} \cdot 100\%.$$

So in these two cases,

$$\text{Percent of change from 10 to 12} = \frac{12 - 10}{10} \cdot 100\% = 20\%$$
$$\text{Percent of change from 100 to 102} = \frac{102 - 100}{100} \cdot 100\% = 2\%$$

18. If an investment decreases by 5% each year, we know that only 95% remains at the end of the first year. After 2 years there will be 95% of 95%, or 0.95^2 left. After 4 years, there will be $0.95^4 \approx 0.81451$ or 81.451% of the investment left; it therefore decreases by about 18.549% altogether.

Problems

19. The starting value is $a = 2200$. The growth rate is $r = -3.2\% = -0.032$, so $b = 1 + r = 0.968$. We have $P = 2200(0.968)^t$.

20. We have $a = 2500$ and $b = 1.0325$, so $r = b - 1 = 0.0325 = 3.25\%$. Thus, the starting value is \$2500 and the percent growth rate is 3.25%.

21. In 2007, the cost of tickets will be 1.1 times their cost in 2006, (i.e. 10% greater). Thus, the price in 2007 is $1.1 \cdot \$73 = \$80.30$. The price in 2008 is then $1.1 \cdot \$80.30 = \$88.33$, and so forth. See Table 3.1.

Table 3.1

Year	2006	2007	2008	2009	2010
Cost (\$)	73	80.30	88.33	97.16	106.88

22. The population is growing at a rate of 1.9% per year. So, at the end of each year, the population is $100\% + 1.9\% = 101.9\%$ of what it had been the previous year. The growth factor is 1.019. If P is the population of this country, in millions, and t is the number of years since 2006, then, after one year,

$$\begin{aligned} & P = 70(1.019). \\ \text{After two years,}\quad & P = 70(1.019)(1.019) = 70(1.019)^2 \\ \text{After three years,}\quad & P = 70(1.019)(1.019)(1.019) = 70(1.019)^3 \\ \text{After } t \text{ years,}\quad & P = 70\underbrace{(1.019)(1.019)\dots(1.019)}_{t \text{ times}} = 70(1.019)^t \end{aligned}$$

23. If it decays at 5.626% per year, its growth factor is $1 - 0.05626 = 0.94374$. So, with t in years and an initial amount of 726 grams, we have:

$$Q = 726(0.94374)^t.$$

We graph it on a graphing calculator or with a computer to obtain the graph in Figure 3.1.

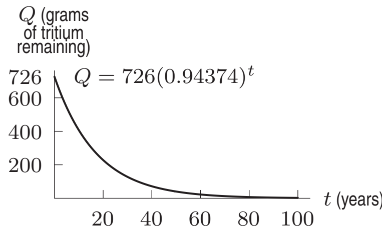


Figure 3.1

24. In year $t = 0$, there are one million organisms, which we take as our initial value. Our growth factor is 0.98, for a decay of 2% per year. Thus:

$$O = 1{,}000{,}000(0.98)^t.$$

25. Since, after one year, 3% of the investment is added on to the original amount, we know that its value is 103% of what it had been a year earlier. Therefore, the growth factor is 1.03.

$$\begin{aligned}
\text{So, after one year,}\quad & V = 100{,}000(1.03) \\
\text{After two years,}\quad & V = 100{,}000(1.03)(1.03) = 100{,}000(1.03)^2 \\
\text{After three years,}\quad & V = 100{,}000(1.03)(1.03)(1.03) = 100{,}000(1.03)^3 = \$109{,}272.70
\end{aligned}$$

26. (a) The initial dose equals the amount of drug in the body when $t = 0$. We have $A(0) = 25(0.85)^0 = 25(1) = 25$ mg.

(b) According to the formula,

$$\begin{aligned}
A(0) &= 25(0.85)^0 = 25 \\
A(1) &= 25(0.85)^1 = 25(0.85) \\
A(2) &= 25(0.85)^2 = 25(0.85)(0.85)
\end{aligned}$$

After each hour, the amount of the drug in the body is the amount at the end of the previous hour multiplied by 0.85. In other words, the amount remaining is 85% of what it had been an hour ago. So, 15% of the drug has left in that time.

(c) After 10 hours, $t = 10$. $A(10) = 4.922$ mg.

(d) Using trial and error, substitute integral values of t into $A(t) = 25(0.85)^t$ to determine the smallest value of t for which $A(t) < 1$. We find that $t = 20$ is the best choice. So, after 20 hours there will be less than one milligram in the body.

27. (a) We assume that the price of a movie ticket increases at the rate of 3.5% per year. This means that the price is rising exponentially, so a formula for p is of the form $p = ab^t$. We have $a = 7.50$ and $b = 1 + r = 1.035$. Thus, a formula for p is

$$p = 7.50(1.035)^t.$$

(b) In 20 years ($t = 20$) we have $p = 7.50(1.035)^{20} \approx 14.92$. Thus, in 20 years, movie tickets will cost almost $\$15$ if the inflation rate remains at 3.5%.

28. (a) Since N is growing by 10% per year, we know that N is an exponential function of t with growth factor $1+0.1 = 1.1$. Since $N = 7.9$ when $t = 0$, we have

$$N = 7.9(1.1)^t.$$

(b) In the year 2010, we have $t = 6$ and

$$N = 7.9(1.1)^6 = 14.0 \text{ million passengers.}$$

In the year 2000, we have $t = -4$ and

$$N = 7.9(1.1)^{-4} = 5.4 \text{ million passengers.}$$

29. (a) We have $C = C_0(1 - r)^t = 100(1 - 0.16)^t = 100(0.84)^t$, so

$$C = 100(0.84)^t.$$

(b) At $t = 5$, we have $C = 100(0.84)^5 = 41.821$ mg

30. **(a)** If gallium-67 decays at the rate of 1.48% each hour, then 98.52% remains at the end of each hour. The growth factor is 0.9852. Since the initial quantity is 100, we have $f(t) = 100(0.9852)^t$, where $f(t)$ represents the number of milligrams of gallium-67 remaining after t hours.

(b) After 24 hours, we have $t = 24$ and

$$f(24) = 100(0.9852)^{24} = 69.92 \text{ mg gallium-67 remaining.}$$

After 1 week, or $7 \cdot 24 = 168$ hours, we have

$$f(168) = 100(0.9852)^{168} = 8.17 \text{ mg gallium-67 remaining.}$$

31. To find a formula for $f(n)$, we start with the number of people infected in 2005, namely P_0. In 2006, only 80% as many people, or $0.8P_0$, were infected. In 2007, again only 80% as many people were infected, which means that 80% of $0.8P_0$ people, or $0.8(0.8P_0)$ people, were infected. Continuing this line of reasoning, we can write

$$f(0) = P_0$$

$$f(1) = \underbrace{(0.80)}_{\substack{\text{one 20\%} \\ \text{reduction}}} P_0 = (0.8)^1 P_0$$

$$f(2) = \underbrace{(0.80)(0.80)}_{\substack{\text{two 20\%} \\ \text{reductions}}} P_0 = (0.8)^2 P_0$$

$$f(3) = \underbrace{(0.80)(0.80)(0.80)}_{\text{three 20\% reductions}} P_0 = (0.8)^3 P_0,$$

and so on, so that n years after 2005 we have

$$f(n) = \underbrace{(0.80)(0.80)\cdots(0.80)}_{n \text{ 20\% reductions}} P_0 = (0.8)^n P_0.$$

We see from its formula that $f(n)$ is an exponential function, because it is of the form $f(n) = ab^n$, with $a = P_0$ and $b = 0.8$. The graph of $y = f(n) = P_0(0.8)^n$, for $n \geq 0$, is given in Figure 3.2. Beginning at the P-axis, the curve decreases sharply at first toward the horizontal axis, but then levels off so that its descent is less rapid.

Figure 3.2 shows that the prevalence of the virus in the population drops quickly at first, and that it eventually levels off and approaches zero. The curve has this shape because in the early years of the vaccination program, there was a relatively large number of infected people. In later years, due to the success of the vaccine, the infection became increasingly rare. Thus, in the early years, a 20% drop in the infected population represented a larger number of people than a 20% drop in later years.

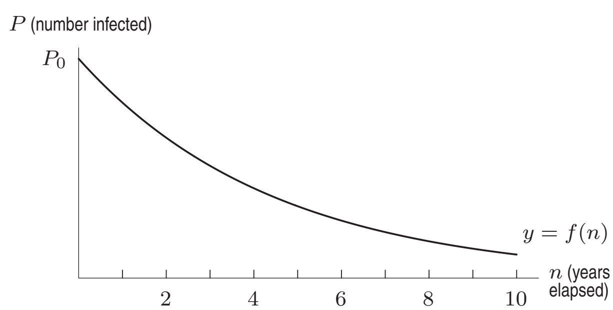


Figure 3.2: The graph of $f(n) = P_0(0.8)^n$ for $n \geq 0$

32. **(a)**

Table 3.2

Month	Balance	Interest	Minimum payment
0	$\$2000.00$	$\$30.00$	$\$50.00$
1	$\$1980.00$	$\$29.70$	$\$49.50$
2	$\$1960.20$	$\$29.40$	$\$49.01$
3	$\$1940.59$	$\$29.11$	$\$48.51$
4	$\$1921.19$	$\$28.82$	$\$48.03$
5	$\$1901.98$	$\$28.53$	$\$47.55$
6	$\$1882.96$	$\$28.24$	$\$47.07$
7	$\$1864.13$	$\$27.96$	$\$46.60$
8	$\$1845.49$	$\$27.68$	$\$46.14$
9	$\$1827.03$	$\$27.41$	$\$45.68$
10	$\$1808.76$	$\$27.13$	$\$45.22$
11	$\$1790.67$	$\$26.86$	$\$44.77$
12	$\$1772.76$		

(b) After one year, your unpaid balance is \$1772.76. You have paid off $\$2000 - \$1772.76 = \$227.24$. The interest you have paid is the sum of the middle column: \$340.84

33. Since each filter removes 85% of the remaining impurities, the rate of change of the impurity level is $r = -0.85$ per filter. Thus, the growth factor is $B = 1 + r = 1 - 0.85 = 0.15$. This means that each time the water is passed through a filter, the impurity level L is multiplied by a factor of 0.15. This makes sense, because if each filter removes 85% of the impurities, it will leave behind 15% of the impurities. We see that a formula for L is

$$L = 420(0.15)^n,$$

because after being passed through n filters, the impurity level will have been multiplied by a factor of 0.15 a total of n times.

34. **(a)** The amount of forest lost is 4.2% of 3843 million, or $0.042 \cdot 3843 \approx 161$ million hectares.

(b) The amount of world forest cover in 2000 is the amount in 1990 minus the amount lost, or $3843 - 161 = 3682$ million hectares.

(c) Let $f(t)$ represent the number of million hectares of natural forest in the world t years after 1990. Since $f(0) = 3843$ and we are assuming exponential decay, we have $f(t) = 3843b^t$ for some base b. Since 4.2% decayed over a 10-year period, we know that $1 - 0.042 = 0.958$ was the growth factor for the 10-year period. We have

$$\begin{aligned} b^{10} &= 0.958 \\ b &= (0.958)^{1/10} = 0.9957. \end{aligned}$$

The formula is $f(t) = 3843(0.9957)^t$.

(d) The annual growth factor is $0.9957 = 1 - 0.0043$ so the world forest cover is decreasing at a rate of 0.43% per year.

35. **(a)** The monthly payment on \$1000 each month at 8% for a loan period of 15 years is \$9.56. For \$60,000, the payment would be $\$9.56 \times 60 = \$573.60$ per month.

(b) The monthly payment on \$1000 each month at 8% for a loan period of 30 years is \$7.34. For \$60,000, the payment would be $\$7.34 \times 60 = \$440.40$ per month.

(c) The monthly payment on \$1000 each month at 10% for a loan period of 15 years is \$10.75. For \$60,000, the payment would be $\$10.75 \times 60 = \$645.00$ per month.

(d) As calculated in part (a), the monthly payment on a \$60,000 loan at 8% for 15 years would be \$573.60 per month. In part (c) we showed that the the monthly payment on a \$60,000 loan at 10% for 15 years would be \$645.00 per month. So taking the loan out at 8% rather that 10% would save the difference:

$$\text{Amount saved} = \$645.00 - \$573.60 = \$71.40 \text{ per month}$$

Since there are $15 \times 12 = 180$ months in 15 years,

$$\text{Total amount saved} = \$71.40 \text{ per month} \times 180 \text{ months} = \$12{,}852.$$

(e) In part (a) we found the monthly payment on an 8% mortgage of \$60,000 for 15 years to be \$573.60. The total amount paid over 15 years is then

$$\$573.60 \text{ per month} \times 180 \text{ months} = \$103,248.$$

In part (b) we found the monthly payment on an 8% mortgage of \$60,000 for 30 years to be \$440.40. The total amount paid over 30 years is then

$$440.40 \text{ per month} \times 360 \text{ months} = \$158,544.$$

The amount saved by taking the mortgage over a shorter period of time is the difference:

$$\$158,544 - \$103,248 = \$55,296.$$

36. (a) $N(0)$ gives the number of teams remaining in the tournament after no rounds have been played. Thus, $N(0) = 64$. After 1 round, half of the original 64 teams remain in the competition, so

$$N(1) = 64(\frac{1}{2}).$$

After 2 rounds, half of these teams remain, so

$$N(2) = 64(\frac{1}{2})(\frac{1}{2}).$$

And, after r rounds, the original pool of 64 teams has been halved r times, so that

$$N(r) = 64 \underbrace{(\frac{1}{2})(\frac{1}{2})\cdots(\frac{1}{2})}_{\text{pool halved } r \text{ times}},$$

giving

$$N(r) = 64(\frac{1}{2})^r.$$

The graph of $y = N(r)$ is given in Figure 3.3. The domain of N is $0 \le r \le 6$, for r an integer. A curve has been dashed in to help you see the overall shape of the function.

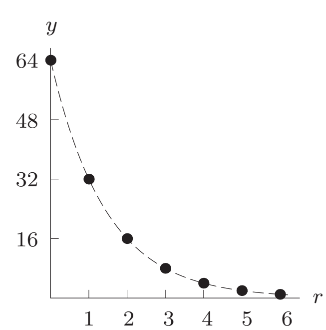


Figure 3.3: The graph of $y = N(r) = 64 \cdot \left(\frac{1}{2}\right)^r$

(b) There will be a winner when there is only one person left. So, $N(r) = 1$.

$$\begin{aligned} 64(\frac{1}{2})^r &= 1 \\ \left(\frac{1}{2}\right)^r &= \frac{1}{64} \\ \frac{1}{2^r} &= \frac{1}{64} \\ 2^r &= 64 \\ r &= 6 \end{aligned}$$

You can solve $2^r = 64$ either by taking successive powers of 2 until you get to 64 or by substituting values for r until you get the one that works.

37. Using the formula for slope, we have

$$\text{Slope} = \frac{f(5) - f(1)}{5 - 1} = \frac{4b^5 - 4b^1}{4} = \frac{4b(b^4 - 1)}{4} = b(b^4 - 1).$$

38. Each time we make a tri-fold, we triple the number of layers of paper, $N(x)$. So $N(x) = 3^x$, where x is the number of folds we make. After 20 folds, the letter would have 3^{20} (almost 3.5 billion!) layers. To find out how high our letter would be, we divide the number of layers by the number of sheets in one inch. So the height, h, is

$$h = \frac{3^{20}\text{sheets}}{150 \text{ sheets/inch}} \approx 23{,}245{,}229.34 \text{ inches.}$$

Since there are 12 inches in a foot and 5280 feet in a mile, this gives

$$\begin{aligned} h &\approx 23245229.34 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)\left(\frac{1 \text{ mile}}{5280 \text{ ft}}\right) \\ &\approx 366.875 \text{ miles.} \end{aligned}$$

39. (a) We have

$$\begin{aligned} \text{Total revenue} &= \text{No. households} \times \text{Rate per household} \\ \text{so} \qquad R &= N \times r. \end{aligned}$$

(b) We have

$$\begin{aligned} \text{Average revenue} &= \frac{\text{Total revenue}}{\text{No. students}} \\ \text{so} \qquad A &= \frac{R}{P} = \frac{Nr}{P}. \end{aligned}$$

(c) We have

$$\begin{aligned} N_{\text{new}} &= N + (2\%)N = 1.02N \\ r_{\text{new}} &= r + (3\%)r = 1.03r \end{aligned}$$

(d) We have

$$\begin{aligned} R_{\text{new}} &= N_{\text{new}} \times r_{\text{new}} = (1.02N)(1.03r) \\ &= 1.0506Nr = 1.0506R. \end{aligned}$$

Thus, R increased by 5.06%, or by just over 5%.

(e) We have

$$
\begin{aligned}
P_{\text{new}} &= P + (8\%)P = 1.08P \\
A_{\text{new}} &= \frac{R_{\text{new}}}{P_{\text{new}}} \\
&= \frac{1.0506R}{1.08P} = (0.9728)\left(\frac{R}{P}\right) \approx (97.3\%)A,
\end{aligned}
$$

so the average revenue fell by 2.7%, despite the fact that the tax rate and the tax base both grew.

40. The vertical intercept of $a_1(b_1)^t$ is greater than that of $a_0(b_0)^t$, so $a_1 > a_0$.

41. The graph $a_0(b_0)^t$ climbs faster than that of $a_1(b_1)^t$, so $b_0 > b_1$.

42. The value of t_0 goes down. To see this, notice that as a_0 increases, the vertical intercept of $a_0(b_0)^t$ goes up, so the point of intersection moves to the left. In other words, as a_0 increases, the graph of $a_0(b_0)^t$ "catches up" to the graph of $a_1(b_1)^t$ earlier. If a_0 rises as high as a_1, the value of t_0 drops to 0, because the two graphs intersect at the vertical axis. If a_0 rises higher than a_1, the value of t_0 becomes negative, because the two graphs intersect to the left of the y-axis.

43. The value of t_0 decreases. To see that, notice that if b_1 is decreased, the graph of $a_1(b_1)^t$ climbs more slowly, and eventually (if b_1 falls below 1) begins to fall. Thus the point of intersection moves to the left, so t_0 goes down. However, since the graph of $a_1(b_1)^t$ intersects the vertical axis above the graph of $a_0(b_0)^t$, t_0 remains positive no matter how small b_1 becomes, so long as $b_1 > 0$. (Recall that the base of an exponential function must be positive.)

Solutions for Section 3.2

Exercises

1. The formula $P_A = 200 + 1.3t$ for City A shows that its population is growing linearly. In year $t = 0$, the city has 200,000 people and the population grows by 1.3 thousand people, or $1{,}300$ people, each year.

The formulas for cities B, C, and D show that these populations are changing exponentially. Since $P_B = 270(1.021)^t$, City B starts with 270,000 people and grows at an annual rate of 2.1%. Similarly, City C starts with $150{,}000$ people and grows at 4.5% annually.

Since $P_D = 600(0.978)^t$, City D starts with 600,000 people, but its population decreases at a rate of 2.2% per year. We find the annual percent rate by taking $b = 0.978 = 1 + r$, which gives $r = -0.022 = -2.2\%$. So City D starts out with more people than the other three but is shrinking.

Figure 3.4 gives the graphs of the three exponential populations. Notice that the P-intercepts of the graphs correspond to the initial populations (when $t = 0$) of the towns. Although the graph of P_C starts below the graph of P_B, it eventually catches up and rises above the graph of P_B, because City C is growing faster than City B.

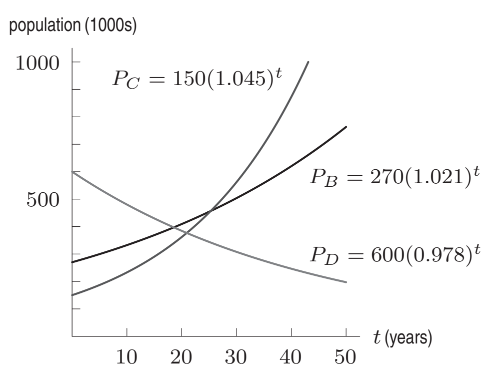


Figure 3.4: The graphs of the three exponentially changing populations

2. **(a)** The population is decreasing linearly, with a slope of -100 people/year, so $P = 5000 - 100t$
(b) The population is decreasing exponentially with "growth" factor $1 - 0.08 = 0.92$, so $P = 5000(1 - 0.08)^t = 5000(0.92)^t$

3. **(a)** $P = 100 + 10t$.
(b) $P = 100(1.10)^t$.
(c) See Figure 3.5.

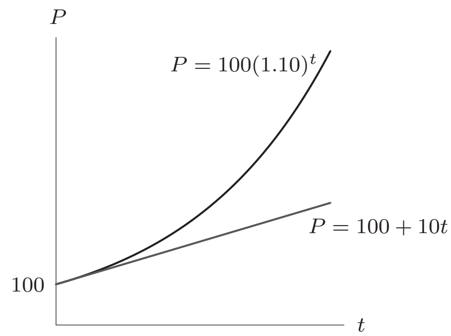


Figure 3.5

4. An exponential function should be used, because an exponential function grows by a constant percentage and a linear function grows by a constant absolute amount.

5. To use the ratio method we must have the y-values given at equally spaced x-values, which they are not. However, some of them are spaced 1 apart, namely, 1 and 2; 4 and 5; and 8 and 9. Thus, we can use these values, and consider

$$\frac{f(2)}{f(1)}, \frac{f(5)}{f(4)}, \text{ and } \frac{f(9)}{f(8)}.$$

We find

$$\frac{f(2)}{f(1)} = \frac{f(5)}{f(4)} = \frac{f(9)}{f(8)} = \frac{1}{4}.$$

With $f(x) = ab^x$ we also have

$$\frac{f(2)}{f(1)} = \frac{f(5)}{f(4)} = \frac{f(9)}{f(8)} = b,$$

so $b = \frac{1}{4}$. Using $f(1) = 4096$ we find $4096 = ab = a\left(\frac{1}{4}\right)$, so $a = 16{,}384$. Thus, $f(x) = 16{,}384\left(\frac{1}{4}\right)^x$.

6. **(a)** If a function is linear, then the differences in successive function values will be constant. If a function is exponential, the ratios of successive function values will remain constant. Now

$$f(1) - f(0) = 13.75 - 12.5 = 1.25$$

while

$$f(2) - f(1) = 15.125 - 13.75 = 1.375.$$

Thus, $f(x)$ is not linear. On the other hand,

$$\frac{f(1)}{f(0)} = \frac{13.75}{12.5} = 1.1$$

and

$$\frac{f(2)}{f(1)} = \frac{15.25}{13.75} = 1.1.$$

Checking the rest of the data, we see that the ratios of differences remains constant, so $f(x)$ is exponential.

(b) We know that f is exponential, so

$$f(x) = ab^x$$

for some constants a and b. We know that $f(0) = 12.5$, so

$$\begin{aligned} 12.5 &= f(0) \\ 12.5 &= ab^0 \\ 12.5 &= a(1). \end{aligned}$$

Thus,

$$a = 12.5.$$

We also know

$$\begin{aligned} 13.75 &= f(1) \\ 13.75 &= 12.5b. \end{aligned}$$

Thus,

$$b = \frac{13.75}{12.5} = 1.1.$$

As a result,

$$f(x) = 12.5(1.1)^x.$$

The graph of $f(x)$ is shown in Figure 3.6.

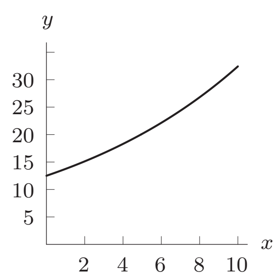


Figure 3.6

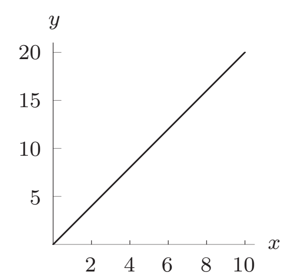


Figure 3.7

7. (a) If a function is linear, then the differences in successive function values will be constant. If a function is exponential, the ratios of successive function values will remain constant. Now

$$g(1) - g(0) = 2 - 0 = 2$$

and

$$g(2) - g(1) = 4 - 2 = 2.$$

Checking the rest of the data, we see that the differences remain constant, so $g(x)$ is linear.

(b) We know that $g(x)$ is linear, so it must be of the form

$$g(x) = b + mx$$

where m is the slope and b is the y-intercept. Since at $x = 0$, $g(0) = 0$, we know that the y-intercept is 0, so $b = 0$. Using the points $(0, 0)$ and $(1, 2)$, the slope is

$$m = \frac{2-0}{1-0} = 2.$$

Thus,

$$g(x) = 0 + 2x = 2x.$$

The graph of $y = g(x)$ is shown in Figure 3.7.

8. (a) If a function is linear, then the differences in successive function values will be constant. If a function is exponential, the ratios of successive function values will remain constant. Now

$$h(1) - h(0) = 12.6 - 14 = -1.4$$

while

$$h(2) - h(1) = 11.34 - 12.6 = -1.26.$$

Thus, $h(x)$ is not linear. On the other hand,

$$\frac{h(1)}{h(0)} = 0.9$$
$$\frac{h(2)}{h(1)} = \frac{11.34}{12.6} = 0.9.$$

Checking the rest of the data, we see that the ratio of differences remains constant, so $h(x)$ is exponential.

(b) We know that $h(x)$ is exponential, so

$$h(x) = ab^x,$$

for some constants a and b. We know that $h(0) = 14$, so

$$14 = h(0)$$
$$14 = ab^0$$
$$14 = a(1).$$

Thus, $a = 14$. Also

$$12.6 = h(1)$$
$$12.6 = 14b.$$

Thus,

$$b = \frac{12.6}{14} = 0.9.$$

So, we have $h(x) = 14(0.9)^x$. The graph of $h(x)$ is shown in Figure 3.8.

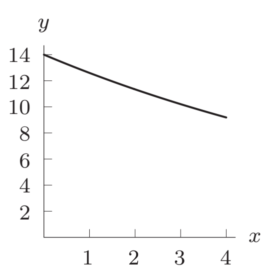


Figure 3.8

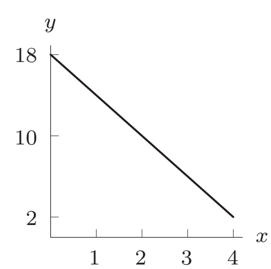


Figure 3.9

9. (a) If a function is linear, then the differences in successive function values will be constant. If a function is exponential, the ratios of successive function values will remain constant. Now

$$i(1) - i(0) = 14 - 18 = -4$$

and

$$i(2) - i(1) = 10 - 14 = -4.$$

Checking the rest of the data, we see that the differences remain constant, so $i(x)$ is linear.

(b) We know that $i(x)$ is linear, so it must be of the form

$$i(x) = b + mx,$$

where m is the slope and b is the y-intercept. Since at $x = 0$, $i(0) = 18$, we know that the y-intercept is 18, so $b = 18$. Also, we know that at $x = 1$, $i(1) = 14$, we have

$$\begin{aligned} i(1) &= b + m \cdot 1 \\ 14 &= 18 + m \\ m &= -4. \end{aligned}$$

Thus, $i(x) = 18 - 4x$. The graph of $i(x)$ is shown in Figure 3.9.

Problems

10. If a function is linear and the x-values are equally spaced, you get from one y-value to the next by adding (or subtracting) the same amount each time. On the other hand, if the function is exponential and the x-values are evenly spaced, you get from one y-value to the next by multiplying by the same factor each time.

11. Since $h(x) = ab^x$, $h(0) = ab^0 = a(1) = a$. We are given $h(0) = 3$, so $a = 3$. If $h(x) = 3b^x$, then $h(1) = 3b^1 = 3b$. But we are told that $h(1) = 15$, so $3b = 15$ and $b = 5$. Therefore $h(x) = 3(5)^x$.

12. Since $f(x) = ab^x$, $f(3) = ab^3$ and $f(-2) = ab^{-2}$. Since we know that $f(3) = -\frac{3}{8}$ and $f(-2) = -12$, we can say

$$ab^3 = -\frac{3}{8}$$

and

$$ab^{-2} = -12.$$

Forming ratios, we have

$$\begin{aligned} \frac{ab^3}{ab^{-2}} &= \frac{-\frac{3}{8}}{-12} \\ b^5 &= -\frac{3}{8} \times -\frac{1}{12} = \frac{1}{32}. \end{aligned}$$

Since $32 = 2^5$, $\frac{1}{32} = \frac{1}{2^5} = (\frac{1}{2})^5$. This tells us that

$$b = \frac{1}{2}.$$

Thus, our formula is $f(x) = a(\frac{1}{2})^x$. Use $f(3) = a(\frac{1}{2})^3$ and $f(3) = -\frac{3}{8}$ to get

$$\begin{aligned} a(\frac{1}{2})^3 &= -\frac{3}{8} \\ a(\frac{1}{8}) &= -\frac{3}{8} \\ \frac{a}{8} &= -\frac{3}{8} \\ a &= -3. \end{aligned}$$

Therefore $f(x) = -3(\dfrac{1}{2})^x$.

13. Since $g(x) = ab^x$, we can say that $g(\frac{1}{2}) = ab^{1/2}$ and $g(\frac{1}{4}) = ab^{1/4}$. Since we know that $g(\frac{1}{2}) = 4$ and $g(\frac{1}{4}) = 2\sqrt{2}$, we can conclude that

$$ab^{1/2} = 4 = 2^2$$

and

$$ab^{1/4} = 2\sqrt{2} = 2 \cdot 2^{1/2} = 2^{3/2}.$$

Forming ratios, we have

$$\begin{aligned}\frac{ab^{1/2}}{ab^{1/4}} &= \frac{2^2}{2^{3/2}}\\ b^{1/4} &= 2^{1/2}\\ (b^{1/4})^4 &= (2^{1/2})^4\\ b &= 2^2 = 4.\end{aligned}$$

Now we know that $g(x) = a(4)^x$, so $g(\frac{1}{2}) = a(4)^{1/2} = 2a$. Since we also know that $g(\frac{1}{2}) = 4$, we can say

$$\begin{aligned}2a &= 4\\ a &= 2.\end{aligned}$$

Therefore $g(x) = 2(4)^x$.

14. Since $g(x) = ab^x$ and $g(0) = 5$, we have $a = 5$, so

$$g(x) = 5b^x.$$

Now $g(-2) = 10$ means that

$$5b^{-2} = 10.$$

Solving for b gives

$$\begin{aligned}\frac{5}{b^2} &= 10\\ \frac{1}{2} &= b^2\\ b &= \frac{1}{\sqrt{2}} = 0.707.\end{aligned}$$

So $g(x) = 5(0.707)^x$.

15. If g is exponential, then $g(x) = ab^x$, so

$$g(1.7) = ab^{1.7} = 6$$

and

$$g(2.5) = ab^{2.5} = 4.$$

We use ratios to see

$$\begin{aligned}\frac{ab^{2.5}}{ab^{1.7}} &= \frac{4}{6} = \frac{g(2.5)}{g(1.7)}\\ b^{0.8} &= \frac{4}{6} = \frac{2}{3}\\ b &= (\frac{2}{3})^{\frac{1}{0.8}} = 0.6024.\end{aligned}$$

Thus, our formula becomes

$$g(x) = a(0.6024)^x.$$

We can use one of our data points to solve for a. For example,

$$\begin{aligned}g(1.7) = a(0.6024)^{1.7} &= 6\\ a &= \frac{6}{0.6024^{1.7}}\\ &\approx 14.20.\end{aligned}$$

Thus, $g(x) = 14.20(0.6024)^x$.

16. We use the exponential formula $f(x) = ab^x$. Since $f(1) = 4$ and $f(3) = d$, we have

$$ab^1 = 4 \quad \text{and} \quad ab^3 = d.$$

Dividing these two equations, we have

$$\frac{ab^3}{ab^1} = \frac{d}{4}.$$

Now we cancel and solve for b in terms of d.

$$\begin{aligned} b^2 &= \frac{d}{4} \\ b &= \frac{d^{0.5}}{2}. \end{aligned}$$

To find a in terms of d, we use the fact that $f(1) = 4$:

$$ab^1 = 4.$$

Substituting for b gives

$$\begin{aligned} a\left(\frac{d^{0.5}}{2}\right) &= 4 \\ a &= \frac{8}{d^{0.5}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} f(x) &= ab^x \\ &= \left(\frac{8}{d^{0.5}}\right)\left(\frac{d^{0.5}}{2}\right)^x \\ &= \frac{8}{2^x} \cdot \frac{d^{0.5x}}{d^{0.5}} \\ &= 2^{3-x} \cdot d^{0.5(x-1)}. \end{aligned}$$

17. **(a)** If f is linear, then $f(x) = b + mx$, where m, the slope, is given by:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(2) - f(-3)}{(2) - (-3)} = \frac{20 - \frac{5}{8}}{5} = \frac{\frac{155}{8}}{5} = \frac{31}{8}.$$

Using the fact that $f(2) = 20$, and substituting the known values for m, we write

$$\begin{aligned} 20 &= b + m(2) \\ 20 &= b + \left(\frac{31}{8}\right)(2) \\ 20 &= b + \frac{31}{4} \end{aligned}$$

which gives

$$b = 20 - \frac{31}{4} = \frac{49}{4}.$$

So, $f(x) = \dfrac{31}{8}x + \dfrac{49}{4}$.

(b) If f is exponential, then $f(x) = ab^x$. We know that $f(2) = ab^2$ and $f(2) = 20$. We also know that $f(-3) = ab^{-3}$ and $f(-3) = \frac{5}{8}$. So

$$\frac{f(2)}{f(-3)} = \frac{ab^2}{ab^{-3}} = \frac{20}{\frac{5}{8}}$$

$$b^5 = 20 \times \frac{8}{5} = 32$$
$$b = 2.$$

Thus, $f(x) = a(2)^x$. Solve for a by using $f(2) = 20$ and (with $b = 2$), $f(2) = a(2)^2$.

$$20 = a(2)^2$$
$$20 = 4a$$
$$a = 5.$$

Thus, $f(x) = 5(2)^x$.

18. If the function is exponential, its formula is of the form $y = ab^x$. Since $(0, 1)$ is on the graph

$$y = ab^x$$
$$1 = ab^0$$

Since $b^0 = 1$,

$$1 = a(1)$$
$$a = 1.$$

Since $(2, 100)$ is on the graph and $a = 1$,

$$y = ab^x$$
$$100 = (1)b^2$$
$$b^2 = 100$$
$$b = 10 \text{ or } b = -10$$

$b = -10$ is excluded, since b must be greater than zero. Therefore, $y = 1(10)^x$ or $y = 10^x$ is a possible formula for this function.

19. The formula for an exponential function is of the form $y = ab^x$. Since $(0, 1)$ is on the graph,

$$y = ab^x$$
$$1 = ab^0.$$

Since $b^0 = 1$,

$$1 = a(1)$$
$$a = 1.$$

Since $(4, 1/16)$ is on the graph and $a = 1$,

$$y = ab^x$$
$$\frac{1}{16} = 1(b)^4$$
$$b^4 = \frac{1}{16}.$$

Since $2 \cdot 2 \cdot 2 \cdot 2 = 16$, we know that $(1/2) \cdot (1/2) \cdot (1/2) \cdot (1/2) = 1/16$, so

$$b = \frac{1}{2}.$$

(Although $b = -1/2$ is also a solution, it is rejected since b must be greater than zero.) Therefore $y = (1/2)^x$ is a possible formula for this function.

20. Since the function is exponential, we know that $y = ab^x$. Since $(0, 1.2)$ is on the graph, we know $1.2 = ab^0$, and that $a = 1.2$. To find b, we use point $(2, 4.8)$ which gives

$$\begin{aligned} 4.8 &= 1.2(b)^2 \\ 4 &= b^2 \\ b &= 2, \text{ since } b > 0. \end{aligned}$$

Thus, $y = 1.2(2)^x$ is a possible formula for this function.

21. The formula is of the form $y = ab^x$. Since the points $(-1, 1/15)$ and $(2, 9/5)$ are on the graph, so

$$\begin{aligned} \frac{1}{15} &= ab^{-1} \\ \frac{9}{5} &= ab^2. \end{aligned}$$

Taking the ratio of the second equation to the first we obtain

$$\begin{aligned} \frac{9/5}{1/15} &= \frac{ab^2}{ab^{-1}} \\ 27 &= b^3 \\ b &= 3. \end{aligned}$$

Substituting this value of b into $\frac{1}{15} = ab^{-1}$ gives

$$\begin{aligned} \frac{1}{15} &= a(3)^{-1} \\ \frac{1}{15} &= \frac{1}{3}a \\ a &= \frac{1}{15} \cdot 3 \\ a &= \frac{1}{5}. \end{aligned}$$

Therefore $y = \frac{1}{5}(3)^x$ is a possible formula for this function.

22. Since the function is exponential, we know that $y = ab^x$. The points $(-2, 45/4)$ and $(1, 10/3)$ are on the graph so,

$$\begin{aligned} \frac{45}{4} &= ab^{-2} \\ \frac{10}{3} &= ab^1 \end{aligned}$$

Taking the ratio of the second equation to the first one we have

$$\frac{10/3}{45/4} = \frac{ab^1}{ab^{-2}}.$$

Since $\frac{10}{3}/\frac{45}{4} = \frac{10}{3} \cdot \frac{4}{45} = \frac{8}{27}$,

$$\frac{8}{27} = b^3.$$

Since $8 = 2^3$ and $27 = 3^3$, we know that $\frac{8}{27} = \frac{2^3}{3^3} = (\frac{2}{3})^3$, so

$$\begin{aligned} (\frac{2}{3})^3 &= b^3 \\ b &= \frac{2}{3}. \end{aligned}$$

Substituting this value of b into the second equation gives

$$\begin{aligned}\frac{10}{3} &= a(\frac{2}{3})^1\\ \frac{2}{3}a &= \frac{10}{3}\\ a &= 5.\end{aligned}$$

Thus, $y = 5\left(\frac{2}{3}\right)^x$.

23. Since the function is exponential, we know that $y = ab^x$. Since the points $(-1, 2.5)$ and $(1, 1.6)$ are on the graph, we know that

$$\begin{aligned}2.5 &= a(b^{-1})\\ 1.6 &= a(b^1)\end{aligned}$$

Dividing the second equation by the first and canceling a gives

$$\begin{aligned}\frac{1.6}{2.5} &= \frac{a\cdot b^1}{a\cdot b^{-1}}\\ \frac{1.6}{2.5} &= b^{1-(-1)} = b^2.\end{aligned}$$

Solving for b and using the fact that $b > 0$ gives

$$b = \sqrt{\frac{1.6}{2.5}} = 0.8.$$

Substituting $b = 0.8$ in the equation $1.6 = a(b^1) = a(0.8)$ and solving for a gives

$$a = \frac{1.6}{0.8} = 2.$$

Thus, $y = 2(0.8)^x$ is a possible formula for this function.

24. Assuming f is linear, we have $f(t) = b + mt$ where $f(5) = 22$ and $f(25) = 6$. This gives

$$m = \frac{f(25) - f(5)}{25 - 5} = \frac{6 - 22}{20} = -0.8.$$

Solving for b, we have

$$\begin{aligned}f(5) &= b - (0.8)5\\ b &= f(5) + (0.8)5 = 22 + (0.8)5 = 26,\end{aligned}$$

so $f(t) = 26 - 0.8t$.

Assuming g is exponential, we have $g(t) = ab^t$, where $g(5) = 22$ and $g(25) = 6$. Using the ratio method, we have

$$\begin{aligned}\frac{ab^{25}}{ab^5} &= \frac{g(25)}{g(5)}\\ b^{20} &= \frac{6}{22}\\ b &= \left(\frac{6}{22}\right)^{1/20} = 0.9371.\end{aligned}$$

Now solve for a:

$$\begin{aligned}a(0.9371)^5 &= 22\\ a &= \frac{22}{(0.9371)^5} = 30.443.\end{aligned}$$

so $g(t) = 30.443(0.9371)^t$.

25. We have $p = f(x) = ab^x$. From the figure, we see that the starting value is $a = 20$ and that the graph contains the point $(10, 40)$. We have

$$\begin{aligned} f(10) &= 40 \\ 20b^{10} &= 40 \\ b^{10} &= 2 \\ b &= 2^{1/10} \\ &= 1.0718, \end{aligned}$$

so $p = 20(1.0718)^x$.

We have $q = g(x) = ab^x$, $g(10) = 40$, and $g(15) = 20$. Using the ratio method, we have

$$\begin{aligned} \frac{ab^{15}}{ab^{10}} &= \frac{g(15)}{g(10)} \\ b^5 &= \frac{20}{40} \\ b &= \left(\frac{20}{40}\right)^{1/5} \\ &= (0.5)^{1/5} = 0.8706. \end{aligned}$$

Now we can solve for a:

$$\begin{aligned} a((0.5)^{1/5})^{10} &= 40 \\ a &= \frac{40}{(0.5)^2} \\ &= 160. \end{aligned}$$

so $q = 160(0.8706)^x$.

26. The average rate of change of this function appears to be constant, and thus it could be linear. Taking any pair of data points, $\Delta p/\Delta r = 3$. So the slope of this linear function should be 3. Using the form $p(r) = b + mr$, we solve for b, substituting in the point $(1, 13)$ (any point will work):

$$\begin{aligned} p(r) &= b + 3r \\ 13 &= b + 3 \cdot 1 \\ b &= 10. \end{aligned}$$

Therefore, $p(r) = 10 + 3r$.

27. The table could represent an exponential function. For every change in x of 3, there is a 10% increase in $q(x)$. We note that there is a 21% increase in $q(x)$ when $\Delta x = 6$, which is the same ($1.1^2 = 1.21$). Using the ratio method, we find b in the form $q(x) = ab^x$.

$$\frac{q(9)}{q(6)} = \frac{110}{100} = 1.1$$

and

$$\frac{q(9)}{q(6)} = \frac{ab^9}{ab^6} = b^3.$$

Thus, $b^3 = 1.1$, and $b = \sqrt[3]{1.1} \approx 1.03228$. We solve for a by substituting $x = 6$ and $q(x) = 100$ into the equation $q(x) = a(1.03228)^x$:

$$\begin{aligned} a \cdot 1.03228^6 &= 100 \\ a &= \frac{100}{1.03228^6} \approx 82.6446. \end{aligned}$$

Thus, $q(x) = 82.6446 \cdot 1.03228^x$.

28. This cannot be linear, since $\Delta f(x)/\Delta x$ is not constant, nor can it be exponential, since between $x = 15$ and $x = 12$, we see that $f(x)$ doubles while $\Delta x = 3$. Between $x = 15$ and $x = 16$, we see that $f(x)$ doubles while $\Delta x = 1$, so the percentage increase is not constant. Thus, the function is neither.

29. This table could represent an exponential function, since for every Δt of 1, the value of $g(t)$ halves. This means that b in the form $g(t) = ab^t$ must be $\frac{1}{2} = 0.5$. We can solve for a by substituting in $(1, 512)$ (or any other point):

$$\begin{aligned} 512 &= a \cdot 0.5^1 \\ 512 &= a \cdot 0.5 \\ 1024 &= a. \end{aligned}$$

Thus, a possible formula to describe the data in the table is $g(t) = 1024 \cdot 0.5^t$.

30. One approach is to graph both functions and to see where the graph of $p(x)$ is below the graph of $q(x)$. From Figure 3.10, we see that $p(x)$ intersects $q(x)$ in two places; namely, at $x \approx -1.69$ and $x = 2$. We notice that $p(x)$ is above $q(x)$ between these two points and below $q(x)$ outside the segment defined by these two points. Hence $p(x) < q(x)$ for $x < -1.69$ and for $x > 2$.

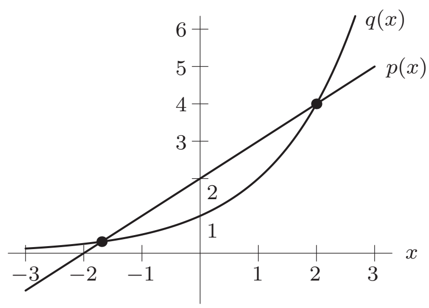


Figure 3.10

31. **(a)** If P is linear, then $P(t) = b + mt$ and

$$m = \frac{\Delta P}{\Delta t} = \frac{P(13) - P(7)}{13 - 7} = \frac{3.75 - 3.21}{13 - 7} = \frac{0.54}{6} = 0.09.$$

So $P(t) = b + 0.09t$ and $P(7) = b + 0.09(7)$. We can use this and the fact that $P(7) = 3.21$ to say that

$$\begin{aligned} 3.21 &= b + 0.09(7) \\ 3.21 &= b + 0.63 \\ 2.58 &= b. \end{aligned}$$

So $P(t) = 2.58 + 0.09t$. The slope is 0.09 million people per year. This tells us that, if its growth is linear, the country grows by $0.09(1{,}000{,}000) = 90{,}000$ people every year.

(b) If P is exponential, $P(t) = ab^t$. So

$$P(7) = ab^7 = 3.21$$

and

$$P(13) = ab^{13} = 3.75.$$

We can say that

$$\begin{aligned} \frac{P(13)}{P(7)} &= \frac{ab^{13}}{ab^7} = \frac{3.75}{3.21} \\ b^6 &= \frac{3.75}{3.21} \\ (b^6)^{1/6} &= \left(\frac{3.75}{3.21}\right)^{1/6} \\ b &= 1.026. \end{aligned}$$

Thus, $P(t) = a(1.026)^t$. To find a, note that

$$P(7) = a(1.026)^7 = 3.21$$
$$a = \frac{3.21}{(1.026)^7} = 2.68.$$

We have $P(t) = 2.68(1.026)^t$. Since $b = 1.026$ is the growth factor, the country's population grows by about 2.6% per year, assuming exponential growth.

32. (a) For the population to decrease linearly, it must change by the same amount over each one year period. If P_0 represents the original population, then it will be reduced by $10\% P_0$ each year. So after one year the remaining population will be $P = P_0 - 0.1P_0 = 0.9P_0$. After two years the remaining population will be $P = 0.9P_0 - 0.1P_0 = 0.8P_0$. At the end of ten years there will be no population remaining. See Figure 3.11.

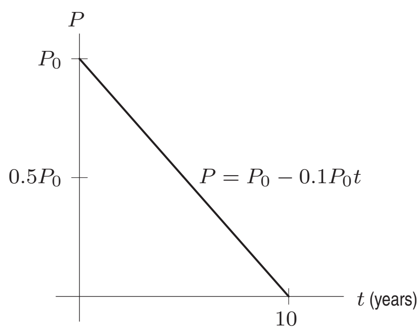


Figure 3.11

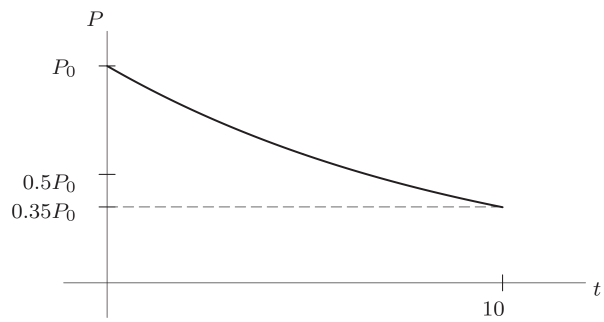


Figure 3.12

(b) In this case the population also decreases each year but not by the same amount. After one year the population remaining is the same as in the linear case:

$$P = P_0 - 0.1P_0 = 0.9P_0.$$

However, after two years, the population remaining is

$$P = 0.9P_0 - 0.1(0.9P_0) = 0.9P_0(1 - 0.1) = P_0(0.9)^2.$$

In general, after t years the remaining population will be given by

$$P = P_0(0.9)^t$$

and after 10 years there will be a population remaining of

$$P = P_0(0.9)^{10} = 0.35P_0 = 35\% P_0.$$

In other words, an exponential decrease of 10% a year will leave 35% of the original population after 10 years. See Figure 3.12.

33. (a) Since this function is exponential, its formula is of the form $f(t) = ab^t$, so

$$f(3) = ab^3$$
$$f(8) = ab^8.$$

From the graph, we know that

$$f(3) = 2000$$
$$f(8) = 5000.$$

So

$$\begin{aligned}
\frac{f(8)}{f(3)} &= \frac{ab^8}{ab^3} = \frac{5000}{2000} \\
b^5 &= \frac{5}{2} = 2.5 \\
(b^5)^{1/5} &= (2.5)^{1/5} \\
b &= 1.20112.
\end{aligned}$$

We now know that $f(t) = a(1.20112)^t$. Using either of the pairs of values on the graph, we can find a. In this case, we use $f(3) = 2000$. According to the formula,

$$\begin{aligned}
f(3) &= a(1.20112)^3 \\
2000 &= a(1.20112)^3 \\
a &= \frac{2000}{(1.20112)^3} \approx 1154.160.
\end{aligned}$$

The formula we want is $f(t) = 1154.160(1.20112)^t$ or $P = 1154.160(1.20112)^t$.

(b) The initial value of the account occurs when $t = 0$.

$$f(0) = 1154.160(1.20112)^0 = 1154.160(1) = \$1154.16.$$

(c) The value of b, the growth factor, is related to the growth rate, r, by

$$b = 1 + r.$$

We know that $b = 1.20112$, so

$$\begin{aligned}
1.20112 &= 1 + r \\
0.20112 &= r
\end{aligned}$$

Thus, in percentage terms, the annual interest rate is 20.112%.

34. (a) We want $N = f(t)$ so we have

$$\text{Slope } = \frac{\Delta N}{\Delta t} = \frac{130 - 84}{2001 - 1990} = \frac{46}{11} = 4.182.$$

Since $N = 84$ when $t = 0$, the vertical intercept is 84 and the linear formula is

$$N = 84 + 4.182t.$$

The slope is 4.182. The number of asthma sufferers has increased, on average, by 4.182 million people per year during this period.

(b) Since $N = 84$ when $t = 0$, we have the exponential function $N = 84b^t$ for some base b. Since $N = 130$ when $t = 11$, we have

$$\begin{aligned}
N &= 84b^t \\
130 &= 84b^{11} \\
b^{11} &= \frac{130}{84} = 1.5476 \\
b &= (1.5476)^{1/11} = 1.0405.
\end{aligned}$$

The exponential formula is

$$N = 84(1.0405)^t.$$

The growth factor is 1.0405. The number of asthma sufferers has increased, on average, by 4.05% per year during this period.

(c) In the year 2010, we have $t = 20$. Using the linear formula, the predicted number in 2010 is

$$N = 84 + 4.182(20) = 167.640 \text{ million asthma sufferers.}$$

Using the exponential formula, the predicted number in 2010 is

$$N = 84(1.0405)^{20} = 185.832 \text{ million asthma sufferers.}$$

35. **(a)** To see if an exponential function fits the data well, we can look at ratios of successive terms. Giving each ratio to two decimal places, we have

$$\frac{138}{91} = 1.52, \quad \frac{210}{138} = 1.52, \quad \frac{320}{210} = 1.52, \quad \frac{485}{320} = 1.52, \quad \frac{738}{485} = 1.52.$$

Since the ratios are all the same, an exponential function fits these data well and the growth factor (or base) is the common ratio 1.52. Since $S = 91$ when $t = 0$, an exponential function to model these data is $S = 91(1.52)^t$.

(b) The number of cell phone subscribers worldwide was growing at a rate of 52% per year during this period.

(c) Using the model $S = 91(1.52)^t$ with $t = 9$ for 2004, we find $S \approx 3941$. The model does not fit the 2004 data; growth has slowed.

36. We use an exponential function of the form $P = ab^t$. Since $P = 1046$ when $t = 0$, we use $P = 1046b^t$ for some base b. Since $P = 338$ when $t = 5$, we have

$$\begin{aligned} P &= 1046b^t \\ 338 &= 1046b^5 \\ b^5 &= \frac{338}{1046} = 0.3231 \\ b &= (0.3231)^{1/5} = 0.798. \end{aligned}$$

An exponential formula for global production of CFCs as a function of t, the number of years since 1989 is

$$P = 1046(0.798)^t.$$

Since $0.798 = 1 - 0.202$, CFC production was decreasing at a rate of 20.2% per year during this time period.

37. Since $N = 10$ when $t = 0$, we use $N = 10b^t$ for some base b. Since $N = 20000$ when $t = 62$, we have

$$\begin{aligned} N &= 10b^t \\ 20000 &= 10b^{62} \\ b^{62} &= \frac{20000}{10} = 2000 \\ b &= (2000)^{1/62} = 1.13. \end{aligned}$$

An exponential formula for the brown tree snake population is

$$N = 10(1.13)^t.$$

The population has been growing by about 13% per year.

38. **(a)** (v) In $k(x) = A(2)^{-x} = A(1/2)^x$, A would be the initial value (0.35). The $(2)^{-x} = 1/2^x$ term tells us that the function is decreasing by half each year.

(b) (iii) In $h(x) = B(0.7)^x$, B is the initial charge. The $(0.7)^x$ term tells us that at the end of each second, the amount of charge is 70% of what it had been the previous second. Therefore, it has decreased by 30%.

(c) (iv) In $j(x) = B(0.3)^x$, B is the initial level of pollutants. The $(0.3)^x = (0.30)^x$ term tells us that 30% of the pollutants remain after each filter.

(d) (ii) In $g(x) = P_0(1 + r)^x$, $P_0 = 3000$ represents the initial population. The $(1 + r)^x$ term represents the growth factor, with $r = 0.10$, a 10% increase, and $0 \leq x \leq 5$ since there are 5 decades between 1950 and 2000.

(e) (i) In $f(x) = P_0 + rx$, $P_0 = 3000$ is the initial population, $r = 250$ is the number by which the town grew every year and $0 \leq x \leq 50$.

39. (a) Take 2006 to be the time $t = 0$ where t is measured in years. If V is the value, and V and t are related exponentially, then

$$V = ab^t.$$

If $V = 61{,}055$ at time $t = 0$, then $a = 61{,}055$, so

$$V = (61{,}055)b^t.$$

We find b by calculating another point that would be on the graph of V. If the car depreciates 46% during its first 7 years, then its value when $t = 7$ is 54% of the initial price. This is $(0.54)(\$61{,}055) = \$32{,}969.70$. So we have the data point $(7, 21262.5)$. To find b:

$$\begin{aligned}32{,}969.7 &= (61{,}055)b^7\\ 0.54 &= b^7\\ b &= (0.54)^{1/7} \approx 0.916.\end{aligned}$$

So the exponential formula relating price and time is:

$$V = (61{,}055)(0.916)^t.$$

(b) If the depreciation is linear, then the value of the car at time t is

$$V = b + mt$$

where b is the value at time $t = 0$ (the year 2006). So $b = 61{,}055$. We already calculated the value of the car after 7 years to be $(0.54)(\$61{,}055) = \$32{,}969.70$. Since $V = 32{,}969.7$ when $t = 7$, and $b = 61{,}055$, we have

$$\begin{aligned}32{,}969.70 &= 61{,}055 + 7m,\\ -28{,}085.3 &= 7m\\ -4012.19 &= m.\end{aligned}$$

So $V = 61{,}055 - 4012.19t$.

(c) Using the exponential model, the value of the car after 4 years would be:

$$V = (61{,}055)(0.916)^4 \approx \$42{,}983.63.$$

Using the linear model, the value would be:

$$V = 61{,}055 - (4012.19)(4) = \$45{,}006.24.$$

So the linear model would result in a higher resale price and would therefore be preferable.

40. (a) Since the human population is growing by a certain percent each year, it can be described by the formula $P = ab^t$. If t is the number of years since 1953, then a represents the population in 1953. If the growth rate is 8%, then each year the population is multiplied by the growth factor 1.08, so $b = 1.08$. Thus,

$$P = a(1.08)^t.$$

We know that in 1993 ($t = 40$) the population was 13 million, so

$$\begin{aligned}13{,}000{,}000 &= a(1.08)^{40}\\ a &= \frac{13{,}000{,}000}{1.08^{40}} \approx 598{,}402.133.\end{aligned}$$

Therefore in 1953, the population of humans in Florida was about 600,000 people.

(b) In 1953 ($t = 0$), the bear population was 11,000, so $a = 11,000$. The population has been decreasing at a rate of 6% a year, so the growth rate is $100\% - 6\% = 94\%$ or 0.94. Thus, the growth function for black bears is

$$P = (11{,}000)(0.94)^t.$$

In 1993, $t = 40$, so

$$P = (11{,}000)(0.94)^{40} \approx 926.$$

(c) To find the year t when the bear population would be 100, we set P equal to 100 in the equation found in part (b) and get an equation involving t:

$$\begin{aligned} P &= (11{,}000)(0.94)^t \\ 100 &= (11000)(0.94)^t \\ \frac{100}{11000} &= (0.94)^t \\ 0.00909 &\approx (0.94)^t. \end{aligned}$$

By looking at the intersection of the graphs $P = 0.00909$ and $P = (0.94)^t$, or by trial and error, we find that $t \approx 75.967$ years. Our model predicts that in 76 years from 1953, which is the year 2029, the population of black bears would fall below 100.

41. (a) Under Penalty A, the total fine is \$1 million for August 2 and \$10 million for each day after August 2 . By August 31, the fine had been increasing for 29 days so the total fine would be $1 + 10(29) = \$291$ million.

Under the Penalty B, the penalty on August 2 is 1 cent. On August 3, it is $1(2)$ cents; on August 4, it is $1(2)(2)$ cents; on August 5, it is $1(2)(2)(2)$ cents. By August 31, the fine has doubled 29 times, so the total fine is $(1) \cdot (2)^{29}$ cents, which is $536{,}870{,}912$ cents or $\$5{,}368{,}709.12$ or, approximately, $\$5.37$ million.

(b) If t represents the number of days after August 2, then the total fine under Penalty A would be $\$1$ million plus the number of days after August 2 times $\$10$ million, or $A(t) = 1 + 10 \cdot t$ million dollars $= (1 + 10t)10^6$ dollars. The total fine under Penalty B would be 1 cent doubled each day after August 2, so $B(t) = 1\,\underbrace{(2)(2)(2)\ldots(2)}_{t \text{ times}}$ cents or $B(t) = 1 \cdot (2)^t$ cents, or $B(t) = (0.01)2^t$ dollars.

(c) We plot $A(t) = (1 + 10t)10^6$ and $B(t) = (0.01)2^t$ on the same set of axes and observe that they intersect at $t \approx 35.032$ days. Another possible approach is to find values of $A(t)$ and $B(t)$ for different values of t, narrowing in on the value for which they are most nearly equal.

Solutions for Section 3.3

Exercises

1. (a) See Table 3.3.

Table 3.3

x	-3	-2	-1	0	1	2	3
$f(x)$	1/8	1/4	1/2	1	2	4	8

(b) For large negative values of x, $f(x)$ is close to the x-axis. But for large positive values of x, $f(x)$ climbs rapidly away from the x-axis. As x gets larger, y grows more and more rapidly. See Figure 3.13.

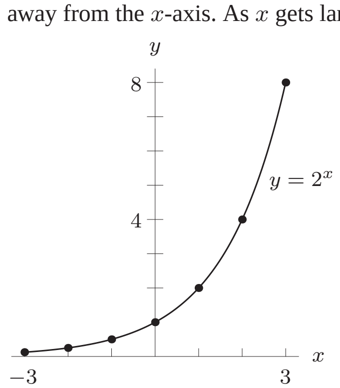


Figure 3.13

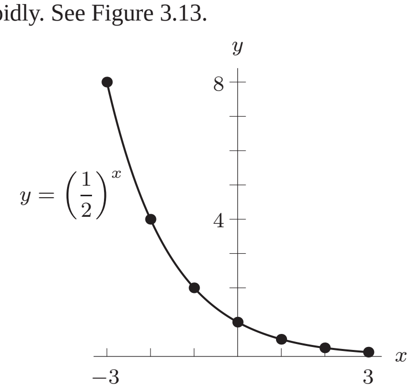


Figure 3.14

2. (a) See Table 3.4.

Table 3.4

x	-3	-2	-1	0	1	2	3
$f(x)$	8	4	2	1	1/2	1/4	1/8

(b) For large positive values of x, $f(x)$ is close to the x-axis. But for large negative values of x, $f(x)$ climbs rapidly away from the x-axis. See Figure 3.14.

3. Let $f(x) = (1.1)^x$, $g(x) = (1.2)^x$, and $h(x) = (1.25)^x$. We note that for $x = 0$,

$$f(x) = g(x) = h(x) = 1;$$

so all three graphs have the same y-intercept. On the other hand, for $x = 1$,

$$f(1) = 1.1, \quad g(1) = 1.2, \quad \text{and} \quad h(1) = 1.25,$$

so $0 < f(1) < g(1) < h(1)$. For $x = 2$,

$$f(2) = 1.21, \quad g(2) = 1.44, \quad \text{and} \quad h(2) = 1.5625,$$

so $0 < f(2) < g(2) < h(2)$. In general, for $x > 0$,

$$0 < f(x) < g(x) < h(x).$$

This suggests that the graph of $f(x)$ lies below the graph of $g(x)$, which in turn lies below the graph of $h(x)$, and that all lie above the x-axis. Alternately, you can consider 1.1, 1.2, and 1.25 as growth factors to conclude $h(x) = (1.25)^x$ is the top function, and $g(x) = (1.2)^x$ is in the middle, $f(x)$ is at the bottom.

4. Let $f(x) = (0.7)^x$, $g(x) = (0.8)^x$, and $h(x) = (0.85)^x$. We note that for $x = 0$,

$$f(x) = g(x) = h(x) = 1.$$

On the other hand, $f(1) = 0.7$, $g(1) = 0.8$, and $h(1) = 0.85$, while $f(2) = 0.49$, $g(2) = 0.64$, and $h(2) = 0.7225$; so

$$0 < f(x) < g(x) < h(x).$$

So the graph of $f(x)$ lies below the graph of $g(x)$, which in turn lies below the graph of $h(x)$.

Alternately, you can consider 0.7, 0.8, and 0.85 as growth factors (decaying). The $f(x) = (0.7)^x$ will be the lowest graph because it is decaying the fastest. The $h(x) = (0.85)^x$ will be the top graph because it decays the least.

5. Since $y = a$ when $t = 0$ in $y = ab^t$, a is the y-intercept. Thus, the function with the greatest y-intercept, D, has the largest a.

6. Since $y = a$ when $t = 0$ in $y = ab^t$, a is the y-intercept. Thus, the two functions with the same y-intercept, A and B, have the same a.

7. The function with the smallest b should be the one that is decreasing the fastest. We note that D approaches zero faster than the others, so D has the smallest b.

8. The function with the largest b should be the one that is increasing the fastest. We note that A increases faster than the others, so A has the largest b.

9. Graphing $y = 46(1.1)^x$ and tracing along the graph on a calculator gives us an answer of $x = 7.158$. See Figure 3.15.

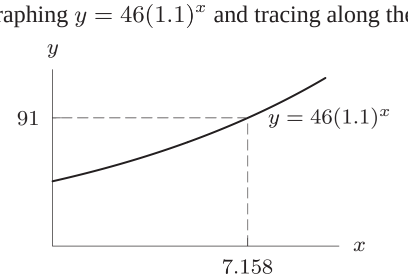


Figure 3.15

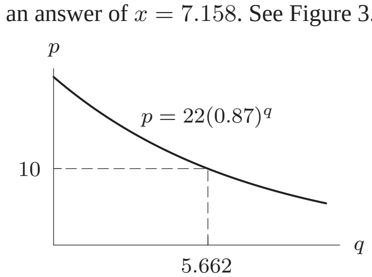


Figure 3.16

10. Graphing $p = 22(0.87)^q$ and tracing along the graph on a calculator gives us an answer of $q = 5.662$. See Figure 3.16.

11. We solve for m to see $m = 4.25(2.3)^w$. Graphing $m = 4.25(2.3)^w$ and tracing along the graph on a calculator gives us an answer of $w = 1.246$. See Figure 3.17.

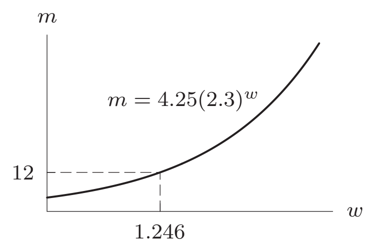


Figure 3.17

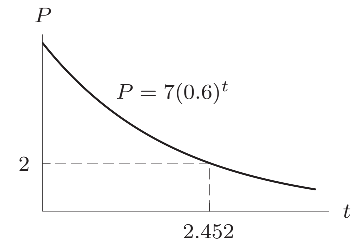


Figure 3.18

12. Solve for P to obtain $P = 7(0.6)^t$. Graphing $P = 7(0.6)^t$ and tracing along the graph on a calculator gives us an answer of $t = 2.452$. See Figure 3.18.

13. As t approaches $-\infty$, the value of ab^t approaches zero for any a, so the horizontal asymptote is $y = 0$ (the x-axis).

14. As t approaches ∞, the value of ab^t approaches zero for any a, so the horizontal asymptote is $y = 0$ (the x-axis).

Problems

15. **(a)** The growth factor is $1 - 0.0075 = 0.9925$ and the initial value is 651, so we have

$$P = 651(0.9925)^t.$$

(b) Using $t = 10$, we have $P = 651(0.9925)^{10} = 603.790$. If the current trend continues, the population of Baltimore is predicted to be 603,790 in the year 2010.

(c) See Figure 3.19. We see that $t = 22.39$ when $P = 550$. The population is expected to be 550 thousand in approximately the year 2023.

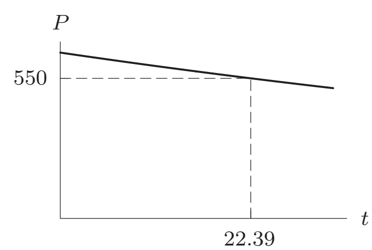


Figure 3.19

16. **(a)** $f(0) = 1000(1.04)^0 = 1000$, which means there are 1000 people in year 0.
$f(10) = 1000(1.04)^{10} \approx 1480.244$, which means there are 1480.244 people in year 10.

(b) For the first 10 years, use $0 \le t \le 10$, $0 \le P \le 1500$. See Figure 3.20. For the first 50 years, use $0 \le t \le 50$, $0 \le P \le 8000$. See Figure 3.21.

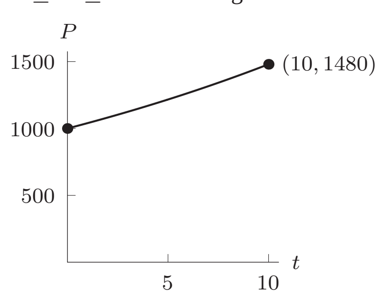


Figure 3.20

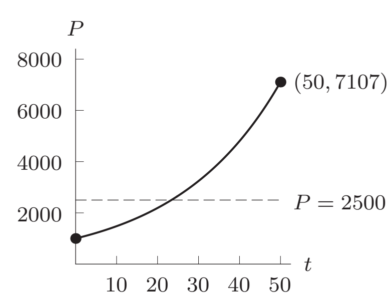


Figure 3.21

(c) The graph of $P = f(t)$ and $P = 2500$ intersect at $t \approx 23.362$. Thus, about 23.362 years after $t = 0$, the population will be 2500.

17. (a) Since the number of cases is reduced by 10% each year, there are 90% as many cases in one year as in the previous one. So, after one year there are 90% of 10000 or $10000(0.90)$ cases, while after two years, there are $10000(0.90)(0.90) = 10000(0.90)^2$ cases. In general, the number of cases after t years is $y = (10000)(0.9)^t$.

(b) Setting $t = 5$, we obtain the number of cases 5 years from now

$$y = (10000) \cdot (0.9)^5 = 5904.9 \approx 5905 \text{ cases.}$$

(c) Plotting $y = (10000) \cdot (0.9)^t$ and approximating the value of t for which $y = 1000$, we obtain $t \approx 21.854$ years.

18. (a) For each kilometer above sea level, the atmospheric pressure is $86\%(= 100\% - 14\%)$ of the pressure one kilometer lower. If P represents the number of millibars of pressure and h represents the number of kilometers above sea level. Table 3.5 leads to the formula $P = 1013(0.86)^h$. So, at 50 km, $P = 1013(0.86)^{50} \approx 0.538$ millibars.

Table 3.5

h	P
0	1013
1	$1013(0.86) = 871.18$
2	$871.18(0.86) = 1013(0.86)(0.86) = 1013(0.86)^2$
3	$1013(0.86)^2 \cdot (0.86) = 1013(0.86)^3$
4	$1013(0.86)^4$
$\dots$	$\dots$
h	$1013(0.86)^h$

(b) If we graph the function $P = 1013(0.86)^h$, we can find the value of h for which $P = 900$. One approach is to see where it intersects the line $P = 900$. Doing so, you will see that at an altitude of $h \approx 0.784$ km, the atmospheric pressure will have dropped to 900 millibars.

19. (a) All constants are positive.

(b) The constant b is definitely between 0 and 1, because $y = a \cdot b^x$ represents a decreasing function.

(c) In addition to b, the constants a, c, p could be between 0 and 1.

(d) Since the curves $y = a \cdot b^x$ and $y = c \cdot d^x$ cross on the y-axis, we must have $a = c$.

(e) The values of a and p are not equal as curves cross y axis at different points. The values of b and d and, likewise, b and q cannot be equal because in each case, one graph climbs while the other falls. However d and q could be equal.

20. As r increases, the graph of $y = a(1 + r)^t$ rises more steeply, so the point of intersection moves to the left and down. However, no matter how steep the graph becomes, the point of intersection remains above and to the right of the y-intercept of the second curve, or the point $(0, b)$. Thus, the value of y_0 decreases but does not reach b.

21. As a increases, the y-intercept of the curve rises, and the point of intersection shifts down and to the left. Thus y_0 decreases. If a becomes larger than b, the point of intersection shifts to the left side of the y-axis, and the value of y_0 continues to decrease. However, y_0 will not decrease to 0, as the point of intersection will always fall above the x-axis.

22. Figure 3.22 shows three different values of b, labeled b_1, b_2, b_3, and the corresponding values of t, labeled t_1, t_2, t_3. As you can see from the figure, as b is decreased, the point of intersection shifts to the left, so the t-coordinate decreases. (Note that if b is decreased to 0 or to a negative number, there is no point of intersection, and the value of t_0 is undefined.)

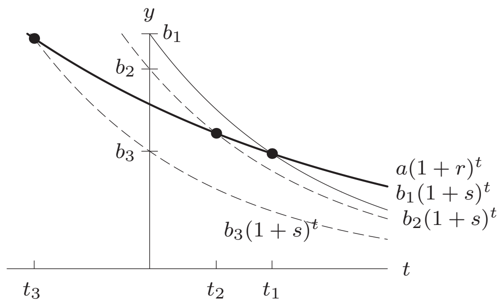


Figure 3.22

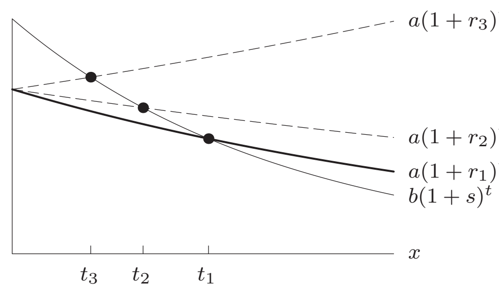


Figure 3.23

23. Figure 3.23 shows three different values of r, labeled r_1, r_2, r_3, and the corresponding values of t, labeled t_1, t_2, t_3. As you can see from the figure, as r is increased, the point of intersection shifts to the left, so the t-coordinate decreases.

24. Increasing: $b > 1$, $a > 0$ or $0 < b < 1$, $a < 0$;
Decreasing: $0 < b < 1$, $a > 0$ or $b > 1$, $a < 0$;
The function is concave up for $a > 0$, $0 < b < 1$ or $b > 1$.

25. Answers will vary, but they should mention that $f(x)$ is increasing and $g(x)$ is decreasing, that they have the same domain, range, and horizontal asymptote. Some may see that $g(x)$ is a reflection of $f(x)$ about the y-axis whenever $b = 1/a$. Graphs might resemble the following:

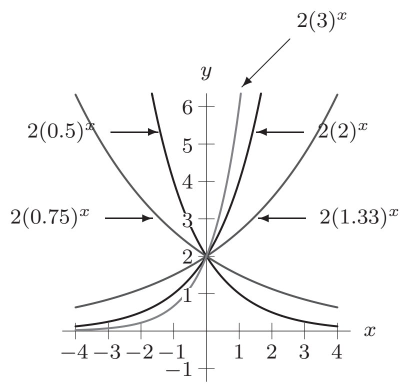


Figure 3.24

26. (a) Note that all the graphs in Figure 3.25 are increasing and concave up. As the value of a increases, the graphs become steeper, but they are all going in the same general direction.

(b) Note that, in this case, while most of the graphs in Figure 3.26 are concave up, some are increasing (when $a > 1$), some are decreasing (when $0 < a < 1$), and one is a constant function (when $a = 1$).

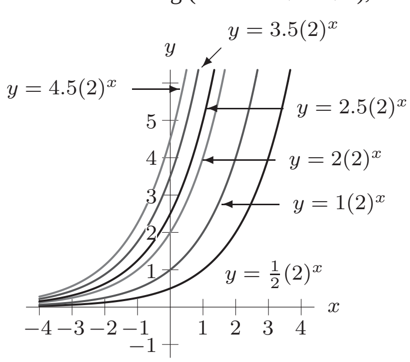


Figure 3.25

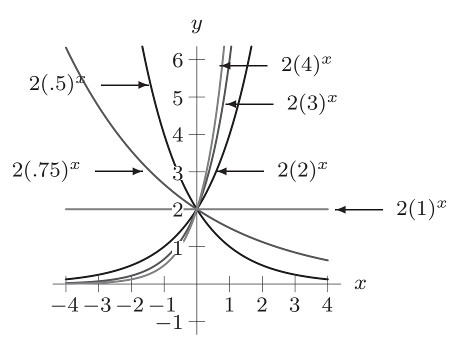


Figure 3.26

27. A possible graph is shown in Figure 3.27.

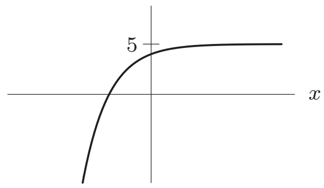


Figure 3.27

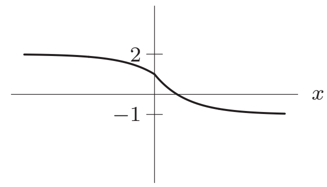


Figure 3.28

28. A possible graph is shown in Figure 3.28.

29. A possible graph is shown in Figure 3.29.

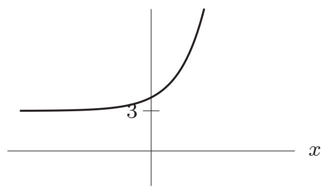


Figure 3.29

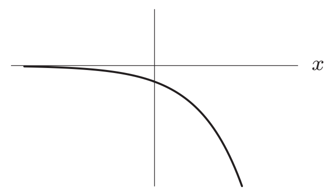


Figure 3.30

30. A possible graph is shown in Figure 3.30.

31. It appears in the graph that

(a) $\lim_{x\to-\infty} f(x) = 5$

(b) $\lim_{x\to\infty} f(x) = -3$.

Of course, we need to be sure that we are seeing all the important features of the graph in order to have confidence in these estimates.

32. It appears in the graph that

(a) $\lim_{x\to-\infty} f(x) = -\infty$

(b) $\lim_{x\to\infty} f(x) = -\infty$.

Of course, we need to be sure that we are seeing all the important features of the graph in order to have confidence in these estimates.

33. **(a)** $\lim_{x\to\infty} 7(0.8)^x = 0$.

(b) $\lim_{t\to-\infty} 5(1.2)^t = 0$.

(c) $\lim_{t\to\infty} 0.7(1-(0.2)^t) = 0.7(1-0) = 0.7$.

34. The value of b is less than 1, so $0 < b < 1$.

35. We see in Figure 3.31 that this function has a horizontal asymptote of $y = 8$.

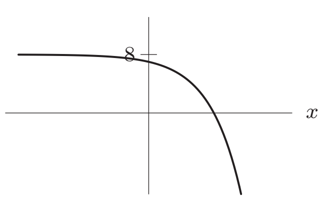


Figure 3.31

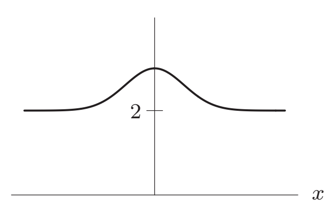


Figure 3.32

36. We see in Figure 3.32 that this function has a horizontal asymptote of $y = 2$.

37. The function, when entered as $y = 1.04\hat{}5x$ is interpreted as $y = (1.04^5)x = 1.217x$. This function's graph is a straight line in all windows. Parentheses must be used to ensure that x is in the exponent.

38. **(a)** Since the population grows exponentially, it can be described by $P = ab^t$, where P is the number of rabbits and t is the number of years which have passed. We know that a represents the initial number of rabbits, so $a = 10$ and $P = 10b^t$. After 5 years, there are 340 rabbits so

$$\begin{aligned} 340 &= 10b^5 \\ 34 &= b^5 \\ (b^5)^{1/5} &= 34^{1/5} \\ b &\approx 2.024. \end{aligned}$$

From this, we know that $P = 10(2.024)^t$.

(b) We want to find t when $P = 1000$. Using a graph of $P = 10(2.024)^t$, we see that $P = 1000$ when $t = 6.53$ years.

39. **(a)** See Figure 3.33. The points appear to represent a function that is increasing and concave up so it makes sense to model these data with an exponential function.

(b) One algorithm used by a calculator or computer gives the exponential regression function as

$$W = 4710(1.306)^t$$

Other algorithms may give different formulas.

(c) Since the base of this exponential function is 1.306, the global wind energy generating capacity was increasing at a rate of about 30.6% per year during this period.

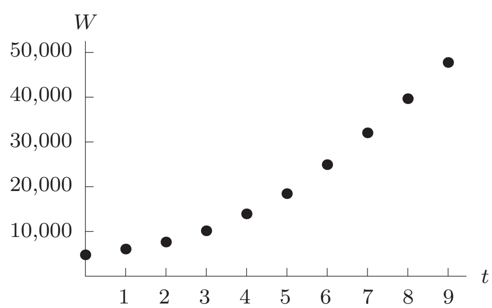


Figure 3.33

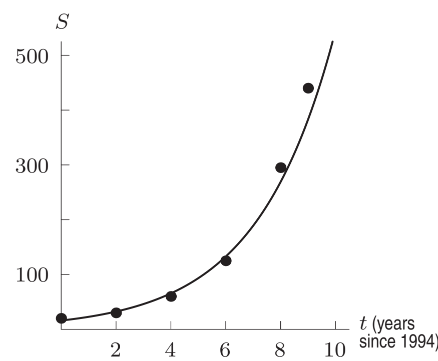


Figure 3.34

40. **(a)** One algorithm used by a calculator or computer gives the exponential regression function as

$$S = 16.6(1.423)^t$$

Other algorithms may give different formulas.

(b) See Figure 3.34. The exponential function appears to fit the points reasonably well.

(c) Since the base of this exponential function is 1.423, sales have been increasing at a rate of about 42.3% per year during this period.

(d) Using $t = 16$, we have $S = 16.6(1.423)^{16} = 4692$ million sales.

41. The domain is all possible t-values, so

Domain: all t-values.

The range is all possible Q-values. Since Q must be positive,

Range: all $Q > 0$.

42. According to Figure 3.35, f seems to approach its horizontal asymptote, $y = 0$, faster. To convince yourself, compare values of f and g for very large values of x.

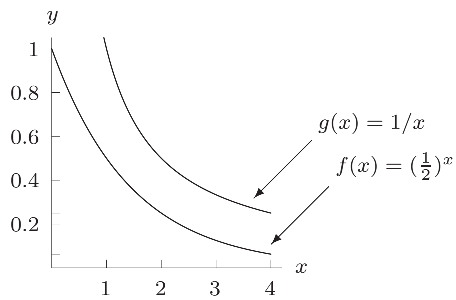


Figure 3.35

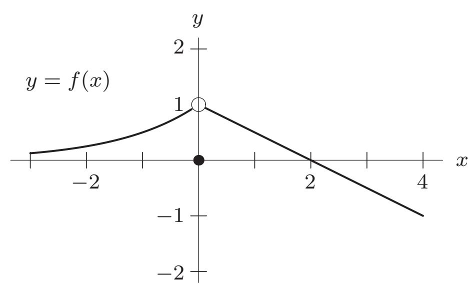


Figure 3.36

43. **(a)** See Figure 3.36

(b) The range of this function is all real numbers less than one — i.e. $f(x) < 1$.

(c) The y-intercept occurs at $(0, 0)$. This point is also an x-intercept. To solve for other x-intercepts we must attempt to solve $f(x) = 0$ for each of the two remaining parts of f. In the first case, we know that the function $f(x) = 2^x$ has no x-intercepts, as there is no value of x for which 2^x is equal to zero. In the last case, for $x > 0$, we set $f(x) = 0$ and solve for x:

$$\begin{aligned} 0 &= 1 - \frac{1}{2}x \\ \frac{1}{2}x &= 1 \\ x &= 2. \end{aligned}$$

Hence $x = 2$ is another x-intercept of f.

(d) As x gets large, the function is defined by $f(x) = 1 - 1/2x$. To determine what happens to f as $x \to +\infty$, find values of f for very large values of x. For example,

$$f(100) = 1 - \frac{1}{2}(100) = -49, \quad f(10000) = 1 - \frac{1}{2}(10000) = -4999$$

$$\text{and} \quad f(1{,}000{,}000) = 1 - \frac{1}{2}(1{,}000{,}000) = -499{,}999.$$

As x becomes larger, $f(x)$ becomes more and more negative. A way to write this is:

$$\text{As } x \to +\infty, \ f(x) \to -\infty.$$

As x gets very negative, the function is defined by $f(x) = 2^x$.

Choosing very negative values of x, we get $f(-100) = 2^{-100} = 1/2^{100}$, and $f(-1000) = 2^{-1000} = 1/2^{1000}$. As x becomes more negative the function values get closer to zero. We write

$$\text{As } x \to -\infty, \ f(x) \to 0.$$

(e) Increasing for $x < 0$, decreasing for $x > 0$.

44. **(a)** Figure 3.37 shows the three populations. From this graph, the three models seem to be in good agreement. Models 1 and 3 are indistinguishable; model 2 appears to rise a little faster. However notice that we cannot see the behavior beyond 50 months because our function values go beyond the top of the viewing window.

(b) Figure 3.38 shows the population differences. The graph of $y = f_2(x) - f_1(x) = 3(1.21)^x - 3(1.2)^x$ grows very rapidly, especially after 40 months. The graph of $y = f_3(x) - f_1(x) = 3.01(1.2)^x - 3(1.2)^x$ is hardly visible on this scale.

(c) Models 1 and 3 are in good agreement, but model 2 predicts a much larger mussel population than does model 1 after only 50 months. We can come to at least two conclusions. First, even small differences in the base of an exponential function can be highly significant, while differences in initial values are not as significant. Second, although two exponential curves can look very similar, they can actually be making very different predictions as time increases.

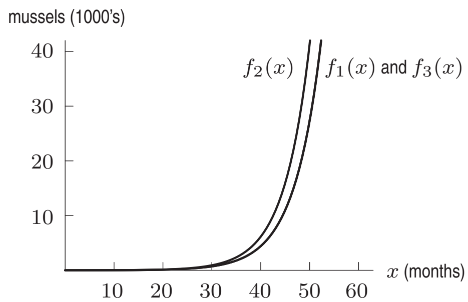


Figure 3.37

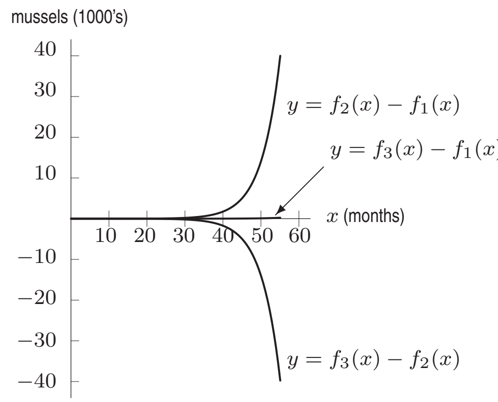


Figure 3.38

Solutions for Section 3.4

Exercises

1. Using the formula $y = ab^x$, each of the functions has the same value for b, but different values for a and thus different y-intercepts.
When $x = 0$, the y-intercept for $y = e^x$ is 1 since $e^0 = 1$.
When $x = 0$, the y-intercept for $y = 2e^x$ is 2 since $e^0 = 1$ and $2(1) = 2$.
When $x = 0$, the y-intercept for $y = 3e^x$ is 3 since $e^0 = 1$ and $3(1) = 3$.
Therefore, $y = e^x$ is the bottom graph, above it is $y = 2e^x$ and the top graph is $y = 3e^x$.

2. We know that $e \approx 2.71828$, so $2 < e < 3$. Since e lies between 2 and 3, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. Since 3^x increases faster than 2^x, the correct matching is shown in Figure 3.39.

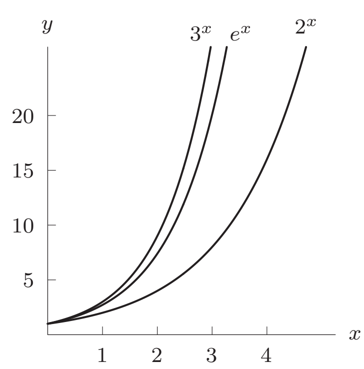


Figure 3.39

3. We know that $e^{0.3t}$ grows faster than $e^{0.25t}$ (because a 30% continuous growth rate is faster than a 25% continuous growth rate). Similarly $(1.25)^t$ grows faster than $(1.2)^t$ (because a 25% annual growth rate is faster than a 20% annual growth rate). In addition, a 25% continuous growth rate is faster than a 25% annual growth rate.

Thus $(1.2)^t$=(c) is (IV); $(1.25)^t$=(b) is (III); $e^{0.25t}$=(a) is (II); $e^{0.3t}$=(d) is (I).

4. Calculating the equivalent continuous rates, we find $e^{0.45} = 1.568$, $e^{0.47} = 1.600$, $e^{0.5} = 1.649$. Thus the functions to be matched are

(a) $(1.5)^x$ **(b)** $(1.568)^x$ **(c)** $(1.6)^x$ **(d)** $(1.649)^x$

So (a) is (IV), (b) is (III), (c) is (II), (d) is (I).

5. $y = e^x$ is an increasing exponential function, since $e > 1$. Therefore, it rises when read from left to right. It matches $g(x)$.

If we rewrite the function $y = e^{-x}$ as $y = (e^{-1})^x$, we can see that in the formula $y = ab^x$, we have $a = 1$ and $b = e^{-1}$. Since $0 < e^{-1} < 1$, this graph has a positive y-intercept and falls when read from left to right. Thus its graph is $f(x)$.

In the function $y = -e^x$, we have $a = -1$. Thus, the vertical intercept is $y = -1$. The graph of $h(x)$ has a negative y-intercept.

6. The functions given in (a) and (c) represent exponential decay while the functions given in (b) and (d) represent exponential growth. Thus, (a) and (c) correspond to (III) and (IV) while (b) and (d) correspond to (I) and (II). The function in (d) grows by 20% per time unit while the function in (b) grows by 5% per time unit. Since (d) is growing faster, formula (d) must correspond to graph (I) while formula (b) corresponds to graph (II). Graphs (III) and (IV) correspond to the exponential decay formulas, with graph (IV) decaying at a more rapid rate. Thus formula (a) corresponds to graph (III) and formula (c) corresponds to graph (IV). We have:

(a) (III)
(b) (II)
(c) (IV)
(d) (I)

7. Since $y = e^x$ is the only increasing function in the list, (a) corresponds to (I).
The other three functions all decrease as x increases, and $y = e^{-x} = 1/e^x$ decreases slowest while $y = e^{-3x} = 1/e^{3x}$ decreases fastest. Thus (b) corresponds to (II), and (c) corresponds to (III), and (d) corresponds to (IV).

8. Tracing along a graph of $V = 1000e^{0.02t}$ until $V = 3000$ gives $t \approx 54.931$ years. See Figure 3.40.

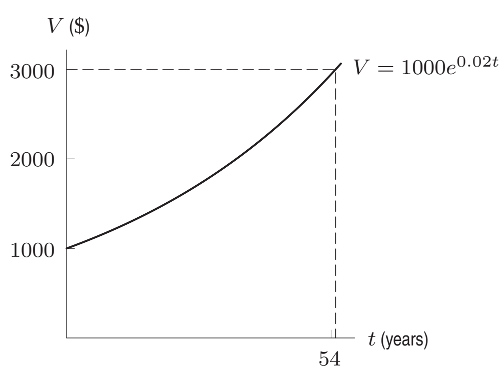


Figure 3.40

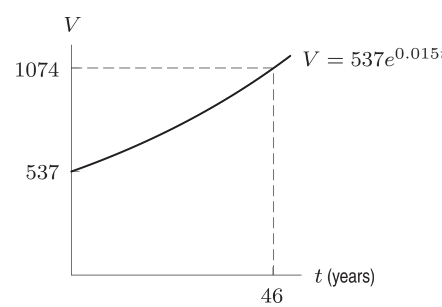


Figure 3.41

9. We want to know when $V = 1074$. Tracing along the graph of $V = 537e^{0.015t}$ gives $t \approx 46.210$ years. See Figure 3.41.

Problems

10. **(a)** Since $\sqrt{2} = 1.414\ldots$ and $e = 2.718\ldots$, we have

$$\sqrt{2} < e < 3,$$

so

$$(\sqrt{2})^{2.2} < e^{2.2} < 3^{2.2}.$$

(b) Note $3^{-2.2} = 1/3^{2.2}$. Thus, since

$$e^{2.2} < 3^{2.2},$$

we know that

$$\frac{1}{e^{2.2}} > \frac{1}{3^{2.2}},$$

which gives

$$3^{-2.2} < e^{-2.2}.$$

11. $\lim\limits_{x\to\infty} e^{-3x} = 0.$

12. $\lim\limits_{t\to-\infty} 5e^{0.07t} = 0.$

13. $\lim\limits_{t\to\infty} (2 - 3e^{-0.2t}) = 2 - 3\cdot 0 = 2.$

14. $\lim\limits_{t\to-\infty} 2e^{-0.1t+6} = \infty.$

15. The values of a and k are both positive.

16. **(a)** At the end of 100 years,

$$B = 1200e^{0.03(100)} = 24{,}102.64 \text{ dollars.}$$

(b) Tracing along a graph of $B = 1200e^{0.03t}$ until $B = 50000$ gives $t \approx 124.323$ years.

17. **(a)** For an annual interest rate of 5%, the balance B after 15 years is

$$B = 2000(1.05)^{15} = 4157.86 \text{ dollars.}$$

(b) For a continuous interest rate of 5% per year, the balance B after 15 years is

$$B = 2000e^{0.05\cdot 15} = 4234.00 \text{ dollars.}$$

18. **(a)** (i) The population, P, in millions, is given by $P = 3.2(1.02)^t$, so a century later

$$P = 3.2(1.02)^{100} = 23.183 \text{ million.}$$

(ii) The population, P, in millions, is given by $P = 3.2e^{0.02t}$, so a century later

$$P = 3.2e^{0.02(100)} = 23.645 \text{ million.}$$

(b) Since $e^{0.02} = 1.0202\ldots$ the growth factor in part (ii) is larger than the growth factor of 1.02 in part (i). Thus we expect the answer to part (ii) to be larger.

19. **(a)** Using $P = P_0e^{kt}$ where $P_0 = 25{,}000$ and $k = 7.5\%$, we have

$$P(t) = 25{,}000e^{0.075t}.$$

(b) We first need to find the growth factor so will rewrite

$$P = 25{,}000e^{0.075t} = 25{,}000(e^{0.075})^t \approx 25{,}000(1.07788)^t.$$

At the end of a year, the population is 107.788% of what it had been at the end of the previous year. This corresponds to an increase of approximately 7.788%. This is greater than 7.5% because the rate of 7.5% per year is being applied to larger and larger amounts. In one instant, the population is growing at a rate of 7.5% per year. In the next instant, it grows again at a rate of 7.5% a year, but 7.5% of a slightly larger number. The fact that the population is increasing in tiny increments continuously results in an actual increase greater than the 7.5% increase that would result from one, single jump of 7.5% at the end of the year.

20. (a) Since $P(t)$ has continuous growth, its formula will be $P(t) = P_0 e^{kt}$. Since P_0 is the initial population, which is 22,000, and k represents the continuous growth rate of 7.1%, our formula is

$$P(t) = 22{,}000e^{0.071t}.$$

(b) While, at any given instant, the population is growing at a rate of 7.1% a year, the effect of compounding is to give us an actual increase of more than 7.1%. To find that increase, we first need to find the growth factor, or b. Rewriting $P(t) = 22{,}000e^{0.071t}$ in the form $P = 22000b^t$ will help us accomplish this. Thus, $P(t) = 22{,}000(e^{0.071})^t \approx 22{,}000(1.07358)^t$. Alternatively, we can equate the two formulas and solve for b:

$$\begin{aligned} 22{,}000e^{0.071t} &= 22{,}000b^t \\ e^{0.071t} &= b^t \quad \text{(dividing both sides by 22,000)} \\ e^{0.071} &= b \quad \text{(taking the } t^{\text{th}} \text{ root of both sides).} \end{aligned}$$

Using your calculator, you can find that $b \approx 1.07358$. Either way, we see that at the end of the year, the population is 107.358% of what it had been at the end of the previous year, and so the population increases by approximately 7.358% each year.

21. (a) The substance decays according to the formula

$$A = 50e^{-0.14t}.$$

(b) At $t = 10$, we have $A = 50e^{-0.14(10)} = 12.330$ mg.

(c) We see in Figure 3.42 that $A = 5$ at approximately $t = 16.45$, which corresponds to the year 2016.

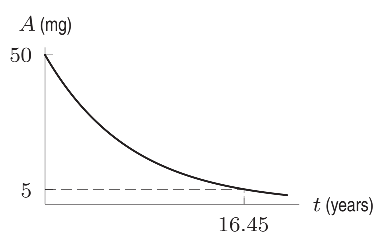


Figure 3.42

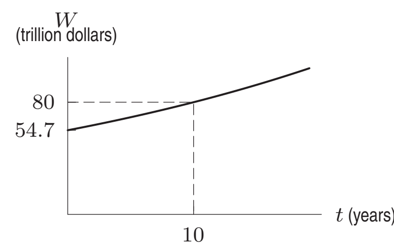


Figure 3.43

22. We have

$$W = 54.7e^{0.038t}.$$

We see in Figure 3.43 that $W = 80$ at approximately $t = 10$ years.

23. (a) Since poultry production is increasing at a constant continuous percent rate, we use the exponential formula $P = ae^{kt}$. Since $P = 77.2$ when $t = 0$, we have $a = 77.2$. Since $k = 0.016$, we have

$$P = 77.2e^{0.016t}.$$

(b) When $t = 6$, we have $P = 77.2e^{0.016(6)} = 84.979$. In the year 2010, the formula predicts that world poultry production will be about 85 million tons.

(c) A graph of $P = 77.2e^{0.016t}$ is given in Figure 3.44. We see that when $P = 90$ we have $t = 9.6$. We expect production to be 90 million tons near the middle of the year 2013.

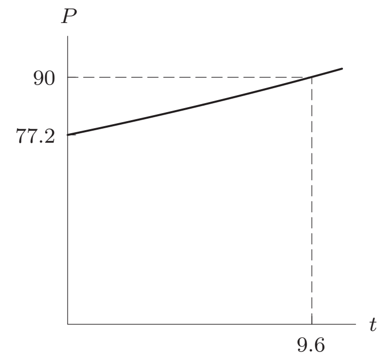


Figure 3.44

24. Since the graphs of ae^{kx} and be^{lx} have the same vertical intercept, we know $a = b$. Since their common intercept is above the vertical intercept of e^x, we know $a = b > 1$.
Since ae^{kx} increases as x increases, we know $k > 0$. But ae^{kx} increases more slowly than e^x, so $0 < k < 1$.
Since be^{lx} decreases as x increases, we know $l < 0$.

25. **(a)** A calculator or computer gives the values in Table 3.6. We see that the values of $(1 + 1/n)^n$ increase as n increases.

Table 3.6

n	$(1+1/n)^n$
1000	2.7169239
10,000	2.7181459
100,000	2.7182682
1,000,000	2.7182805

(b) Extending Table 3.6 gives Table 3.7. Since, correct to 6 decimal places, $e = 2.718282$, we need approximately $n = 10^7$ to achieve an estimate for e that is correct to 6 decimal places.

Table 3.7

n	$(1+1/n)^n$	Correct to 6 decimal places
10^5	2.71826824	2.718268
10^6	2.71828047	2.718280
10^7	2.71828169	2.718282
10^8	2.71828179	2.718282

(c) Using most calculators, when $n = 10^{16}$ the computed value of $(1 + 1/n)^n$ is 1. The reason is that calculators use only a limited number of decimal places, so the calculator finds that $1 + 1/10^{16} = 1$.

26. **(a)** See Figure 3.45.

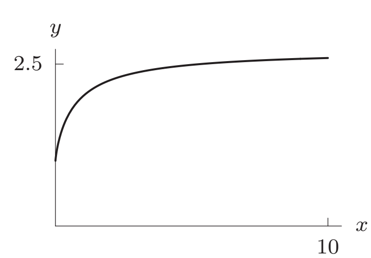


Figure 3.45

(b) The y values are increasing.

(c) The values of $(1+1/x)^x$ appear to approach a limiting value as x gets larger.

(d) Figures 3.46 and 3.47 show $y=(1+1/x)^x$ for $1 \le x \le 100$ and $1 \le x \le 1000$, respectively. We see y appears to approach a limiting value of slightly above 2.5

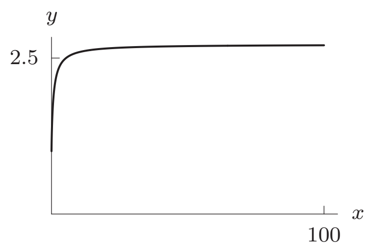


Figure 3.46

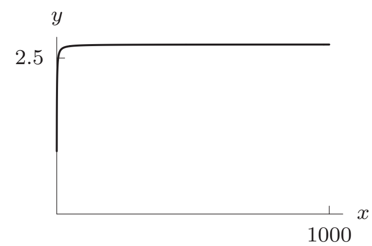


Figure 3.47

(e) The graphs of $y=(1+1/x)^x$ and $y=e$ are indistinguishable in Figure 3.48, suggesting that $(1+1/x)^x$ approaches e as x gets larger. However, a graph cannot tell us that $(1+1/x)^x$ approaches e exactly as x gets larger—only that $(1+1/x)^x$ gets very close to e.

(f) Table 3.8 shows that the value of $(1+1/x)^x$ agrees with $e=2.718281828 \approx 2.7183$ for $x=50{,}000$ and above.

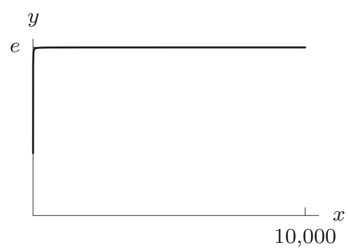
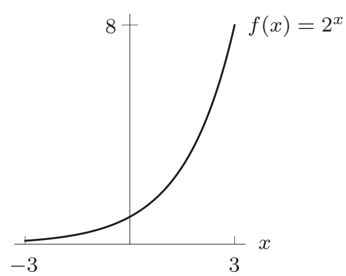


Figure 3.48

Table 3.8

x	$(1+1/x)^x$	Correct to 4 decimal places
10,000	2.718146	2.7181
20,000	2.718214	2.7182
30,000	2.718237	2.7182
40,000	2.718248	2.7182
50,000	2.718255	2.7183
60,000	2.718262	2.7183

27. (a)



(b) The point $(0,1)$ is on the graph. So is $(0.01, 1.00696)$. Taking $\dfrac{y_2-y_1}{x_2-x_1}$, we get an estimate for the slope of 0.696. We may zoom in still further to find that $(0.001, 1.000693)$ is on the graph. Using this and the point $(0,1)$ we would get a slope of 0.693. Zooming in still further we find that the slope stabilizes at around 0.693; so, to two digits of accuracy, the slope is 0.69.

(c) Using the same method as in part (b), we find that the slope is ≈ 1.10.

(d) We might suppose that the slope of the tangent line at $x=0$ increases as b increases. Trying a few values, we see that this is the case. Then we can find the correct b by trial and error: $b=2.5$ has slope around 0.916, $b=3$ has slope around 1.1, so $2.5 < b < 3$. Trying $b=2.75$ we get a slope of 1.011, just a little too high. $b=2.7$ gives a slope of 0.993, just a little too low. $b=2.72$ gives a slope of 1.0006, which is as good as we can do by giving b to two decimal places. Thus $b \approx 2.72$.

In fact, the slope is exactly 1 when $b=e=2.718\ldots$.

28. **(a)** The sum is 2.708333333.
(b) The sum of $1+\frac{1}{1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{1\cdot2\cdot3\cdot4\cdot5}+\frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$ is 2.718055556.
(c) 2.718281828 is the calculator's internal value for e. The sum of the first five terms has two digits correct, while the sum of the first seven terms has four digits correct.
(d) One approach to finding the number of terms needed to approximate e is to keep a running sum. We already have the total for seven terms displayed, so we can add the eighth term, $\frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}$, and compare the result with 2.718281828. Repeat this process until you get the required degree of accuracy. Using this process, we discover that 13 terms are required.

Solutions for Section 3.5

Exercises

1. **(a)** If the interest is compounded annually, there will be $\$500\cdot 1.01=\$505$ after one year.
(b) If the interest is compounded weekly, after one year, there will be $\$500\cdot(1+0.01/52)^{52}=\$505.02$.
(c) If the interest is compounded every minute, after one year, there will be $\$500\cdot(1+0.01/525{,}600)^{525{,}600}=\$505.03$.
(d) If the interest is compounded continuously, after one year, there will be $\$500\cdot e^{0.01}=\$505.03$.

2. **(a)** If the interest is compounded annually, there will be $\$500\cdot 1.03=\$515$ after one year.
(b) If the interest is compounded weekly, there will be $500\cdot(1+0.03/52)^{52}=\$515.22$ after one year.
(c) If the interest is compounded every minute, there will be $500\cdot(1+0.03/525{,}600)^{525{,}600}=\$515.23$ after one year.
(d) If the interest is compounded continuously, there will be $500\cdot e^{0.03}=\$515.23$ after one year.

3. **(a)** If the interest is compounded annually, there will be $\$500\cdot 1.05=\$525$ after one year.
(b) If the interest is compounded weekly, there will be $500\cdot(1+0.05/52)^{52}=\$525.62$ after one year.
(c) If the interest is compounded every minute, there will be $500\cdot(1+0.05/525{,}600)^{525{,}600}=\$525.64$ after one year.
(d) If the interest is compounded continuously, there will be $500\cdot e^{0.05}=\$525.64$ after one year.

4. **(a)** If the interest is compounded annually, there will be $\$500\cdot 1.08=\$540$ after one year.
(b) If the interest is compounded weekly, there will be $500\cdot(1+0.08/52)^{52}=\$541.61$ after one year.
(c) If the interest is compounded every minute, there will be $500\cdot(1+0.08/525{,}600)^{525{,}600}=\$541.64$ after one year.
(d) If the interest is compounded continuously, there will be $500\cdot e^{0.08}=\$541.64$ after one year.

5. With continuous compounding, the interest earns interest during the year, so the balance grows faster with continuous compounding than with annual compounding. Curve A corresponds to continuous compounding and curve B corresponds to annual compounding. The initial amount in both cases is the vertical intercept, $\$500$.

6. The continuous growth rate is $k=0.19=19\%$ per year. To calculate the effective annual yield, we rewrite the function in the form $Q=5500e^{0.19t}=5500b^t$. Thus,

$$b=e^{0.19}=1.20925,$$

Writing

$$Q=5500(1.209250)^t$$

indicates that the effective annual yield is 20.925% per year.

7. We let B represent the balance in the account after t years.
(a) If interest is compounded annually, we have $B=5000(1.04)^t$. After ten years, the amount in the account is $5000(1.04)^{10}=\$7401.22$.
(b) If interest is compounded continuously, we have $B=5000e^{0.04t}$. After ten years, the amount in the account is $5000e^{0.04(10)}=\$7459.12$. As expected, the account contains more money if the interest is compounded continuously.

8. (a) The nominal interest rate is 8%, so the interest rate per month is $0.08/12$. Therefore, at the end of 3 years, or 36 months,

$$\text{Balance} = \$1000\left(1+\frac{0.08}{12}\right)^{36} = \$1270.24.$$

(b) There are 52 weeks in a year, so the interest rate per week is $0.08/52$. At the end of $52 \times 3 = 156$ weeks,

$$\text{Balance} = \$1000\left(1+\frac{0.08}{52}\right)^{156} = \$1271.01.$$

(c) Assuming no leap years, the interest rate per day is $0.08/365$. At the end of 3×365 days

$$\text{Balance} = \$1000\left(1+\frac{0.08}{365}\right)^{3\cdot 365} = \$1271.22.$$

(d) With continuous compounding, after 3 years

$$\text{Balance} = \$1000e^{0.08(3)} = \$1271.25$$

9. The value of the deposit is given by

$$V = 1000e^{0.05t}.$$

To find the effective annual rate, we use the fact that $e^{0.05t} = (e^{0.05})^t$ to rewrite the function as

$$V = 1000(e^{0.05})^t.$$

Since $e^{0.05} = 1.05127$, we have

$$V = 1000(1.05127)^t.$$

This tells us that the effective annual rate is 5.127%.

10. (a) $B = B_0(1.042)^1 = B_0(1.042)$, so the effective annual yield is 4.2%.

(b) $B = B_0\left(1+\frac{.042}{12}\right)^{12} \approx B_0(1.0428)$, so the effective annual yield is approximately 4.28%.

(c) $B = B_0e^{0.042(1)} \approx B_0(1.0429)$, so the effective annual yield is approximately 4.29%.

11. (a) The nominal rate is the stated annual interest without compounding, thus 1%.

The effective annual rate for an account paying 1% compounded annually is 1%.

(b) The nominal rate is the stated annual interest without compounding, thus 1%.

With quarterly compounding, there are four interest payments per year, each of which is $1/4 = 0.25\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.0025^4 = 1.01004$, which is 1.004%.

(c) The nominal rate is the stated annual interest without compounding, thus 1%.

With daily compounding, there are 365 interest payments per year, each of which is $(1/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1+0.01/365)^{365} = 1.01005$, which is 1.005%.

(d) The nominal rate is the stated annual interest without compounding, thus 1%.

The effective annual rate for an account paying 1% compounded continuously is $e^{0.01} = 1.01005$, which is 1.005%.

12. (a) The nominal rate is the stated annual interest without compounding, thus 100%.

The effective annual rate for an account paying 1% compounded annually is 100%.

(b) The nominal rate is the stated annual interest without compounding, thus 100%.

With quarterly compounding, there are four interest payments per year, each of which is $100/4 = 25\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.25^4 = 2.44141$, which is 144.141%.

(c) The nominal rate is the stated annual interest without compounding, thus 100%.

With daily compounding, there are 365 interest payments per year, each of which is $(100/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1 + 1/365)^{365} = 2.71457$, which is 171.457%.

(d) The nominal rate is the stated annual interest without compounding, thus 100%.

The effective annual rate for an account paying 100% compounded continuously is $e^{1.00} = 2.71828$, which is 171.828%.

13. (a) The nominal rate is the stated annual interest without compounding, thus 3%.
The effective annual rate for an account paying 1% compounded annually is 3%.

(b) The nominal rate is the stated annual interest without compounding, thus 3%.
With quarterly compounding, there are four interest payments per year, each of which is $3/4 = 0.75\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.0075^4 = 1.03034$, which is 3.034%.

(c) The nominal rate is the stated annual interest without compounding, thus 3%.
With daily compounding, there are 365 interest payments per year, each of which is $(3/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1+0.03/365)^{365} = 1.03045$, which is 3.045%.

(d) The nominal rate is the stated annual interest without compounding, thus 3%.
The effective annual rate for an account paying 3% compounded continuously is $e^{0.03} = 1.03045$, which is 3.045%.

14. (a) The nominal rate is the stated annual interest without compounding, thus 6%.
The effective annual rate for an account paying 1% compounded annually is 6%.

(b) The nominal rate is the stated annual interest without compounding, thus 6%.
With quarterly compounding, there are four interest payments per year, each of which is $6/4 = 1.5\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.015^4 = 1.06136$, which is 6.136%.

(c) The nominal rate is the stated annual interest without compounding, thus 6%.
With daily compounding, there are 365 interest payments per year, each of which is $(6/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1+0.06/365)^{365} = 1.06183$, which is 6.183%.

(d) The nominal rate is the stated annual interest without compounding, thus 6%.
The effective annual rate for an account paying 6% compounded continuously is $e^{0.06} = 1.06184$, which is 6.184%.

Problems

15. (a) (i) $B = B_0\left(1+\dfrac{.06}{4}\right)^4 \approx B_0(1.0614)$, so the APR is approximately 6.14%.

(ii) $B = B_0\left(1+\dfrac{.06}{12}\right)^{12} \approx B_0(1.0617)$, so the APR is approximately 6.17%.

(iii) $B = B_0\left(1+\dfrac{.06}{52}\right)^{52} \approx B_0(1.0618)$, so the APR is approximately 6.18%.

(iv) $B = B_0\left(1+\dfrac{.06}{365}\right)^{365} \approx B_0(1.0618)$, so the APR is approximately 6.18%.

(b) $e^{0.06} \approx 1.0618$. No matter how often we compound interest, we'll never get more than $\approx 6.18\%$ APR.

16. (a) Let x be the amount of money you will need. Then, at 5% annual interest, compounded annually, after 6 years you will have the following dollar amount:

$$x\,(1+0.05)^6 = x(1.05)^6.$$

If this needs to equal $\$25{,}000$, then we have

$$\begin{aligned} x(1.05)^6 &= 25{,}000 \\ x &= \frac{25{,}000}{(1.05)^6} \approx \$18{,}655.38. \end{aligned}$$

(b) At 5% annual interest, compounded monthly, after 6 years, or $6\cdot 12 = 72$ months, you will have the following dollar amount:

$$x\left(1+\frac{0.05}{12}\right)^{72}.$$

If this needs to equal $\$25{,}000$, then we have

$$\begin{aligned} x\left(1+\frac{0.05}{12}\right)^{72} &= 25{,}000 \\ x &= \frac{25{,}000}{(1+\frac{0.05}{12})^{72}} \approx \$18{,}532.00. \end{aligned}$$

(c) At 5% annual interest, compounded daily, after 6 years, or $6 \cdot 365 = 2190$ days, you will have the following dollar amount:

$$x\left(1+\frac{0.05}{365}\right)^{2190} = x(1.000136986)^{2190}.$$

If this needs to equal $\$25{,}000$, then we have

$$\begin{aligned} x(1.000136986)^{2190} &= 25{,}000 \\ x &= \frac{25{,}000}{(1.000136986)^{2190}} \approx \$18{,}520.84. \end{aligned}$$

(d) The effective yield on an account increases with the number of times of compounding. So, as the number of times increases, the amount of money you need to begin with in order to end up with $25{,}000$ in 6 years decreases.

17. (i) Equation (b). Since the growth factor is 1.12, or 112%, the annual interest rate is 12%.

(ii) Equation (a). An account earning at least 1% monthly will have a monthly growth factor of at least 1.01, which means that the annual (12-month) growth factor will be at least

$$(1.01)^{12} \approx 1.1268.$$

Thus, an account earning at least 1% monthly will earn at least 12.68% yearly. The only account that earns this much interest is account (a).

(iii) Equation (c). An account earning 12% annually compounded semi-annually will earn 6% twice yearly. In t years, there are $2t$ half-years.

(iv) Equations (b), (c) and (d). An account that earns 3% each quarter ends up with a yearly growth factor of $(1.03)^4 \approx 1.1255$. This corresponds to an annual percentage rate of 12.55%. Accounts (b), (c) and (d) earn less than this. Check this by determining the growth factor in each case.

(v) Equations (a) and (e). An account that earns 6% every 6 months will have a growth factor, after 1 year, of $(1+0.06)^2 = 1.1236$, which is equivalent to a 12.36% annual interest rate, compounded annually. Account (a), earning 20% each year, clearly earns more than 6% twice each year, or 12.36% annually. Account (e), which earns 3% each quarter, earns $(1.03)^2 = 1.0609$, or 6.09% every 6 months, which is greater than 6%.

18. **(a)** For investment A, we have

$$P = 875(1+\frac{0.135}{365})^{365(2)} = \$1146.16.$$

For investment B,

$$P = 1000(e^{0.067(2)}) = \$1143.39.$$

For investment C,

$$P = 1050(1+\frac{0.045}{12})^{12(2)} = \$1148.69.$$

(b) A comparison of final balances does not reflect the fact that the initial investment amounts are different. One way to take initial amount into consideration is to look at the overall growth in the account. Comparing final balance to initial deposit for each account we find

$$\text{Investment A: } \frac{1146.16}{875} \approx 1.31$$

$$\text{Investment B: } \frac{1143.39}{1000} \approx 1.143$$

$$\text{Investment C: } \frac{1148.69}{1050} \approx 1.093.$$

Thus, in the two year period Investment A has grown by approximately 31%, followed by Investment B (14.3%) and finally Investment C (9.3%). From best to worst, we have A, B, C.

[Note: Comparing the effective annual rates for each account would be a more efficient way to solve the problem and would give the same result.]

19. To see which investment is best after 1 year, we compute the effective annual yield:
For Bank A, $P = P_0(1 + \frac{0.07}{365})^{365(1)} \approx 1.0725P_0$
For Bank B, $P = P_0(1 + \frac{0.071}{12})^{12(1)} \approx 1.0734P_0$
For Bank C, $P = P_0(e^{0.0705(1)}) \approx 1.0730P_0$
Therefore, the best investment is with Bank B, followed by Bank C and then Bank A.

20. Since $e^{0.053} = 1.0544$, the effective annual yield of the account paying 5.3% interest compounded continuously is 5.44%. Since this is less than the effective annual yield of 5.5% from the 5.5% compounded annually, we see that the account paying 5.5% interest compounded annually is slightly better.

21. **(a)** The effective annual rate is the rate at which the account is actually increasing in one year. According to the formula, $M = M_0(1.07763)^t$, at the end of one year you have $M = 1.07763M_0$, or 1.07763 times what you had the previous year. The account is 107.763% larger than it had been previously; that is, it increased by 7.763%. Thus the effective rate being paid on this account each year is about 7.763%.

(b) Since the money is being compounded each month, one way to find the nominal annual rate is to determine the rate being paid each month. In t years there are $12t$ months, and so, if b is the monthly growth factor, our formula becomes

$$M = M_0 b^{12t} = M_0(b^{12})^t.$$

Thus, equating the two expressions for M, we see that

$$M_0(b^{12})^t = M_0(1.07763)^t.$$

Dividing both sides by M_0 yields

$$(b^{12})^t = (1.07763)^t.$$

Taking the t^{th} root of both sides, we have

$$b^{12} = 1.07763$$

which means that

$$b = (1.07763)^{1/12} \approx 1.00625.$$

Thus, this account earns 0.625% interest every month, which amounts to a nominal interest rate of about $12(0.625\%) = 7.5\%$.

22. Let r represent the nominal annual rate. Since the interest is compounded quarterly, the investment earns $\frac{r}{4}$ each quarter. So, at the end of the first quarter, the investment is $850\left(1 + \frac{r}{4}\right)$, and at the end of the second quarter is $850\left(1 + \frac{r}{4}\right)^2$. By the end of 40 quarters (which is 10 years), it is $850\left(1 + \frac{r}{4}\right)^{40}$. But we are told that the value after 10 years is $\$1{,}000$, so

$$\begin{aligned}
1000 &= 850\left(1 + \frac{r}{4}\right)^{40} \\
\frac{1000}{850} &= \left(1 + \frac{r}{4}\right)^{40} \\
\frac{20}{17} &= \left(1 + \frac{r}{4}\right)^{40} \\
\left(\frac{20}{17}\right)^{\frac{1}{40}} &= 1 + \frac{r}{4} \\
1.00407 &\approx 1 + \frac{r}{4} \\
0.00407 &\approx \frac{r}{4} \\
r &\approx 0.01628.
\end{aligned}$$

We see that the nominal interest rate is 1.628%.

23. To find the fee for six hours, we convert 6 hours to years: (6)·(1 year/365 days)·(1 day/24 hours)= $\frac{6}{(365)(24)}$ years.

Since the interest is being compounded continuously, the total amount of money is given by $P = P_0e^{kt}$, where, in this case, $k = 0.20$ is the continuous annual rate and t is the number of years. So

$$P = 200{,}000{,}000e^{0.20(\frac{6}{365\cdot 24})} = 200{,}027{,}399.14$$

The value of the money at the end of the six hours was $\$200{,}027{,}399.14$, so the fee for that time was $\$27{,}399.14$.

24. If the investment is growing by 3% per year, we know that, at the end of one year, the investment will be worth 103% of what it had been the previous year. At the end of two years, it will be 103% of $103\% = (1.03)^2$ as large. At the end of 10 years, it will have grown by a factor of $(1.03)^{10}$, or 1.34392. The investment will be 134.392% of what it had been, so we know that it will have increased by 34.392%. Since $(1.03)^{10} \approx 1.34392$, it increases by 34.392%.

25. If the annual growth factor is b, then we know that, at the end of 5 years, the investment will have grown by a factor of b^5. But we are told that it has grown by 30%, so it is 130% of its original size. So

$$\begin{aligned} b^5 &= 1.30 \\ b &= 1.30^{\frac{1}{5}} \approx 1.05387. \end{aligned}$$

Since the investment is 105.387% as large as it had been the previous year, we know that it is growing by about 5.387% each year.

26. Let b represent the growth factor, since the investment decreases, $b < 1$. If we start with an investment of P_0, then after 12 years, there will be $P_0 b^{12}$ left. But we know that since the investment has decreased by 60% there will be 40% remaining after 12 years. Therefore,

$$\begin{aligned} P_0 b^{12} &= P_0 0.40 \\ b^{12} &= 0.40 \\ b &= (0.40)^{1/12} = 0.92648. \end{aligned}$$

This tells us that the value of the investment will be 92.648% of its value the previous year, or that the value of the investment decreases by approximately 7.352% each year, assuming a constant percent decay rate.

Solutions for Chapter 3 Review

Exercises

1. Yes. Writing the function as

$$g(w) = 2\left(2^{-w}\right) = 2\left(2^{-1}\right)^w = 2\left(\frac{1}{2}\right)^w,$$

we have $a = 2$ and $b = 1/2$.

2. Yes. Writing the function as

$$m(t) = (2 \cdot 3^t)^3 = 2^3 \cdot (3^t)^3 = 8 \cdot 3^{3t} = 8(3^3)^t = 8(27)^t,$$

we have $a = 8$ and $b = 27$.

3. Yes. Writing the function as

$$f(x) = \frac{3^{2x}}{4} = \frac{1}{4}(3^{2x}) = \frac{1}{4}(3^2)^x = \frac{1}{4}(9)^x,$$

we have $a = 1/4$ and $b = 9$.

4. No. The base must be a constant.

5. Yes. Writing the function as

$$q(r) = \frac{-4}{3^r} = -4\left(\frac{1}{3^r}\right) = -4\left(\frac{1^r}{3^r}\right) = -4\left(\frac{1}{3}\right)^r,$$

we have $a = -4$ and $b = 1/3$.

6. Yes. Writing the function as

$$j(x) = 2^x 3^x = (2 \cdot 3)^x = 6^x,$$

we have $a = 1$ and $b = 6$.

7. Yes. Writing the function as

$$Q(t) = 8^{t/3} = 8^{(1/3)t} = \left(8^{1/3}\right)^t = 2^t,$$

we have $a = 1$ and $b = 2$.

8. Yes. Writing the function as

$$K(x) = \frac{2^x}{3 \cdot 3^x} = \frac{1}{3}\left(\frac{2^x}{3^x}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^x,$$

we have $a = 1/3$ and $b = 2/3$.

9. No. The two terms cannot be combined into the form b^r.

10. If the population is growing or shrinking at a constant rate of m people per year, the formula is linear. Since the vertical intercept is 3000, we have $P = 3000 + mt$.
If the population is growing or shrinking at a constant percent rate of r percent per year, the formula is exponential in the form $P = a(1 + r)^t$. Since the vertical intercept is 3000, we have $P = 3000(1 + r)^t$.
If the population is growing or shrinking at a constant continuous percent rate of k percent per year, the formula is exponential in the form $P = ae^{kt}$. Since the vertical intercept is 3000, we have $P = 3000e^{kt}$.
We have:

(a) $P = 3000 + 200t$.
(b) $P = 3000(1.06)^t$.
(c) $P = 3000e^{0.06t}$.
(d) $P = 3000 - 50t$.
(e) $P = 3000(0.96)^t$.
(f) $P = 3000e^{-0.04t}$.

11. **(a)** The population starts at 200 and grows by 2.8% per year.
(b) The population starts at 50 and shrinks at a continuous rate of 17% per year.
(c) The population starts at 1000 and shrinks by 11% per year.
(d) The population starts at 600 and grows at a continuous rate of 20% per year.
(e) The population starts at 2000 and shrinks by 300 animals per year.
(f) The population starts at 600 and grows by 50 animals per year.

12. By graphing both functions in a window centered at the origin we get Figure 3.49 with graphs of f and g for $-1 \leq x \leq 1$ and $-1 \leq y \leq 2$. We see an intersection point to the left of the origin. So $g(x) < f(x)$ for $x > x_1$. Using a computer or graphing calculator, it can be found as $x_1 \approx -0.587$.

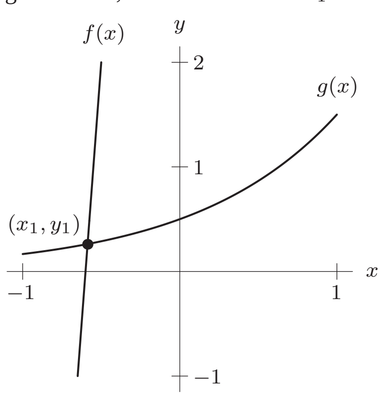


Figure 3.49: Graph for $-1 \leq x \leq 1$, $-1 \leq y \leq 2$

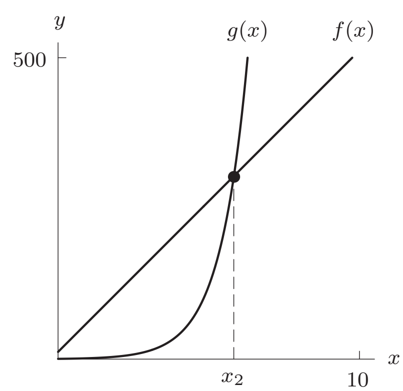


Figure 3.50: Graph for $0 \leq x \leq 10$, $0 \leq y \leq 500$

To the left of our viewing window there can be no more intersections because $f(x)$ will get more and more negative while $g(x)$ remains positive. So for $x < -1$, we must have $f(x) < 0 < g(x)$. Since an exponential function will eventually grow greater than any linear function, there must be another intersection point to the right of our first viewing window. See Figure 3.50. We find $g(x) < f(x)$ for $x < x_2$, where x_2 can be found, using a computer or a graphing calculator, to be $x_2 \approx 4.911$.

Thus, $g(x) < f(x)$ for $x_1 < x < x_2$, that is, the approximate interval

$$-0.587 < x < 4.911.$$

13. We use $y = ab^x$. Since $y = 10$ when $x = 0$, we have $y = 10b^x$. We use the point $(3, 20)$ to find the base b:

$$\begin{aligned} y &= 10b^x \\ 20 &= 10b^3 \\ b^3 &= 2 \\ b &= 2^{1/3} = 1.260. \end{aligned}$$

The formula is

$$y = 10(1.260)^x.$$

14. We use $y = ab^x$. Since $y = 50$ when $x = 0$, we have $y = 50b^x$. We use the point $(5, 20)$ to find the base b:

$$\begin{aligned} y &= 50b^x \\ 20 &= 50b^5 \\ b^5 &= 0.4 \\ b &= 0.4^{1/5} = 0.833. \end{aligned}$$

The formula is

$$y = 50(0.833)^x.$$

15. Since the function is exponential, we know $y = ab^x$. We also know that $(0, 1/2)$ and $(3, 1/54)$ are on the graph of this function, so $1/2 = ab^0$ and $1/54 = ab^3$. The first equation implies that $a = 1/2$. Substituting this value in the second equation gives $1/54 = (1/2)b^3$ or $b^3 = 1/27$, or $b = 1/3$. Thus, $y = \frac{1}{2}\left(\frac{1}{3}\right)^x$.

16. Since this function is exponential, we know $y = ab^x$. We also know that $(-2, 8/9)$ and $(2, 9/2)$ are on the graph of this function, so

$$\frac{8}{9} = ab^{-2}$$

and

$$\frac{9}{2} = ab^2.$$

From these two equations, we can say that

$$\frac{\frac{9}{2}}{\frac{8}{9}} = \frac{ab^2}{ab^{-2}}.$$

Since $(9/2)/(8/9) = 9/2 \cdot 9/8 = 81/16$, we can re-write this equation to be

$$\frac{81}{16} = b^4.$$

Keeping in mind that $b > 0$, we get

$$b = \sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}.$$

Substituting $b = 3/2$ in $ab^2 = 9/2$, we get

$$\begin{aligned} \frac{9}{2} &= a(\frac{3}{2})^2 = \frac{9}{4}a \\ a &= \frac{\frac{9}{2}}{\frac{9}{4}} = \frac{9}{2} \cdot \frac{4}{9} = \frac{4}{2} = 2. \end{aligned}$$

Thus, $y = 2(3/2)^x$.

17. The formula for this function must be of the form $y = ab^x$. We know that $(-2, 400)$ and $(1, 0.4)$ are points on the graph of this function, so

$$400 = ab^{-2}$$

and

$$0.4 = ab^1.$$

This leads us to

$$\begin{aligned}\frac{0.4}{400} &= \frac{ab^1}{ab^{-2}}\\ 0.001 &= b^3\\ b &= 0.1.\end{aligned}$$

Substituting this value into $0.4 = ab^1$, we get

$$\begin{aligned}0.4 &= a(0.1)\\ a &= 4.\end{aligned}$$

So our formula for this function is $y = 4(0.1)^x$. Since $0.1 = 10^{-1}$ we can also write $y = 4(10^{-1})^x = 4(10)^{-x}$.

18. The graph contains the points $(40, 80)$ and $(120, 20)$. Using the ratio method, we have

$$\begin{aligned}\frac{ab^{120}}{ab^{40}} &= \frac{20}{80}\\ b^{80} &= \frac{20}{80}\\ b &= \left(\frac{20}{80}\right)^{1/80} = 0.98282.\end{aligned}$$

Now we can solve for a:

$$\begin{aligned}a(0.98282)^{40} &= 80\\ a &= \frac{80}{(0.98282)^{40}} = 160.\end{aligned}$$

so $y = 160(0.983)^x$.

19. To match formula and graph, we keep in mind the effect on the graph of the parameters a and b in $y = ab^t$.

If $a > 0$ and $b > 1$, then the function is positive and increasing.
If $a > 0$ and $0 < b < 1$, then the function is positive and decreasing.
If $a < 0$ and $b > 1$, then the function is negative and decreasing.
If $a < 0$ and $0 < b < 1$, then the function is negative and increasing.

(a) $y = 0.8^t$. So $a = 1$ and $b = 0.8$. Since $a > 0$ and $0 < b < 1$, we want a graph that is positive and decreasing. The graph in (ii) satisfies the conditions.

(b) $y = 5(3)^t$. So $a = 5$ and $b = 3$. The graph in (i) is both positive and increasing.

(c) $y = -6(1.03)^t$. So $a = -6$ and $b = 1.03$. Here, $a < 0$ and $b > 1$, so we need a graph which is negative and decreasing. The graph in (iv) satisfies these conditions.

(d) $y = 15(3)^{-t}$. Since $(3)^{-t} = (3)^{-1\cdot t} = (3^{-1})^t = (\frac{1}{3})^t$, this formula can also be written $y = 15(\frac{1}{3})^t$. $a = 15$ and $b = \frac{1}{3}$. A graph that is both positive and decreasing is the one in (ii).

(e) $y = -4(0.98)^t$. So $a = -4$ and $b = 0.98$. Since $a < 0$ and $0 < b < 1$, we want a graph which is both negative and increasing. The graph in (iii) satisfies these conditions.

(f) $y = 82(0.8)^{-t}$. Since $(0.8)^{-t} = (\frac{8}{10})^{-t} = (\frac{8}{10})^{-1\cdot t} = ((\frac{8}{10})^{-1})^t = (\frac{10}{8})^t = (1.25)^t$ this formula can also be written as $y = 82(1.25)^t$. So $a = 82$ and $b = 1.25$. A graph which is both positive and increasing is the one in (i).

20. To match formula and graph, we keep in mind the effect on the graph of the parameters a and b in $y = ab^t$.
If $a > 0$ and $b > 1$, then the function is positive and increasing.
If $a > 0$ and $0 < b < 1$, then the function is positive and decreasing.
If $a < 0$ and $b > 1$, then the function is negative and decreasing.
If $a < 0$ and $0 < b < 1$, then the function is negative and increasing.

(a) $y = 8.3e^{-t}$, so $a = 8.3$ and $b = e^{-1}$. Since $a > 0$ and $0 < b < 1$, we want a graph which is positive and decreasing. The graph in (ii) satisfies this condition.

(b) $y = 2.5e^{t}$, so $a = 2.5$ and $b = e$. Since $a > 0$ and $b > 1$, we want a graph which is positive and increasing, such as (i).

(c) $y = -4e^{-t}$, so $a = -4$ and $b = e^{-1}$. Since $a < 0$ and $0 < b < 1$, we want a graph which is negative and increasing, such as (iii).

Problems

21. A possible graph is shown in Figure 3.51.

22. A possible graph is shown in Figure 3.52.

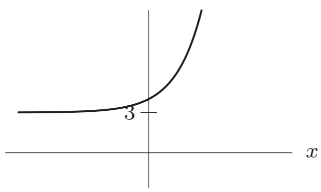


Figure 3.51

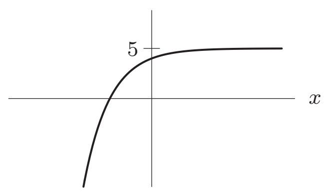


Figure 3.52

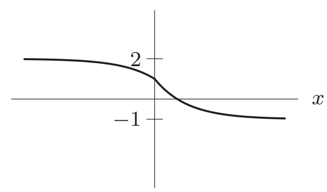


Figure 3.53

23. A possible graph is shown in Figure 3.53.

24. A possible graph is shown in Figure 3.54.

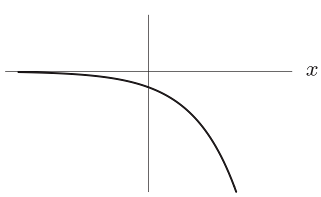


Figure 3.54

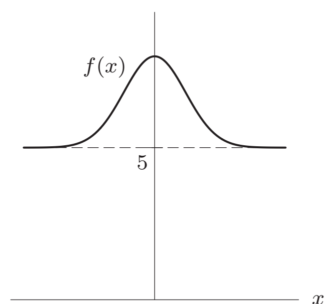


Figure 3.55

25. A possible graph is shown in Figure 3.55. There are many possible answers.

26. We have $\lim\limits_{x\to\infty} 257(0.93)^x = 0$.

27. We have $\lim\limits_{t\to\infty} 5.3e^{-0.12t} = 0$.

28. We have $\lim\limits_{x\to-\infty} (15 - 5e^{3x}) = 15 - 5\cdot 0 = 15$.

29. We have $\lim\limits_{t\to-\infty} (21(1.2)^t + 5.1) = 21(0) + 5.1 = 5.1$.

30. We have $\lim\limits_{x\to\infty}(7.2-2e^{3x}) = -\infty$.

31. We have $\lim\limits_{x\to-\infty}(5e^{-7x}+1.5) = \infty$.

32. **(a)** Suppose $\$1$ is put in the account. The interest rate per month is $0.08/12$. At the end of a year,

$$\text{Balance} = \left(1+\frac{0.08}{12}\right)^{12} = \$1.08300,$$

which is 108.3% of the original amount. So the effective annual yield is 8.300%.

(b) With weekly compounding, the interest rate per week is $0.08/52$. At the end of a year,

$$\text{Balance} = \left(1+\frac{0.08}{52}\right)^{52} = \$1.08322,$$

which is 108.322% of the original amount. So the effective annual yield is 8.322%.

(c) Assuming it is not a leap year, the interest rate per day is $0.08/365$. At the end of a year

$$\text{Balance} = \left(1+\frac{0.08}{365}\right)^{365} = \$1.08328,$$

which is 108.328% of the original amount. So the effective annual yield is 8.328%.

(d) For continuous compounding, at the end of the year

$$\text{Balance} = e^{0.08} = \$1.08329,$$

which is 108.329% of the original amount. So the effective annual yield is 8.329%.

33. We have $g(10) = 50$ and $g(30) = 25$. Using the ratio method, we have

$$\begin{aligned}\frac{ab^{30}}{ab^{10}} &= \frac{g(30)}{g(10)}\\ b^{20} &= \frac{25}{50}\\ b &= \left(\frac{25}{50}\right)^{1/20} \approx 0.965936.\end{aligned}$$

Now we can solve for a:

$$\begin{aligned}a(0.965936)^{10} &= 50\\ a &= \frac{50}{(0.965936)^{10}} \approx 70.711.\end{aligned}$$

so $Q = 70.711(0.966)^t$.

34. We have $f(x) = ab^x$. Using our two points, we have

$$f(-8) = ab^{-8} = 200$$

and

$$f(30) = ab^{30} = 580.$$

Taking ratios, we have

$$\begin{aligned}\frac{580}{200} &= \frac{ab^{30}}{ab^{-8}} = b^{38}\\ \frac{580}{200} &= b^{38}.\end{aligned}$$

This gives

$$b = (\frac{580}{200})^{1/38} \approx 1.0284.$$

We now solve for a. We know that $f(30) = 580$ and $f(x) = a(1.0284)^x$, so we have

$$\begin{aligned} f(30) &= a(1.0284)^{30} \\ 580 &= a(1.0284)^{30} \\ a &= \frac{580}{1.0284^{30}} \\ &\approx 250.4. \end{aligned}$$

Thus, $f(x) = 250.4(1.0284)^x$.

35. Since f is exponential, $f(x) = ab^x$. We know that

$$f(2) = ab^2 = \frac{2}{9}$$

and

$$f(-3) = ab^{-3} = 54,$$

so

$$\frac{2/9}{54} = \frac{ab^2}{ab^{-3}} = b^5.$$

$$\begin{aligned} b^5 &= \frac{1}{243} \\ b &= \left(\frac{1}{243}\right)^{1/5} = \frac{1}{3}. \end{aligned}$$

Thus, $f(x) = a\left(\frac{1}{3}\right)^x$. Since $f(2) = \frac{2}{9}$ and $f(2) = a(\frac{1}{3})^2$, we have

$$\begin{aligned} a\left(\frac{1}{3}\right)^2 &= \frac{2}{9} \\ \frac{a}{9} &= \frac{2}{9} \\ a &= 2. \end{aligned}$$

Thus, $f(x) = 2\left(\frac{1}{3}\right)^x$.

36. We know that $f(x) = ab^x$. Taking the ratio of $f(2)$ to $f(-1)$ we have

$$\begin{aligned} \frac{f(2)}{f(1)} &= \frac{1/27}{27} = \frac{ab^2}{ab^{-1}} \\ \frac{1}{(27)^2} &= b^3 \\ b^3 &= \frac{1}{27^2} \\ b &= (\frac{1}{27^2})^{\frac{1}{3}}. \end{aligned}$$

Thus, $b = \frac{1}{9}$. Therefore, $f(x) = a(\frac{1}{9})^x$.

Using the fact that $f(-1) = 27$, we have

$$f(-1) = a\left(\frac{1}{9}\right)^{-1} = a \cdot 9 = 27,$$

which means $a = 3$. Thus,

$$f(x) = 3\left(\frac{1}{9}\right)^x.$$

37. Let the equation of the exponential curve be $Q = ab^t$. Since this curve passes through the points $(-1, 2)$, $(1, 0.3)$, we have

$$\begin{aligned} 2 &= ab^{-1} \\ 0.3 &= ab^1 = ab \end{aligned}$$

So,

$$\frac{0.3}{2} = \frac{ab}{ab^{-1}} = b^2,$$

that is, $b^2 = 0.15$, thus $b = \sqrt{0.15} \approx 0.3873$ because b is positive. Since $2 = ab^{-1}$, we have $a = 2b = 2 \cdot 0.3873 = 0.7746$, and the equation of the exponential curve is

$$Q = 0.7746 \cdot (0.3873)^t.$$

38. We have $V = ae^{kt}$ where $a = 12{,}000$ and $k = 0.042$, so $V = 12{,}000e^{0.042t}$.

39. We have $p(20) = 300$ and $p(50) = 40$. Using the ratio method, we have

$$\begin{aligned} \frac{ab^{50}}{ab^{20}} &= \frac{p(50)}{p(20)} \\ b^{30} &= \frac{40}{300} \\ b &= \left(\frac{40}{300}\right)^{1/30} \approx 0.935. \end{aligned}$$

Now we can solve for a:

$$\begin{aligned} a\left(\left(\frac{40}{300}\right)^{1/30}\right)^{20} &= 300 \\ a &= \frac{300}{((40/300)^{1/30})^{20}} \\ &= 1149.4641, \end{aligned}$$

so $Q = 1149.4641(0.935)^t$. We can also write this in the form $Q = ae^{kt}$ where

$$k = \ln b = -0.06716.$$

40. At $x = 50$,

$$y = 5000e^{-50/40} = 1432.5240.$$

At $x = 150$,

$$y = 5000e^{-150/40} = 117.5887.$$

We have $q(50) = 1432.524$ and $q(150) = 117.5887$. This gives $y = b + mx$ where

$$m = \frac{q(150) - q(50)}{150 - 50} = \frac{117.5887 - 1432.524}{100} = -13.1.$$

Solving for b, we have

$$\begin{aligned} q(50) &= b - 13.1(50) \\ b &= q(50) + 13.1(50) \\ &= 1432.524 + 13.1(50) \\ &= 2090, \end{aligned}$$

so $y = -13.1x + 2090$.

41. The starting value is $a = 2500$, and the continuous growth rate is $k = 0.042$, so $V = 2500e^{0.042t}$.

42. We have $a = 12{,}000$. This tells us that in year $t = 0$ the population begins with 12,000 members. The constant $k = -0.122 = -12.2\%$. This tells us that the population is decreasing at a continuous annual rate of 12.2%. We have $b = e^k = e^{-0.122} = 0.8851$. This is the annual growth factor; since it is less than 1, we know the population is decreasing. The constant $r = b - 1 = -0.1149 = -11.49\%$. This tells us that the population decreases by 11.49% each year.

43. If a function is linear, then the rate of change is constant. For $Q(t)$,

$$\frac{8.70 - 7.51}{10 - 3} = 0.17.$$

and

$$\frac{9.39 - 8.7}{14 - 10} = 0.17.$$

So this function appears to be very close to linear. Thus, $Q(t) = b + mt$ where $m = 0.17$ as shown above. We solve for b by using the point $(3, 7.51)$.

$$\begin{aligned} Q(t) &= b + 0.17t \\ 7.51 &= b + 0.17(3) \\ 7.51 - 0.51 &= b \\ b &= 7. \end{aligned}$$

Therefore, $Q(t) = 7 + 0.17t$.

44. Testing the rates of change for $R(t)$, we find that

$$\frac{2.61 - 2.32}{9 - 5} = 0.0725$$

and

$$\frac{3.12 - 2.61}{15 - 9} = 0.085,$$

so we know that $R(t)$ is not linear. If $R(t)$ is exponential, then $R(t) = ab^t$, and

$$R(5) = a(b)^5 = 2.32$$

and

$$R(9) = a(b)^9 = 2.61.$$

So

$$\begin{aligned} \frac{R(9)}{R(5)} &= \frac{ab^9}{ab^5} = \frac{2.61}{2.32} \\ \frac{b^9}{b^5} &= \frac{2.61}{2.32} \\ b^4 &= \frac{2.61}{2.32} \\ b &= (\frac{2.61}{2.32})^{\frac{1}{4}} \approx 1.030. \end{aligned}$$

Since

$$\begin{aligned} R(15) = a(b)^{15} &= 3.12 \\ \frac{R(15)}{R(9)} = \frac{ab^{15}}{ab^9} &= \frac{3.12}{2.61} \\ b^6 &= \frac{3.12}{2.61} \\ b &= \left(\frac{3.12}{2.61}\right)^{\frac{1}{6}} \approx 1.030. \end{aligned}$$

Since the growth factor, b, is constant, we know that $R(t)$ could be an exponential function and that $R(t) = ab^t$. Taking the ratios of $R(5)$ and $R(9)$, we have

$$\begin{aligned} \frac{R(9)}{R(5)} = \frac{ab^9}{ab^5} &= \frac{2.61}{2.32} \\ b^4 &= 1.125 \\ b &= 1.030. \end{aligned}$$

So $R(t) = a(1.030)^t$. We now solve for a by using $R(5) = 2.32$,

$$\begin{aligned} R(5) &= a(1.030)^5 \\ 2.32 &= a(1.030)^5 \\ a &= \frac{2.32}{1.030^5} \approx 2.001. \end{aligned}$$

Thus, $R(t) = 2.001(1.030)^t$.

45. Testing rates of change for $S(t)$, we find that

$$\frac{6.72 - 4.35}{12 - 5} = 0.339$$

and

$$\frac{10.02 - 6.72}{16 - 12} = 0.825.$$

Since the rates of change are not the same we know that $S(t)$ is not linear.

Testing for a possible constant growth factor we see that

$$\begin{aligned} \frac{S(12)}{S(5)} = \frac{ab^{12}}{ab^5} &= \frac{6.72}{4.35} \\ b^7 &= \frac{6.72}{4.35} \\ b &\approx 1.064 \end{aligned}$$

and

$$\begin{aligned} \frac{S(16)}{S(12)} = \frac{ab^{16}}{ab^{12}} &= \frac{10.02}{6.72} \\ b^4 &= \frac{10.02}{6.72} \\ b &\approx 1.105. \end{aligned}$$

Since the growth factors are different, $S(t)$ is not an exponential function.

46. **(a)** At a time t years after 2005, the population P in millions is $P = 36.8(1.013)^t$.
In 2030, we have $t = 25$, so

$$P = 36.8(1.013)^{25} = 50.826 \text{ million.}$$

Between 2005 and 2030,

$$\text{Increase } = 50.826 - 36.8 = 14.026 \text{ million.}$$

In 2055, we have $t = 50$, so

$$P = 36.8(1.013)^{50} = 70.197 \text{ million.}$$

Between 2030 and 2055,

$$\text{Increase } = 70.197 - 50.826 = 19.371 \text{ million.}$$

(b) The increase between 2030 and 2055 is expected to be larger than the increase between 2005 and 2030 because the exponential function is concave up. Both increases are over 25 year periods, but since the graph of the function bends upward, the increase in the later time period is larger.

47. (a) We use $N = b + mt$. Since $N = 178.8$ when $t = 0$, we have $b = 178.8$. We find the slope:

$$m = \frac{\Delta N}{\Delta t} = \frac{187.2 - 178.8}{10 - 0} = 0.84.$$

The formula is

$$N = 178.8 + 0.84t.$$

(b) We use $N = ab^t$. Since $N = 178.8$ when $t = 0$, we have $a = 178.8$. We find the base b using the fact that $N = 187.2$ when $t = 10$:

$$\begin{aligned} N &= 178.8b^t \\ 187.2 &= 178.8b^{10} \\ b^{10} &= \frac{187.2}{178.8} = 1.047 \\ b &= (1.047)^{1/10} = 1.0046. \end{aligned}$$

The formula is

$$N = 178.8(1.0046)^t.$$

48. (a) Since

$$\begin{aligned} \text{New population } &= 1.134 \cdot (\text{old population}) \\ &= 113.4\% \text{ of old population} \\ &= 100\% \text{ of old population } + 13.4\% \text{ of old population,} \end{aligned}$$

so the town has increased in size by 13.4%.

(b) Let b be the annual growth factor. Then since 1.134 is the two-year growth factor,

$$\begin{aligned} b^2 &= 1.134 \\ b &= \sqrt{1.134} \approx 1.06489. \end{aligned}$$

With this result, we know that after one year the town is 106.489% of its size from the previous year. Thus, this town grew at an annual rate of 6.489%.

49. (a) In this account, the initial balance in the account is $\$1100$ and the effective yield is 5 percent each year.

(b) In this account, the initial balance in the account is $\$1500$ and the effective yield is approximately 5.13%, because $e^{0.05} \approx 1.0513$.

50. (a) Accion's interest rate $= (1160 - 1000)/1000 = 0.16 = 16\%$.

(b) Payment to loan shark $= 1000 + 22\% \cdot 1000 = \$1220$.

(c) The one from Accion, since the interest rate is lower.

51. The y-intercept of f is greater than that of g, so $a > c$.

52. In (a), we see that the graph of g starts out below the graph of f. In (c), we see that at some point, the lower graph rises to intersect the higher graph, which tells us that g grows faster than f. This means that $d > b$.

53. In (a), we can see the y-intercept of f, which starts above g. The graph of g leaves the window in (a) to the right, not at the top, so the values of g are less than the values of f throughout the interval $0 \le x \le x_1$. Thus, the point of intersection is to the right of x_1, so x_1 is the smallest of these three values. The x-value of the point of intersection of the graphs in (b) is closer to x_3 than the x-value of the point of intersection in (c) is to x_2. Thus $x_3 < x_2$, so we have $x_1 < x_3 < x_2$.

54.
- f matches (ii) and (iv).
- g matches (i) and (iii)

55. For the following, let Q be the quantity after t years, and Q_0 be the initial amount.

(a) If Q doubles in size every 7 years, we have

$$\begin{aligned} 2Q_0 &= Q_0(b)^7 \\ b^7 &= 2 \\ b &= 2^{\frac{1}{7}} \approx 1.10409 \end{aligned}$$

and so Q grows by 10.409% per year.

(b) If Q triples in size every 11 years, we have

$$\begin{aligned} 3Q_0 &= Q_0 b^{11} \\ b^{11} &= 3 \\ b &= (3^{1/11}) \approx 1.10503 \end{aligned}$$

and so Q grows by 10.503% per year.

(c) If Q grows by 3% per month, we have

$$\begin{aligned} Q &= Q_0(1.03)^{12t} \quad \text{(because } 12t \text{ is number of months)} \\ &= Q_0(1.03^{12})^t \approx Q_0(1.42576)^t, \end{aligned}$$

and so the quantity grows by 42.576% per year.

(d) In t years there are $12t$ months. Thus, the number of 5-month periods in $12t$ months is $\frac{12}{5}t$. So, if Q grows by 18% every 5 months, we have

$$\begin{aligned} Q &= Q_0(1.18)^{\frac{12}{5}t} \\ &= Q_0(1.18^{12/5})^t \approx Q_0(1.48770)^t. \end{aligned}$$

Thus, Q grows by 48.770% per year.

56. (a) Assuming linear growth at 250 per year, the population in 2005 would be

$$18{,}500 + 250 \cdot 10 = 21{,}000.$$

Assuming exponential growth at a constant percent rate, the percent rate would be $250/18{,}500 \approx 0.013514 = 1.351\%$ per year, so after 10 years the population would be

$$18{,}500(1.013514)^{10} \approx 21{,}158.$$

The town's growth is poorly modeled by both linear and exponential functions.

(b) We do not have enough information to make even an educated guess about a formula.

57. (a) For a linear model, we assume that the population increases by the same amount every year. Since it grew by 4.14% in the first year, the town had a population increase of $0.0414(20{,}000) = 828$ people in one year. According to a linear model, the population in 2005 would be $20{,}000 + 10 \cdot 828 = 28{,}280$. Using an exponential model, we assume that the population increases by the same percent every year, so the population in 2005 would be $20{,}000 \cdot (1.0414)^{10} = 30{,}006$. Clearly the exponential model is a better fit.

(b) Assuming exponential growth at 4.14% a year, the formula for the population is

$$P(t) = 20{,}000(1.0414)^t.$$

58. Let r be the percentage by which the substance decays each year. Every year we multiply the amount of radioactive substance by $1 - r$ to determine the new amount. If a is the amount of the substance on hand originally, we know that after five years, there have been five yearly decreases, by a factor of $1 - r$. Since we know that there will be 60% of a, or $0.6a$, remaining after five years (because 40% of the original amount will have decayed), we know that

$$a \cdot \underbrace{(1-r)^5}_{\substack{\text{five annual decreases} \\ \text{by a factor of } 1-r}} = 0.6a.$$

Dividing both sides by a, we have $(1-r)^5 = 0.6$, which means that

$$1 - r = (0.6)^{\frac{1}{5}} \approx 0.9029$$

so

$$r \approx 0.09712 = 9.712\%.$$

Each year the substance decays by 9.712%.

59. (a) After the first hour all C values are measured at a common 2 hour interval, so we can estimate b by looking at the ratio of successive concentrations after the $t = 0$ concentration, namely,

$$\frac{7}{10} = 0.7, \quad \frac{5}{7} \approx 0.714, \quad \frac{3.5}{5} = 0.7, \quad \frac{2.5}{3.5} \approx 0.714.$$

These are nearly equal, the average being approximately 0.707, so $b^2 \approx 0.707$ and $b \approx 0.841$. Using the data point $(0, 12)$ we estimate $a = 12$. This gives $C = 12\,(0.841)^t$. Figure 3.56 shows these data plotted against time with this exponential function, which seems in good agreement.

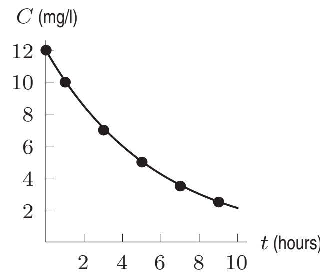


Figure 3.56: Drug concentration versus time with exponential fit

(b) One algorithm used by a calculator or computer gives the exponential regression function as

$$C = 11.914(0.840)^t$$

Other algorithms may give different formulas. This function is very similar to the answer to part (a).

60. (a) Since, for $t < 0$, we know that the voltage is a constant 80 volts, $V(t) = 80$ on that interval.

For $t \geq 0$, we know that $v(t)$ is an exponential function, so $V(t) = ab^t$. According to this formula, $V(0) = ab^0 = a(1) = a$. According to the graph, $V(0) = 80$. From these two facts, we know that $a = 80$, so $V(t) = 80b^t$. If $V(10) = 80b^{10}$ and $V(10) = 15$ (from the graph), then

$$\begin{aligned} 80b^{10} &= 15 \\ b^{10} &= \frac{15}{80} \\ (b^{10})^{\frac{1}{10}} &= (\frac{15}{80})^{\frac{1}{10}} \\ b &\approx 0.8459 \end{aligned}$$

so that $V(t) = 80(0.8459)^t$ on this interval. Combining the two pieces, we have

$$V(t) = \begin{cases} 80 & \text{for } t < 0 \\ 80(0.8459)^t & \text{for } t \geq 0. \end{cases}$$

(b) Using a computer or graphing calculator, we can find the intersection of the line $y = 0.1$ with $y = 80(0.8459)^t$. We find $t \approx 39.933$ seconds.

61. (a) The data points are approximately as shown in Table 3.9. This results in $a \approx 15.269$ and $b \approx 1.122$, so $E(t) = 15.269(1.122)^t$.

Table 3.9

t (years)	0	1	2	3	4	5	6	7	8	9	10	11	12
$E(t)$ (thousands)	22	18	20	20	22	22	19	30	45	42	62	60	65

(b) In 1997 we have $t = 17$ so $E(17) = 15.269(1.122)^{17} \approx 108{,}066$.

(c) The model is probably not a good predictor of emigration in the year 2010 because Hong Kong was transferred to Chinese rule in 1997. Thus, conditions which affect emigration in 2010 may be markedly different than they were in the period from 1989 to 1992, for which data is given. In 2000, emigration was about 12,000.

62. (a) Let p_0 be the price of an item at the beginning of 2000. At the beginning of 2001, its price will be 103.4% of that initial price or $1.034p_0$. At the beginning of 2002, its price will be 102.8% of the price from the year before, that is:

$$\text{Price beginning 2002} = (1.028)(1.034p_0).$$

By the beginning of 2003, the price will be 101.6% of its price the previous year.

$$\begin{aligned}\text{Price beginning 2003} &= 1.016(\text{price beginning 2002})\\ &= 1.016(1.028)(1.034p_0).\end{aligned}$$

Continuing this process,

$$\begin{aligned}(\text{Price beginning 2005}) &= (1.027)(1.023)(1.016)(1.028)(1.034)p_0\\ &\approx 1.135p_0.\end{aligned}$$

So, the cost at the beginning of 2005 is 113.5% of the cost at the beginning of 2000 and the total percent increase is 13.5%.

(b) If r is the average inflation rate for this time period, then $b = 1 + r$ is the factor by which the population on the average grows each year. Using this average growth factor, if the price of an item is initially p_0, at the end of a year its value would be p_0b, at the end of two years it would be $(p_0b)b = p_0b^2$, and at the end of five years p_0b^5. According to the answer in part (a), the price at the end of five years is $1.135p_0$. So

$$\begin{aligned}p_0b^5 &= 1.135p_0\\ b^5 &= 1.135\\ b &= (1.135)^{1/5} \approx 1.026.\end{aligned}$$

If $b = 1.026$, then $r = 0.026$ or 2.6%, the average annual inflation rate.

(c) We assume that the average rate of 2.6% inflation for 2000 through 2004 holds through the beginning of 2010. So, on average, the price of the shower curtain is 102.6% of what it was the previous year for ten years. Then the price of the shower curtain would be $20(1.026)^{10} \approx \$25.85$.

63. (a) Because the time intervals are equally spaced at $t = 1$ units apart, we can estimate b by looking at the ratio of successive populations, namely,

$$\frac{0.901}{0.755} \approx 1.193, \quad \frac{1.078}{0.901} \approx 1.196, \quad \frac{1.285}{1.078} \approx 1.192.$$

These are nearly equal, the average being approximately 1.194, so $b \approx 1.194$. Using the data point $(0, 0.755)$ we estimate $a = 0.755$. Figure 3.57 shows the population data as well as the function $P = 0.755\,(1.194)^t$, where t is the number of 5-year intervals since 1975.

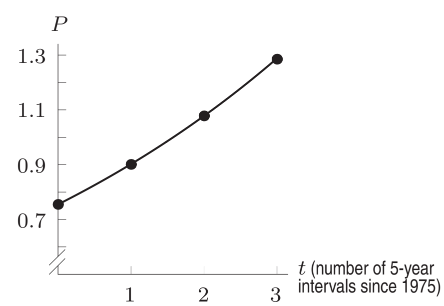


Figure 3.57: Actual and theoretical population of Botswana

(b) To find when the population doubles, we need to find the time t when $P = 2 \cdot 0.755$, that is, when

$$2 \cdot 0.755 = 0.755\,(1.194)^t\,,$$

$$2 = (1.194)^t\,.$$

By using a calculator to compute $(1.194)^t$ for $t = 1, 2, 3$, and so on, we find

$$(1.194)^4 \approx 2.03.$$

(Recall that t is measured in 5-year intervals.) This means the population doubles about every $5 \cdot 4 = 20$ years. Continuing in this way we see that

$$0.755\,(1.194)^{32} \approx 219.8.$$

Thus, according to this model, about $5 \cdot 32 = 160$ years from 1975, or 2135, the population of Botswana will exceed the 1975 population of the United States.

64. (a) See Figure 3.58.

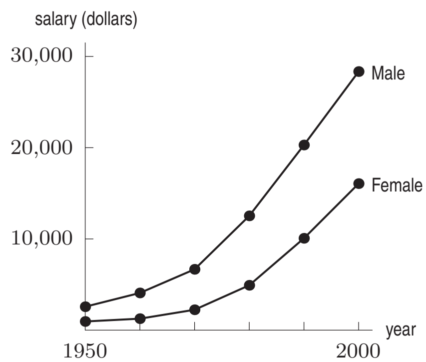


Figure 3.58

(b) For females, $W(1950) = ae^{b(0)} = a = 953$. Using trial and error, we find a value for $b = 0.062$ which approximates values in the table. So a possible formula for the median income of women is $W_F(t) = 953e^{0.062(t-1950)}$.
For males, $W(1950) = ae^{b(0)} = a = 2570$. A possible value for b is 0.051. These values give us the formula for median income of men of $W_M(t) = 2570e^{0.051(t-1950)}$.

(c) Through the year 2000, women's incomes trail behind those of men. See Figure 3.59. However Figure 3.60 shows women's incomes eventually overtake men's.

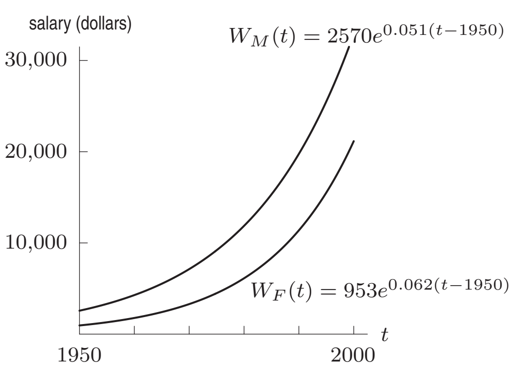


Figure 3.59

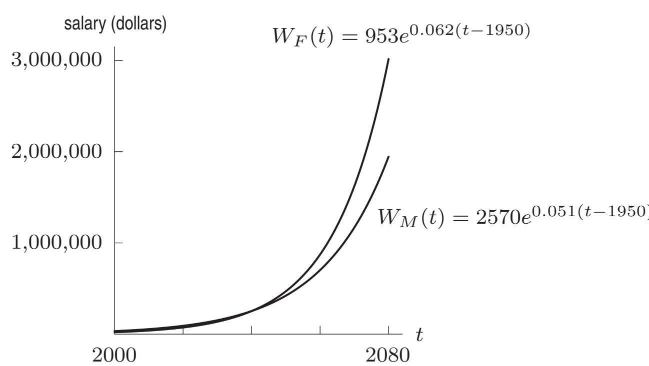


Figure 3.60

(d) The graph over a larger interval predicts a more promising outlook for equality in incomes. If we look carefully at each formula, $W_F(t) = 953e^{0.062(t-1950)}$ and $W_M(t) = 2570e^{0.051(t-1950)}$, we observe that women should ultimately catch up with, and even surpass, the income of men. The justification for this is the fact that the exponent in W_F is larger than the exponent in W_M, so W_F increases more quickly than W_M. See Figure 3.60.

The graph suggests that women will earn approximately the same amount as men in about the year 2060.

(e) The trends observed may not continue into the future, so the model may not apply. Thus these predictions are not reliable.

65. (a) Since f is an exponential function, we can write $f(x) = ab^x$, where a and b are constants. Since the blood alcohol level of a non-drinker is zero, we know that $f(0) = p_0$. Since $f(x) = ab^x$,

$$f(0) = ab^0 = a \cdot 1 = a,$$

and so $a = p_0$ and $f(x) = p_0 b^x$. With a BAC of 0.15, the probability of an accident is $25p_0$, so

$$f(0.15) = 25p_0.$$

From the formula we know that $f(0.15) = p_0 b^{0.15}$, so

$$\begin{aligned} p_0 b^{0.15} &= 25p_0 \\ b^{0.15} &= 25 \\ (b^{0.15})^{1/0.15} &= 25^{1/0.15} \\ b &\approx 2{,}087{,}372{,}982. \end{aligned}$$

Thus, $f(x) = p_0(2{,}087{,}372{,}982)^x$.

(b) Since

$$f(x) = p_0(2{,}087{,}372{,}982)^x,$$

using our formula, we see that $f(0.1) = p_0(2{,}087{,}372{,}982)^{0.1} \approx 8.55p_0$. This means that a legally intoxicated person is about 8.55 times as likely as a nondrinker to be involved in a single-car accident.

(c) If the probability of an accident is only three times the probability for a non-drinker, then we need to find the value of x for which

$$f(x) = 3p_0.$$

Since

$$f(x) = p_0(2{,}087{,}372{,}982)^x,$$

we have

$$p_0(2{,}087{,}372{,}982)^x = 3p_0$$

and

$$2{,}087{,}372{,}982^x = 3.$$

Using a calculator or computer, we find that $x \approx 0.051$. This is about half the BAC currently used in the legal definition.

CHECK YOUR UNDERSTANDING

1. True. If the constant rate is r then the formula is $f(t) = a \cdot (1 + r)^t$. The function decreases when $0 < 1 + r < 1$ and increases when $1 + r > 1$.
2. True. The exponential function formula $f(t) = a \cdot b^t$ shows that the independent variable, t, is in the exponent.
3. True. This fits the general exponential form $y = a \cdot b^t$ with $a = 40$ and $b = 1.05$.
4. False. When x increases from 1 to 2, the value of y doubles, but when x increase from 4 to 5, the value of y does not double.
5. False. The annual growth factor would be 1.04, so $S = S_0(1.04)^t$.
6. False. Evaluate $f(2) = 4(2)^2 = 16$.

7. True. The growth factor is $(2/5)$, which is less than 1, so the function values are decreasing.

8. True. Evaluate $Q = f(3) = 1000(0.5)^3 = 1000/8 = 125$. Since f is decreasing, there is no second value of t where $Q = f(t) = 3$.

9. True. The initial value means the value of Q when $t = 0$, so $Q = f(0) = a \cdot b^0 = a \cdot 1 = a$.

10. True. Using two data points, the parameters a and b can be found for the general linear function $y = b + ax$. In addition, new a and b can be calculated from the same data points for the exponential function $y = ab^x$.

11. False. The correct formula is $P = 1000(1 - 0.10)^t$ or $P = 1000(0.90)^t$.

12. True. The exponential function increases over intervals of length 1 by multiplication by a constant greater than 1, so it increases at an increasing rate. A linear function grows at a constant rate.

13. False. This is the formula of a linear function.

14. False. Suppose the initial population is 1 million, then $P = f(t) = 1(1.5)^t$ and $f(2) = 2.25$ so the population has increased from 1 million to 2.25 million, which is more than 100%.

15. True. The graph crosses the Q-axis when $t = 0$, and there $Q = ab^0 = a$.

16. False. For example, the graph of $Q = 2(0.5)^t$ falls as we read from left to right because $b = 0.5$ and $0 < b < 1$.

17. True. The irrational number $e = 2.71828\cdots$ has this as a good approximation.

18. True. As the x values get large the values of $f(x)$ gets close to k.

19. False. The graph of $y = a \cdot b^x$ is concave up when $a > 0$ but is concave down when $a < 0$.

20. False. The quantity is given by $Q = 110(1 - 0.03)^t = 110(0.97)^t$ grams.

21. True. The initial value is 200 and the growth factor is 1.04.

22. False. Since $0.2 = 20\%$, we see the given formula is for 20% continuous growth.

23. False. Since $-0.90 = -90\%$, we see the given formula is for 90% continuous decay.

24. False. Since for $t = 5$, we have $Q = 3e^{0.2 \cdot 5} = 3e = 8.15$.

25. True. Since k is the continuous growth rate and negative, Q is decreasing.

26. True. You will earn interest on the interest of each previous month.

27. False. The formula is nearly correct but 6% should be in its decimal form of 0.06:

$$B = 500\left(1 + \frac{0.06}{4}\right)^{3 \cdot 4}.$$

28. True. The formula is correct for the continuous compounding calculation.

29. True. The interest from any quarter is compounded in subsequent quarters.

30. False. The rate makes a difference. It is better to invest at 10% annually than 5% continuously.

31. False. For a 5% nominal rate, no matter how many times the interest is compounded, the earnings can never exceed the continuous rate of 5%. In twenty years, the investment cannot grow in value above $\$10{,}000e^{(0.05)20} = \$27{,}183$.

32. False. The rule of 70 estimates that it takes $70/5.5$ or about 13 years. You can also find that in 18 years your investment would be worth $\$1000e^{(0.055)18} = \$2691$.

Solutions to Tools for Chapter 3

1. $4^3 = 4 \cdot 4 \cdot 4 = 64$

2. $(-5)^2 = (-5)(-5) = 25$

3. $11^2 = 11 \cdot 11 = 121$

4. $10^4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

5. $(-1)^{12} = \underbrace{(-1)(-1)\cdots(-1)}_{12\ factors} = 1$

6. $(-1)^{13} = \underbrace{(-1)(-1)\cdots(-1)}_{13 \text{ factors}} = -1$

7. $\frac{5^3}{5^2} = 5^{3-2} = 5^1 = 5$

8. $\dfrac{5^3}{5} = 5^{3-1} = 5^2 = 5 \cdot 5 = 25$

9. $\frac{10^8}{10^5} = 10^{8-5} = 10^3 = 10 \cdot 10 \cdot 10 = 1{,}000$

10. $\frac{6^4}{6^4} = 6^{4-4} = 6^0 = 1$

11. $8^0 = 1$

12. $\sqrt{4} = 2$

13. $\sqrt{4^2} = 4$

14. $\sqrt{4^3} = \sqrt{64} = 8$

15. $\sqrt{4^4} = \sqrt{256} = 16$

16. $\sqrt{(-4)^2} = \sqrt{16} = 4$

17. Since $\dfrac{1}{7^{-2}}$ is the same as 7^2, we obtain $7 \cdot 7$ or 49.

18. Since the base of 2 is the same in both numerator and denominator, we have $\dfrac{2^7}{2^3} = 2^{7-3} = 2^4$ or $2 \cdot 2 \cdot 2 \cdot 2$ or 16.

19. In this example, a negative base is raised to an odd power. The answer will thus be negative. Therefore $(-1)^{445} = -1$.

20. The order of operations tells us we have to square 11 first (giving 121), then take the negative. Thus $-11^2 = -\left(11^2\right) = -121$.

21. The order of operations tells us to square 3 first (giving 9) and then multiply by -2. Therefore $(-2)3^2 = (-2)9 = -18$.

22. First we see that $5^0 = 1$. Then $\left(5^0\right)^3 = 1^3 = 1$.

23. The order of operations tells us to find 10^3 and then multiply by 2.1. Therefore $(2.1)\left(10^3\right) = (2.1)(1,000) = 2,100$.

24. $32^{1/5} = (2^5)^{1/5} = 2^1 = 2$

25. $16^{1/2} = (2^4)^{1/2} = 2^2 = 4$

26. $16^{1/4} = (2^4)^{1/4} = 2^1 = 2$

27. $16^{3/4} = (2^4)^{3/4} = 2^3 = 8$

28. $16^{5/4} = (2^4)^{5/4} = 2^5 = 32$

29. $16^{5/2} = (2^4)^{5/2} = 2^{10} = 1024$

30. $100^{5/2} = (\sqrt{100})^5 = 10^5 = 100{,}000$

31. Since $(-5)^3 = -125$, we have $\sqrt[3]{-125} = -5$.

32. First we see within the radical that $(-4)^2 = 16$. Therefore $\sqrt{(-4)^2} = \sqrt{16} = 4$.

33. Exponentiation is done first, with the result that $(-1)^3 = -1$. Therefore $(-1)^3\sqrt{36} = (-1)\sqrt{36} = (-1)(6) = -6$.

34. Since the exponent is $\dfrac{1}{2}$, we can write $(0.04)^{1/2} = \sqrt{0.04} = 0.2$.

35. We can obtain the answer to $(-8)^{2/3}$ in two different ways: either by finding the cube root of $(-8)^2$ yielding $\sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$, or by finding the square of $\sqrt[3]{-8}$ yielding $\left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$.

36. $3^{-1} = \frac{1}{3}^1 = \frac{1}{3}$

37. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

38. $3^{-3/2} = \frac{1}{3^{3/2}} = \frac{1}{(3^3)^{1/2}} = \frac{1}{(27)^{1/2}} = \frac{1}{(9\cdot 3)^{1/2}} = \frac{1}{9^{1/2}\cdot 3^{1/2}} = \frac{1}{3\sqrt{3}}$

39. $25^{-1} = \frac{1}{25}$

40. $25^{-2} = \frac{1}{25}^2 = \frac{1}{625}$

41. $25^{-3/2} = \frac{1}{(25)^{3/2}} = \frac{1}{(25^{1/2})^3} = \frac{1}{5^3} = \frac{1}{125}$

42. For this example, we have $\left(\frac{1}{27}\right)^{-1/3} = (27)^{1/3} = 3$. This is because $\left(\frac{1}{27}\right)^{-1/3} = \left(\left(\frac{1}{27}\right)^{-1}\right)^{1/3} = \left(\frac{27}{1}\right)^{1/3} = 3$.

43. The cube root of 0.125 is 0.5. Therefore $(0.125)^{1/3} = \sqrt[3]{0.125} = 0.5$.

44. $\sqrt{x^4} = (x^4)^{1/2} = x^{4/2} = x^2$

45. $\sqrt{y^8} = (y^8)^{1/2} = y^{8/2} = y^4$

46. $\sqrt{w^8z^4} = (w^8z^4)^{1/2} = (w^8)^{1/2} \cdot (z^4)^{1/2} = w^{8/2} \cdot z^{4/2} = w^4z^2$

47. $\sqrt{x^5y^4} = (x^5 \cdot y^4)^{1/2} = x^{5/2} \cdot y^{4/2} = x^{5/2}y^2$

48. $\sqrt{16x^3} = (16x^3)^{1/2} = 16^{1/2}x^{3/2} = 4x^{3/2}$

49. $\sqrt{49w^9} = (49w^9)^{1/2} = 49^{1/2} \cdot w^{9/2} = 7w^{9/2}$

50. $\sqrt{25x^3z^4} = (25x^3z^4)^{1/2} = 25^{1/2} \cdot x^{3/2} \cdot z^{4/2} = 5x^{3/2}z^2$

51. $\sqrt{r^2} = (r^2)^{1/2} = |r^1| = |r|$

52. $\sqrt{r^3} = (r^3)^{1/2} = r^{3/2}$

53. $\sqrt{r^4} = (r^4)^{1/2} = r^{4/2} = r^2$

54. $\sqrt{36t^2} = (36t^2)^{1/2} = 36^{1/2} \cdot (t^2)^{1/2} = 6|t^1| = 6|t|$

55. $\sqrt{64s^7} = (64s^7)^{1/2} = 64^{1/2} \cdot s^{7/2} = 8s^{7/2}$

56.

$$\begin{aligned}
\sqrt{50x^4y^6} &= 50^{1/2} \cdot (x^4)^{1/2} \cdot (y^6)^{1/2} \\
&= 50^{1/2}x^2y^3 \\
&= (25 \cdot 2)^{1/2}x^2y^3 \\
&= 25^{1/2} \cdot 2^{1/2} \cdot x^2 \cdot y^3 \\
&= 5\sqrt{2}x^2y^3
\end{aligned}$$

57.

$$\begin{aligned}
\sqrt{48u^{10}v^{12}y^5} &= (48)^{1/2} \cdot (u^{10})^{1/2} \cdot (v^{12})^{1/2} \cdot (y^5)^{1/2} \\
&= (16 \cdot 3)^{1/2}u^5v^6y^{5/2} \\
&= 16^{1/2} \cdot 3^{1/2} \cdot u^5v^6y^{5/2} \\
&= 4\sqrt{3}u^5v^6y^{5/2}
\end{aligned}$$

58.

$$\begin{aligned}
\sqrt{8m}\sqrt{2m^3} &= (8m)^{1/2} \cdot (2m^3)^{1/2} = 8^{1/2} \cdot m^{1/2} \cdot 2^{1/2} \cdot (m^3)^{1/2} \\
&= (2^3)^{1/2} \cdot m^{1/2} \cdot 2^{1/2} \cdot m^{3/2} \\
&= 2^{3/2} \cdot m^{1/2} \cdot 2^{1/2} \cdot m^{3/2} \\
&= 2^{4/2} \cdot m^{4/2} \\
&= 2^2m^2 \\
&= 4m^2
\end{aligned}$$

or $\sqrt{8m} \cdot \sqrt{2m^3} = \sqrt{16m^4} = 4m^2$.

59.

$$\begin{aligned}\sqrt{6s^2t^3v^5}\sqrt{6st^5v^3} &= \sqrt{36s^3t^8v^8}\\ &= (36)^{1/2}\cdot(s^3)^{1/2}\cdot(t^8)^{1/2}\cdot(v^8)^{1/2}\\ &= 6s^{3/2}t^4v^4\end{aligned}$$

60. Raising (0.1) and $\left(4xy^2\right)$ to the second power yields $(0.1)^2=(0.01)$ and $\left(4xy^2\right)^2=16x^2y^4$. Therefore $(0.1)^2\left(4xy^2\right)^2=(0.01)\left(16x^2y^4\right)=.16x^2y^4$.

61. First we raise $3^{x/2}$ to the second power and multiply this result by 3. Therefore $3\left(3^{x/2}\right)^2=3\left(3^x\right)=3^1\left(3^x\right)=3^{x+1}$.

62. If we expand $\left(4L^{2/3}P\right)^{3/2}$, we obtain $4^{3/2}\cdot L^1\cdot P^{3/2}$ and then multiplying by $P^{-3/2}$ yields $\left(4^{3/2}\cdot L^1\cdot P^{3/2}\right)P^{-3/2}=8LP^0=8L$.

63. In this example the same variable base w occurs in two separate factors: $w^{1/2}$ and $w^{1/3}$. Since we are multiplying these factors, we need to add the exponents, namely $1/2$ and $1/3$. This requires a common denominator of 6. Therefore

$$7\left(5w^{1/2}\right)\left(2w^{1/3}\right)=70\cdot w^{1/2}\cdot w^{1/3}=70w^{3/6}\cdot w^{2/6}=70w^{5/6}.$$

64. $\left(S\sqrt{16xt^2}\right)^2=S^2(\sqrt{16xt^2})^2=S^2\cdot 16xt^2=16S^2xt^2$

65. $\sqrt{e^{2x}}=(e^{2x})^{\frac{1}{2}}=e^{2x\cdot\frac{1}{2}}=e^x$

66.

$$(3AB)^{-1}\left(A^2B^{-1}\right)^2=\left(3^{-1}\cdot A^{-1}\cdot B^{-1}\right)\left(A^4\cdot B^{-2}\right)=\frac{A^4}{3^1\cdot A^1\cdot B^1\cdot B^2}=\frac{A^3}{3B^3}.$$

67. Since we are multiplying numbers with the same base, e, we need only add the exponents. Thus, $e^{kt}\cdot e^3\cdot e^1=e^{kt+4}$.

68. First we write the radical exponentially. Therefore, $\sqrt{m+2}(2+m)^{3/2}=(m+2)^{1/2}(2+m)^{3/2}$ or $(m+2)^{1/2}\cdot(m+2)^{3/2}$. Then since the base is the same and we are multiplying, we simply add the exponents, or $(m+2)^{1/2}(m+2)^{3/2}=(m+2)^2$.

69. Inside the parenthesis we write the radical as an exponent, which results in

$$\left(3x\sqrt{x^3}\right)^2=\left(3x\cdot x^{3/2}\right)^2.$$

Then within the parenthesis we write

$$\left(3x^1\cdot x^{3/2}\right)^2=\left(3x^{5/2}\right)^2=3^2(x^{5/2})^2=9x^5.$$

70. $x^e\left(x^e\right)^2=x^e\cdot x^{2e}=x^{e+2e}=x^{3e}$

71. $\left(y^{-2}e^y\right)^2=y^{-4}\cdot e^{2y}=\dfrac{e^{2y}}{y^4}$

72. Be careful to realize that in the numerator only x (and not 4) is raised to the power of $3\pi+1$. Then since we are dividing and the same base of x appears, we can subtract exponents. Therefore,

$$\frac{4x^{(3\pi+1)}}{x^2}=4\cdot x^{(3\pi+1-2)}=4x^{(3\pi-1)}.$$

73. $\dfrac{4A^{-3}}{(2A)^{-4}}=\dfrac{4/A^3}{1/(2A)^4}=\dfrac{4}{A^3}\cdot\dfrac{(2A)^4}{1}=\dfrac{4}{A^3}\cdot\dfrac{2^4A^4}{1}=64A.$

74. $\dfrac{a^{n+1}3^{n+1}}{a^n3^n}=a^{n+1-n}3^{n+1-n}=a^1\cdot 3^1=3a$

75. $\dfrac{12u^3}{3(uv^2w^4)^{-1}}=\dfrac{12u^3(uv^2w^4)^1}{3}=4u^4v^2w^4$

76. $\left(a^{-1}+b^{-1}\right)^{-1}=\dfrac{1}{\frac{1}{a}+\frac{1}{b}}=\dfrac{1}{\frac{b+a}{ab}}=\dfrac{ab}{b+a}$.

77. First we divide within the larger parentheses. Therefore,

$$\left(\frac{35(2b+1)^9}{7(2b+1)^{-1}}\right)^2=\left(5(2b+1)^{9-(-1)}\right)^2=\left(5(2b+1)^{10}\right)^2.$$

Then we expand to obtain

$$25(2b+1)^{20}.$$

78. $(-32)^{3/5}=(\sqrt[5]{-32})^3=(-2)^3=-8$

79. $-32^{3/5}=-(\sqrt[5]{32})^3=-(2)^3=-8$

80. $-625^{3/4}=-(\sqrt[4]{625})^3=-(5)^3=-125$

81. $(-625)^{3/4}=(\sqrt[4]{-625})^3$. Since $\sqrt[4]{-625}$ is not a real number, $(-625)^{3/4}$ is undefined.

82. $(-1728)^{4/3}=(\sqrt[3]{-1728})^4=(-12)^4=20,736$

83. $64^{-3/2}=(\sqrt{64})^{-3}=(8)^{-3}=\left(\dfrac{1}{8}\right)^3=\dfrac{1}{512}$

84. $-64^{3/2}=-(\sqrt{64})^3=-(8)^3=-512$

85. $(-64)^{3/2}=(\sqrt{-64})^3$. Since $\sqrt{-64}$ is not a real number, $(-64)^{3/2}$ is undefined.

86. $81^{5/4}=(\sqrt[4]{81})^5=3^5=243$.

87. We have

$$\begin{aligned}10x^{5-2}&=2\\10x^3&=2\\x^3&=0.2\\x&=(0.2)^{1/3}=0.585.\end{aligned}$$

88. We have

$$\begin{aligned}\frac{5}{x^2}&=125\\\frac{1}{x^2}&=25\\x^2&=1/25\\x&=\pm(1/25)^{1/2}=\pm1/5=\pm0.2.\end{aligned}$$

89. We have

$$\begin{aligned}\sqrt{4x^3}&=5\\2x^{3/2}&=5\\x^{3/2}&=2.5\\x&=(2.5)^{2/3}=1.842.\end{aligned}$$

90. We have

$$\begin{aligned}7x^4&=20x^2\\\frac{x^4}{x^2}&=\frac{20}{7}\end{aligned}$$

$$
\begin{aligned}
x^2 &= 20/7 \\
x &= \pm(20/7)^{1/2} = \pm 1.690.
\end{aligned}
$$

91. We have

$$
\begin{aligned}
\frac{5}{x^2} &= 500 \\
\frac{1}{x^2} &= 100 \\
x^2 &= 1/100 \\
x &= \pm(1/100)^{1/2} = \pm 0.1.
\end{aligned}
$$

92. We have

$$
\begin{aligned}
2(x+2)^3 &= 100 \\
(x+2)^3 &= 50 \\
x+2 &= (50)^{1/3} = 3.684. \\
x &= 1.684.
\end{aligned}
$$

93. The point of intersection occurs where the curves have the same x and y values. We set the two formulas equal and solve:

$$
\begin{aligned}
0.8x^4 &= 5x^2 \\
\frac{x^4}{x^2} &= \frac{5}{0.8} \\
x^2 &= 6.25 \\
x &= (6.25)^{1/2} = 2.5.
\end{aligned}
$$

The x coordinate of the point of intersection is 2.5. We use either formula to find the y-coordinate:

$$y = 5(2.5)^2 = 31.25,$$

or

$$y = 0.8(2.5)^4 = 31.25.$$

The coordinates of the point of intersection are $(2.5, 31.25)$.

94. The point of intersection occurs where the curves have the same x and y values. We set the two formulas equal and solve:

$$
\begin{aligned}
2x^3 &= 100\sqrt{x} \\
\frac{x^3}{x^{1/2}} &= \frac{100}{2} \\
x^{5/2} &= 50 \\
x &= (50)^{2/5} = 4.78176.
\end{aligned}
$$

The x coordinate of the point of intersection is about 4.782. We use either formula to find the y-coordinate:

$$y = 100\sqrt{4.78176} = 218.672,$$

or

$$y = 2(4.78176)^3 = 218.672.$$

The coordinates of the point of intersection are $(4.782, 218.672)$.

95. False

96. False

97. False

98. False

99. True

100. True

101. False

102. False

103. True

104. False

105. We have

$$\begin{aligned} 2^x &= 35 \\ &= 5 \cdot 7 \\ &= 2^r \cdot 2^s \\ &= 2^{r+s}, \end{aligned}$$

so $x = r + s$.

106. We have

$$\begin{aligned} 2^x &= 140 \\ &= 5 \cdot 7 \cdot 4 \\ &= 2^r \cdot 2^s \cdot 2^2 \\ &= 2^{r+s+2}, \end{aligned}$$

so $x = r + s + 2$.

107. We have

$$\begin{aligned} 5^x &= 32 \\ (2^a)^x &= 2^5 \\ 2^{ax} &= 2^5 \\ ax &= 5 \\ x &= \frac{5}{a}. \end{aligned}$$

108. We have

$$\begin{aligned} 7^x &= \frac{1}{8} \\ \left(2^b\right)^x &= \frac{1}{2^3} \\ 2^{bx} &= 2^{-3} \\ bx &= -3 \\ x &= \frac{-3}{b}. \end{aligned}$$

109. We have

$$\begin{aligned} 25^x &= 64 \\ \left(5^2\right)^x &= 64 \end{aligned}$$

$$\begin{aligned}
5^{2x} &= 64 \\
\left(2^a\right)^{2x} &= 64 \\
2^{2ax} &= 2^6 \\
2ax &= 6 \\
x &= \frac{3}{a}.
\end{aligned}$$

110. We have

$$\begin{aligned}
14^x &= 16 \\
(2 \cdot 7)^x &= 16 \\
\left(2 \cdot 2^b\right)^x &= 16 \\
\left(2^{b+1}\right)^x &= 16 \\
2^{(b+1)x} &= 2^4 \\
(b+1)x &= 4 \\
x &= \frac{4}{b+1}.
\end{aligned}$$

111. We have

$$\begin{aligned}
5^x &= 7 \\
\left(2^a\right)^x &= 2^b \\
2^{ax} &= 2^b \\
ax &= b \\
x &= \frac{b}{a}.
\end{aligned}$$

112.

$$\begin{aligned}
0.4^x &= 49 \\
\left(\frac{2}{5}\right)^x &= 7^2 \\
\left(2 \cdot 5^{-1}\right)^x &= 7^2 \\
\left(2\left(2^a\right)^{-1}\right)^x &= 7^2 \\
\left(2\left(2^{-a}\right)\right)^x &= 7^2 \\
\left(2^{-a+1}\right)^x &= \left(2^b\right)^2 \\
2^{(1-a)x} &= 2^{2b} \\
(1-a)x &= 2b \\
x &= \frac{2b}{1-a}.
\end{aligned}$$