FIND SOLUTIONS ON NEXT PAGE

CHAPTER FOUR

Solutions for Section 4.1

Exercises

1. The statement is equivalent to $19 = 10^{1.279}$.

2. The statement is equivalent to $4 = 10^{0.602}$.

3. The statement is equivalent to $26 = e^{3.258}$.

4. The statement is equivalent to $0.646 = e^{-0.437}$.

5. The statement is equivalent to $P = 10^t$.

6. The statement is equivalent to $q = e^z$.

7. The statement is equivalent to $8 = \log 100{,}000{,}000$.

8. The statement is equivalent to $-4 = \ln(0.0183)$.

9. The statement is equivalent to $v = \log \alpha$.

10. The statement is equivalent to $a = \ln b$.

11. We are solving for an exponent, so we use logarithms. We can use either the common logarithm or the natural logarithm. Since $2^3 = 8$ and $2^4 = 16$, we know that x must be between 3 and 4. Using the log rules, we have

$$\begin{aligned} 2^x &= 11 \\ \log(2^x) &= \log(11) \\ x\log(2) &= \log(11) \\ x &= \frac{\log(11)}{\log(2)} = 3.459. \end{aligned}$$

If we had used the natural logarithm, we would have

$$x = \frac{\ln(11)}{\ln(2)} = 3.459.$$

12. We are solving for an exponent, so we use logarithms. We can use either the common logarithm or the natural logarithm. Using the log rules, we have

$$\begin{aligned} 1.45^x &= 25 \\ \log(1.45^x) &= \log(25) \\ x\log(1.45) &= \log(25) \\ x &= \frac{\log(25)}{\log(1.45)} = 8.663. \end{aligned}$$

If we had used the natural logarithm, we would have

$$x = \frac{\ln(25)}{\ln(1.45)} = 8.663.$$

13. We are solving for an exponent, so we use logarithms. Since the base is the number e, it makes the most sense to use the natural logarithm. Using the log rules, we have

$$\begin{aligned} e^{0.12x} &= 100 \\ \ln(e^{0.12x}) &= \ln(100) \\ 0.12x &= \ln(100) \\ x &= \frac{\ln(100)}{0.12} = 38.376. \end{aligned}$$

14. We begin by dividing both sides by 22 to isolate the exponent:

$$\frac{10}{22} = (0.87)^q.$$

We then take the log of both sides and use the rules of logs to solve for q:

$$\begin{aligned}
\log\frac{10}{22} &= \log(0.87)^q \\
\log\frac{10}{22} &= q\log(0.87) \\
q &= \frac{\log\frac{10}{22}}{\log(0.87)} = 5.662.
\end{aligned}$$

15. We begin by dividing both sides by 17 to isolate the exponent:

$$\frac{48}{17} = (2.3)^w.$$

We then take the log of both sides and use the rules of logs to solve for w:

$$\begin{aligned}
\log\frac{48}{17} &= \log(2.3)^w \\
\log\frac{48}{17} &= w\log(2.3) \\
w &= \frac{\log\frac{48}{17}}{\log(2.3)} = 1.246.
\end{aligned}$$

16. We take the log of both sides and use the rules of logs to solve for t:

$$\begin{aligned}
\log\frac{2}{7} &= \log(0.6)^{2t} \\
\log\frac{2}{7} &= 2t\log(0.6) \\
\frac{\log\frac{2}{7}}{\log(0.6)} &= 2t \\
t &= \frac{\frac{\log\frac{2}{7}}{\log(0.6)}}{2} = 1.226.
\end{aligned}$$

17. We take the log of both sides and use the rules of logs to solve for m:

$$\begin{aligned}
\log(0.00012) &= \log(0.001)^{m/2} \\
\log(0.00012) &= \frac{m}{2}\log(0.001) \\
\log(0.00012) &= \frac{m}{2}(-3) \\
\frac{\log(0.00012)}{-3} &= \frac{m}{2} \\
m &= 2\frac{\log(0.00012)}{-3} = 2.614.
\end{aligned}$$

19. To do these problems, keep in mind that we are looking for a power of 10. For example, $\log 10{,}000$ is asking for the power of 10 which will give 10,000. Since $10^4 = 10{,}000$, we know that $\log 10{,}000 = 4$.

(a) Since $1 = 10^0$, $\log 1 = 0$.
(b) Since $0.1 = \frac{1}{10} = 10^{-1}$, we know that $\log 0.1 = \log 10^{-1} = -1$.
(c) In this problem, we can use the identity $\log 10^N = N$. So $\log 10^0 = 0$. We can check this by observing that $10^0 = 1$, similar to what we saw in (a), that $\log 1 = 0$.
(d) To find the $\log \sqrt{10}$ we need to recall that $\sqrt{10} = 10^{1/2}$. Now we can use our identity and say $\log \sqrt{10} = \log 10^{1/2} = \dfrac{1}{2}$.
(e) Using the identity, we get $\log 10^5 = 5$.
(f) Using the identity, we get $\log 10^2 = 2$.
(g) $\log \dfrac{1}{\sqrt{10}} = \log 10^{-1/2} = -\dfrac{1}{2}$

For the last three problems, we'll use the identity $10^{\log N} = N$.

(h) $10^{\log 100} = 100$
(i) $10^{\log 1} = 1$
(j) $10^{\log 0.01} = 0.01$

20. **(a)** Since $1 = e^0$, $\ln 1 = 0$.
(b) Using the identity $\ln e^N = N$, we get $\ln e^0 = 0$. Or we could notice that $e^0 = 1$, so using part (a), $\ln e^0 = \ln 1 = 0$.
(c) Using the identity $\ln e^N = N$, we get $\ln e^5 = 5$.
(d) Recall that $\sqrt{e} = e^{1/2}$. Using the identity $\ln e^N = N$, we get $\ln \sqrt{e} = \ln e^{1/2} = \dfrac{1}{2}$.
(e) Using the identity $e^{\ln N} = N$, we get $e^{\ln 2} = 2$.
(f) Since $\dfrac{1}{\sqrt{e}} = e^{-1/2}$, $\ln \dfrac{1}{\sqrt{e}} = \ln e^{-1/2} = -\dfrac{1}{2}$

Problems

21. See Table 4.1. From the table, we see that the value of 10^n draws quite close to 3000 as n draws close to 3.47712, and so we estimate that $\log 3000 \approx 3.47712$. Using a calculator, we see that $\log 3000 = 3.4771212\ldots$.

Table 4.1

n	3	3.5	3.48	3.477	3.4771	3.47712
10^n	1000	3162.278	3019.952	2999.163	2999.853	2999.991

22. $10^{1.3} \approx 20$ tells us that $\log 20 \approx 1.3$. Using one of the properties of logarithms, we can find $\log 200$:

$$\log 200 = \log(10 \cdot 20) = \log 10 + \log 20 \approx 1 + 1.3 = 2.3.$$

23. **(a)** $\log(\log 10) = \log 1 = 0$.
(b) Substituting 10^2 for 100 we have

$$\sqrt{\log 100} - \log\sqrt{100} = \sqrt{\log 10^2} - \log\sqrt{10^2}$$

Since $\log 10^2 = 2$ and $\sqrt{10^2} = 10$ we have

$$\sqrt{\log 10^2} - \log \sqrt{10^2} = \sqrt{2} - \log 10$$

But $\log 10 = 1$, so

$$\sqrt{2} - \log 10 = \sqrt{2} - 1.$$

(c) We will first simplify the expression $\sqrt{10}\sqrt[3]{10}\sqrt[5]{10}$ by using exponents instead of radicals:

$$\begin{aligned}\sqrt{10}\sqrt[3]{10}\sqrt[5]{10} &= 10^{\frac{1}{2}} \cdot 10^{\frac{1}{3}} \cdot 10^{\frac{1}{5}} \\ &= 10^{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}} \quad \text{(using an exponent rule)} \\ &= 10^{\frac{15+10+6}{30}} \quad \text{(finding an LCD)} \\ &= 10^{31/30}.\end{aligned}$$

Thus,

$$\log \sqrt{10}\sqrt[3]{10}\sqrt[5]{10} = \log 10^{31/30} = \frac{31}{30}.$$

(d)

$$\begin{aligned}1000^{\log 3} &= (10^3)^{\log 3} \\ &= (10^{\log 3})^3 \quad \text{(using an exponent rule)} \\ &= 3^3 \quad \text{(definition of } \log 3\text{)} \\ &= 27.\end{aligned}$$

(e)

$$\begin{aligned}0.01^{\log 2} &= \left(\frac{1}{100}\right)^{\log 2} \\ &= (10^{-2})^{\log 2} \\ &= (10^{\log 2})^{-2} \\ &= 2^{-2} = \frac{1}{4}.\end{aligned}$$

(f)

$$\begin{aligned}\frac{1}{\log \frac{1}{\log \sqrt[10]{10}}} &= \frac{1}{\log \frac{1}{\log 10^{1/10}}} \\ &= \frac{1}{\log \frac{1}{(1/10)}} \quad \text{(because } \log 10^{1/10} = \tfrac{1}{10}\text{)} \\ &= \frac{1}{\log 10} \quad \text{(since } \tfrac{1}{1/10} = 10\text{)} \\ &= 1 \quad \text{(because } \log 10 = 1\text{)}\end{aligned}$$

24. (a)

$$\begin{aligned}\log 100^x &= \log(10^2)^x \\ &= \log 10^{2x}.\end{aligned}$$

Since $\log 10^N = N$, then

$$\log 10^{2x} = 2x.$$

(b)

$$\begin{aligned} 1000^{\log x} &= (10^3)^{\log x} \\ &= (10^{\log x})^3 \end{aligned}$$

Since $10^{\log x} = x$ we know that

$$(10^{\log x})^3 = (x)^3 = x^3.$$

(c)

$$\begin{aligned} \log 0.001^x &= \log\left(\frac{1}{1000}\right)^x \\ &= \log(10^{-3})^x \\ &= \log 10^{-3x} \\ &= -3x. \end{aligned}$$

25. **(a)** Using the identity $\ln e^N = N$, we get $\ln e^{2x} = 2x$.
(b) Using the identity $e^{\ln N} = N$, we get $e^{\ln(3x+2)} = 3x + 2$.
(c) Since $\dfrac{1}{e^{5x}} = e^{-5x}$, we get $\ln\left(\dfrac{1}{e^{5x}}\right) = \ln e^{-5x} = -5x$.
(d) Since $\sqrt{e^x} = (e^x)^{1/2} = e^{\frac{1}{2}x}$, we have $\ln\sqrt{e^x} = \ln e^{\frac{1}{2}x} = \frac{1}{2}x$.

26. **(a)** True.
(b) False. $\frac{\log A}{\log B}$ cannot be rewritten.
(c) False. $\log A \log B = \log A \cdot \log B$, not $\log A + \log B$.
(d) True.
(e) True. $\sqrt{x} = x^{1/2}$ and $\log x^{1/2} = \frac{1}{2}\log x$.
(f) False. $\sqrt{\log x} = (\log x)^{1/2}$.

27. **(a)** False. It is true that $\ln(ab)^t = t\ln(ab)$.
(b) True. We have $\ln(1/a) = \ln 1 - \ln a = 0 - \ln a = -\ln a$.
(c) False. It is true that $\ln(a) + \ln(b) = \ln(a \cdot b)$.
(d) False. It is true that $\ln a - \ln b = \ln(a/b)$.

28. **(a)** $\log AB = \log A + \log B = x + y$
(b) $\log(A^3 \cdot \sqrt{B}) = \log A^3 + \log\sqrt{B} = 3\log A + \log B^{\frac{1}{2}} = 3\log A + \frac{1}{2}\log B = 3x + \frac{1}{2}y$
(c) $\log(A - B) = \log(10^x - 10^y)$ because $A = 10^{\log A} = 10^x$ and $B = 10^{\log B} = 10^y$, and this can't be further simplified.
(d) $\dfrac{\log A}{\log B} = \dfrac{x}{y}$
(e) $\log\left(\dfrac{A}{B}\right) = \log A - \log B = x - y$
(f) $AB = 10^x \cdot 10^y = 10^{x+y}$

29. **(a)** Since $p = \log m$, we have $m = 10^p$.
(b) Since $q = \log n$, we have $n = 10^q$, and so

$$n^3 = (10^q)^3 = 10^{3q}.$$

(c) By parts (a) and (b), we have

$$\begin{aligned} \log(mn^3) &= \log(10^p \cdot 10^{3q}) \\ &= \log(10^{p+3q}) \end{aligned}$$

Using the identity $\log 10^N = N$, we have

$$\log(mn^3) = p + 3q.$$

(d) Since $\sqrt{m} = m^{1/2}$,

$$\log \sqrt{m} = \log m^{1/2}.$$

Using the identity $\log a^b = b \cdot \log a$ we have

$$\log m^{1/2} = \frac{1}{2} \log m.$$

Since $p = \log m$

$$\begin{aligned}\frac{1}{2} \log m &= \frac{1}{2} p, \\ \log \sqrt{m} &= \frac{p}{2}.\end{aligned}$$

30. **(a)** We have $\ln(nm^4) = \ln n + 4 \ln m = q + 4p$.
(b) We have $\ln(1/n) = \ln 1 - \ln n = 0 - \ln n = -q$.
(c) We have $(\ln m)/(\ln n) = p/q$.
(d) We have $\ln(n^3) = 3 \ln n = 3q$.

32. **(a)** The initial value of Q is 10. The quantity is decaying at a continuous rate of 15% per time unit.
(b) We see on the graph that $Q = 2$ at approximately $t = 10.5$. We can use graphing technology to estimate t as accurately as we like.
(c) We substitute $Q = 2$ and use the natural logarithm to solve for t:

$$\begin{aligned}Q &= 10e^{-0.15t} \\ 2 &= 10e^{-0.15t} \\ 0.2 &= e^{-0.15t} \\ \ln(0.2) &= -0.15t \\ t &= \frac{\ln(0.2)}{-0.15} = 10.730.\end{aligned}$$

33. To find a formula for S, we find the points labeled (x_0, y_1) and (x_1, y_0) in Figure 4.1. We see that $x_0 = 4$ and that $y_1 = 27$. From the graph of R, we see that

$$y_0 = R(4) = 5.1403(1.1169)^4 = 8.$$

To find x_1 we use the fact that $R(x_1) = 27$:

$$\begin{aligned}5.1403(1.1169)^{x_1} &= 27 \\ 1.1169^{x_1} &= \frac{27}{5.1403} \\ x_1 &= \frac{\log(27/5.1403)}{\log 1.1169} \\ &= 15.\end{aligned}$$

We have $S(4) = 27$ and $S(15) = 8$. Using the ratio method, we have

$$\begin{aligned}\frac{ab^{15}}{ab^4} &= \frac{S(15)}{S(4)}\\ b^{11} &= \frac{8}{27}\\ b &= \left(\frac{8}{27}\right)^{1/11} \approx 0.8953.\end{aligned}$$

Now we can solve for a:

$$\begin{aligned}a(0.8953)^4 &= 27\\ a &= \frac{27}{(0.8953)^4}\\ &\approx 42.0207.\end{aligned}$$

so $S(x) = 42.0207(0.8953)^x$.

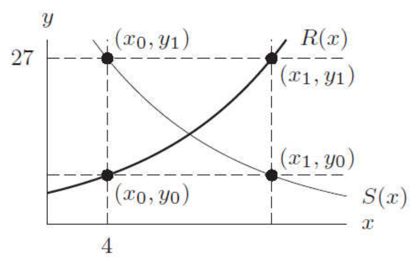


Figure 4.1

34. Using the log rules, we have

$$\begin{aligned}5(1.031)^x &= 8\\ 1.031^x &= \frac{8}{5}\\ \log(1.031)^x &= \log\frac{8}{5}\\ x\log 1.031 &= \log\frac{8}{5} = \log 1.6\\ x &= \frac{\log 1.6}{\log 1.031}.\end{aligned}$$

Checking the answer with a calculator, we get

$$x = \frac{\log 1.6}{\log 1.031} \approx 15.395 \qquad \text{and} \qquad 5(1.031)^{15.4} \approx 8.001.$$

37. First, we isolate the power on one side of the equation:

$$\begin{aligned} 3 \cdot 2^x &= 17 \\ 2^x &= \frac{17}{3}. \end{aligned}$$

Taking the log of both sides of the equation gives

$$\begin{aligned} \log\left(2^x\right) &= \log\frac{17}{3} \\ x \cdot \log 2 &= \log\frac{17}{3} \\ x &= \frac{\log(17/3)}{\log 2} \qquad \text{(dividing by log 2).} \end{aligned}$$

38. Taking natural logs, we get

$$\begin{aligned} e^{x+4} &= 10 \\ \ln e^{x+4} &= \ln 10 \\ x+4 &= \ln 10 \\ x &= \ln 10 - 4 \end{aligned}$$

39. Taking natural logs, we get

$$\begin{aligned} e^{x+5} &= 7\cdot 2^x \\ \ln e^{x+5} &= \ln(7\cdot 2^x) \\ x+5 &= \ln 7 + \ln 2^x \\ x+5 &= \ln 7 + x\ln 2 \\ x - x\ln 2 &= \ln 7 - 5 \\ x(1-\ln 2) &= \ln 7 - 5 \\ x &= \frac{\ln 7 - 5}{1-\ln 2} \end{aligned}$$

40. Taking logs and using the log rules:

$$\begin{aligned} \log b^x &= \log c \\ x\log b &= \log c \\ x &= \frac{\log c}{\log b}. \end{aligned}$$

41. Taking logs and using the log rules:

$$\begin{aligned} \log(ab^x) &= \log c \\ \log a + \log b^x &= \log c \\ \log a + x\log b &= \log c \\ x\log b &= \log c - \log a \\ x &= \frac{\log c - \log a}{\log b}. \end{aligned}$$

42. Taking logs and using the log rules

$$\begin{aligned} \log(Pa^x) &= \log(Qb^x) \\ \log P + \log a^x &= \log Q + \log b^x \\ \log P + x\log a &= \log Q + x\log b \\ x\log a - x\log b &= \log Q - \log P \\ x(\log a - \log b) &= \log Q - \log P \\ x &= \frac{\log Q - \log P}{\log a - \log b}. \end{aligned}$$

43. Take natural logs and use the log rules

$$\begin{aligned} \ln(Pe^{kx}) &= \ln Q \\ \ln P + \ln(e^{kx}) &= \ln Q \\ \ln P + kx &= \ln Q \\ kx &= \ln Q - \ln P \\ x &= \frac{\ln Q - \ln P}{k}. \end{aligned}$$

44.

$$
\begin{aligned}
121e^{-0.112t} &= 88\\
e^{-0.112t} &= \frac{88}{121}\\
\ln e^{-0.112t} &= \ln(\frac{88}{121})\\
-0.112t &= \ln(\frac{88}{121})\\
t &= \frac{\ln(88/121)}{-0.112}.
\end{aligned}
$$

45.

$$
\begin{aligned}
58e^{4t+1} &= 30\\
e^{4t+1} &= \frac{30}{58}\\
\ln e^{4t+1} &= \ln(\frac{30}{58})\\
4t+1 &= \ln(\frac{30}{58})\\
t &= \frac{1}{4}\left(\ln(\frac{30}{58})-1\right).
\end{aligned}
$$

47.

$$
\begin{aligned}
44e^{0.15t} &= 50(1.2)^t\\
\ln(44e^{0.15t}) &= \ln(50(1.2)^t)\\
\ln 44 + \ln e^{0.15t} &= \ln 50 + \ln 1.2^t\\
\ln 44 + 0.15t &= \ln 50 + t\ln 1.2\\
0.15t - (\ln 1.2)t &= \ln 50 - \ln 44\\
(0.15 - \ln 1.2)t &= \ln 50 - \ln 44\\
t &= \frac{\ln 50 - \ln 44}{0.15 - \ln 1.2}.
\end{aligned}
$$

48. Using $\log a - \log b = \log\left(\frac{a}{b}\right)$ we can rewrite the left side of the equation to read

$$\log\left(\frac{1-x}{1+x}\right) = 2.$$

This logarithmic equation can be rewritten as

$$10^2 = \frac{1-x}{1+x},$$

since if $\log a = b$ then $10^b = a$. Multiplying both sides of the equation by $(1+x)$ yields

$$\begin{aligned} 10^2(1+x) &= 1-x \\ 100+100x &= 1-x \\ 101x &= -99 \\ x &= -\frac{99}{101} \end{aligned}$$

Check your answer:

$$\begin{aligned} \log\left(1-\frac{-99}{101}\right) - \log\left(1+\frac{-99}{101}\right) &= \log\left(\frac{101+99}{101}\right) - \log\left(\frac{101-99}{101}\right) \\ &= \log\left(\frac{200}{101}\right) - \log\left(\frac{2}{101}\right) \\ &= \log\left(\frac{200}{101}/\frac{2}{101}\right) \\ &= \log 100 = 2 \end{aligned}$$

49. We have $\log(2x+5)\cdot\log(9x^2) = 0$.
In order for this product to equal zero, we know that one or both terms must be equal to zero. Thus, we will set each of the factors equal to zero to determine the values of x for which the factors will equal zero. We have

$$\begin{aligned} \log(2x+5) &= 0 \\ 2x+5 &= 1 \\ 2x &= -4 \\ x &= -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \log(9x^2) &= 0 \\ 9x^2 &= 1 \\ x^2 &= \frac{1}{9} \\ x &= \frac{1}{3} \text{ or } x = -\frac{1}{3}. \end{aligned}$$

Checking and substituting back into the original equation, we see that the three solutions work. Thus our solutions are $x = -2, \frac{1}{3}$, or $-\frac{1}{3}$.

51. **(a)** We combine like terms and then use properties of logs.

$$\begin{aligned} e^{2x}+e^{2x} &= 1 \\ 2e^{2x} &= 1 \\ e^{2x} &= 0.5 \\ 2x &= \ln(0.5) \\ x &= \frac{\ln(0.5)}{2} = -0.347. \end{aligned}$$

(b) We combine like terms and then use properties of logs.

$$\begin{aligned}
2e^{3x} + e^{3x} &= b \\
3e^{3x} &= b \\
e^{3x} &= \frac{b}{3} \\
3x &= \ln(b/3) \\
x &= \frac{\ln(b/3)}{3}.
\end{aligned}$$

52. Here are some observations:

- $A < 1$ means $\log A < 0$.
- $B > 1$ means $\log B > 0$.
- Since $AB < 1$, we know that $\log(AB) < 0$, so $\log A + \log B < 0$.
- Since $\log B > 0$, the sum $\log A + \log B$ is greater than $\log A$.
- Since $\log(B^A) = A \log B$, and since both A and $\log B$ are positive, we know that $\log(B^A) > 0$.
- Since $A < 1$, we know that the product $A \log B$ is less than $\log B$.

Putting this all together, we have

$$\log A < \log A + \log B < 0 < \log(B^A) < \log B.$$

53. Since $\log A < 0$, we know that $0 < A < 1$. Since $\log B > 1$, we know that $B > 10$, so $B^2 > 100$. We know that $0 < \log AB < 1$, so $1 < AB < 10$. Since $A^2B^2 = (AB)^2$, this means that $1 < (AB)^2 < 100$. Putting all this together, we have

$$0 < A < 1 < A^2B^2 < 100 < B^2.$$

54. Solving:

$$\begin{aligned}
11 \cdot 3^x &= 5 \cdot 7^x \\
\frac{11}{5} &= \frac{7^x}{3^x} = \left(\frac{7}{3}\right)^x \\
\log \frac{11}{5} &= \log \left(\frac{7}{3}\right)^x \\
\log \frac{11}{5} &= x \log \frac{7}{3} \\
x &= \frac{\log \frac{11}{5}}{\log \frac{7}{3}}
\end{aligned}$$

Notice that, using log rules, we have

$$x = \frac{\log \frac{11}{5}}{\log \frac{7}{3}} = \frac{\log 11 - \log 5}{\log 7 - \log 3},$$

and so the first student's answer is the same as the third's. By multiplying this fraction by $\frac{-1}{-1}$, we have

$$x = \frac{-(\log 11 - \log 5)}{-(\log 7 - \log 3)} = \frac{\log 5 - \log 11}{\log 3 - \log 7}.$$

But $\log 5 - \log 11 = \log \frac{5}{11}$ and $\log 3 - \log 7 = \log \frac{3}{7}$ so

$$x = \frac{\log \frac{5}{11}}{\log \frac{3}{7}}.$$

The second student's answer is the same as the other two! All three are correct.

Solutions for Section 4.2

Exercises

1. We have $a = 230$, $b = 1.182$, $r = b - 1 = 18.2\%$, and $k = \ln b = 0.1672 = 16.72\%$.

2. We have $a = 0.181$, $b = e^{0.775} = 2.1706$, $r = b - 1 = 1.1706 = 117.06\%$, and $k = 0.775 = 77.5\%$.

3. We have $a = 0.81$, $b = 2$, $r = b - 1 = 1 = 100\%$, and $k = \ln 2 = 0.6931 = 69.31\%$.

4. Writing this as $Q = 5 \cdot (2^{\frac{1}{8}})^t$, we have $a = 5$, $b = 2^{\frac{1}{8}} = 1.0905$, $r = b - 1 = 0.0905 = 9.05\%$, and $k = \ln b = 0.0866 = 8.66\%$.

5. Writing this as $Q = 12.1(10^{-0.11})^t$, we have $a = 12.1$, $b = 10^{-0.11} = 0.7762$, $r = b - 1 = -22.38\%$, and $k = \ln b = -25.32\%$.

6. We can rewrite this as

$$\begin{aligned} Q &= 40e^{t/12-5/12} \\ &= 40e^{-5/12}\left(e^{1/12}\right)^t. \end{aligned}$$

We have $a = 40e^{-5/12} = 26.3696$, $b = e^{1/12} = 1.0869$, $r = b - 1 = 8.69\%$, and $k = 1/12 = 8.333\%$.

7. We can use exponent rules to place this in the form ae^{kt}:

$$\begin{aligned} 2e^{(1-3t/4)} &= 2e^1 e^{-3t/4} \\ &= (2e)e^{-\frac{3}{4}t}, \end{aligned}$$

so $a = 2e = 5.4366$ and $k = -3/4$. To find b and r, we have

$$\begin{aligned} b &= e^k = e^{-3/4} = 0.4724 \\ r &= b - 1 = -0.5276 = -52.76\%. \end{aligned}$$

8. We can use exponent rules to place this in the form ab^t:

$$\begin{aligned} 2^{-(t-5)/3} &= 2^{-(1/3)(t-5)} \\ &= 2^{5/3-(1/3)t} \\ &= 2^{5/3} \cdot 2^{-(1/3)t} \\ &= 2^{5/3} \cdot \left(2^{-1/3}\right)^t, \end{aligned}$$

so $a = 2^{5/3} = 3.1748$ and $b = 2^{(-1/3)} = 0.7937$. To find k and r, we use the fact that:

$$\begin{aligned} r &= b - 1 = -0.2063 = -20.63\% \\ k &= \ln b = -0.2310 = -23.10\%. \end{aligned}$$

9. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 4 \cdot 7^t$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k \cdot 0} &= 4 \cdot 7^0 \\ a \cdot 1 &= 4 \cdot 1 \\ a &= 4. \end{aligned}$$

Thus, we have $4e^{kt} = 4 \cdot 7^t$, and we solve for k:

$$\begin{aligned} 4e^{kt} &= 4 \cdot 7^t \\ e^{kt} &= 7^t \\ \left(e^k\right)^t &= 7^t \\ e^k &= 7 \\ \ln e^k &= \ln 7 \\ k &= \ln 7 \approx 1.946. \end{aligned}$$

Therefore, the equation is $Q = 4e^{1.946t}$.

10. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 2 \cdot 3^t$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k \cdot 0} &= 2 \cdot 3^0 \\ a \cdot 1 &= 2 \cdot 1 \\ a &= 2. \end{aligned}$$

Thus, we have $2e^{kt} = 2 \cdot 3^t$, and we solve for k:

$$\begin{aligned} 2e^{kt} &= 2 \cdot 3^t \\ e^{kt} &= 3^t \\ \left(e^k\right)^t &= 3^t \\ e^k &= 3 \\ \ln e^k &= \ln 3 \\ k &= \ln 3 \approx 1.099. \end{aligned}$$

Therefore, the equation is $Q = 2e^{1.099t}$.

11. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 4 \cdot 8^{1.3t}$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k \cdot 0} &= 4 \cdot 8^0 \\ a \cdot 1 &= 4 \cdot 1 \\ a &= 4. \end{aligned}$$

Thus, we have $4e^{kt} = 4 \cdot 8^{1.3t}$, and we solve for k:

$$\begin{aligned} 4e^{kt} &= 4 \cdot 8^{1.3t} \\ e^{kt} &= \left(8^{1.3}\right)^t \\ \left(e^k\right)^t &= 14.929^t \\ e^k &= 14.929 \\ \ln e^k &= \ln 14.929 \\ k &= 2.703. \end{aligned}$$

Therefore, the equation is $Q = 4e^{2.703t}$.

12. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 973 \cdot 6^{2.1t}$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k \cdot 0} &= 973 \cdot 6^0 \\ a \cdot 1 &= 973 \cdot 1 \\ a &= 973. \end{aligned}$$

Thus, we have $973e^{kt} = 973 \cdot 6^{2.1t}$, and we solve for k:

$$\begin{aligned}
973e^{kt} &= 973 \cdot 6^{2.1t}\\
e^{kt} &= \left(6^{2.1}\right)^t\\
\left(e^k\right)^t &= 43.064^t\\
e^k &= 43.064\\
\ln e^k &= \ln 43.064\\
k &= 3.763.
\end{aligned}$$

Therefore, the equation is $Q = 973e^{3.763t}$.

13. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is 7.

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = 4e^{7t}$. At $t = 0$, we can solve for a:

$$\begin{aligned}
ab^0 &= 4e^{7\cdot 0}\\
a \cdot 1 &= 4 \cdot 1\\
a &= 4.
\end{aligned}$$

Thus, we have $4b^t = 4e^{7t}$, and we solve for b:

$$\begin{aligned}
4b^t &= 4e^{7t}\\
b^t &= e^{7t}\\
b^t &= \left(e^7\right)^t\\
b &= e^7 \approx 1096.633.
\end{aligned}$$

Therefore, the equation is $Q = 4 \cdot 1096.633^t$.

14. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is 0.7.

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = 0.3e^{0.7t}$. At $t = 0$, we can solve for a:

$$\begin{aligned}
ab^0 &= 0.3e^{0.7\cdot 0}\\
a \cdot 1 &= 0.3 \cdot 1\\
a &= 0.3.
\end{aligned}$$

Thus, we have $0.3b^t = 0.3e^{0.7t}$, and we solve for b:

$$\begin{aligned}
0.3b^t &= 0.3e^{0.7t}\\
b^t &= e^{0.7t}\\
b^t &= \left(e^{0.7}\right)^t\\
b &= e^{0.7} \approx 2.014.
\end{aligned}$$

Therefore, the equation is $Q = 0.3 \cdot 2.014^t$.

15. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is 0.03.

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = \frac{14}{5}e^{0.03t}$. At $t = 0$, we can solve for a:

$$\begin{aligned}
ab^0 &= \frac{14}{5}e^{0.03\cdot 0}\\
a \cdot 1 &= \frac{14}{5} \cdot 1\\
a &= \frac{14}{5}.
\end{aligned}$$

Thus, we have $\frac{14}{5}b^t = \frac{14}{5}e^{0.03t}$, and we solve for b:

$$\begin{aligned}\frac{14}{5}b^t &= \frac{14}{5}e^{0.03t}\\ b^t &= e^{0.03t}\\ b^t &= \left(e^{0.03}\right)^t\\ b &= e^{0.03} \approx 1.030.\end{aligned}$$

Therefore, the equation is $Q = \frac{14}{5} \cdot 1.030^t$.

16. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is -0.02.

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = e^{-0.02t}$. At $t = 0$, we can solve for a:

$$\begin{aligned}ab^0 &= e^{-0.02\cdot 0}\\ a\cdot 1 &= 1\\ a &= 1.\end{aligned}$$

Thus, we have $1b^t = e^{-0.02t}$, and we solve for b:

$$\begin{aligned}1b^t &= e^{-0.02t}\\ b^t &= e^{-0.02t}\\ b^t &= \left(e^{-0.02}\right)^t\\ b &= e^{-0.02} \approx 0.980.\end{aligned}$$

Therefore, the equation is $Q = 1(0.980)^t$.

17. We want $25e^{0.053t} = 25(e^{0.053})^t = ab^t$, so we choose $a = 25$ and $b = e^{0.053} = 1.0544$. The given exponential function is equivalent to the exponential function $y = 25(1.0544)^t$. The annual percent growth rate is 5.44% and the continuous percent growth rate per year is 5.3% per year.

18. We want $100e^{-0.07t} = 100(e^{-0.07})^t = ab^t$, so we choose $a = 100$ and $b = e^{-0.07} = 0.9324$. The given exponential function is equivalent to the exponential function $y = 100(0.9324)^t$. Since $1 - 0.9324 = 0.0676$, the annual percent decay rate is 6.76% and the continuous percent decay rate per year is 7% per year.

19. We want $6000(0.85)^t = ae^{kt} = a(e^k)^t$ so we choose $a = 6000$ and we find k so that $e^k = 0.85$. Taking logs of both sides, we have $k = \ln(0.85) = -0.1625$. The given exponential function is equivalent to the exponential function $y = 6000e^{-0.1625t}$. The annual percent decay rate is 15% and the continuous percent decay rate per year is 16.25% per year.

20. We want $5(1.12)^t = ae^{kt} = a(e^k)^t$ so we choose $a = 5$ and we find k so that $e^k = 1.12$. Taking logs of both sides, we have $k = \ln(1.12) = 0.1133$. The given exponential function is equivalent to the exponential function $y = 5e^{0.1133t}$. The annual percent growth rate is 12% and the continuous percent growth rate per year is 11.33% per year.

21. Let t be the doubling time, then the population is $2P_0$ at time t, so

$$\begin{aligned}2P_0 &= P_0e^{0.2t}\\ 2 &= e^{0.2t}\\ 0.2t &= \ln 2\\ t &= \frac{\ln 2}{0.2} \approx 3.466.\end{aligned}$$

22. Since the growth factor is $1.26 = 1 + 0.26$, the formula for the city's population, with an initial population of a and time t in years, is

$$P = a(1.26)^t.$$

The population doubles for the first time when $P = 2a$. Thus, we solve for t after setting P equal to $2a$ to give us the doubling time:

$$\begin{aligned}2a &= a(1.26)^t\\ 2 &= (1.26)^t\\ \log 2 &= \log(1.26)^t\\ \log 2 &= t\log(1.26)\\ t &= \frac{\log 2}{\log(1.26)} = 2.999.\end{aligned}$$

So the doubling time is about 3 years.

23. Since the growth factor is $1.027 = 1 + 0.027$, the formula for the bank account balance, with an initial balance of a and time t in years, is

$$B = a(1.027)^t.$$

The balance doubles for the first time when $B = 2a$. Thus, we solve for t after putting B equal to $2a$ to give us the doubling time:

$$\begin{aligned}2a &= a(1.027)^t\\ 2 &= (1.027)^t\\ \log 2 &= \log(1.027)^t\\ \log 2 &= t\log(1.027)\\ t &= \frac{\log 2}{\log(1.027)} = 26.017.\end{aligned}$$

So the doubling time is about 26 years.

24. Since the growth factor is 1.12, the formula for the company's profits, Π, with an initial annual profit of a and time t in years, is

$$\Pi = a(1.12)^t.$$

The annual profit doubles for the first time when $\Pi = 2a$. Thus, we solve for t after putting Π equal to $2a$ to give us the doubling time:

$$\begin{aligned}2a &= a(1.12)^t\\ 2 &= (1.12)^t\\ \log 2 &= \log(1.12)^t\\ \log 2 &= t\log(1.12)\\ t &= \frac{\log 2}{\log(1.12)} = 6.116.\end{aligned}$$

So the doubling time is about 6.1 years.

25. The growth factor for Tritium should be $1 - 0.05471 = 0.94529$, since it is decaying by 5.471% per year. Therefore, the decay equation starting with a quantity of a should be:

$$Q = a(0.94529)^t,$$

where Q is quantity remaining and t is time in years. The half life will be the value of t for which Q is $a/2$, or half of the initial quantity a. Thus, we solve the equation for $Q = a/2$:

$$\begin{aligned}\frac{a}{2} &= a(0.94529)^t\\ \frac{1}{2} &= (0.94529)^t\\ \log(1/2) &= \log(0.94529)^t\end{aligned}$$

$$\log(1/2) = t\log(0.94529)$$
$$t = \frac{\log(1/2)}{\log(0.94529)} = 12.320.$$

So the half-life is about 12.3 years.

26. The growth factor for Einsteinium-253 should be $1 - 0.03406 = 0.96594$, since it is decaying by 3.406% per day. Therefore, the decay equation starting with a quantity of a should be:

$$Q = a(0.96594)^t,$$

where Q is quantity remaining and t is time in days. The half life will be the value of t for which Q is $a/2$, or half of the initial quantity a. Thus, we solve the equation for $Q = a/2$:

$$\begin{aligned}
\frac{a}{2} &= a(0.96594)^t \\
\frac{1}{2} &= (0.96594)^t \\
\log(1/2) &= \log(0.96594)^t \\
\log(1/2) &= t\log(0.96594) \\
t &= \frac{\log(1/2)}{\log(0.96594)} = 20.002.
\end{aligned}$$

So the half-life is about 20 days.

27. Let $Q(t)$ be the mass of the substance at time t, and Q_0 be the initial mass of the substance. Since the substance is decaying at a continuous rate, we know that $Q(t) = Q_0e^{kt}$ where $k = -0.11$ (This is an 11% decay). So $Q(t) = Q_0e^{-0.11t}$. We want to know when $Q(t) = \frac{1}{2}Q_0$.

$$\begin{aligned}
Q_0e^{-0.11t} &= \frac{1}{2}Q_0 \\
e^{-0.11t} &= \frac{1}{2} \\
\ln e^{-0.11t} &= \ln\left(\frac{1}{2}\right) \\
-0.11t &= \ln\left(\frac{1}{2}\right) \\
t &= \frac{\ln\frac{1}{2}}{-0.11} \approx 6.301
\end{aligned}$$

So the half-life is 6.301 minutes.

Problems

28. We have

$$\begin{aligned}
V &= 2500\left(1 + \frac{3.25\%}{4}\right)^{4t} \\
&= 2500\left((1.008125)^4\right)^t \\
&= 2500 \cdot 1.03290^t,
\end{aligned}$$

so $a = 2500$ and $b = 1.03290$. We can also place this in the form ae^{kt} by using the fact that

$$k = \ln b = 0.03237,$$

so $V = 2500e^{0.03237t}$.

29. We have $g(-50) = 20$ and $g(120) = 70$. Using the ratio method, we have

$$\begin{aligned}\frac{ab^{120}}{ab^{-50}} &= \frac{g(120)}{g(-50)}\\ b^{170} &= \frac{70}{20}\\ b &= \left(\frac{70}{20}\right)^{1/170}\\ &\approx 1.0074.\end{aligned}$$

Now we can solve for a:

$$\begin{aligned}a(1.0074)^{-50} &= 20\\ a &= \frac{20}{(1.0074)^{-50}}\\ &\approx 28.9101.\end{aligned}$$

We can also place this in the form ae^{kt} by using the fact that

$$k = \ln b = \ln(1.0074) = 0.007369,$$

so $g(t) = 28.9101e^{0.007396t}$.

30. The starting value is $a = \$10{,}000$. After 5 years, the value halves to \$5000, so

$$\begin{aligned}10{,}000b^5 &= 5000\\ b^5 &= \frac{1}{2}\\ b &= \left(\frac{1}{2}\right)^{1/5} = 0.871.\end{aligned}$$

Thus, $V = 10{,}000 \cdot 0.871^t$.

31. **(a)** To find the annual growth rate, we need to find a formula which describes the population, $P(t)$, in terms of the initial population, a, and the annual growth factor, b. In this case, we know that $a = 11{,}000$ and $P(3) = 13{,}000$. But $P(3) = ab^3 = 11{,}000b^3$, so

$$\begin{aligned}13000 &= 11000b^3\\ b^3 &= \frac{13000}{11000}\\ b &= \left(\frac{13}{11}\right)^{\frac{1}{3}} \approx 1.05726.\end{aligned}$$

Since b is the growth factor, we know that, each year, the population is about 105.726% of what it had been the previous year, so it is growing at the rate of 5.726% each year.

(b) To find the continuous growth rate, we need a formula of the form $P(t) = ae^{kt}$ where $P(t)$ is the population after t years, a is the initial population, and k is the rate we are trying to determine. We know that $a = 11{,}000$ and, inthis case, that $P(3) = 11{,}000e^{3k} = 13{,}000$. Therefore,

$$\begin{aligned}e^{3k} &= \frac{13000}{11000}\\ \ln e^{3k} &= \ln\left(\frac{13}{11}\right)\\ 3k &= \ln(\frac{13}{11}) \qquad \text{(because } \ln e^{3k} = 3k\text{)}\\ k &= \frac{1}{3}\ln(\frac{13}{11}) \approx 0.05568.\end{aligned}$$

So our continuous annual growth rate is 5.568%.

(c) The annual growth rate, 5.726%, describes the actual percent increase in one year. The continuous annual growth rate, 5.568%, describes the percentage increase of the population at any given instant, and so should be a smaller number.

32. **(a)** Let $P(t) = P_0 b^t$ describe our population at the end of t years. Since P_0 is the initial population, and the population doubles every 15 years, we know that, at the end of 15 years, our population will be $2P_0$. But at the end of 15 years, our population is $P(15) = P_0 b^{15}$. Thus

$$\begin{aligned} P_0 b^{15} &= 2P_0 \\ b^{15} &= 2 \\ b &= 2^{\frac{1}{15}} \approx 1.04729 \end{aligned}$$

Since b is our growth factor, the population is, yearly, 104.729% of what it had been the previous year. Thus it is growing by 4.729% per year.

(b) Writing our formula as $P(t) = P_0 e^{kt}$, we have $P(15) = P_0 e^{15k}$. But we already know that $P(15) = 2P_0$. Therefore,

$$\begin{aligned} P_0 e^{15k} &= 2P_0 \\ e^{15k} &= 2 \\ \ln e^{15k} &= \ln 2 \\ 15k \ln e &= \ln 2 \\ 15k &= \ln 2 \\ k &= \frac{\ln 2}{15} \approx 0.04621. \end{aligned}$$

This tells us that we have a continuous annual growth rate of 4.621%.

33. We have $P = ab^t$ where $a = 5.2$ and $b = 1.031$. We want to find k such that

$$P = 5.2e^{kt} = 5.2(1.031)^t,$$

so

$$e^k = 1.031.$$

Thus, the continuous growth rate is $k = \ln 1.031 \approx 0.03053$, or 3.053% per year.

34. If 17% of the substance decays, then 83% of the original amount of substance, P_0, remains after 5 hours. So $P(5) = 0.83P_0$. If we use the formula $P(t) = P_0 e^{kt}$, then

$$P(5) = P_0 e^{5t}.$$

But $P(5) = 0.83P_0$, so

$$\begin{aligned} 0.83P_0 &= P_0 e^{k(5)} \\ 0.83 &= e^{5k} \\ \ln 0.83 &= \ln e^{5k} \\ \ln 0.83 &= 5k \\ k &= \frac{\ln 0.83}{5} \approx -0.037266. \end{aligned}$$

Having a formula $P(t) = P_0 e^{-0.037266t}$, we can find its half-life. This is the value of t for which $P(t) = \frac{1}{2}P_0$. To do this, we will solve

$$\begin{aligned} P_0 e^{-0.037266t} &= \frac{1}{2}P_0 \\ e^{-0.037266t} &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
\ln e^{-0.037266t} &= \ln \frac{1}{2} \\
-0.037266t &= \ln \frac{1}{2} \\
t &= \frac{\ln \frac{1}{2}}{-0.037266} \approx 18.583.
\end{aligned}$$

So the half-life of this substance is about 18.6 hours

35. We know the y-intercept is 0.8 and that the y-value doubles every 12 units. We can make a quick table and then plot points. See Table 4.2 and Figure 4.2.

Table 4.2

t	y
0	0.8
12	1.6
24	3.2
36	6.4
48	12.8
60	25.6

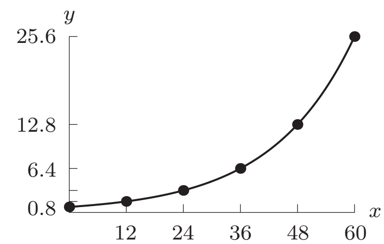


Figure 4.2

36. Get all expressions containing x on one side of the equation and everything else on the other side. To do this we divide both sides of the equation by 1.7 and by $(4.5)^x$.

$$\begin{aligned}
1.7(2.1)^{3x} &= 2(4.5)^x \\
\frac{(2.1)^{3x}}{(4.5)^x} &= \frac{2}{1.7}
\end{aligned}$$

We know that $(2.1)^{3x} = [(2.1)^3]^x = (9.261)^x$. Therefore,

$$\begin{aligned}
\frac{(9.261)^x}{(4.5)^x} &= \frac{2}{1.7} \\
\left(\frac{9.261}{4.5}\right)^x &= \frac{2}{1.7} \\
\log\left(\frac{9.261}{4.5}\right)^x &= \log\left(\frac{2}{1.7}\right) \\
x\log\left(\frac{9.261}{4.5}\right) &= \log\left(\frac{2}{1.7}\right) \\
x &= \frac{\log(2/1.7)}{\log(9.261/4.5)}.
\end{aligned}$$

37. Take logarithms:

$$\begin{aligned}
3^{(4\log x)} &= 5 \\
\log 3^{(4\log x)} &= \log 5 \\
(4\log x)\log 3 &= \log 5 \\
4\log x &= \frac{\log 5}{\log 3} \\
\log x &= \frac{\log 5}{4\log 3} \\
x &= 10^{(\log 5)/(4\log 3)}.
\end{aligned}$$

38. This equation cannot be solved analytically. Using a graphing calculator, we find $t \approx -6.166$ and $t \approx 61.916$ are solutions.

39. Since the variable is both in a power and in a linear expression, we cannot use logarithms to solve this equation. Instead, we graph the two functions and try to find the point of intersection. The graph of the functions $y = 12(1.221)^t$ and $y = t + 3$ in Figure 4.3 shows no point of intersection. The concavity of the graphs ensures that there is no point of intersection outside the window shown. Thus, there is no solution.

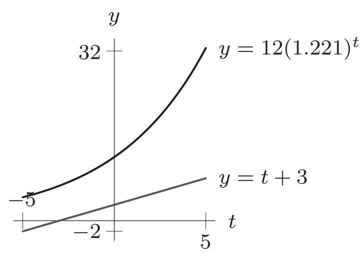


Figure 4.3

40. First rewrite $10e^{3t} - e = 2e^{3t}$ as

$$8e^{3t} = e.$$

Then take natural logs and use the log rules

$$\begin{aligned}
\ln(8e^{3t}) &= \ln e \\
\ln 8 + \ln e^{3t} &= 1 \\
\ln 8 + 3t &= 1 \\
3t &= 1 - \ln 8 \\
t &= \frac{1 - \ln 8}{3} = -0.360.
\end{aligned}$$

41. Using $\log a + \log b = \log(ab)$, we can rewrite the equation as

$$\begin{aligned}
\log(x(x-1)) &= \log 2 \\
x(x-1) &= 2 \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0 \\
x &= 2 \text{ or } -1
\end{aligned}$$

but $x \neq -1$ since $\log x$ is undefined at $x = -1$. Thus $x = 2$.

42. We have $P = ab^t$ and $2P = ab^{t+d}$. Using the algebra rules of exponents, we have

$$2P = ab^{t+d} = ab^t \cdot b^d = Pb^d.$$

Since P is nonzero, we can divide through by P, and we have

$$2 = b^d.$$

Notice that the time it takes an exponential growth function to double does not depend on the initial quantity a and does not depend on the time t. It depends only on the growth factor b.

43. **(a)** For a function of the form $N(t) = ae^{kt}$, the value of a is the population at time $t = 0$ and k is the continuous growth rate. So the continuous growth rate is $0.013 = 1.3\%$.

(b) In year $t = 0$, the population is $N(0) = a = 5.4$ million.

(c) We want to find t such that the population of 5.4 million triples to 16.2 million. So, for what value of t does $N(t) = 5.4e^{0.013t} = 16.2$?

$$\begin{aligned}
5.4e^{0.013t} &= 16.2 \\
e^{0.013t} &= 3 \\
\ln e^{0.013t} &= \ln 3 \\
0.013t &= \ln 3 \\
t &= \frac{\ln 3}{0.013} \approx 84.509
\end{aligned}$$

So the population will triple in approximately 84.5 years.

(d) Since $N(t)$ is in millions, we want to find t such that $N(t) = 0.000001$.

$$\begin{aligned}
5.4e^{0.013t} &= 0.000001 \\
e^{0.013t} &= \frac{0.000001}{5.4} \approx 0.000000185 \\
\ln e^{0.013t} &\approx \ln(0.000000185) \\
0.013t &\approx \ln(0.000000185) \\
t &\approx \frac{\ln(0.000000185)}{0.013} \approx -1192.455
\end{aligned}$$

According to this model, the population of Washington State was 1 person 1192.455 years ago. It is unreasonable to suppose the formula extends so far into the past.

45. Let $P = ab^t$ where P is the number of bacteria at time t hours since the beginning of the experiment. a is the number of bacteria we're starting with.

(a) Since the colony begins with 3 bacteria we have $a = 3$. Using the information that $P = 100$ when $t = 3$, we can solve the following equation for b:

$$P = 3b^t$$

$$\begin{aligned}100 &= 3b^3\\ \sqrt[3]{\frac{100}{3}} &= b\\ b &= \left(\frac{100}{3}\right)^{1/3} \approx 3.218\end{aligned}$$

Therefore, $P = 3(3.218)^t$.

(b) We want to find the value of t for which the population triples, going from three bacteria to nine. So we want to solve:

$$\begin{aligned}9 &= 3(3.218)^t\\ 3 &= (3.218)^t\\ \log 3 &= \log(3.218)^t\\ &= t\log(3.218).\end{aligned}$$

Thus,

$$t = \frac{\log 3}{\log(3.218)} \approx 0.940 \text{ hours}.$$

47. **(a)** If we let $y =$ the number of cases of sepsis in year t, then we have $y = ae^{kt}$. We find k using the fact that the number of cases doubles in 5 years:

$$\begin{aligned}y &= ae^{kt}\\ 2a &= ae^{k\cdot 5}\\ 2 &= e^{5k}\\ \ln 2 &= 5k\\ k &= \frac{\ln 2}{5} = 0.139.\end{aligned}$$

The number of cases of sepsis has been growing at a continuous rate of about 13.9% per year.

(b) We have:

$$\begin{aligned} y &= ae^{0.139t} \\ 3a &= ae^{0.139t} \\ 3 &= e^{0.139t} \\ \ln 3 &= 0.139t \\ t &= \frac{\ln 3}{0.139} = 7.904. \end{aligned}$$

It will take about 7.9 years for the number of cases to triple.

49. (a) Use $o(t)$ to describe the number of owls as a function of time. After 1 year, we see that the number of owls is 103% of 245, or $o(1) = 245(1.03)$. After 2 years, the population is 103% of that number, or $o(2) = (245(1.03)) \cdot 1.03 = 245(1.03)^2$. After t years, it is $o(t) = 245(1.03)^t$.

(b) We will use $h(t)$ to describe the number of hawks as a function of time. Since $h(t)$ doubles every 10 years, we know that its growth factor is constant and so it is an exponential function with a formula of the form $h(t) = ab^t$. In this case the initial population is 63 hawks, so $h(t) = 63b^t$. We are told that the population in 10 years is twice the current population, that is

$$63b^{10} = 126.$$

Thus,

$$\begin{aligned} b^{10} &= 2 \\ b &= 2^{1/10} \approx 1.072. \end{aligned}$$

The number of hawks as a function of time is

$$h(t) = 63(2^{1/10})^t = 63 \cdot 2^{t/10} \approx 63 \cdot (1.072)^t.$$

(c) Looking at Figure 4.4 we see that it takes about 34.2 years for the populations to be equal.

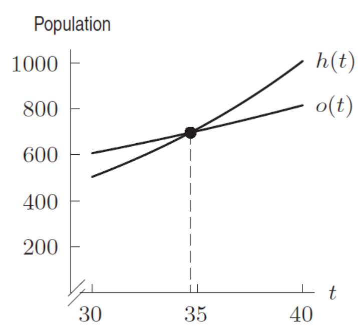


Figure 4.4

50. (a) Since $f(x)$ is exponential, its formula will be $f(x) = ab^x$. Since $f(0) = 0.5$,

$$f(0) = ab^0 = 0.5.$$

But $b^0 = 1$, so

$$\begin{aligned} a(1) &= 0.5 \\ a &= 0.5. \end{aligned}$$

We now know that $f(x) = 0.5b^x$. Since $f(1) = 2$, we have

$$\begin{aligned} f(1) = 0.5b^1 &= 2 \\ 0.5b &= 2 \\ b &= 4 \end{aligned}$$

So $f(x) = 0.5(4)^x$.

We will find a formula for $g(x)$ the same way.

$$g(x) = ab^x.$$

Since $g(0) = 4$,

$$\begin{aligned} g(0) = ab^0 &= 4 \\ a &= 4. \end{aligned}$$

Therefore,

$$g(x) = 4b^x.$$

We'll use $g(2) = \frac{4}{9}$ to get

$$\begin{aligned} g(2) = 4b^2 &= \frac{4}{9} \\ b^2 &= \frac{1}{9} \\ b &= \pm\frac{1}{3}. \end{aligned}$$

Since $b > 0$,

$$g(x) = 4\left(\frac{1}{3}\right)^x.$$

Since $h(x)$ is linear, its formula will be

$$h(x) = b + mx.$$

We know that b is the y-intercept, which is 2, according to the graph. Since the points $(a, a+2)$ and $(0, 2)$ lie on the graph, we know that the slope, m, is

$$\frac{(a+2)-2}{a-0} = \frac{a}{a} = 1,$$

so the formula is

$$h(x) = 2 + x.$$

(b) We begin with

$$\begin{aligned} f(x) &= g(x) \\ \frac{1}{2}(4)^x &= 4\left(\frac{1}{3}\right)^x. \end{aligned}$$

Since the variable is an exponent, we need to use logs, so

$$\begin{aligned} \log\left(\frac{1}{2}\cdot 4^x\right) &= \log\left(4\cdot\left(\frac{1}{3}\right)^x\right) \\ \log\frac{1}{2} + \log(4)^x &= \log 4 + \log\left(\frac{1}{3}\right)^x \\ \log\frac{1}{2} + x\log 4 &= \log 4 + x\log\frac{1}{3}. \end{aligned}$$

Now we will move all expressions containing the variable to one side of the equation:

$$x\log 4 - x\log\frac{1}{3} = \log 4 - \log\frac{1}{2}.$$

Factoring out x, we get

$$\begin{aligned} x(\log 4 - \log \frac{1}{3}) &= \log 4 - \log \frac{1}{2} \\ x \log\left(\frac{4}{1/3}\right) &= \log\left(\frac{4}{1/2}\right) \\ x \log 12 &= \log 8 \\ x &= \frac{\log 8}{\log 12}. \end{aligned}$$

This is the exact value of x. Note that $\frac{\log 8}{\log 12} \approx 0.837$, so $f(x) = g(x)$ when x is exactly $\frac{\log 8}{\log 12}$ or about 0.837.

(c) Since $f(x) = h(x)$, we want to solve

$$\frac{1}{2}(4)^x = x + 2.$$

The variable does not occur only as an exponent, so logs cannot help us solve this equation. Instead, we need to graph the two functions and note where they intersect. The points occur when $x \approx 1.378$ or $x \approx -1.967$.

51. Since the half-life of carbon-14 is $5{,}728$ years, and just a little more than 50% of it remained, we know that the man died nearly $5{,}700$ years ago. To obtain a more precise date, we need to find a formula to describe the amount of carbon-14 left in the man's body after t years. Since the decay is continuous and exponential, it can be described by $Q(t) = Q_0 e^{kt}$. We first find k. After $5{,}728$ years, only one-half is left, so

$$Q(5{,}728) = \frac{1}{2}Q_0.$$

Therefore,

$$\begin{aligned} Q(5{,}728) = Q_0 e^{5728k} &= \frac{1}{2}Q_0 \\ e^{5728k} &= \frac{1}{2} \\ \ln e^{5728k} &= \ln \frac{1}{2} \\ 5728k &= \ln \frac{1}{2} = \ln 0.5 \\ k &= \frac{\ln 0.5}{5728} \end{aligned}$$

So, $Q(t) = Q_0 e^{\frac{\ln 0.5}{5728} t}$.

If 46% of the carbon-14 has decayed, then 54% remains, so that $Q(t) = 0.54Q_0$.

$$\begin{aligned} Q_0 e^{\left(\frac{\ln 0.5}{5728}\right)t} &= 0.54Q_0 \\ e^{\left(\frac{\ln 0.5}{5728}\right)t} &= 0.54 \\ \ln e^{\left(\frac{\ln 0.5}{5728}\right)t} &= \ln 0.54 \\ \frac{\ln 0.5}{5728} t &= \ln 0.54 \\ t &= \frac{(\ln 0.54)\cdot(5728)}{\ln 0.5} = 5092.013 \end{aligned}$$

So the man died about 5092 years ago.

53. We have

$$Q = 0.1e^{-(1/2.5)t},$$

and need to find t such that $Q = 0.04$. This gives

$$\begin{aligned}
0.1e^{-\frac{t}{2.5}} &= 0.04\\
e^{-\frac{t}{2.5}} &= 0.4\\
\ln e^{-\frac{t}{2.5}} &= \ln 0.4\\
-\frac{t}{2.5} &= \ln 0.4\\
t &= -2.5\ln 0.4 \approx 2.291.
\end{aligned}$$

It takes about 2.3 hours for their BAC to drop to 0.04.

54. **(a)** The probability of failure within 6 months is

$$P(6) = 1 - e^{(-0.016)(6)} \approx 0.09154 = 9.154\%.$$

In order to find the probability of failure in the second six months, we must first find the probability of its failure in the first 12 months and then subtract the probability of failure in the first six months. The probability of failure within the first 12 months is

$$P(12) = 1 - e^{(-0.016)(12)} \approx 0.17469 = 17.469\%.$$

Therefore, the probability of failure within the second 6 months is

$$17.469 - 9.154 = 8.315\%.$$

(b) We want to find t such that

$$\begin{aligned}
1 - e^{-0.016t} &= 99.99\%\\
1 - e^{-0.016t} &= 0.9999\\
e^{-0.016t} &= 0.0001\\
\frac{1}{e^{0.016t}} &= \frac{1}{10{,}000}\\
10{,}000 &= e^{0.016t}.
\end{aligned}$$

Taking the $\ln$ of both sides and solving for t we get

$$t = \frac{\ln 10{,}000}{0.016}.$$

We see that $t \approx 575.646$ months, or 47.971 years.

55. Setting the balances equal,

$$\underbrace{4000(1.06)^t}_{\substack{\text{your}\\ \text{balance}}} = \underbrace{3500e^{0.0595t}}_{\substack{\text{your friend's}\\ \text{balance}}}$$

$$\begin{aligned}
\ln(4000(1.06)^t) &= \ln(3500e^{0.0595t})\\
\ln 4000 + \ln(1.06)^t &= \ln 3500 + \ln e^{0.0595t}\\
\ln 4000 + t\ln 1.06 &= \ln 3500 + 0.0595t\\
\ln 4000 - \ln 3500 &= 0.0595t - t\ln 1.06 = t(0.0595 - \ln 1.06)\\
t &= \frac{\ln 4000 - \ln 3500}{0.0595 - \ln 1.06} \approx 108.466
\end{aligned}$$

Yes, the balances will eventually be equal, but only after 109 years!

56. **(a)** If $P(t)$ is the investment's value after t years, we have $P(t) = P_0e^{0.04t}$. We want to find t such that $P(t)$ is three times its initial value, P_0. Therefore, we need to solve:

$$\begin{aligned}
P(t) &= 3P_0\\
P_0e^{0.04t} &= 3P_0\\
e^{0.04t} &= 3\\
\ln e^{0.04t} &= \ln 3\\
0.04t &= \ln 3\\
t &= (\ln 3)/0.04 \approx 27.465 \text{ years.}
\end{aligned}$$

With continuous compounding, the investment should triple in about $27\frac{1}{2}$ years.

(b) If the interest is compounded only once a year, the formula we will use is $P(t) = P_0b^t$ where b is the percent value of what the investment had been one year earlier. If it is earning 4% interest compounded once a year, it is 104% of what it had been the previous year, so our formula is $P(t) = P_0(1.04)^t$. Using this new formula, we will now solve

$$\begin{aligned}
P(t) &= 3P_0\\
P_0(1.04)^t &= 3P_0\\
(1.04)^t &= 3\\
\log(1.04)^t &= \log 3\\
t\log 1.04 &= \log 3\\
t &= \frac{\log 3}{\log 1.04} \approx 28.011 \text{ years.}
\end{aligned}$$

So, compounding once a year, it will take a little more than 28 years for the investment to triple.

57. **(a)** At time $t = 0$ we see that the temperature is given by

$$\begin{aligned}
H &= 70 + 120(1/4)^0\\
&= 70 + 120(1)\\
&= 190.
\end{aligned}$$

At time $t = 1$, we see that the temperature is given by

$$H = 70 + 120(1/4)^1$$

$$\begin{aligned}&= 70 + 120(1/4)\\&= 70 + 30\\&= 100.\end{aligned}$$

At $t = 2$, we see that the temperature is given by

$$\begin{aligned}H &= 70 + 120(1/4)^2\\&= 70 + 120(1/16)\\&= 70 + 7.5\\&= 77.5.\end{aligned}$$

(b) We solve for t to find when the temperature reaches $H = 90°$F:

$$\begin{aligned}70 + 120(1/4)^t &= 90\\120(1/4)^t &= 20 \quad \text{subtracting}\\(1/4)^t &= 20/120 \quad \text{dividing}\\\log(1/4)^t &= \log(1/6) \quad \text{taking logs}\\t\log(1/4) &= \log(1/6) \quad \text{using a log property}\\t &= \frac{\log(1/6)}{\log(1/4)} \quad \text{dividing}\\&= 1.292,\end{aligned}$$

so the coffee temperature reaches $90°$F after about 1.292 hours. Similar calculations show that the temperature reaches $75°$F after about 2.292 hours.

58. (a) Applying the given formula,

$$\text{Number toads in year 0 is } P = \frac{1000}{1 + 49(1/2)^0} = 20$$
$$\text{Number toads in year 5 is } P = \frac{1000}{1 + 49(1/2)^5} = 395$$
$$\text{Number toads in year 10 is } P = \frac{1000}{1 + 49(1/2)^{10}} = 954.$$

(b) We set up and solve the equation $P = 500$:

$$\begin{aligned}\frac{1000}{1 + 49(1/2)^t} &= 500\\500\left(1 + 49(1/2)^t\right) &= 1000 \quad \text{multiplying by denominator}\\1 + 49(1/2)^t &= 2 \quad \text{dividing by 500}\\49(1/2)^t &= 1\\(1/2)^t &= 1/49\\\log(1/2)^t &= \log(1/49) \quad \text{taking logs}\\t\log(1/2) &= \log(1/49) \quad \text{using a log rule}\\t &= \frac{\log(1/49)}{\log(1/2)}\\&= 5.615,\end{aligned}$$

and so it takes about 5.6 years for the population to reach 500. A similar calculation shows that it takes about 7.2 years for the population to reach 750.

(c) The graph in Figure 4.5 suggests that the population levels off at about 1000 toads. We can see this algebraically by using the fact that $(1/2)^t \to 0$ as $t \to \infty$. Thus,

$$P = \frac{1000}{1+49(1/2)^t} \to \frac{1000}{1+0} = 1000 \text{ toads} \qquad \text{as} \quad t \to \infty.$$

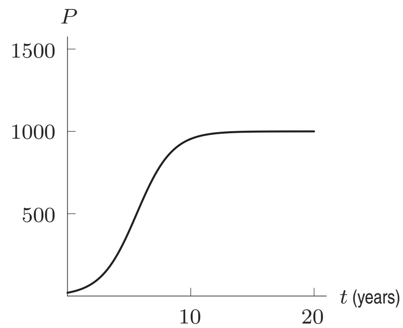


Figure 4.5: Toad population, P, against time, t

59. As a is increased, the y-intercept of the graph of $y = ae^{rt}$ rises, so the point of intersection of the two graphs shifts to the left. Therefore, t_0 decreases.

60. As s decreases (becomes more negative), the decay rate of be^{st} increases, so the graph drops more steeply. This means the point of intersection shifts to the left, so t_0 decreases.

61. Since a and b are fixed, the y-intercepts of the two graphs remain fixed. Since r increases, the growth rate of $y = ae^{rt}$ increases, and the graph of this function climbs more steeply. This will shift the point of intersection to the left unless the graph of be^{st} descends less steeply—that is, unless the value of s increases (becomes less negative/more positive).

Solutions for Section 4.3

Exercises

1. The graphs of $y = 10^x$ and $y = 2^x$ both have horizontal asymptotes, $y = 0$. The graph of $y = \log x$ has a vertical asymptote, $x = 0$.

2. The graphs of both $y = e^x$ and $y = e^{-x}$ have the same horizontal asymptote. Their asymptote is the x-axis, whose equation is $y = 0$. The graph of $y = \ln x$ is asymptotic to the y-axis, hence the equation of its asymptote is $x = 0$.

3. A is $y = 10^x$, B is $y = e^x$, C is $y = \ln x$, D is $y = \log x$.

4. A is $y = 3^x$, B is $y = 2^x$, C is $y = \ln x$, D is $y = \log x$, E is $y = e^{-x}$.

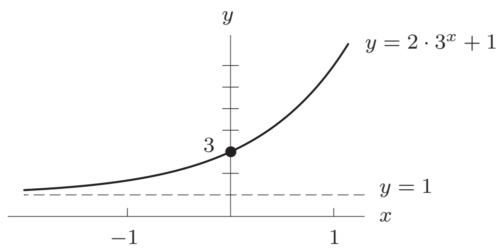
5. **(a)** $10^{-x} \to 0$ as $x \to \infty$.
(b) The values of $\log x$ get more and more negative as $x \to 0^+$, so

$$\log x \to -\infty.$$

6. **(a)** $e^x \to 0$ as $x \to -\infty$.
(b) The values of $\ln x$ get more and more negative as $x \to 0^+$, so

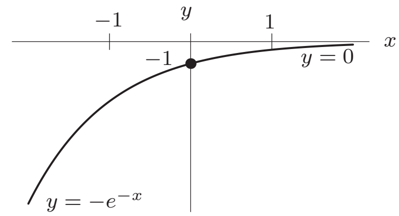
$$\ln x \to -\infty.$$

7. See Figure 4.6. The graph of $y = 2 \cdot 3^x + 1$ is the graph of $y = 3^x$ stretched vertically by a factor of 2 and shifted up by 1 unit.



(a)

Figure 4.6

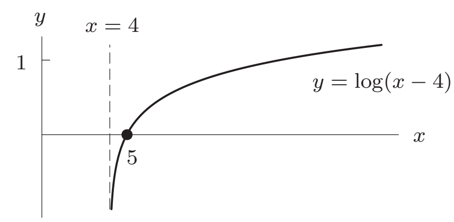


(b)

Figure 4.7

8. See Figure 4.7. The graph of $y = -e^{-x}$ is the graph of $y = e^x$ flipped over the y-axis and then over the x-axis.

9. See Figure 4.8. The graph of $y = \log(x - 4)$ is the graph of $y = \log x$ shifted to the right 4 units.



(c)

Figure 4.8

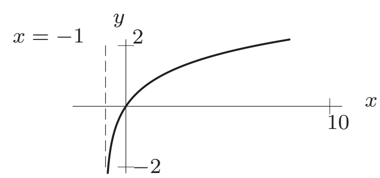


Figure 4.9

10. See Figure 4.9. The vertical asymptote is $x = -1$; there is no horizontal asymptote.

11. A graph of this function is shown in Figure 4.10. We see that the function has a vertical asymptote at $x = 3$. The domain is $(3, \infty)$.

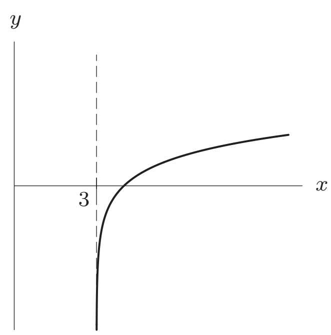


Figure 4.10

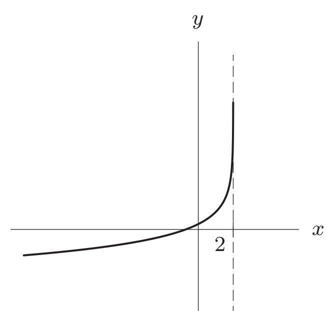


Figure 4.11

12. A graph of this function is shown in Figure 4.11. We see that the function has a vertical asymptote at $x = 2$. The domain is $(-\infty, 2)$.

13. We know, by the definition of pH, that $13 = -\log[H^+]$. Therefore, $-13 = \log[H^+]$, and $10^{-13} = [H^+]$. Thus, the hydrogen ion concentration is 10^{-13} moles per liter.

14. We know, by the definition of pH, that $1 = -\log[H^+]$. Therefore, $-1 = \log[H^+]$, and $10^{-1} = [H^+]$. Thus, the hydrogen ion concentration is $10^{-1} = 0.1$ moles per liter.

15. We know, by the definition of pH, that $8.3 = -\log[H^+]$. Therefore, $-8.3 = \log[H^+]$, and $10^{-8.3} = [H^+]$. Thus, the hydrogen ion concentration is $10^{-8.3} = 5.012 \times 10^{-9}$ moles per liter.

16. We know, by the definition of pH, that $4.5 = -\log[H^+]$. Therefore, $-4.5 = \log[H^+]$, and $10^{-4.5} = [H^+]$. Thus, the hydrogen ion concentration is $10^{-4.5} = 3.162 \times 10^{-5}$ moles per liter.

17. We know, by the definition of pH, that $0 = -\log[H^+]$. Therefore, $-0 = \log[H^+]$, and $10^{-0} = [H^+]$. Thus, the hydrogen ion concentration is $10^{-0} = 10^0 = 1$ mole per liter.

18. **(a)** Since $\log x$ becomes more and more negative as x decreases to 0 from above,

$$\lim_{x\to 0^+} \log x = -\infty.$$

(b) Since $-x$ is positive if x is negative and $-x$ decreases to 0 as x increases to 0 from below,

$$\lim_{x\to 0^-} \ln(-x) = -\infty.$$

Problems

19. **(a)** The graph in (III) has a vertical asymptote at $x = 0$ and $f(x) \to -\infty$ as $x \to 0^+$.
(b) The graph in (IV) goes through the origin, so $f(x) \to 0$ as $x \to 0^-$.
(c) The graph in (I) goes upward without bound, that is $f(x) \to \infty$, as $x \to \infty$.
(d) The graphs in (I) and (II) tend toward the x-axis, that is $f(x) \to 0$, as $x \to -\infty$.

20. **(a)** Let the functions graphed in (a), (b), and (c) be called $f(x)$, $g(x)$, and $h(x)$ respectively. Looking at the graph of $f(x)$, we see that $f(10) = 3$. In the table for $r(x)$ we note that $r(10) = 1.699$ so $f(x) \neq r(x)$. Similarly, $s(10) = 0.699$, so $f(x) \neq s(x)$. The values describing $t(x)$ do seem to satisfy the graph of $f(x)$, however. In the graph, we note that when $0 < x < 1$, then y must be negative. The data point $(0.1, -3)$ satisfies this. When $1 < x < 10$, then $0 < y < 3$. In the table for $t(x)$, we see that the point $(2, 0.903)$ satisfies this condition. Finally, when $x > 10$ we see that $y > 3$. The values $(100, 6)$ satisfy this. Therefore, $f(x)$ and $t(x)$ could represent the same function.

(b) For $g(x)$, we note that

$$\begin{cases} \text{when } 0 < x < 0.2, & \text{then } y < 0; \\ \text{when } 0.2 < x < 1, & \text{then } 0 < y < 0.699; \\ \text{when } x > 1, & \text{then } y > 0.699. \end{cases}$$

All the values of x in the table for $r(x)$ are greater than 1 and all the corresponding values of y are greater than 0.699, so $g(x)$ could equal $r(x)$. We see that, in $s(x)$, the values $(0.5, -0.060)$ do not satisfy the second condition so $g(x) \neq s(x)$. Since we already know that $t(x)$ corresponds to $f(x)$, we conclude that $g(x)$ and $r(x)$ correspond.

(c) By elimination, $h(x)$ must correspond to $s(x)$. We see that in $h(x)$,

$$\begin{cases} \text{when } x < 2, & \text{then } y < 0; \\ \text{when } 2 < x < 20, & \text{then } 0 < y < 1; \\ \text{when } x > 20, & \text{then } y > 1. \end{cases}$$

Since the values in $s(x)$ satisfy these conditions, it is reasonable to say that $h(x)$ and $s(x)$ correspond.

21. The log function is increasing but is concave down and so is increasing at a decreasing rate. It is not a compliment—growing exponentially would have been better. However, it is most likely realistic because after you are proficient at something, any increase in proficiency takes longer and longer to achieve.

22. A possible formula is $y = \log x$.

23. This graph could represent exponential decay, so a possible formula is $y = b^x$ with $0 < b < 1$.

24. This graph could represent exponential growth, with a y-intercept of 2. A possible formula is $y = 2b^x$ with $b > 1$.

25. A possible formula is $y = \ln x$.

26. This graph could represent exponential decay, with a y-intercept of 0.1. A possible formula is $y = 0.1b^x$ with $0 < b < 1$.

27. This graph could represent exponential "growth", with a y-intercept of -1. A possible formula is $y = (-1)b^x = -b^x$ for $b > 1$.

28. We need $x^2 > 0$, which is true as long as $x \neq 0$, so the domain is all $x \neq 0$.

29. Since $\ln(x)$ is defined only for $x > 0$, $x > 0$.

30. Since $\ln x$ is only defined for $x > 0$, $\ln(\ln x)$ is only defined for $\ln x > 0$. If you look at a graph of $y = \ln x$, you will see that $\ln x > 0$ when $x > 1$. Therefore the domain of f is all $x > 1$.

31. Since the domain of $\ln x$ is $x > 0$, the domain of $\ln(x-3)$ is $(x-3) > 0$, or $x > 3$.

32. **(a)** (i) $\text{pH} = -\log x = 2$ so $\log x = -2$ so $x = 10^{-2}$

(ii) $\text{pH} = -\log x = 4$ so $\log x = -4$ so $x = 10^{-4}$

(iii) $\text{pH} = -\log x = 7$ so $\log x = -7$ so $x = 10^{-7}$

(b) Solutions with high pHs have low concentrations and so are less acidic.

33. **(a)** We are given the number of H^+ ions in 12 oz of coffee, and we need to find the number of moles of ions in 1 liter of coffee. So we need to convert numbers of ions to moles of ions, and ounces of coffee to liters of coffee. Finding the number of moles of H^+, we have:

$$2.41 \cdot 10^{18} \text{ ions} \cdot \frac{1 \text{ mole of ions}}{6.02 \cdot 10^{23} \text{ ions}} = 4 \cdot 10^{-6} \text{ ions.}$$

Finding the number of liters of coffee, we have:

$$12 \text{ oz} \cdot \frac{1 \text{ liter}}{30.3 \text{ oz}} = 0.396 \text{ liters.}$$

Thus, the concentration, $[H^+]$, in the coffee is given by

$$\begin{aligned}[H^+] &= \frac{\text{Number of moles } H^+ \text{ in solution}}{\text{Number of liters solution}} \\ &= \frac{4 \cdot 10^{-6}}{0.396} \\ &= 1.01 \cdot 10^{-5} \text{ moles/liter.}\end{aligned}$$

(b) We have

$$\begin{aligned}\text{pH} &= -\log[H^+] \\ &= -\log\left(1.01 \cdot 10^{-5}\right) \\ &= -(-4.9957) \\ &\approx 5.\end{aligned}$$

Thus, the pH is about 5. Since this is less than 7, it means that coffee is acidic.

34. **(a)** The pH is 2.3, which, according to our formula for pH, means that

$$-\log\left[H^+\right] = 2.3.$$

This means that

$$\log\left[H^+\right] = -2.3.$$

This tells us that the exponent of 10 that gives $[H^+]$ is -2.3, so

$$\begin{aligned}[H^+] &= 10^{-2.3} && \text{because } -2.3 \text{ is exponent of } 10 \\ &= 0.005 \text{ moles/liter.}\end{aligned}$$

(b) From part (a) we know that 1 liter of lemon juice contains 0.005 moles of H^+ ions. To find out how many H^+ ions our lemon juice has, we need to convert ounces of juice to liters of juice and moles of ions to numbers of ions. We have

$$2 \text{ oz} \times \frac{1 \text{ liter}}{30.3 \text{ oz}} = 0.066 \text{ liters.}$$

We see that

$$0.066 \text{ liters juice} \times \frac{0.005 \text{ moles H}^+ \text{ ions}}{1 \text{ liter}} = 3.3 \times 10^{-4} \text{ moles H}^+ \text{ ions.}$$

There are 6.02×10^{23} ions in one mole, and so

$$3.3 \times 10^{-4} \text{ moles} \times \frac{6.02 \times 10^{23} \text{ ions}}{\text{mole}} = 1.987 \times 10^{20} \text{ ions.}$$

35. Let I_A and I_B be the intensity of sound A and sound B, respectively. We know that $I_B = 5I_A$ and, by the definition of the decibel rating, we know that $10\log(I_A/I_0) = 30$. We have:

$$\begin{aligned}
\text{Decibel rating of B } &= 10\log(\frac{I_B}{I_0}) \\
&= 10\log(\frac{5I_A}{I_0}) \\
&= 10\log 5 + 10\log(\frac{I_A}{I_0}) \\
&= 10\log 5 + 30 \\
&= 10(0.699) + 30 \\
&\approx 37.
\end{aligned}$$

Notice that although sound B is 5 times as loud as sound A, the decibel rating only goes from 30 to 37.

36. (a) We know that $D_1 = 10\log\left(\frac{I_1}{I_0}\right)$ and $D_2 = 10\log\left(\frac{I_2}{I_0}\right)$. Thus

$$\begin{aligned}
D_2 - D_1 &= 10\log\left(\frac{I_2}{I_0}\right) - 10\log\left(\frac{I_1}{I_0}\right) \\
&= 10\left(\log\left(\frac{I_2}{I_0}\right) - \log\left(\frac{I_1}{I_0}\right)\right) \quad \text{factoring} \\
&= 10\log\left(\frac{I_2/I_0}{I_1/I_0}\right) \quad \text{using a log property}
\end{aligned}$$

and so

$$D_2 - D_1 = 10\log\left(\frac{I_2}{I_1}\right).$$

(b) Suppose the sound's initial intensity is I_1 and that its new intensity is I_2. Then here we have $I_2 = 2I_1$. If D_1 is the original decibel rating and D_2 is the new rating then

$$\begin{aligned}
\text{Increase in decibels } &= D_2 - D_1 \\
&= 10\log\left(\frac{I_2}{I_1}\right) \quad \text{using formula from part (a)} \\
&= 10\log\left(\frac{2I_1}{I_1}\right) \\
&= 10\log 2 \\
&\approx 3.01.
\end{aligned}$$

Thus, the sound increases by 3 decibels when it doubles in intensity.

37. (a) We know $M_1 = \log\left(\frac{W_1}{W_0}\right)$ and $M_2 = \log\left(\frac{W_2}{W_0}\right)$. Thus,

$$\begin{aligned}
M_2 - M_1 &= \log\left(\frac{W_2}{W_0}\right) - \log\left(\frac{W_1}{W_0}\right) \\
&= \log\left(\frac{W_2}{W_1}\right).
\end{aligned}$$

(b) Let $M_2 = 8.7$ and $M_1 = 7.1$, so

$$M_2 - M_1 = \log\left(\frac{W_2}{W_1}\right)$$

becomes

$$\begin{aligned} 8.7 - 7.1 &= \log\left(\frac{W_2}{W_1}\right) \\ 1.6 &= \log\left(\frac{W_2}{W_1}\right) \end{aligned}$$

so

$$\frac{W_2}{W_1} = 10^{1.6} \approx 40.$$

Thus, the seismic waves of the 2005 Sumatran earthquake were about 40 times as large as those of the 1989 California earthquake.

38. (a) In the formula, D is divided by S, so provided the same units are used for both, the units will cancel. For instance, if $D = 10$ cm and $S = 2$ cm, then

$$\frac{D}{S} = \frac{10}{2} = 5.$$

If instead D and S are measured in millimeters, the ratio remains the same:

$$\frac{D}{S} = \frac{100}{20} = 5.$$

The same is true for other units of measure. From a design point of view, it is useful not to have to consider units of length, since the same image appears differently on screens of different sizes and resolutions.

(b) We have

$$\begin{aligned} T &= a + b\log\left(\frac{D}{S} + 1\right) \\ &= 50 + 500\log\left(\frac{15}{3} + 1\right) \\ &= 50 + 500\log 6 = 439 \text{ ms.} \end{aligned}$$

(c) See Figure 4.12.

(d) At $D = 0$,

$$T = 50 + 500\log\left(\frac{0}{3} + 1\right) = 50 + 500\log 1 = 50 \text{ ms.}$$

This suggests it takes 50 ms for a user to position the cursor over a target without having to move it. One possible explanation is that 50 ms (or 1/20 of a second) is required for the user to recognize the fact that the cursor does not need to be moved.

(e) If the distance moved doubles from D to $2D$, then

$$T_{\text{old}} = a + b\log\left(\frac{D}{S} + 1\right)$$

and

$$T_{\text{new}} = a + b\log\left(\frac{2D}{S} + 1\right).$$

Now,

$$\frac{2D}{S} + 1 < 2\left(\frac{D}{S} + 1\right),$$

so

$$\log\left(\frac{2D}{S} + 1\right) < \log\left(2\left(\frac{D}{S} + 1\right)\right).$$

This means that, if the distance moved doubles from D to $2D$, then we can rewrite this as

$$\begin{aligned}
T_{\text{new}} &< a + b\log\left(2\left(\frac{D}{S}+1\right)\right)\\
T_{\text{new}} &< a + b\left(\log\left(\frac{D}{S}+1\right)+\log 2\right)\\
T_{\text{new}} &< a + b\log\left(\frac{D}{S}+1\right)+b\log 2\\
T_{\text{new}} &< T_{\text{old}} + b\log 2\\
T_{\text{new}} &< T_{\text{old}} + 500\log 2.
\end{aligned}$$

Thus, the change in T is less than $500\log 2 = 151$ ms.

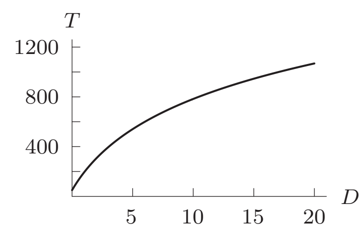


Figure 4.12

Solutions for Section 4.4

Exercises

1. Using a linear scale, the wealth of everyone with less than a million dollars would be indistinguishable because all of them are less than one one-thousandth of the wealth of the average billionaire. A log scale is more useful.
2. In all cases, the average number of diamonds owned is probably less than 100 (probably less than 20). Therefore, the data will fit neatly into a linear scale.
3. As nobody eats fewer than zero times in a restaurant per week, and since it's unlikely that anyone would eat more than 50 times per week in a restaurant, a linear scale should work fine.
4. This should be graphed on a log scale. Someone who has never been exposed presumably has zero bacteria. Someone who has been slightly exposed has perhaps one thousand bacteria. Someone with a mild case may have ten thousand bacteria, and someone dying of tuberculosis may have hundreds of thousands or millions of bacteria. Using a linear scale, the data points of all the people not dying of the disease would be too close to be readable.
5. **(a)**

Table 4.3

n	1	2	3	4	5	6	7	8	9
$\log n$	0	0.3010	0.4771	0.6021	0.6990	0.7782	0.8451	0.9031	0.9542

Table 4.4

n	10	20	30	40	50	60	70	80	90
$\log n$	1	1.3010	1.4771	1.6021	1.6990	1.7782	1.8451	1.9031	1.9542

(b) The first tick mark is at $10^0 = 1$. The dot for the number 2 is placed $\log 2 = 0.3010$ of the distance from 1 to 10. The number 3 is placed at $\log 3 = 0.4771$ units from 1, and so on. The number 30 is placed 1.4771 units from 1, the number 50 is placed 1.6989 units from 1, and so on.

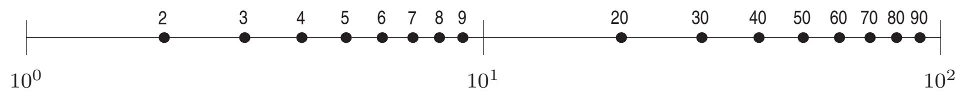


Figure 4.13

6. (a) Using linear regression we find that the linear function $y = 48.097 + 0.803x$ gives a correlation coefficient of $r = 0.9996$. We see from the sketch of the graph of the data that the estimated regression line provides an excellent fit. See Figure 4.14.

(b) To check the fit of an exponential we make a table of x and $\ln y$:

x	30	85	122	157	255	312
$\ln y$	4.248	4.787	4.977	5.165	5.521	5.704

Using linear regression, we find $\ln y = 4.295 + 0.0048x$. We see from the sketch of the graph of the data that the estimated regression line fits the data well, but not as well as part (a). See Figure 4.15. Solving for y to put this into exponential form gives

$$\begin{aligned} e^{\ln y} &= e^{4.295+0.0048x} \\ y &= e^{4.295} e^{0.0048x} \\ y &= 73.332e^{0.0048x}. \end{aligned}$$

This gives us a correlation coefficient of $r \approx 0.9728$. Note that since $e^{0.0048} = 1.0048$, we could have written $y = 73.332(1.0048)^x$.

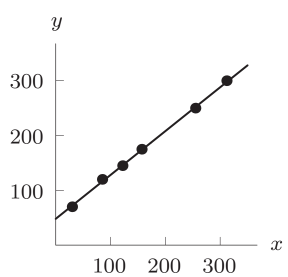


Figure 4.14

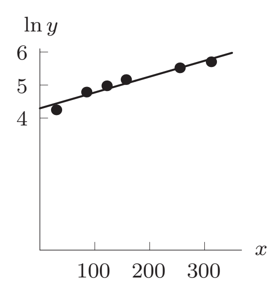


Figure 4.15

(c) Both fits are good. The linear equation gives a slightly better fit.

7. (a) Run a linear regression on the data. The resulting function is $y = -3582.145 + 236.314x$, with $r \approx 0.7946$. We see from the sketch of the graph of the data that the estimated regression line provides a reasonable but not excellent fit. See Figure 4.16.

(b) If, instead, we compare x and $\ln y$ we get

$$\ln y = 1.568 + 0.200x.$$

We see from the sketch of the graph of the data that the estimated regression line provides an excellent fit with $r \approx 0.9998$. See Figure 4.17. Solving for y, we have

$$\begin{aligned} e^{\ln y} &= e^{1.568+0.200x} \\ y &= e^{1.568}e^{0.200x} \\ y &= 4.797e^{0.200x} \\ \text{or} \quad y &= 4.797(e^{0.200})^x \approx 4.797(1.221)^x. \end{aligned}$$

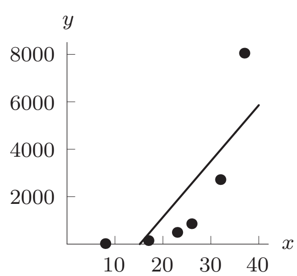


Figure 4.16

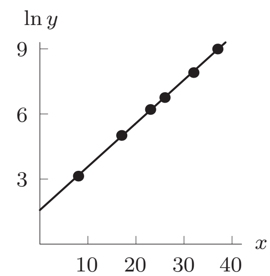


Figure 4.17

(c) The linear equation is a poor fit, and the exponential equation is a better fit.

8. (a) Using linear regression on x and y, we find $y = -169.331 + 57.781x$, with $r \approx 0.9707$. We see from the sketch of the graph of the data that the estimated regression line provides a good fit. See Figure 4.18.

(b) Using linear regression on x and $\ln y$, we find $\ln y = 2.258 + 0.463x$, with $r \approx 0.9773$. We see from the sketch of the graph of the data that the estimated regression line provides a good fit. See Figure 4.19. Solving as in the previous problem, we get $y = 9.566(1.589)^x$.

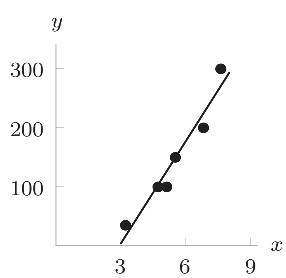


Figure 4.18

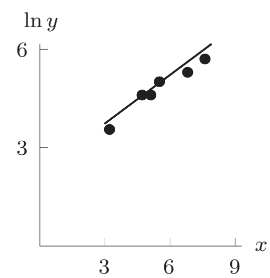


Figure 4.19

(c) The exponential function $y = 9.566(1.589)^x$ gives a good fit, and so does the linear function $y = -169.331 + 57.781x$.

Problems

9. The Declaration of Independence was signed in 1776, about 225 years ago. We can write this number as

$$\frac{225}{1{,}000{,}000} = 0.000225 \text{ million years ago.}$$

This number is between $10^{-4} = 0.0001$ and $10^{-3} = 0.001$. Using a calculator, we have

$$\log 0.000225 \approx -3.65,$$

which, as expected, lies between -3 and -4 on the log scale. Thus, the Declaration of Independence is placed at

$$10^{-3.65} \approx 0.000224 \text{ million years ago} = 224 \text{ years ago}.$$

10. (a) An appropriate scale is from 0 to 70 at intervals of 10. (Other answers are possible.) See Figure 4.20. The points get more and more spread out as the exponent increases.

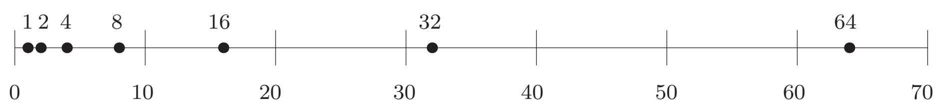


Figure 4.20

(b) If we want to locate 2 on a logarithmic scale, since $2 = 10^{0.3}$, we find $10^{0.3}$. Similarly, $8 = 10^{0.9}$ and $32 = 10^{1.5}$, so 8 is at $10^{0.9}$ and 32 is at $10^{1.5}$. Since the values of the logs go from 0 to 1.8, an appropriate scale is from 0 to 2 at intervals of 0.2. See Figure 4.21. The points are spaced at equal intervals.

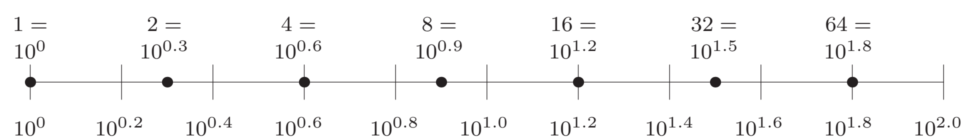


Figure 4.21

11. (a) A log scale is necessary because the numbers are of such different magnitudes. If we used a small scale (such as 0, 10, 20,...) we could see the small numbers but would never get large enough for the big numbers. If we used a large scale (such as counting by 100, 000s), we would not be able to differentiate between the small numbers. In order to see all of the values, we need to use a log scale.

(b) See Table 4.5.

Table 4.5 *Deaths due to various causes in the US in 2002*

Cause	Log of number of deaths
Scarlet fever	0.30
Whooping cough	1.26
Asthma	3.63
HIV	4.15
Kidney Diseases	4.61
Accidents	5.03
Malignant neoplasms	5.75
Cardiovascular Disease	5.96
All causes	6.39

(c) See Figure 4.22.

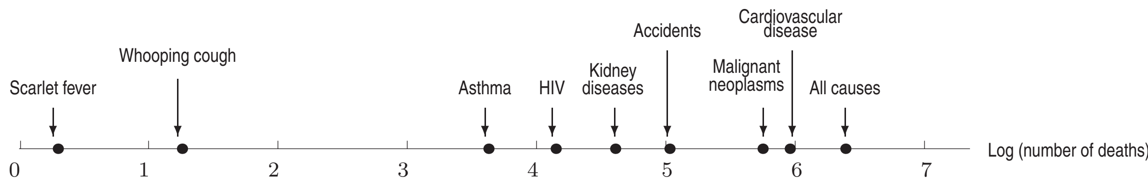


Figure 4.22

12. (a) Table 4.6 is the completed table.

Table 4.6 *The mass of various animals in kilograms*

Animal	Body mass	log of body mass
Blue Whale	91000	4.96
African Elephant	5450	3.74
White Rhinoceros	3000	3.48
Hippopotamus	2520	3.40
Black Rhinoceros	1170	3.07
Horse	700	2.85
Lion	180	2.26
Human	70	1.85
Albatross	11	1.04
Hawk	1	0.00
Robin	0.08	−1.10
Hummingbird	0.003	−2.52

(b) Figure 4.23 shows Table 4.6 plotted on a linear scale.

Figure 4.23: On a linear scale, all masses except that of the blue whale are very close together

(c) Figure 4.24 shows Table 4.6 plotted on a logarithmic scale.

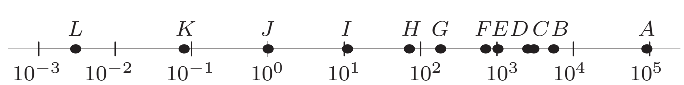


Figure 4.24

(d) Figure 4.24 gives more information than Figure 4.23.

13. The figure represents populations using logs, which are exponents of 10. For instance, Greasewood, AZ corresponds to the logarithm 2.3. This means that

$$\text{Population of Greasewood} = 10^{2.3} = 200 \qquad \text{after rounding}$$

The population of the other places are in Table 4.7.

Table 4.7 *Approximate populations of eleven different localities*

Locality	Exponent	Approx. population
Lost Springs, Wy	0.6	4
Greasewood, Az	2.3	200
Bar Harbor, Me	3.4	2,500
Abilene, Tx	5.1	130, 000
Worcester, Ma	5.6	400,000
Massachusetts	6.8	6,300, 000
Chicago	6.9	7,900,000
New York	7.3	20,000,000
California	7.5	32,000,000
US	8.4	250,000,000
World	9.8	6,300,000,000

14. For the pack of gum, $\log(0.50) = -0.30$, so the pack of gum is plotted at -0.3. For the movie ticket, $\log(9) = 0.95$, so the ticket is plotted at 0.95, and so on. See Figure 4.25.

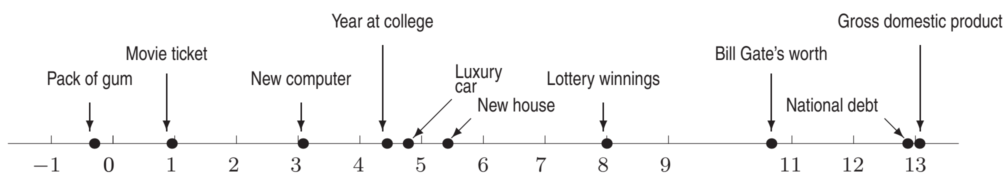


Figure 4.25: Log (dollar value)

15. Table 4.8 gives the logs of the sizes of the various organisms. The log values from Table 4.8 have been plotted in Figure 4.26.

Table 4.8 *Size (in cm) and* log*(size) of various organisms*

Animal	Size	log(size)	Animal	Size	log(size)
Virus	0.0000005	−6.3	Cat	60	1.8
Bacterium	0.0002	−3.7	Wolf	200	2.3
Human cell	0.002	−2.7	Shark	600	2.8
Ant	0.8	−0.1	Squid	2200	3.3
Hummingbird	12	1.1	Sequoia	7500	3.9

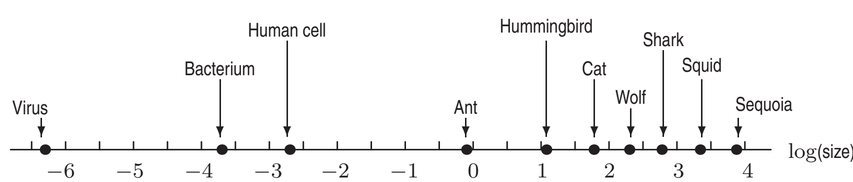


Figure 4.26: The $\log(\text{sizes})$ of various organisms (sizes in cm)

16. (a)

Table 4.9

x	0	1	2	3	4	5
$y = 3^x$	1	3	9	27	81	243

(b)

Table 4.10

x	0	1	2	3	4	5
$y = \log(3^x)$	0	0.477	0.954	1.431	1.908	2.386

The differences between successive terms are constant(≈ 0.477), so the function is linear.

(c)

Table 4.11

x	0	1	2	3	4	5
$f(x)$	2	10	50	250	1250	6250

Table 4.12

x	0	1	2	3	4	5
$g(x)$	0.301	1	1.699	2.398	3.097	3.796

We see that $f(x)$ is an exponential function (note that it is increasing by a constant growth factor of 5), while $g(x)$ is a linear function with a constant rate of change of 0.699.

(d) The resulting function is linear. If $f(x) = a \cdot b^x$ and $g(x) = \log(a \cdot b^x)$ then

$$\begin{aligned} g(x) &= \log(ab^x) \\ &= \log a + \log b^x \\ &= \log a + x \log b \\ &= k + m \cdot x, \end{aligned}$$

where the y intercept $k = \log a$ and $m = \log b$. Thus, g will be linear.

17.

Table 4.13

x	0	1	2	3	4	5
$y = \ln(3^x)$	0	1.0986	2.1972	3.2958	4.3944	5.4931

Table 4.14

x	0	1	2	3	4	5
$g(x) = \ln(2 \cdot 5^x)$	0.6931	2.3026	3.9120	5.5215	7.1309	8.7403

Yes, the results are linear.

18. (a) We obtain the linear approximation $y = 2237 + 2570x$ using linear regression.

(b) Table 4.15 gives the natural log of the cost of imports, rather than the cost of imports itself.

Table 4.15

Year	x	$\ln y$
1985	0	8.259
1986	1	8.470
1987	2	8.747
1988	3	9.049
1989	4	9.392
1990	5	9.632
1991	6	9.851

Using linear regression we get $\ln y = 8.227 + 0.2766x$ as an approximation.

(c) To find a formula for the cost and not for the natural log of the cost, we need to solve

$$\begin{aligned} \ln y &= 8.227 + 0.2766x \quad \text{for } y. \\ e^{\ln y} &= e^{8.227+0.2766x} \\ y &= e^{8.227}e^{0.2766x} \\ y &= 3741e^{0.2766x} \end{aligned}$$

19. (a) Using linear regression we get $y = 14.227 - 0.233x$ as an approximation for the percent share x years after 1950. Table 4.16 gives $\ln y$:

Table 4.16

Year	x	$\ln y$
1950	0	2.773
1960	10	2.380
1970	20	2.079
1980	30	1.902
1990	40	1.758
1992	42	1.609

(b) Using linear regression on the values in Table 4.16 we get $\ln y = 2.682 - 0.0253x$.

(c) Taking e to the power of both sides we get $y = e^{2.682-0.0253x} = e^{2.682}(e^{-0.0253x}) \approx 14.614e^{-.0253x}$ as an exponential approximation for the percent share.

20. (a) Find the values of $\ln t$ in the table, use linear regression on a calculator or computer with $x = \ln t$ and $y = P$. The line has slope -7.787 and P-intercept 86.283 ($P = -7.787 \ln t + 86.283$). Thus $a = -7.787$ and $b = 86.283$.

(b) Figure 4.27 shows the data points plotted with P against $\ln t$. The model seems to fit well.

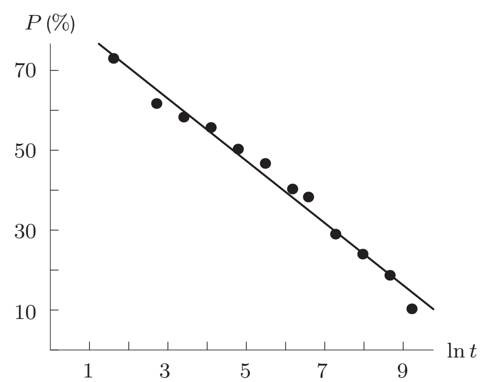


Figure 4.27: Plot of P against $\ln t$ and the line with slope -7.787 and intercept 86.283

(c) The subjects will recognize no words when $P = 0$, that is, when $-7.787 \ln t + 86.283 = 0$. Solving for t:

$$\begin{aligned} -7.787 \ln t &= -86.283 \\ \ln t &= \frac{86.283}{7.787} \end{aligned}$$

Taking both sides to the e power,

$$e^{\ln t} = e^{\frac{86.283}{7.787}}$$
$$t \approx 64{,}918.342,$$

so $t \approx 45$ days.

The subject recognized all the words when $P = 100$, that is, when $-7.787 \ln t + 86.283 = 100$. Solving for t:

$$-7.787 \ln t = 13.717$$
$$\ln t = \frac{13.717}{-7.787}$$
$$t \approx 0.172,$$

so $t \approx 0.172$ minutes (≈ 10 seconds) from the start of the experiment.

(d)

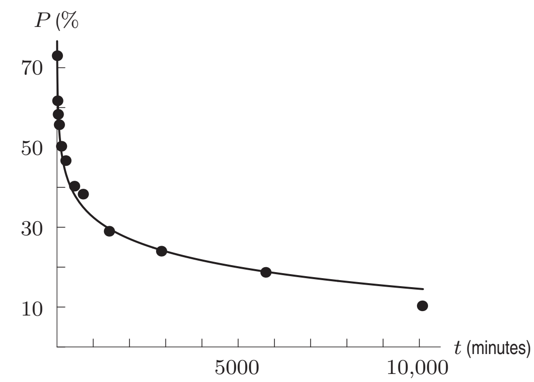


Figure 4.28: The percentage P of words recognized as a function of t, the time elapsed and the function $P = -7.787 \ln t + 86.283$

21. (a) Table 4.17 gives values of $L = \ln \ell$ and $W = \ln w$. The data in Table 4.17 have been plotted in Figure 4.29, and a line of best fit has been drawn in. See part (b).

Table 4.17 $L = \ln \ell$ *and* $W = \ln w$ *for 16 different fish*

Type	1	2	3	4	5	6	7	8
L	2.092	2.208	2.322	2.477	2.501	2.625	2.695	2.754
W	1.841	2.262	2.451	2.918	3.266	3.586	3.691	3.857
Type	9	10	11	12	13	14	15	16
L	2.809	2.874	2.929	2.944	3.025	3.086	3.131	3.157
W	4.184	4.240	4.336	4.413	4.669	4.786	5.131	5.155

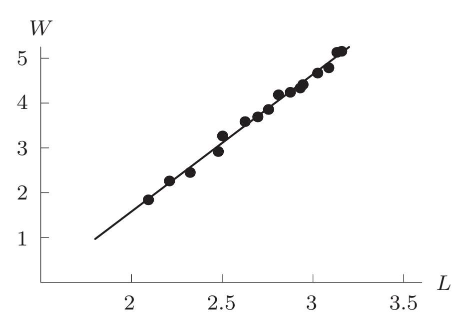


Figure 4.29: Plot of data in Table 4.17 together with line of best fit

(b) The formula for the line of best fit is $W = 3.06L - 4.54$, as determined using a spreadsheet. However, you could also obtain comparable results by fitting a line by eye.

(c) We have

$$W = 3.06L - 4.54$$

$$\begin{aligned}
\ln w &= 3.06 \ln \ell - 4.54 \\
\ln w &= \ln \ell^{3.06} - 4.54 \\
w &= e^{\ln \ell^{3.06} - 4.54} \\
&= \ell^{3.06} e^{-4.54} \approx 0.011 \ell^{3.06}.
\end{aligned}$$

(d) Weight tends to be directly proportional to volume, and in many cases volume tends to be proportional to the cube of a linear dimension (e.g., length). Here we see that w is in fact very nearly proportional to the cube of ℓ.

22. (a) After converting the I values to $\ln I$, we use linear regression on a computer or calculator with $x = \ln I$ and $y = F$. We find $a \approx 4.26$ and $b \approx 8.95$ so that $F = 4.26 \ln I + 8.95$. Figure 4.30 shows a plot of F against $\ln I$ and the line with slope 4.26 and intercept 8.95.

(b) See Figure 4.30.

(c) Figure 4.31 shows a plot of $F = 4.26 \ln I + 8.95$ and the data set in Table 4.19. The model seems to fit well.

(d) Imagine the units of I were changed by a factor of $\alpha > 0$ so that $I_{\text{old}} = \alpha I_{\text{new}}$.

Then

$$\begin{aligned}
F &= a_{\text{old}} \ln I_{\text{old}} + b_{\text{old}} \\
&= a_{\text{old}} \ln(\alpha I_{\text{new}}) + b_{\text{old}} \\
&= a_{\text{old}} (\ln \alpha + \ln I_{\text{new}}) + b_{\text{old}} \\
&= a_{\text{old}} \ln \alpha + a_{\text{old}} \ln I_{\text{new}} + b_{\text{old}}.
\end{aligned}$$

Rearranging and matching terms, we see:

$$F = \underbrace{a_{\text{old}} \ln I_{\text{new}}}_{a_{\text{new}} \ln I_{\text{new}}} + \underbrace{a_{\text{old}} \ln \alpha + b_{\text{old}}}_{b_{\text{new}}}$$

so

$$a_{\text{new}} = a_{\text{old}} \quad \text{and} \quad b_{\text{new}} = b_{\text{old}} + a_{\text{old}} \ln \alpha.$$

We can also see that if $\alpha > 1$ then $\ln \alpha > 0$ so the term $a_{\text{old}} \ln \alpha$ is positive and $b_{\text{new}} > b_{\text{old}}$. If $\alpha < 1$ then $\ln \alpha < 0$ so the term $a_{\text{old}} \ln \alpha$ is negative and $b_{\text{new}} < b_{\text{old}}$.

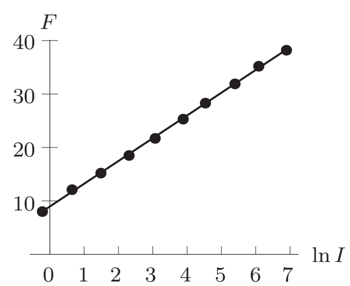


Figure 4.30

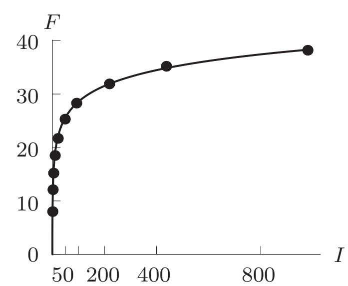


Figure 4.31

23. (a) Based on Figure 4.32, a log function seems as though it might give a good fit to the data in the table.

(b)

z	-1.56	-0.60	0.27	1.17	1.64	2.52
y	-11	-2	6.5	16	20.5	29

(c)

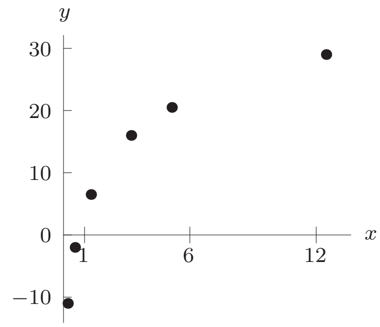


Figure 4.32

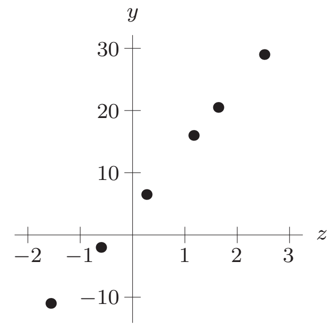


Figure 4.33

As you can see from Figure 4.33, the transformed data falls close to a line. Using linear regression, we see that $y = 4 + 9.9z$ gives an excellent fit to the data.

(d) Since $z = \ln x$, we see that the logarithmic function $y = 4 + 9.9 \ln x$ gives an excellent fit to the data.

(e) Solving $y = 4 + 9.9 \ln x$ for x, we have

$$\begin{aligned} y - 4 &= 9.9 \ln x \\ \ln x &= \frac{y}{9.9} - \frac{4}{9.9} \\ e^{\ln x} &= e^{\frac{y}{9.9} - \frac{4}{9.9}} \\ x &= (e^{y/9.9})(e^{-4/9.9}). \end{aligned}$$

Since $e^{-\frac{4}{9.9}} \approx 0.67$ and $1/9.9 \approx 0.1$, we have

$$x \approx 0.67e^{0.1y}.$$

Thus, x is an exponential function of y.

Solutions for Chapter 4 Review

Exercises

1. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 12(0.9)^t$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k\cdot 0} &= 12(0.9)^0 \\ a \cdot 1 &= 12 \cdot 1 \\ a &= 12. \end{aligned}$$

Thus, we have $12e^{kt} = 12(0.9)^t$, and we solve for k:

$$\begin{aligned} 12e^{kt} &= 12(0.9)^t \\ e^{kt} &= (0.9)^t \\ \left(e^k\right)^t &= (0.9)^t \\ e^k &= 0.9 \\ \ln e^k &= \ln 0.9 \\ k &= \ln 0.9 \approx -0.105. \end{aligned}$$

Therefore, the equation is $Q = 12e^{-0.105t}$.

2. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 16(0.487)^t$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k\cdot 0} &= 16(0.487)^0 \\ a\cdot 1 &= 16\cdot 1 \\ a &= 16. \end{aligned}$$

Thus, we have $16e^{kt} = 16(0.487)^t$, and we solve for k:

$$\begin{aligned} 16e^{kt} &= 16(0.487)^t \\ e^{kt} &= (0.487)^t \\ \left(e^k\right)^t &= (0.487)^t \\ e^k &= 0.487 \\ \ln e^k &= \ln 0.487 \\ k &= \ln 0.487 \approx -0.719. \end{aligned}$$

Therefore, the equation is $Q = 16e^{-0.719t}$.

3. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 14(0.862)^{1.4t}$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k\cdot 0} &= 14(0.862)^0 \\ a\cdot 1 &= 14\cdot 1 \\ a &= 14. \end{aligned}$$

Thus, we have $14e^{kt} = 14(0.862)^{1.4t}$, and we solve for k:

$$\begin{aligned} 14e^{kt} &= 14(0.862)^{1.4t} \\ e^{kt} &= \left(0.862^{1.4}\right)^t \\ \left(e^k\right)^t &= (0.812)^t \\ e^k &= 0.812 \\ \ln e^k &= \ln 0.812 \\ k &= -0.208. \end{aligned}$$

Therefore, the equation is $Q = 14e^{-0.208t}$.

4. To convert to the form $Q = ae^{kt}$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and k. Thus, we have $ae^{kt} = 721(0.98)^{0.7t}$. At $t = 0$, we can solve for a:

$$\begin{aligned} ae^{k\cdot 0} &= 721(0.98)^0 \\ a\cdot 1 &= 721\cdot 1 \\ a &= 721. \end{aligned}$$

Thus, we have $721e^{kt} = 721(0.98)^{0.7t}$, and we solve for k:

$$\begin{aligned} 721e^{kt} &= 721(0.98)^{0.7t} \\ e^{kt} &= \left(0.98^{0.7}\right)^t \\ \left(e^k\right)^t &= (0.986)^t \\ e^k &= 0.986 \\ \ln e^k &= \ln 0.986 \\ k &= -0.0141. \end{aligned}$$

Therefore, the equation is $Q = 721e^{-0.0141t}$.

5. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is -10.

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = 7e^{-10t}$. At $t = 0$, we can solve for a:

$$\begin{aligned} ab^0 &= 7e^{-10\cdot 0} \\ a \cdot 1 &= 7 \cdot 1 \\ a &= 7. \end{aligned}$$

Thus, we have $7b^t = 7e^{-10t}$, and we solve for b:

$$\begin{aligned} 7b^t &= e^{-10t} \\ b^t &= e^{-10t} \\ b^t &= \left(e^{-10}\right)^t \\ b &= e^{-10} \approx 0.0000454. \end{aligned}$$

Therefore, the equation is $Q = 7(0.0000454)^t$.

6. The continuous percent growth rate is the value of k in the equation $Q = ae^{kt}$, which is 1 (since $t \cdot 1 = 1$).

To convert to the form $Q = ab^t$, we first say that the right sides of the two equations equal each other (since each equals Q), and then we solve for a and b. Thus, we have $ab^t = 5e^t$. At $t = 0$, we can solve for a:

$$\begin{aligned} ab^0 &= 5e^0 \\ a \cdot 1 &= 5 \cdot 1 \\ a &= 5. \end{aligned}$$

Thus, we have $5b^t = 5e^t$, and we solve for b:

$$\begin{aligned} 5b^t &= 5e^t \\ b^t &= e^t \\ b &= e \approx 2.718. \end{aligned}$$

Therefore, the equation is $Q = 5e^t$ or $Q = 5 \cdot 2.718^t$.

7. We are solving for an exponent, so we use logarithms. We can use either the common logarithm or the natural logarithm. Using the log rules, we have

$$\begin{aligned} 3^t &= 50 \\ \ln(3^t) &= \ln(50) \\ t\ln(3) &= \ln(50) \\ t &= \frac{\ln(50)}{\ln(3)} = 3.561. \end{aligned}$$

8. We are solving for an exponent, so we use logarithms. Using the log rules, we have

$$\begin{aligned} e^{0.15t} &= 25 \\ \ln(e^{0.15t}) &= \ln(25) \\ 0.15t &= \ln(25) \\ t &= \frac{\ln(25)}{0.15} = 21.459. \end{aligned}$$

9. We are solving for an exponent, so we use logarithms. We first divide both sides by 40 and then use logs:

$$\begin{aligned} 40e^{-0.2t} &= 12 \\ e^{-0.2t} &= 0.3 \\ \ln(e^{-0.2t}) &= \ln(0.3) \\ -0.2t &= \ln(0.3) \\ t &= \frac{\ln(0.3)}{-0.2} = 6.020. \end{aligned}$$

10. We are solving for an exponent, so we use logarithms. We first divide both sides by 5 and then use logs:

$$\begin{aligned} 5 \cdot (2^t) &= 100 \\ 2^t &= 20 \\ \ln(2^t) &= \ln(20) \\ t\ln 2 &= \ln(20) \\ t &= \frac{\ln 20}{\ln 2} = 4.322. \end{aligned}$$

11. Writing this as $0.05(1.13)^t$, we have $a = 0.05$, $b = 1.13$, and $r = b - 1 = 0.13 = 13\%$. The value of k is given by $k = \ln b = \ln 1.13 = 0.1222$.

12. We rewrite this as

$$\begin{aligned} \frac{200}{e^{0.177(t-2)}} &= 200e^{-0.177(t-2)} \\ &= 200e^{-0.177t+2(0.177))} \\ &= 200e^{-0.177t}e^{0.354} \\ &= \left(200e^{0.354}\right)e^{-0.177t} \\ &= 284.9510e^{-0.177t}. \end{aligned}$$

Thus, $a = 284.9510, k = -0.177, b = e^k = 0.8378, r = b - 1 = -0.1622 = -16.22\%$.

13. **(a)** $P(t) = 51(1.03)^t = 51e^{t\ln 1.03} \approx 51e^{0.0296t}$. The population starts at 51 million with a 3 percent annual growth rate and a continuous annual growth rate of about 2.96 percent.

(b) $P(t) = 15e^{0.03t} = 15(e^{0.03})^t \approx 15(1.0305)^t$. The population starts at 15 million with an approximate 3.05% annual growth rate and a 3 percent continuous annual growth rate.

(c) $P(t) = 7.5(0.94)^t = 7.5e^{t\ln 0.94} \approx 7.5e^{-0.0619t}$. The population starts at 7.5 million with an annual percent reduction of 6% and a continuous annual decay rate of about 6.19%.

(d) $P(t) = 16e^{-0.051t} = 16(e^{-0.051})^t \approx 16(0.9503)^t$. The population starts at 16 million with an approximate 4.97% annual rate of decrease and a 5.1% continuous annual decay rate.

(e) $P(t) = 25(2^{1/18})^t = 25(e^k)^t$. Find $e^k = 2^{1/18}$, so $k = \ln(2^{1/18})$ and $P(t) \approx 25(1.0393)^t$ for an approximate annual growth rate of 3.93%, and $P(t) \approx 25e^{0.0385t}$ for an approximate continuous annual growth rate of 3.85%. (The initial population is 25 million.)

(f) Find k when $P(t) = 10((1/2)^{1/25})^t = 10(e^k)^t$. So $k = \ln\left((1/2)^{1/25}\right)$ and $P(t) \approx 10(0.9727)^t$ for an approximate annual reduction of 2.73%, and $P(t) \approx 10e^{-0.0277t}$, for an approximate continuous annual decay rate of 2.77%. (The initial population is 10 million.)

14. **(a)** If $P(t) = ab^t$, then $P(8) = ab^8$ and $P(15) = ab^{15}$. But we are told that $P(8) = 20$ and $P(15) = 28$, so

$$\begin{aligned} 28 &= ab^{15} \\ \text{and} \quad 20 &= ab^8. \\ \text{Dividing gives} \quad \frac{28}{20} &= \frac{ab^{15}}{ab^8} = b^7 \end{aligned}$$

$$\text{so} \qquad b = \left(\frac{28}{20}\right)^{\frac{1}{7}} \approx 1.04924.$$

Since $ab^8 = 20$, we now have

$$\begin{aligned} a(1.04924)^8 &= 20 \\ a &= \frac{20}{(1.04924)^8} \approx 13.615. \end{aligned}$$

Therefore, $P(t) = 13.615(1.04924)^t$.

(b) We already know that $P(t) = ae^{kt} = 13.615e^{kt}$ and that $P(t) = 13.615(1.04924)^t$, so

$$\begin{aligned} 13.615e^{kt} &= 13.615(1.04924)^t \\ e^{kt} &= 1.04924^t \\ e^k &= 1.04924 \\ \ln e^k &= \ln 1.04924 \\ \text{but } \ln e^k = k \ln e = k, \text{ so} \qquad k &= \ln 1.04924 \approx 0.04807. \end{aligned}$$

While the annual growth rate $(b-1)$ is about 4.9%, the continuous annual growth rate, k, is about 4.8%.

15. We have

$$\begin{aligned} 400(1.112)^t &= 1328 \\ 1.112^t &= \frac{1328}{400} \\ t \ln 1.112 &= \ln\left(\frac{1328}{400}\right) \\ t &= \frac{\ln(1328/400)}{\ln 1.112} \\ &= 11.3033. \end{aligned}$$

16. We have

$$\begin{aligned} 0.007e^{-1.22t} &= 0.002 \\ e^{-1.22t} &= \frac{0.002}{0.007} \\ -1.22t &= \ln\left(\frac{0.002}{0.007}\right) \\ t &= \frac{\ln(0.002/0.007)}{-1.22} \\ &= 1.0269. \end{aligned}$$

17. We have

$$\begin{aligned} 55e^{0.571t} &= 28e^{0.794t} \\ \frac{e^{0.571t}}{e^{0.794t}} &= \frac{28}{55} \\ e^{0.571t-0.794t} = e^{-0.223t} &= \frac{28}{55} \\ -0.223t &= \ln\left(\frac{28}{55}\right) \\ t &= \frac{\ln(28/55)}{-0.223} \\ &= 3.0275. \end{aligned}$$

18. Rewriting each side to the base 10 gives

$$\begin{aligned}(10^2)^{2x+3} &= \left(10^4\right)^{1/3}\\ 10^{4x+6} &= 10^{4/3}.\end{aligned}$$

Since the base of each side is the same, we can equate the exponents:

$$\begin{aligned}4x+6 &= \frac{4}{3}\\ 12x+18 &= 4\\ 12x &= -14\\ x &= -\frac{14}{12} = -\frac{7}{6}.\end{aligned}$$

19. We have

$$\begin{aligned}5(1.1)^x &= 55\\ 1.1^x &= \frac{55}{5}\\ x\ln 1.1 &= \ln\left(\frac{55}{5}\right)\\ x &= \frac{\ln(55/5)}{\ln 1.1},\end{aligned}$$

or approximately $x = 25.1589$.

20. We have

$$\begin{aligned}7e^{2t} &= 2e^{0.9t}\\ \frac{e^{2t}}{e^{0.9t}} &= \frac{2}{7}\\ e^{2t-0.9t} = e^{1.1t} &= \frac{2}{7}\\ 1.1t &= \ln\left(\frac{2}{7}\right)\\ t &= \frac{\ln(2/7)}{1.1},\end{aligned}$$

or approximately $t = -1.1389$.

21. Rewriting both sides to the base 10 gives:

$$\begin{aligned}\left(10^{-2}\right)^x &= 10^{-0.5}\\ 10^{-2x} &= 10^{-0.5}\\ -2x &= -0.5\\ x &= 0.25.\end{aligned}$$

22. Dividing by 16.3 and taking logs gives

$$\begin{aligned}16.3(1.072)^t &= 18.5\\ 1.072^t &= \frac{18.5}{16.3}\\ t\ln 1.072 &= \ln(18.5/16.3)\\ t &= \frac{\ln(18.5/16.3)}{\ln 1.072} \approx 1.821.\end{aligned}$$

23. Dividing by 13 and 25 before taking logs gives

$$\begin{aligned}
13e^{0.081t} &= 25e^{0.032t} \\
\frac{e^{0.081t}}{e^{0.032t}} &= \frac{25}{13} \\
e^{0.081t-0.032t} &= \frac{25}{13} \\
\ln e^{0.049t} &= \ln\left(\frac{25}{13}\right) \\
0.049t &= \ln\left(\frac{25}{13}\right) \\
t &= \frac{1}{0.049}\ln\left(\frac{25}{13}\right) \approx 13.345.
\end{aligned}$$

24. This equation cannot be solved analytically. Graphing $y = 87e^{0.066t}$ and $y = 3t + 7$ it is clear that these graphs will not intersect, which means $87e^{0.66t} = 3t + 7$ has no solution. The concavity of the graphs ensures that they will not intersect beyond the portions of the graphs shown in Figure 4.34.

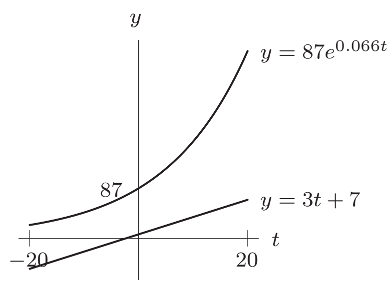


Figure 4.34

25.

$$\begin{aligned}
\frac{\log x^2 + \log x^3}{\log(100x)} &= 3 \\
\log x^2 + \log x^3 &= 3\log(100x) \\
2\log x + 3\log x &= 3(\log 100 + \log x) \\
5\log x &= 3(2 + \log x) \\
5\log x &= 6 + 3\log x \\
2\log x &= 6 \\
\log x &= 3 \\
x &= 10^3 = 1000.
\end{aligned}$$

To check, we see that

$$\begin{aligned}
\frac{\log x^2 + \log x^3}{\log(100x)} &= \frac{\log(1000^2) + \log(1000^3)}{\log(100 \cdot 1000)} \\
&= \frac{\log(1{,}000{,}000) + \log(1{,}000{,}000{,}000)}{\log(100{,}000)} \\
&= \frac{6+9}{5} \\
&= 3,
\end{aligned}$$

as required.

26.

$$
\begin{aligned}
220e^{1.323t} &= 500e^{1.118t} \\
\frac{e^{1.323t}}{e^{1.118t}} &= \frac{500}{220} \\
e^{1.323t-1.118t} = e^{0.205t} &= \frac{500}{220} \\
0.205t &= \ln\left(\frac{500}{220}\right) \\
t &= \frac{\ln(500/220)}{0.205}.
\end{aligned}
$$

An approximate solution is 4.0048.

27. $\log\left(100^{x+1}\right) = \log\left((10^2)^{x+1}\right) = 2(x+1)$.

28. $\ln\left(e \cdot e^{2+M}\right) = \ln\left(e^{3+M}\right) = 3 + M$.

29. Using the fact that $A^{-1} = 1/A$ and the log rules:

$$
\begin{aligned}
\ln(A+B) - \ln(A^{-1}+B^{-1}) &= \ln(A+B) - \ln\left(\frac{1}{A} + \frac{1}{B}\right) \\
&= \ln(A+B) - \ln\frac{A+B}{AB} \\
&= \ln\left((A+B)\cdot\frac{AB}{A+B}\right) \\
&= \ln(AB).
\end{aligned}
$$

Problems

30. We have $f(t) = ae^{kt}$ where the starting value is $a = 22{,}000$ and the continuous growth rate is $k = 0.0176$, giving $f(t) = 22{,}000e^{0.0176t}$.

31. We have $g(-30) = 200$ and $g(20) = 60$. Using the ratio method, we have

$$
\begin{aligned}
\frac{ab^{20}}{ab^{-30}} &= \frac{g(20)}{g(-30)} \\
b^{50} &= \frac{60}{200} \\
b &= \left(\frac{60}{200}\right)^{1/50} = (0.3)^{1/50} \approx 0.9762.
\end{aligned}
$$

Now we can solve for a:

$$
\begin{aligned}
a((0.3)^{1/50})^{-30} &= 200 \\
a &= \frac{200}{((0.3)^{1/50})^{-30}} \approx 97.1187.
\end{aligned}
$$

so $Q = 97.1187(0.9762)^t$. We can also write this as $g(t) = ae^{kt}$ by letting $k = \ln b = \ln 0.9762 = -0.0241$, so $g(t) = 97.1187e^{-0.0241t}$.

32. We have

$$
\begin{aligned}
V &= 650\left(1 + \frac{2.95\%}{12}\right)^{12t} \\
&= 650(1.002458)^{12t} \\
&= 650(1.02990)^t.
\end{aligned}
$$

33. We know that the starting value is $a = 5000$. After 30 years, the value doubles, so $p(30) = 10{,}000$. Writing $p(t) = ab^t$, we see that

$$\begin{aligned} p(30) &= 10{,}000 \\ 5000b^{30} &= 10{,}000 \\ b^{30} &= 2 \\ b &= 2^{1/30} = 1.0234, \end{aligned}$$

so $p(t) = 5000(1.0234)^t$.

34. **(a)** For $f(x) = 10^x$,

$$\text{Domain of } f(x) \text{ is all } x$$

$$\text{Range of } f(x) \text{ is all } y > 0.$$

There is one asymptote, the horizontal line $y = 0$.

(b) Since $g(x) = \log x$ is the inverse function of $f(x) = 10^x$, the domain of $g(x)$ corresponds to range of $f(x)$ and range of $g(x)$ corresponds to domain of $g(x)$.

$$\text{Domain of } g(x) \text{ is all } x > 0$$

$$\text{Range of } g(x) \text{ is all } y.$$

The asymptote of $f(x)$ becomes the asymptote of $g(x)$ under reflection across the line $y = x$. Thus, $g(x)$ has one asymptote, the line $x = 0$.

35. The quadratic $y = x^2 - x - 6 = (x-3)(x+2)$ has zeros at $x = -2, 3$. It is positive outside of this interval and negative within this interval. Therefore, the function $y = \ln(x^2 - x - 6)$ is undefined on the interval $-2 \le x \le 3$, so the domain is all x not in this interval.

36. **(a)** After 5 years, $B = 5000(1.12)^5 = \$8{,}811.71$. After 10 years, $B = 5000(1.12)^{10} = \$15{,}529.24$.

(b) We need to find the value of t such that $5000(1.12)^t = 10{,}000$:

$$\begin{aligned} 5000(1.12)^t &= 10{,}000 \\ 1.12^t &= 2 && \text{dividing by 5000} \\ \log\left(1.12^t\right) &= \log 2 && \text{taking logs} \\ t \cdot \log 1.12 &= \log 2 && \text{using a log property} \\ t &= \frac{\log 2}{\log 1.12} && \text{dividing} \\ &= 6.12. \end{aligned}$$

This means it will take just over 6 years for the balance to reach \$10,000. We can use a similar approach to find out how long it takes the balance to reach \$20,000:

$$\begin{aligned} 5000(1.12)^t &= 20{,}000 \\ 1.12^t &= 4 && \text{dividing by 5000} \\ \log\left(1.12^t\right) &= \log 4 && \text{taking logs} \\ t \cdot \log 1.12 &= \log 4 && \text{using a log property} \\ t &= \frac{\log 4}{\log 1.12} && \text{dividing} \\ &= 12.2. \end{aligned}$$

This means it will take just over 12 years for the balance to reach \$20,000.

37. We need to find the value of t for which

$$\begin{aligned}
P_1 &= P_2\\
51(1.031)^t &= 63(1.052)^t\\
\frac{(1.031)^t}{(1.052)^t} &= \frac{63}{51}\\
\left(\frac{1.031}{1.052}\right)^t &= \frac{63}{51}\\
\log\left(\frac{1.031}{1.052}\right)^t &= \log\frac{63}{51}\\
t\log\left(\frac{1.031}{1.052}\right) &= \log\frac{63}{51}\\
t &= \frac{\log\frac{63}{51}}{\log\left(\frac{1.031}{1.052}\right)} = -10.480.
\end{aligned}$$

So, the populations are the same 10.5 years before 1980, in the middle of 1969.

38. (a)

$$\begin{aligned}
\text{If } B = 5000(1.06)^t &= 5000e^{kt},\\
1.06^t &= (e^k)^t\\
\text{we have } e^k &= 1.06.
\end{aligned}$$

Use the natural log to solve for k,

$$k = \ln(1.06) \approx 0.0583.$$

This means that at a continuous growth rate of $5.83\%/\text{year}$, the account has an effective annual yield of 6%.

(b)

$$\begin{aligned}
7500e^{0.072t} &= 7500b^t\\
e^{0.072t} &= b^t\\
e^{0.072} &= b\\
b &\approx 1.0747
\end{aligned}$$

This means that an account earning 7.2% continuous annual interest has an effective yield of 7.47%.

39. (a) If t represents the number of years since 2005, let $W(t) =$ population of Erehwon at time t, in millions of people, and let $C(t) =$ population of Ecalpon at time t, in millions of people. Since the population of both Erehwon and Ecalpon are increasing by a constant percent, we know that they are both exponential functions. In Erehwon, the growth factor is 1.029. Since its population in 2005 (when $t = 0$) is 50 million people, we know that

$$W(t) = 50(1.029)^t.$$

In Ecalpon, the growth factor is 1.032, and the population starts at 45 million, so

$$C(t) = 45(1.032)^t.$$

(b) The two countries will have the same population when $W(t) = C(t)$. We therefore need to solve:

$$\begin{aligned}
50(1.029)^t &= 45(1.032)^t\\
\frac{1.032^t}{1.029^t} = \left(\frac{1.032}{1.029}\right)^t &= \frac{50}{45} = \frac{10}{9}\\
\log\left(\frac{1.032}{1.029}\right)^t &= \log\left(\frac{10}{9}\right)\\
t\log\left(\frac{1.032}{1.029}\right) &= \log\left(\frac{10}{9}\right)\\
t &= \frac{\log(10/9)}{\log(1.032/1.029)} = 36.191.
\end{aligned}$$

So the populations are equal after about 36.191 years.

(c) The population of Ecalpon is double the population of Erehwon when

$$C(t) = 2W(t)$$

that is, when

$$45(1.032)^t = 2 \cdot 50(1.029)^t.$$

We use logs to solve the equation.

$$\begin{aligned}
45(1.032)^t &= 100(1.029)^t \\
\frac{(1.032)^t}{(1.029)^t} &= \frac{100}{45} = \frac{20}{9} \\
\left(\frac{1.032}{1.029}\right)^t &= \frac{20}{9} \\
\log\left(\frac{1.032}{1.029}\right)^t &= \log\left(\frac{20}{9}\right) \\
t\log\left(\frac{1.032}{1.029}\right) &= \log\left(\frac{20}{9}\right) \\
t &= \frac{\log(20/9)}{\log(1.032/1.029)} = 274.287 \text{ years.}
\end{aligned}$$

So it will take about 274 years for the population of Ecalpon to be twice that of Erehwon.

40. **(a)** Let t be the number of years in a man's age above 30 (i.e. let t =the man's age-30) and let M_0 denote his bone mass at age 30. If he is losing 2% per year, then 98% remains after each year, and thus we can say that $M(t) = M_0(0.98)^t$, where $M(t)$ represents the man's bone mass t years after age 30. But we want a formula describing bone mass in terms of a, his age. Since t is number of years in his age over 30, $t = a - 30$. So, we can substitute $a - 30$ for t in our formula to find an expression in terms of a:

$$M(a) = M_0(0.98)^{(a-30)}.$$

(b) We want to know for what value of a

$$\begin{aligned}
M(a) &= \frac{1}{2}M_0 \\
\text{Therefore, we will solve}\quad M_0(0.98)^{(a-30)} &= \frac{1}{2}M_0 \\
(0.98)^{(a-30)} &= \frac{1}{2} \\
\log\left((0.98)^{(a-30)}\right) &= \log\frac{1}{2} = \log 0.5 \\
(a-30)\log(0.98) &= \log 0.5 \\
a - 30 &= \frac{\log 0.5}{\log 0.98} \\
a &= 30 + \frac{\log 0.5}{\log(0.98)} \approx 64.3
\end{aligned}$$

The average man will have lost half his bone mass at approximately 64.3 years of age.

41. **(a)** The number of bacteria present after 1/2 hour is

$$N = 1000e^{0.69(1/2)} \approx 1412.$$

If you notice that $0.69 \approx \ln 2$, you could also say

$$N = 1000e^{0.69/2} \approx 1000e^{\frac{1}{2}\ln 2} = 1000e^{\ln 2^{1/2}} = 1000e^{\ln\sqrt{2}} = 1000\sqrt{2} \approx 1412.$$

(b) We solve for t in the equation

$$\begin{aligned}
1{,}000{,}000 &= 1000e^{0.69t} \\
e^{0.69t} &= 1000 \\
0.69t &= \ln 1000 \\
t &= \left(\frac{\ln 1000}{0.69}\right) \approx 10.011 \text{ hours.}
\end{aligned}$$

(c) The doubling time is the time t such that $N = 2000$, so

$$\begin{aligned}
2000 &= 1000e^{0.69t} \\
e^{0.69t} &= 2 \\
0.69t &= \ln 2 \\
t &= \left(\frac{\ln 2}{0.69}\right) \approx 1.005 \text{ hours.}
\end{aligned}$$

If you notice that $0.69 \approx \ln 2$, you see why the half-life turns out to be 1 hour:

$$\begin{aligned}
e^{0.69t} &= 2 \\
e^{t\ln 2} &\approx 2 \\
e^{\ln 2^t} &\approx 2 \\
2^t &\approx 2 \\
t &\approx 1
\end{aligned}$$

42. (a) If $Q(t) = Q_0 b^t$ describes the number of gallons left in the tank after t hours, then Q_0, the amount we started with, is 250, and b, the percent left in the tank after 1 hour, is 96%. Thus $Q(t) = 250(0.96)^t$. After 10 hours, there are $Q(10) = 250(0.96)^{10} \approx 166.208$ gallons left in the tank. This $\frac{166.208}{250} = 0.66483 = 66.483\%$ of what had initially been in the tank. Therefore approximately 33.517% has leaked out. It is less than 40% because the loss is 4% of 250 only during the first hour; for each hour after that it is 4% of whatever quantity is left.

(b) Since $Q_0 = 250$, $Q(t) = 250e^{kt}$. But we can also define $Q(t) = 250(0.96)^t$, so

$$\begin{aligned}
250e^{kt} &= 250(0.96)^t \\
e^{kt} &= 0.96^t \\
e^k &= 0.96 \\
\ln e^k &= \ln 0.96 \\
k\ln e &= \ln 0.96 \\
k &= \ln 0.96 \approx -0.04082.
\end{aligned}$$

Since k is negative, we know that the value of $Q(t)$ is decreasing by 4.082% per hour. Therefore, k is the continuous hourly decay rate.

43. (a) The population has increased by $34{,}000 - 30{,}000 = 4{,}000$ people in that time period, so its total percent increase is $\frac{4{,}000}{30{,}000} = 0.13333 = 13.333\%$.

(b) If b represents the annual growth factor, then in five years the population will have grown by a factor of b^5. We learned in part (a) that the population has increased by 13.333% in that time, so it is 113.333% of what it had been five years earlier. Thus

$$\begin{aligned}
b^5 &= 1.13333 \\
b &= (1.13333)^{\frac{1}{5}} \approx 1.02535.
\end{aligned}$$

If, at the end of each year, the population is 102.5345% of what it had been at the beginning of the year, then the rate of growth is about 2.534% per year.

(c) The continuous annual growth rate is represented by k in the formula $P(t) = P_0e^{kt}$. Since we know that the initial population is 30,000 and the growth factor is 102.5345 (from (b)), we can say that $P(t) = 30{,}000(1.02535)^t$ defines this function. To find k, we can equate the two formulas:

$$\begin{aligned}30{,}000(1.02535)^t &= P_0e^{kt} = 30{,}000e^{kt}\\ 1.02535^t &= e^{kt}\\ 1.02535 &= e^k\\ \ln 1.02535 &= \ln e^k\\ \ln 1.02535 &= k\\ k &\approx 0.02503.\end{aligned}$$

Thus, while the population is growing at 2.534% per year, it is growing at a rate of 2.503% at any given instant.

44. For what value of t will $Q(t) = 0.23Q_0$?

$$\begin{aligned}0.23Q_0 &= Q_0e^{-0.000121t}\\ 0.23 &= e^{-0.000121t}\\ \ln 0.23 &= \ln e^{-0.000121t}\\ \ln 0.23 &= -0.000121t\\ t &= \frac{\ln 0.23}{-0.000121} = 12146.082.\end{aligned}$$

So the skull is about 12,146 years old.

45. (a) Since the drug is being metabolized continuously, the formula for describing the amount left in the bloodstream is $Q(t) = Q_0e^{kt}$. We know that we start with 2 mg, so $Q_0 = 2$, and the rate of decay is 4%, so $k = -0.04$. (Why is k negative?) Thus $Q(t) = 2e^{-0.04t}$.

(b) To find the percent decrease in one hour, we need to rewrite our equation in the form $Q = Q_0b^t$, where b gives us the percent left after one hour:

$$Q(t) = 2e^{-0.04t} = 2(e^{-0.04})^t \approx 2(0.96079)^t.$$

We see that $b \approx 0.96079 = 96.079\%$, which is the percent we have left after one hour. Thus, the drug level decreases by about 3.921% each hour.

(c) We want to find out when the drug level reaches 0.25 mg. We therefore ask when $Q(t)$ will equal 0.25.

$$\begin{aligned}2e^{-0.04t} &= 0.25\\ e^{-0.04t} &= 0.125\\ -0.04t &= \ln 0.125\\ t &= \frac{\ln 0.125}{-0.04} \approx 51.986.\end{aligned}$$

Thus, the second injection is required after about 52 hours.

(d) After the second injection, the drug level is 2.25 mg, which means that Q_0, the initial amount, is now 2.25. The decrease is still 4% per hour, so when will the level reach 0.25 again? We need to solve the equation

$$2.25e^{-0.04t} = 0.25,$$

where t is now the number of hours since the second injection.

$$\begin{aligned}e^{-0.04t} &= \frac{0.25}{2.25} = \frac{1}{9}\\ -0.04t &= \ln(1/9)\\ t &= \frac{\ln(1/9)}{-0.04} \approx 54.931.\end{aligned}$$

Thus the third injection is required about 55 hours after the second injection, or about $52 + 55 = 107$ hours after the first injection.

46. (a) (i) $0.4 = \frac{2}{5}$, so $\log(2/5) = \log 2 - \log 5 = u - v$

(ii) $\log 0.25 = \log\left(\frac{1}{4}\right) = \log(2^{-2}) = -2\log 2 = -2u$

(iii) $\log 40 = \log(2^3 \cdot 5) = \log 2^3 + \log 5 = 3\log 2 + \log 5 = 3u + v$

(iv) $\log\sqrt{10} = \log 10^{\frac{1}{2}} = \log(2 \cdot 5)^{\frac{1}{2}} = \frac{1}{2}(\log 2 + \log 5) = \frac{1}{2}(u+v)$

(b)

$$\begin{aligned}
\frac{1}{2}(u+2v) &= \frac{1}{2}(\log 2 + 2\log 5)\\
&= \frac{1}{2}\log(2 \cdot 5^2)\\
&= \log\sqrt{50}\\
&\approx \log\sqrt{49}\\
&= \log 7.
\end{aligned}$$

47. (a)

$$\begin{aligned}
e^{x+3} &= 8\\
\ln e^{x+3} &= \ln 8\\
x + 3 &= \ln 8\\
x &= \ln 8 - 3 \approx -0.9206
\end{aligned}$$

(b)

$$\begin{aligned}
4(1.12^x) &= 5\\
1.12^x &= \frac{5}{4} = 1.25\\
\log 1.12^x &= \log 1.25\\
x\log 1.12 &= \log 1.25\\
x &= \frac{\log 1.25}{\log 1.12} \approx 1.9690
\end{aligned}$$

(c)

$$\begin{aligned}
e^{-0.13x} &= 4\\
\ln e^{-0.13x} &= \ln 4\\
-0.13x &= \ln 4\\
x &= \frac{\ln 4}{-0.13} \approx -10.6638
\end{aligned}$$

(d)

$$\begin{aligned}
\log(x-5) &= 2\\
x - 5 &= 10^2\\
x &= 10^2 + 5 = 105
\end{aligned}$$

(e)

$$\begin{aligned}
2\ln(3x) + 5 &= 8\\
2\ln(3x) &= 3\\
\ln(3x) &= \frac{3}{2}\\
3x &= e^{\frac{3}{2}}\\
x &= \frac{e^{\frac{3}{2}}}{3} \approx 1.4939
\end{aligned}$$

(f)

$$
\begin{aligned}
\ln x - \ln(x-1) &= \frac{1}{2} \\
\ln\left(\frac{x}{x-1}\right) &= \frac{1}{2} \\
\frac{x}{x-1} &= e^{\frac{1}{2}} \\
x &= (x-1)e^{\frac{1}{2}} \\
x &= xe^{\frac{1}{2}} - e^{\frac{1}{2}} \\
e^{\frac{1}{2}} &= xe^{\frac{1}{2}} - x \\
e^{\frac{1}{2}} &= x(e^{\frac{1}{2}} - 1) \\
\frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}} - 1} &= x \\
x &\approx 2.5415
\end{aligned}
$$

Note: (g) (h) and (i) can not be solved analytically, so we use graphs to approximate the solutions.

(g) From Figure 4.35 we can see that $y = e^x$ and $y = 3x + 5$ intersect at $(2.534, 12.601)$ and $(-1.599, 0.202)$, so the values of x which satisfy $e^x = 3x+5$ are $x = 2.534$ or $x = -1.599$. We also see that $y_1 \approx 12.601$ and $y_2 \approx 0.202$.

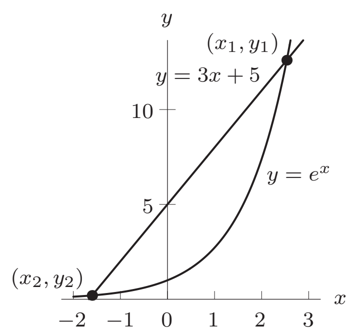


Figure 4.35

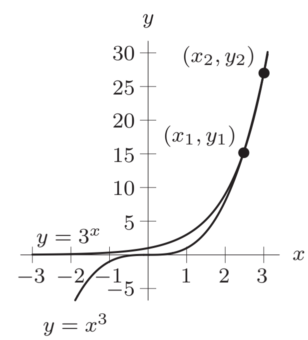


Figure 4.36

(h) The graphs of $y = 3^x$ and $y = x^3$ are seen in Figure 4.36. It is very hard to see the points of intersection, though $(3, 27)$ would be an immediately obvious choice (substitute 3 for x in each of the formulas). Using technology, we can find a second point of intersection, $(2.478, 15.216)$. So the solutions for $3^x = x^3$ are $x = 3$ or $x = 2.478$.

Since the points of intersection are very close, it is difficult to see these intersections even by zooming in. So, alternatively, we can find where $y = 3^x - x^3$ crosses the x-axis. See Figure 4.37.

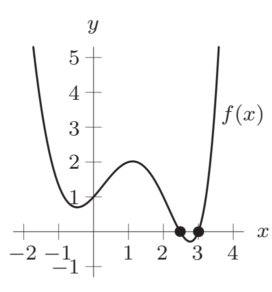


Figure 4.37

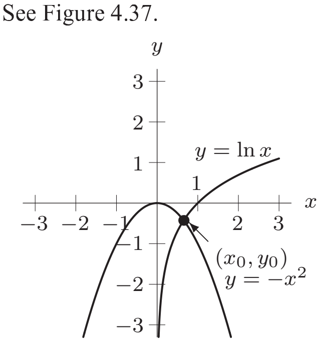


Figure 4.38

(i) From the graph in Figure 4.38, we see that $y = \ln x$ and $y = -x^2$ intersect at $(0.6529, -0.4263)$, so $x = 0.6529$ is the solution to $\ln x = -x^2$.

48. (a) Solving for x exactly:

$$\begin{aligned}
\frac{3^x}{5^{(x-1)}} &= 2^{(x-1)} \\
3^x &= 5^{x-1} \cdot 2^{x-1} \\
3^x &= (5 \cdot 2)^{x-1} \\
3^x &= 10^{x-1} \\
\log 3^x &= \log 10^{x-1} \\
x \log 3 &= (x-1)\log 10 = (x-1)(1) \\
x \log 3 &= x - 1 \\
x \log 3 - x &= -1 \\
x(\log 3 - 1) &= -1 \\
x &= \frac{-1}{\log 3 - 1} = \frac{1}{1 - \log 3}
\end{aligned}$$

(b)

$$\begin{aligned}
-3 + e^{x+1} &= 2 + e^{x-2} \\
e^{x+1} - e^{x-2} &= 2 + 3 \\
e^x e^1 - e^x e^{-2} &= 5 \\
e^x(e^1 - e^{-2}) &= 5 \\
e^x &= \frac{5}{e - e^{-2}} \\
\ln e^x &= \ln\left(\frac{5}{e - e^{-2}}\right) \\
x &= \ln\left(\frac{5}{e - e^{-2}}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\ln(2x-2) - \ln(x-1) &= \ln x \\
\ln\left(\frac{2x-2}{x-1}\right) &= \ln x \\
\frac{2x-2}{x-1} &= x \\
\frac{2(x-1)}{(x-1)} &= x \\
2 &= x
\end{aligned}$$

(d) Let $z = 3^x$, then $z^2 = (3^x)^2 = 9^x$, and so we have

$$\begin{aligned}
z^2 - 7z + 6 &= 0 \\
(z-6)(z-1) &= 0 \\
z = 6 \quad &\text{or} \quad z = 1.
\end{aligned}$$

Thus, $3^x = 1$ or $3^x = 6$, and so $x = 0$ or $x = \ln 6/\ln 3$.

(e)

$$\begin{aligned}
\ln\left(\frac{e^{4x}+3}{e}\right) &= 1 \\
e^1 &= \frac{e^{4x}+3}{e}
\end{aligned}$$

$$
\begin{aligned}
e^2 &= e^{4x} + 3 \\
e^2 - 3 &= e^{4x} \\
\ln(e^2 - 3) &= \ln\left(e^{4x}\right) \\
\ln(e^2 - 3) &= 4x \\
\frac{\ln(e^2 - 3)}{4} &= x
\end{aligned}
$$

(f)

$$
\begin{aligned}
\frac{\ln(8x) - 2\ln(2x)}{\ln x} &= 1 \\
\ln(8x) - 2\ln(2x) &= \ln x \\
\ln(8x) - \ln\left((2x)^2\right) &= \ln x \\
\ln\left(\frac{8x}{(2x)^2}\right) &= \ln x \\
\ln\left(\frac{8x}{4x^2}\right) &= \ln x \\
\frac{8x}{4x^2} &= x \\
8x &= 4x^3 \\
4x^3 - 8x &= 0 \\
4x(x^2 - 2) &= 0 \\
x &= 0, \sqrt{2}, -\sqrt{2}
\end{aligned}
$$

Only $\sqrt{2}$ is a valid solution, because when $-\sqrt{2}$ and 0 are substituted into the original equation we are taking the logarithm of negative numbers and 0, which is undefined.

49. At an annual growth rate of 1%, the Rule of 70 tells us this investment doubles in $70/1 = 70$ years. At a 2% rate, the doubling time should be about $70/2 = 35$ years. The doubling times for the other rates are, according to the Rule of 70,

$$
\frac{70}{5} = 14 \text{ years}, \qquad \frac{70}{7} = 10 \text{ years}, \qquad \text{and} \qquad \frac{70}{10} = 7 \text{ years}.
$$

To check these predictions, we use logs to calculate the actual doubling times. If V is the dollar value of the investment in year t, then at a 1% rate, $V = 1000(1.01)^t$. To find the doubling time, we set $V = 2000$ and solve for t:

$$
\begin{aligned}
1000(1.01)^t &= 2000 \\
1.01^t &= 2 \\
\log(1.01^t) &= \log 2 \\
t\log 1.01 &= \log 2 \\
t = \frac{\log 2}{\log 1.01} &\approx 69.661.
\end{aligned}
$$

This agrees well with the prediction of 70 years. Doubling times for the other rates have been calculated and recorded in Table 4.18 together with the doubling times predicted by the Rule of 70.

Table 4.18 *Doubling times predicted by the Rule of 70 and actual values*

Rate (%)	1	2	5	7	10
Predicted doubling time (years)	70	35	14	10	7
Actual doubling time (years)	69.661	35.003	14.207	10.245	7.273

The Rule of 70 works reasonably well when the growth rate is small. The Rule of 70 does not give good estimates for growth rates much higher than 10%. For example, at an annual rate of 35%, the Rule of 70 predicts that the doubling time is $70/35 = 2$ years. But in 2 years at 35% growth rate, the \$1000 investment from the last example would be not worth \$2000, but only

$$1000(1.35)^2 = \$1822.50.$$

50. Let $Q = ae^{kt}$ be an increasing exponential function, so that k is positive. To find the doubling time, we find how long it takes Q to double from its initial value a to the value $2a$:

$$\begin{aligned} ae^{kt} &= 2a \\ e^{kt} &= 2 \qquad \text{(dividing by } a\text{)} \\ \ln e^{kt} &= \ln 2 \\ kt &= \ln 2 \qquad \text{(because } \ln e^x = x \text{ for all } x\text{)} \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Using a calculator, we find $\ln 2 = 0.693 \approx 0.70$. This is where the 70 comes from.

If, for example, the continuous growth rate is $k = 0.07 = 7\%$, then

$$\text{Doubling time} = \frac{\ln 2}{0.07} = \frac{0.693}{0.07} \approx \frac{0.70}{0.07} = \frac{70}{7} = 10.$$

If the growth rate is $r\%$, then $k = r/100$. Therefore

$$\text{Doubling time} = \frac{\ln 2}{k} = \frac{0.693}{k} \approx \frac{0.70}{r/100} = \frac{70}{r}.$$

51. We need to solve

$$P = \frac{\frac{r}{12}L}{1-(1+\frac{r}{12})^{-m}},$$

for m, where $P = 330$, $L = 25,000$ and $r = 0.069$.

$$\begin{aligned} 330 &= \frac{(\frac{0.069}{12})25,000}{1-(1+\frac{0.069}{12})^{-m}} \\ 330 &= \frac{143.75}{1-(1.00575)^{-m}} \\ 330\cdot(1-(1.00575)^{-m}) &= 143.75 \\ 1-(1.00575)^{-m} &= \frac{143.75}{330} \\ 1-(1.00575)^{-m} &\approx 0.436 \\ -(1.00575)^{-m} &\approx -0.564 \\ (1.00575)^{-m} &\approx 0.564 \\ \log\left((1.00575)^{-m}\right) &\approx \log(0.564) \\ -m\log(1.00575) &\approx \log(0.564) \\ -m &\approx \frac{\log(0.564)}{\log(1.00575)} \\ -m &\approx \frac{-0.249}{0.00249} \\ -m &\approx -99.764 \\ m &\approx 99.764 \end{aligned}$$

So it will take about 100 months, or 8 years and 4 months, to pay off the loan.

52. **(a)** Table 4.19 describes the height of the ball after n bounces:

Table 4.19

n	$h(n)$
0	6
1	90% of $6 = 6(0.9) = 5.4$
2	90% of $5.4 = 5.4(0.9) = 6(0.9)(0.9) = 6(0.9)^2$
3	90% of $6(0.9)^2 = 6(0.9)^2 \cdot (0.9) = 6(0.9)^3$
4	$6(0.9)^3 \cdot (0.9) = 6(0.9)^4$
5	$6(0.9)^5$
$\vdots$	$\vdots$
n	$6(0.9)^n$

so $h(n) = 6(0.9)^n$.

(b) We want to find the height when $n = 12$, so we will evaluate $h(12)$:

$$h(12) = 6(0.9)^{12} \approx 1.695 \text{ feet (about 1 ft 8.3 inches).}$$

(c) We are looking for the values of n for which $h(n) \leq 1$ inch $= \frac{1}{12}$ foot. So

$$\begin{aligned} h(n) &\leq \frac{1}{12} \\ 6(0.9)^n &\leq \frac{1}{12} \\ (0.9)^n &\leq \frac{1}{72} \\ \log(0.9)^n &\leq \log\frac{1}{72} \\ n\log(0.9) &\leq \log\frac{1}{72} \end{aligned}$$

Using your calculator, you will notice that $\log(0.9)$ is negative. This tells us that when we divide both sides by $\log(0.9)$, we must reverse the inequality. We now have

$$n \geq \frac{\log\frac{1}{72}}{\log(0.9)} \approx 40.591$$

So, the ball will rise less than 1 inch by the 41^{st} bounce.

53. If $P(t)$ describes the number of people in the store t minutes after it opens, we need to find a formula for $P(t)$. Perhaps the easiest way to develop this formula is to first find a formula for $P(k)$ where k is the number of 40-minute intervals since the store opened. After the first such interval there are $500(2) = 1{,}000$ people; after the second interval, there are $1{,}000(2) = 2{,}000$ people. Table 4.20 describes this progression:

Table 4.20

k	$P(k)$
0	500
1	$500(2)$
2	$500(2)(2) = 500(2)^2$
3	$500(2)(2)(2) = 500(2)^3$
4	$500(2)^4$
$\vdots$	$\vdots$
k	$500(2)^k$

From this, we conclude that $P(k) = 500(2)^k$. We now need to see how k and t compare. If $t = 120$ minutes, then we know that $k = \frac{120}{40} = 3$ intervals of 40 minutes; if $t = 187$ minutes, then $k = \frac{187}{40}$ intervals of 40 minutes. In general, $k = \frac{t}{40}$. Substituting $k = \frac{t}{40}$ into our equation for $P(k)$, we get an equation for the number of people in the store t minutes after the store opens:

$$P(t) = 500(2)^{\frac{t}{40}}.$$

To find the time when we'll need to post security guards, we need to find the value of t for which $P(t) = 10{,}000$.

$$\begin{aligned} 500(2)^{t/40} &= 10{,}000 \\ 2^{t/40} &= 20 \\ \log\left(2^{t/40}\right) &= \log 20 \\ \frac{t}{40}\log(2) &= \log 20 \\ t(\log 2) &= 40\log 20 \\ t &= \frac{40\log 20}{\log 2} \approx 172.877 \end{aligned}$$

The guards should be commissioned about 173 minutes after the store is opened, or 12:53 pm.

54. **(a)** Using $B = 4.250$ and $T = 2.5$, $R = \log\left(\frac{a}{T}\right) + B$ becomes

$$R = \log\left(\frac{a}{2.5}\right) + 4.25.$$

If $R = 6.1$, then we want to solve

$$\begin{aligned} 6.1 &= \log\left(\frac{a}{2.5}\right) + 4.250 \\ 1.850 &= \log\left(\frac{a}{2.5}\right). \end{aligned}$$

If $y = \log x$, then $10^y = x$. So we can rewrite this equation to get

$$\begin{aligned} 10^{1.850} &= \frac{a}{2.5} \\ (2.5)(10^{1.850}) &= a \\ a &\approx (2.5)(70.8) \approx 176.986 \text{ microns.} \end{aligned}$$

(b) Another way to find the value of a is to first solve the equation for a

$$\begin{aligned} R &= \log\left(\frac{a}{T}\right) + B \\ R - B &= \log\left(\frac{a}{T}\right). \end{aligned}$$

Writing in exponential form:

$$\begin{aligned} 10^{(R-B)} &= \frac{a}{T} \\ a &= 10^{(R-B)}(T) \end{aligned}$$

In this case,

$$a = 10^{(7.1-4.250)}(2.5) \approx 1769.864 \text{ microns.}$$

(c) The values of R differ by 1 $(= 7.1 - 6.1)$, but the values of a differ by a factor of 10 $(= 1769.864/176.986)$.

55. (a) We will divide this into two parts and first show that $1 < \ln 3$. Since

$$e < 3$$

and ln is an increasing function, we can say that

$$\ln e < \ln 3.$$

But $\ln e = 1$, so

$$1 < \ln 3.$$

To show that $\ln 3 < 2$, we will use two facts:

$$3 < 4 = 2^2 \quad \text{and} \quad 2^2 < e^2.$$

Combining these two statements, we have $3 < 2^2 < e^2$, which tells us that $3 < e^2$. Then, using the fact that $\ln e^2 = 2$, we have

$$\ln 3 < \ln e^2 = 2$$

Therefore, we have

$$1 < \ln 3 < 2.$$

(b) To show that $1 < \ln 4$, we use our results from part (a), that $1 < \ln 3$. Since

$$3 < 4,$$

we have

$$\ln 3 < \ln 4.$$

Combining these two statements, we have

$$1 < \ln 3 < \ln 4, \quad \text{so} \quad 1 < \ln 4.$$

To show that $\ln 4 < 2$ we again use the fact that $4 = 2^2 < e^2$. Since $\ln e^2 = 2$, and ln is an increasing function, we have

$$\ln 4 < \ln e^2 = 2.$$

CHECK YOUR UNDERSTANDING

1. True. If x is a positive number, $\log x$ is defined and $10^{\log x} = x$.
2. True. This is the definition of a logarithm.
3. False. Since $10^{-k} = 1/10^k$ we see the value is positive.
4. True. The log function outputs the power of 10 which in this case is n.
5. True. The value of $\log n$ is the exponent to which 10 is raised to get n.
6. False. If a and b are positive, $\log\left(\dfrac{a}{b}\right) = \log a - \log b$.
7. False. If a and b are positive, $\ln a + \ln b = \ln(ab)$. There is no simple formula for $\ln(a+b)$.
8. False. For example, $\log 10 = 1$, but $\ln 10 \approx 2.3$.
9. True. The natural log function and the e^x function are inverses.
10. False. The log function has a vertical asymptote at $x = 0$.
11. True. As x increases, $\log x$ increases but at a slower and slower rate.
12. True. The two functions are inverses of one another.
13. True. Since $y = \log\sqrt{x} = \log(x^{1/2}) = \frac{1}{2}\log x$.
14. False. Consider $b = 10$ and $t = 2$, then $\log(10^2) = 2$, but $(\log 10)^2 = 1^2 = 1$. For $b > 0$, the correct formula is $\log(b^t) = t\log b$.

15. True. Think of these as $\ln e^1 = 1$ and $\log 10^1 = 1$.
16. False. Taking the natural log of both sides we see $t = \ln 7.32$.
17. True. Divide both sides of the first equation by 50. Then take the log of both sides and finally divide by $\log 0.345$ to solve for t.
18. True. Divide both sides of the first equation by a. Then take the log of both sides and finally divide by $\log b$ to solve for t.
19. False. It is the time it takes for the Q-value to double.
20. True. This is the definition of half-life.
21. False. Since $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$, it takes only two half- life periods. That is 10 hours.
22. True. Replace the base 3 in the first equation with $3 = e^{\ln 3}$.
23. False. Since for $2P = Pe^{20r}$, we have $r = \frac{\ln 2}{20} = 0.035$ or 3.5%.
24. True. Solve for t by dividing both sides by Q_0, taking the ln of both sides and then dividing by k.
25. True. For example, astronomical distances.
26. True. Both scales are calibrated with powers of 10, which is a log scale.
27. False. An order of magnitude is a power of 10. They differ by a multiple of 1000 or three orders of magnitude.
28. False. There is no simple relation between the values of A and B and the data set.
29. False. The fit will not be as good as $y = x^3$ but an exponential function can be found.

Solutions to Tools for Chapter 4

1. $\log(\log 10) = \log(1) = 0.$
2. $\ln(\ln e) = \ln(1) = 0.$
3. $\log 0.0001 = \log 10^{-4} = -4 \log 10 = -4.$
4. $2 \ln e^4 = 2(4 \ln e) = 2(4) = 8.$
5. $\dfrac{\log 100^6}{\log 100^2} = \dfrac{6 \log 100}{2 \log 100} = \dfrac{6}{2} = 3.$
6. $\ln\left(\dfrac{1}{e^5}\right) = \ln 1 - \ln e^5 = \ln 1 - 5 \ln e = 0 - 5 = -5.$
7. $\dfrac{\log 1}{\log 10^5} = \dfrac{0}{5 \log 10} = \dfrac{0}{5} = 0.$
8. $e^{\ln 3} - \ln e = 3 - 1 = 2.$
9. $\sqrt{\log 10{,}000} = \sqrt{\log 10^4} = \sqrt{4 \log 10} = \sqrt{4} = 2.$
10. By definition, $10^{\log 7} = 7.$
11. The equation $10^5 = 100{,}000$ is equivalent to $\log 100{,}000 = 5.$
12. The equation $10^{-4} = 0.0001$ is equivalent to $\log 0.0001 = -4.$
13. The equation $10^{0.477} = 3$ is equivalent to $\log 3 = 0.477.$
14. The equation $e^2 = 7.389$ is equivalent to $\ln 7.389 = 2.$
15. The equation $e^{-2} = 0.135$ is equivalent to $\ln 0.135 = -2.$
16. The equation $e^{2x} = 7$ is equivalent to $2x = \ln 7.$
17. The equation $\log 0.01 = -2$ is equivalent to $10^{-2} = 0.01.$
18. The equation $\log(x + 3) = 2$ is equivalent to $10^2 = x + 3.$
19. The equation $\ln x = -1$ is equivalent to $e^{-1} = x.$
20. The equation $\ln 4 = x^2$ is equivalent to $e^{x^2} = 4.$

21. Rewrite the logarithm of the product as a sum, $\log 2x = \log 2 + \log x$.

22. Rewrite the expression as a sum and then use the power property,

$$\ln(x(7-x)^3) = \ln x + \ln(7-x)^3 = \ln x + 3\ln(7-x).$$

23. The expression is not the logarithm of a quotient, so it cannot be rewritten using the properties of logarithms.

24. The logarithm of a quotient rule applies, so $\log\left(\frac{x}{5}\right) = \log x - \log 5$.

25. The logarithm of a quotient and the power property apply, so

$$\begin{aligned}\log\left(\frac{x^2+1}{x^3}\right) &= \log(x^2+1) - \log x^3\\ &= \log(x^2+1) - 3\log x.\end{aligned}$$

26. Rewrite the power,

$$\ln\sqrt{\frac{x-1}{x+1}} = \ln\frac{(x-1)^{1/2}}{(x+1)^{1/2}}.$$

Use the quotient and power properties,

$$\ln\frac{(x-1)^{1/2}}{(x+1)^{1/2}} = \ln(x-1)^{1/2} - \ln(x+1)^{1/2} = (1/2)\ln(x-1) - (1/2)\ln(x+1).$$

27. There is no rule for the logarithm of a sum, it cannot be rewritten.

28. Using the properties of logarithms we have

$$\begin{aligned}\ln\left(\frac{xy^2}{z}\right) &= \ln xy^2 - \ln z\\ &= \ln x + \ln y^2 - \ln z\\ &= \ln x + 2\ln y - \ln z.\end{aligned}$$

29. In general the logarithm of a difference cannot be simplified. In this case we rewrite the expression so that it is the logarithm of a product.

$$\log(x^2-y^2) = \log((x+y)(x-y)) = \log(x+y) + \log(x-y).$$

30. The expression is a product of logarithms, not a logarithm of a product, so it cannot be simplified.

31. The expression is a quotient of logarithms, not a logarithm of a quotient, but we can use the properties of logarithms to rewrite it without the x^2 term:

$$\frac{\ln x^2}{\ln(x+2)} = \frac{2\ln x}{\ln(x+2)}.$$

32. Rewrite the sum as $\log 12 + \log x = \log 12x$.

33. Rewrite the sum as $\ln x^3 + \ln x^2 = \ln(x^3 \cdot x^2) = \ln x^5$.

34. Rewrite the difference as

$$\ln x^2 - \ln(x+10) = \ln\frac{x^2}{x+10}.$$

35. Rewrite with powers and combine,

$$\frac{1}{2}\log x + 4\log y = \log\sqrt{x} + \log y^4 = \log(\sqrt{x}y^4).$$

36. Rewrite with powers and combine,

$$\log 3 + 2\log\sqrt{x} = \log 3 + \log(\sqrt{x})^2 = \log 3 + \log x = \log 3x.$$

37. Rewrite with powers and combine,

$$\begin{aligned}\frac{1}{3}\log 8 - \frac{1}{2}\log 25 &= \log 8^{1/3} - \log 25^{1/2}\\ &= \log 2 - \log 5\\ &= \log\frac{2}{5}.\end{aligned}$$

38. Rewrite with powers and combine,

$$\begin{aligned}3\left(\log(x+1) + \frac{2}{3}\log(x+4)\right) &= 3\log(x+1) + 2\log(x+4)\\ &= \log(x+1)^3 + \log(x+4)^2\\ &= \log\left((x+1)^3(x+4)^2\right)\end{aligned}$$

39. Rewrite as

$$\ln x + \ln\left(\frac{y}{2}(x+4)\right) + \ln z^{-1} = \ln x + \ln\left(\frac{xy+4y}{2}\right) - \ln z = \ln\left(\frac{(x^2)y+4xy}{2}\right) - \ln z = \ln\left(\frac{(x^2)y+4xy}{2z}\right).$$

40. Rewrite with powers and combine,

$$\begin{aligned}2\log(9-x^2) - (\log(3+x) + \log(3-x)) &= \log(9-x^2)^2 - (\log(3+x)(3-x))\\ &= \log(9-x^2)^2 - \log(9-x^2)\\ &= \log\frac{(9-x^2)^2}{(9-x^2)}\\ &= \log(9-x^2).\end{aligned}$$

41. Rewrite as $10^{-\log 5x} = 10^{\log(5x)^{-1}} = (5x)^{-1}$.

42. Rewrite as $e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$.

43. Rewrite as $2\ln e^{\sqrt{x}} = 2\sqrt{x}$.

44. The logarithm of a sum cannot be simplified.

45. Rewrite as $t\ln e^{t/2} = t(t/2) = t^2/2$.

46. Rewrite as $10^{2+\log x} = 10^2 \cdot 10^{\log x} = 100x$.

47. Rewrite as $\log(10x) - \log x = \log(10x/x) = \log 10 = 1$.

48. Rewrite as $2\ln x^{-2} + \ln x^4 = 2(-2)\ln x + 4\ln x = 0$.

49. Rewrite as $\ln\sqrt{x^2+16} = \ln(x^2+16)^{1/2} = \frac{1}{2}\ln(x^2+16)$.

50. Rewrite as $\log 100^{2z} = 2z\log 100 = 2z(2) = 4z$.

51. Rewrite as $\dfrac{\ln e}{\ln e^2} = \dfrac{\ln e}{2\ln e} = \dfrac{1}{2}$.

52. Rewrite as $\ln \dfrac{1}{e^x+1} = \ln 1 - \ln(e^x+1) = -\ln(e^x+1)$.

53. Taking logs of both sides we get
$$\log(4^x) = \log 9.$$
This gives
$$x \log 4 = \log 9$$
or in other words
$$x = \frac{\log 9}{\log 4} \approx 1.585.$$

54. Taking logs of both sides we get
$$\log(12^x) = \log 7.$$
This gives
$$x \log 12 = \log 7$$
or in other words
$$x = \frac{\log 7}{\log 12} \approx 0.783.$$

55. We divide both sides by 3 to get
$$5^x = 3.$$
Taking logs of both sides we get
$$\log(5^x) = \log 3.$$
This gives
$$x \log 5 = \log 3$$
or in other words
$$x = \frac{\log 3}{\log 5} \approx 0.683.$$

56. We divide both sides by 4 to get
$$13^{3x} = \frac{17}{4}.$$
Taking logs of both sides we get
$$\log(13^{3x}) = \log\left(\frac{17}{4}\right).$$
This gives
$$3x \log 13 = \log\left(\frac{17}{4}\right)$$
or in other words
$$x = \frac{\log(17/4)}{3\log 13} \approx 0.188.$$

57. Taking natural logs of both sides we get
$$\ln(e^x) = \ln 8.$$
This gives
$$x = \ln 8 \approx 2.079.$$

58. Dividing both sides by 2 gives
$$e^x = \frac{13}{2}$$
Taking natural logs of both sides we get
$$\ln(e^x) = \ln\left(\frac{13}{2}\right).$$
This gives
$$x = \ln(13/2) \approx 1.872.$$

59. Taking natural logs of both sides we get
$$\ln(e^{-5x}) = \ln 9.$$
This gives
$$\begin{aligned} -5x &= \ln 9 \\ x &= -\frac{\ln 9}{5} \approx -0.439. \end{aligned}$$

60. Taking natural logs of both sides we get
$$\ln(e^{7x}) = \ln(5e^{3x}).$$
Since $\ln(MN) = \ln M + \ln N$, we then get
$$\begin{aligned} 7x &= \ln 5 + \ln e^{3x} \\ 7x &= \ln 5 + 3x \\ 4x &= \ln 5 \\ x &= \frac{\ln 5}{4} \approx 0.402. \end{aligned}$$

61. Taking logs of both sides we get
$$\log 12^{5x} = \log(3 \cdot 15^{2x}).$$
This gives
$$\begin{aligned} 5x \log 12 &= \log 3 + \log 15^{2x} \\ 5x \log 12 &= \log 3 + 2x \log 15 \\ 5x \log 12 - 2x \log 15 &= \log 3 \\ x(5 \log 12 - 2 \log 15) &= \log 3 \\ x &= \frac{\log 3}{5 \log 12 - 2 \log 15} \approx 0.157. \end{aligned}$$

62. Taking logs of both sides we get
$$\log 19^{6x} = \log(77 \cdot 7^{4x}).$$
This gives
$$\begin{aligned} 6x \log 19 &= \log 77 + \log 7^{4x} \\ 6x \log 19 &= \log 77 + 4x \log 7 \\ 6x \log 19 - 4x \log 7 &= \log 77 \\ x(6 \log 19 - 4 \log 7) &= \log 77 \\ x &= \frac{\log 77}{6 \log 19 - 4 \log 7} \approx 0.440. \end{aligned}$$

63. We begin by converting to exponential form:

$$\begin{aligned}
\log(2x+7) &= 2 \\
10^{\log(2x+7)} &= 10^2 \\
2x+7 &= 100 \\
2x &= 93 \\
x &= \frac{93}{2}.
\end{aligned}$$

64. We first re-arrange the equation so that the log is alone on one side, and we then convert to exponential form:

$$\begin{aligned}
3\log(4x+9)-6 &= 2 \\
3\log(4x+9) &= 8 \\
\log(4x+9) &= \frac{8}{3} \\
10^{\log(4x+9)} &= 10^{8/3} \\
4x+9 &= 10^{8/3} \\
4x &= 10^{8/3}-9 \\
x &= \frac{10^{8/3}-9}{4} \approx 113.790.
\end{aligned}$$

65. We first re-arrange the equation so that the log is alone on one side, and we then convert to exponential form:

$$\begin{aligned}
4\log(9x+17)-5 &= 1 \\
4\log(9x+17) &= 6 \\
\log(9x+17) &= \frac{3}{2} \\
10^{\log(9x+17)} &= 10^{3/2} \\
9x+17 &= 10^{3/2} \\
9x &= 10^{3/2}-17 \\
x &= \frac{10^{3/2}-17}{9} \approx 1.625.
\end{aligned}$$

66. We first convert to exponential form and then use the properties of exponents:

$$\begin{aligned}
\log(2x) &= \log(x+10) \\
10^{\log(2x)} &= 10^{\log(x+10)} \\
2x &= x+10 \\
x &= 10.
\end{aligned}$$

67. We begin by converting to exponential form:

$$\begin{aligned}
\ln(3x+4) &= 5 \\
e^{\ln(3x+4)} &= e^5 \\
3x+4 &= e^5 \\
3x &= e^5-4 \\
x &= \frac{e^5-4}{3} \approx 48.138.
\end{aligned}$$

68. We first re-arrange the equation so that the natural log is alone on one side, and we then convert to exponential form:

$$\begin{aligned}
2\ln(6x-1)+5 &= 7\\
2\ln(6x-1) &= 2\\
\ln(6x-1) &= 1\\
e^{\ln(6x-1)} &= e^1\\
6x-1 &= e\\
6x &= e+1\\
x &= \frac{e+1}{6} \approx 0.620.
\end{aligned}$$