Differentiable Implies Continuous

Theorem: If f is differentiable at x_0, then f is continuous at x_0.

We need to prove this theorem so that we can use it to find general formulas for products and quotients of functions.

We begin by writing down what we need to prove; we choose this carefully to make the rest of the proof easier. We want to show that:

$$\lim_{x \to x_0} f(x) - f(x_0) = 0.$$

This is the same as saying that the function is continuous, because to prove that a function was continuous we'd show that $\lim_{x \to x_0} f(x) = f(x_0)$.

We prove $\lim_{x \to x_0} f(x) - f(x_0) = 0$ by multiplying and dividing it by the same number – this won't change its value.

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\ &= f'(x) \cdot 0 \\ &= 0. \end{aligned}$$

(Notice that we used our assumption that f was differentiable when we wrote down $f'(x)$.)

But wait! When we multiplied and divided by $x - x_0$ weren't we multiplying and dividing by zero? We know from our algebra classes that this never works! It turns out that we're safe because we're using limits. Although x gets closer and closer to x_0, it never actually equals x_0, and so we never quite divide by 0. That's what limits are for; $x - x_0$ may be small, but it's always non-zero.

So this calculation is valid, it's true that $\lim_{x \to x_0} f(x) - f(x_0) = 0$, and it's true that differentiable functions are continuous.