# Markov's inequality

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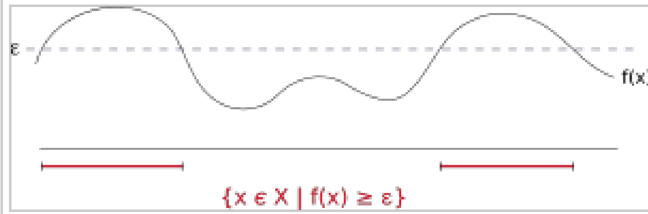
In probability theory, **Markov's inequality** gives an upper bound for the probability that a non-negative function of a random variable is greater than or equal to some positive constant. It is named after the Russian mathematician Andrey Markov, although it appeared earlier in the work of Pafnuty Chebyshev (Markov's teacher), and many sources, especially in analysis, refer to it as Chebyshev's inequality or Bienaymé's inequality.



Markov's inequality gives an upper bound for the measure of the set (indicated in red) where $f(x)$ exceeds a given level $\epsilon$. The bound combines the level $\epsilon$ with the average value of $f$.

Markov's inequality (and other similar inequalities) relate probabilities to expectations, and provide (frequently loose but still useful) bounds for the cumulative distribution function of a random variable.

An example of an application of Markov's inequality is the fact that (assuming incomes are non-negative) no more than 1/5 of the population can have more than 5 times the average income.

## Contents



## Statement

If $X$ is any nonnegative random variable and $a > 0$, then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

In the language of measure theory, Markov's inequality states that if $(X, \Sigma, \mu)$ is a measure space, $f$ is a measurable extended real-valued function, and $\epsilon > 0$, then

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| \, d\mu.$$

(This measure theoretic definition may sometimes be referred to as Chebyshev's inequality .[1])

## Corollary: Chebyshev's inequality

Chebyshev's inequality uses the variance to bound the probability that a random variable deviates far from the mean. Specifically:

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2},$$

for any $a>0$. Here Var(X) is the variance of X, defined as:

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2].$$

Chebyshev's inequality follows from Markov's inequality by considering the random variable

$$(X - \mathbb{E}(X))^2$$

for which Markov's inequality reads

$$\mathbb{P}((X - \mathbb{E}(X))^2 \geq a^2) \leq \frac{\mathrm{Var}(X)}{a^2},$$

# Proofs

We separate the case in which the measure space is a probability space from the more general case because the probability case is more accessible for the general reader.

## Proof In the language of probability theory

For any event $E$, let $I_E$ be the indicator random variable of $E$, that is, $I_E = 1$ if $E$ occurs and $I_E = 0$ otherwise.

Using this notation, we have $I_{(X \geq a)} = 1$ if the event $X \geq a$ occurs, and $I_{(X \geq a)} = 0$ if $X < a$. Then, given $a > 0$,

$$aI_{(X \geq a)} \leq X$$

which is clear if we consider the two possible values of $I_{(X \geq a)}$. If $X < a$, then $I_{(X \geq a)} = 0$, and so $aI_{(X \geq a)} = 0 \leq X$. Otherwise, we have $X \geq a$, for which $I_{(X \geq a)} = 1$ and so $aI_{(X \geq a)} = \mathrm{a} \leq X$.

Therefore

$$\mathbb{E}(aI_{(X \geq a)}) \leq \mathbb{E}(X).$$

Now, using linearity of expectations, the left side of this inequality is the same as

$$a\mathbb{E}(I_{(X \geq a)}) = a(1 \cdot \mathbb{P}(X \geq a) + 0 \cdot \mathbb{P}(X < a)) = a\mathbb{P}(X \geq a).$$

Thus we have

$$a\mathbb{P}(X \geq a) \leq \mathbb{E}(X)$$

and since $a > 0$, we can divide both sides by $a$.

## In the language of measure theory

We may assume that the function $f$ is non-negative, since only its absolute value enters in the equation. Now, consider the real-valued function *s* on *X* given by

$$s(x) = \begin{cases} \epsilon, & \text{if } f(x) \geq \epsilon \\ 0, & \text{if } f(x) < \epsilon \end{cases}$$

Then $s$ is a simple function such that $0 \leq s(x) \leq f(x)$. By the definition of the Lebesgue integral

$$\int_X f(x)\,d\mu \geq \int_X s(x)\,d\mu = \epsilon\mu(\{x \in X:\, f(x) \geq \epsilon\})$$

and since $\epsilon > 0$, both sides can be divided by $\epsilon$, obtaining

$$\mu(\{x \in X:\, f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon}\int_X f\,d\mu.$$

Q.E.D.

# Matrix-valued Markov

Let $M \succeq 0$ be a self adjoint matrix-valued random variable and $a > 0$. Then

$$\mathbb{P}(M \npreceq a \cdot I) \leq \frac{\mathrm{tr}\,(E(M))}{a}.$$

# Examples

- Markov's inequality is used to prove Chebyshev's inequality.
- Markov's inequality can be used to show that, for a nonnegative random variable, the mean $\mu$ and a median $m$ are such that $m \leq 2\mu$.

# See also

- McDiarmid's inequality
- Bernstein inequalities (probability theory)

# References


1. ^ E.M. Stein, R. Shakarchi, "Real Analysis, Measure Theory, Integration, & Hilbert Spaces", vol. 3, 1st ed., 2005, p.91


# External links

- The formal proof of Markov's inequality (http://mws.cs.ru.nl/mwiki/random_1.html#T36) in the Mizar system.