

A “Harder” Problem

People say that calculus is hard, but the example we just saw — computing the derivative of $f(x) = \frac{1}{x}$ — was not very difficult. What makes calculus seem hard is the context calculus problems appear in. For example, the problem we are about to solve combines algebra, geometry and problem solving with calculus. Because we use calculus to solve it, it is “a calculus problem”. And although it is a harder problem, it’s not the calculus that makes it hard.

So far all we’ve talked about is geometry, so our example problem must be geometric.

Problem: Find the area of the triangle formed by the x and y -axes and the tangent to the graph of $y = \frac{1}{x}$.

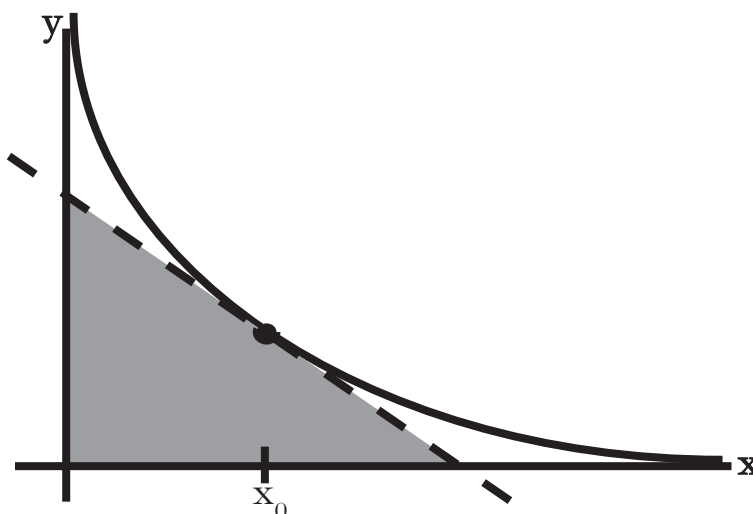


Figure 1: Triangle formed by axes and tangent line

We start by drawing a picture. As we draw the picture, we realize we’ve assumed that the triangle lies in the first quadrant. The solution we find might or might not depend on this assumption — we’ll have to do some work later if we want to be sure it’s true when x and $\frac{1}{x}$ are negative.

Because the problem refers to a tangent line it is a calculus problem, but as you’ll see, the calculus is the easy part.

The next step in our solution is labeling the picture. We label the graph $y = \frac{1}{x}$. We’d like to put some labels on the triangle, like the lengths of its sides, but we don’t know how to find those numbers. What we do know is that the hypotenuse of the triangle is tangent to the graph, and we can label the point of tangency $(x_0, \frac{1}{x_0})$ (which lets us label the points $(x_0, 0)$ and $(0, y_0)$ on the axes).

The area of a triangle is $\frac{1}{2}b \cdot h$, so we’d like to find the lengths of the base

and height of this triangle. If we can find the x -coordinate of the point where the tangent line intersects the x -axis, we'll know the length of the base of the triangle. Similarly, finding the y -intercept of the tangent line will give us the triangle's height.

In order to find the x - and y -intercepts of the tangent line, we first must find the equation of the line. We start by writing down the "point-slope" form of the equation for a line:

$$y - y_0 = m(x - x_0).$$

The part of this problem that requires calculus is finding the slope m of the tangent line. Luckily, we already did this calculation in the previous example: $m = -\frac{1}{x_0^2}$.

$$y - y_0 = -\frac{1}{x_0^2}(x - x_0)$$

We've finished all the calculus in the problem, but we still need to do some work to find the area of the triangle. The point (x_0, y_0) is the point where the hypotenuse is tangent to the graph of $y = \frac{1}{x}$. We leave x_0 as is, and replace y_0 by $\frac{1}{x_0}$.

$$y - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0).$$

First we calculate the x -intercept of the tangent line, which will give us the length of the base of the triangle. The line crosses the x -axis when $y = 0$. Setting $y = 0$ in the equation for the tangent line, we get:

$$\begin{aligned} 0 - \frac{1}{x_0} &= -\frac{1}{x_0^2}(x - x_0) \\ \frac{-1}{x_0} &= \frac{-1}{x_0^2}x + \frac{1}{x_0} \\ \frac{1}{x_0^2}x &= \frac{2}{x_0} \\ x &= x_0^2\left(\frac{2}{x_0}\right) = 2x_0 \end{aligned}$$

So, the x -intercept of this tangent line is at $x = 2x_0$.

Next we could find the y -intercept using a very similar calculation — replacing x by 0 and solving for y (because the y -intercept is the point on the line with x coordinate 0). If we don't want to do the calculation all over again, or if we like using clever tricks to make problems easier, we can use the symmetry of the graph to take a short cut.

Since $y = \frac{1}{x}$ and $x = \frac{1}{y}$ are identical equations, the graph of $y = \frac{1}{x}$ is symmetric when x and y are exchanged. By symmetry, then, we could swap the

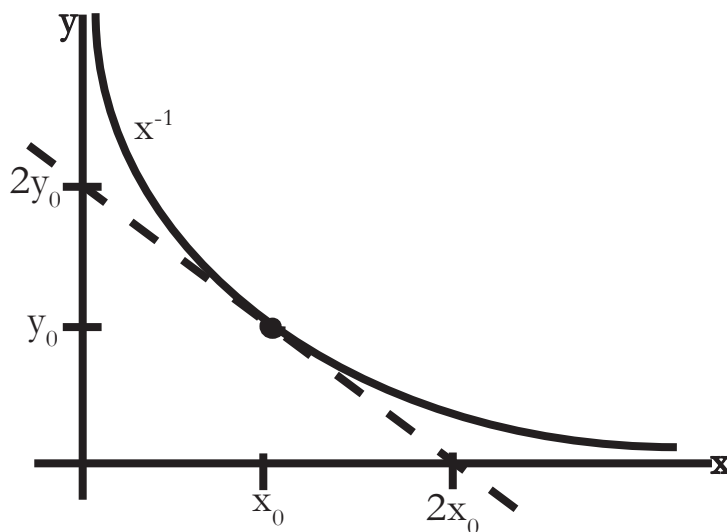


Figure 2: Triangle formed by axes and tangent line, labeled

x 's in the calculation above with the y 's to conclude that the y -intercept is at $y = 2y_0 = \frac{2}{x_0}$.

Finally,

$$\text{Area} = \frac{1}{2}b \cdot h = \frac{1}{2}(2x_0)(2y_0) = 2x_0y_0 = 2x_0\left(\frac{1}{x_0}\right) = 2 \text{ (see Fig. 2)}$$

Curiously, the area of the triangle is *always* 2, no matter where on the graph we draw the tangent line!

Remark: We call it “one variable calculus”, but we just used *four* variables: x , y , x_0 , and y_0 . We could have had more! This makes things complicated, and it's something that you'll have to get used to.

Another complicated thing that we do is reuse variables. In this problem, there are (at least) three different possible interpretations of the variable y . When we said $y = \frac{1}{x}$, we were thinking of y as the vertical position of a point on the hyperbola. When we said $y - y_0 = \frac{-1}{x_0^2}(x - x_0)$ we were thinking of y as the vertical position of a point on the tangent line. And when we said $y = 0$ we were talking about the vertical position of all the points on the x -axis. Once you've practiced calculus for a while you will know from context which meaning y has, just as you can tell in conversation whether a person is saying “sea” or “see”. Until you reach that point, be sure you understand the meaning of an equation before using it in a calculation.