Chapter Two

FUNCTIONS

In this chapter, we investigate properties and notation common to all functions. We begin with a closer look at function notation, including an introduction to inverse functions. The ideas of domain, range, and piecewise defined functions are addressed. Concavity is then introduced and applied to quadratic functions.

The Tools Section on page 99 reviews quadratic equations.

2.1 INPUT AND OUTPUT

Finding Output Values: Evaluating a Function

Evaluating a function means calculating the value of a function's output from a particular value of the input.

In the housepainting example on page 4, the notation $n = f(A)$ indicates that n is a function of A. The expression $f(A)$ represents the output of the function—specifically, the amount of paint required to cover an area of A ft^2. For example $f(20{,}000)$ represents the number of gallons of paint required to cover a house of $20{,}000$ ft^2.

Example 1   Using the fact that 1 gallon of paint covers 250 ft^2, evaluate the expression $f(20{,}000)$.

Solution   To evaluate $f(20{,}000)$, calculate the number of gallons required to cover $20{,}000$ ft^2:

$$f(20{,}000) = \frac{20{,}000 \text{ ft}^2}{250 \text{ ft}^2/\text{gallon}} = 80 \text{ gallons of paint.}$$

Evaluating a Function Using a Formula

If we have a formula for a function, we evaluate it by substituting the input value into the formula.

Example 2   The formula for the area of a circle of radius r is $A = q(r) = \pi r^2$. Use the formula to evaluate $q(10)$ and $q(20)$. What do your results tell you about circles?

Solution   In the expression $q(10)$, the value of r is 10, so

$$q(10) = \pi \cdot 10^2 = 100\pi \approx 314.$$

Similarly, substituting $r = 20$, we have

$$q(20) = \pi \cdot 20^2 = 400\pi \approx 1257.$$

The statements $q(10) \approx 314$ and $q(20) \approx 1257$ tell us that a circle of radius 10 cm has an area of approximately 314 cm^2 and a circle of radius 20 cm has an area of approximately 1257 cm^2.

Example 3   Let $g(x) = \dfrac{x^2 + 1}{5 + x}$. Evaluate the following expressions.

(a) $g(3)$ (b) $g(-1)$ (c) $g(a)$

Solution   (a)   To evaluate $g(3)$, replace every x in the formula with 3:

$$g(3) = \frac{3^2 + 1}{5 + 3} = \frac{10}{8} = 1.25.$$

(b)   To evaluate $g(-1)$, replace every x in the formula with (-1):

$$g(-1) = \frac{(-1)^2 + 1}{5 + (-1)} = \frac{2}{4} = 0.5.$$

(c)   To evaluate $g(a)$, replace every x in the formula with a:

$$g(a) = \frac{a^2 + 1}{5 + a}.$$

Evaluating a function may involve algebraic simplification, as the following example shows.

Example 4   Let $h(x) = x^2 - 3x + 5$. Evaluate and simplify the following expressions.

(a) $h(2)$ (b) $h(a-2)$ (c) $h(a) - 2$ (d) $h(a) - h(2)$

Solution   Notice that x is the input and $h(x)$ is the output. It is helpful to rewrite the formula as

$$\text{Output} = h(\text{Input}) = (\text{Input})^2 - 3 \cdot (\text{Input}) + 5.$$

(a) For $h(2)$, we have Input $= 2$, so

$$h(2) = (2)^2 - 3 \cdot (2) + 5 = 3.$$

(b) In this case, Input $= a - 2$. We substitute and multiply out

$$\begin{aligned} h(a-2) &= (a-2)^2 - 3(a-2) + 5 \\ &= a^2 - 4a + 4 - 3a + 6 + 5 \\ &= a^2 - 7a + 15. \end{aligned}$$

(c) First input a, then subtract 2:

$$\begin{aligned} h(a) - 2 &= a^2 - 3a + 5 - 2 \\ &= a^2 - 3a + 3. \end{aligned}$$

(d) Since we found $h(2) = 3$ in part (a), we subtract from $h(a)$:

$$\begin{aligned} h(a) - h(2) &= a^2 - 3a + 5 - 3 \\ &= a^2 - 3a + 2. \end{aligned}$$

Finding Input Values: Solving Equations

Given an input, we evaluate the function to find the output. Sometimes the situation is reversed; we know the output and we want to find the corresponding input. If the function is given by a formula, the input values are solutions to an equation.

Example 5   Use the cricket function $T = \frac{1}{4}R + 40$, introduced on page 3, to find the rate, R, at which the snowy tree cricket chirps when the temperature, T, is 76°F.

Solution   We want to find R when $T = 76$. Substitute $T = 76$ into the formula and solve the equation

$$76 = \frac{1}{4}R + 40$$

$$36 = \frac{1}{4}R \qquad \text{subtract 40 from both sides}$$

$$144 = R. \qquad \text{multiply both sides by 4}$$

The cricket chirps at a rate of 144 chirps per minute when the temperature is 76°F.

Example 6   Suppose $f(x) = \dfrac{1}{\sqrt{x-4}}$.

(a) Find an x-value that results in $f(x) = 2$.
(b) Is there an x-value that results in $f(x) = -2$?

Solution   (a) To find an x-value that results in $f(x) = 2$, solve the equation

$$2 = \frac{1}{\sqrt{x-4}}.$$

Square both sides

$$4 = \frac{1}{x-4}.$$

Now multiply by $(x-4)$

$$4(x-4) = 1$$
$$4x - 16 = 1$$
$$x = \frac{17}{4} = 4.25.$$

The x-value is 4.25. (Note that the simplification $(x-4)/(x-4) = 1$ in the second step was valid because $x - 4 \neq 0$.)

(b) Since $\sqrt{x-4}$ is nonnegative if it is defined, its reciprocal, $f(x) = \dfrac{1}{\sqrt{x-4}}$ is also nonnegative if it is defined. Thus, $f(x)$ is not negative for any x input, so there is no x-value that results in $f(x) = -2$.

In the next example, we solve an equation for a quantity that is being used to model a physical quantity; we must choose the solutions that make sense in the context of the model.

Example 7   Let $A = q(r)$ be the area of a circle of radius r, where r is in cm. What is the radius of a circle whose area is 100 cm^2?

Solution   The output $q(r)$ is an area. Solving the equation $q(r) = 100$ for r gives the radius of a circle whose area is 100 cm^2. Since the formula for the area of a circle is $q(r) = \pi r^2$, we solve

$$q(r) = \pi r^2 = 100$$
$$r^2 = \frac{100}{\pi}$$
$$r = \pm\sqrt{\frac{100}{\pi}} = \pm 5.642.$$

We have two solutions for r, one positive and one negative. Since a circle cannot have a negative radius, we take $r = 5.642$ cm. A circle of area 100 cm^2 has a radius of 5.642 cm.

Finding Output and Input Values From Tables and Graphs

The following two examples use function notation with a table and a graph respectively.

Example 8   Table 2.1 shows the revenue, $R = f(t)$, received or expected, by the National Football League,[1] NFL, from network TV as a function of the year, t, since 1975.

(a) Evaluate and interpret $f(25)$.   (b) Solve and interpret $f(t) = 1159$.

Table 2.1

Year, t (since 1975)	0	5	10	15	20	25	30
Revenue, R (million $)	201	364	651	1075	1159	2200	2200

[1] *Newsweek*, January 26, 1998.

Solution
(a) The table shows $f(25) = 2200$. Since $t = 25$ in the year 2000, we know that NFL's projected revenue from TV was \$2200 million in the year 2000.
(b) Solving $f(t) = 1159$ means finding the year in which TV revenues were \$1159 million; it is $t = 20$. In 1995, NFL's TV revenues were \$1159 million.

Example 9   A man drives from his home to a store and back. The entire trip takes 30 minutes. Figure 2.1 gives his velocity $v(t)$ (in mph) as a function of the time t (in minutes) since he left home. A negative velocity indicates that he is traveling away from the store back to his home.
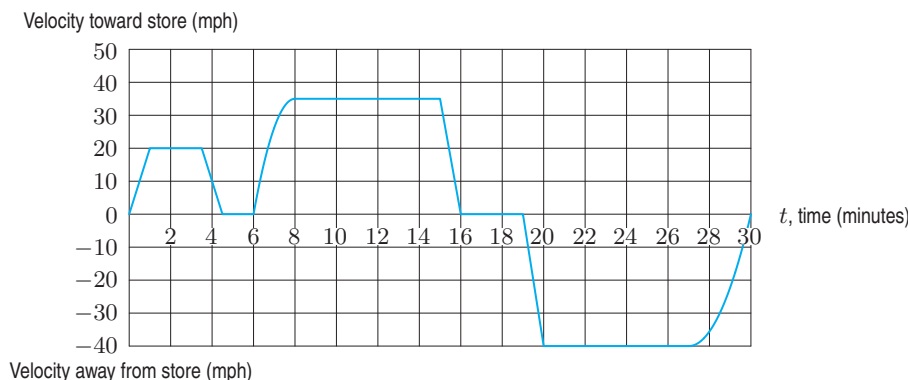

Figure 2.1: Velocity of a man on a trip to the store and back

Evaluate and interpret:

(a) $v(5)$ (b) $v(24)$ (c) $v(8) - v(6)$ (d) $v(-3)$

Solve for t and interpret:

(e) $v(t) = 15$ (f) $v(t) = -20$ (g) $v(t) = v(7)$

Solution
(a) To evaluate $v(5)$, look on the graph where $t = 5$ minutes. Five minutes after he left home, his velocity is 0 mph. Thus, $v(5) = 0$. Perhaps he had to stop at a light.
(b) The graph shows that $v(24) = -40$ mph. After 24 minutes, he is traveling at 40 mph away from the store.
(c) From the graph, $v(8) = 35$ mph and $v(6) = 0$ mph. Thus, $v(8) - v(6) = 35 - 0 = 35$. This shows that the man's speed increased by 35 mph in the interval between $t = 6$ minutes and $t = 8$ minutes.
(d) The quantity $v(-3)$ is not defined since the graph only gives velocities for nonnegative times.
(e) To solve for t when $v(t) = 15$, look on the graph where the velocity is 15 mph. This occurs at $t \approx 0.75$ minute, 3.75 minutes, 6.5 minutes, and 15.5 minutes. At each of these four times the man's velocity was 15 mph.
(f) To solve $v(t) = -20$ for t, we see that the velocity is -20 mph at $t \approx 19.5$ and $t \approx 29$ minutes.
(g) First we evaluate $v(7) \approx 27$. To solve $v(t) = 27$, we look for the values of t making the velocity 27 mph. One such t is of course $t = 7$; the other t is $t \approx 15$ minutes.

Exercises and Problems for Section 2.1

Exercises

In Exercises 1–2, evaluate the function for $x = -7$.

1. $f(x) = x/2 - 1$ **2.** $f(x) = x^2 - 3$

For Exercises 3–6, calculate exactly the values of y when $y = f(4)$ and of x when $f(x) = 6$.

3. $f(x) = \dfrac{6}{2 - x^3}$ **4.** $f(x) = \sqrt{20 + 2x^2}$

5. $f(x) = 4x^{3/2}$ **6.** $f(x) = x^{-3/4} - 2$

7. If $f(x) = 2x+1$, (a) Find $f(0)$ (b) Solve $f(x) = 0$.

8. If $f(t) = t^2 - 4$, (a) Find $f(0)$ (b) Solve $f(t) = 0$.

9. If $g(x) = x^2 - 5x + 6$, (a) Find $g(0)$ (b) Solve $g(x) = 0$.

10. If $g(t) = \dfrac{1}{t+2} - 1$, (a) Find $g(0)$ (b) Solve $g(t) = 0$.

11. If $f(x) = \dfrac{x}{1 - x^2}$, find $f(-2)$.

12. If $h(x) = ax^2 + bx + c$, find $h(0)$.

13. If $P(t) = 170 - 4t$, find $P(4) - P(2)$.

14. If $g(x) = -\frac{1}{2}x^{1/3}$, find $g(-27)$.

15. Let $h(x) = 1/x$. Find **(a)** $h(x+3)$ **(b)** $h(x)+ h(3)$

16. Let $f(x) = \dfrac{2x + 1}{x + 1}$. For what value of x is $f(x) = 0.3$?

17. (a) Using Table 2.2, evaluate $f(1)$, $f(-1)$, and $-f(1)$.
     (b) Solve $f(x) = 0$ for x.

Table 2.2

x	-1	0	1	2
$f(x)$	0	-1	2	1

18. (a) In Figure 2.2, estimate $f(0)$.
     (b) For what x-value(s) is $f(x) = 0$?
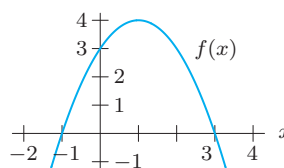   **(c)** For what x-value(s) is $f(x) > 0$?

Figure 2.2

19. Let $F = g(t)$ be the number of foxes in a park as a function of t, the number of months since January 1. Evaluate $g(9)$ using Table 1.3 on page 5. What does this tell us about the fox population?

20. Let $F = g(t)$ be the number of foxes in month t in the national park described in Example 5 on page 5. Solve the equation $g(t) = 75$. What does your solution tell you about the fox population?

Problems

21. If $V = \frac{1}{3}\pi r^2 h$ gives the volume of a cylinder, what is the value of V when $r = 3$ inches and $h = 2$ inches? Give units.

22. Let $q(x) = 3 - x^2$. Evaluate and simplify:

   (a) $q(5)$ **(b)** $q(a)$
   (c) $q(a - 5)$ **(d)** $q(a) - 5$
   (e) $q(a) - q(5)$

23. Let $p(x) = x^2 + x + 1$. Find $p(-1)$ and $-p(1)$. Are they equal?

24. Let $f(x) = 3 + 2x^2$. Find $f\left(\dfrac{1}{3}\right)$ and $\dfrac{f(1)}{f(3)}$. Are they equal?

25. Chicago's average monthly rainfall, $R = f(t)$ inches, is given as a function of month, t, in Table 2.3. (January is $t = 1$.) Solve and interpret:

   (a) $f(t) = 3.7$ **(b)** $f(t) = f(2)$

Table 2.3

t	1	2	3	4	5	6	7	8
R	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4

26. Let $g(x) = x^2 + x$. Find formulas for the following functions. Simplify your answers.

    (a) $g(-3x)$ **(b)** $g(1-x)$ **(c)** $g(x+\pi)$
    (d) $g(\sqrt{x})$ **(e)** $g(1/(x+1))$ **(f)** $g(x^2)$

27. Let $f(x) = \dfrac{x}{x-1}$.

    (a) Find and simplify

        (i) $f\left(\dfrac{1}{t}\right)$ (ii) $f\left(\dfrac{1}{t+1}\right)$

    (b) Solve $f(x) = 3$.

28. **(a)** Find a point on the graph of $h(x) = \sqrt{x+4}$ whose x-coordinate is 5.
    (b) Find a point on the graph whose y-coordinate is 5.
    (c) Graph $h(x)$ and mark the points in parts (a) and (b).
    (d) Let $p = 2$. Calculate $h(p+1) - h(p)$.

29. Use the graph of $f(x)$ in Figure 2.3 to estimate:

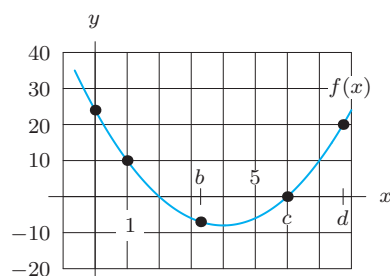   **(a)** $f(0)$ **(b)** $f(1)$ **(c)** $f(b)$ **(d)** $f(c)$ **(e)** $f(d)$

Figure 2.3

30. **(a)** Using Figure 2.4, fill in Table 2.4.

Table 2.4

x	-2	-1	0	1	2	3
$h(x)$						

    (b) Evaluate $h(3) - h(1)$ **(c)** Evaluate $h(2) - h(0)$
    (d) Evaluate $2h(0)$ **(e)** Evaluate $h(1) + 3$

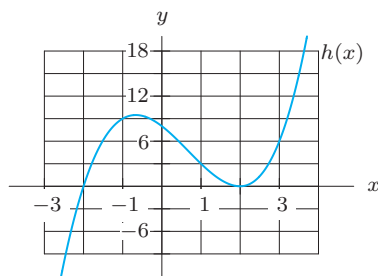**Figure 2.4**

31. Let $f(x) = \sqrt{x^2 + 16} - 5$.

    (a) Find $f(0)$
    (b) For what values of x is $f(x)$ zero?
    (c) Find $f(3)$
    (d) What is the vertical intercept of the graph of $f(x)$?
    (e) Where does the graph cross the x-axis?

32. Use the letters a, b, c, d, e, h in Figure 2.5 to answer the following questions.

    (a) What are the coordinates of the points P and Q?
    (b) Evaluate $f(b)$.
    (c) Solve $f(x) = e$ for x.
    (d) Suppose $c = f(z)$ and $z = f(x)$. What is x?
    (e) Suppose $f(b) = -f(d)$. What additional information does this give you?
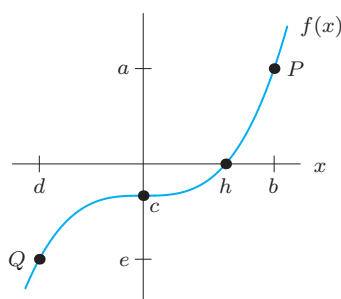

Figure 2.5

33. A ball is thrown up from the ground with initial velocity 64 ft/sec. Its height at time t is

$$h(t) = -16t^2 + 64t.$$

    (a) Evaluate $h(1)$ and $h(3)$. What does this tell us about the height of the ball?
    (b) Sketch this function. Using a graph, determine when the ball hits the ground and the maximum height of the ball.

34. Let $v(t) = t^2 - 2t$ be the velocity, in ft/sec, of an object at time t, in seconds.

    (a) What is the initial velocity, $v(0)$?
    (b) When does the object have a velocity of zero?
    (c) What is the meaning of the quantity $v(3)$? What are its units?

35. Let $s(t) = 11t^2 + t + 100$ be the position, in miles, of a car driving on a straight road at time t, in hours. The car's velocity at any time t is given by $v(t) = 22t + 1$.

(a) Use function notation to express the car's position after 2 hours. Where is the car then?

(b) Use function notation to express the question, "When is the car going 65 mph?"

(c) Where is the car when it is going 67 mph?

36. New York state income tax is based on taxable income, which is part of a person's total income. The tax owed to the state is calculated using the taxable income (not total income). In 2005, for a single person with a taxable income between $20,000 and $100,000, the tax owed was $973 plus 6.85% of the taxable income over $20,000.

(a) Compute the tax owed by a lawyer whose taxable income is $68,000.

(b) Consider a lawyer whose taxable income is 80% of her total income, $\$x$, where x is between $85,000 and $120,000. Write a formula for $T(x)$, the taxable income.

(c) Write a formula for $L(x)$, the amount of tax owed by the lawyer in part (b).

(d) Use $L(x)$ to evaluate the tax liability for $x = 85,000$ and compare your results to part (a).

37. (a) The Fibonacci sequence is a sequence of numbers that begins 1, 1, 2, 3, 5, ... Each term in the sequence is the sum of the two preceding terms. For example,

$$2 = 1 + 1, \quad 3 = 2 + 1, \quad 5 = 2 + 3, \ldots.$$

Based on this observation, complete the following table of values for $f(n)$, the nth term in the Fibonacci sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	2	3	5							

(b) The table of values in part (a) can be completed even though we don't have a formula for $f(n)$. Does the fact that we don't have a formula mean that $f(n)$ is not a function?

(c) Are you able to evaluate the following expressions using parts (a) and (b)? If so, do so; if not, explain why not.

$$f(0), \quad f(-1), \quad f(-2), \quad f(0.5).$$

38. (a) Complete Table 2.5 using

$$f(x) = 2x(x-3) - x(x-5) \quad \text{and} \quad g(x) = x^2 - x.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

(b) Complete Table 2.6 using

$$h(x) = x^5 - 5x^3 + 6x + 1 \quad \text{and} \quad j(x) = 2x + 1.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

Table 2.5

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

Table 2.6

x	-2	-1	0	1	2
$h(x)$					
$j(x)$					

39. A psychologist conducts an experiment to determine the effect of sleep loss on job performance. Let $p = f(t)$ be the number of minutes it takes the average person to complete a particular task if they have lost t minutes of sleep, where $t = 0$ means they have had exactly 8 hours of sleep. For instance, $f(60)$ is the amount of time it takes the average person to complete the task after sleeping for only 7 hours. Let $p_0 = f(0)$ and let t_1, t_2, and t_3 be positive constants. Explain what the following statements tell you about sleep loss and job performance.

(a) $f(30) = p_0 + 5$

(b) $f(t_1) = 2p_0$

(c) $f(2t_1) = 1.5f(t_1)$

(d) $f(t_2 + 60) = f(t_2 + 30) + 10$

Problems 40–41 concern $v = r(s)$, the eyewall wind profile of a hurricane at landfall, where v is the eyewall wind speed (in mph) as a function of s, the height (in meters) above the ground. (The eyewall is the band of clouds that surrounds the eye of the storm.) Let s_0 be the height at which the wind speed is greatest, and let $v_0 = r(s_0)$. Interpret the following in terms of the hurricanes.

40. $r(0.5s_0)$

41. $r(s) = 0.75v_0$

2.2 DOMAIN AND RANGE

In Example 4 on page 5, we defined R to be the average monthly rainfall at Chicago's O'Hare airport in month t. Although R is a function of t, the value of R is not defined for every possible value of t. For instance, it makes no sense to consider the value of R for $t = -3$, or $t = 8.21$, or $t = 13$ (since a year has 12 months). Thus, although R is a function of t, this function is defined only for certain values of t. Notice also that R, the output value of this function, takes only the values $\{1.8, 2.1, 2.4, 2.5, 2.7, 3.1, 3.2, 3.4, 3.5, 3.7\}$.

A function is often defined only for certain values of the independent variable. Also, the dependent variable often takes on only certain values. This leads to the following definitions:

> If $Q = f(t)$, then
> - the **domain** of f is the set of input values, t, which yield an output value.
> - the **range** of f is the corresponding set of output values, Q.

Thus, the domain of a function is the set of input values, and the range is the set of output values.

If the domain of a function is not specified, we usually assume that it is as large as possible—that is, all numbers that make sense as inputs for the function. For example, if there are no restrictions, the domain of the function $f(x) = x^2$ is the set of all real numbers, because we can substitute any real number into the formula $f(x) = x^2$. Sometimes, however, we may restrict the domain to suit a particular application. If the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to positive numbers.

If a function is being used to model a real-world situation, the domain and range of the function are often determined by the constraints of the situation being modeled, as in the next example.

Example 1   The house painting function $n = f(A)$ in Example 2 on page 4 has domain $A > 0$ because all houses have some positive area. There is a practical upper limit to A because houses cannot be infinitely large, but in principle, A can be as large or as small as we like, as long as it is positive. Therefore we take the domain of f to be $A > 0$.

The range of this function is $n \geq 0$, because we cannot use a negative amount of paint.

Choosing Realistic Domains and Ranges

When a function is used to model a real situation, it may be necessary to modify the domain and range.

Example 2   Algebraically speaking, the formula

$$T = \frac{1}{4}R + 40$$

can be used for all values of R. If we know nothing more about this function than its formula, its domain is all real numbers. The formula for $T = \frac{1}{4}R + 40$ can return any value of T when we choose an appropriate R-value (See Figure 2.6.) Thus, the range of the function is also all real numbers. However, if we use this formula to represent the temperature, T, as a function of a cricket's chirp rate, R, as we did in Example 1 on page 2, some values of R cannot be used. For example, it does not make sense to talk about a negative chirp rate. Also, there is some maximum chirp rate $R_{\max}$

that no cricket can physically exceed. Thus, to use this formula to express T as a function of R, we must restrict R to the interval $0 \leq R \leq R_{max}$ shown in Figure 2.7.
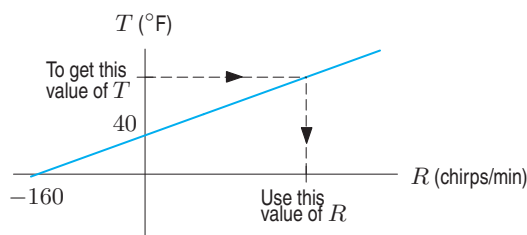

Figure 2.6: Graph showing that any T value can be obtained from some R value
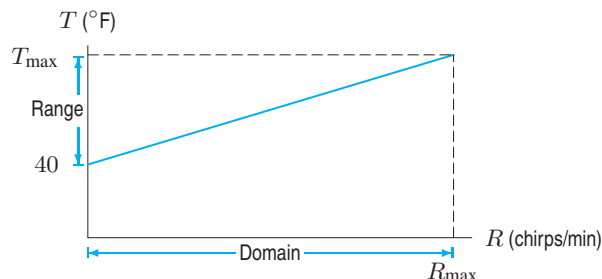

Figure 2.7: Graph showing that if $0 \leq R \leq R_{max}$, then $40 \leq T \leq T_{max}$

The range of the cricket function is also restricted. Since the chirp rate is nonnegative, the smallest value of T occurs when $R = 0$. This happens at $T = 40$. On the other hand, if the temperature gets too hot, the cricket will not be able to keep chirping faster. If the temperature T_{max} corresponds to the chirp rate R_{max}, then the values of T are restricted to the interval $40 \leq T \leq T_{max}$.

Using a Graph to Find the Domain and Range of a Function

A good way to estimate the domain and range of a function is to examine its graph. The domain is the set of input values on the horizontal axis which give rise to a point on the graph; the range is the corresponding set of output values on the vertical axis.

Example 3   A sunflower plant is measured every day t, for $t \geq 0$. The height, $h(t)$ centimeters, of the plant[2] can be modeled by using the *logistic function*

$$h(t) = \frac{260}{1 + 24(0.9)^t}.$$

(a) Using a graphing calculator or computer, graph the height over 80 days.
(b) What is the domain of this function? What is the range? What does this tell you about the height of the sunflower?

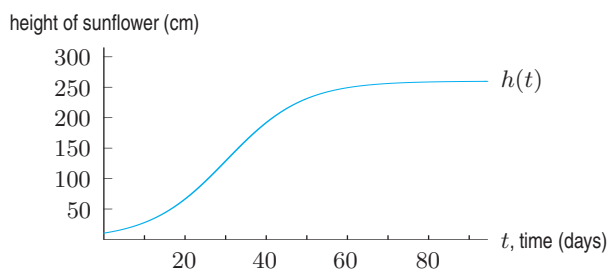**Solution**   (a) The logistic function is graphed in Figure 2.8.

Figure 2.8: Height of sunflower as a function of time

---

[2] Adapted from H.S. Reed and R.H. Holland, "Growth of an Annual Plant Helianthus" *Proc. Nat. Acad. Sci.*, 5, 1919.

(b) The domain of this function is $t \geq 0$. If we consider the fact that the sunflower dies at some point, then there is an upper bound on the domain, $0 \leq t \leq T$, where T is the day on which the sunflower dies.

　　To find the range, notice that the smallest value of h occurs at $t = 0$. Evaluating gives $h(0) = 10.4$ cm. This means that the plant was 10.4 cm high when it was first measured on day $t = 0$. Tracing along the graph, $h(t)$ increases. As t-values get large, $h(t)$-values approach, but never reach, 260. This suggests that the range is $10.4 \leq h(t) < 260$. This information tells us that sunflowers typically grow to a height of about 260 cm.

Using a Formula to Find the Domain and Range of a Function

When a function is defined by a formula, its domain and range can often be determined by examining the formula algebraically.

Example 4　　State the domain and range of g, where

$$g(x) = \frac{1}{x}.$$

Solution　　The domain is all real numbers except those which do not yield an output value. The expression $1/x$ is defined for any real number x except 0 (division by 0 is undefined). Therefore,

Domain: all real x,　$x \neq 0$.

The range is all real numbers that the formula can return as output values. It is not possible for $g(x)$ to equal zero, since 1 divided by a real number is never zero. All real numbers except 0 are possible output values, since all nonzero real numbers have reciprocals. Thus

Range: all real values,　$g(x) \neq 0$.

The graph in Figure 2.9 indicates agreement with these values for the domain and range.
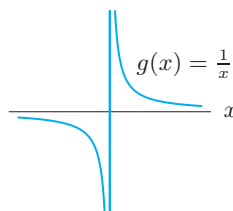

$g(x) = \frac{1}{x}$

x

Figure 2.9: Domain and range of $g(x) = 1/x$

Example 5　　Find the domain of the function $f(x) = \dfrac{1}{\sqrt{x-4}}$ by examining its formula.

Solution　　The domain is all real numbers except those for which the function is undefined. The square root of a negative number is undefined (if we restrict ourselves to real numbers), and so is division by zero. Therefore we need

$$x - 4 > 0.$$

Thus, the domain is all real numbers greater than 4.

Domain:　$x > 4$.

　　In Example 6 on page 63, we saw that for $f(x) = 1/\sqrt{x-4}$, the output, $f(x)$, cannot be negative. Note that $f(x)$ cannot be zero either. (Why?) The range of $f(x) = 1/\sqrt{x-4}$ is $f(x) > 0$. See Problem 16.

Exercises and Problems for Section 2.2

Exercises

In Exercises 1–4, use a graph to find the range of the function on the given domain.

1. $f(x) = \dfrac{1}{x}$, $-2 \leq x \leq 2$

2. $f(x) = \dfrac{1}{x^2}$, $-1 \leq x \leq 1$

3. $f(x) = x^2 - 4$, $-2 \leq x \leq 3$

4. $f(x) = \sqrt{9 - x^2}$, $-3 \leq x \leq 1$

Graph and give the domain and range of the functions in Exercises 5–12.

5. $f(x) = (x - 4)^3$

6. $f(x) = x^2 - 4$

7. $f(x) = 9 - x^2$
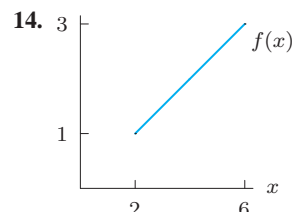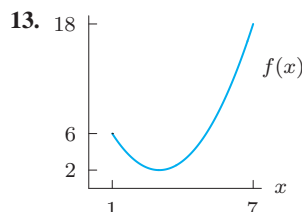
8. $f(x) = x^3 + 2$

9. $f(x) = \sqrt{8 - x}$

10. $f(x) = \sqrt{x - 3}$

11. $f(x) = \dfrac{-1}{(x + 1)^2}$

12. $f(x) = \dfrac{1}{x^2}$

In Exercises 13–14, estimate the domain and range of the function. Assume the entire graph is shown.

13.

14.

Problems

Find the domain and range of functions in Exercises 15–18 algebraically.

15. $q(x) = \sqrt{x^2 - 9}$

16. $f(x) = \dfrac{1}{\sqrt{x - 4}}$

17. $m(x) = 9 - x$

18. $n(x) = 9 - x^4$

In Exercises 19–24, find the domain and range.

19. $f(x) = -x^2 + 7$

20. $f(x) = \sqrt[3]{x + 77}$

21. $f(x) = x^2 + 2$

22. $f(x) = 1/(x + 1) + 3$

23. $f(x) = x - 3$

24. $f(x) = (x - 3)^2 + 2$

25. Give a formula for a function whose domain is all non-negative values of x except $x = 3$.

26. Give a formula for a function that is undefined for $x = 8$ and for $x < 4$, but is defined everywhere else.

27. A restaurant is open from 2 pm to 2 am each day, and a maximum of 200 clients can fit inside. If $f(t)$ is the number of clients in the restaurant t hours after 2 pm each day, what are a reasonable domain and range for $f(t)$?

28. What is the domain of the function f giving average monthly rainfall at Chicago's O'Hare airport? (See Table 1.2 on page 5)

29. A movie theater seats 200 people. For any particular show, the amount of money the theater makes is a function of the number of people, n, in attendance. If a ticket costs $4.00, find the domain and range of this function. Sketch its graph.

30. A car gets the best mileage at intermediate speeds. Graph the gas mileage as a function of speed. Determine a reasonable domain and range for the function and justify your reasoning.

31. **(a)** Use Table 2.7 to determine the number of calories that a person weighing 200 lb uses in a half–hour of walking.[3]

(b) Table 2.7 illustrates a relationship between the number of calories used per minute walking and a person's weight in pounds. Describe in words what is true about this relationship. Identify the dependent and independent variables. Specify whether it is an increasing or decreasing function.

(c) (i) Graph the linear function for walking, as described in part (b), and estimate its equation.

(ii) Interpret the meaning of the vertical intercept of the graph of the function.

(iii) Specify a meaningful domain and range for your function.

(iv) Use your function to determine how many calories per minute a person who weighs 135 lb uses per minute of walking.

Table 2.7 *Calories per minute as a function of weight*

Activity	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9
Bicycling	5.4	6.5	8.1	9.2	10.8	11.9
Swimming	5.8	6.9	8.7	9.8	11.6	12.7

[3]Source: 1993 World Almanac. Speeds assumed are 3 mph for walking, 10 mph for bicycling, and 2 mph for swimming.

In Exercises 32–33, find the domain and range.

32. $g(x) = a + 1/x$, where a is a constant

33. $q(r) = (x - b)^{1/2} + 6$, where b is a constant

34. The last digit, d, of a phone number is a function of n, its position in the phone book. Table 2.8 gives d for the first 10 listings in the 1998 Boston telephone directory. The table shows that the last digit of the first listing is 3, the last digit of the second listing is 8, and so on. In principle we could use a phone book to figure out other values of d. For instance, if $n = 300$, we could count down to the 300th listing in order to determine d. So we write $d = f(n)$.

(a) What is the value of $f(6)$?

(b) Explain how you could use the phone book to find the domain of f.

(c) What is the range of f?

Table 2.8

n	1	2	3	4	5	6	7	8	9	10
d	3	8	4	0	1	8	0	4	3	5

35. In month $t = 0$, a small group of rabbits escapes from a ship onto an island where there are no rabbits. The island rabbit population, $p(t)$, in month t is given by

$$p(t) = \frac{1000}{1 + 19(0.9)^t}, \quad t \geq 0.$$

(a) Evaluate $p(0)$, $p(10)$, $p(50)$, and explain their meaning in terms of rabbits.

(b) Graph $p(t)$ for $0 \leq t \leq 100$. Describe the graph in words. Does it suggest the growth in population you would expect among rabbits on an island?

(c) Estimate the range of $p(t)$. What does this tell you about the rabbit population?

(d) Explain how you can find the range of $p(t)$ from its formula.

36. Bronze is an alloy or mixture of the metals copper and tin. The properties of bronze depend on the percentage of copper in the mix. A chemist decides to study the properties of a given alloy of bronze as the proportion of copper is varied. She starts with 9 kg of bronze that contain 3 kg of copper and 6 kg of tin and either adds or removes copper. Let $f(x)$ be the percentage of copper in the mix if x kg of copper are added ($x > 0$) or removed ($x < 0$).

(a) State the domain and range of f. What does your answer mean in the context of bronze?

(b) Find a formula in terms of x for $f(x)$.

(c) If the formula you found in part (b) was not intended to represent the percentage of copper in an alloy of bronze, but instead simply defined an abstract mathematical function, what would be the domain and range of this function?

2.3 PIECEWISE DEFINED FUNCTIONS

A function may employ different formulas on different parts of its domain. Such a function is said to be *piecewise defined*. For example, the function graphed in Figure 2.10 has the following formulas:

$$\begin{aligned} y &= x^2 & \text{for } x \leq 2 \\ y &= 6 - x & \text{for } x > 2 \end{aligned} \quad \text{or more compactly} \quad y = \begin{cases} x^2 & \text{for } x \leq 2 \\ 6 - x & \text{for } x > 2. \end{cases}$$
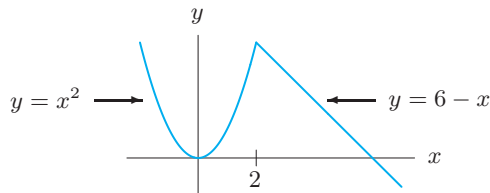

Figure 2.10: Piecewise defined function

Example 1 Graph the function $y = g(x)$ given by the following formulas:

$$g(x) = x + 1 \quad \text{for} \quad x \leq 2 \qquad \text{and} \qquad g(x) = 1 \quad \text{for} \quad x > 2.$$

Using bracket notation, this function is written:

$$g(x) = \begin{cases} x + 1 & \text{for } x \leq 2 \\ 1 & \text{for } x > 2. \end{cases}$$

Solution   For $x \leq 2$, graph the line $y = x + 1$. The solid dot at the point $(2, 3)$ shows that it is included in the graph. For $x > 2$, graph the horizontal line $y = 1$. See Figure 2.11. The open circle at the point $(2, 1)$ shows that it is not included in the graph. (Note that $g(2) = 3$, and $g(2)$ cannot have more than one value.)
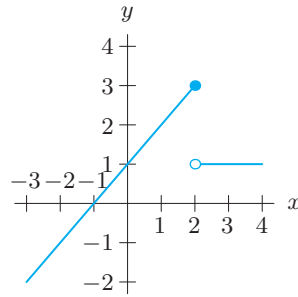

Figure 2.11: Graph of the piecewise defined function g

Example 2   A long-distance calling plan charges 99 cents for any call up to 20 minutes in length and 7 cents for each additional minute or part of a minute.

(a) Use bracket notation to write a formula for the cost, C, of a call as a function of its length t in minutes.
(b) Graph the function.
(c) State the domain and range of the function

Solution   (a) For $0 < t \leq 20$, the value of C is 99 cents. If $t > 20$, we subtract 20 to find the additional minutes and multiply by the rate 7 cents per minute.[4] The cost function in cents is thus

$$C = f(t) = \begin{cases} 99 & \text{for } 0 < t \leq 20 \\ 99 + 7(t - 20) & \text{for } t > 20, \end{cases}$$

or, after simplifying,

$$C = f(t) = \begin{cases} 99 & \text{for } 0 < t \leq 20 \\ 7t - 41 & \text{for } t > 20. \end{cases}$$

(b) See Figure 2.12.
(c) Because negative and zero call lengths do not make sense, the domain is $t > 0$. From the graph, we see that the range is $C \geq 99$.
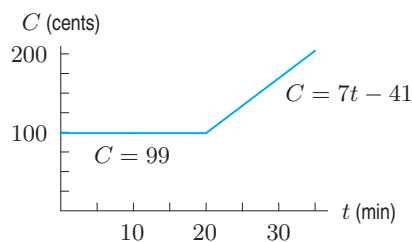

Figure 2.12: Cost of a long-distance phone call

---

[4]In actuality, most calling plans round the call length to whole minutes or specified fractions of a minute.

Example 3   The Ironman Triathlon is a race that consists of three parts: a 2.4 mile swim followed by a 112 mile bike race and then a 26.2 mile marathon. A participant swims steadily at 2 mph, cycles steadily at 20 mph, and then runs steadily at 9 mph.[5] Assuming that no time is lost during the transition from one stage to the next, find a formula for the distance d, covered in miles, as a function of the elapsed time t in hours, from the beginning of the race. Graph the function.

Solution   For each leg of the race, we use the formula Distance $=$ Rate $\cdot$ Time. First, we calculate how long it took for the participant to cover each of the three parts of the race. The first leg took $2.4/2 = 1.2$ hours, the second leg took $112/20 = 5.6$ hours, and the final leg took $26.2/9 \approx 2.91$ hours. Thus, the participant finished the race in $1.2 + 5.6 + 2.91 = 9.71$ hours.

During the first leg, $t \leq 1.2$ and the speed is 2 mph, so

$$d = 2t \qquad \text{for} \qquad 0 \leq t \leq 1.2.$$

During the second leg, $1.2 < t \leq 1.2 + 5.6 = 6.8$ and the speed is 20 mph. The length of time spent in the second leg is $(t - 1.2)$ hours. Thus, by time t,

$$\text{Distance covered in the second leg} = 20(t - 1.2) \quad \text{for } 1.2 < t \leq 6.8.$$

When the participant is in the second leg, the total distance covered is the sum of the distance covered in the first leg (2.4 miles) plus the part of the second leg that has been covered by time t.

$$d = 2.4 + 20(t - 1.2)$$
$$= 20t - 21.6 \qquad \text{for } 1.2 < t \leq 6.8.$$

In the third leg, $6.8 < t \leq 9.71$ and the speed is 9 mph. Since 6.8 hours were spent on the first two parts of the race, the length of time spent on the third leg is $(t - 6.8)$ hours. Thus, by time t,

$$\text{Distance covered in the third leg} = 9(t - 6.8) \quad \text{for } 6.8 < t \leq 9.71.$$

When the participant is in the third leg, the total distance covered is the sum of the distances covered in the first leg (2.4 miles) and the second leg (112 miles), plus the part of the third leg that has been covered by time t:

$$d = 2.4 + 112 + 9(t - 6.8)$$
$$= 9t + 53.2 \qquad \text{for } 6.8 < t \leq 9.71.$$

The formula for d is different on different intervals of t:

$$d = \begin{cases} 2t & \text{for} & 0 \leq t \leq 1.2 \\ 20t - 21.6 & \text{for} & 1.2 < t \leq 6.8 \\ 9t + 53.2 & \text{for} & 6.8 < t \leq 9.71. \end{cases}$$

Figure 2.13 gives a graph of the distance covered, d, as a function of time, t. Notice the three pieces.
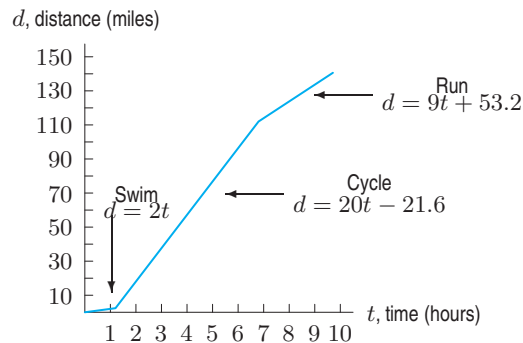

Figure 2.13: Ironman Triathlon: d as a function of t

[5]Data supplied by Susan Reid, Athletics Department, University of Arizona.

The Absolute Value Function

The absolute value of a x, written $|x|$, is defined piecewise

$$\text{For positive } x, \quad |x| = x.$$

$$\text{For negative } x, \quad |x| = -x.$$

(Remember that $-x$ is a positive number if x is a negative number.) For example, if $x = -3$, then

$$|-3| = -(-3) = 3.$$

For $x = 0$, we have $|0| = 0$. This leads to the following two-part definition:

The **Absolute Value Function** is defined by

$$f(x) = |x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ -x & \text{for} \quad x < 0 \end{cases}.$$

Table 2.9 gives values of $f(x) = |x|$ and Figure 2.14 shows a graph of $f(x)$.

Table 2.9   *Absolute value function*

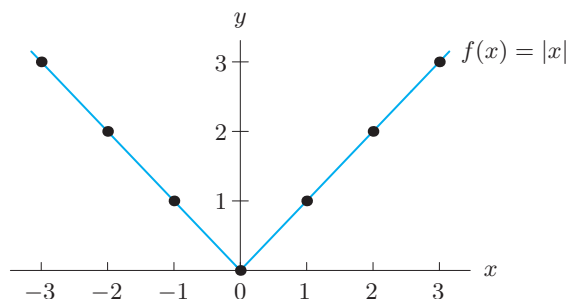| x	$|x|$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

Figure 2.14: Graph of absolute value function

Exercises and Problems for Section 2.3

Exercises

Graph the piecewise defined functions in Exercises 1–4. Use an open circle to represent a point which is not included and a solid dot to indicate a point which is on the graph.

1. $f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

2. $f(x) = \begin{cases} x + 1, & -2 \leq x < 0 \\ x - 1, & 0 \leq x < 2 \\ x - 3, & 2 \leq x < 4 \end{cases}$

3. $f(x) = \begin{cases} x + 4, & x \leq -2 \\ 2, & -2 < x < 2 \\ 4 - x, & x \geq 2 \end{cases}$
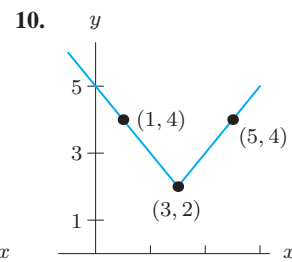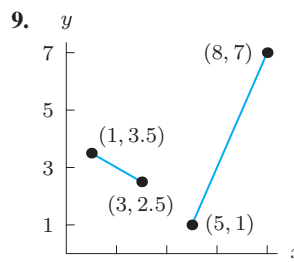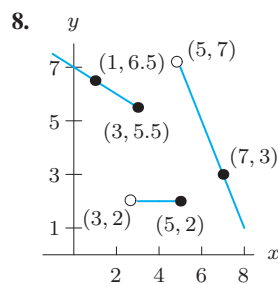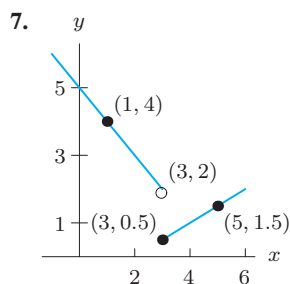
4. $f(x) = \begin{cases} x^2, & x \leq 0 \\ \sqrt{x}, & 0 < x < 4 \\ x/2, & x \geq 4 \end{cases}$

For Exercises 5–6, find the domain and range.

5. $G(x) = \begin{cases} x + 1 & \text{for} \quad x < -1 \\ x^2 + 3 & \text{for} \quad x \geq -1 \end{cases}$

6. $F(x) = \begin{cases} x^3 & \text{for} \quad x \leq 1 \\ 1/x & \text{for} \quad x > 1 \end{cases}$

In Exercises 7–10, write formulas for the functions.

7.

8.

9.

10.

Problems

11. Consider the graph in Figure 2.15. An open circle represents a point which is not included.

    (a) Is y a function of x? Explain.

    (b) Is x a function of y? Explain.

    (c) The domain of $y = f(x)$ is $0 \leq x < 4$. What is the range of $y = f(x)$?
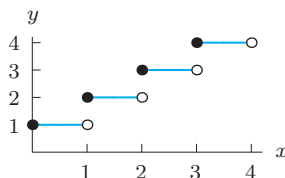

Figure 2.15

12. Many people believe that $\sqrt{x^2} = x$. We will investigate this claim graphically and numerically.

    (a) Graph the two functions x and $\sqrt{x^2}$ in the window $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. Based on what you see, do you believe that $\sqrt{x^2} = x$? What function does the graph of $\sqrt{x^2}$ remind you of?

    (b) Complete Table 2.10. Based on this table, do you believe that $\sqrt{x^2} = x$? What function does the table for $\sqrt{x^2}$ remind you of? Is this the same function you found in part (a)?

Table 2.10

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
$\sqrt{x^2}$											

    (c) Explain how you know that $\sqrt{x^2}$ is the same as the function $|x|$.

    (d) Graph the function $\sqrt{x^2} - |x|$ in the window $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. Explain what you see.

13. **(a)** Graph $u(x) = |x|/x$ in the window $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. Explain what you see.

    (b) Complete Table 2.11. Does this table agree with what you found in part (a)?

Table 2.11

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
$|x|/x$											

    (c) Identify the domain and range of $u(x)$.

    (d) Comment on the claim that $u(x)$ can be written as

$$u(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

14. The charge for a taxi ride in New York City is $2.50 for the first $1/4$ of a mile, and $0.40 for each additional $1/4$ of a mile (rounded up to the nearest $1/4$ mile).

    (a) Make a table showing the cost of a trip as a function of its length. Your table should start at zero and go up to two miles in $1/4$-mile intervals.

    (b) What is the cost for a 1.25-mile trip?

    (c) How far can you go for $5.30?

    (d) Graph the cost function in part (a).

15. A museum charges $40 for a group of 10 or fewer people. A group of more than 10 people must, in addition to the $40, pay $2 per person for the number of people above 10. For example, a group of 12 pays $44 and a group of 15 pays $50. The maximum group size is 50.

    (a) Draw a graph that represents this situation.

    (b) What are the domain and range of the cost function?

16. A floor-refinishing company charges $1.83 per square foot to strip and refinish a tile floor for up to 1000 square feet. There is an additional charge of $350 for toxic waste disposal for any job which includes more than 150 square feet of tile.

    (a) Express the cost, y, of refinishing a floor as a function of the number of square feet, x, to be refinished.

    (b) Graph the function. Give the domain and range.

17. A contractor purchases gravel one cubic yard at a time.

    (a) A gravel driveway L yards long and 6 yards wide is to be poured to a depth of 1 foot. Find a formula for $n(L)$, the number of cubic yards of gravel the contractor buys, assuming that he buys 10 more cubic yards of gravel than are needed (to be sure he'll have enough).

    (b) Assuming no driveway is less than 5 yards long, state the domain and range of $n(L)$. Graph $n(L)$ showing the domain and range.

    (c) If the function $n(L)$ did not represent an amount of gravel, but was a mathematical relationship defined by the formula in part (a), what is its domain and range?

18. At a supermarket checkout, a scanner records the prices of the foods you buy. In order to protect consumers, the state of Michigan passed a "scanning law" that says something similar to the following:

> If there is a discrepancy between the price marked on the item and the price recorded by the scanner, the consumer is entitled to receive 10 times the difference between those prices; this amount given must be at least $1 and at most $5. Also, the consumer will be given the difference between the prices, in addition to the amount calculated above.

For example: If the difference is 5¢, you should receive $1 (since 10 times the difference is only 50¢ and you are to receive at least $1), plus the difference of 5¢. Thus, the total you should receive is $1.00 + $0.05 = $1.05, If the difference is 25¢, you should receive 10 times the difference in addition to the difference, giving $(10)(0.25) + 0.25 = 2.75. If the difference is 95¢, you should receive $5 (because $10(.95) = 9.50 is more than $5, the maximum penalty), plus 95¢, giving $5 + 0.95 = 5.95.

    (a) What is the lowest possible refund?

    (b) Suppose x is the difference between the price scanned and the price marked on the item, and y is the amount refunded to the customer. Write a formula for y in terms of x. (Hints: Look at the sample calculations.)

    (c) What would the difference between the price scanned and the price marked have to be in order to obtain a $9.00 refund?

    (d) Graph y as a function of x.

19. Many printing presses are designed with large plates that print a fixed number of pages as a unit. Each unit is called a signature. A particular press prints signatures of 16 pages each. Suppose $C(p)$ is the cost of printing a book of p pages, assuming each signature printed costs $0.14.

    (a) What is the cost of printing a book of 128 pages? 129 pages? p pages?

    (b) What are the domain and range of C?

    (c) Graph $C(p)$ for $0 \le p \le 128$.

20. Gore Mountain is a ski resort in the Adirondack mountains in upstate New York. Table 2.12 shows the cost of a weekday ski-lift ticket for various ages and dates.

    (a) Graph cost as a function of age for each time period given. (One graph will serve for times when rates are identical).

    (b) For which age group does the date affect cost?

    (c) Graph cost as a function of date for the age group mentioned in part (b).

    (d) Why does the cost fluctuate as a function of date?

Table 2.12   *Ski-lift ticket prices at Gore Mountain, 1998–1999*[6]

Age	Opening-Dec 12	Dec 13-Dec 24	Dec 25-Jan 3	Jan 4-Jan 15	Jan 16-Jan 18
Up to 6	Free	Free	Free	Free	Free
7–12	$19	$19	$19	$19	$19
13–69	$29	$34	$39	$34	$39
70+	Free	Free	Free	Free	Free
Age	Jan 19-Feb 12	Feb 13-Feb 21	Feb 22-Mar 28	Mar 29-Closing	
Up to 6	Free	Free	Free	Free	
7–12	$19	$19	$19	$19	
13–69	$34	$39	$34	$29	
70+	Free	Free	Free	Free	

[6]The Olympic Regional Development Authority.

2.4 COMPOSITE AND INVERSE FUNCTIONS

Composition of Functions

Two functions may be connected by the fact that the output of one is the input of the other. For example, to find the cost, C, in dollars, to paint a room of area A square feet, we need to know the number, n, of gallons of paint required. Since one gallon covers 250 square feet, we have the function $n = f(A) = A/250$. If paint is \$30.50 a gallon, we have the function $C = g(n) = 30.5n$. We substitute $n = f(A)$ into $g(n)$ to find the cost C as a function of A.

Example 1   Find a formula for cost, C, as a function of area, A.

Solution   Since we have

$$C = 30.5n \quad \text{and} \quad n = \frac{A}{250},$$

substituting for n in the formula for C gives

$$C = 30.5\frac{A}{250} = 0.122A.$$

We say that C is a "function of a function", or *composite function*. If the function giving C in terms of A is called h, so $C = h(A)$, then we write

$$C = h(A) = g(f(A)).$$

The function h is said to be the *composition* of the functions f and g. We say f is the *inside* function and g is the *outside* function. In this example, the composite function $C = h(A) = g(f(A))$ tells us the cost of painting an area of A square feet.

Example 2   The air temperature, T, in °F, is given in terms of the chirp rate, R, in chirps per minute, of a snowy tree cricket by the function

$$T = f(R) = \frac{1}{4}R + 40.$$

Suppose one night we record the chirp rate and find that it varies with time, x, according to the function

$$R = g(x) = 20 + x^2 \qquad \text{where } x \text{ is in hours since midnight and } 0 \le x \le 10.$$

Find how temperature varies with time by obtaining a formula for h, where $T = h(x)$.

Solution   Since $f(R)$ is a function of R and $R = g(x)$, we see that g is the inside function and f is the outside function. Thus we substitute $R = g(x)$ into f:

$$T = f(R) = f(g(x)) = \frac{1}{4}g(x) + 40 = \frac{1}{4}(20 + x^2) + 40 = \frac{1}{4}x^2 + 45.$$

Thus, for $0 \le x \le 10$, we have

$$T = h(x) = \frac{1}{4}x^2 + 45.$$

Example 3 shows another example of composition.

Example 3   If $f(x) = x^2$ and $g(x) = 2x + 1$, find

(a) $f(g(x))$ (b) $g(f(x))$

Solution   (a) We have

$$f(g(x)) = f(2x + 1) = (2x + 1)^2.$$

(b) We have

$$g(f(x)) = g(x^2) = 2(x^2) + 1 = 2x^2 + 1.$$

Notice that $f(g(x))$ is not equal to $g(f(x))$ in this case.

Inverse Functions

The roles of a function's input and output can sometimes be reversed. For example, the population, P, of birds is given, in thousands, by $P = f(t)$, where t is the number of years since 2007. In this function, t is the input and P is the output. If the population is increasing, knowing the year enables us to calculate the population. Thus we can define a new function, $t = g(P)$, which tells us the value of t given the value of P instead of the other way round. For this function, P is the input and t is the output. The functions f and g are called *inverses* of each other. A function which has an inverse is said to be *invertible*.

The fact that f and g are inverse functions means that they go in "opposite directions." The function f takes t as input and outputs P, while g takes P as input and outputs t.

Inverse Function Notation

In the preceding discussion, there was nothing about the names of the two functions that stressed their special relationship. If we want to emphasize that g is the inverse of f, we call it f^{-1} (read "f-inverse"). To express the fact that the population of birds, P, is a function of time, t, we write

$$P = f(t).$$

To express the fact that the time t is also determined by P, so that t is a function of P, we write

$$t = f^{-1}(P).$$

The symbol f^{-1} is used to represent the function that gives the output t for a given input P.

Warning: The -1 which appears in the symbol f^{-1} for the inverse function is not an exponent. Unfortunately, the notation $f^{-1}(x)$ might lead us to interpret it as $\frac{1}{f(x)}$. The two expressions are not the same in general: $f^{-1}(x)$ is the output when x is fed into the inverse of f, while $\frac{1}{f(x)}$ is the reciprocal of the number we get when x is fed into f.

Example 4   Using $P = f(t)$, where P represents the population, in thousands, of birds on an island and t is the number of years since 2007:

(a)   What does $f(4)$ represent?   (b)   What does $f^{-1}(4)$ represent?

Solution   (a) The expression $f(4)$ is the bird population (in thousands) in the year 2011.

(b) Since f^{-1} is the inverse function, f^{-1} is a function which takes population as input and returns time as output. Therefore, $f^{-1}(4)$ is the number of years after 2007 at which there were 4,000 birds on the island.

Finding a Formula for the Inverse Function

In the next example, we find the formula for an inverse function.

Example 5   The cricket function, which gives temperature, T, in terms of chirp rate, R, is

$$T = f(R) = \frac{1}{4} \cdot R + 40.$$

Find a formula for the inverse function, $R = f^{-1}(T)$.

Solution   The inverse function gives the chirp rate in terms of the temperature, so we solve the following equation for R:

$$T = \frac{1}{4} \cdot R + 40,$$

giving

$$T - 40 = \frac{1}{4} \cdot R$$
$$R = 4(T - 40).$$

Thus, $R = f^{-1}(T) = 4(T - 40)$.

Domain and Range of an Inverse Function

The input values of the inverse function f^{-1} are the output values of the function f. Thus, the domain of f^{-1} is the range of f. For the cricket function, $T = f(R) = \frac{1}{4}R + 40$, if a realistic domain is $0 \leq R \leq 160$, then the range of f is $40 \leq T \leq 80$. See Figure 2.16.
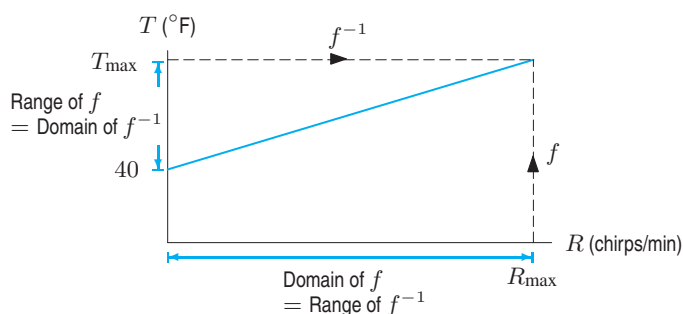
Figure 2.16

A Function and its Inverse Undo Each Other

Example 6   Calculate the composite functions $f^{-1}(f(R))$ and $f(f^{-1}(T))$ for the cricket example. Interpret the results.

Solution   Since $f(R) = \frac{1}{4}R + 40$. and $f^{-1}(T) = 4(T - 40)$, we have

$$f^{-1}(f(R)) = f^{-1}\left(\frac{1}{4} \cdot R + 40\right) = 4\left(\left(\frac{1}{4} \cdot R + 40\right) - 40\right) = R.$$
$$f(f^{-1}(T)) = f(4(T - 40)) = \frac{1}{4}(4(T - 40)) + 40 = T.$$

To interpret these results, we use the fact that $f(R)$ gives the temperature corresponding to chirp rate R, and $f^{-1}(T)$ gives the chirp rate corresponding to temperature T. Thus $f^{-1}(f(R))$ gives the chirp rate at temperature $f(R)$, which is R. Similarly, $f(f^{-1}(T))$ gives the temperature at chirp rate $f^{-1}(T)$, which is T.

Example 6 illustrates the following result, which we see is true in general in Chapter **??**.

The functions f and f^{-1} are called inverses because they "undo" each other when composed.

Exercises and Problems for Section 2.4

Exercises

In Exercises 1–4, give the meaning and units of the composite function.

1. $A(f(t))$, where $r = f(t)$ is the radius, in centimeters, of a circle at time t minutes, and $A(r)$ is the area, in square centimeters, of a circle of radius r centimeters.

2. $R(f(p))$, where $Q = f(p)$ is the number of barrels of oil sold by a company when the price is p dollars/barrel and $R(Q)$ is the revenue earned in millions of dollars.

3. $a(g(w))$, where $F = g(w)$ is the force, in newtons, on a rocket when the wind speed is w meters/sec and $a(F)$ is the acceleration, in meters/sec², when the force is F newtons.

4. $P(f(t))$, where $l = f(t)$ is the length, in centimeters, of a pendulum at time t minutes, and $P(l)$ is the period, in seconds, of a pendulum of length l.

In Exercises 5–12, use $f(x) = x^2 + 1$, $g(x) = 2x + 3$.

5. $f(g(0))$ **6.** $f(g(1))$ **7.** $g(f(0))$ **8.** $g(f(1))$

9. $f(g(x))$ **10.** $g(f(x))$ **11.** $f(f(x))$ **12.** $g(g(x))$

In Exercises 13–17, give the meaning and units of the inverse function. (Assume f is invertible.)

13. $P = f(t)$ is population in millions in year t.

14. $T = f(H)$ is time in minutes to bake a cake at $H°$F.

15. $N = f(t)$ number of inches of snow in the first t days of January.

16. $V = f(t)$ is the speed in km/hr of an accelerating car t seconds after starting.

17. $I = f(r)$ is the interest earned, in dollars, on a $10,000 deposit at an interest rate of r% per year, compounded annually.

In Exercises 18–21, find the inverse function.

18. $y = f(t) = 2t + 3$

19. $Q = f(x) = x^3 + 3$

20. $y = g(t) = \sqrt{t} + 1$

21. $P = f(q) = 14q - 2$

22. Use the graph in Figure 2.17 to fill in the missing values:

(a) $f(0) =?$ (b) $f(?) = 0$
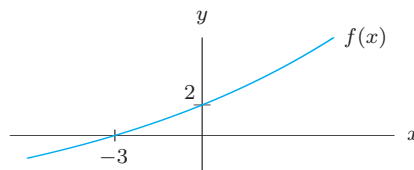(c) $f^{-1}(0) =?$ (d) $f^{-1}(?) = 0$

Figure 2.17

23. Use the graph in Figure 2.18 to fill in the missing values:

(a) $f(0) =?$ (b) $f(?) = 0$
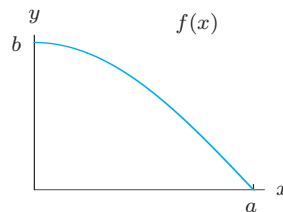(c) $f^{-1}(0) =?$ (d) $f^{-1}(?) = 0$

Figure 2.18

Problems

24. Let $n = f(A) = A/250$, where n is the number of gallons of paint and A is the area to be painted. Find a formula for the inverse function $A = g(n)$.

25. Calculate the composite functions $f^{-1}(f(A))$ and $f(f^{-1}(n))$ from Problem 24. Explain the results.

26. Table 2.13 gives values of an invertible function, f.

(a) Using the table, fill in the missing values:

(i) $f(0) =$?   (ii) $f(?) = 0$
(iii) $f^{-1}(0) =$?   (iv) $f^{-1}(?) = 0$

(b) How do the answers to (i)–(iv) in part (a) relate to one another? In particular, how could you have obtained the answers to (iii) and (iv) from the answers to (i) and (ii)?

Table 2.13

x	-2	-1	0	1	2
$f(x)$	5	4	2	0	-3

27. Use the values of the invertible function in Table 2.14 to find as many values of g^{-1} as possible.

Table 2.14

t	1	2	3	4	5
$y = g(t)$	7	12	13	19	22

28. The cost (in dollars) of producing x air conditioners is $C = g(x) = 600 + 45x$. Find a formula for the inverse function $g^{-1}(C)$.

29. The formula $V = f(r) = \frac{4}{3}\pi r^3$ gives the volume of a sphere of radius r. Find a formula for the inverse function, $f^{-1}(V)$, giving radius as a function of volume.

30. The cost, C, in thousands of dollars, of producing q kg of a chemical is given by $C = f(q) = 100 + 0.2q$. Find and interpret

(a) $f(10)$ (b) $f^{-1}(200)$ (c) $f^{-1}(C)$

31. The perimeter of a square of side s is given by $P = f(s) = 4s$. Find and interpret

(a) $f(3)$ (b) $f^{-1}(20)$ (c) $f^{-1}(P)$

32. The gross domestic product (GDP) of the US is given by $G(t)$ where t is the number of years since 1990 and the units of G are billions of dollars.

(a) What does $G(11)$ represent?
(b) What does $G^{-1}(9873)$ represent?

33. The formula for the volume of a cube with side s is $V = s^3$. The formula for the surface area of a cube is $A = 6s^2$.

(a) Find and interpret the formula for the function $s = f(A)$.
(b) If $V = g(s)$, find and interpret the formula for $g(f(A))$.

34. Interpret and evaluate $f(100)$ and $f^{-1}(100)$ for the house-painting function, $n = f(A) = A/250$. (See Problem 24.)

35. The cost of producing q thousand loaves of bread is $C(q)$ dollars. Interpret the following statements in terms of bread; give units.

(a) $C(5) = 653$
(b) $C^{-1}(80) = 0.62$
(c) The solution to $C(q) = 790$ is 6.3
(d) The solution to $C^{-1}(x) = 1.2$ is 150.

36. The area, $A = f(s)$ ft^2, of a square wooden deck is a function of the side s feet. A can of stain costs \$29.50 and covers 200 square feet of wood.

(a) Write the formula for $f(s)$.
(b) Find a formula for $C = g(A)$, the cost in dollars of staining an area of A ft^2.
(c) Find and interpret $C = g(f(s))$.
(d) Evaluate and interpret, giving units:

(i) $f(8)$ (ii) $g(80)$ (iii) $g(f(10))$

37. The radius, r, in centimeters, of a melting snowball is given by $r = 50 - 2.5t$, where t is time in hours. The snowball is spherical, with volume $V = \frac{4}{3}\pi r^3$ cm^3. Find a formula for $V = f(t)$, the volume of the snowball as a function of time.

38. A circular oil slick is expanding with radius, r in yards, at time t in hours given by $r = 2t - 0.1t^2$, for t in hours, $0 \le t \le 10$. Find a formula for the area in square yards, $A = f(t)$, as a function of time.

39. Carbon dioxide is one of the "greenhouse" gases that are believed to affect global warming. Between January 1998 and January 2003, the concentration of carbon dioxide in the earth's atmosphere increased steadily from 365 parts per million (ppm) to 375 ppm. Let $C(t)$ be the concentration in ppm of carbon dioxide t years after 1998.

(a) State the domain and range of $C(t)$.
(b) What is the practical meaning of $C(4)$?
(c) What does $C^{-1}(370)$ represent?

40. Table 2.15 shows the cost, $C(m)$, of a taxi ride as a function of the number of miles, m, traveled.

(a) Estimate and interpret $C(3.5)$ in practical terms.
(b) Assume C is invertible. What does $C^{-1}(3.5)$ mean in practical terms? Estimate $C^{-1}(3.5)$.

Table 2.15

m	0	1	2	3	4	5
$C(m)$	0	2.50	4.00	5.50	7.00	8.50

41. The perimeter, in meters, of a square whose side is s meters is given by $P = 4s$.

(a) Write this formula using function notation, where f is the name of the function.
(b) Evaluate $f(s + 4)$ and interpret its meaning.
(c) Evaluate $f(s) + 4$ and interpret its meaning.
(d) What are the units of $f^{-1}(6)$?

2.5 CONCAVITY

Concavity and Rates of Change

The graph of a linear function is a straight line because the average rate of change is a constant. However, not all graphs are straight lines; they may bend up or down. Consider the salary function $S(t)$ shown in Table 2.16 and Figure 2.19, where t is time in years since being hired. Since the rate of change increases with time, the slope of the graph increases as t increases, so the graph bends upward. We say such graphs are *concave up*.

Table 2.16   *Salary: Increasing rate of change*

t (years)	S ($1000s)	Rate of change $\Delta S/\Delta t$
0	40	
		3.2
10	72	
		5.6
20	128	
		10.2
30	230	
		18.1
40	411	
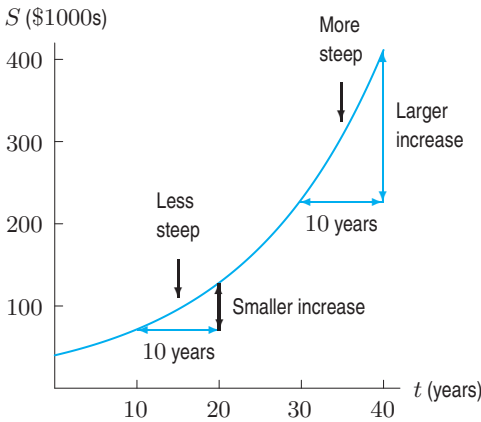
Figure 2.19: Graph of salary function is concave up because rate of change increases

The next example shows that a decreasing function can also be concave up.

Example 1   Table 2.17 shows Q, the quantity of carbon-14 (in μg) in a 200 μg sample remaining after t thousand years. We see from Figure 2.20 that Q is a decreasing function of t, so its rate of change is always negative. What can we say about the concavity of the graph, and what does this mean about the rate of change of the function?

Table 2.17 *Carbon-14: Increasing rate of change*

t (thousand years)	Q (μg)	Rate of change $\Delta Q/\Delta t$
0	200	
		-18.2
5	109	
		-9.8
10	60	
		-5.4
15	33	
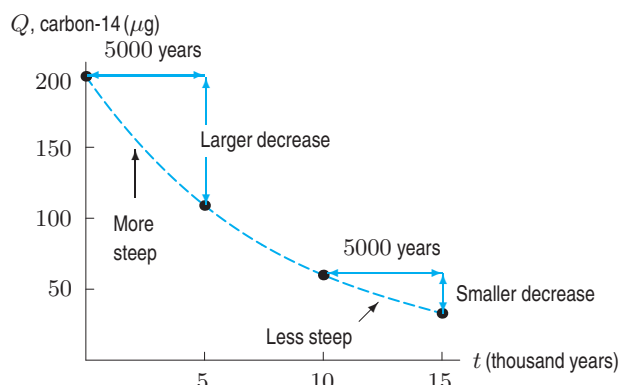
Figure 2.20: Graph of the quantity of carbon-14 is concave up

Solution   The graph bends upward, so it is concave up. Table 2.18 shows that the rate of change of the function is increasing, because the rate is becoming less negative. Figure 2.20 shows how the increasing rate of change can be visualized on the graph: the slope is negative and increasing.

Graphs can bend downward; we call such graphs *concave down*.

Example 2   Table 2.18 gives the distance traveled by a cyclist, Karim, as a function of time. What is the concavity of the graph? Was Karim's speed (that is, the rate of change of distance with respect to time) increasing, decreasing, or constant?

Solution   Table 2.18 shows Karim's speed was decreasing throughout the trip. Figure 2.21 shows how the decreasing speed leads to a decreasing slope and a graph which bends downward; thus the graph is concave down.

Table 2.18 *Karim's distance as a function of time, with the average speed for each hour*

t, time (hours)	d, distance (miles)	Average speed, $\Delta d/\Delta t$ (mph)
0	0	
		20 mph
1	20	
		15 mph
2	35	
		10 mph
3	45	
		7 mph
4	52	
		5 mph
5	57	

Figure 2.21: Karim's distance as a function of time

Summary: Increasing and Decreasing Functions; Concavity

Figure 2.22:
Increasing and
concave down

Figure 2.23:
Decreasing and
concave down

Figure 2.24:
Decreasing and
concave up

Figure 2.25:
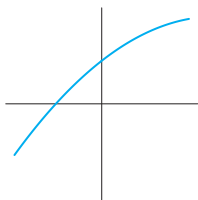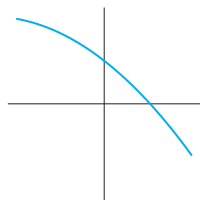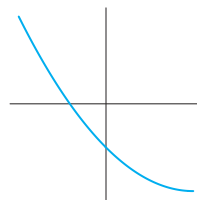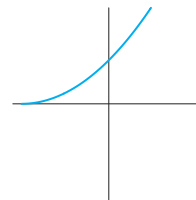Increasing and
concave up

Figures 2.22–2.25 reflect the following relationships between concavity and rate of change:

- If f is a function whose rate of change increases (gets less negative or more positive as we move from left to right[7]), then the graph of f is **concave up**. That is, the graph bends upward.
- If f is a function whose rate of change decreases (gets less positive or more negative as we move from left to right), then the graph of f is **concave down**. That is, the graph bends downward.

If a function has a constant rate of change, its graph is a line and it is neither concave up nor concave down.

Exercises and Problems for Section 2.5

Exercises

Do the graphs of the functions in Exercises 1–8 appear to be concave up, concave down, or neither?

1.

x	0	1	3	6
$f(x)$	1.0	1.3	1.7	2.2

2.

t	0	1	2	3	4
$f(t)$	20	10	6	3	1

3.

4.
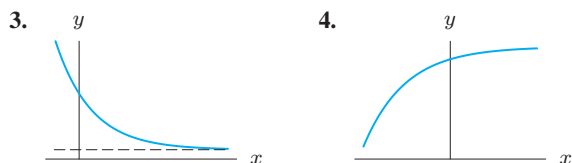
5. $y = x^2$

6. $y = -x^2$

7. $y = x^3, x > 0$

8. $y = x^3, x < 0$

9. Calculate successive rates of change for the function, $p(t)$, in Table 2.19 to decide whether you expect the graph of $p(t)$ to be concave up or concave down.

Table 2.19

t	0.2	0.4	0.6	0.8
$p(t)$	-3.19	-2.32	-1.50	-0.74

10. Calculate successive rates of change for the function, $H(x)$, in Table 2.20 to decide whether you expect the graph of $H(x)$ to be concave up or concave down.

Table 2.20

x	12	15	18	21
$H(x)$	21.40	21.53	21.75	22.02

11. Sketch a graph which is everywhere negative, increasing, and concave down.

12. Sketch a graph which is everywhere positive, increasing, and concave up.

---

[7]In fact, we need to take the average rate of change over an arbitrarily small interval.

Problems

Are the functions in Problems 13–17 increasing or decreasing? What does the scenario tell you about the concavity of the graph modeling it?

13. When money is deposited in the bank, the amount of money increases slowly at first. As the size of the account increases, the amount of money increases more rapidly, since the account is earning interest on the new interest, as well as on the original amount.

14. After a cup of hot chocolate is poured, the temperature cools off very rapidly at first, and then cools off more slowly, until the temperature of the hot chocolate eventually reaches room temperature.

15. When a rumor begins, the number of people who have heard the rumor increases slowly at first. As the rumor spreads, the rate of increase gets greater (as more people continue to tell their friends the rumor), and then slows down again (when almost everyone has heard the rumor).

16. When a drug is injected into a person's bloodstream, the amount of the drug present in the body increases rapidly at first. If the person receives daily injections, the body metabolizes the drug so that the amount of the drug present in the body continues to increase, but at a decreasing rate. Eventually, the quantity levels off at a saturation level.

17. When a new product is introduced, the number of people who use the product increases slowly at first, and then the rate of increase is faster (as more and more people learn about the product). Eventually, the rate of increase slows down again (when most people who are interested in the product are already using it).

18. Match each story with the table and graph which best represent it.

(a) When you study a foreign language, the number of new verbs you learn increases rapidly at first, but slows almost to a halt as you approach your saturation level.

(b) You board an airplane in Philadelphia heading west. Your distance from the Atlantic Ocean, in kilometers, increases at a constant rate.

(c) The interest on your savings plan is compounded annually. At first your balance grows slowly, but its rate of growth continues to increase.

(E)

x	0	5	10	15	20	25
y	20	275	360	390	395	399

(F)

x	0	5	10	15	20	25
y	20	36	66	120	220	400

(G)

x	0	5	10	15	20	25
y	20	95	170	245	320	395

(I)   (II)   (III)
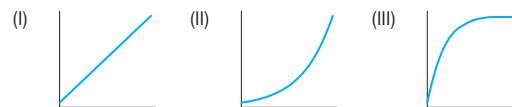
19. Match each of the following descriptions with an appropriate graph and table of values.

(a) The weight of your jumbo box of Fruity Flakes decreases by an equal amount every week.

(b) The machinery depreciated rapidly at first, but its value declined more slowly as time went on.

(c) In free fall, your distance from the ground decreases faster and faster.

(d) For a while it looked like the decline in profits was slowing down, but then they began declining ever more rapidly.

(E)

x	0	1	2	3	4	5
y	400	384	336	256	144	0

(F)

x	0	1	2	3	4	5
y	400	320	240	160	80	0

(G)

x	0	1	2	3	4	5
y	400	184	98	63	49	43

(H)

x	0	1	2	3	4	5
y	412	265	226	224	185	38

(I)   (II)   (III)   (IV)
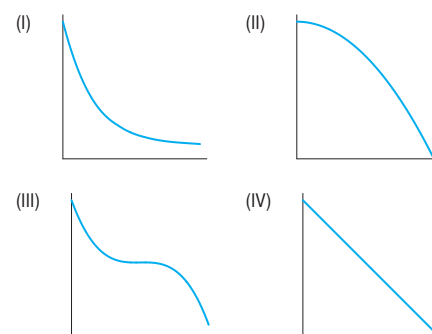
20. An incumbent politician running for reelection declared that the number of violent crimes is no longer rising and is presently under control. Does the graph shown in Figure 2.26 support this claim? Why or why not?
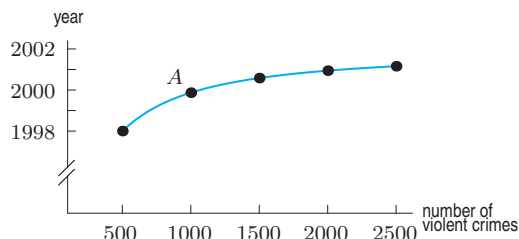
Figure 2.26

21. The rate at which water is entering a reservoir is given for time $t > 0$ by the graph in Figure 2.27. A negative rate means that water is leaving the reservoir. For each of the following statements, give the largest interval on which:

(a) The volume of water is increasing.
(b) The volume of water is constant.
(c) The volume of water is increasing fastest.
(d) The volume of water is decreasing.
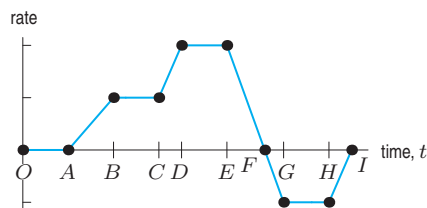
Figure 2.27

22. The relationship between the swimming speed U (in cm/sec) of a salmon to the length l of the salmon (in cm) is given by the function[8]

$$U = 19.5\sqrt{l}.$$

(a) If one salmon is 4 times the length of another salmon, how are their swimming speeds related?
(b) Graph the function $U = 19.5\sqrt{l}$. Describe the graph using words such as increasing, decreasing, concave up, concave down.
(c) Using a property that you described in part (b), answer the question "Do larger salmon swim faster than smaller ones?"
(d) Using a property that you described in part (b), answer the question "Imagine four salmon—two small and two large. The smaller salmon differ in length by 1 cm, as do the two larger. Is the difference in speed between the two smaller fish, greater than, equal to, or smaller than the difference in speed between the two larger fish?"

2.6 QUADRATIC FUNCTIONS

A baseball is "popped" straight up by a batter. The height of the ball above the ground is given by the function $y = f(t) = -16t^2 + 64t + 3$, where t is time in seconds after the ball leaves the bat and y is in feet. See Figure 2.28. Although the path of the ball is straight up and down, the graph of its height as a function of time is concave down. The ball goes up fast at first and then more slowly because of gravity.

Figure 2.28: The height of a ball t seconds after being "popped up". (Note: This graph does not represent the ball's path)

[8] From K. Schmidt-Nielsen, *Scaling, Why is Animal Size so Important?* (Cambridge: CUP, 1984).

The baseball height function is an example of a *quadratic function*, whose general form is $y = ax^2 + bx + c$.

Finding the Zeros of a Quadratic Function

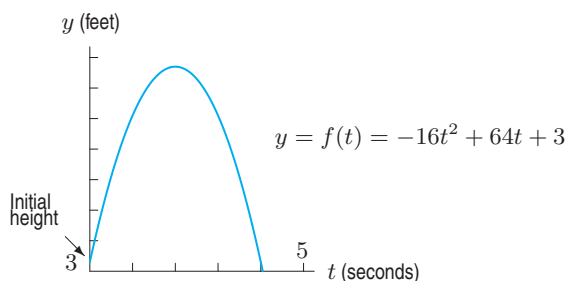A natural question to ask is when the ball hits the ground. The graph suggests that $y = 0$ when $t \approx 4$. (See Problem 25 on page 92.) We can phrase the question symbolically: For what value of t does $f(t) = 0$? Input values of t which make the output $f(t) = 0$ are called *zeros* of f. It is easy to find the zeros of a quadratic function if its formula can be factored (see the Tools section to review factoring).

Example 1   Find the zeros of $f(x) = x^2 - x - 6$.

Solution   To find the zeros, set $f(x) = 0$ and solve for x by factoring:

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0.$$

Thus the zeros are $x = 3$ and $x = -2$.

Some quadratic functions can be expressed in *factored form*,

$$q(x) = a(x - r)(x - s),$$

where a, r, and s are constants, $a \neq 0$. Note that r and s are zeros of the function q. The factored form of the function f in Example 1 is $f(x) = (x - 3)(x + 2)$.

We can also find the zeros of a quadratic function by using the quadratic formula. (See the Tools section to review the quadratic formula.)

Example 2   Find the zeros of $f(x) = x^2 - x - 6$ by using the quadratic formula.

Solution   We must solve the equation $x^2 - x - 6 = 0$. For this equation, $a = 1$, $b = -1$, and $c = -6$. Thus

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{25}}{2} = 3 \text{ or } -2.$$

The zeros are $x = 3$ and $x = -2$, the same as we found by factoring.

The zeros of a function occur at the x-intercepts of its graph. Not every quadratic function has x-intercepts, as we see in the next example.

Example 3   Figure 2.29 shows a graph of $h(x) = -\frac{1}{2}x^2 - 2$. What happens if we try to use algebra to find its zeros?
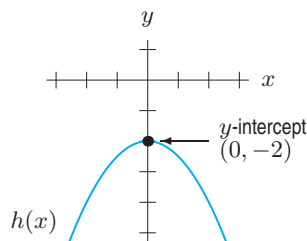
Figure 2.29

Solution   To find the zeros, we solve the equation

$$-\frac{1}{2}x^2 - 2 = 0$$
$$-\frac{1}{2}x^2 = 2$$
$$x^2 = -4$$
$$x = \pm\sqrt{-4}.$$

Since $\sqrt{-4}$ is not a real number, there are no real solutions, so h has no real zeros. This corresponds to the fact that the graph of h in Figure 2.29 does not cross the x-axis.

Concavity and Quadratic Functions

Unlike a linear function, whose graph is a straight line, a quadratic function has a graph which is either concave up or concave down.

Example 4   Let $f(x) = x^2$. Find the average rate of change of f over the intervals of length 2 between $x = -4$ and $x = 4$. What do these rates tell you about the concavity of the graph of f?

Solution   Between $x = -4$ and $x = -2$, we have

$$\begin{array}{c} \text{Average rate of change} \\ \text{of } f \end{array} = \frac{f(-2) - f(-4)}{-2 - (-4)} = \frac{(-2)^2 - (-4)^2}{-2 + 4} = -6.$$

Between $x = -2$ and $x = 0$, we have

$$\begin{array}{c} \text{Average rate of change} \\ \text{of } f \end{array} = \frac{f(0) - f(-2)}{0 - (-2)} = \frac{0^2 - (-2)^2}{0 + 2} = -2.$$

Between $x = 0$ and $x = 2$, we have

$$\begin{array}{c} \text{Average rate of change} \\ \text{of } f \end{array} = \frac{f(2) - f(0)}{2 - 0} = \frac{2^2 - 0^2}{2 - 0} = 2.$$

Between $x = 2$ and $x = 4$, we have

$$\begin{array}{c} \text{Average rate of change} \\ \text{of } f \end{array} = \frac{f(4) - f(2)}{4 - 2} = \frac{4^2 - 2^2}{4 - 2} = 6.$$

Since these rates are increasing, we expect the graph of f to be bending upward. Figure 2.30 confirms that the graph is concave up.

Figure 2.30

Example 5   A high diver jumps off a 10-meter springboard. For h in meters and t in seconds after the diver leaves the board, her height above the water is in Figure 2.31 and given by

$$h = f(t) = -4.9t^2 + 8t + 10.$$

(a) Find and interpret the domain and range of the function and the intercepts of the graph.
(b) Identify the concavity.
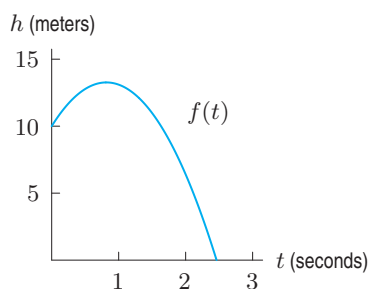
Figure 2.31: Height of diver as a function of time

Solution   (a) The diver enters the water when her height is 0. This occurs when

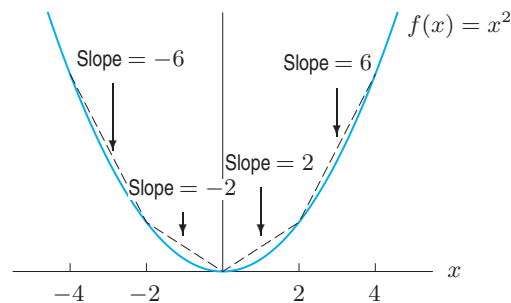$$h = f(t) = -4.9t^2 + 8t + 10 = 0.$$

Using the quadratic formula to solve this equation, we find $t = 2.462$ seconds. The domain is the interval of time the diver is in the air, namely $0 \le t \le 2.462$. To find the range of f, we look for the largest and smallest outputs for h. From the graph, the diver's maximum height appears to occur at about $t = 1$, so we estimate the largest output value for f to be

$$f(1) = -4.9 \cdot 1^2 + 8 \cdot 1 + 10 = 13.1 \text{ meters}$$

Thus, the range of f is approximately $0 \le f(t) \le 13.1$.
    The vertical intercept of the graph is

$$f(0) = -4.9 \cdot 0^2 + 8 \cdot 0 + 10 = 10 \text{ meters}.$$

The diver's initial height is 10 meters (the height of the springboard). The horizontal intercept is the point where $f(t) = 0$, which we found in part (a). The diver enters the water approximately 2.462 seconds after leaving the springboard.
(b) In Figure 2.31, we see that the graph is bending downward over its entire domain, so it is concave down. This is confirmed by Table 2.21, which shows that the rate of change, $\Delta h / \Delta t$, is decreasing.

Table 2.21

t (sec)	h (meters)	Rate of change $\Delta h / \Delta t$
0	10	
		5.55
0.5	12.775	
		0.65
1.0	13.100	
		−4.25
1.5	10.975	
		−9.15
2.0	6.400	

Exercises and Problems for Section 2.6

Exercises

Are the functions in Exercises 1–7 quadratic? If so, write the function in the form $f(x) = ax^2 + bx + c$.

1. $f(x) = 2(7 - x)^2 + 1$

2. $L(P) = (P + 1)(1 - P)$

3. $g(m) = m(m^2 - 2m) + 3\left(14 - \dfrac{m^3}{3}\right) + \sqrt{3}m$

4. $h(t) = -16(t - 3)(t + 1)$

5. $R(q) = \dfrac{1}{q^2}(q^2 + 1)^2$

6. $K(x) = 13^2 + 13^x$

7. $T(n) = \sqrt{5} + \sqrt{3n^4} - \sqrt{\dfrac{n^4}{4}}$

8. Find the zeros of $Q(r) = 2r^2 - 6r - 36$ by factoring.

9. Find the zeros of $Q(x) = 5x - x^2 + 3$ using the quadratic formula.

10. Find two quadratic functions with zeros $x = 1$, $x = 2$.

11. Solve for x using the quadratic formula and demonstrate your solution graphically:

   (a) $6x - \frac{1}{3} = 3x^2$ **(b)** $2x^2 + 7.2 = 5.1x$

12. Without a calculator, graph $y = 3x^2 - 16x - 12$ by factoring and plotting zeros.

In Exercises 13–24, find the zeros (if any) of the function algebraically.

13. $y = (2 - x)(3 - 2x)$ **14.** $y = 2x^2 + 5x + 2$

15. $y = 4x^2 - 4x - 8$ **16.** $y = 7x^2 + 16x + 4$

17. $y = 9x^2 + 6x + 1$ **18.** $y = 6x^2 - 17x + 12$

19. $y = 5x^2 + 2x - 1$ **20.** $y = 3x^2 - 2x + 6$

21. $y = -17x^2 + 23x + 19$ **22.** $y = 89x^2 + 55x + 34$

23. $y = x^4 + 5x^2 + 6$ **24.** $y = x - \sqrt{x} - 12$

Problems

25. Use the quadratic formula to find the time at which the baseball in Figure 2.28 on page 88 hits the ground.

26. Is there a quadratic function with zeros $x = 1$, $x = 2$ and $x = 3$?

27. Determine the concavity of the graph of $f(x) = 4 - x^2$ between $x = -1$ and $x = 5$ by calculating average rates of change over intervals of length 2.

28. Graph a quadratic function which has all the following properties: concave up, y-intercept is -6, zeros at $x = -2$ and $x = 3$.

29. Without a calculator, graph the following function by factoring and plotting zeros:

$$y = -4cx + x^2 + 4c^2 \quad \text{for} \quad c > 0$$

30. A ball is thrown into the air. Its height (in feet) t seconds later is given by $h(t) = 80t - 16t^2$.

   (a) Evaluate and interpret $h(2)$.
   (b) Solve the equation $h(t) = 80$. Interpret your solutions and illustrate them on a graph of $h(t)$.

31. Let $V(t) = t^2 - 4t + 4$ represent the velocity of an object in meters per second.

   (a) What is the object's initial velocity?
   (b) When is the object not moving?
   (c) Identify the concavity of the velocity graph.

32. The percentage of schools with interactive videodisc players[9] each year from 1992 to 1996 is shown in Table 2.22. If x is in years since 1992, show that this data set can be approximated by the quadratic function $p(x) = -0.8x^2 + 8.8x + 7.2$. What does this model predict for the year 2004? How good is this model for predicting the future?

Table 2.22

Year	1992	1993	1994	1995	1996
Percentage	8	14	21	29.1	29.3

---

[9] Data from R. Famighetti, ed. *The World Almanac and Book of Facts: 1999.* (New Jersey: Funk and Wagnalls, 1998).

33. Let $f(x) = x^2$ and $g(x) = x^2 + 2x - 8$.

   (a) Graph f and g in the window $-10 \leq x \leq 10$, $-10 \leq y \leq 10$. How are the two graphs similar? How are they different?

   (b) Graph f and g in the window $-10 \leq x \leq 10$, $-10 \leq y \leq 100$. Why do the two graphs appear more similar on this window than on the window from part (a)?

   (c) Graph f and g in the window $-20 \leq x \leq 20$, $-10 \leq y \leq 400$, the window $-50 \leq x \leq 50$, $-10 \leq y \leq 2500$, and the window $-500 \leq x \leq 500$, $-2500 \leq y \leq 250{,}000$. Describe the change in appearance of f and g on these three successive windows.

34. A relief package is dropped from a moving airplane. Since the package is initially released with a forward horizontal velocity, it follows a parabolic path (instead of dropping straight down). Figure 2.32 shows the height of the package, h, in km, as a function of the horizontal distance, d, in meters, it has traveled since it was dropped.

   (a) From what height was the package released?

   (b) How far away from the spot above which it was released does the package hit the ground?

   (c) Write a formula for $h(d)$. [Hint: The package starts falling at the highest point on the parabola].
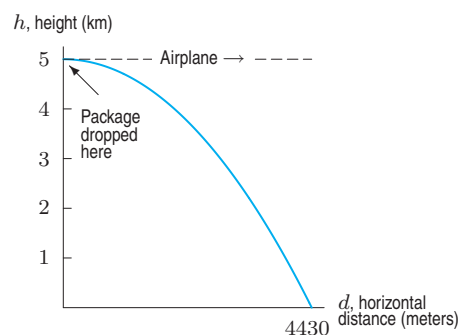

_h_, height (km)

Airplane → ----

Package dropped here

_d_, horizontal distance (meters)

4430

Figure 2.32

35. **(a)** Fit a quadratic function to the first three data points in Table 2.23. Use the fifth point as a check.

   (b) Find the formula for a linear function that passes through the second two data points.

   (c) Compare the value of the linear function at $x = 3$ to the value of the quadratic at $x = 3$.

   (d) Compare the values of the linear and quadratic functions at $x = 50$.

   (e) For approximately what x values do the quadratic and linear function values differ by less than 0.05? Using a calculator or computer, graph both functions on the same axes and estimate an answer.

Table 2.23

x	0	1	2	3	50
y	1.0	3.01	5.04	7.09	126.0

CHAPTER SUMMARY

- **Input and Output**
  Evaluating functions: finding $f(a)$ for given a.
  Solving equations: finding x if $f(x) = b$ for given b.

- **Domain and Range**
  Domain: set of input values.
  Range: set of output values
  Piecewise functions: different formulas on different intervals.

- **Inverse functions**
  If $y = f(x)$, then $f^{-1}(y) = x$.
  Evaluating $f^{-1}(b)$. Interpretation of $f^{-1}(b)$. Formula for $f^{-1}(y)$ given formula for $f(x)$.

- **Concavity**
  Concave up: increasing rate of change.
  Concave down: decreasing rate of change.

- **Quadratic Functions**
  Standard form for quadratic functions:

  $$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

  Factored form gives zeros r, s of quadratic:

  $$f(x) = a(x - r)(x - s).$$

  Quadratic formula:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TWO

Exercises

If $p(r) = r^2 + 5$, evaluate the expressions in Exercises 1–2.

1. $p(7)$

2. $p(x) + p(8)$

3. Let $h(x) = x^2 + bx + c$. Evaluate and simplify:

    (a) $h(1)$ **(b)** $h(b+1)$

4. If $g(x) = x\sqrt{x} + 100x$, evaluate without a calculator

    (a) $g(100)$ **(b)** $g(4/25)$ **(c)** $g(1.21 \cdot 10^4)$

5. Find the zeros of $s(l) = 7l - l^2$.

6. (a) How can you tell from the graph of a function that an x-value is not in the domain? Sketch an example.
    (b) How can you tell from the formula for a function that an x-value is not in the domain? Give an example.

In Exercises 7–10, state the domain and range.

7. $h(x) = x^2 + 8x$

8. $f(x) = \sqrt{x - 4}$

9. $r(x) = \sqrt{4 - \sqrt{x - 4}}$

10. $g(x) = \dfrac{4}{4 + x^2}$

11. Let $g(x) = x^2 + x$. Evaluate and simplify the following.

    (a) $-3g(x)$ **(b)** $g(1) - x$
    (c) $g(x) + \pi$ **(d)** $\sqrt{g(x)}$
    (e) $g(1)/(x+1)$ **(f)** $(g(x))^2$

12. Let $f(x) = 1 - x$. Evaluate and simplify the following.

    (a) $2f(x)$ **(b)** $f(x) + 1$ **(c)** $f(1 - x)$
    (d) $(f(x))^2$ **(e)** $f(1)/x$ **(f)** $\sqrt{f(x)}$

In Exercises 13–14, let $f(x) = 3x - 7$, $g(x) = x^3 + 1$ to find a formula for the function.

13. $f(g(x))$

14. $g(f(x))$

In Exercises 15–16, find the inverse function.

15. $y = f(x) = 3x - 7$

16. $y = g(x) = x^3 + 1$

17. Use the graph in Figure 2.33 to fill in the missing values:

    (a) $f(0) =?$ **(b)** $f(?) = 0$
    (c) $f^{-1}(0) =?$ **(d)** $f^{-1}(?) = 0$
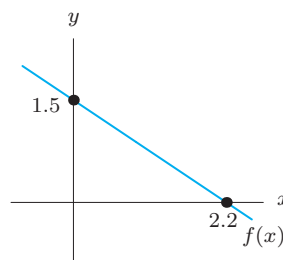
Figure 2.33

In Exercises 18–20, let $P = f(t)$ be the population, in millions, of a country at time t in years and let $E = g(P)$ be the daily electricity consumption, in megawatts, when the population is P. Give the meaning and units of the function. Assume both f and g are invertible.

18. $g(f(t))$ **19.** $f^{-1}(P)$ **20.** $g^{-1}(E)$

Problems

In Figure 2.34, show the coordinates of the point(s) representing the statements in Problems 21–24.
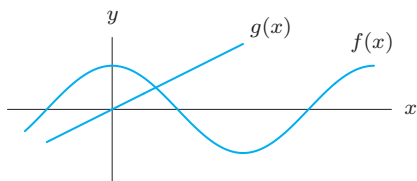
Figure 2.34

21. $f(0) = 2$

22. $f(-3) = f(3) = f(9) = 0$

23. $f(2) = g(2)$

24. $g(x) > f(x)$ for $x > 2$

In Problems 25–27, if $f(x) = \dfrac{ax}{a + x}$, find and simplify

25. $f(a)$ **26.** $f(1 - a)$ **27.** $f\left(\dfrac{1}{1 - a}\right)$

28. (a) Find the side, $s = f(d)$, of a square as function of its diagonal d.
(b) Find the area, $A = g(s)$, of a square as function of its side s.
(c) Find the area $A = h(d)$ as a function of d.
(d) What is the relation between f, g, and h?

29. The cost of producing q thousand loaves of bread is $C(q)$ dollars. Interpret the following statements in terms of bread; give units.

(a) $C(5) = 653$
**(b)** $C^{-1}(80) = 0.62$
(c) The solution to $C(q) = 790$ is 6.3
(d) The solution to $C^{-1}(x) = 1.2$ is 150.

In Exercises 30–32, let $H = f(t) = \frac{5}{9}(t - 32)$, where H is temperature in degrees Celsius and t is in degrees Fahrenheit.

30. Find and interpret the inverse function, $f^{-1}(H)$.

31. Using the results of Exercise 30, evaluate and interpret:

(a) $f(0)$
(b) $f^{-1}(0)$
(c) $f(100)$
(d) $f^{-1}(100)$

32. The temperature, $t = g(n) = 68 + 10 \cdot 2^{-n}$, in degrees Fahrenheit of a room is a function of the number, n, of hours that the air conditioner has been running. Find and interpret $f(g(n))$. Give units.

33. The period, T, of a pendulum of length l is given by $T = f(l) = 2\pi\sqrt{l/g}$, where g is a constant. Find a formula for $f^{-1}(T)$ and explain its meaning.

34. The area, in square centimeters, of a circle whose radius is r cm is given by $A = \pi r^2$.

(a) Write this formula using function notation, where f is the name of the function.
(b) Evaluate $f(0)$.
(c) Evaluate and interpret $f(r + 1)$.
(d) Evaluate and interpret $f(r) + 1$.
(e) What are the units of $f^{-1}(4)$?

35. An epidemic of influenza spreads through a city. Figure 2.35 is the graph of $I = f(w)$, where I is the number of individuals (in thousands) infected w weeks after the epidemic begins.

(a) Evaluate $f(2)$ and explain its meaning in terms of the epidemic.
(b) Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form $f(a) = b$.
(c) Solve $f(w) = 4.5$ and explain what the solutions mean in terms of the epidemic.
(d) The graph used $f(w) = 6w(1.3)^{-w}$. Use the graph to estimate the solution of the inequality

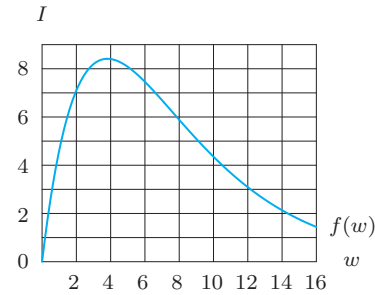$6w(1.3)^{-w} \geq 6$. Explain what the solution means in terms of the epidemic.

Figure 2.35

36. Let t be time in seconds and let $r(t)$ be the rate, in gallons/second, that water enters a reservoir:

$$r(t) = 800 - 40t.$$

(a) Evaluate the expressions $r(0)$, $r(15)$, $r(25)$, and explain their physical significance.
(b) Graph $y = r(t)$ for $0 \leq t \leq 30$, labeling the intercepts. What is the physical significance of the slope and the intercepts?
(c) For $0 \leq t \leq 30$, when does the reservoir have the most water? When does it have the least water?
(d) What are the domain and range of $r(t)$?

37. Suppose that $f(x)$ is invertible and that both f and f^{-1} are defined for all values of x. Let $f(2) = 3$ and $f^{-1}(5) = 4$. Evaluate the following expressions, or, if the given information is insufficient, write unknown.

(a) $f^{-1}(3)$
(b) $f^{-1}(4)$
(c) $f(4)$

38. Suppose that $j(x) = h^{-1}(x)$ and that both j and h are defined for all values of x. Let $h(4) = 2$ and $j(5) = -3$. Evaluate if possible:

(a) $j(h(4))$
(b) $j(4)$
(c) $h(j(4))$
(d) $j(2)$
(e) $h^{-1}(-3)$
(f) $j^{-1}(-3)$
(g) $h(5)$
(h) $(h(-3))^{-1}$
(i) $(h(2))^{-1}$

39. Values of f and g are given in Table 2.24.

(a) Evaluate $f(1)$ and $g(3)$.
(b) Describe in full sentences the patterns you see in the values for each function.
(c) Assuming that the patterns you observed in part (b) hold true for all values of x, calculate $f(5)$, $f(-2)$, $g(5)$, and $g(-2)$.
(d) Find possible formulas for $f(x)$ and $g(x)$.

Table 2.24

x	−1	0	1	2	3	4
$f(x)$	−4	−1	2	5	8	11
$g(x)$	4	1	0	1	4	9

40. Let $k(x) = 6 - x^2$.

   (a) Find a point on the graph of $k(x)$ whose x-coordinate is -2.

   (b) Find two points on the graph whose y-coordinates are -2.

   (c) Graph $k(x)$ and locate the points in parts (a) and (b).

   (d) Let $p = 2$. Calculate $k(p) - k(p - 1)$.

41. Let $f(a)$ be the cost in dollars of a pounds of organic apples at the Gourmet Garage in New York City in January 2006. What do the following statements tell you? What are the units of each of the numbers?

   (a) $f(2) = 4.98$ **(b)** $f(0.5) = 1.25$

   (c) $f^{-1}(0.62) = 0.25$ **(d)** $f^{-1}(12.45) = 5$

42. Let $t(x)$ be the time required, in seconds, to melt 1 gram of a compound at $x°\mathrm{C}$.

   (a) Express the following statement as an equation using $t(x)$: It takes 272 seconds to melt 1 gram of the compound at $400°\mathrm{C}$.

   (b) Explain the following equations in words:

     (i) $t(800) = 136$ **(ii)** $t^{-1}(68) = 1600$

   (c) Above a certain temperature, doubling the temperature, x, halves the melting time. Express this fact with an equation involving $t(x)$.

43. Table 2.25 shows $N(s)$, the number of sections of Economics 101, as a function of s, the number of students in the course. If s is between two numbers listed in the table, then $N(s)$ is the higher number of sections.

Table 2.25

s	50	75	100	125	150	175	200
$N(s)$	4	4	5	5	6	6	7

   (a) Evaluate and interpret:

     (i) $N(150)$ **(ii)** $N(80)$ **(iii)** $N(55.5)$

   (b) Solve for s and interpret:

     (i) $N(s) = 4$ **(ii)** $N(s) = N(125)$

Problems 44–45 concern studies which indicate that as carbon dioxide (CO_2) levels rise, hurricanes will become more intense.[10] Hurricane intensity is measured in terms of the minimum central pressure (in mb): the lower the pressure, the more powerful the storm. Since warm ocean waters fuel hurricanes, P is a decreasing function of H, sea surface temperature in $°\mathrm{C}$. Let $P = n(H)$ be the hurricane-intensity function for present-day CO_2 levels, and let $P = N(H)$ be the hurricane-intensity function for future projected CO_2 levels. If H_0 is the average temperature in the Caribbean Sea, what do the following quantities tell you about hurricane intensity?

44. $N(H_0) - n(H_0)$ **45.** $n(H_0 + 1) - n(H_0)$

46. Table 2.26 shows the population, P, in millions, of Ireland[11] at various times between 1780 and 1910, with t in years since 1780.

   (a) When was the population increasing? Decreasing?

   (b) For each successive time interval, construct a table showing the average rate of change of the population.

   (c) From the table you constructed in part (b), when is the graph of the population concave up? Concave down?

   (d) When was the average rate of change of the population the greatest? The least? How is this related to part (c)? What does this mean in human terms?

   (e) Graph the data in Table 2.26 and join the points by a curve to show the trend in the data. From this graph identify where the curve is increasing, decreasing, concave up and concave down. Compare your answers to those you got in parts (a) and (c). Identify the region you found in part (d).

   (f) Something catastrophic happened in Ireland between 1780 and 1910. When? What happened in Ireland at that time to cause this catastrophe?

Table 2.26   *The population of Ireland from 1780 to 1910, where $t = 0$ corresponds to 1780*

t	0	20	40	60	70	90	110	130
P	4.0	5.2	6.7	8.3	6.9	5.4	4.7	4.4

[10]Journal of Climate, September 14, 2004, pages 3477–3495.

[11]Adapted from D. N. Burghes and A. D. Wood, Ellis Horwood, *Mathematical Models in the Social, Management and Life Science,* p. 104 (Ellis Horwood, 1980).

47. The surface area of a cylindrical aluminum can is a measure of how much aluminum the can requires. If the can has radius r and height h, its surface area A and its volume V are given by the equations:

$$A = 2\pi r^2 + 2\pi rh \quad \text{and} \quad V = \pi r^2 h.$$

    (a) The volume, V, of a 12 oz cola can is 355 cm^3. A cola can is approximately cylindrical. Express its surface area A as a function of its radius r, where r is measured in centimeters. [Hint: First solve for h in terms of r.]

    (b) Graph $A = s(r)$, the surface area of a cola can whose volume is 355 cm^3, for $0 \le r \le 10$.

    (c) What is the domain of $s(r)$? Based on your graph, what, approximately, is the range of $s(r)$?

    (d) The manufacturers wish to use the least amount of aluminum (in cm^2) necessary to make a 12 oz cola can. Use your answer in (c) to find the minimum amount of aluminum needed. State the values of r and h that minimize the amount of aluminum used.

    (e) The radius of a real 12 oz cola can is about 3.25 cm. Show that real cola cans use more aluminum than necessary to hold 12 oz of cola. Why do you think real cola cans are made to this way?

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–51 true or false? Give an explanation for your answer.

1. If $f(t) = 3t^2 - 4$ then $f(2) = 0$.

2. If $f(x) = x^2 - 9x + 10$ then $f(b) = b^2 - 9b + 10$.

3. If $f(x) = x^2$ then $f(x + h) = x^2 + h^2$.

4. If $q = \dfrac{1}{\sqrt{z^2 + 5}}$ then the values of z that make $q = \frac{1}{3}$ are $z = \pm 2$.

5. If $W = \dfrac{t + 4}{t - 4}$ then when $t = 8$, $W = 1$.

6. If $f(t) = t^2 + 64$ then $f(0) = 64$.

7. If $f(x) = 0$ then $x = 0$.

8. If $f(x) = x^2 + 2x + 7$ then $f(-x) = f(x)$.

9. If $g(x) = \dfrac{3}{\sqrt{x^2 + 4}}$ then $g(x)$ can never be zero.

10. If $h(p) = -6p + 9$ then $h(3) + h(4) = h(7)$.

11. The domain of a function is the set of input values.

12. If a function is being used to model a real world situation, the domain and range are often determined by the constraints of the situation being modeled.

13. The domain of $f(x) = \dfrac{4}{x - 3}$ consists of all real numbers x, $x \ne 0$.

14. If $f(x) = \sqrt{2 - x}$, the domain of f consists of all real numbers $x \ge 2$.

15. The range of $f(x) = \dfrac{1}{x}$ is all real numbers.

16. The range of $y = 4 - \dfrac{1}{x}$ is $0 < y < 4$.

17. If $f(x) = \frac{2}{5}x + 6$ and its domain is $15 \le x \le 20$ then the range of f is $12 \le x \le 14$.

18. The domain of $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$ is all real numbers.

19. The graph of the absolute value function $y = |x|$ has a V shape.

20. The domain of $f(x) = |x|$ is all real numbers.

21. If $f(x) = |x|$ and $g(x) = |-x|$ then for all x, $f(x) = g(x)$.

22. If $f(x) = |x|$ and $g(x) = -|x|$ then for all x, $f(x) = g(x)$.

23. If $y = \dfrac{x}{|x|}$ then $y = 1$ for $x \ne 0$.

24. If $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 4 \\ 7 & \text{if } x > 4 \end{cases}$, then $f(3) = 0$.

25. Let $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 4 \\ -x & \text{if } x > 4 \end{cases}$ If $f(x) = 4$ then $x = 2$.

26. If $f(3) = 5$ and f is invertible, then $f^{-1}(3) = 1/5$.

27. If $h(7) = 4$ and h is invertible, then $h^{-1}(4) = 7$.

28. If $f(x) = \frac{3}{4}x - 6$ then $f^{-1}(8) = 0$.

29. If $R = f(S) = \frac{2}{3}S + 8$ then $S = f^{-1}(R) = \frac{3}{2}(R - 8)$.

30. In general $f^{-1}(x) = (f(x))^{-1}$.

31. If $f(x) = \dfrac{x}{x + 1}$ then $f(t^{-1}) = \dfrac{1/t}{1/t + 1}$.

32. The units of the output of a function are the same as the units of output of its inverse.

33. The functions $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}x - 1$ are inverses.

34. If $q = f(x)$ is the quantity of rice in tons required to feed x million people for a year and $p = g(q)$ is the cost, in dollars, of q tons of rice, then $g(f(x))$ is the dollar cost of feeding x million people for a year.

35. If $f(t) = t+2$ and $g(t) = 3t$, then $g(f(t)) = 3(t+2) = 3t + 6$.

36. A fireball has radius $r = f(t)$ meters t seconds after an explosion. The volume of the ball is $V = g(r)$ meter^3 when it has radius r meters. Then the units of measurement of $g(f(t))$ are meter^3/sec.

37. If the graph of a function is concave up, then the average rate of change of a function over an interval of length 1 increases as the interval moves from left to right.

38. The function f in the table could be concave up.

x	−2	0	2	4
$f(x)$	5	6	8	12

39. The function g in the table could be concave down.

t	−1	1	3	5
$g(t)$	9	8	6	3

40. A straight line is concave up.

41. A function can be both decreasing and concave down.

42. If a function is concave up, it must be increasing.

43. The quadratic function $f(x) = x(x + 2)$ is in factored form.

44. If $f(x) = (x + 1)(x + 2)$, then the zeros of f are 1 and 2.

45. A quadratic function whose graph is concave up has a maximum.

46. All quadratic equations have the form $f(x) = ax^2$.

47. If the height above the ground of an object at time t is given by $s(t) = at^2 + bt + c$, then $s(0)$ tells us when the object hits the ground.

48. To find the zeros of $f(x) = ax^2 + bx + c$, solve the equation $ax^2 + bx + c = 0$ for x.

49. Every quadratic equation has two real solutions.

50. There is only one quadratic function with zeros at $x = -2$ and $x = 2$.

51. A quadratic function has exactly two zeros.

TOOLS FOR CHAPTER 2: QUADRATIC EQUATIONS

Expanding an Expression

The *distributive property* for real numbers a, b, and c tells us that

$$a(b + c) = ab + ac,$$

and

$$(b + c)a = ba + ca.$$

We use the distributive property and the rules of exponents to multiply algebraic expressions involving parentheses. This process is sometimes referred to as *expanding* the expression.

Example 1 Multiply the following expressions and simplify.

(a) $3x^2 \left(x + \dfrac{1}{6}x^{-3} \right)$ (b) $\left((2t)^2 - 5 \right) \sqrt{t}$

Solution (a) $3x^2 \left(x + \dfrac{1}{6}x^{-3} \right) = (3x^2)(x) + (3x^2)\left(\dfrac{1}{6}x^{-3} \right) = 3x^3 + \dfrac{1}{2}x^{-1}$

(b) $\left((2t)^2 - 5 \right) \sqrt{t} = (2t)^2(\sqrt{t}) - 5\sqrt{t} = (4t^2)\left(t^{1/2} \right) - 5t^{1/2} = 4t^{5/2} - 5t^{1/2}$

If there are two terms in each factor, then there are four terms in the product:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

The following special cases of the above product occur frequently. Learning to recognize their forms aids in factoring.

$$(a + b)(a - b) = a^2 - b^2$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

Example 2 Expand the following and simplify by gathering like terms.

(a) $(5x^2 + 2)(x - 4)$ (b) $(2\sqrt{r} + 2)(4\sqrt{r} - 3)$ (c) $\left(3 - \dfrac{1}{2}x \right)^2$

Solution (a) $(5x^2 + 2)(x - 4) = (5x^2)(x) + (5x^2)(-4) + (2)(x) + (2)(-4) = 5x^3 - 20x^2 + 2x - 8$

(b) $(2\sqrt{r} + 2)(4\sqrt{r} - 3) = (2)(4)(\sqrt{r})^2 + (2)(-3)(\sqrt{r}) + (2)(4)(\sqrt{r}) + (2)(-3) = 8r + 2\sqrt{r} - 6$

(c) $\left(3 - \dfrac{1}{2}x \right)^2 = 3^2 - 2(3)\left(\dfrac{1}{2}x \right) + \left(-\dfrac{1}{2}x \right)^2 = 9 - 3x + \dfrac{1}{4}x^2$

Factoring

To write an expanded expression in factored form, we "un-multiply" the expression. Some techniques for factoring are given in this section. We can check factoring by remultiplying.

Removing a Common Factor

It is sometimes useful to factor out the same factor from each of the terms in an expression. This is basically the distributive law in reverse:

$$ab + ac = a(b + c).$$

One special case is removing a factor of -1, which gives

$$\boxed{-a - b = -(a + b)}$$

Another special case is

$$\boxed{(a - b) = -(b - a)}$$

Example 3   Factor the following:

(a) $\dfrac{2}{3}x^2y + \dfrac{4}{3}xy$ (b) $(2p + 1)p^3 - 3p(2p + 1)$ (c) $-\dfrac{s^2t}{8w} - \dfrac{st^2}{16w}$

Solution   (a) $\dfrac{2}{3}x^2y + \dfrac{4}{3}xy = \dfrac{2}{3}xy(x + 2)$

(b) $(2p + 1)p^3 - 3p(2p + 1) = (p^3 - 3p)(2p + 1) = p(p^2 - 3)(2p + 1)$
(Note that the expression $(2p + 1)$ was one of the factors common to both terms.)

(c) $-\dfrac{s^2t}{8w} - \dfrac{st^2}{16w} = -\dfrac{st}{8w}\left(s + \dfrac{t}{2}\right).$

Grouping Terms

Even though all the terms may not have a common factor, we can sometimes factor by first grouping the terms and then removing a common factor.

Example 4   Factor $x^2 - hx - x + h$.

Solution $x^2 - hx - x + h = \left(x^2 - hx\right) - (x - h) = x(x - h) - (x - h) = (x - h)(x - 1)$

Factoring Quadratics

One way to factor quadratics is to mentally multiply out the possibilities.

Example 5   Factor $t^2 - 4t - 12$.

Solution   If the quadratic factors, it will be of the form

$$t^2 - 4t - 12 = (t + ?)(t + ?).$$

We are looking for two numbers whose product is -12 and whose sum is -4. By trying combinations, we find

$$t^2 - 4t - 12 = (t - 6)(t + 2).$$

Example 6   Factor $4 - 2M - 6M^2$.

Solution   By a similar method as in the previous example, we find $4 - 2M - 6M^2 = (2 - 3M)(2 + 2M)$.

Perfect Squares and the Difference of Squares

Recognition of the special products $(x + y)^2$, $(x - y)^2$ and $(x + y)(x - y)$ in expanded form is useful in factoring. Reversing the results in the last section, we have

$$a^2 + 2ab + b^2 = (a + b)^2,$$
$$a^2 - 2ab + b^2 = (a - b)^2,$$
$$a^2 - b^2 = (a - b)(a + b).$$

When we can see that terms in an expression we want to factor are squares, it often makes sense to look for one of these forms. The difference of squares identity (the third one listed above) is especially useful.

Example 7   Factor: (a) $16y^2 - 24y + 9$ (b) $25S^2R^4 - T^6$ (c) $x^2(x - 2) + 16(2 - x)$

Solution   (a) $16y^2 - 24y + 9 = (4y - 3)^2$
(b) $25S^2R^4 - T^6 = \left(5SR^2\right)^2 - \left(T^3\right)^2 = \left(5SR^2 - T^3\right)\left(5SR^2 + T^3\right)$
(c) $x^2(x - 2) + 16(2 - x) = x^2(x - 2) - 16(x - 2) = (x - 2)\left(x^2 - 16\right) = (x - 2)(x - 4)(x + 4)$

Solving Quadratic Equations

Example 8   Give exact and approximate solutions to $x^2 = 3$.

Solution   The exact solutions are $x = \pm\sqrt{3}$; approximate ones are $x \approx \pm 1.73$, or $x \approx \pm 1.732$, or $x \approx \pm 1.73205$. (since $\sqrt{3} = 1.732050808\ldots$). Notice that the equation $x^2 = 3$ has only two exact solutions, but many possible approximate solutions, depending on how much accuracy is required.

Solving by Factoring
Some equations can be put into factored form such that the product of the factors is zero. Then we solve by using the fact that if $a \cdot b = 0$, then either a or b (or both) is zero.

Example 9   Solve $(x + 1)(x + 3) = 15$ for x.

Solution   Do not make the mistake of setting $x + 1 = 15$ and $x + 3 = 15$. It is not true that $a \cdot b = 15$ means that $a = 15$ or $b = 15$ (or both). (Although it is true that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, or both.) So, we must expand the left-hand side and set the equation equal to zero:

$$x^2 + 4x + 3 = 15,$$
$$x^2 + 4x - 12 = 0.$$

Then, factoring gives
$$(x - 2)(x + 6) = 0.$$

Thus $x = 2$ and $x = -6$ are solutions.

Example 10   Solve $2(x + 3)^2 = 5(x + 3)$.

Solution   You might be tempted to divide both sides by $(x + 3)$. However, if you do this you will overlook one of the solutions. Instead, write

$$2(x + 3)^2 - 5(x + 3) = 0$$
$$(x + 3)\,(2(x + 3) - 5) = 0$$
$$(x + 3)(2x + 6 - 5) = 0$$
$$(x + 3)(2x + 1) = 0.$$

Thus, $x = -\dfrac{1}{2}$ and $x = -3$ are solutions.

Solving with the Quadratic Formula

Alternatively, we can solve the equation $ax^2 + bx + c = 0$ by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula is derived by completing the square for $y = ax^2 + bx + c$. See page 239 in Tools for Chapter 5.

Example 11   Solve $11 + 2x = x^2$.

Solution   The equation is

$$-x^2 + 2x + 11 = 0.$$

The expression on the left does not factor using integers, so we use

$$x = \frac{-2 + \sqrt{4 - 4(-1)(11)}}{2(-1)} = \frac{-2 + \sqrt{48}}{-2} = \frac{-2 + \sqrt{16 \cdot 3}}{-2} = \frac{-2 + 4\sqrt{3}}{-2} = 1 - 2\sqrt{3},$$

$$x = \frac{-2 - \sqrt{4 - 4(-1)(11)}}{2(-1)} = \frac{-2 - \sqrt{48}}{-2} = \frac{-2 - \sqrt{16 \cdot 3}}{-2} = \frac{-2 - 4\sqrt{3}}{-2} = 1 + 2\sqrt{3}.$$

The exact solutions are $x = 1 - 2\sqrt{3}$ and $x = 1 + 2\sqrt{3}$.

The decimal approximations to these numbers $x = 1 - 2\sqrt{3} = -2.464$ and $x = 1 + 2\sqrt{3} = 4.464$ are approximate solutions to this equation. The approximate solutions could be found directly from a graph or calculator.

Exercises on Tools for Chapter 2

For Exercises 1–18, expand and simplify.

1. $3(x + 2)$

2. $5(x - 3)$

3. $2(3x - 7)$

4. $-4(y + 6)$

5. $12(x + y)$

6. $-7(5x - 8y)$

7. $x(2x + 5)$

8. $3z(2x - 9z)$

9. $-10r(5r + 6rs)$

10. $x(3x - 8) + 2(3x - 8)$

11. $5z(x - 2) - 3(x - 2)$

12. $(x + 1)(x + 3)$

13. $(x - 2)(x + 6)$

14. $(5x - 1)(2x - 3)$

15. $(x + 2)(3x - 8)$

16. $(y + 1)(z + 3)$

17. $(12y - 5)(8w + 7)$

18. $(5z - 3)(x - 2)$

Multiply and write the expressions in Problems 19–27 without parentheses. Gather like terms.

19. $-(x - 3) - 2(5 - x)$

20. $(x - 5)6 - 5(1 - (2 - x))$

21. $(3x - 2x^2) 4 + (5 + 4x)(3x - 4)$

22. $(t^2 + 1) 50t - (25t^2 + 125) 2t$

23. $P(p - 3q)^2$

24. $(A^2 - B^2)^2$

25. $4(x - 3)^2 + 7$

26. $-(\sqrt{2x} + 1)^2$

27. $u(u^{-1} + 2^u) 2^u$

For Exercises 28–76, factor completely if possible.

28. $2x + 6$

29. $3y + 15$

30. $5z - 30$

31. $4t - 6$

32. $10w - 25$

33. $u^2 - 2u$

34. $3u^4 - 4u^3$

35. $3u^7 + 12u^2$

36. $12x^3y^2 - 18x$

37. $14r^4s^2 - 21rst$

38. $x^2 + 3x + 2$

39. $x^2 + 3x - 2$

40. $x^2 - 3x + 2$

41. $x^2 - 3x - 2$

42. $x^2 + 2x + 3$

43. $x^2 - 2x - 3$

44. $x^2 - 2x + 3$

45. $x^2 + 2x - 3$

46. $2x^2 + 5x + 2$

47. $3x^2 - x - 4$

48. $2x^2 - 10x + 12$

49. $x^2 + 3x - 28$

50. $x^3 - 2x^2 - 3x$

51. $x^3 + 2x^2 - 3x$

52. $ac + ad + bc + bd$

53. $x^2 + 2xy + 3xz + 6yz$

54. $x^2 - 1.4x - 3.92$

55. $a^2x^2 - b^2$

56. $\pi r^2 + 2\pi rh$

57. $B^2 - 10B + 24$

58. $c^2 + x^2 - 2cx$

59. $x^2 + y^2$

60. $a^4 - a^2 - 12$

61. $(t + 3)^2 - 16$

62. $x^2 + 4x + 4 - y^2$

63. $a^3 - 2a^2 + 3a - 6$

64. $b^3 - 3b^2 - 9b + 27$

65. $c^2d^2 - 25c^2 - 9d^2 + 225$

66. $hx^2 + 12 - 4hx - 3x$

67. $r(r - s) - 2(s - r)$

68. $y^2 - 3xy + 2x^2$

69. $x^2e^{-3x} + 2xe^{-3x}$

70. $t^2e^{5t} + 3te^{5t} + 2e^{5t}$

71. $(s + 2t)^2 - 4p^2$

72. $P(1 + r)^2 + P(1 + r)^2 r$

73. $x^2 - 6x + 9 - 4z^2$

74. $dk + 2dm - 3ek - 6em$

75. $\pi r^2 - 2\pi r + 3r - 6$

76. $8gs - 12hs + 10gm - 15hm$

Solve the equations in Exercises 77–108.

77. $x^2 + 7x + 6 = 0$

78. $y^2 - 5y - 6 = 0$

79. $2w^2 + w - 10 = 0$

80. $4s^2 + 3s - 15 = 0$

81. $\dfrac{2}{x} + \dfrac{3}{2x} = 8$

82. $\dfrac{3}{x - 1} + 1 = 5$

83. $\sqrt{y - 1} = 13$

84. $\sqrt{5y + 3} = 7$

85. $\sqrt{2x - 1} + 3 = 9$

86. $\dfrac{21}{z - 5} - \dfrac{13}{z^2 - 5z} = 3$

87. $-16t^2 + 96t + 12 = 60$

88. $r^3 - 6r^2 = 5r - 30$

89. $g^3 - 4g = 3g^2 - 12$

90. $8 + 2x - 3x^2 = 0$

91. $2p^3 + p^2 - 18p - 9 = 0$

92. $N^2 - 2N - 3 = 2N(N - 3)$

93. $\dfrac{1}{64}t^3 = t$

94. $x^2 - 1 = 2x$

95. $4x^2 - 13x - 12 = 0$

96. $60 = -16t^2 + 96t + 12$

97. $n^5 + 80 = 5n^4 + 16n$

98. $5a^3 + 50a^2 = 4a + 40$

99. $y^2 + 4y - 2 = 0$

100. $\dfrac{2}{z - 3} + \dfrac{7}{z^2 - 3z} = 0$

101. $\dfrac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = 0$

102. $4 - \dfrac{1}{L^2} = 0$

103. $2 + \dfrac{1}{q + 1} - \dfrac{1}{q - 1} = 0$

104. $\sqrt{r^2 + 24} = 7$

105. $\dfrac{1}{\sqrt[3]{x}} = -2$

106. $3\sqrt{x} = \dfrac{1}{2}x$

107. $10 = \sqrt{\dfrac{v}{7\pi}}$

108. $\dfrac{(3x + 4)(x - 2)}{(x - 5)(x - 1)} = 0$

In Exercises 109–112, solve for the indicated variable.

109. $T = 2\pi\sqrt{\dfrac{l}{g}}$, for l.

110. $Ab^5 = C$, for b.

111. $|2x + 1| = 7$, for x.

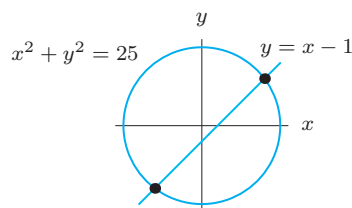**112.** $\dfrac{x^2 - 5mx + 4m^2}{x - m} = 0$, for x

Solve the systems of equations in Exercises 113–117.

113. $\begin{cases} y = 2x - x^2 \\ y = -3 \end{cases}$

114. $\begin{cases} y = 1/x \\ y = 4x \end{cases}$

115. $\begin{cases} x^2 + y^2 = 36 \\ y = x - 3 \end{cases}$

116. $\begin{cases} y = 4 - x^2 \\ y - 2x = 1 \end{cases}$

117. $\begin{cases} y = x^3 - 1 \\ y = e^x \end{cases}$

118. Let ℓ be the line of slope 3 passing through the origin. Find the points of intersection of the line ℓ and the parabola whose equation is $y = x^2$. Sketch the line and the parabola, and label the points of intersection.

Determine the points of intersection for Problems 119–120.

119.

120.