Limits

Last class we talked about a series of secant lines approaching the "limit" of a tangent line, and about how as Δx approaches zero, $\frac{\Delta y}{\Delta x}$ approaches the "limit" $y' = \frac{dy}{dx}$. Now we want to talk about limits more carefully; this will include some of our first steps towards our goal of being able to differentiate every function you know.

Some limits are easy to compute:

$$\lim_{x \to 3} \frac{x^2 + x}{x + 1} = \frac{3^2 + 3}{3 + 1} = \frac{12}{4} = 3$$

With an easy limit, you can get a meaningful answer just by plugging in the limiting value. This is because when x is close to 3, the value of the function $f(x) = \frac{x^2 + x}{x + 1}$ is close to f(3). Some limits are not easy to compute. For example, the definition of the

derivative:

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is never an easy limit, because the denominator $\Delta x = 0$ is not allowed. (The limit $x \to x_0$ is computed under the implicit assumption that $x \neq x_0$.) We'll always need to cancel Δx before we can make sense out of the limit.

Other "hard" limits would be:

$$\lim_{x\to -1}\frac{x^2+x}{x+1}\quad \text{and}\quad \lim_{x\to \infty}\frac{x^2+x}{x+1}.$$

Any limit involving infinity or division by zero is going to be harder to compute; sometimes the answer will will be that there is no limit.

To complete our discussion of limits, we need just one more piece of notation the concepts of left hand and right hand limits.

The limit

$$\lim_{x \to x_0^+} f(x)$$

is known as the right-hand limit and means that you should use values of x that are greater than x_0 (to the right of x_0 on the number line) to compute the limit. Shown below is the graph of the function:

$$f(x) = \begin{cases} x+1 & x>0\\ -x & x \le 0 \end{cases}$$

The right-hand limit $\lim_{x\to 0^+} f(x)$ equals 1. The *left-hand limit*

$$\lim_{x \to x_0^-} f(x)$$

is found by looking at values of f(x) when x is less than x_0 (to the left of x_0 on the number line). For this function, $\lim_{x \to 0} f(x) = 0$.

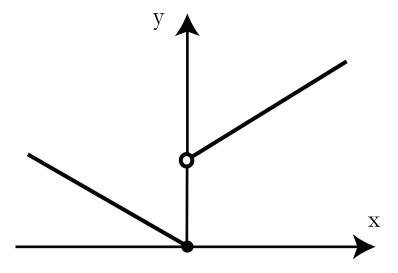


Figure 1: Graph of f(x)

The notions of left- and right- hand limits will make things much easier for us as we discuss continuity, next.

Let's talk more about the example graphed above. To calculate

$$\lim_{x \to x_0^+} f(x)$$

we use only values of x that are greater than 0. When x > 0, f(x) is defined to equal x + 1. So we plugged x = 0 into the expression x + 1 to calculate the right-hand limit.

When calculating

$$\lim_{x \to x_0^-} f(x),$$

we have x < 0. Here f(x) is defined to equal -x; when we plug x = 0 into this expression we get $\lim_{x \to x_0^-} f(x) = 0$.

Notice that it doesn't matter that f(0) = 0. Our calculations would have been exactly the same if f(0) were 1 or even if f(0) = 2.