A geometric proof that the derivative of $\sin x$ is $\cos x$.

At the start of the lecture we saw an algebraic proof that the derivative of $\sin x$ is $\cos x$. While this proof was perfectly valid, it was somewhat abstract – it did not make use of the definition of the sine function.

The proof that $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$ did use the unit circle definition of the sine of an angle. It also showed that when $x = 0$ the derivative of $\sin x$ is 1:

$$\begin{aligned} \frac{d}{dx} \sin x|_{x=0} &= \lim_{\Delta x \to 0} \frac{\sin(0 + \Delta x) - \sin 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin \Delta x - 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \\ &= 1. \end{aligned}$$

We'll now prove that the derivative of $\sin \theta$ is $\cos \theta$ directly from the definition of the sine function as the ratio $\dfrac{|\text{opposite}|}{|\text{hypotenuse}|}$ of the side lengths of a right triangle.

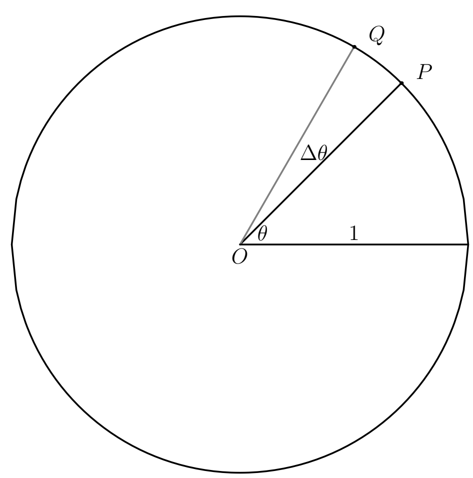

Figure 1: Point P has vertical position $\sin \theta$.

We start with a point P on the unit circle centered at O and the angle θ associated with P. As indicated in Figure 1, $\sin \theta$ is the vertical distance between P and the x-axis. Next, we add a small amount $\Delta\theta$ to angle θ; let Q be the point on the unit circle at angle $\theta + \Delta\theta$. The y-coordinate of Q is $\sin(\theta + \Delta\theta)$. To find the rate of change of $\sin \theta$ with respect to θ we just need to find the rate of change of $y = \sin \theta$.

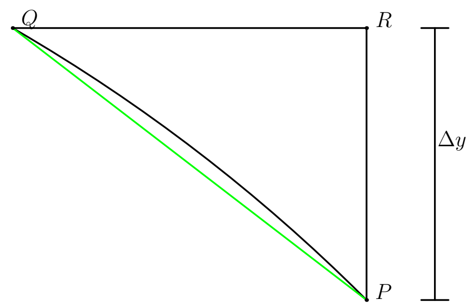

Figure 2: When $\Delta\theta$ is small, $\overset{\frown}{PQ} \approx \overline{PQ}$. Find $\frac{dy}{d\theta}$.

As shown in Figure 2, $\Delta y = |PR|$ and segment PQ is a straight line approximation of the circular arc PQ. If $\Delta\theta$ is small enough, segment PQ and arc PQ are practically the same, so $|PQ| \approx \Delta\theta$.

We're trying to find Δy. Since we know the length of the hypotenuse PQ, all we need is the measure of $\angle QPR$ to solve for $\Delta y = |PR|$.

Since $\Delta\theta$ is small, segment PQ is (nearly) tangent to the circle, and so angle $\angle OPQ$ is (nearly) a right angle. We know that PR is vertical, we know that θ is the angle OP makes with the horizontal, and we can combine these facts to prove that $\angle RPQ$ and θ are (nearly) congruent angles.[1]

The arc length $\Delta\theta$ is approximately equal to the length $|PR|$ of the hypotenuse and angle RPQ is approximately equal to θ. By the definition of the cosine function we get $\cos\theta \approx \frac{|PR|}{\Delta\theta}$. But $|PR|$ is just the vertical distance between Q and P, which is just the difference between $\sin(\theta + \Delta\theta)$ and $\sin\theta$. In other words, when $\Delta\theta$ is very small,

$$\cos\theta \approx \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}.$$

As $\Delta\theta$ approaches 0, segment QP gets closer and closer to arc QP and angle QPO gets closer and closer to a right angle, so the value of $\frac{(\sin(\theta+\Delta\theta)-\sin\theta)}{\Delta\theta}$ gets closer and closer to $\cos\theta$. We conclude that:

$$\lim_{\Delta\theta\to 0} \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta} = \cos\theta$$

and thus that the derivative of $\sin\theta$ is $\cos\theta$.

[1] Professor Jerison does this by rotating and translating angle θ to coincide with angle RPQ. Another way to see this is to extend segment RP until it intersects the horizontal line through O at point S, then note that $m\angle RPQ + m\angle QPO + m\angle OPS = \pi$ and also $\theta + m\angle PSO + m\angle OPS = \pi$. Since $m\angle QPO \cong m\angle PSO$, we get $m\angle RPQ \cong \theta$. (If $\theta > \pi/2$ a different, but similar, argument applies.)