

## Derivative of a Sum

One of our examples of a general derivative formula was:

$$(u + v)'(x) = u'(x) + v'(x).$$

(Remember that by  $(u + v)(x)$  we mean  $u(x) + v(x)$ .)

In other words, the derivative of the sum of two functions is just the sum of their derivatives. We'll now prove that this is true for any pair of functions  $u$  and  $v$ , provided that those functions have derivatives. Since we don't know in advance what functions  $u$  and  $v$  are, we can't use any specific information about the functions or the slopes of their graphs; all we have to work with is the formal definition of the derivative.

When we apply the definition of the derivative to the function  $(u + v)(x)$  we get:

$$(u + v)'(x) = \lim_{\Delta x \rightarrow 0} \frac{(u + v)(x + \Delta x) - (u + v)(x)}{\Delta x}$$

Since  $(u + v)(x)$  is just  $u(x) + v(x)$ ,

$$(u + v)'(x) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) + v(x + \Delta x) - u(x) - v(x)}{\Delta x}.$$

Combining like terms, we see that:

$$(u + v)'(x) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x) + v(x + \Delta x) - v(x)}{\Delta x}$$

or:

$$(u + v)'(x) = \lim_{\Delta x \rightarrow 0} \left\{ \frac{u(x + \Delta x) - u(x)}{\Delta x} + \frac{v(x + \Delta x) - v(x)}{\Delta x} \right\}.$$

Because  $u$  and  $v$  are differentiable (and therefore continuous), the limit of the sum is the sum of the limits. Therefore:

$$(u + v)'(x) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x}.$$

The two limits above match the definition of the derivatives of  $u$  and  $v$ , so we've shown that  $(u + v)'(x) = u'(x) + v'(x)$ .