Moebius Toolbox

Let S be the set of infinite sequences $a = (a_1, a_2, \dots)$ where $a_i \in \mathbf{N} = \{0, 1, 2, 3, \dots\}$ and only a finite number of a_i's are nonzero.

You can think of S as giving prime factorizations. The sequence a corresponds to the product $p(a) = \prod_i p_i^{a_i}$ where p_i is the ith prime number. The product $p(a) * p(b)$ of integers corresponds to the sum $a+b$ of sequences. Under addition, S is a free semigroup. We will give S a partial order and define a moebius-like function on it. The Moebius problem is to show that the partial order on S can be extended in a unique way to a total order where moebius on S is compatible with the known values of the moebius function μ on integers.

This document suggests some elements of a software toolbox to facilitate experimentation.

1 Operating on S and its subsets

1.0.1 A moebius function for S

1. We desire a function $\mu_S : S \to \{0, 1, -1\}$ such that
   (a) $\mu_S(a) = -1$ if an odd number of a_i's are 1 and all the other a_i's are 0.
   (b) $\mu_S(a) = 1$ if an even number of a_i's are 1 and all the other a_i's are 0.
   (c) $\mu_S(a) = 0$ if one or more a_i's is 2 or greater.
2. We desire functions that takes as input a finite subset X of S and produces a list of the elements of X with specified μ_X values.

1.0.2 Partial ordering on S

Given $a = (a_i)$ and $b = (b_i)$ in S, we say $a \le b$ if and only if $a_i \le b_i$ for all i. We say $a < b$ if $a \le b$ and $a \ne b$.

We desire a function

$$\mathrm{ord} : S \times S \to \{0, 1, -1, 99\}$$

such that

$$\mathrm{ord}(a, b) = \begin{cases} 1 & \text{if } a < b \\ 0 & \text{if } a = b \\ -1 & \text{if } b < a \\ 99 & \text{otherwise} \end{cases}.$$

1.0.3 Comparable pairs

We desire a function that has

Input: A finite subset of S.

Output: A list of all pairs (a, b) such that $a \in S$, $b \in S$, and $a < b$.

1.0.4 Saturation

A finite subset $X \subset S$ is saturated if and only if the conditions $b \in X$, $a \in S$, and $a \leq b$ together imply that $a \in X$.

We desire a function

$$\text{sat} : \text{Finite subsets of } S \to \{0, 1\}$$

such that

$$\text{sat}(X) = \begin{cases} 1 & \text{if } X \text{ is saturated} \\ 0 & \text{otherwise} \end{cases}.$$

1.0.5 Options

Given a saturated finite subset $X \subset S$, we define options$(X) \subset S$ to be the set of elements $s \in S$ such that

1. $s \notin X$
2. $X \cup \{s\}$ is saturated.

In other words, $s \in$ options (X) if and only if $s = (s_i) \in S$, $s \notin X$, and there exists $a = (a_i) \in S$ such that $s_i = a_i$ for all i but one value $i = I$ and $s_I = a_I + 1$.

We desire a function taking as input a saturated set X and listing the elements of options(X).

2 Functions from $[1, N]$ to S

2.0.6 1-to-1

We desire a function with

Input: a function $f : \{1, 2, 3, \ldots, N\} \to S$.

Output: YES if f is 1-to-1 ($a \neq b$ implies $f(a) \neq f(b)$); NO if f is not 1-to-1.

2.0.7 Image

We desire a function with

Input: a function $f : \{1, 2, 3, \ldots, N\} \to S$.

Output: a listing of the elements of $f(\{1, \ldots N\})$, the image of f.

2.0.8 Inverse

Input: a 1-to-1 function $f : \{1, 2, 3, \ldots, N\} \to S$ and an element a of $X = f(\{1, \ldots N\}) = \text{image}(f)$.

Output: $n = f_{\text{inv}}(a) \in \{1, 2, \ldots N\}$ defined by the property $f(n) = a$, so $f(f_{\text{inv}}(a)) = a$.

2.0.9 Testing

Input: function $f : \{1, \ldots, N\} \to S$.

Output: 1 if f is acceptable, 0 if it is not.

Acceptable means that

1. f is 1-to-1.
2. $f(1) = (0, 0, 0, \ldots)$ is the zero sequence.
3. Define the length of a sequence $a = (a_1, a_2, \ldots) \in S$ to be the largest index i such that $a_i > 0$. Define the length of f to be the largest length of any sequence $f(n) \in S$ for $1 \leq n \leq N$. Acceptable requires that if L is the length of f, then for every index $1 \leq i \leq L$ there is at least one integer n with $1 \leq n \leq N$ such that $f(n)_i > 0$. (Intuitively, we do not skip primes.)
4. $f(\{1, \ldots, N\})$ is saturated.
5. $\mu(n) = \mu_S(f(n))$ for $1 \leq n \leq N$.
6. If $a, b \in f(\{1, \ldots, N\})$ and $a < b$ then $f_{\text{inv}}(a) < f_{\text{inv}}(b)$.

2.0.10 Questions

For given N we would like to find out the following statistics.

1. Given N, how many acceptable functions $f : [0, N] \to S$ are there?

   We know for example that there is just 1 for $1 \leq N \leq 7$ and exactly 2 if $8 \leq N \leq 13$. It might be 216 for $N = 36$.

2. Let $A \subset [1, N]$ be the set of integers n such that $f(n)$ has the same same value for all acceptable functions $f : [0, N] \to S$. For example, A will contain at least $\{1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13\}$ but will not contain 8 and 9 till N is large enough to resolve the 2^3, 3^2 confusion. I would like to be able to list the elements of A for a given N.

3. Upon input N and an integer n with $1 \leq n \leq N$, output is a list of all elements $f(n) \in S$ where f ranges over all acceptable functions $f : [0, N] \to S$. This is a list of a subset of L. (giving potential factorizations of n.)

4. Given n to find the smallest $N \geq n$ such that that $f(n) \in S$ has the same value for all acceptable functions $f : [0, N] \to S$.