Calculations for Moebius: 1 through 36

1. $\mu = 1$. The factorization is just 1, by definition.

2. $\mu = -1$. Can only be prime.

3. $\mu = -1$. Can only be prime; we have not had 3 primes yet.

4. $\mu = 0$. First number divisible by a square. Has to be 2.2

5. $\mu = -1$. Can only be prime; we have not had 3 primes yet.

6. $\mu = +1$. Smallest with $\mu = 1$ is product of two smallest primes, 2.3.

7. $\mu = -1$. Can't be 2.3.5 since we have not even had 2.5 yet. So 7 is prime.

8. $\mu = 0$. Options: $2^2.2$, 3^2. Can't tell which.

9. $\mu = 0$. If $8 = 2^2.2$, options are $2^2.3$ and 3^2 and 3^2 is smaller because $3 < 2^2$. If $8 = 3^2$, options are $2^2.2$, $3^2.2$ and 5^2 of which $2^2.2 = 2^3$ is smallest.

Conclusion: 8 and 9 are 2^3 and 3^2 but we do not know which is which.

10. $\mu = +1$. Can't be a product of 4 primes since we have not even had 2.3.5 yet, so it is a product of two primes. Options are 2.5 and 3.5 (organizing by smallest prime) so $10 = 2.5$, the smaller.

11. $\mu = -1$. Prime. Not 2.3.5 because we have not had 3.5.

12. $\mu = 0$. Options are $2^2.3$, $3^2.2$ (the same as preceding), and 5^2 (which is too big, because $2^2.3 < 5.5$). So $12 = 2^2.3$.

13. $\mu = -1$. Prime. Not 2.3.5 because we have not had 3.5 yet.

14. $\mu = 1$. Options are 2.7, 3.5, 5.7 and it is not 5.7 which is larger than the others, so 14 is 2.7 or 3.5.

15. $\mu = 1$. If $14 = 2.7$ options are 2.11, 3.5, 5.7 and it can not be 2.11 because $3.5 < (2.2).5 < 2.11$, nor is it 5.7 since $3.5 < 5.7$. If $14 = 3.5$ options for 15 are 2.7, 3.7, 5.7 the smallest of which is 2.7.

Conclusion: 14 and 15 are 2.7 and 3.5 but we do not know which is which.

INTERLUDE to sort out 14 and 15. Let $Z(x)$ be the number of integers n such that $1 \leq n \leq x$ and $\mu(n) = 0$.

Compute $Z(2.3.5)$.

We need to list the elements n such that (1) $1 \leq n \leq 2.3.5$ and (2) $\mu(n) = 0$. We can list them in factored form.

First batch are of the form $2^2.a$ where a an integer such that $2^2.a \leq 2.3.5$, so $2.a \leq 3.5$. The products 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 all ok. For the last, $2.6 = 2.2.3 < 3.5$ because $2.2 < 5$ (we already verified this). How about 2.7? We do not know, because it depends which comes first, 2.7 or 3.5, which is just the question we are trying to resolve. But 2.8 is too big, since the possible factorizations of 2.8, which are 2.2^3 and 2.3^2, are greater than 3.5 because they are not any of the numbers 1 through 15. So there are 6 (if $2.7 > 3.5$) or 7 (if $2.7 < 2.5$) numbers in our list so far.

Next batch are of the form $3^2.b$ where b is an integer such that $3^2.b \leq 2.3.5$ and that is not on the $2^2.a$ list we already made. Thus $3.b \leq 2.5$. The products 3.1, 3.2, 3.3 ok (we know that $3^2 < 2.5$) but $3.4 = 3.2^2$ is too big because

$3.2^2 = 2.(2.3) > 2.5$. We have just three numbers to add to the list of 6 or 7 we already had.

Finally, there are numbers $5^2.c$ such that $5^2.c \leq 2.3.5$, or equivalently $5.c \leq 2.3$, so there is only $c = 1$.

We do not need to go up to 7^2 because $7.7 > (2.3).5$.

So we have $Z(2.3.5) = 6 + 3 + 1 = 10$ (if $2.7 > 3.5$) or $7 + 3 + 1 = 11$ (if $2.7 < 3.5$)

Now we look at the known values of the moebius function. Based on counting how many values are zero, we have $Z(26) = 9$, $Z(27) = 10$, $Z(28) = 11$.

We can not have $Z(2.3.5) = 10$ because that would imply $2.3.5 = 27$ (27 is the only number with $Z = 10$), but $\mu(2.3.7) = -1$ and $\mu(27) = 0$. Therefore $Z(2.3.5) = 11$ which we already saw implies $2.7 < 3.5$.

Conclusion: $14 = 2.7$ and $15 = 3.5$ and $2.3.5$ is 29, 30, or 31 (the only numbers n with $Z(n) = 11$ and $\mu(n) = -1$. .

INTERLUDE to sort out 8 and 9.

Since $3.11 > 2.3.5 \geq 29$ and $\mu(3.11) = 1$, we have $3.11 \geq 33$. Since $2^2.3^2 > 3.11 \geq 33$ and $\mu(2^2.3^2) = 0$, we have $2^2.3^2 \geq 36$, and hence $Z(2^2.3^2) \geq Z(36) = 13$.

But we can also figure out $Z(2^2.3^2)$ by listing its elements.

First batch: $2^2.a$ where $1 \leq a \leq 3^2$ so we have so far found 8 elements if $3^2 = 8$ and 9 elements if $3^2 = 9$.

Next batch: $3^2.1$, 3^2, $3^2.3$, (but $3^2.4$ is already in the first batch, and $3^2.5$, is too big), so 3 more elements to count.

Next batch. 5^2. ($5^2.2$ is too big). This is one more element contributing to $Z(2^2.3^2)$.

Thus: $Z(2^2.3^2) = 8 + 3 + 1 = 12$ if $3^2 = 8$ and $Z(2^2.3^2) = 9 + 3 + 1 = 13$ if $3^2 = 9$. We already know that $Z(2^2.3^3) \geq 13$.

Conclusion: $8 = 2^3$, $9 = 3^2$, $Z(2^2.3^2) = 13$ and hence $36 = 2^2.3^2$.

PROGRESS report: We have factored everything up through 15, also factored 36, and have figured out that $2.3.5$ is one of 29, 30, or 31.

16. $\mu = 0$. So far for $\mu = 0$ we have had $2^2 = 4$, $2^2.2 = 2^3 = 8$, $2^2.3 = 12$, and $3^2 = 9$. Since 16 is the next, it must be one of $2^4.4 = 2^4$, $3^2.2$, or 5^2. We know that $2^4 < 2.3^2$ (because $2^3 < 3^2$) and that $2^4 = 2^2.2^2 < 5.5 = 5^2$, so $16 = 2^4$.

17. $\mu = -1$. Can't be $2.3.5$ because we have not had $2.3.3$ yet. Hence 17 is prime.

18. $\mu = 0$. Options are $2^2.5$ or $3^2.2$ or 5^2; but $3^2.2 < (5.2).2 = 2^2.5 < 5^2$ and we have not had $2^2.5$ yet so $18 = 2.3^2$.

19. $\mu = -1$. Can't be $2.3.5$ because we have not had 3.7 yet, so 19 is prime.

20. $\mu = 0$. Must be $2^2.5$ or $3^2.3$ or 5^2. We have $2^2.5 < 2^2.6 = 2^3.3 < 3^2.3$ because $2^3 < 3^2$, and $2^2.5 < 5^2$, so $20 = 2^2.5$.

21. $\mu = 1$. Candidates are 2.11, 3.7, 5.7. (Can't have 4 factors because we have not had $2.3.5$ yet.) Can't be 5.7 because $3.7 < 5.7$. Hence 21 is 2.11 or 3.7 but we don't know which.

22. $\mu = 1$. If $21 = 3.7$ the options are 2.11, 3.11, 5.7. But $2.11 < 3.11$ and we have not had $5.5 < 5.7$ yet, so in this case $22 = 2.11$. If $21 = 2.11$ the options are 2.13. 3.7, 5.7. We have not had 5.5 yet, so 5.7 is ruled out. Also, $3.7 < 2^3.3 = 2.12 < 2.13$ so 2.13 is ruled out.

Conclusion: 21 and 22 are 3.7 and 2.11 but we do not know which is which.

23. $\mu = -1$. Prime. Can't be $2.3.5$ because we have not had $5.5 < 2.3.5$.

24. $\mu = 0$. The options are $2^2.6 = 2^3.3$, $3^2.3 = 3^3$ and 5^2. Since $2^3.3 < 3^2.3$ (because $2^3 < 3^2$), we see that 24 is $2^3.3$ or 5^2.

25. $\mu = 0$. If $24 = 2^3.3 = 2^2.6$ the options are $2^2.7$, $3^2.3 = 3^3$, and 5^2. We have not had $2.13 < 2^2.7$ yet so we can remove $2^2.7$ from the list. In this case 24 is 3^3 or 5^2.

If $24 = 5^2$ the options are $2^2.6 = 2^3.3$, $3^2.3 = 3^3$, $5^2.2$, and 7^2. We rule out $5^2.2 > 3.5.2$ and $7^2 > 6^2 = 36$. Thus in this case 24 is $2^3.3$ or 3^3.

Conclusion: $(24, 25)$ is one of $(2^3.3, 3^3)$, $(2^3.3, 5^2)$, $(5^2, 2^3.3)$, $(5^2, 3^3)$.

26. $\mu = 1$. The options are 2.13, 3.11, 5.7. But $3.11 > 3.10 = 3.2.5 \geq 29$ and $5.7 > 5.6 = 2.3.5 \geq 29$ so $26 = 2.13$.

We now know that $2^3.3 = 2.12 < 2.13 = 26$, so $2^3.3$ is 24 or 25. This removes $(5^2, 3^3)$ from the list of possible pairs for $(24, 25)$.

Conclusion: $(24, 25)$ is one of $(2^3.3, 3^3)$, $(2^3.3, 5^2)$, $(5^2, 2^3.3)$,

27 and 28.

Note that $2^3.3$, 5^2, 3^3, and $2^2.7$ are all less than $2.3.5$ (which is 29, 30, or 31). Since μ is zero for these four numbers, they are all less than or equal to 28.

Hence 24, 25, 27, 28 are (in some order) $2^3.3$, 5^2, 3^3, and $2^2.7$. There are six possibilities:

Conclusion: $(24, 25, 27, 28)$ is one of the following six quadruples:
$(2^3.3, 3^3, 5^2, 2^2.7)$
$(2^3.3, 3^3, 2^2.7, 5^2)$
$(2^3.3, 5^2, 3^3, 2^2.7)$
$(2^3.3, 5^2, 2^2.7, 3^3)$
$(5^2, 2^3.3, 3^3, 2^2.7)$
$(5^2, 2^3.3, 2^2.7, 3^3)$.

29, 30, 31. $\mu = -1$. One of them is $2.3.5$. The other two are prime because the next product of 3 primes is $2.3.7 > 2.3.6 = 2^2.3^2 = 36$.

32. $\mu = 0$. The options are $2^2.8 = 2^5$, $3^2.4 = 36$, $5^2.2 > 4^2.2 = 2^5$, $7^2 > 6^2 = 36$. All but 2^5 are too large, so $32 = 2^5$.

33, 34, 35. $\mu = 1$. Can't be a product of 4 primes $2.3.5.7$ because we have not even had $2.3.7$. These three numbers must include the factorizations $3.11 < 3.12 = 36$, $2.17 < 2.18 = 36$ and another product of 2 primes. The next products on the list are $2.19 > 2.18 = 36$, $3.13 > 3.12 = 36$, and 5.7, so 5.7 must be one of 33, 34, 35.

Conclusion: 33, 34,and 35 are 3.11, 2.17, and 5.7 in some but some (as yet unknown) order.

36. $\mu = 0$. We already know that $36 = 2^2.3^2$.

HOW MANY BRANCHES SO FAR?

How many branches through 36?

21 and 22: 2 options: 3.7 and 2.11 in either order.

24, 25, 27, 28: 6 options listed above.

29, 30, 31: 3 options: prime, prime, 2.3.5 where 2.3.5 can be in any of the three positions.

33, 34, 35: 6 options: 3.11, 2.17, 5.7 in any order.

TOTAL $2 * 6 * 3 * 6 = 216$ options.