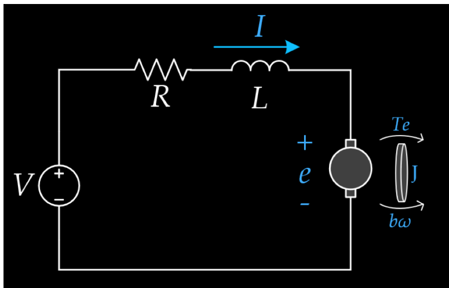


Figure 1: DC motor problem.

I : current
ω : Angular Velocity of motor

Moment of Inertia

Derive the state space model for the dc motor in Fig. 1. Assume the states are $x = \begin{pmatrix} I \\ \omega \end{pmatrix}$, the output is $y = \omega$, and the input is $u = V$. Place it into the form:

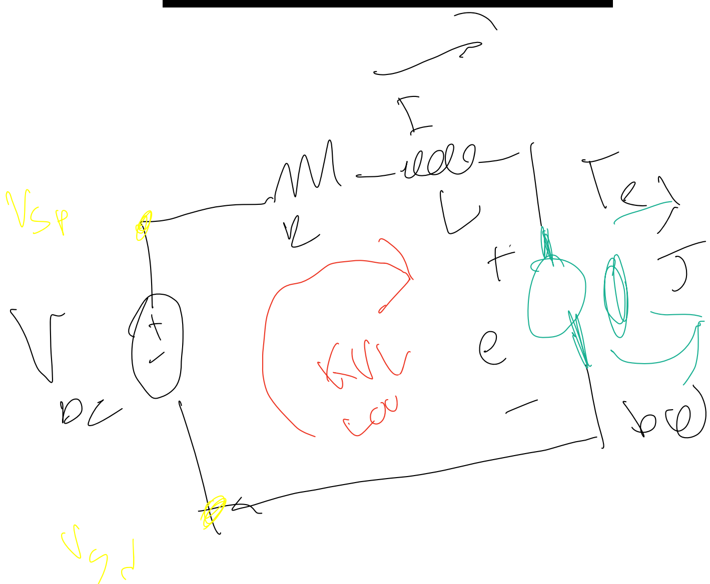
$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{1}$$

Assume the back-emf of the motor is given by $e = k_e\omega$ and the electrical torque is $T_e = k_t I$, where k_e and k_t are constants. Use variables only for this part.

Hint: To derive the equation for the motor speed, you can use Newton's second law $J\dot{w}$ = sum of the torques. Assume clockwise is positive direction. To derive the equation for the current, use KVL.

B

KVL



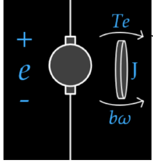
$$-V_{DC} + V_R + V_L + e = 0$$

$$-V_{DC} + IR + L\frac{dI}{dt} + K_e\omega = 0 \qquad -L\frac{dI}{dt}$$

$$-L\frac{dI}{dt} = \frac{-V_{DC} + IR + K_e\omega}{-1}$$

$$L\frac{dI}{dt} = \frac{V_{DC} - IR - K_e\omega}{L}$$

$$\frac{dI}{dt} = \frac{V_{DC}}{L} - \frac{IR}{L} - \frac{K_e\omega}{L}$$



$$+\ \sum \tau = T_e - b\omega = \dot{\omega} J$$

$$\dot{\omega} = \frac{T_e}{J} - \frac{b}{J}\omega$$

$T_e = k_t I,$

$$\dot{\omega} = \frac{K_T I}{J} - \frac{b}{J}\omega$$

$$\frac{dI}{dt} = \frac{V_{dc}}{L} - \frac{IR}{L} - \frac{K_e}{L}\omega$$

$$\frac{d\omega}{dt} = \frac{K_T I}{J} - \frac{b\omega}{J}$$

$$\begin{pmatrix} \dot{I} \\ \dot{\omega} \end{pmatrix} = \overbrace{\begin{pmatrix} -\frac{R}{L} & -\frac{K_e}{L} \\ \frac{K_T}{J} & -\frac{b}{J} \end{pmatrix}}^{A} \begin{pmatrix} I \\ \omega \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{B} V_{dc}(t)$$

$$y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} I \\ \omega \end{pmatrix}$$

b) Use the following values for the parameters: $R = 1\ \Omega$, $L = 0.5$ H, $b = 0.1$ Nms, $J = 0.01$ kgm^2, and $k_e = k_t = 0.01$.

Use Matlab (m-file or Simulink) to plot the states when $x(0) = \begin{pmatrix} I(0) \\ \omega(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $V = 5$ V are applied. Simulate for 4 s using a discrete time solver with timestep $T_s = 1$ ms.

$$\begin{pmatrix} \dot{I} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} -\frac{1}{0.5} & -\frac{0.01}{0.5} \\ \frac{0.01}{0.01} & -\frac{0.1}{0.01} \end{pmatrix} \begin{pmatrix} I \\ \omega \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 5$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ \omega \end{pmatrix}$$

$$\begin{pmatrix} I(0) \\ \omega(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Problem 4

Define state variables such that the n^{th} order linear time-varying differential equation:

$$y^{(n)} + a_{n-1}t^{-1}y^{(n-1)} + \cdots + a_1 t^{-n+1}\dot{y} + a_0 t^{-n} y = 0 \tag{2}$$

can be written as a time varying linear system of the form:

$$\dot{x}(t) = A(t)x(t) \tag{3}$$

where $x \in \mathbb{R}^n$ and $A(t) = t^{-1}A_0$ where $A_0 \in \mathbb{R}^{n\times n}$ is a constant matrix. The coefficients $a_0, \cdots, a_{n-1} \in \mathbb{R}$ are constant, but notice that t is explicitly part of the differential equation.

Hint: Begin with $n = 2$, write the ODE, and place it in the form required as shown in (3). Repeat for $n = 3$ and see if you can find a pattern to write for a general n.

$n = 2$

$$\cancel{\ddot{y} + a_1 t^{-1}\dot{y} + a_0 t^{-2} y = 0}$$

$$\cancel{-a_1 t^{-1}\dot{y} - a_0 t^{-2} y}$$

$$\ddot{y} = -a_1 t^{-1}\dot{y} - a_0 t^{-2} y$$

$y_1 = y$ $\qquad \dfrac{dy_1}{dt} = \dot{y} = y_2$

$y_2 = \dot{y}$ $\qquad \dfrac{dy_2}{dt} = \ddot{y} = -a_1 t^{-1} y_2 - a_0 t^{-2} y_1$

$y^{(n)} + a_{n-1}t^{-1}y^{(n-1)} + \cdots + a_1 t^{-n+1}\dot{y} + a_0 t^{-n} y = 0$

a_1

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a_1 & -a_0 T^{-1} \end{pmatrix} \cdot T^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y^{(3)} + a_2 T^{-1} y^{(2)} + a_1 T^{-2} \dot{y} + a_0 T^{-3} y = 0$$

$$\dddot{y} = -a_2 T^{-1} \ddot{y} - a_1 T^{-2} \dot{y} - a_0 T^{-3} y$$

$y_1 = y$ $\quad \frac{dy_1}{dT} = \dot{y} = y_2$

$y_2 = \dot{y}$ $\quad \frac{dy_2}{dT} = \ddot{y} = y_3$

$y_3 = \ddot{y}$ $\quad \frac{dy_3}{dT} = \dddot{y} = -a_2 T^{-1} y_3 - a_1 T^{-2} y_2 - a_0 T^{-3} y_1$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -a_2 & -a_1 T^{-1} & -a_0 T^{-2} \end{pmatrix} T^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

propose:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ -a_n & -a_{n-1} T^{-1} & -a_2 T^{-n-2} & -a_1 T^{-n-1} & & -a_0 T^{-n} \end{pmatrix} T^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$