PROOFS: MULTI-AGENT OPTIMIZATION OF NON-COOPERATIVE MULTIMODAL MOBILITY SYSTEMS

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Lemma 1. *The optimal solution* (q^T) *of problem* (1) *are the equilibrium solutions of the mode choice of travelers with multinomial logit model* (3) *given the ridesourcing prices* η^R_{rs} *and* $\eta^R_{rh(r,s)}$.

Proof. We know, traveler's convex optimization model,

$$\underset{q^T \geq 0}{\text{minimize}} \quad \sum_{(r,s)\in\mathscr{RS},k\in\mathscr{K}} q^T_{rsk}(\ln q^T_{rsk} - 1 - U^T_{rsk}) \tag{1a}$$

$$\text{subject to} \quad \sum_{k\in\mathscr{K}} q^T_{rsk} = d^T_{rs} \quad \forall (r,s)\in\mathscr{RS} \tag{1b}$$

If we expand traveler's optimization model, we can write as follows,

$$\begin{aligned}\underset{q^T \geq 0}{\text{minimize}} \quad & \sum_{(r,s)\in\mathscr{RS}} \{q^T_{rs1}[\ln q^T_{rs1} - 1 - \beta^T_{0,1} + \beta^T_{11}(t_{rs}+t^P_s) + \beta^T_2(c^C_{rs}+c^P_s)] \\ & + q^T_{rs2}[\ln q^T_{rs2} - 1 - \beta^T_{0,2} + \beta^T_{12}(t_{rs}) + \beta^T_2(\eta^R_{rs})] \\ & + q^T_{rs3}[\ln q^T_{rs3} - 1 - \beta^T_{0,3} + \beta^T_{13}(t_{rh(r,s)} + t^B_{h(r,s)s}) + \beta^{T\,\prime\prime}_1(w^B_{h(r,s)s}) + \beta^T_2(\eta^R_{rh(r,s)} + \rho^B_{h(r,s)s})]\}\end{aligned} \tag{2a}$$

$$\text{subject to} \quad \sum_{k\in\mathscr{K}} q^T_{rsk} = d^T_{rs} \quad \forall (r,s)\in\mathscr{RS} \tag{2b}$$

The flow of travelers from r to s using mode k,

$$q^T_{rsk} = d^T_{rs}P^T_{rsk} \tag{3a}$$

If we write the Lagrangian function of traveler's optimization problem (1), the equation would be as follows, where ζ^T_{rs} is the Lagrangian multiplier of the constraint (2b).

$$\mathscr{L} = \sum_{(r,s)\in\mathscr{RS},k\in\mathscr{K}} q^T_{rsk}(\ln q^T_{rsk} - 1 - U^T_{rsk}) - \zeta^T_{rs}(\sum_{k\in\mathscr{K}} q^T_{rsk} - d^T_{rs}) \tag{4a}$$

If we take the partial derivative of equation (4) with respect to q^R_{rsk}, the equation will be as follows as follow,

$$\frac{\partial\mathscr{L}}{\partial q^R_{rsk}} = \ln q^T_{rsk} - 1 - U^T_{rsk} + 1 - \zeta^T_{rs} = \ln q^T_{rsk} - U^T_{rsk} - \zeta^T_{rs} \tag{5a}$$

According to the Karush–Kuhn–Tucker (KKT) conditions,

$$0 \leq q^T_{rsk} \perp \frac{\partial\mathscr{L}}{\partial q^T_{rsk}} \geq 0$$

Now, It is reasonable to consider a positive flow of rider from r to s using mode k, so $q^T_{rsk} > 0$. In that case,

$$\frac{\partial\mathscr{L}}{\partial q^T_{rsk}} = 0 \tag{6a}$$

$$\ln q^T_{rsk} - U^T_{rsk} - \zeta^T_{rs} = 0 \tag{6b}$$

$$q^T_{rsk} = e^{U^T_{rsk}+\zeta^T_{rs}} \tag{6c}$$

$$q^T_{rsk} = e^{U^T_{rsk}} \cdot e^{\zeta^T_{rs}} \tag{6d}$$

$$\sum_{k\in\mathscr{K}} q^T_{rsk} = \sum_k e^{U^T_{rsk}} \cdot e^{\zeta^T_{rs}} \tag{6e}$$

$$\sum_{k\in\mathcal{K}} q^T_{rsk} = e^{\zeta^T_{rs}} \cdot \sum_k e^{U^T_{rsk}} \tag{6f}$$

Now, if we insert the value of $\sum_k q^R_{rsk}$ from Equation in the above Equation,

$$d^T_{rs} = e^{\zeta^T_{rs}} \cdot \sum_{k\in\mathcal{K}} e^{U^T_{rsk}} \tag{7a}$$

$$e^{\zeta^T_{rs}} = \frac{d^T_{rs}}{\sum_{k\in\mathcal{K}} e^{U^T_{rsk}}} \tag{7b}$$

Now let's plug in the value of $e^{\zeta^T_{rs}}$ from the abve Equation back in Equation (6d), we get,

$$q^T_{rsk} = \frac{e^{U^T_{rsk}}}{\sum_{k\in\mathcal{K}} e^{U^T_{rsk}}} \cdot d^T_{rs} \tag{8a}$$

$$q^T_{rsk} = d^T_{rs} \cdot P^T_{rsk} \quad \forall (r,s) \in \mathcal{RS}, k \in \mathcal{K} \tag{8b}$$

We can see that Equation (8b) is the same as the multinomial logit model derived in Equation (3). Hence, optimal solution of the model (1) is the same as the solution of the multinomial logit model of (3) at given ridesouricng price η^R_{rs} and $\eta^R_{rh(r,s)}$. □

Lemma 2. *The optimal solutions (q^D, Q_n) of problem* (9) *are the equilibrium solutions of the choice decision of drivers with multinomial logit model* (10a) *and* (10b) *given the driver's ride sourcing prices $\rho^R_{rs'}$.*

Proof. Driver's equivalent optimization probelm,

$$\begin{aligned}\underset{q^D, Q_n \geq 0}{\text{minimize}} \quad & \sum_{n\in\mathcal{N}} [\sum_{(r,s')\in\overline{\mathcal{RS}}} q^D_{nrs'} (\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr} - \beta^D_3 \rho^R_{rs'}) \\ & + q^D_{nH} (\ln q^D_{nH} - 1 - \beta^D_{0,H})] - \sum_{n\in\mathcal{N}} \lambda_n (Q_n - \Delta Q^D_n) \qquad (9a) \\ \text{subject to} \quad & \sum_{(r,s')\in\overline{\mathcal{RS}}} q^D_{nrs'} + q^D_{nH} = Q_n \quad \forall n \in \mathcal{N} \qquad (9b)\end{aligned}$$

Also, Driver's flow,

$$q^D_{nrs'} = P^D_{nrs'} \cdot Q_n \quad \forall n \in \mathcal{N}, (r,s') \in \overline{\mathcal{RS}} \tag{10a}$$

$$q^D_{nH} = P^D_{nH} \cdot Q_n \quad \forall n \in \mathcal{N}, (r,s') \in \overline{\mathcal{RS}} \tag{10b}$$

If we write the Lagrangian function for Problem (9), the equation would be as follows, where γ is the lagrangian multiplier for the constraints of the driver's model

$$\begin{aligned}\mathcal{L} = & \sum_{n\in\mathcal{N}} [\sum_{(r,s')\in\overline{\mathcal{RS}}} q^D_{nrs'} (\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr} - \beta^D_3 \rho^R_{rs'}) + q^D_{nH} (\ln q^D_{nH} - 1 - \beta^D_{0,H})] \\ & - \sum_{n\in\mathcal{N}} \lambda_n (Q_n - \Delta Q^D_n) - \sum_{n\in\mathcal{N}} \gamma^D_n (\sum_{(r,s')\in\overline{\mathcal{RS}}} q^D_{nrs'} + q^D_{nH} - Q_n) \qquad (11a)\end{aligned}$$

If we take the partial derivative of equation (11) with respect to $q^D_{nrs'}$ the equation will be as follows as follows,

$$\frac{\partial \mathcal{L}}{\partial q^D_{nrs'}} = (\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr} - \beta^D_3 \rho^R_{rs'}) + 1 - \gamma^D_n \tag{12a}$$

$$\frac{\partial \mathcal{L}}{\partial q^D_{nrs'}} = (\ln q^D_{nrs'} - 1 - U^D_{nrs'}) + 1 - \gamma^D_n \tag{12b}$$

If we take the partial derivative of equation (11) with respect to q^D_{nH}, the equation will be as follows as follows,

$$\frac{\partial \mathscr{L}}{\partial q^D_{nH}} = (\ln q^D_{nH} - 1 - \beta^D_{0,H}) + 1 - \gamma^D_n \tag{13a}$$

$$\frac{\partial \mathscr{L}}{\partial q^D_{nH}} = (\ln q^D_{nrs'} - 1 - U^D_{nH}) + 1 - \gamma^D_n \tag{13b}$$

If we take the partial derivative of equation (11) with respect to Q_n, the equation will be as follows as follows,

$$\frac{\partial \mathscr{L}}{\partial Q_n} = \gamma^D_n - \lambda_n \tag{14a}$$

According to the KKT condition,

$$0 \leq q^D_{nrs'} \perp \frac{\partial \mathscr{L}}{\partial q^D_{nrs'}} \geq 0 \tag{15a}$$

$$0 \leq q^D_{nH} \perp \frac{\partial \mathscr{L}}{\partial q^D_{nH}} \geq 0 \tag{15b}$$

$$0 \leq Q_n \perp \frac{\partial \mathscr{L}}{\partial Q_n} \geq 0 \tag{15c}$$

Now, If we consider a positive flow of driver from n to r to pickup rider and drop off at s', we can write $q^D_{nrs'} > 0$. In that case,

$$\frac{\partial \mathscr{L}}{\partial q^D_{nrs'}} = 0 \tag{16a}$$

$$\ln q^D_{nrs'} - U^D_{nrs'} - \gamma^D_n = 0 \tag{16b}$$

$$q^D_{nrs'} = e^{U^D_{nrs'} + \gamma^D_n} \tag{16c}$$

$$q^D_{nrs'} = e^{U^D_{nrs'}} \cdot e^{\gamma^D_n} \tag{16d}$$

$$\sum_{(r,s') \in \overline{\mathscr{R}\mathscr{S}}} q^D_{nrs'} = \sum_{(r,s') \in \overline{\mathscr{R}\mathscr{S}}} e^{U^D_{nrs'}} \cdot e^{\gamma^D_n} \tag{16e}$$

Again, If we consider non-zero number of drivers signing off from location n , we can write $q^D_{nH} > 0$. In that case,

$$\frac{\partial \mathscr{L}}{\partial q^D_{nH}} = 0 \tag{17a}$$

$$\ln q^D_{nH} - U^D_{nH} - \gamma^D_n = 0 \tag{17b}$$

$$q^D_{nH} = e^{U^D_{nH} + \gamma^D_n} \tag{17c}$$

$$q^D_{nH} = e^{U^D_{nH}} \cdot e^{\gamma^D_n} \tag{17d}$$

Again, If we consider non-zero number of drivers available at location n , we can write $Q_n > 0$. In that case, $\lambda_n = \gamma^D_n$

If we add (16e) and (17d), we get,

$$\sum_{(r,s') \in \overline{\mathscr{R}\mathscr{S}}} q^D_{nrs'} + q^D_{nH} = \sum_{(r,s') \in \overline{\mathscr{R}\mathscr{S}}} e^{U^D_{nrs'}} \cdot e^{\gamma^D_n} + e^{U^D_{nH}} \cdot e^{\gamma^D_n} \tag{18a}$$

Now, if we insert the value of $(\sum_{(r,s')\in\overline{\mathscr{RS}}} q^D_{nrs'} + q^D_{nH})$ from Equation (22c) in the above Equation,

$$Q_n = \sum_{(r,s')\in\overline{\mathscr{RS}}} e^{U^D_{nrs'}} \cdot e^{\gamma^D_n} + e^{U^D_{nH}} \cdot e^{\gamma^D_n} \tag{19a}$$

$$e^{\gamma^D_n} = \frac{Q_n}{\sum_{(r,s')\in\overline{\mathscr{RS}}} e^{U^D_{nrs'}} + e^{U^D_{nH}}} \tag{19b}$$

Now let's plug in the value of $e^{\gamma^D_n}$ from the above Equation back in Equation (16d), we get,

$$q^D_{nrs'} = \frac{e^{U^D_{nrs'}}}{\sum_{(r,s')\in\overline{\mathscr{RS}}} e^{U^D_{nrs'}} + e^{U^D_{nH}}} \cdot Q_n \tag{20a}$$

$$q^D_{nrs'} = P^D_{nrs'} \cdot Q_n \quad \forall n \in \mathscr{N}, \forall (r,s') \in \overline{\mathscr{RS}} \tag{20b}$$

Also let's plug in the value of $e^{\gamma^D_n}$ from the above Equation back in Equation (17d), we get,

$$q^D_{nH} = \frac{e^{U^D_{nH}}}{\sum_{(r,s')\in\overline{\mathscr{RS}}} e^{U^D_{nrs'}} + e^{U^D_{nH}}} \cdot Q_n \tag{21a}$$

$$q^D_{nH} = P^D_{nH} \cdot Q_n \quad \forall n \in \mathscr{N}, \forall (r,s') \in \overline{\mathscr{RS}} \tag{21b}$$

We can see that Equation (20b) and (21b) is the same as the multinomial logit model derived in Equation (10a) and (10b). Hence, optimal solution of the model (9) is the same as the solution of the multinomial logit model of (10a) and (10b) at given the driver's ridesouricng price $\rho^R_{rs'}$. $\square$

Theorem 1. *If both traveler's model and driver's model are convex, the equilibrium states of agents interactions in a perfectly competitive market i.e.* (2), (9) *and equilibrium conditions* (23) *are equivalent to solving a single level convex optimization problem, formulated as* (22).

$$\begin{aligned}
\underset{q^T, q^D, Q_n \geq 0}{\textit{minimize}} \quad & \frac{1}{\beta^T_2} \sum_{(r,s)\in\mathscr{RS}} \{ q^T_{rs1}[\ln q^T_{rs1} - 1 - \beta^T_{0,1} + \beta^T_{11}(t_{rs} + t^P_s) + \beta^T_2(c^C_{rs} + c^P_s)] \\
& + q^T_{rs2}[\ln q^T_{rs2} - 1 - \beta^T_{0,2} + \beta^T_{12}(t_{rs})] \\
& + q^T_{rs3}[\ln q^T_{rs3} - 1 - \beta^T_{0,3} + \beta^T_{13}(t_{rh(r,s)} + t^B_{h(r,s)s}) + \beta^{T\,\prime\prime}_1(w^B_{h(r,s)s}) + \beta^T_2(\rho^B_{h(r,s)s})]\} \\
& + \frac{1}{\beta^D_3} \sum_{n\in\mathscr{N}} [\sum_{(r,s')\in\overline{\mathscr{RS}}} q^D_{nrs'}(\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr}) + q^D_{nH}(\ln q^D_{nH} - 1 - \beta^D_{0,H})] && \text{(22a)} \\
\textit{subject to} \quad & \sum_{k\in\mathscr{K}} q^T_{rsk} = d^T_{rs} \quad \forall (r,s) \in \mathscr{RS} && \text{(22b)} \\
& \sum_{(r,s')\in\overline{\mathscr{RS}}} q^D_{nrs'} + q^D_{nH} = Q_n \quad \forall n \in \mathscr{N} && \text{(22c)} \\
(\rho^R_{rs}) \quad & \sum_{n\in\mathscr{N}} q^D_{nrs} = q^T_{rs2} \quad \forall (r,s) \in \mathscr{R}S && \text{(22d)} \\
(\rho^R_{rh(r,s)}) \quad & \sum_{n\in\mathscr{N}} q^D_{nrh(r,s)} = q^T_{rs3} \quad \forall (r,s) \in \mathscr{R}S && \text{(22e)} \\
(\lambda_n) \quad & Q_n = \sum_{r\in\mathscr{R}} q^T_{rn2} + \Delta Q^D_n \quad \forall n \in \mathscr{S} && \text{(22f)}
\end{aligned}$$

$$(\lambda_n) \qquad Q_n = \sum_{(r,s)\in\mathscr{RS}(n)} q^T_{rs3} + \Delta Q^D_n \quad \forall n \in \mathscr{H} \tag{22g}$$

$$(\lambda_n) \qquad Q_n = \Delta Q^D_n \quad \forall n \in \mathscr{N} \setminus (\mathscr{S} \cup \mathscr{H}) \tag{22h}$$

Proof. We know, the equilibrium conditions of the model are,

$$(\rho^R_{rs}) \qquad \sum_{n\in\mathscr{N}} q^D_{nrs} = q^T_{rs2} \quad \forall (r,s) \in \mathscr{RS} \tag{23a}$$

$$(\rho^R_{rh(r,s)}) \qquad \sum_{n\in\mathscr{N}} q^D_{nrh(r,s)} = q^T_{rs3} \quad \forall (r,s) \in \mathscr{RS} \tag{23b}$$

$$(\lambda_n) \qquad Q_n = \sum_{r\in\mathscr{R}} q^T_{rn2} + \Delta Q^D_n \quad \forall n \in \mathscr{S} \tag{23c}$$

$$(\lambda_n) \qquad Q_n = \sum_{(r,s)\in\mathscr{RS}(n)} q^T_{rs3} + \Delta Q^D_n \quad \forall n \in \mathscr{H} \tag{23d}$$

$$(\lambda_n) \qquad Q_n = \Delta Q^D_n \quad \forall n \in \mathscr{N} \setminus (\mathscr{S} \cup \mathscr{H}) \tag{23e}$$

Now, the Lagrangian function of the reformulation model (22) can be written as follows after relaxing the equilibrium constraints (23a), (23b), (23c), (23d) and (23e).

$$\begin{aligned}
\mathscr{L} =& \frac{1}{\beta^T_2} \sum_{(r,s)\in\mathscr{RS}} \{q^T_{rs1}[\ln q^T_{rs1} - 1 - \beta^T_{0,1} + \beta^T_{11}(t_{rs} + t^P_s) + \beta^T_2(c^C_{rs} + c^P_s)] \\
& + q^T_{rs2}[\ln q^T_{rs2} - 1 - \beta^T_{0,2} + \beta^T_{12}(t_{rs})] \\
& + q^T_{rs3}[\ln q^T_{rs3} - 1 - \beta^T_{0,3} + \beta^T_{13}(t_{rh(r,s)} + t^B_{h(r,s)s}) + \beta^{T\prime\prime}_1(w^B_{h(r,s)s}) + \beta^T_2(\rho^B_{h(r,s)s})]\} \\
& + \frac{1}{\beta^D_3} \sum_{n\in\mathscr{N}} [\sum_{(r,s')\in\overline{\mathscr{RS}}} q^D_{nrs'}(\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr}) + q^D_{nH}(\ln q^D_{nH} - 1 - \beta^D_{0,H})] \\
& - \sum_{(r,s)} \rho^R_{rs}(\sum_n q^D_{nrs} - q^T_{rs2}) - \sum_{(r,s)} \rho^R_{rh(r,s)}(\sum_n q^D_{nrh(r,s)} - q^T_{rs3}) \\
& - \sum_{n\in\mathscr{S}} \lambda_n(Q_n - \sum_{r\in\mathscr{R}} q^T_{rn2} - \Delta Q^D_n) - \sum_{n\in\mathscr{H}} \lambda_n(Q_n - \sum_{(r,s)\in\mathscr{RS}(n)} q^T_{rs3} - \Delta Q^D_n) - \sum_{n\in\mathscr{N}\setminus(\mathscr{S}\cup\mathscr{H})} \lambda_n(Q_n - \Delta Q^D_n) \\
=& \frac{1}{\beta^T_2} \sum_{(r,s)\in\mathscr{RS}} \{q^T_{rs1}[\ln q^T_{rs1} - 1 - \beta^T_{0,1} + \beta^T_{11}(t_{rs} + t^P_s) + \beta^T_2(c^C_{rs} + c^P_s)] \\
& + q^T_{rs2}[\ln q^T_{rs2} - 1 - \beta^T_{0,2} + \beta^T_{12}(t_{rs})] + \beta^T_2 \rho^R_{rs}(q^T_{rs2}) + \beta^T_2 \lambda_s(q^T_{rs2}) \\
& + q^T_{rs3}[\ln q^T_{rs3} - 1 - \beta^T_{0,3} + \beta^T_{13}(t_{rh(r,s)} + t^B_{h(r,s)s}) + \beta^{T\prime\prime}_1(w^B_{h(r,s)s}) + \beta^T_2(\rho^B_{h(r,s)s})] \\
& + \beta^T_2 \rho^R_{rh(r,s)}(q^T_{rs3}) + \beta^T_2 \lambda_{h(r,s)}(q^T_{rs3})\} \\
& + \frac{1}{\beta^D_3} \sum_{n\in\mathscr{N}} [\sum_{(r,s')\in\overline{\mathscr{RS}}} q^D_{nrs'}(\ln q^D_{nrs'} - 1 - \beta^D_{0,r} + \beta^D_1 t_{nr}) + q^D_{nH}(\ln q^D_{nH} - 1 - \beta^D_{0,H})] \\
& - (\sum_{(r,s)} \rho^R_{rs} \sum_n q^D_{nrs} + \sum_{(r,s)} \rho^R_{rh(r,s)} \sum_n q^D_{nrh(r,s)})] \\
& - (\sum_{n\in\mathscr{S}} \lambda_n(Q_n - \Delta Q^D_n) + \sum_{n\in\mathscr{H}} \lambda_n(Q_n - \Delta Q^D_n) + \sum_{n\in\mathscr{N}\setminus(\mathscr{S}\cup\mathscr{H})} \lambda_n(Q_n - \Delta Q^D_n)) \\
=& \frac{1}{\beta^T_2} \sum_{(r,s)\in\mathscr{RS}} \{q^T_{rs1}[\ln q^T_{rs1} - 1 - \beta^T_{0,1} + \beta^T_{11}(t_{rs} + t^P_s) + \beta^T_2(c^C_{rs} + c^P_s)]
\end{aligned}$$

$$
\begin{aligned}
&+q^T_{rs2}[\ln q^T_{rs2}-1-\beta^T_{0,2}+\beta^T_{12}(t_{rs}))+\beta^T_2(\rho^R_{rs}+\lambda_s)]\\
&+q^T_{rs3}[\ln q^T_{rs3}-1-\beta^T_{0,3}+\beta^T_{13}(t_{rh(r,s)}+t^B_{h(r,s)s})+\beta^{T}_1{}''(w^B_{h(r,s)s})+\beta^T_2(\rho^R_{rh(r,s)}+\lambda_{h(r,s)}+\rho^B_{h(r,s)s})]\}\\
&+\frac{1}{\beta^D_3}\sum_{n\in\mathcal{N}}[\sum_{(r,s')\in\overline{\mathcal{RS}}}q^D_{nrs'}(\ln q^D_{nrs'}-1-\beta^D_{0,r}+\beta^D_1 t_{nr})+q^D_{nH}(\ln q^D_{nH}-1-\beta^D_{0,H})]\\
&-(\sum_{(r,s')}\rho^R_{rs'}\sum_n q^D_{nrs'})-\sum_{n\in\mathcal{N}}\lambda_n(Q_n-\Delta Q^D_n)\\
&=\frac{1}{\beta^T_2}\sum_{(r,s)\in\mathcal{RS}}\{q^T_{rs1}[\ln q^T_{rs1}-1-\beta^T_{0,1}+\beta^T_{11}(t_{rs}+t^P_s)+\beta^T_2(c^C_{rs}+c^P_s)]\\
&+q^T_{rs2}[\ln q^T_{rs2}-1-\beta^T_{0,2}+\beta^T_{12}(t_{rs})+\beta^T_2(\rho^R_{rs}+\lambda_s)]\\
&+q^T_{rs3}[\ln q^T_{rs3}-1-\beta^T_{0,3}+\beta^T_{13}(t_{rh(r,s)}+t^B_{h(r,s)s})+\beta^{T}_1{}''(w^B_{h(r,s)s})+\beta^T_2(\rho^R_{rh(r,s)}+\lambda_{h(r,s)}+\rho^B_{h(r,s)s})]\}\\
&+\frac{1}{\beta^D_3}[\sum_{n\in\mathcal{N}}[\sum_{(r,s')\in\overline{\mathcal{RS}}}q^D_{nrs'}(\ln q^D_{nrs'}-1-\beta^D_{0,r}+\beta^D_1 t_{nr}-\beta^D_3\rho^R_{rs'})+q^D_{nH}(\ln q^D_{nH}-1-\beta^D_{0,H})]\\
&-\sum_{n\in\mathcal{N}}\lambda_n(Q_n-\Delta Q^D_n)] \qquad (24a)\\
&=\frac{1}{\beta^T_2}\sum_{(r,s)\in\mathcal{RS}}\{q^T_{rs1}[\ln q^T_{rs1}-1-\beta^T_{0,1}+\beta^T_{11}(t_{rs}+t^P_s)+\beta^T_2(c^C_{rs}+c^P_s)]\\
&+q^T_{rs2}[\ln q^T_{rs2}-1-\beta^T_{0,2}+\beta^T_{12}(t_{rs})+\beta^T_2\eta^R_{rs}]\\
&+q^T_{rs3}[\ln q^T_{rs3}-1-\beta^T_{0,3}+\beta^T_{13}(t_{rh(r,s)}+t^B_{h(r,s)s})+\beta^{T}_1{}''(w^B_{h(r,s)s})+\beta^T_2(\eta^R_{rh(r,s)}+\rho^B_{h(r,s)s})]\}\\
&+\frac{1}{\beta^D_3}[\sum_{n\in\mathcal{N}}[\sum_{(r,s')\in\overline{\mathcal{RS}}}q^D_{nrs'}(\ln q^D_{nrs'}-1-\beta^D_{0,r}+\beta^D_1 t_{nr}-\beta^D_3\rho^R_{rs'})+q^D_{nH}(\ln q^D_{nH}-1-\beta^D_{0,H})]\\
&-\sum_{n\in\mathcal{N}}\lambda_n(Q_n-\Delta Q^D_n)] \qquad (24b)
\end{aligned}
$$

We can see that, if we relax the equilibrium constraints (23a), (23b), (23c), (23d) and (23e) in the reformulation model (22) using Lagrangian relaxation, we can separate the reformulation model (22) into traveler's model (2) and driver's model (9). Now, let's define the feasible set $\mathcal{X}=\{q^T,q^D,Q_n\geq 0\}\mid(26b),(27a)$. Also, Model (2), (9) and (22) are all convex optimization problem and they all satisfy linearity constraint qualifications. Thus, the strong duality holds. The reformulated model (22) is equivalent to (25) due to strong duality.

$$\underset{(q^T,q^D,Q_n)\in X}{min}\quad\underset{\rho^R_{rs},\rho^R_{rh(r,s)},\lambda_s,\lambda_{h(r,s)}}{max}\mathcal{L}=\underset{\rho^R_{rs},\rho^R_{rh(r,s)},\lambda_s,\lambda_{h(r,s)}}{max}\quad\underset{(q^T,q^D,Q_n)\in X}{min}\mathcal{L} \qquad (25a)$$

Now, $\underset{(q^T,q^D,Q_n)\in X}{min}\mathcal{L}$ is equivalent to (2) and (9) for any given value of $\rho^R_{rs},\rho^R_{rh(r,s)},\lambda_s,\lambda_{h(r,s)}$. In addition, (23) holds for optimal $\rho^R_{rs},\rho^R_{rh(r,s)},\lambda_s,\lambda_{h(r,s)}$. □

Corollary 1. *The reformulation model* (22) *is separable to traveler's optimization model* (2) *and driver's optimization model* (9) *if the following conditions hold.*

$$\eta^R_{rs}=\rho^R_{rs}+\lambda_s \quad \forall(r,s)\in\mathcal{RS} \qquad (26a)$$

$$\eta^R_{rh(r,s)}=\rho^R_{rh(r,s)}+\lambda_{h(r,s)} \quad \forall(r,s)\in\mathcal{RS} \qquad (26b)$$

Proof. Notice that (24a) can only be written as (24b) i.e., separable to traveler's optimization model (2) and driver's optimization model (9) if and only if (26a) and (26b) holds. □

Corollary 2. *If traveler's optimization model* (2) *and driver's optimization model* (9) *are strictly convex functions, the system equilibrium exists and is unique.*

Proof. If the traveler's optimization model (2) and the driver's optimization model (9) are strictly convex, the reformulation model (22) is a strictly convex optimization problem, having unique optimal solution. According to Theorem 1, the system equilibrium therefore exists and is unique. □