$\boxed{P \wedge Q}$

14 Axiom Systems for Propositional Logic

- notions of axiomatic system, deduction
- particular axiom system for propositional logic
- use of this system to establish tautologies, equivalences, consistency, argument validity

Basic Idea (Kelly Ch 4)

- so far ...
  - *truth tables* used to define
    - tautologies, (in)consistency, validity, ...
    - **model theory** approach
  - gives the *semantics* of propositional logic
    - semantics = considerations involving "truth" (meaning)
- now ...
  - a **proof theory** approach
    - axioms, inference rules, proofs, theorems, ...
- later – *connections* between the semantic and proof theory approaches

$\boxed{P \wedge Q}$ 14.1 Axiomatic Systems (Kelly 4.2)

- **Proof**:
  - set of initial hypotheses
  - each step generates a new consequence of the hypotheses
  - until the desired proposition is reached.
- In Propositional Logic
  - start with **axioms** (simple tautological propositions)
  - each step uses a **rule of inference**

  (Axioms $\approx$ Prolog program; conclusion $\approx$ query.)
- Infinitely many possible axiom systems; we use **AL**

$\boxed{P \wedge Q}$ 14.1.1 Syntactic Description

Axiomatic system has 4 parts:

Σ	**alphabet** of symbols; we use $\neg, \rightarrow, (,), P, Q, R, \ldots$
WF	the **well formed formulae** (**wff**)
Ax	the set of axioms, a subset of WF
R	the set of rules of deduction

Well Formed Formulae (wff):

- Any propositional letter (*e.g.*, P, Q, R) is a wff
- If W and V are wff then $W \rightarrow V$ and $\neg W$ are wff
- Nothing else is a wff

Usually want to assume infinite number of propositional letters; assume we can subscript them

14.1.2 Axioms

There are infinitely many axioms, but all have one of these forms:

Axiom schemas — allow any wff for A, B, and C

Ax1 $A \rightarrow (B \rightarrow A)$

Ax2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Ax3 $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

Examples:

- $P \rightarrow (Q \rightarrow P)$ is an *instance* of Ax1.
- $(\neg(D \rightarrow E) \rightarrow \neg S) \rightarrow (S \rightarrow (D \rightarrow E))$ is an instance of Ax3.

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14.1.3 Inference Rule

One rule of deduction, or **inference rule**:

modus ponens from A and $A \rightarrow B$ infer B

Given a set of hypotheses H, where $H \subseteq WF$, a **deduction** is a sequence of wff $F_1, F_2, \ldots, F_n$ where each F_i is either:

- An axiom ($F_i \in Ax$); or
- A hypothesis ($F_i \in H$); or
- Follows from *earlier* steps by a (the) rule of inference

$\boxed{P \wedge Q}$ Inference Rule (2)

We say a deduction ending in F_n is a deduction of F_n from H or that F_n is a deductive consequence of H, and write

$$H \vdash F_n$$

When $H = \emptyset$, we write

$$\vdash F_n$$

and say F_n is a *theorem*

$\boxed{P \wedge Q}$ 14.1.4 Trivial Examples

- A boring example with a very short proof

$$\vdash P \rightarrow (Q \rightarrow P) \qquad \text{Ax1}$$

- Another boring example

$$\vdash (\neg(D \rightarrow E) \rightarrow \neg S) \rightarrow (S \rightarrow (D \rightarrow E)) \qquad \text{Ax3}$$

$\boxed{P \wedge Q}$ 14.1.5 Short Proof (Kelly Theorem 4.1)

$$\vdash A \rightarrow A$$

1.	$A \rightarrow ((B \rightarrow A) \rightarrow A)$	Ax1
2.	$(A \rightarrow ((B \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A))$	Ax2
3.	$(A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A)$	MP 1, 2
4.	$A \rightarrow (B \rightarrow A)$	Ax1
5.	$A \rightarrow A$	MP 3, 4

Exercise: Prove the following Simplification MetaTheorems:

$$\vdash (A \wedge B) \rightarrow A$$

$$\vdash (A \wedge B) \rightarrow B$$

Here $A \wedge B$ is *defined* as $\neg(A \rightarrow \neg B)$.

$\boxed{P \wedge Q}$ 14.2 Meta-theorems

We have proved $\vdash A \to A$, but what about $\vdash P \to P$?

We haven't proved it, but we could: duplicate proof, replacing A with P

Works because for every axiom with A, there's one with P

Modus ponens works for P as well as for A

Take our theorem as a *meta*-theorem — a theorem template

As a shortcut, we could use $A \to A$ as if it were an axiom schema, because we could always replace it by its proof

Like using subroutines when programming

$\boxed{P \wedge Q}$

14.2.1 Using $A \rightarrow A$

Prove $B \rightarrow (A \rightarrow A)$

1.	$A \rightarrow A$	Theorem 4.1
2.	$(A \rightarrow A) \rightarrow (B \rightarrow (A \rightarrow A))$	Ax1
3.	$B \rightarrow (A \rightarrow A)$	MP 1, 2

Could replace Theorem 4.1 line with proof of Theorem 4.1:

1.	$A \rightarrow ((B \rightarrow A) \rightarrow A)$	Ax1
2.	$(A \rightarrow ((B \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A))$	Ax2
3.	$(A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A)$	MP 1, 2
4.	$A \rightarrow (B \rightarrow A)$	Ax1
5.	$A \rightarrow A$	MP 3, 4
6.	$(A \rightarrow A) \rightarrow (B \rightarrow (A \rightarrow A))$	Ax1
7.	$B \rightarrow (A \rightarrow A)$	MP 5, 6

$\boxed{P \wedge Q}$ Exercise: Transitive Implication

Fill in the blanks in the following proof of

$$\{A \to B, B \to C\} \vdash A \to C$$

1.	$B \to C$	________
2.	$(B \to C) \to (A \to (B \to C))$	________
3.	____________________	MP 1, 2
4.	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	________
5.	$(A \to B) \to (A \to C)$	________
6.	$A \to B$	Hypothesis
7.	____________________	MP 5, 6

This proves implication is transitive; call it TI

$\boxed{P \wedge Q}$ 14.2.2 **Using TI**

Can use meta-theorems with hypotheses as rules of inference

First prove instances of hypotheses, then use meta-theorem to conclude instance of conclusion

Prove $\vdash \neg A \to (A \to B)$

1.	$\neg A \to (\neg B \to \neg A)$	Ax1
2.	$(\neg B \to \neg A) \to (A \to B)$	Ax3
3.	$\neg A \to (A \to B)$	TI 1, 2

Could turn into proper proof by replacing step 3 with proof of TI, with $\neg A$ substituted for A, $(\neg B \to \neg A)$ for B and $(A \to B)$ for C, and lines 1 and 2 for the hypothesis lines

But this proof is much easier!

$\boxed{P \wedge Q}$ 14.2.3 **More Meta-theorems**

Kelly proves other useful Meta-theorems

Deduction meta-theorem:

$$\text{If} \quad H \cup \{A\} \vdash B \quad \text{then} \quad H \vdash A \rightarrow B$$

Proof is more complex (by induction)

$\boxed{P \wedge Q}$ More Meta-theorems (2)

Use deduction meta-theorem to prove
$\{A \to (B \to C)\} \vdash B \to (A \to C)$:

1.	$A \to (B \to C)$	Hypothesis
2.	A	Hypothesis (for sake of proof
3.	$B \to C$	MP 1, 2
4.	B	Hypothesis (for sake of proof
5.	C	MP 3, 4
So,	$\{A \to (B \to C), A, B\} \vdash C$	
then	$\{A \to (B \to C), B\} \vdash A \to C$	Deduction theorem
then	$\{A \to (B \to C)\} \vdash B \to (A \to C)$	Deduction theorem

$\boxed{P \wedge Q}$ 14.3 Finding axiomatic proofs

- no firm search strategy
- proofs generally hard to find
  - usually a mixture of seeing what you can do with the hypotheses you have, and seeing what might lead to the desired conclusion.

$\boxed{P \wedge Q}$ 14.4 Truth and Deduction (Kelly 4.5)

What is the relationship between *truth* ($\models$, see Kelly chapter 1) and *deduction* ($\vdash$, see Kelly chapter 4)?

What makes an axiom system useful?

1. **Consistency**: if $\vdash A$ then $\nvdash \neg A$
   Is it impossible to prove a wff and its negation?

2. **Soundness**: $\vdash A$ implies $\models A$
   Is it impossible to prove a wff that is not a tautology?

3. **Completeness**: $\models A$ implies $\vdash A$
   Can everything that is a tautology be proved?

4. **Decidability**:
   Is there an algorithm to always decide if $\vdash A$?

Truth and Deduction (Kelly 4.5) (2)

The *AL* system of Kelly chapter 4 has all these properties

1. *Consistency*: follows from *soundness* (below), and that $\models$ is consistent

2. *Soundness*: proof by induction; axioms are tautologies, and modus ponens only yields tautologies from tautologies

3. *Completeness*: complex proof, see Kelly pp. 87–90

4. *Decidability*: *AL* is complete, so it is enough to decide if $\models A$, which can be decided by truth table

$\boxed{P \wedge Q}$ 14.5 Summary

- proof system based on axioms and inference rules defines a computational procedure for verifying deductions ... though not necessarily practical for *generating* deductions

- correctness of proof system with respect to semantics is guaranteed by soundness, completeness