Neural Networks and Biological Modeling

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Correction Question Set 8

Exercise 1: Continuous population model

We study the system with lateral connection $w(x-y)$ described by the equation

$$\tau \frac{\partial h(x,t)}{\partial t} = -h(x,t) + \int w(x-y) F[h(y,t)] dy + I_{ext}(x,t) \tag{1}$$

where $F[h(x,t)] = A(x,t)$ is the population activity at point x and at time t.

1.1 With the following conditions,

$$\begin{aligned} I_{ext}(t) &= const. \\ h(x,t) &= h_0 \end{aligned}$$

the equation (1) becomes

$$\begin{aligned} 0 &= -h_0 + I_{ext} + F(h_0) \underbrace{\int w(x-y) dy}_{=\bar{w}} \qquad (2) \\ &= \bar{w} A_0 - h_0 + I_{ext}. \end{aligned}$$

Therefore,

$$A_0 = \frac{h_0 - I_{ext}}{\bar{w}} \tag{3}$$

1.2 Linearizing (1) around h_0, we find

$$\tau \frac{\partial}{\partial t} \Delta h(x,t) = -\Delta h(x,t) + \int w(x-y) F'(h_0) \Delta h(y,t) dy + O(\Delta h^2)$$

where we used (2) to get rid of h_0. Using the following Fourier transform formula for the convolution

$$\left(\int f(x-y) g(y) dy \right)^* = f^*(k) g^*(k)$$

where f^* is the Fourier transform of f, $f^*(k) = \int e^{-ikx} f(x) dx$, we have

$$\tau \frac{\partial}{\partial t} \Delta h^*(k,t) = -\Delta h^*(k,t) + F'(h_0) w^*(k) \Delta h^*(k,t) = (F'(h_0) w^*(k) - 1) \Delta h^*(k,t) \,.$$

Integration once through time,

$$\Delta h^*(k,t) = C(k) e^{-(1 - F'(h_0) w^*(k)) t/\tau} = C(k) e^{-\kappa(k) t/\tau} \,,$$

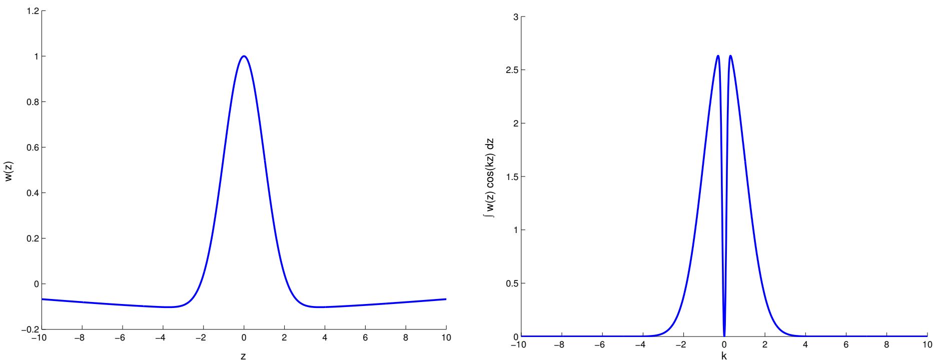
and taking the inverse of the transform,

$$\Delta h(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(k) e^{ikx} e^{-\kappa(k) t/\tau} dk \,.$$

The perturbation evolves as a superposition of modes which has the form $\cos(kx+\varphi(t))e^{-\Re(\kappa(k))t/\tau}$. The stationary state is stable if $\Re(\kappa(k)) > 0$ for all k.

1.3 The function $w(z)$ is shown of figure 1a. It's an excitatory interaction at short distance and an inhibitory at long distance. There is an equilibrium between excitation and inhibition in the sense that $\int_{-\infty}^{+\infty} w(z)dz = 0$. The typical form of this function is often called "mexican hat".

The real part of the Fourier transform, $\int w(z)\cos(kz)dz$ is shown on figure 1b. we see that this function is positive everywhere and its maximal value is about 2.5. From stability condition $\Re(\kappa) > 0$, we deduce that the uniform steady state stability is only standing if the susceptibility $f'(h_0)$ is small enough, i.e. of the order of 0.4. For a better understanding, note that the derivative $f'(h_0)$ represent the variation of the activity due to a small perturbation of the potential: if f' is big, a small perturbation of the potential leads to a big perturbation in the activity of what leads the instability.



(a) Graph of $w(z)$ with $\sigma_1 = 1$ and $\sigma_2 = 10$

(b) Graph of $\int w(z)cos(kz)dz$ with $\sigma_1 = 1$ and $\sigma_2 = 10$.

Figure 1:

Exercise 2: Stationary state in a network with lateral connections

Justification of the hypothesis that $d > \sigma$: Consider the potential, $h(x)$, at the boundary of the group of active neurons, where $x_1 \lesssim 2d \lesssim x_2$. We find that, if $d \leq \sigma$, then, $h(x_1) = h(x_2) = d$, but as we suppose that the neuron x_1 is active and the one in x_2 is not, this leads to $\theta \leq d < \theta$ which shows that d cannot be less than σ.

2.1 The input potential at x_0 is:

$$\begin{aligned} h(x_0) = &\int_{-\infty}^{0} w(x_0,x')A(x')dx' + \int_{0}^{x_0-\sigma} w(x_0,x')A(x')dx' + \int_{x_0-\sigma}^{x_0} w(x_0,x')A(x')dx' \\ &+ \int_{x_0}^{2d} w(x_0,x')A(x')dx' + \int_{2d}^{\infty} w(x_0,x')A(x')dx' \\ &= 0 + (x_0-\sigma)(-b) + \sigma\cdot 1 + (2d-x_0)\cdot 1 + 0 \\ &= (x_0-\sigma)(-b) + \sigma + 2d - x_0 \\ &= \sigma(1+b) - x_0(1+b) + 2d. \end{aligned} \tag{4}$$

2.2 It follows from the definition of $F(h)$:

$$\begin{aligned} h(2d) = \Theta &= (2d-\sigma)(-b) + \sigma + 2d - 2d \\ &= \sigma(1+b) - 2db \end{aligned} \tag{5}$$

We can now calculate d:

$$d = \frac{-\Theta + \sigma(1 + b)}{2b} \tag{6}$$

2.3 Neglecting the trivial solution of $d = 0$ the bump's size cannot be infinite (in that case each neuron would receive close to infinite inhibition and it would therefore not be active, which leads to inconsistency). Moreover the $2d$ bump could appear in any location (it is translation-invariant).