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Master Project



Low-dimensional population dynamics of spiking neurons via eigenfunction expansion

Author:
Noé GALLICE

Directors:
Prof. Wulfram GERSTNER
Prof. Matthieu WYART

Supervisor:
Dr. Tilo SWCHALGER

Abstract

TODO

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1 Introduction

1.1 Firing rate models

-phenomenological, not derived from spiking neuron [?]

1.2 Renewal spiking neuron model

Renewal processes keep memory of the last event, las firing time \hat{t} . For those processes the spikes are generated according to a stochastic intensity called the hazard rate

$$\rho(t|\hat{t}) = \rho(\tau) \quad (1.1)$$

which depends on the age of the neuron τ , i.e the time since the last spike $\tau = t - \hat{t}$. $\rho(\tau)$ define the probability to spike between $t + \Delta t$ knowing that there were no spike bewteen t and \hat{t} . The renewal theory allows to define the probability of the next event given the age of the system, to calculate the interspike-interval (ISI) distribution, i.e the probability to spike at age τ and no spike before.

$$P(\tau) = P(\hat{t} + \tau|\hat{t}) \quad (1.2)$$

The ISI distribution satisfy

$$\int_0^\infty P(\tau) d\tau = 1 \quad (1.3)$$

1.2.1 Moment of the interspike interval distribution $P(\tau)$

The interval distribution allows to compute the k -th moment

$$\langle \tau^k \rangle = \int_0^\infty \tau^k P(\tau) d\tau \quad (1.4)$$

It is usefull to introduce the Laplace transform

$$P_L(\lambda) = \int_0^\infty d\tau e^{-\lambda\tau} P(\tau) \quad (1.5)$$

from which the k -th ISI moment can be generated

$$\langle \tau^k \rangle = (-1)^k \frac{d^k P_L}{d\lambda^k} \Big|_{\lambda=0} \quad (1.6)$$

Hence, $P_L(\lambda)$ is called the ISI moment generating function. We can generated from this function the ISI cumulants defined by

$$\kappa_k = (-1)^k \frac{d^k \ln P_L}{d\lambda^k} \Big|_{\lambda=0} \quad (1.7)$$

The first four ISI cumulants are related to the ISI moments by

$$\kappa_1 = \langle \tau \rangle \quad (1.8)$$

$$\kappa_2 = \langle \tau^2 \rangle - \langle \tau \rangle^2 \quad (1.9)$$

$$\kappa_3 = \langle \tau^3 \rangle - 3 \langle \tau^2 \rangle \langle \tau \rangle + 2 \langle \tau \rangle^3 \quad (1.10)$$

$$\kappa_4 = \langle \tau^4 \rangle - 4 \langle \tau^3 \rangle \langle \tau \rangle - 3 \langle \tau^2 \rangle^2 + 12 \langle \tau^2 \rangle \langle \tau \rangle^2 - \langle \tau \rangle^4 \quad (1.11)$$

$$(1.12)$$

The cumulant can be used to characterize the shape of the ISI density. The rate of a process R is given by

$$R = \langle \tau \rangle^{-1} = \kappa_1^{-1} \quad (1.13)$$

An important measure that quantify the variability of ISI distribution is the coefficient of variation C_V defined as

$$C_V = \sqrt{\frac{\langle \tau^2 \rangle}{\langle \tau \rangle^2} - 1} = \frac{\sqrt{\kappa_2}}{\kappa_1} \quad (1.14)$$

A C_V equal to 1 indicates a highly irregular spike train whereas $C_V = 0$ indicates a perfectly regular spike train.

TODO ADD SKEWNESS AND KURTOSIS

1.2.2 Interval distribution and Survivor function

the interval distribution $P(\tau)$ is a probability density, which implies that integration of $P(\tau)$ over time yields a probability. The probability that neuron which has fired a spike at \hat{t} and fires the next spike at between \hat{t} and t is given by $\int_0^\tau P(s)ds$.

The interspike-interval (ISI) distribution can be linked to the survivor function

$$S(\tau) = 1 - \int_0^\tau P(s)ds \quad (1.15)$$

The survivor function $S(\tau)$ define the probability that a neuron reach the age τ , so that a neuron "survive" without firing between \hat{t} and t . $P(\tau)$ describes the probability to spike at age τ and no spike before. This is given by the product of the probability to survive until age τ times the momentary hazard $\rho(\tau)$

$$P(\tau) = \rho(\tau)S(\tau) \quad (1.16)$$

The derivation of Eq.(1.15) yields to

$$P(\tau) = -\frac{d}{d\tau}S(\tau) \quad (1.17)$$

Inserting Eq.(1.17) in Eq.(1.15), we find that the hazard rate $\rho(\tau)$ corresponds to the rate of decay of the survivor function:

$$\rho(\tau) = -\frac{\frac{d}{d\tau}S(\tau)}{S(\tau)} \quad (1.18)$$

Integrating eq.1.18 yields to the survivor function:

$$S(\tau) = \exp\left[-\int_0^\tau \rho(s)ds\right] \quad (1.19)$$

Inserting Eq.(1.19) in Eq.(1.16) The interval distribution can be explicitly express in terms of the hazard, and is by itself normalized:

$$P(\tau) = -\frac{d}{d\tau}S(\tau) = \rho(\tau) \exp\left[-\int_0^\tau \rho(s)ds\right] \quad (1.20)$$

1.2.3 Examples

Interval distribution and hazard functions have been measured in many experiments. Here are some examples widely used.

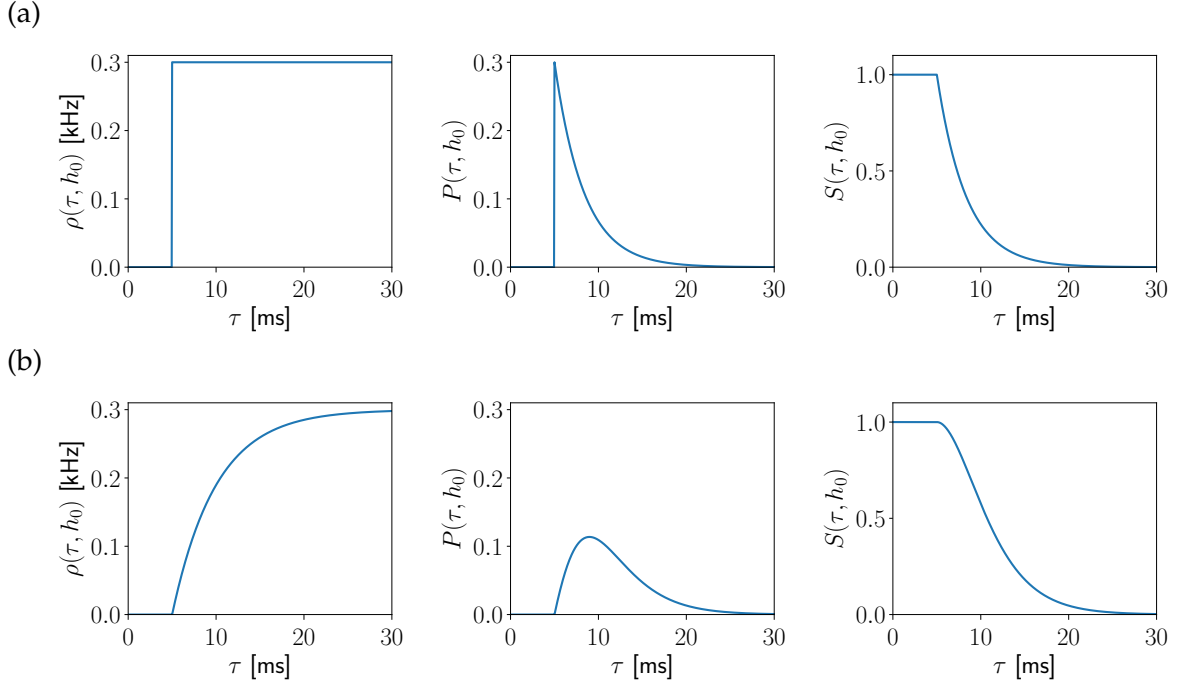


Figure 1: Hazard rate $\rho(\tau, h)$ (left), interval distribution $P(\tau, h)$ (middle) and survivor function $S(\rho(\tau, h))$ (right) for different recovery function $g(\tau)$. (a) Recovery function corresponds to a Poisson neuron with absolute refractoriness Δ , with $\Delta = 5$ ms, $h = h_0$, $\nu_{max} = 0.6$ kHz. (b) Recovery function defined by $g(\tau) = [1 - \exp(-\lambda(\tau - \Delta))] \theta(\tau - \Delta)$ Poisson neuron with absolute refractoriness Δ , with $\Delta = 5$ ms, $h = h_0$, $\nu_{max} = 0.6$ kHz.

Simple model with recovery function In the previous section we were implicitly considering stationary renewal system using the notation $\rho(t|\hat{t})$. In this section we will use the notation $\rho(\tau, h)$, with h a time dependent parameter, to show explicitly that $\rho(\tau)$ can change in time.

$$\tau_m \dot{h} = -h + \mu(t) \quad (1.21)$$

Where $\mu(t)$ is a time dependent external input.

The hazard rate, can be expressed using a recovery function $g(\tau)$

$$\rho(\tau, h) = \Phi(h)g(\tau) \quad (1.22)$$

With

$$\Phi(h) = \frac{\nu_{max}}{1 + \exp[-\beta(h - h_0)]} \quad (1.23)$$

The hazard rate, the survival probability, and the interval distribution are shown in Fig.1 for two Examples of recovery function g . Fig.1(a) corresponds to a poisson process with absolute refractoriness:

$$g(\tau) = \theta(\tau - \Delta) \quad (1.24)$$

The recovery function for Fig.1(b) is given by

$$g(\tau) = [1 - \exp(-\lambda(\tau - \Delta))] \theta(\tau - \Delta) \quad (1.25)$$

The main difference is that for the poisson neuron with absolute refractoriness the recovery function Eq.(1.24) make a jump, whereas in Eq.(1.25) the transition is smooth.

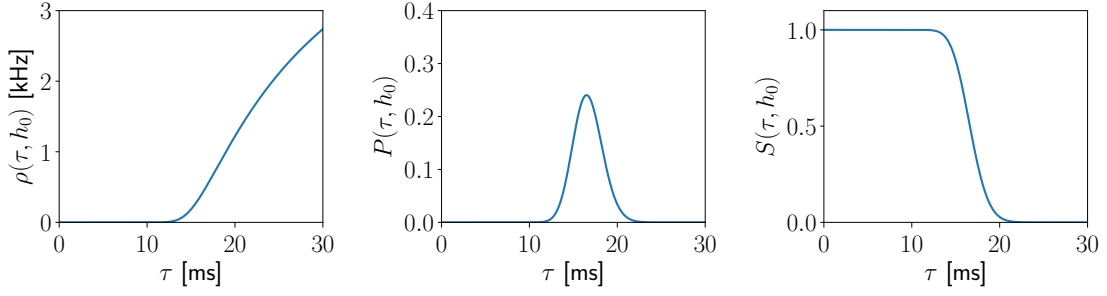


Figure 2: Hazard rate $\rho(\tau)$ (left), interval distribution $P(\tau)$ (middle) and survivor function $S(\rho(\tau))$ (right) for a Gamma Process with $\beta = 5\text{kHz}$ $\gamma = 100$

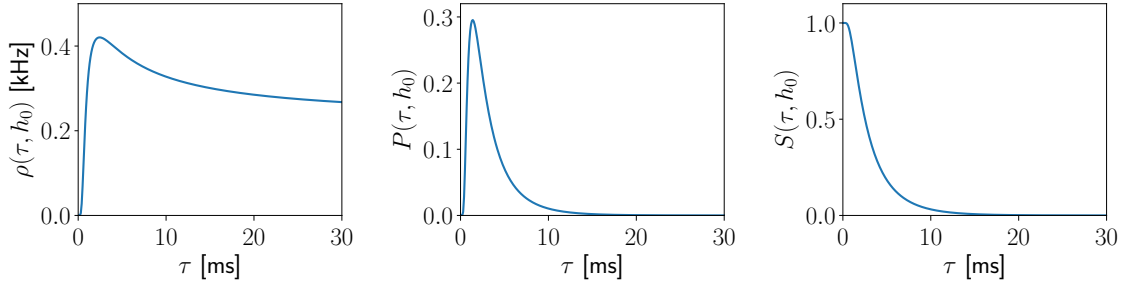


Figure 3: Hazard rate $\rho(\tau)$ (left), interval distribution $P(\tau)$ (middle) and survivor function $S(\rho(\tau))$ (right) for the PIF model.

Gamma process The gamma process is often used to model spike trains as it is one of the easiest non-Poisson process to analyze. The interspike distribution is given by:

$$P(\tau) = \frac{\beta^\gamma}{\Gamma(\gamma)} \tau^{\gamma-1} e^{-\beta\tau} \quad (1.26)$$

The rate of this process is $R = \beta/\gamma$, And the coefficient of variation is given by $C_V = \gamma^{-\frac{1}{2}}$. For $\gamma = 1$ this corresponds to a Poisson process. For $C_V > 1$ the Interspike distribution diverges as τ goes to 0. If gamma is an integer, one can see the gamma process as a chain of γ poisson neuron with rate β before, this implies that the global rate of the total chain is $R = \beta/\gamma$, and induces refractoriness as shown on Fig.2

Perfect integrate-and-fire model driven by white noise The perfect integrating-and fire model has been used to explain statistics of of single neuron. The membrane potential can be seen as a Brownian-motion with drift μ (mean current) driven by a white noise

$$\dot{V} = \mu + \sqrt{2D}\xi(t), \quad \text{if } V = V_{th} \quad V \rightarrow V_r \quad (1.27)$$

$$\langle \xi(t)\xi(s) \rangle = \delta(t-s) \quad (1.28)$$

1.3 Populations of neurons and refractory density equations

Network model - population activity definition

- definition of network : all-to-all coupling

-external input $\mu(t)$

$$\mu(t) = V_{rest} + RI_{ext}(t) + RI_{syn}(t) \quad (1.29)$$

- and/or synaptic input

$$RI_{syn}(t) = \tau_m JA(t) \quad (1.30)$$

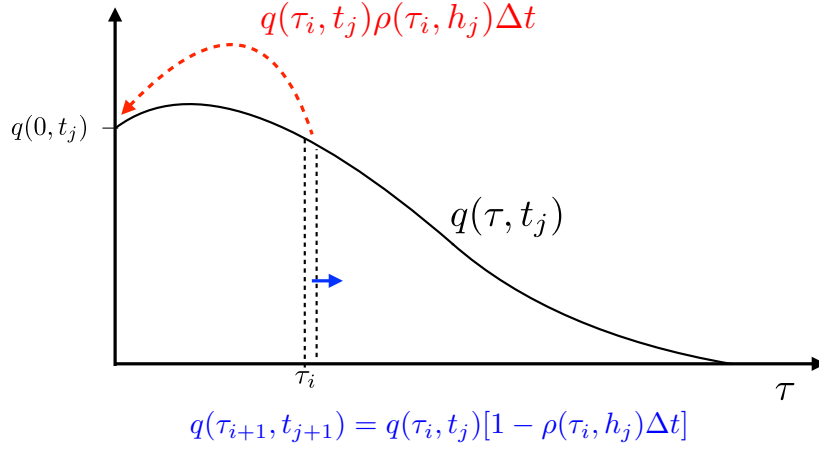


Figure 4: TODO

Refractory density equation An approach on the refractory density $q(\tau, t)$ can be used to analyze the dynamic in a neuron network. $q(\tau, t)$ defined the density of neuron at time t with age $\tau = t - \hat{t}$ where \hat{t} . The activity of a network is thus given by

$$A(t) = q(0, t) \quad (1.31)$$

As long as a neuron does not fire, the age τ increase at the speed of $\frac{d\tau}{dt} = 1$, therefore the flux along the refractory variable τ is simply $q(\tau, t)$ and the continuity equation is given by

$$\frac{\partial q}{\partial t} = -\frac{\partial q}{\partial \tau} \quad (1.32)$$

When a neuron fires, the trajectory along the refractory variable τ stops at the current value and "reappears" at $\tau = 0$. The instantaneous probability to fire is given by the hazard function. Therefore the loss per unit time is given by $-\rho(\tau, h)q(\tau, t)$ and the full dynamic is then given by the master equation:

$$\partial_t q(\tau, t) = -\partial_\tau q(\tau, t) - \rho(\tau, h)q(\tau, t) \quad (1.33)$$

The sum of all trajectory that "disappear" at time t due to firing, are "reappearing" at $\tau = 0$ and neuron will fire at some finite time, therefore the boundary condition are:

$$q(0, t) = \int_0^\infty \rho(\tau, h)q(\tau, t)d\tau = A(t) \quad (1.34)$$

$$q(\infty, t) = 0 \quad (1.35)$$

Additionally, q is normalized

$$\int_0^\infty q(\tau, t)d\tau = 1 \quad (1.36)$$

and has the initial density

$$q(\tau, 0) = q_0(\tau) \quad (1.37)$$

where $q_0(\tau)$ is some function that satisfies the conditions Eq.(1.34),(1.36)

[?, ?]

1.4 Spectral decomposition method

The eigenfunction expansion is a method applied on partial differential equation. The main idea is identify an operator and expand the equation in terms of the operator eigenfunction times time dependent coefficient.

For concreteness, and to recall the basic property of this methods, we will introduce the spectral expansion of the Fokker-Planck operator.

1.4.1 Spectral decomposition of the Fokker-Planck equation

An approach on the membrane potential densities $p(v, t)$ can be used to analyse the dynamics of networks of integrate and fire neurons. We assume that a neuron can be described by one state variable e.g. the membrane potential v .

For a large homogeneous network of IF neurons, the evolution of probability density function $p(v, t)$ is described by the Fokker-Planck equation (TODO add reference).

$$\partial_t p(v, t) = \mathcal{L}p(v, t) \quad (1.38)$$

where \mathcal{L} is the Fokker-Planck operator, which takes the general form

$$\mathcal{L}(v, t) = -\partial_v A(v, t) + \partial_v^2 B(v, t) \quad (1.39)$$

The Fokker-Planck operator is in general non linear because it depend on activity $A(t)$ given by the flux of realizations crossing the emission threshold θ .

1.4.2 General properties of the Fokker-Planck operator

The Fokker-Planck operator has a set of eigenfunctions and associated eigenvalues:

$$\mathcal{L}|\phi_n\rangle = \lambda_n |\phi_n\rangle \quad (1.40)$$

If the eigenvalues λ_n are complex, the complex conjugate of an eigenvalue $\bar{\lambda}_n$ is also an eigenvalue of \mathcal{L} , with eigenfunction $|\bar{\phi}_n\rangle$ because \mathcal{L} is a real operator. We will use the notation $\bar{\lambda}_n = \lambda_{-n}$, so that the sums over the spectrum of \mathcal{L} range over all integer numbers. Note that $\lambda_0 = 0$ is always an eigenvalue of the operator \mathcal{L} and corresponds to the stationary solution.

Because \mathcal{L} cannot be generally brought to an Hermitian form we also need the eigenfunction $|\psi_n\rangle$ of the adjoint operator \mathcal{L}^+

$$\mathcal{L}^+|\psi_n\rangle = \tilde{\lambda}_n |\psi_n\rangle \quad (1.41)$$

Defining the inner product one can show that the eigenvalues of Eq.(1.40) and Eq.(1.41) are the same:

$$\langle\psi|\phi\rangle = \int \psi(v, t)\phi(v, t)dv \quad (1.42)$$

$$\begin{aligned} \lambda_n \langle\psi_n|\phi_n\rangle &= \int \psi(v, t)\mathcal{L}\phi(v, t)dv \\ &= \langle\psi_n|\mathcal{L}\phi_n\rangle \\ &= \langle\mathcal{L}^+\psi_n|\phi_n\rangle \\ &= \int \mathcal{L}^+\psi_n(v, t)\phi_n(v, t)dv \\ &= \tilde{\lambda}_n \langle\psi_n, |\phi_n\rangle \end{aligned} \quad (1.43)$$

Eq.(1.43) implies that $\lambda_n = \tilde{\lambda}_n$ and

$$\mathcal{L}^+ |\psi_n\rangle = \lambda_n |\psi_n\rangle \quad (1.44)$$

For different eigenvalues, the eigenfunctions ψ_i and ϕ_j are orthogonal

$$\begin{aligned} \lambda_j \langle \psi_i | \phi_j \rangle &= \langle \psi_i | \mathcal{L} \phi_j \rangle \\ &= \langle \mathcal{L}^+ \psi_i | \phi_j \rangle \\ &= \lambda_i \langle \psi_i | \phi_j \rangle \end{aligned} \quad (1.45)$$

And with an appropriate normalization the two set of eigenfunctions are biorthonormal

$$\langle \psi_i | \phi_j \rangle = \delta_{ij} \quad (1.46)$$

1.5 Expansion and rate equation

The spectrum $\{\lambda_n\}$ of the Fokker-Planck operator provide a moving basis $\{|\phi_n\rangle\}$. The density $p(v, t)$ can be expressed in terms of this basis

$$|p\rangle = \sum_n a_n |\phi_n\rangle \quad (1.47)$$

where $a_n = \langle \psi_n | p \rangle$ are the time dependent coefficients of the modal expansion. Since p is real $\bar{a}_n = a_{-n}$. The dynamics of the a_n can be determined directly from the Fokker-Plank equation

$$\begin{aligned} \dot{a}_n &= \langle \psi_n | \partial_t p \rangle + \langle \partial_t \psi_n | p \rangle \\ &= \langle \psi_n | \mathcal{L} p \rangle + \dot{\nu} \sum_m a_m \langle \partial_\nu \psi_n | \phi_m \rangle \\ &= \lambda_n a_n + \dot{\nu} \sum_m a_m \langle \partial_\nu \psi_n | \phi_m \rangle \end{aligned} \quad (1.48)$$

Here we have use the fact that the time dependence of ψ is due to the moment of ν . An emission rate equation result

$$\begin{aligned} \dot{\vec{a}} &= (\mathbf{\Lambda} + \mathbf{C} \dot{h}) \vec{a} + \vec{c} \dot{h} \\ \nu &= \Phi + \vec{f} \cdot \vec{a} \end{aligned} \quad (1.49)$$

where $\nu(t)$ is the instantaneous firing rate. $\vec{a} = \{a_n\}_{n \neq 0}$. \vec{f} are the flux in θ for non stationary modes. The synamptic coupling are expressed in the in the vector $\vec{c} c_n = \langle \partial_h \psi_n | \phi_0 \rangle$, $\forall n \neq 0$ and the matrix \mathbf{C} : $C_{nm} = \langle \partial_h \psi_n | \phi_m \rangle$, $\forall n \neq 0$. And $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues of \mathcal{L} : $\Lambda_{nm} = \lambda_n \delta_{nm}$

2 Theory

2.1 Operator of the refractory density, and eigenvalue spectrum $\{\lambda_n\}$

The master equation Eq.(1.33) can be rewritten introducing the operator :

$$\mathcal{L} = -\partial_\tau - \rho(\tau, h) \quad (2.1)$$

$$\partial_t q(\tau, t) = \mathcal{L} q(\tau, t) \quad (2.2)$$

The set of eigenfunctions and associated eigenvalues of \mathcal{L} obeys to

$$\mathcal{L} |\phi_n\rangle = \lambda_n |\phi_n\rangle \quad (2.3)$$

And respect the boundary conditions Eq.(1.34)

$$\phi_n(0, h) = \int_0^\infty \rho(\tau, h) \phi_n(\tau, h) d\tau \quad (2.4)$$

$$\phi_n(\infty, h) = 0 \quad (2.5)$$

$$(2.6)$$

The solution of Eq.(2.3) is

$$\begin{aligned} \phi_n(\tau, h) &= \phi_n(0, h) \exp(-\lambda_n(h)\tau - \int_0^\tau \rho(s, h) ds) \\ &= \phi_n(0, h) e^{-\lambda_n(h)\tau} S(\tau, h) \end{aligned} \quad (2.7)$$

Inserting the Eq.(2.7) in the boundary condition Eq.(2.4), we find the condition

$$\phi_n(0, h) = \int_0^\infty \rho(\tau, h) \phi_n(0, h) \exp(-\lambda_n(h)\tau - \int_0^\tau \rho(s, h) ds) d\tau \quad (2.8)$$

which can be written as

$$1 = \int_0^\infty e^{-\lambda_n(h)\tau} P(\tau, h) d\tau = P_L(\lambda_n) \quad (2.9)$$

The condition Eq.(2.9) was already derived by Tilo Schwalger in an unpublished paper. It states that the Laplace transform of the ISI density $P_L(\lambda)$ at the (complex) arguments λ_n must be unity.

We can conclude some properties of the spectrum λ_n imposed by Eq.(2.9)

As expected, the eigenvalue $\lambda_0 = 0$ fulfilled the condition because the ISI density is normalized. This eigenvalue corresponds to the stationary density.

The real part of λ_n cannot be positive, as expected for physical reason. Indeed the solution of the refractory density equation is directly related to the eigenvalues of \mathcal{L} and is expected to converge to ϕ_0 , instead of exploding which would be the case for positive eigenvalues. In fact for $\Re(\lambda_n) > 0$,

$$\int_0^\infty e^{-\lambda_n(h)\tau} P(\tau, h) d\tau < \int_0^\infty |e^{-\lambda_n(h)\tau}| P(\tau, h) d\tau < \int_0^\infty P(\tau, h) d\tau = 1 \quad (2.10)$$

which contradict Eq.(2.9).

2.2 Adjoint operator \mathcal{L}^+ , and normalization

Because \mathcal{L} cannot be generally brought to an Hermitian form we also need the eigenfunction ψ_n of the adjoint operator \mathcal{L}^+

$$\mathcal{L}^+ |\psi_n\rangle = \lambda_n |\psi_n\rangle \quad (2.11)$$

We can find the adjoint operator \mathcal{L} , using the integration by part

$$\begin{aligned}
\langle \psi | \mathcal{L} \phi \rangle &= \int_0^\infty \psi(\tau, h) \mathcal{L} \phi(\tau, h) d\tau \\
&= \int_0^\infty \psi(\tau, h) [-\partial_\tau - \rho(\tau, h)] \phi(\tau, h) d\tau \\
&= -[\psi(\tau, h) \phi(\tau, h)]_0^\infty + \int_0^\infty \partial_\tau \psi(\tau, h) \phi(\tau, h) d\tau - \int_0^\infty \rho(\tau, h) \psi(\tau, h) \phi(\tau, h) d\tau \\
&= \psi(0, h) \phi(0, h) + \int_0^\infty [\partial_\tau - \rho(\tau, h)] \psi(\tau, h) \phi(\tau, h) d\tau \\
&= \int_0^\infty \psi(0, h) \rho(\tau, h) \phi(\tau, h) d\tau + \int_0^\infty [\partial_\tau - \rho(\tau, h)] \psi(\tau, h) \phi(\tau, h) d\tau \\
&= \int_0^\infty \{ [\partial_\tau - \rho(\tau, h)] \psi(\tau, h) + \psi(0, h) \rho(\tau, h) \} \phi(\tau, h) d\tau \\
&= \langle \mathcal{L}^+ \psi | \phi \rangle
\end{aligned} \tag{2.12}$$

As we will normalize the eigenfunction to obtain a biorthonormal basis we can set $\psi(0, h) = 1$, and the adjoint operator can be express as

$$\mathcal{L}^+ \psi(\tau, h) = [\partial_\tau - \rho(\tau, h)] \psi(\tau, h) + \rho(\tau, h) \tag{2.13}$$

The solution of Eq.(2.11) is

$$\begin{aligned}
\psi_n(\tau, h) &= \exp(\lambda_n(h)\tau + \int_0^\tau \rho(s, h) ds) \left[1 - \int_0^\tau \rho(x, h) \exp(-\lambda_n(h)x - \int_0^x \rho(s, h) ds) dx \right] \\
&= e^{\lambda_n(h)\tau} S^{-1}(\tau, h) \left[1 - \int_0^\tau P(x, h) e^{-\lambda_n(h)x} dx \right]
\end{aligned} \tag{2.14}$$

In particular we have

$$\psi_0(\tau, h) = 1 \tag{2.15}$$

The a biorthonormal basis is obtained inserting Eq.(2.7) and Eq.(2.14) in Eq.(1.46), teh normalization is then given by

$$1 = \int_0^\infty \phi_n(0, h) \left[1 - \int_0^\tau P(x, h) \exp(-\lambda_n(h)x) dx \right] d\tau \tag{2.16}$$

$$\phi_n(0, h) = \frac{1}{\int_0^\infty \left[1 - \int_0^\tau P(x, h) e^{-\lambda_n(h)x} dx \right] d\tau} \tag{2.17}$$

In particular for $n = 0$, $\lambda_0 = 0$ and we recover the relation:

$$\phi_0(0, h) = \frac{1}{\int_0^\infty S(\tau, h) d\tau} \tag{2.18}$$

2.3 Emission rate equation

As the density $p(v, t)$ with the Fokker-Planck equation, the refractory density $q(\tau, t)$ can be expressed in the eigenfunction $\{|\phi_n\rangle\}$. T

$$|q\rangle = \sum_n a_n |\phi_n\rangle \tag{2.19}$$

where $a_n = \langle \psi_n | q \rangle$ are the time dependent coefficients of the modal expansion. In particular from Eq.(2.15), and Eq.(1.46) we have $a_0(t) = 1$. The dynamics of the a_n can be determined directly using Eq.(2.1), and Eq.(2.2)

$$\begin{aligned}
\dot{a}_n &= \langle \psi_n | \partial_t q \rangle + \langle \partial_t \psi_n | q \rangle \\
&= \langle \psi_n | \mathcal{L} q \rangle + \hbar \sum_m a_m \langle \partial_h \psi_n | \phi_m \rangle \\
&= \lambda_n a_n + \hbar \sum_m a_m \langle \partial_h \psi_n | \phi_m \rangle
\end{aligned} \tag{2.20}$$

Defining $C_{nm} = \langle \partial_h \psi_n | \phi_m \rangle$ we can rewrite

$$\dot{a}_n = \lambda_n a_n + \hbar \sum_m C_{nm} a_m \tag{2.21}$$

We can finally express the activity $A(t) = q(0, t)$ as

$$A(t) = \sum_n a_n(t) \phi_n(0) \tag{2.22}$$

Keeping only the first mode, and using the fact that $\text{ket} \phi_{-n} = \text{ket} \bar{\phi}_n$ and $a_{-n} = \bar{a}_n$, Eq.(2.22) become

$$\begin{aligned}
A(t) &= \phi_0(0) + a_1 \phi_1(0) + a_{-1} \phi_{-1}(0) \\
&= \phi_0(0) + 2[\Re(a_1) \Re(\phi_1(0)) - \Im(a_1) \Im(\phi_1(0))]
\end{aligned} \tag{2.23}$$

And the dynamics of the a_1 is given by

$$\dot{a}_1 = \lambda_1 a_1 + \hbar [C_{10} + C_{11} a_1 + C_{1-1} a_{-1}]$$

Separating explicitly the real part $X(t)$ and the imaginary part $Y(t)$ of $a_1(t)$

$$a_1(t) = X(t) + iY(t) \tag{2.24}$$

we derived from Eq.(2.24) two non linear differential equation

$$\dot{X}(t) = \Re[f(t)]X(t) - \Im[g(t)]Y(t) + \Re[C_{10}]\hbar \tag{2.25}$$

$$\dot{Y}(t) = \Re[g(t)]Y(t) + \Im[f(t)]X(t) + \Im[C_{10}]\hbar \tag{2.26}$$

$$\tag{2.27}$$

with

$$f(t) = \lambda_1 + \hbar(c_{11} + c_{1-1}) \tag{2.28}$$

$$g(t) = \lambda_1 + \hbar(c_{11} - c_{1-1}) \tag{2.29}$$

We have finally a set of three non linear differential equation $\dot{h}(t), \dot{X}(t), \dot{Y}(t)$

2.4 Full Mattia 2002 system

2.5 Low-dimensional dynamics

2.5.1 Truncation Full Mattia 2002 system

2.5.2 Schaffer like

2.5.3 Mattias equation

3 Spectral properties of specific models

3.1 Poisson neuron with absolute refractoriness

3.2 Gamma process

3.3 PIF neuron

4 General renewal neuron

4.1 Population response to time-dependent input

–for uncoupled neurons –susceptibility

5 Population dynamics of coupled neurons

- one population

- two populations (E-I net)