Low-dimensional population dynamics of spiking neurons via eigenfunction expansion

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	Complex system	**Simplified model**	**Math. tractable / comput. efficiency**
Neuron:	Biophysical model	Hodgkin-Huxley model	IF model

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Aim: Derive a low dimensional dynamics taking into account the slowest modes of the expansion of the refractory density.

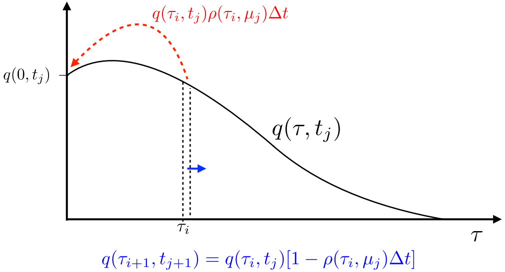
Overview

1. Theory

1.1 Refractory density equation

In a large homogeneous population of neurons, spikes are generated at time t according to a hazard function $\rho(\tau, \mu)$

The state variable τ is the age of the neuron.



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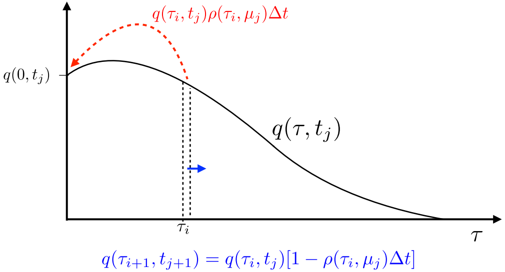
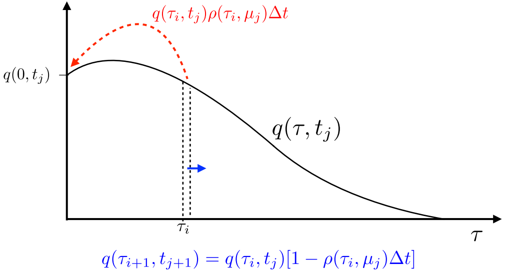


The refractory density $q(\tau, t)$ obeys the master equation:

$$\partial_t q(\tau, t) + \partial_\tau q(\tau, t) = -\rho(\tau, \mu) q(\tau, t)$$

1.1 Refractory density equation



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With boundary conditions:

$$q(0, t) = \int_0^\infty \rho(\tau, t) q(\tau, t) d\tau = A(t)$$

$$q(\infty, t) = 0$$

1.2 Eigenfunction expansion of the refractory density

We consider first the time homogeneous case: $\rho(\tau,\mu) = \rho(\tau)$

$$\begin{aligned}\partial_t q(\tau,t) &= -\partial_\tau q(\tau,t) - \rho(\tau) q(\tau,t)\\ &= \mathcal{L}\, q(\tau,t)\end{aligned}$$

with $\mathcal{L} = -\partial_\tau - \rho(\tau)$

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$$\partial_t q(\tau, t) = \mathcal{L}\, q(\tau, t) \qquad \text{with} \qquad \mathcal{L} = -\partial_\tau - \rho(\tau)$$

We can expand the refractory density as:

$$q(\tau, t) = \textstyle\sum_n a_n(t)\phi_n(\tau)$$

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where $\phi_n(\tau)$ are the eigenfunctions of the operator $\mathcal{L}$

$$\mathcal{L}\phi_n = \lambda_n \phi_n$$

With eigenvalues λ_n and boundary conditions:

$$\phi_n(0) = \int_0^\infty \rho(\tau)\phi_n(\tau)d\tau$$

$$\phi_n(\infty) = 0$$

1.2 Eigenfunction expansion of the refractory density

Solving $[-\partial_\tau - \rho(\tau)]\phi_n = \lambda_n \phi_n$ we have:

$$\phi_n(\tau) = \phi_n(0) \exp(-\lambda_n \tau - \int_0^\tau \rho(s) ds)$$

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which can be written as:

$$\boxed{1 = \int_0^\infty e^{-\lambda_n \tau} P(\tau) d\tau}$$

with ISI density $P(\tau) = \rho(\tau) \exp(- \int_0^\tau \rho(s) ds)$

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$\Rightarrow \lambda_0 = 0$ fulfilled the condition, it corresponds to the stationary density

1.3 Definition and property of the adjoint operator $\mathcal{L}^+$

To recover the activity we will need the eigenfunctions ψ_n of the adjoint operator $\mathcal{L}^+$:

$$\mathcal{L}^+\psi_n = \lambda_n\psi_n$$

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ψ_n, ϕ_n form a biorthonormal basis:

$$(\psi_i, \phi_j) = \delta_{ij}$$

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From which we obtained:

$$\begin{aligned}\frac{da_n}{dt} &= (\psi_n, \partial_t q) \\ &= (\psi_n, \mathcal{L}q) \\ &= (\mathcal{L}^+\psi_n, q) \\ &= \lambda_n a_n\end{aligned}$$

$$a_n(t) = a_n(0)\exp(\lambda_n t)$$

The activity is given by:

$$A(t) = q(t,0) = \sum_n a_n(t)\phi_n(0)$$

Keeping the two first modes, one can obtain a second order differential equation for the firing rate :

$$\ddot{A}(t) = [2Re(\tfrac{1}{\lambda_1})\dot{A}(t) - A(t) + A_\infty]|\lambda_1|^2$$

M. Mattia, *Low-dimensional firing rate dynamic of spiking neuron networks* (2016)

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Knowing the eigenvalues and the hazard function we can analytically define the eigenfunctions ϕ_n and ψ_n.

Thanks to those eigenfunctions we can recover the activity:

$$A(t) = \sum_n a_n(t)\phi_n(0)$$

Overview

2 Spectral expansion for different processes

2.1 Gamma process

The ISI distribution is given by:

$$P(\tau) = \frac{\beta^\gamma}{(\gamma-1)!}\tau^{\gamma-1}e^{-\beta\tau} \text{ for integer } \gamma \text{ and } \beta > 0.$$

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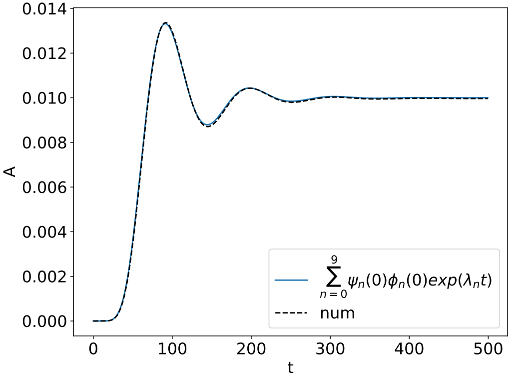
$$\bar{P}(\lambda) = (\frac{\beta}{\beta+\lambda})^\gamma$$

Solving $\bar{P}(\lambda_n) = 1$ we find:

$$\lambda_n = \beta(\exp(\frac{2\pi i}{\gamma}n) - 1),\ n = 0, ..., \gamma - 1$$

2.1 Gamma process

Initial condition $q(\tau, 0) = \delta(\tau) \rightarrow A(t) = \sum_{n=0}^{\gamma-1} \psi_n(0)\phi_n(0)exp(\lambda_n t)$



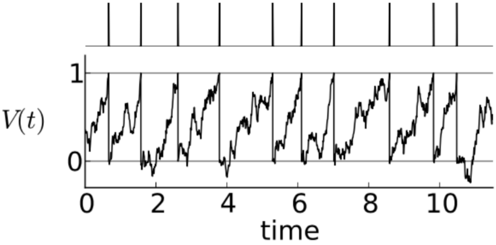
$\beta = 0.1$, $\gamma = 10$

2.2 Inverse Gaussian process

The perfect integrate fire model driven by a Gaussian white noise:

$$\dot{V} = \mu + \sqrt{2D}\xi(t) \qquad < \xi(t)\xi(s) >= \delta(t-s)$$

if $V = V_{th} : V \to V_r$



$\mu = 1$, $D = 0.125$ and $V_{th} = 1$

figure: T. Schwalger, The interspike-interval statistics of non-renewal neuron models (2013)

2.2 Inverse Gaussian process

The ISI distribution is given by:

$$P(\tau) = \frac{V_{th}}{\sqrt{4\pi D\tau^3}} \exp(-\frac{(\mu\tau - V_{th})^2}{4D\tau})$$

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$$\bar{P}(\lambda) = \exp(\frac{\mu V_{th}}{2D}[1 - \sqrt{1 + \frac{4D\lambda}{\mu^2}}])$$

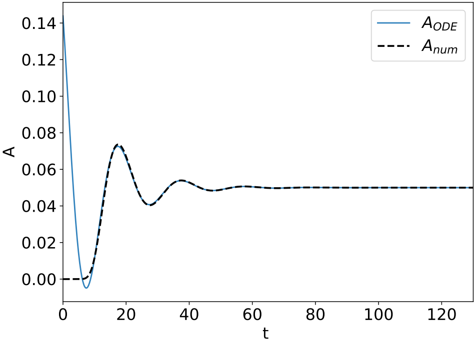
Solving $\bar{P}(\lambda_n) = 1$ we find:

$$\lambda_n = -\frac{2\pi\mu}{V_{th}} n (\frac{2\pi D}{\mu V_{th}} n + i)$$

2.2 Inverse Gaussian process

Recover the activity solving the second order differential equation:

$$\ddot{A}(t) = [2Re(\frac{1}{\lambda_1})\dot{A}(t) - A(t) + A_\infty]|\lambda_1|^2$$



$$\mu = 0.05 \ , D = 0.002, \ V_{th} = 1$$

4. Summary and Future Work

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Future work:

- Derive a low-dimensional ordinary differential equation for the firing rate in the case of a coupled network.

$$\frac{da_n}{dt} = \lambda_n a_n + \frac{d\mu}{dt}\left(\frac{\partial \psi_n}{\partial \mu}, q\right)$$

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Future work:

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$$\frac{da_n}{dt} = \lambda_n a_n + \frac{d\mu}{dt}(\frac{\partial \psi_n}{\partial \mu}, q)$$
- Find a efficient method to approximate the first eigenvalue for any hazard rate $\rho(\tau, \mu)$
$$1 = \int_0^\infty e^{-\lambda_n \tau} P(\tau) d\tau$$

Thanks for your attention

one can show that for different eigenvalues, the eigenfunctions ψ_i and ϕ_j are orthogonal:

$$\begin{aligned}\lambda_j(\psi_i, \phi_j) &= (\psi_i, \mathcal{L}\phi_j) \\ &= (\mathcal{L}^+\psi_i, \phi_j) \\ &= \lambda_i(\psi_i, \phi_j)\end{aligned}$$

2.2 Definition and property of the adjoint operator $\mathcal{L}^+$

$$
\begin{aligned}
(\psi, \mathcal{L}\phi) &= \int_0^\infty \psi(\tau)\mathcal{L}\phi(\tau)d\tau \\
&= \int_0^\infty \psi(\tau)[-\partial_\tau - \rho(\tau)]\phi(\tau)d\tau \\
&= -[\psi(\tau)\phi(\tau)]_0^\infty + \int_0^\infty \partial_\tau\psi(\tau)\phi(\tau)d\tau - \int_0^\infty \rho(\tau)\psi(\tau)\phi(\tau)d\tau \\
&= \psi(0)\phi(0) + \int_0^\infty [\partial_\tau - \rho(\tau)]\psi(\tau)\phi(\tau)d\tau \\
&= \int_0^\infty \psi(0)\rho(\tau)\phi(\tau)d\tau + \int_0^\infty [\partial_\tau - \rho(\tau)]\psi(\tau)\phi(\tau)d\tau \\
&= \int_0^\infty \{[\partial_\tau - \rho(\tau)]\psi(\tau) + \psi(0)\rho(\tau)\}\phi(\tau)d\tau \\
&= (\mathcal{L}^+\psi, \phi)
\end{aligned}
$$

with $\mathcal{L}^+\psi(\tau) = [\partial_\tau - \rho(\tau)]\psi(\tau) + \psi(0)\rho(\tau)$

2.2 Definition and property of the adjoint operator $\mathcal{L}^+$

$$\begin{aligned}\mathcal{L}^+\psi_n(\tau) &= [\partial_\tau - \rho(\tau)]\psi_n(\tau) + \psi_n(0)\rho(\tau) \\ &= \lambda_n\psi_n(\tau)\end{aligned}$$

The solution of this equation is:

$\psi_n(\tau) = \psi_n(0)\exp(\lambda_n\tau)S^{-1}(\tau)\big[1 - \int_0^\tau P(x)\exp\big(-\lambda_n x)dx\big]$

$$1 = \int_0^\infty \phi_n(0)\psi_n(0)\big[1 - \int_0^\tau P(x)\exp\big(-\lambda_n x)dx\big]d\tau$$

$$\phi_n(0)\psi_n(0) = \frac{1}{\int_0^\infty \big[1 - \int_0^\tau P(x)\exp\big(-\lambda_n x)dx\big]d\tau}$$

Second order differential equation for the firing rate for uncoupled neurons

M. Mattia, *Low-dimensional firing rate dynamic of spiking neuron networks* (2016)

$$\ddot{A}(t) = [2Re(\frac{1}{\lambda_1})\dot{A}(t) - A(t) + A_\infty]|\lambda_1|^2$$