# LAPLACE TRANSFORMS OF PROBABILITY DENSITY FUNCTIONS WITH SERIES REPRESENTATIONS

by

Joseph Abate[1]
AT&T retired

*Ward Whitt*[2]
AT&T Labs

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[1]900 Hammond Road, Ridgewood, NJ 07450-2908

[2]Room A117, AT&T Labs, 180 Park Avenue, Building 103, Florham Park, NJ 07932-0971;
email: wow@research.att.com

*Abstract*

In order to numerically invert Laplace transforms to calculate probability distributions in queueing and related models, we need to be able to calculate the Laplace transform values. In many cases the desired Laplace transform values (e.g., of a waiting-time distribution) can be computed when the Laplace transform values of component probability density functions (pdf's) (e.g., of a service-time pdf) can be computed. However, in some cases explicit expressions for Laplace transforms of component pdf's are not available. Hence, we propose the construction of infinite-series representations for Laplace transforms of pdf's and show how they can be used to calculate transform values. We use the Laplace transforms of Laguerre functions, Erlang pdf's and exponential pdf's as basis elements in the series representation. We develop several specific parametric families of pdf's in this infinite series framework. We show how to determine the asymptotic form of the pdf from the series representation and how to truncate so as to preserve the asymptotic form for a time of interest.

*Keywords*: Laplace transforms, numerical inversion of Laplace transforms, moments, Laguerre polynomials, Laguerre-series representation, Laguerre transform, infinite series, long-tail probability distributions, Erlang transform, Bell numbers, asymptotics for tail probabilities.

## 1. Introduction

Many descriptive quantities of interest in queueing models and other probability models arising in operations research can be effectively computed by numerically inverting Laplace transforms; e.g., by using the Fourier-series method or the Laguerre-series method; see Hosono (1984), Abate and Whitt (1992, 1995), Choudhury, Lucantoni and Whitt (1994), Abate, Choudhury and Whitt (1996, 1997, 1998) and references therein. The most difficult step in performing the numerical inversion, if there is any difficulty at all, is usually computing the Laplace transform values; e.g., see the application to polling models in Choudhury and Whitt (1996).

In many cases, numerical inversion is straightforward provided that Laplace transforms are available for component probability density functions (pdf's). A familiar example is the steady-state waiting-time distribution in the M/G/1 queue. Numerical inversion can be applied directly to the classical Pollaczek-Khintchine transform provided that the Laplace transform of the service-time pdf is available. More generally, the steady-state waiting-time distribution in the GI/G/1 queue can be computed by numerical transform inversion provided that the Laplace transforms of both the interarrival-time pdf and the service-time pdf are available; see Abate, Choudhury and Whitt (1993, 1994). In the GI/G/1 case, an extra numerical integration is required to calculate the required waiting-time transform values. For these inversion algorithms to be effective, the pdf's also need to be suitably smooth; otherwise preliminary smoothing may need to be performed; see Section 6 of Abate and Whitt (1992). In this paper, all the pdf's considered will be continuous.

Another example is the renewal function. Since the Laplace transform of the renewal function can be expressed directly in terms of the Laplace transform of the interrenewal-time pdf, the renewal function can be computed by numerical inversion provided that the Laplace transform of the interrenewal-time pdf is available; see Section 13 of Abate and Whitt (1992).

A remaining difficulty, however, is that convenient closed-form Laplace transforms are not readily available for some pdf's. This is particularly true for long-tail (also called heavy-tail) pdf's, which are currently of interest to model features of communication networks. To address this problem, Abate, Choudhury and Whitt (1994) introduced a family of long-tail distributions with convenient Laplace transforms, the Pareto mixtures of exponential (PME) distributions. To provide additional possibilities, Abate and Whitt (1996) initiated the development of an operational calculus for manipulating Laplace transforms, and Abate and Whitt

(1998) introduced two classes of beta mixtures of exponential (BME and $B_2$ME) distributions. Feldmann and Whitt (1997) also provided an algorithm for approximating probabilities density functions with decreasing failure rate, which include the Pareto, Weibull and other long-tail distributions, by hyperexponential distributions (finite mixtures of exponential distributions), whose transforms are readily available. Asmussen, Nerman and Olsson (1996) and Turin (1996) showed that the expectation-maximization (EM) algorithm can often be used to fit phase-type distributions to general probability distributions. The phase-type distributions also have relatively convenient Laplace transforms.

In this paper we make further contributions in this direction. We suggest a general approach for constructing probability density functions for which the Laplace transform values can be computed. In particular, we suggest representing the Laplace transform as an infinite series and then numerically calculating the sum, using acceleration methods if necessary. Perhaps the main point is to recognize that for the inversion we do not actually need a convenient closed-form expression for the transform. It suffices to have an algorithm to compute the transform value $\hat{f}(s)$ for the required arguments $s$. Thus an infinite series can be a satisfactory representation.

The first method we propose is the Laguerre-series representation. In Abate, Choudhury and Whitt (1996) we studied the Laguerre-series representation as a tool to numerically invert Laplace transforms, given that the Laplace transform values are available. Our purpose here is different. Here we want to compute Laplace transform values of a Laplace transform of a component pdf, such as a service-time pdf in a queueing model. We do so as an intermediate step to compute the final Laplace transform values of some other pdf or cdf, perhaps of a waiting-time distribution, which will be used to perform a numerical inversion. The final inversion may be done by any method, possibly but not necessarily by the Laguerre-series method.

Given that we decide to seek a Laguerre-series representation, there are two problems: (1) calculating the Laguerre coefficients, and (2) calculating the sum for the Laplace transform. It turns out that both problems can be addressed very effectively when the pdf is short-tailed, in particular, when the rightmost singularity of its Laplace transform is strictly less than 0. For this we use two approaches. The first way is to compute the coefficients by numerical integration using the standard integral representation of the coefficients, assuming that the pdf is known. The second way is to compute the coefficients from the moments, assuming that the moments are known. In exploiting the moments, we follow Keilson and

Nunn (1979), who showed that the Laguerre coefficients can be calculated from the moments under a regularity condition. However, we obtain a more elementary connection between the Laguerre coefficients and moments by altering the way the Laguerre coefficients are defined. The resulting Laguerre-series representation for the Laplace transform is convenient because the series tends to converge geometrically fast, even when the corresponding Laguerre-series in the time domain converges slowly. Moreover, it is sometimes possible to analytically determine the Laguerre coefficients, as we illustrate here.

In this work, as in Abate, Choudhury and Whitt (1996), we draw on previous studies of Laguerre-series representation by Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981), Sumita (1984) and Sumita and Kijima (1988), but we do not focus on manipulations that can be performed on the Laguerre coefficients, i.e., the Laguerre transform calculus. Applications of the Laguerre transform calculus are described by Keilson and Sumita (1982) and Litko (1989). These authors show that the Laguerre transform calculus can be effective, but much can also be done with Laplace transforms.

We also propose two other series representations for Laplace transforms of pdf's that are useful for long-tail pdf's. The first is the Erlang-series representation, which was introduced by Keilson and Nunn (1979) as the Erlang transform. Then the $n^{\text{th}}$ basis function is an Erlang pdf of order $n$ ($E_n$) or its Laplace transform $(1+s)^{-n}$. The coefficients of the Erlang transform are obtained from the coefficients of a power series representation of the pdf, assuming that it exists. The Erlang-series representation is a convenient alternative to the Laguerre-series representation because it often applies when the Laguerre-series representation does not and because the Erlang series for the Laplace transform tends to converge even more rapidly.

The final series representation we consider is the exponential-series representations, in which the $n^{\text{th}}$ basis element is an exponential pdf with mean $a_n$ or its Laplace transform $(1+a_n s)^{-1}$, where $a_n \to \infty$ as $n \to \infty$. Since the Laplace transform has poles at $s = -1/a_n$ for all $n$, the Laplace transform must have its rightmost singularity at 0, so these are long-tail distributions. We show how the exponential-series representation enables us to construct pdf's with given moment sequences. To illustrate, we construct pdf's with the famous Bell numbers and the ordered Bell numbers as moment sequences; e.g., see pp. 20, 42, 175 of Wilf (1994).

We also show how to determine the asymptotic form of a pdf represented by an infinite series and how to truncate the infinite series so that the asymptotic form is preserved for times of interest. In particular, we apply the Euler-Maclaurin formula and Laplace method to determine the asymptotics and the appropriate truncation point (as a function of time) to capture this asymptotic form.

Here is how the rest of this paper is organized. In Section 2 we present the Laguerre-series representation that we will use and derive our new algorithm for computing the Laguerre coefficients from the moments. In Section 3 we discuss the application of this algorithm to develop Laguerre-series representations of Laplace transforms of mixtures of exponential pdf's. An important class for which explicit results hold is the beta mixture of exponential (BME) pdf's. Properties of BME distributions, including this representation, were established in Abate and Whitt (1998). The BME example is very nice because we are able to apply the moment algorithm analytically to obtain simple expressions for the Laguerre coefficients. The moment algorithm is genuinely useful for BME pdf's because neither the pdf nor its transform has a convenient closed form.

In Section 4 we discuss the Erlang-series representation for pdf's. We illustrate in Section 5 by establishing results for mixtures of exponential distributions, including an explicit Erlang-series expansion for a third class of beta mixtures of exponentials. This class includes a class of Pareto mixtures of exponentials pdf's with a pure power tail, i.e., for which the next term in the asymptotic expansion decays exponentially. We consider the exponential-series representations in Section 6. In Section 7 we show how to determine the asymptotic form of the pdf and how to truncate the infinite series to preserve the asymptotic form at times of interest. Finally, we draw conclusions in Section 8.

We close this introduction by mentioning that multivariate extensions are possible. For work on multidimensional Laguerre-series representations, see Sumita (1984), Sumita and Kijima (1985) and Abate, Choudhury and Whitt (1997).

## 2. Laguerre-Series Representations

Let $f$ be a pdf and $\hat{f}$ its Laplace transform, i.e.,

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt \ , \tag{2.1}$$

which is well defined and analytic for all complex $s$ with $\mathrm{Re}(s) > 0$; see Doetsch (1974). If $\hat{f}$ is analytic for $\mathrm{Re}(s) > -s^*$ for $s^* > 0$, then $\hat{f}$ is analytic at 0 and has the *power series* (about the origin)

$$\hat{f}(s) = \sum_{n=0}^\infty s^n \frac{f^{(n)}(0)}{n!} = \sum_{n=0}^\infty (-s)^n \frac{m_n}{n!} \ , \tag{2.2}$$

where $m_n$ is the $n^{\mathrm{th}}$ moment of $f$. Unfortunately, though, even if (2.2) is valid and the moments $m_n$ are known, (2.2) is usually not a good way to compute $\hat{f}(s)$. Indeed, we may want to compute $\hat{f}(s)$ for values of $s$ outside the radius of convergence of (2.2). For all $s$ with $\mathrm{Re}(s) > -s^*$, $\hat{f}(s)$ will be analytic, so that a power-series representation for $\hat{f}(s)$ about each of those points $s$ is possible, but the domain of convergence for the power series about the origin need not include these points.

A Laguerre-series representation may be a much better series representation for computation of the Laplace transform. The *Laguerre-series representation* of $f$ is

$$f(t) = \sum_{n=0}^{\infty} q_n L_n(t), \quad t \geq 0 \ , \tag{2.3}$$

where $q_n$ are real numbers and

$$L_n(t) \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{(-t)^k}{k!}, \quad t \geq 0 \ , \tag{2.4}$$

are the *Laguerre polynomials*, e.g., see 22.3.9 of Abramowitz and Stegan (1972). The Laguerre polynomials form a complete orthonormal basis for the space $L_2[0,\infty)$ of real-valued functions on the nonnegative real line $[0,\infty)$ that are square integrable with respect to the weight function $e^{-t}$, using the inner product

$$\langle f_1, f_2 \rangle = \int_0^{\infty} e^{-t} f_1(t) f_2(t) dt \tag{2.5}$$

and norm $\|f\| = \langle f, f \rangle^{1/2}$. Then (2.3) is valid in the sense of convergence in $L_2[0,\infty)$ with

$$q_n = \langle f, L_n \rangle \equiv \int_0^{\infty} e^{-t} f(t) L_n(t) dt \tag{2.6}$$

being the *Laguerre coefficient* in (2.3); e.g., see Chapter 22 of Abramowitz and Stegun (1972) and Szegö (1975). Assuming that $f$ is continuous, the series (2.3) converges pointwise for all $t$ as well. The Laguerre polynomial $L_n(t)$ can be calculated in a numerically stable way via the recursion

$$L_n(t) = \left(\frac{2n-1-t}{n}\right) L_{n-1}(t) - \left(\frac{n-1}{n}\right) L_{n-2}(t) \ . \tag{2.7}$$

Our interest, however, is in the associated series representation for the Laplace transform $\hat{f}$. Since the Laplace transform of $L_n(t)$ has the simple form

$$\hat{L}_n(s) = \int_0^{\infty} e^{-st} L_n(t) dt = \frac{(s-1)^n}{s^{n+1}} \ , \tag{2.8}$$

the associated Laguerre-series representation for $\hat{f}$ is

$$\hat{f}(s) = \sum_{n=0}^{\infty} q_n \int_0^{\infty} e^{-st} L_n(t) dt = \sum_{n=0}^{\infty} q_n \frac{(s-1)^n}{s^{n+1}} \ , \tag{2.9}$$

where again $q_n$ is the Laguerre coefficient in (2.3) and (2.6).

Above we have used the Laguerre polynomials and the weight $e^{-t}$. An essentially equivalent approach is to use the associated orthonormal Laguerre functions $e^{-t/2}L_n(t)$ without a weight. Although that approach is essentially the same, the resulting coefficients $q_n$ are different.

In order to exploit the Laguerre-series representation for computing the Laplace transform, we primarily work with the Laguerre series representation of $e^t f(t)$. This requires that $e^t f(t)$ be square integrable with respect to the weight $e^{-t}$ or, equivalently, that $e^{t/2}f(t)$ be square integrable directly. That, in turn, clearly requires that $f(t)$ be asymptotically dominated by the exponential $e^{-t/2}$, which means that its Laplace transform $\hat{f}(s)$ should have its rightmost singularity $-s^*$ satisfy $-s^* < -1/2$. This method thus applies to short-tail pdf's, but not long-tail pdf's. In the terminology of Abate, Choudhury and Whitt (1994) and Abate and Whitt (1997), the algorithm applies to pdf's in classes I and II but not class III. Class I contains pdf's with a pure-exponential tail, in particular, pdf's $f$ for which the rightmost singularity of its Laplace transform $\hat{f}$ is at $-s^* < 0$ and $\hat{f}(-s^*) = \infty$. For class II, again $-s^* < 0$, but $\hat{f}(-s^*) < \infty$. The rightmost singularity of $\hat{f}(s)$ is 0 for class III. Class II routinely arises for busy-period distributions and low-priority waiting-time distributions in queueing; e.g., see Abate and Whitt (1997).

**Remark 2.1.** Some pdf's $f$ such as the gamma pdf with shape parameter 1/2, i.e., $\gamma(1/2;t) \equiv e^{-t}/\sqrt{\pi t}$, are not square integrable with respect to the exponential weight because $f(t) \sim t^p$ as $t \to 0$ for $1/2 \le p < 1$. As a consequence, such pdf's do not fit directly into the $L_2$ theory. Indeed, then $\sum_{n=1}^{\infty} q_n^2 = \infty$, as can be seen from Examples 2.1–2.4 of Abate, Choudhury and Whitt (1996). however, the $L_2$ theory can be applied by using the generalized Laguerre polynomials $L_n^{(\alpha)}(t) \equiv \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}t^k/k!$ with respect to the weight $t^\alpha e^{-t}$ for $\alpha = 1$. The expansion for $e^t f(t)$ can then be re-expressed in terms of the standard Laguerre polynomials and coefficients, i.e.,

$$e^t f(t) = \sum_{n=1}^{\infty} q_n^{(1)} L_n^{(1)}(t) = \sum_{n=1}^{\infty} q_n L_n(t) \ , \tag{2.10}$$

using relationships among the Laguerre coefficients and polynomials, i.e., $q_n^{(1)} = q_n - q_{n+1}$, $L_n^{(1)} = -L'_{n+1}(t)$ and $L'_{n+1}(t) = L'_n(t) - L_n(t)$, see p. 241 of Magnus et al. (1966). As a consequence, the direct Laguerre-series representations are still valid, with the understanding that the $L_2$ properties require the generalized Laguerre polynomials. ∎

Even for class I and II pdf's, we find it necessary to scale appropriately so that the rightmost singularity $-s^*$ of the Laplace transform $f(s)$ satisfies $-s^* < -1/2$. Such scaling can easily be done for classes I and II, but cannot be done for class III. Class III can be transformed to satisfy this condition too, e.g., by exponential damping as discussed in Abate, Choudhury and Whitt (1994) and Abate and Whitt (1996), but then the moments and Laguerre coefficients are altered, so we are unable to exploit the transformation. (This point is made in Section 7 of Keilson and Nunn (1979) too.) In contrast, the moments are simply scaled under our linear scaling of class I and II pdf's.

Assuming that we are indeed considering the Laguerre-series representation for $e^t f(t)$, then the Laguerre coefficient $q_n$ becomes

$$q_n = \int_0^\infty f(t) L_n(t) dt \ . \tag{2.11}$$

Since $L_n(t)$ is a polynomial of degree $n$, we see that $q_n$ is a linear combination of the first $n$ moments of $f$. Indeed, we can combine (2.4) and (2.11) to obtain the following result.

**Theorem 2.1.** *If the function $e^t f(t)$ has a Laguerre-series representation with respect to the weight $e^{-t}$ then all moments of $f$ are finite and the $n^{\text{th}}$ Laguerre coefficient $q_n$ of $e^t f(t)$ can be expressed as*

$$q_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{m_k}{k!} \ , \tag{2.12}$$

*where $m_k$ is the $k^{\text{th}}$ moment of $f$ with $m_0 = 1$.*

Since the rightmost singularity is less than $-1/2$, all moments are finite.

**Remark 2.2.** In establishing a connection between moments and Laguerre coefficients, we follow Section 7 of Keilson and Nunn (1979). However, (2.12) is a more elementary relation than their infinite series. We obtain the simplicity by letting $q_n$ be the Laguerre coefficient of $e^t f(t)$ with respect to the weight $e^{-t}$. Both approaches require that the rightmost singularity of the Laplace transform $\hat{f}(s)$ satisfy $-s^* < -1/2$.

**Example 2.1.** To better understand the condition requiring that the rightmost singularity of $\hat{f}(s)$ be less than $-1/2$, consider the case of an exponential pdf with mean 3. The Laplace transform is

$$\hat{f}(s) = (1 + 3s)^{-1} = \sum_{n=0}^{\infty} 3^n (-s)^n \tag{2.13}$$

the rightmost singularity is $-1/3$ and the series has radius of convergence $1/3$. If we were to apply (2.12), we would get

$$q_n = \sum_{k=0}^{n} (-3)^k \binom{n}{k} = (1-3)^n = (-2)^n \ , \tag{2.14}$$

which is growing geometrically in $n$. Moreover, the Laguerre-series representation for $\hat{f}(s)$ would be

$$\hat{f}(s) = \frac{1}{1+s} \sum_{n=0}^{\infty} (-2)^n \left( \frac{s}{1+s} \right)^n \ , \tag{2.15}$$

which is absolutely convergent for $|s| < 1/3$, but fails to converge for real $s$ with $s \geq 1$. ■

With the Laguerre coefficients as in (2.11) and (2.12), we obtain

$$f(t) = e^{-t} \sum_{n=0}^{\infty} q_n L_n(t) \tag{2.16}$$

and

$$\hat{f}(s) = \sum_{n=0}^{\infty} q_n \frac{s^n}{(s+1)^{n+1}} \ . \tag{2.17}$$

It is noteworthy that the convergence of the transform series (2.17) tends to be faster than the convergence of the basic Laguerre series (2.16), because $L_n(t)$ converges to 0 slowly as $n \to \infty$. Indeed,

$$L_n(t) \sim \frac{e^{t/2}}{(\pi^2 nt)^{1/4}} \cos(2\sqrt{nt} - \pi/4) \quad \text{as} \quad n \to \infty \ , \tag{2.18}$$

see p. 245 of Magnus, Oberhettinger and Soni (1966). The Laguerre series in (2.17) tends to converge geometrically for all $s$ with $\mathrm{Re}(s) > 0$ because the Laguerre coefficients are bounded and $|s/(s+1)| < 1$ for $\mathrm{Re}(s) > 0$. The Laguerre coefficients are bounded because $|L_n(t)| \leq e^{t/2}$ for $t \geq 0$, see 22.14.12 on p. 786 of Abramowitz and Stegun (1972), and

$$|q_n| \leq \int_0^{\infty} |f(t)||L_n(t)|dt \leq \int_0^{\infty} e^{t/2}|f(t)|dt \equiv M < \infty \ . \tag{2.19}$$

Using the Hilbert space theory of the function space $L_2[0,\infty)$, we know that, with $g(t) = e^t f(t)$,

$$||g||_2^2 \equiv \int_0^{\infty} e^t f(t)^2 dt = \sum_{n=0}^{\infty} q_n^2 < \infty \ , \tag{2.20}$$

which implies that $q_n \to 0$. Hence the series (2.17) indeed converges geometrically fast.

It is important to note, however, that when we do numerical inversion, we need to compute the transform $\hat{f}(s)$ not for a single value of $s$ but for several values of $s$. If we use the Fourier-series method in Abate and Whitt (1992, 1995), then we need to consider values of $s = u + iv$ with $u = A/2t$ and $v = k\pi/t$ for $A = 30$, say, and $k \geq 1$. A serious difficulty occurs because $|s/(1+s)| \to 1$ as $v \to \infty$ and, thus, as $k \to \infty$. Thus, the Euler summation used in the Fourier-series method helps greatly, because it enables us to restrict attention to $k \leq 40$, say. Then $u = 15/t$ and $v \approx 125/t$.

We now give a bound on the error from truncation.

**Theorem 2.2.** *Let $M$ be the bound on $|q_n|$ in (2.19). For $s = u + iv$ with $u > 0$ and each $N \geq 1$,*

$$|\hat{f}(s) - \hat{f}_N(s)| \leq M \left|\frac{1}{s+1}\right| \frac{\epsilon^N}{1-\epsilon} , \tag{2.21}$$

*where*

$$f_N(s) = \sum_{n=0}^{N} \frac{1}{s+1} q_n \left(\frac{s}{s+1}\right)^n \tag{2.22}$$

*with $q_n$ begin the Laguerre coefficients in (2.11) and (2.12), and*

$$\epsilon = \left(\frac{\beta^4 + \beta^2(1+\alpha^2) + \alpha^2}{\beta^4 + 2\beta^2 + 1}\right)^{1/2} < 1 \tag{2.23}$$

*with $\beta = v/(1+u)$ and $\alpha = u/(1+u)$.*

**Proof.** If $s = u + vi$ for real $u$ and $v$, then

$$z \equiv \frac{s}{s+1} = \frac{A + Bi}{C + B_i} \tag{2.24}$$

for $A = u$, $B = v$ and $C = u + 1$. If (2.24) holds with $0 \leq |A| < C$, then

$$|z|^2 = \frac{\beta^4 + \beta^2(1+\alpha^2) + \alpha^2}{\beta^4 + 2\beta^2 + 1} < 1 \tag{2.25}$$

for $\beta = B/C$ and $\alpha = |A|/C$. ■

Now let us consider again the Fourier-series method of numerical transform inversion with Euler summation. We have observed that we need to consider $s = u + iv$ with $u = 15/t$ and $v \approx 125/t$. From (2.23) we see that

$$\left|\frac{s}{s+1}\right| \approx 1 - \frac{u}{v^2} \quad \text{when} \quad u << v , \tag{2.26}$$

so that here

$$\left|\frac{s}{s+1}\right| \approx 1 - t10^{-3} \tag{2.27}$$

in the worst case. Hence, to have truncation error $|\hat{f}_N(s) - \hat{f}(s)| \leq \delta M$, we need

$$N \approx \frac{\log(\delta t 10^{-3})}{\log(1 - t10^{-3})} \approx \frac{-\log(\delta t 10^{-3})}{t10^{-3}} . \tag{2.28}$$

For example, for $\delta = 10^{-15}$ and $M = 1$, we require $N \approx 4.4t^{-1} \times 10^4$. Thus, for $t = 1$, we require $N = 44,000$. The resulting computation is feasible, but not easy. Hence, if alternative methods are available, then they might well be perferred.

To directly apply (2.17) for computing transform values given a pdf $f$, we can compute the Laguerre coefficients $q_n$ associated with $e^t f(t)$ for the given pdf $f$ by numerically integrating the integral in (2.11) and then approximate the sum in (2.17) by $\hat{f}_N(s)$ in (2.22) to achieve desired error in (2.21). However, it is often possible to obtain the coefficients more easily from the moments of $f$ using (2.12).

To apply Theorem 2.1, we can first scale the transform so that it has rightmost singularity $-1$. This is achieved by considering the scaled Laplace transform $\hat{f}_{s^*}(s) \equiv \hat{f}(s^*s)$. In terms of a random variable $X$ with pdf $f$ and moments $m_n$, this corresponds to replacing $X$ by $X/s^*$, which has pdf $s^*f(s^*t)$, $t \geq 0$, and moments $m_n/(s^*)^n$. For the exponential pdf with mean 3 in Example 2.1, the rightmost singularity is $-s^* = -1/3$. The scaling converts the pdf to an exponential pdf with mean 1. However, for nonexponential pdf's, we usually do *not* have $s^* = 1/m_1$.

To carry out the scaling, we need to determine the rightmost singularity $-s^*$. Fortunately, algorithms to compute $s^*$ from the moments are contained in Abate, Choudhury, Lucantoni and Whitt (1995), where $s^*$ is denoted $\eta$. Under considerable generality,

$$s^* = \lim_{n\to\infty} \frac{nm_{n-1}}{m_n} \ , \tag{2.29}$$

but the proposed algorithms often do much better by exploiting Richardson extrapolation; see p. 987 of Abate, Choudhury, Lucantoni and Whitt (1995).

It is convenient that we can relate the Laguerre coefficients of a pdf $f(t)$ to its complementary cdf (ccdf) $F^c(t) \equiv 1 - F(t)$ directly.

**Theorem 2.3.** *If $e^t f(t)$ has a Laguerre-series representation for a pdf $f(t)$, then so does $e^t F^c(t)$ for the associated ccdf $F^c(t)$, and the expansions are related by*

$$f(t) = e^{-t} \sum_{n=0}^{\infty} q_n L_n(t) \tag{2.30}$$

*and*

$$F^c(t) = e^{-t} \sum_{n=0}^{\infty} (q_n - q_{n+1}) L_n(t) \ . \tag{2.31}$$

**First Proof.** Using Laplace transforms,

$$\begin{aligned} \hat{F}^c(s) &= \frac{1-\hat{f}(s)}{s} = \frac{1}{s} + \hat{f}(s) - \left(\frac{s+1}{s}\right)\hat{f}(s) \\ &= \frac{1}{1+s}\sum_{n=0}^{\infty} q_n \left(\frac{s}{s+1}\right)^n - \frac{1}{1+s}\sum_{n=0}^{\infty} q_{n+1}\left(\frac{s}{s+1}\right)^n \end{aligned}$$

using the fact that

$$q_0 = \int_0^\infty f(t)L_0(t)dt = \int_0^\infty f(t)dt = 1 \ .$$

**Second Proof.** Use one of the basic differentiation formulas for Laguerre polynomials

$$L_n'(t) = \frac{nL_n(t) - nL_{n-1}(t)}{t} \tag{2.32}$$

or

$$L_n'(t) - L_{n+1}'(t) = L_n(t) \tag{2.33}$$

e.g., see p. 241 of Magnus, Oberhettinger and Soni (1966). ■

We now give two examples illustrating how the Laguerre-series representation of the transform obtained from the moments can be helpful.

**Example 2.2.** In Theorem 5.4 of Abate and Whitt (1998) we indicated how new pdf's can be constructed from others using their moment sequences. In particular, suppose that we have two pdf's $f(t)$ and $g(t)$ where their moments are related by

$$m_n(G) = \frac{m_n(F)}{2n+1}, \quad n \geq 1 \ . \tag{2.34}$$

Then their Laplace transform are related by

$$\hat{g}(s) = \frac{1}{2\sqrt{s}} \int_0^s \frac{\hat{f}(z)}{\sqrt{z}} dz \ . \tag{2.35}$$

If both the moments and the Laplace transform of $f$ are known, it may be easier to compute $\hat{g}(s)$ using the Laguerre-series representation based on Theorem 2.1 and (2.34) than to perform the integration in (2.35).

**Example 2.3.** An interesting example from p. 86 of Abate and Whitt (1996) and p. 986 of Abate, Choudhury, Lucantoni and Whitt (1995) is the Cayley-Einstein-Polya (CEP) pdf with mean 1 and Laplace transform satisfying the functional equation

$$\hat{f}(s) = \exp(-s\hat{f}(s)) \ . \tag{2.36}$$

Evidently, the associated pdf is unknown, but the moments are $m_n = (n+1)^{n-1}$, $n \geq 1$. The ccdf has the asymptotic form

$$F^c(t) \sim \sqrt{\frac{e^5}{2\pi}} t^{-3/2} e^{-\eta t} \quad \text{as} \quad t \to \infty \tag{2.37}$$

for $\eta = 1/e$. Even though the mean is $m_1 = 1$, the rightmost singularity of $\hat{f}(s)$ is $-s^* = -e^{-1} > -1/2$, so that scaling is needed in this case. As indicated earlier, the final decay rate has to be at least 1/2. From numerical experience, we found that the scaling becomes more effective in this example as the final decay rate approaches 1/2. Hence, we use the scale parameter $\sigma = e/2$, making the ccdf $G^c(t) \equiv F^c(\sigma t)$ with moments $m_n(G) = (n+1)^{n+1}(2/e)^n$. Then

$$G^c(t) \sim \frac{2e}{\sqrt{\pi}} t^{-3/2} e^{-t/2} \quad \text{as} \quad t \to \infty \tag{2.38}$$

and

$$m_n(G) \sim \frac{2e}{\sqrt{\pi}} (2n)^{-3/2} 2^n \quad \text{as} \quad n \to \infty \ . \tag{2.39}$$

For the ccdf, we obtain the Laguerre-series expansion

$$G^c(t) = \sum_{n=0}^{\infty} (q_n - q_{n+1}) L_n(t) \tag{2.40}$$

where $q_n$ can be obtained from the moments using (2.12).

We now compute $G^c(t)$ for several values of $t$ using the method of moments. As in Abate, Choudhury and Whitt (1996), we also use Wynn's (1956, 1966) $\epsilon$-algorithm to accelerate convergence. In particular, we use the fourth-order version; i.e., we use $e^n_{2m}$ for $m = 4$ and $n = 80$. This requires that we compute $q_n$ for $0 \le n \le 89$. To determine the correct digits, we redo the computation for $n = 100$, and keep only the digits that agree.

The numerical results are displayed in Table 1. Table 1 shows that we obtain four correct digits except for the largest values of $t$ considered, $t = 30$. The asymptotic values from (2.38) are also displayed in Table 1. As should be anticipated, the asymptotics do not perform too well for the displayed times because the next term is smaller only by a factor $t^{-1}$; see Abate and Whitt (1997) for further discussion. However, the asymptotics would be useful for larger times.

| time $t$ | $G^c(t)$ | asymptotic values in (2.38) |
|---|---|---|
| .1 | .7563 | |
| .5 | .4110 | |
| 1 | .2346 | |
| 2 | .9334 $e-1$ | |
| 3 | .4178 $e-1$ | |
| 5 | .9843 $e-2$ | |
| 7 | .2599 $e-2$ | .501 $e-2$ |
| 10 | .3957 $e-3$ | .654 $e-3$ |
| 15 | .2029 $e-4$ | .292 $e-4$ |
| 20 | .1169 $e-5$ | .156 $e-5$ |
| 25 | .7224 $e-7$ | .914 $e-7$ |
| 30 | .466 $e-8$ | .571 $e-8$ |

Table 1: Numerical values of the CEP ccdf computed from the Laguerre-series representation using the method of moments.

Because of the combinatorial sum, the calculation of $q_n$ from the moments requires about $(n^{-2}2^{5n/2}+15)$ digits of precision. Hence, for $n=80$, we need about 70 digits precision. For this purpose, we used UBASIC; see Kida (1990).

## 3. Mixtures of Exponential Distributions

In this section we obtain some results about Laguerre-series representations for Laplace transforms of completely monotone pdf's, i.e., for pdf's that are mixtures of exponential pdf's. We first obtain some general results. Then we describe an application of the method of moments in Theorem 2.1 to obtain a Laguerre-series representation for a class of pdf's called a *beta mixture of exponential* (BME) pdf's. These BME pdf's are investigated further in Abate and Whitt (1998). The important point here is that the method of moments can be applied *analytically* to determine simple expressions for the Laguerre coefficients.

We first establish integral representations for the Laguerre coefficients for general mixtures of exponentials.

**Theorem 3.1.** *If a pdf $f$ can be represented as*

$$f(t)=\int_a^b xe^{-xt}dG(x) \tag{3.1}$$

*for a cdf $G$, where $1/2<a<b\le\infty$, then $e^t f(t)$ has a Laguerre-series representation with Laguerre coefficients*

$$q_n=\int_a^b\left(\frac{x-1}{x}\right)^n dG(x)\ . \tag{3.2}$$

**Proof.** First, because of the exponentials in (3.1), the rightmost singularity of $\hat{f}(s)$ satisfies $-s^*<-1/2$. We exploit the fact that the Laplace transform of $L_n(t)$ is $(s-1)^n/s^{n+1}$. We change the order of integration to obtain

$$\begin{aligned}
q_n &= \int_0^\infty e^{-t}(e^t f(t))L_n(t)dt \\
&= \int_a^b\left[\int_0^\infty xe^{-xt}L_n(t)dt\right]dG(x) \\
&= \int_a^b\left(\frac{x-1}{x}\right)^n dG(x)\ .
\end{aligned}$$

**Corollary.** *If a pdf $f$ can be represented as*

$$f(t)=\int_c^d y^{-1}e^{-t/y}w(y)dy \tag{3.3}$$

*for $0 \leq c < d < 2$, where $w$ is a pdf, then $e^t f(t)$ has a Laguerre-series representation with Laguerre coefficients*

$$q_n = \int_{d^{-1}}^{c^{-1}} \left(\frac{x}{1-x}\right)^n x^{-2} w(1/x) dx \ . \tag{3.4}$$

**Proof.** Make the change of variables $x = 1/y$ to go from the mixture in (3.3) to

$$f(t) = \int_{d^{-1}}^{c^{-1}} x e^{-xt} x^{-2} w(1/x) dx \ .$$

Then apply Theorem 3.1.

We next characterize Laguerre coefficients of mixtures of exponentials where the mixing pdf has support $[0,1]$. Recall that mixtures can be represented as products of independent random variables.

**Theorem 3.2.** *Let $X$ and $Y$ be independent random variables with $X$ exponentially distributed having mean 1 and $Y$ having support on $[0,1]$. If $f$ is the pdf of $XY$, then $e^t f(t)$ has a Laguerre-series representation with the $n^{\text{th}}$ Laguerre coefficient $q_n$ equal to the $n^{\text{th}}$ moment of $(1-Y)$. Hence $1 = q_0 \geq q_n \geq q_{n+1}$ for all $n$ and $q_n \to 0$.*

**Proof.** The rightmost singularity of the Laplace transform is less than or equal to $-1$, so that the Laguerre-series representation exists. By Theorem 2.1, the Laguerre coefficient is

$$q_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{m_k}{k!} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k y_k$$

where $m_k$ $(y_k)$ is the $k^{\text{th}}$ moment of $XY$ $(Y)$. The $n^{\text{th}}$ moment of $1-Y$ clearly has the final expression too.

**Example 3.1.** Consider the pdf

$$f(t) = \int_0^1 y^{-1} e^{-t/y} w(y) dy \ , \tag{3.5}$$

where $w(y) = -\log(1-y)$, $0 \leq y \leq 1$. The $n^{\text{th}}$ moment of the pdf $w(1-y) = -\log y$ has $n^{\text{th}}$ moment $(n+1)^{-2}$ by 4.1.50 of Abramowitz and Stegan (1972), so that the Laguerre coefficient of $e^t f(t)$ is $q_n = (n+1)^{-2}$ by Theorem 3.2.

**Corollary.** *If, in addition to the assumptions of Theorem 3.2, the mixing pdf $w(y)$ of $Y$ is symmetric on $[0,1]$, i.e., if $w(y) = w(1-y)$, $0 \le y \le 1$, then the $n^{\text{th}}$ Laguerre coefficient $q_n$ of $e^t f(t)$ equals the $n^{\text{th}}$ moment of $Y$.*

**Example 3.2.** Consider the pdf (3.5), where the mixing pdf $w(y)$ is uniform on $[0,1]$. By the Corollary to Theorem 3.2, $q_n = (n+1)^{-1}$.

We now obtain even more explicit representations for the Laguerre coefficients of $e^t f(t)$ for the class of BME pdf's. A BME pdf can be expressed as

$$v(p,q;t) = \int_0^1 y^{-1} e^{-t/y} b(p,q;y)dy, \quad t \ge 0 , \tag{3.6}$$

where $b(p,q;y)$ is the standard beta pdf, i.e.,

$$b(p,q;y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 \le y \le 1 , \tag{3.7}$$

and $\Gamma(x)$ is the gamma function. A BME pdf has three parameters $p > 0$, $q > 0$ and a third positive scale parameter, which has been omitted from (3.6). (The parameter $q$ should not be confused with the Laguerre coefficients $q_n$, which have the subscript.) In general, a more explicit form for the BME pdf is not known, but special cases are described in Abate and Whitt (1998). Especially tractable are the BME pdf's where both $p$ and $q$ are integer multiples of 1/2.

The Laplace transform $\hat{v}(p,q;s)$ has rightmost singularity $-s^* = -1$. The BME pdf is class II with asymptotic form

$$v(p,q;t) \sim \frac{\Gamma(p+q)e^{-t}}{\Gamma(p)t^q} \quad \text{as} \quad t \to \infty \tag{3.8}$$

and $n^{\text{th}}$ moment

$$m_n(p,q) = \frac{(p)_n n!}{(p+q)_n} \tag{3.9}$$

where $(x)_n = x(x+1)\ldots(x+n-1)$ is the Pochammer symbol with $(x)_0 = 1$.

Theorem 2.1, (3.9) and a combinatorial identity, (7.1) on p. 58 of Gould (1972), yields the Laguerre coefficients in (2.11), as shown in Theorem 2.2 and 2.3 of Abate and Whitt (1998); in particular,

$$q_n = \frac{(q)_n}{(p+q)_n}, \quad n \ge 0 . \tag{3.10}$$

Since $1-Y$ has a beta pdf with parameter pair $(q,p)$ when $Y$ has a beta pdf with parameter pair $(p,q)$, we could also apply Theorem 3.2 to obtain (3.10).

Given the explicit expression for the Laguerre coefficients $q_n$ in (3.10), we can apply the series representation (2.17) to compute the BME transform values, but in this instance it turns out to be more effective to use continued fractions; see Abate and Whitt (1998a).

We can use the Laguerre-series representation for the BME pdf's to obtain Laguerre-series representations for some related pdf's. In Abate and Whitt (1998) a second class of beta mixtures of exponential ($B_2$ME) is defined with pdf

$$v_2(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\infty y^{-1} e^{-t/y} y^{p-1} (1+y)^{-(p+q)} dy \ . \tag{3.11}$$

In (3.11) the mixing pdf is a beta pdf of the second kind.

However, by (1.21) of Abate and Whitt (1998), the $B_2$ME pdf is an exponentially undamped version of a BME pdf, i.e.,

$$v_2(p,q;t) = \frac{q}{p+q} e^t v(p,q+1,t), \quad t \ge 0 \ , \tag{3.12}$$

so that

$$v_2(p,q;t) \sim \frac{q\Gamma(p+q)}{\Gamma(p)t^{q+1}} \quad \text{as} \quad t \to \infty \ . \tag{3.13}$$

From (3.13), we see that the $B_2$ME pdf $v_2(p,q;t)$ is a long-tail (class III) pdf. From (3.13) we can obtain Laguerre-series representations for $v_2(p,q;t)$ and its Laplace transform $\hat{v}_2(p,q;s)$; in particular, from (3.10) and (3.12), we obtain

$$v_2(p,q,t) = \sum_{n=0}^\infty \left(\frac{q}{p+q}\right) \frac{(q+1)_n}{(p+q+1)_n} L_n(t) \tag{3.14}$$

and

$$\hat{v}_2(p,q;s) = \sum_{n=0}^\infty \left(\frac{q}{p+q}\right) \frac{(q+1)_n}{(p+q+1)_n} \frac{(s-1)^n}{s^{n+1}} \ . \tag{3.15}$$

Unfortunately, however, as noted in Section 2, the Laguerre-series representation (3.15) is not as useful as (2.17) with (3.10). Because of the undamping, $s$ in (2.17) has been shifted to $s-1$.

## 4. The Erlang-Series Representation

In this section, following Section 4 of Keilson and Nunn (1979), we consider another series representation for the Laplace transform $\hat{f}$ of a pdf $f$ based on a Taylor series expansion of $g(t) \equiv e^t f(t)$, i.e.,

$$g(t) \equiv e^t f(t) = \sum_{n=0}^\infty g^{(n)}(0) \frac{t^n}{n!}, \quad t \ge 0 \ , \tag{4.1}$$

where $g^{(n)}(0)$ is the $n^{\text{th}}$ (right) derivative of $g$ evaluated at 0 and the coefficients $f^{(n)}(0)$ and $g^{(n)}(0)$ are related by

$$g^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(0) \ . \tag{4.2}$$

Then $f(t)$ can be rewritten as

$$f(t) = \sum_{n=1}^{\infty} g^{(n-1)}(0) e(n;t), \quad t \geq 0 \ , \tag{4.3}$$

where, for $n \geq 1$, $e_n(t)$ is the Erlang ($E_n$) pdf

$$e(n;t) = \frac{e^{-t} t^{n-1}}{(n-1)!}, \quad t \geq 0 \ , \tag{4.4}$$

which has mean $n$, variance $n$ and Laplace transform $\hat{e}(n;s) = (1+s)^{-n}$. As $n$ increases $e(n;t)$ approaches a normal cdf with mean and variance $n$. The squared coefficient of variation (SCV, variance divided by the square of the mean) of $e(n;t)$ is $1/n$, showing that the variability is asymptotically negligible compared to the mean.

From (4.3) we immediately obtain an associated Erlang-series representation for the Laplace transform, i.e.,

$$\hat{f}(s) = \sum_{n=1}^{\infty} g^{(n-1)}(0)(1+s)^{-n} \ . \tag{4.5}$$

Closely paralleling Theorem 2.3, we can easily relate the Erlang-series representation of pdf's and cdf's.

**Theorem 4.1.** *Let $g$ be a pdf with ccdf $G^c$. Then the following are equivalent:*

*(i)* $G^c(t) = \sum\limits_{n=1}^{\infty} c_n e_n(n;t)$

*(ii)* $g(t) = \sum\limits_{n=1}^{\infty} (c_n - c_{n+1}) e(n;t)$.

**Proof.** Note that $\hat{g}(s) = 1 + G^c(s) - (s+1)G^c(s)$. Hence,

$$G^c(s) = \sum_{n=1}^{\infty} c_n (1+s)^{-n}$$

if and only if

$$\hat{g}^c(s) = \sum_{n=1}^{\infty} d_n (1+s)^{-n}$$

with $\sum_{n=1}^{\infty} d_n = c_1$ for the coefficients as indicated.

Remarkably, many exponential pdf's can be represented as a mixtures of Erlangs.

**Theorem 4.2.** *The Erlang coefficients are $c_n = \rho(1-\rho)^{n-1}$ for $\rho > 0$, if and only if the pdf $f$ is exponential with mean $1/\rho$.*

**Proof.** Starting with the exponential pdf $\rho e^{-\rho t}$ with $\rho \neq 1$ we get the expansion for the Laplace transform

$$\hat{f}(s) = \frac{\rho}{1-\rho} \sum_{n=1}^{\infty} \frac{(1-\rho)^n}{(1+s)^n} = \frac{\rho}{\rho + s} \ , \tag{4.6}$$

and vice versa. For $\rho = 1$, we get $c_1 = 1$ and $c_n = 0$ for $n \geq 1$. ■

For $\rho > 1$, the coefficients in (4.6) alternate in sign; otherwise they are probabilities.

**Corollary.** *A pdf $f$ is a geometric probabilistic mixture of Erlang pdf's if and only if $f$ is an exponential pdf with mean $m_1 > 1$, i.e., $f(t) = \rho e^{-\rho t}$ for $0 < \rho < 1$.*

In general it is necessary to check that the Erlang-series representation for $\hat{f}(s)$ in (4.5) converges.

**Example 4.1.** Consider the pdf

$$f(t) = (\log 2)^{-1} e^{-t}/(2 - e^{-t}), \quad t \geq 0 \ . \tag{4.7}$$

Using the Taylor series expansion of $(2 - e^{-t})^{-1}$, we would obtain the associated Erlang-series representation

$$f(t) = (\log 2)^{-1} \sum_{k=1}^{\infty} (-1)^{k-1} \tilde{b}(k-1) e(k;t) \ , \tag{4.8}$$

where $\tilde{b}(k)$ are the ordered Bell numbers; see 5.27 on p. 175 of Wilf (1994). The ordered Bell number $\tilde{b}(n)$ is the number of ordered partitions of a set of $n$ elements, where the order of the classes is counted but not the order of the elements in the classes. The associated term-by-term Erlang-series representation for the Laplace transform would be

$$\hat{f}(s) = (\log 2)^{-1} \sum_{k=1}^{\infty} (-1)^{k-1} \tilde{b}(k-1)(1+s)^{-n} \ . \tag{4.9}$$

However, neither (4.8) nor (4.9) is convergent. Thus, we see that the Erlang-series representation for the Laplace transform is not effective in this example.

**Example 4.2.** As a second more favorable example, consider the pdf

$$f(t) = 2e^{-t}(1 - \cos t), \quad t \geq 0 \ , \tag{4.10}$$

as on p. 990 of Abate, Choudhury, Lucantoni and Whitt (1995). Using the Taylor series representation of $1 - \cos t$, we obtain

$$f(t) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} e(2k+1;t), \quad t \geq 0 \ , \tag{4.11}$$

and

$$\hat{f}(s) = 2\sum_{k=1}^{\infty}(-1)^{k+1}(1+s)^{-(2k+1)} \ . \tag{4.12}$$

Unlike in Example 4.1, the Erlang-series representation in (4.12) is (conditionally) convergent for all $s$ with $\mathrm{Re}(s) \geq 0$.

## 5. Mixtures of Exponentials Again

In this section we obtain results about Erlang-series representations when the pdf is a mixture of exponential pdf's. We start by showing that every mixture of exponential pdf's (spectral representation) where the mixing pdf has support $[0, b]$ for $b < \infty$ has an Erlang-series representation and identify the coefficients. First, we note that it suffices to consider mixing pdf's with support $[0, 1]$, because we can always rescale; i.e., if $Z = YX$ where $Y$ has support on $[0, b]$, we consider $(Y/b)$ with support $[0, 1]$ and let $Z^* = (Y/b)X$. We then consider $Z = bZ^* = YX$.

**Theorem 5.1.** *Consider a completely monotone pdf $f(t)$ that has a Laguerre-series representation as in Theorem 3.1, i.e.,*

$$f(t) = \int_a^b y^{-1}e^{-t/y}dH(y) = e^{-t}\sum_{n=0}^{\infty} q_n L_n(t) \ , \tag{5.1}$$

*where $H$ is a cdf with support in $[a, b]$ with $0 \leq a < b < 2$. If*

$$G^c(t) = \int_a^b e^{-tx}dH(x), \quad t \geq 0 \ , \tag{5.2}$$

*Then $G^c(t)$ has the Erlang-series representation*

$$G^c(t) = \sum_{k=1}^{\infty} q_{k-1}e(k;t), \quad t \geq 0 \ , \tag{5.3}$$

*and $G^c(t)$ has a pdf $g(t)$ with*

$$g(t) = \sum_{k=1}^{\infty}(q_{k-1} - q_k)e(k;t) \ . \tag{5.4}$$

To prove Theorem 5.1, we use the following lemma.

**Lemma 5.1.** *If (5.2) and the first relation (5.1) hold, where $0 \leq a < b \leq \infty$, then $\hat{G}^c(s) = s^{-1}\hat{f}(s^{-1})$.*

**Proof.** Note that

$$\hat{f}(s) = \int_a^b (1+sy)^{-1} dH(y)$$

and

$$G^c(s) = \int_a^b (x+s)^{-1} dH(y) = \frac{1}{s}\int_a^b (1+xs^{-1})^{-1} dH(x) \ .$$

**Proof of Theorem 5.1.** From Lemma 5.1,

$$\hat{G}^c(s) = s^{-1}\hat{f}(s^{-1}) = \int_0^\infty (s^{-1}e^{-x/s}) f(x)dx \ ,$$

so that

$$G^c(t) = \int_0^\infty \mathcal{L}^{-1}(s^{-1}e^{-x/s}) f(x)dx = \int_0^\infty J_0(2\sqrt{xt}) f(x)dx \tag{5.5}$$

where

$$J_0(2\sqrt{xt}) = e^{-t}\sum_{n=0}^\infty \frac{t^n}{n!} L_n(t) \ ; \tag{5.6}$$

see 29.3.75 and 22.9.16 of Abramowitz and Stegun (1972). Therefore,

$$G^c(t) = e^{-t}\sum_{k=0}^\infty \frac{q_k t^k}{k!} = \sum_{k=1}^\infty q_{k-1} e(k;t)$$

for $q_k$ in (2.11).

**Remark 5.1.** We use the condition $b < 2$ to apply Theorem 3.1 to have the Laguerre-series representation in (5.1). As remarked before Theorem 5.1, we can rescale to have $b < 2$ or $b = 1$ if $b < \infty$. By Theorem 3.2, for the special case in which $b \leq 1$, we have $1 = q_0 \geq q_1 \geq q_n \geq q_{n+1}$ for all $n$.

**Second Proof of Theorem 5.1 for $b = 1$.** Working with the density $g(t)$ of $G^c(t)$ in (5.2), we have

$$\begin{aligned} g(t) &= e^{-t}\int_0^1 xe^{-(1-x)t} h(x)dx \\ &= e^{-t}\int_0^1 \sum_{n=0}^\infty \frac{x(1-x)^n t^n}{n!} h(x)dx \\ &= \sum_{n=1}^\infty \frac{e^{-t}t^{n-1}}{(n-1)!}\int_0^1 x(1-x)^n h(x)dx \\ &= \sum_{n=1}^\infty e(n;t)\int_0^1 (1-x)x^n h(1-x)dx \ . \end{aligned} \tag{5.7}$$

On the other hand, the Laplace transform $\hat{f}(s)$ of $f$ in (5.1) can be expressed as

$$\begin{aligned}\hat{f}(s) &= \int_0^\infty e^{-st} \int_0^1 y^{-1} e^{-t/y} h(y) dy = \int_0^1 \frac{1}{1+sy} h(y) dy \\ &= \int_0^1 \frac{1}{1+s(1-z)} h(1-z) dz = \frac{1}{1+s} \int_0^1 \frac{1}{1+z\left(\frac{-s}{1+s}\right)} h(1-z) dz \\ &= \sum_{n=0}^\infty \frac{1}{1+s} \left(\frac{s}{1+s}\right)^n \int_0^1 z^n h(1-z) dz \ .\end{aligned}$$

Hence,

$$f(t) = \sum_{n=0}^\infty q_n e^{-t} L_n(t)$$

for

$$q_n = \int_0^1 (1-z)^n h(z) dz \ . \tag{5.8}$$

The result follows by comparing the coefficients. ∎

**Remark 5.2.** We need the condition that the pdf $f(t)$ be completely monotone in order for $G^c(t)$ to be a bonafide ccdf. To see this, suppose that

$$f(t) = \sum_{n=0}^\infty \frac{f^{(n)}(0) t^n}{n!} \tag{5.9}$$

and that its transform can be computed term by term, i.e.,

$$\hat{f}(s) = \sum_{n=0}^\infty f^{(n)}(0) s^{-(n+1)} \ ; \tag{5.10}$$

see Section 30 of Doetsch in (1974) for conditions. Also suppose that $\hat{G}^c(s)$ is analytic at $s = 0$, so that

$$G^c(s) = \sum_{n=0}^\infty \frac{m_{n+1}}{(n+1)!} (-s)^n \ . \tag{5.11}$$

The representations (5.10) and (5.11) imply that

$$(-1)^n f^{(n)}(0) = \frac{m_{n+1}(G)}{(n+1)!} > 0 \tag{5.12}$$

so that $f(t)$ must be completely monotone at 0.

**Example 5.1.** In this example, the conditions of Theorem 5.1 and Lemma 5.1 are not satisfied, so we run into difficulties. Consider Feller's second Bessel pdf for the case $r = 0$, as discussed in Section 11 of Abate and Whitt (1996),

$$f(t) = e^{-(1+t)} I_0(2\sqrt{t}), \quad t \geq 0 \ , \tag{5.13}$$

which is not completely monotone, because $f''(t) \sim -(1-4t/3)/2$ as $t \to 0$. We can see that $f(t)$ has a generalized Laguerre-series representation from the Laplace transform

$$\hat{f}(s) = \frac{1}{1+s}\exp(-s/(1+s)) = \frac{1}{1+s}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{s}{1+s}\right)^n \ . \tag{5.14}$$

We can construct the transform

$$\hat{G}^c(s) = s^{-1}\hat{f}(s^{-1}) = \frac{1}{1+s}\exp(-1/(1+s)) \tag{5.15}$$

which has inverse

$$G^c(t) = e^{-t}J_0(2\sqrt{t}), \quad t \geq 0 \ , \tag{5.16}$$

which is not monotone and thus not a bonafide ccdf.

**Example 5.2.** As a quick illustration of Theorem 5.1, we draw on Example 3.1 and let the mixing density be $w(x) = -\log(1-x)$, $0 \leq x \leq 1$. Then (5.1) and (5.3) are valid with $q_n = 1/(n+1)^2$. ■

**Example 5.3.** We now illustrate how we can apply Theorem 5.1 to characterize the dual ccdf $G^c(t)$ defined in (5.2). Suppose that

$$f(t) = \sqrt{2}e^{-3t/2}I_0(t/2), \quad t \geq 0 \ , \tag{5.17}$$

where $I_0(t)$ is the Bessel function. We can then show that

$$f(t) = \int_{1/2}^{1} y^{-1}e^{-t/y}\frac{dy}{\pi\sqrt{1-y}\sqrt{y-1/2}} = e^{-t}\sum_{n=0}^{\infty}8^{-n}\binom{2n}{n}L_n(t) \ . \tag{5.18}$$

Thus the dual ccdf is

$$G^c(t) = \int_{1/2}^{1} e^{-tx}\frac{dx}{\pi\sqrt{1-x}\sqrt{x-1/2}} = \int_{1}^{2} e^{-t/y}\frac{dy}{\pi y\sqrt{y-1}\sqrt{1-y/2}} \ . \tag{5.19}$$

From Theorem 5.1,

$$G^c(t) = \sum_{k=1}^{\infty}8^{-(k-1)}\binom{2k-2}{k-1}e(k;t) \ . \tag{5.20}$$

By Lemma 5.1,

$$\hat{G}^c(s) = s^{-1}\hat{f}(s^{-1}) = \frac{1}{\sqrt{1+s}\sqrt{(1/2)+s}} \tag{5.21}$$

from which we can deduce that

$$G^c(t) = e^{-3t/4}I_0(t/4), \quad t \geq 0 \tag{5.22}$$

see 29.3.49 of Abramowitz and Stegun (1972).

Next we apply Theorem 5.1 to develop an Erlang-series representation for the third beta mixture of exponential ($B_3$ME) pdf

$$v_3(p,q;t) = \int_0^1 xe^{-tx} b(p,q;x)dx \ , \tag{5.23}$$

for $b(p,q;x)$ in (3.7), which is dual to the BME pdf in (3.6); i.e., it is dual in that the exponential means are averaged in (3.6) while the rates are averaged in (5.23), both with respect to the same standard beta pdf. The BME pdf (3.6) has a Laguerre-series representation, but no Erlang-series representation.

Paralleling the connection between the BME pdf and the Tricomi confluent hypergeometric function $U(a,b,z)$ established in Theorem 1.7 of Abate and Whitt (1998), we now establish a connection between the $B_3$ME pdf and the Kummer confluent hypergeometric function $M(a,b,z)$; see Chapter 13 of Abramowitz and Stegun (1972). From the integral representation 13.2.1 there, we obtain the following result.

**Theorem 5.2.** *For all $p > 0$ and $q > 0$,*

$$v_3(p,q;t) = \frac{p}{p+q} M(p+1, p+q+1, -t), \quad t \geq 0 \ . \tag{5.24}$$

**Proof.** Starting from the definition (5.23), we obtain

$$\begin{aligned} v_3(p,q;t) &= \int_0^1 xe^{-tx} b(p,q;x)dx \\ &= \frac{p}{p+q} e^{-t} M(q, p+q+1, t) \\ &= \frac{p}{p+q} M(p+1, p+q+1, -t) \ . \end{aligned}$$

**Remark 6.1.** The $B_3$ME pdf in the form (5.24) was introduced by Mathai and Saxena (1966); see (67) on p. 32 of Johnson and Kotz (1970).

We can apply Theorem 5.2 to describe the asymptotic form of the $B_3$ME tail probabilities. We see that a $B_3$ME pdf has a long tail.

**Corollary.** *For each $p > 0$ and $q > 0$,*

$$v_3(p,q;t) \sim \frac{\Gamma(p+q)p}{\Gamma(q)t^{p+1}} \quad as \quad t \to \infty \ . \tag{5.25}$$

As before, it is significant that we can evaluate the coefficients analytically. The transform series also tend to converge quite rapidly.

**Theorem 5.3.** *For each $p > 0$ and $q > 0$, $v_3(p,q;t)$ has an Erlang-series representation, in particular,*

$$v_3(p,q;t) = p\sum_{n=1}^{\infty} \frac{(q)_{n-1}}{(p+q)_n} e(n;t) \tag{5.26}$$

*and*

$$\hat{v}_3(p,q;s) = p\sum_{n=1}^{\infty} \frac{(q)_{n-1}}{(p+q)_n} (1+s)^{-n} \ . \tag{5.27}$$

**First Proof.** Make the change of variables $x = 1 - y$ in (5.23) to obtain

$$v_3(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} e^{-t} \int_0^1 e^{ty} y^{q-1}(1-y)^p dy \ . \tag{5.28}$$

Now expand $e^{ty}$ in a power series and integrate term by term.

**Second Proof.** Apply (5.4) in Theorem 5.1 with (3.10). We get the Erlang-series representation with

$$q_{n-1} - q_n = \frac{(q)_{n-1}}{(p+1)_{n-1}} - \frac{(q)_n}{(p_q)_n} = \frac{p(q)_{n-1}}{(p+q)_n} \ . \tag{5.29}$$

Let $v_{3e}(p,q;t) \equiv m(p,q)^{-1} V_3^c(p,q;t)$ denote the stationary-excess pdf associated with $v_3(p,q;t)$. Just as for the BME pdf, we can integrate to see that the $\mathrm{B_3ME}$ stationary-excess pdf is again a $\mathrm{B_3ME}$ pdf with different parameters.

**Theorem 5.4.** *For each $p > 1$ and $q > 0$,*

$$v_{3e}(p,q;t) = v_3(p-1,q;t), \quad t \geq 0 \ . \tag{5.30}$$

**Example 5.4.** Consider the BME pdf

$$v(2,1;t) = \int_0^1 y^{-1} e^{-t/y}(2y)dy = e^{-t} \sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)} L_n(t) \ .$$

The dual cdf is

$$V_3^c(2,1;t) = \int_0^1 e^{-tx}(2x)dx = \int_1^{\infty} e^{-t/y} 2y^{-3} dy \ .$$

From Theorem 5.1,

$$V_3^c(2,1;t) = \sum_{k=1}^{\infty} \frac{2}{k(k+1)} e(k;t) \ .$$

Also, since $\hat{v}(2,1;s) = 2s^{-1} - 2s^{-2}\log(1+s)$,

$$\hat{V}_3^c(2,1;s) = 2 - 2s\log(1+s^{-1}) \ ,$$

so that

$$V_3(2,1;t) = 2(1-(1+t)e^{-t}), \quad t \geq 0 \ .$$

We now observe that the special case of a $B_3$ME pdf with $q = 1$ corresponds to a Pareto mixture of exponentials (PME); the PME is a minor modification of PME considered in Abate, Choudhury and Whitt (1994); a scale factor there made the mean 1. For each $r > 0$, let the Pareto ccdf be

$$F^c(r; y) = y^{-r}, \quad y > 1 \ . \tag{5.31}$$

The associated Pareto pdf is

$$f(r; y) = ry^{-(r+1)}, \quad y > 1 \ . \tag{5.32}$$

The $n^{\text{th}}$ moment is $m_n(F) = r/(r-n)$ for $n < r$.

**Theorem 5.5.** *For $q = 1$, the $B_3$ME pdf is the PME pdf with $p = r$, i.e.,*

$$v_3(p, 1; t) = \frac{p\gamma(p+1; t)}{t^{p+1}} = \int_1^\infty y^{-1} e^{-t/y} f_p(y) dy \ , \quad t \geq 0 \ . \tag{5.33}$$

*where $\gamma(p, t) \equiv \int_0^t \mu^{p-1} e^{-\mu} d\mu$ is the incomplete gamma function, and its moments are $m_n(p, 1) = p(n!)/(p-n)$.*

**Proof.** To prove the first relation in (5.33), compare the power series for $\gamma(p+1; t)$ with

$$v_3(p, 1; t) = p \sum_{n=1}^\infty \frac{(n-1)!}{(p+1)_n} e(n; t) = pe^{-t} \sum_{n=1}^\infty \frac{t^{n-1}}{(p+1)_n} \ ; \tag{5.34}$$

see 6.5.4 and 6.5.29 of Abramowitz and Stegun (1972). For the second relation, apply 6.5.2 of Abramowitz and Stegun (1972) to get

$$\begin{aligned} v_3(p, 1; t) &= \frac{p}{t^{p+1}} \int_0^t \mu^p e^{-\mu} d\mu = p \int_0^1 x^p e^{-tx} dx \\ &= p \int_1^\infty e^{-t/y} y^{-(r+2)} dy \ . \quad \blacksquare \end{aligned} \tag{5.35}$$

We now give the full asymptotic form as $t \to \infty$. It is significant that the second term is exponentially damped.

**Corollary.** *For all $p > 0$*

$$v_3(p, 1; t) \sim \frac{p\Gamma(p+1)}{t^{p+1}} - \frac{pe^{-t}}{t} \left( 1 + \frac{p}{t} + \frac{p(p-1)}{t^2} + \cdots \right) \quad \text{as} \quad t \to \infty \ . \tag{5.36}$$

**Proof.** Apply 6.5.3 and 6.5.32 of Abramowitz and Stegun (1972).

We now relate the ccdf to the pdf.

**Theorem 5.6.** *For each* $p > 1$,

$$V_3^c(p,1;t) = \frac{p}{p-1} v_3(p-1,1;t) = \frac{t}{p} v_3(p,1;t) + e^{-t}, \quad t \geq 0 \ . \tag{5.37}$$

**Proof.** Start with the second relation in (5.33). Then use the first relation in (5.33) with 6.5.22 of Abramowitz and Stegun (1972). ∎

## 6. Exponential-Series Representations

We now consider a simple alternative to the Erlang-series representation in Section 5: an exponential-series representation, which is just a countably infinite mixture of exponential pdf's, i.e.,

$$f(t) = \sum_{k=1}^{\infty} p_k \frac{e^{-t/a_k}}{a_k}, \quad t \geq 0 \ , \tag{6.1}$$

and

$$\hat{f}(s) = \sum_{k=1}^{\infty} p_k (1 + s a_k)^{-1} \ , \tag{6.2}$$

where $\{p_k : k \geq 1\}$ is a probability mass function (pmf) and $\{a_k : k \geq 1\}$ is the sequence of means of the component exponential pdf's. The standard case is $a_k < a_{k+1}$ and $a_k \to \infty$ as $k \to \infty$, so that $\hat{f}(s)$ has poles at $-1/a_k$ for all $k$, implying that $\hat{f}(s)$ has a singularity at 0, so that it is a long-tail pdf.

If we truncate the infinite series in (6.1) or (6.2), we obtain a finite mixture of exponential distributions, i.e., a hyperexponential distribution. We consider how to truncate in the next section with the objective of capturing the tail asymptotics at times of interest. Thus this section together with the next constitutes an alternative approach to Feldmann and Whitt (1997).

It is significant that the $n^{\text{th}}$ moment of $f$ in (6.1) can be simply expressed as

$$m_n = n! \sum_{k=1}^{\infty} p_k a_k^n \ . \tag{6.3}$$

By choosing appropriate sequences $\{p_k\}$ and $\{a_k\}$, we can produce desired moment sequences. In many cases we can describe the asymptotic behavior of the pdf and/or its associated ccdf distribution function $F^c(t) \equiv 1 - F(t)$ from the asymptotic behavior of the moments, e.g., by invoking Abate, Choudhury, Lucantoni and Whitt (1995).

**Example 6.1.** Let $p_k = e^{-1}/k!$ and $a_k = k$. Then

$$\frac{m_n}{n!} = e^{-1} \sum_{k=1}^{\infty} \frac{k^n}{k!} = b(n) \ , \tag{6.4}$$

where $b(n) = \{1, 2, 5, 15, 52, 203, \ldots\}$ are the Bell numbers; see p. 20 of Wilf (1994). The Bell number $b(n)$ is the number of ways a set of $n$ elements can be partitioned. The Bell numbers themselves have the relatively simple exponential generating function

$$B(z) \equiv \sum_{n=0}^{\infty} \frac{b(n)}{n!} z^n = e^{e^z - 1} \ , \tag{6.5}$$

from which we can deduce the recurrence relation

$$b(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} b(k) \tag{6.6}$$

with $b(0) = 1$; see p. 23 of Wilf (1994). The discrete distribution with mass $p_k = e^{-1}/k!$ at $k$ has Laplace transform $\hat{b}(s) = B(-s)$; see (3.9) on p. 86 of Abate and Whitt (1996).

We can easily generalize to obtain a two-parameter family; e.g., keep $p_k = e^{-1}/k!$ but now let $a_k = c(k - 1 + a)$. The parameter $c$ is a scale parameter; $m_1 = 1$ if $c = 1/(1+a)$. In general, for this "generalized Bell pdf", the $n^{\text{th}}$ moment is

$$m_n = \frac{n! c^n}{e} \sum_{k=1}^{\infty} \frac{(k+1-a)^n}{k!} \ . \tag{6.7}$$

**Example 6.2.** Let $p^k = 2^{-(k+1)}$ and $a_k = k$, $k \geq 0$. Then the $n^{\text{th}}$ moment is

$$m_n = n! \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = n! \tilde{b}(n) \ , \tag{6.8}$$

where $\{\tilde{b}(n)\}$ is the sequence of ordered Bell numbers as in Example 4.1.

## 7. Truncating Infinite Series

The Erlang-series and exponential-series representations in Sections 4–6 produce infinite series of pdf's, which in applications we will want to truncate. In this section we consider how to truncate. Appropriate truncations depend on the asymptotic form of the pdf or ccdf and the times of interest. We now show how the asymptotic form and the appropriate truncation point (as a function of the time of interest) can be identified by applying the Euler-Maclaurin summation formula to approximate the sum by an integral and then Laplace's method to determine the asymptotic behavior of the integral; see pp. 80 and 285 of Olver (1974).

We start with the ccdf

$$F^c(t) = \sum_{n=1}^{\infty} p_n F_n^c(t), \quad t \geq 0 , \tag{7.1}$$

where $F_n^c(t)$ is a ccdf for each $n$, which we rewrite as

$$F^c(t) = \sum_{n=1}^{\infty} e^{-\phi(n,t)} , \tag{7.2}$$

where

$$\phi(n,t) = -\log p_n - \log F_n^c(t) . \tag{7.3}$$

We now regard $\phi(x,t)$ as a continuous function of $x$ and assume that we can approximate the sum by an integral (invoking the Euler-Maclaurin formula to quantify the error), obtaining

$$F^c(t) \approx \int_0^{\infty} e^{-\phi(x,t)} dx . \tag{7.4}$$

We now use Laplace's method on the integral in (7.4). We assume that there is a unique $x^*(t)$ that minimizes $\phi(x,t)$, i.e., for which $\phi'(x^*(t),t) = 0$ and $\phi''(x^*(t),t) > 0$. The idea is that, for suitably large $t$, the integral will be dominated by the contribution in the neighborhood of $x^*(t)$. Then Laplace's method yields

$$\int_0^{\infty} e^{-\phi(x,t)} dx \sim \sqrt{\frac{2\pi}{\phi''(x^*(t),t)}} e^{-\phi(x^*(t),t)} \quad \text{as} \quad t \to \infty . \tag{7.5}$$

In summary, the asymptotic form is given by (7.5) and the truncation point as a function of $t$ is $x^*(t)$. In particular, if we are interested in time $t_0$, then the truncation point should be at least $x^*(t_0)$.

**Example 7.1.** Suppose that the weights in (7.1) are $p_n = (1-\rho)\rho^n$ (a geometric pmf with an atom $(1-\rho)$ at 0) and $F_n$ is exponential with mean $a_n = bn^c$. For the special case $\rho = 1/2$, $b = 1$ and $c = 1$, we obtain the second Bell pdf in Example 6.2. In general, by (7.3),

$$\phi(n,t) = -\log(1-\rho) - n \log \rho + \frac{t}{bn^c} . \tag{7.6}$$

As indicated above, $x^*(t)$ is the root of

$$\phi'(x,t) = -\log \rho - \frac{ct}{bx^{c+1}} = 0 , \tag{7.7}$$

so that

$$x^*(t) = \left(\frac{ct}{b(-\log \rho)}\right)^{1/(c+1)} , \tag{7.8}$$

$$\phi''(x,t) = \frac{c(c+1)t}{bx^{(c+2)}} , \tag{7.9}$$

$$\phi''(x^*(t),t) = (1+c)\left(\frac{b(-\log\rho)^{2+c}}{ct}\right)^{1/(1+c)}, \tag{7.10}$$

and

$$F^c(t) \sim At^{1/2(1+c)}e^{-Bt^{1/(1+c)}} \quad \text{as} \quad t\to\infty, \tag{7.11}$$

where

$$A = (1-\rho)\left(\frac{2\pi c^{1/(1+c)}}{(1+c)b^{1/(1+c)}(-\log\rho)^{(2+c)/(1+c)}}\right)^{1/2} \tag{7.12}$$

and

$$B = \left(1+\frac{1}{c}\right)\left(\frac{(-\log\rho)^c c}{b}\right)^{1/(1+c)}. \tag{7.13}$$

In the special case $\rho = 1/2$, $b = c = 1$ corresponding to Example 6.2,

$$F^c(t) \sim At^{1/4}e^{-Bt^{1/2}} \quad \text{as} \quad t\to\infty, \tag{7.14}$$

where

$$A = \frac{\sqrt{\pi}}{2(\log 2)^{3/4}} \quad \text{and} \quad B = 2\sqrt{\log 2}. \tag{7.15}$$

For the case of Example 6.2 we can verify the asymptotic by applying Section 6 of Abate, Choudhury, Lucantoni and Whitt (1995), but we must correct errors in formulas (6.8) and (6.9) there. Assuming that $F^c(t) \sim \alpha t^\beta e^{-\eta t^\delta}$ as $t\to\infty$, we obtain estimators for the parameters $\alpha$, $\beta$, $\eta$ and $\delta$ from the moments. First, by p. 176 of Wilf (1994), the ordered Bell numbers satisfy $\tilde{b}(n) \sim n!/2a^{n+1}$ as $n\to\infty$ for $a = \log 2$. Therefore, by (6.8),

$$m_n \sim \frac{1}{2\log 2}\frac{(n!)^2}{(\log 2)^n} \quad \text{as} \quad n\to\infty, \tag{7.16}$$

$$r_n \equiv \frac{m_n}{m_{n-1}} \sim \frac{n^2}{a} \quad \text{as} \quad n\to\infty, \tag{7.17}$$

$$\delta_n \equiv \frac{r_n}{r(r_{n+1}-r_n)} \to \frac{1}{2} \quad \text{as} \quad n\to\infty, \tag{7.18}$$

$$\eta_n \equiv \frac{n}{\delta r_n^\delta} \to 2\sqrt{a} \quad \text{as} \quad n\to\infty, \tag{7.19}$$

$$\beta_n \equiv \delta\left((\eta\delta/n)^{1/\delta}nr_n - n + (\delta^{-1}-1)/2\right) \to 1/4 \quad \text{as} \quad n\to\infty, \tag{7.20}$$

$$\alpha_n \equiv \frac{m_n\eta^{(n+\beta)\delta}}{\Gamma((n+\beta)/\delta+1)} \to \frac{\sqrt{\pi}}{2a^{3/4}} \quad \text{as} \quad n\to\infty. \tag{7.21}$$

Note that (7.18)–(7.21) agree with (7.15). Formulas (7.20) and (7.21) here correct (6.8) and (6.9) in Abate, Choudhury, Lucantoni and Whitt (1995).

**Example 7.2.** We now produce a pdf with a power tail. Suppose that $a_n = n^\gamma$ for $\gamma > 0$ and $p_n = \alpha n^{-\beta}$ for $\alpha > 0$ and $\beta > 1$. Then

$$\phi(x,t) = -\log\alpha + \beta\log x + \frac{t}{x^\gamma} \ , \tag{7.22}$$

so that

$$x^*(t) = (\gamma t/\beta)^{1/\gamma} \ , \tag{7.23}$$

$$\phi''(x,t) = \frac{-\beta}{x^2} + \frac{\gamma(\gamma+1)t}{x^{\gamma+2}} \ < \tag{7.24}$$

$$\phi''(x^*(t),t) = \left(\frac{\beta}{\gamma t}\right)^{2/\gamma} (\beta\gamma) > 0 \tag{7.25}$$

and

$$F^c(t) \sim At^{-(\beta-1)/\gamma} \quad \text{as} \quad t \to \infty \tag{7.26}$$

for

$$A = \alpha e^{-\beta/\gamma}(\gamma/\beta)^{-(\beta-1)/\gamma} \ . \tag{7.27}$$

If $\gamma = 1$, then

$$F^c(t) \sim \alpha e^{-\beta}\left(\frac{\beta}{t}\right)^{\beta-1} \quad \text{as} \quad t \to \infty \tag{7.28}$$

and

$$m_n = \alpha n! \sum_{k=1}^{\infty} \beta^{n-\beta} = \alpha n! \zeta(\beta - n) \ , \tag{7.29}$$

where $\zeta(z)$ is the Riemann zeta function; e.g., see p. 19 of Magnus et al. (1966). The moment $m_n$ is finite for $n < \beta - 1$ and otherwise infinite.

**Example 7.3.** An exponential series representation for a cdf associated with the M/M/1 queue was recently derived and applied to determine the asymptotic form by Guillemin and Pinchon (1998). They focused on the total time spent in the system by all customers in a busy period (the area under the queue length process) in the M/M/1 queue. Daley and Jacobs (1969) had shown that the transform can be expressed as the ratio of two Bessel functions, i.e.,

$$\hat{g}(s) = \frac{1}{\sqrt{\rho}} \frac{J_{\nu+1}(2\sqrt{\rho}/s)}{J_\nu(2\sqrt{\rho}/s)} \quad \text{for} \quad \nu = (1+\rho)/s \ , \tag{7.30}$$

where $\rho$ is the traffic intensity. Guillemin and Pinchon showed that the distribution has an exponential series representation, where the exponentials satisfy $a_k \sim k/(1+\rho)$ as $k \to \infty$. Moreover, they applied the Laplace method directly to the discrete sums to derive the asymptotic form of the ccdf,

$$G^c(t) \sim \frac{1-\rho}{\rho\sqrt{2\pi}}\left(\frac{b}{4\theta t}\right)^{1/4} \exp(-\sqrt{4\theta t/b}) \quad \text{as} \quad t \to \infty \ , \tag{7.31}$$

where $b \equiv (1+\rho)^{-1}$ and $\theta \equiv -\log \rho - 2(1-\rho)/(1+\rho)$.

We now develop a relatively simple approximation for this ccdf. We propose the three-parameter exponential-series ccdf

$$G_a^c(t) = A \sum_{k=1}^{\infty} \frac{r^k}{k} \exp(-t/bk) \ , \tag{7.32}$$

which would be proper if $A = -1/\log(1-r)$. Using the results of Example 7.1, we have

$$G_a^c(t) \sim \sqrt{2\pi} A \left(\frac{b}{4\omega t}\right)^{1/4} \exp(1 - \sqrt{4\omega t/b}) \quad \text{as} \quad t \to \infty \tag{7.33}$$

where $\omega \equiv -\log r$. Hence, we can make the asymptotic values agree by letting $A = (1-\rho)/2\pi\rho$ and $w = \theta$, which gives $r = \rho \exp(2(1-\rho)/(1+\rho))$. Moreover, if the total probability mass is less than 1, we can make the approximating ccdf proper by adding an atom at 0.

Numerical results for the case $\rho = 0.6$ are displayed in Table 2. The exact values are taken from Table 3 of Guillemin and Pinchon (1998). From Table 2, we see that the complicated exact ccdf can be reasonably approximated by the relatively simple exponential-series ccdf $G_a^c(t)$ in (7.32).

| time $t$ | exact | approximation | asymptote |
|---|---|---|---|
| 0 | 1.000 | 1.000 | $\infty$ |
| 0+ | 1.000 | 0.481 | $\infty$ |
| 1 | 0.482 | 0.313 | 0.398 |
| 10 | 0.117 | 0.113 | 0.127 |
| 20 | $0.656\ e-1$ | $0.688\ e-1$ | $0.755\ e-1$ |
| 40 | $0.324\ e-1$ | $0.362\ e-1$ | $0.390\ e-1$ |
| 60 | $0.199\ e-1$ | $0.229\ e-1$ | $0.242\ e-1$ |
| 80 | $0.134\ e-1$ | $0.157\ e-1$ | $0.165\ e-1$ |
| 200 | $0.274\ e-2$ | $0.322\ e-2$ | $0.333\ e-2$ |
| 300 | $0.109\ e-2$ | $0.127\ e-2$ | $0.130\ e-2$ |
| 400 | $0.507\ e-3$ | $0.586\ e-3$ | $0.600\ e-3$ |
| 2000 | $0.555\ e-7$ | $0.598\ e-7$ | $0.598\ e-7$ |
| 4000 | $0.365\ e-9$ | $0.384\ e-9$ | $0.384\ e-9$ |

Table 2: A comparison of the approximate exponential-series ccdf $G_a^c(t)$ in (7.32) with the exact ccdf and the asymptotic in (7.31) for Example 7.3.

## 8. Conclusions

We have shown how pdf's and their transforms can be constructed via infinite-series representations, in particular, as Laguerre series (Sections 2 and 3), Erlang series (Sections 4 and

5) and exponential series (Section 6). These series representations make it possible to compute the Laplace transform values in applications. At the same time, other properties of the pdf's can be deduced, such as moments and the asymptotic form, so that the pdf can be chosen to satisfy desired properties in applications. For example, exponential series representations always yield completely monotone pdf's whose moments can be computed. We showed how to determine the asymptotic form in Section 7.

By imposing additional structure, we obtain relatively simple parametric families of pdf's such as the BME pdf's in (3.6), the $B_3$ME pdf's in (5.23), the PME pdf's in (5.33), the generalized Bell pdf's in Example 6.1, and the geometric mixture of exponential pdf's in Example 7.1. Since the Laplace transform values for these pdf's can be computed, these pdf's are natural candidates to use in applications. Then the parameters can be chosen to match the moments, the asymptotic form and other features of interest. Toward that end, we studied the BME pdf's further in Abate and Whitt (1998). The connection to the confluent hypergeometric function $M(a, b, z)$ established in Theorem 5.2 makes it possible to establish analogous results for $B_3$ME pdf's.

Over all, our analysis help extend the analyst's toolkit.

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