Neural Networks and Biological Modeling

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Correction Question Set 7

Exercise 1

1.1 The fixed point h_0 of the activity is defined by a loop of two closed equations: First, the mean firing rate $f = g(h_0)$ and second the population activity in the stationary state $A(t) = A_0 = g(h_0)$ (which is valid because the network is homogeneous and we have asynchronous firing).
For each neuron in the population we must have $h_0 = RI_0$ in the stationary state. For each neuron's current we have

$$\begin{aligned} I_i(t) &= I^{\text{ext}}(t) + \sum_j \sum_f w_{ij}\alpha(t - t_j^f) \\ I_i(t) &= I^{\text{ext}}(t) + \frac{J_0}{N} N \int \alpha(s)A(t-s)ds \\ I_i(t) &= I^{\text{ext}}(t) + J_0 A_0 \end{aligned}$$

Above we used the fact that the network is in the stationary state, has large N and the fact that we have all-to-all connectivity with same weights.There are two ways to see the last step: We have
$\int \alpha(s)A(t-s)ds$
which is just the definition of the *filtered population activity* $\overline{A}(t)$. For large populations, the fluctuations go to 0 and we have $\overline{A}(t) = A_0(t)$.
A second interpretation is to replace $A(t-s)$ by the constant (stationary) population activity A_0 and pull that constant out of the integral.
$A_0 \int \alpha(s)ds$
The integral is assumed to be normalized to 1.

For I_0 we have

$$\begin{aligned} I_0 &= I^{\text{ext}}(t) + J_0 g(h_0) \\ g(h_0) &= \frac{I_0 - I^{\text{ext}}(t)}{J_0} \\ g(h_0) &= \frac{h_0 - RI^{\text{ext}}(t))}{RJ_0} \end{aligned}$$

The fixed point of the activity is therefore given by the intersection between the curve $f = g(h_0)$ and the straight line defined by the last equation.

1.2 For $h_1 = 1$ and $h_2 = 2$ we have

$$f = g(h) = \begin{cases} 0 & , \quad h < 1 \\ h - 1 & , \quad 1 \le h \le 2 \\ 1 & , \quad 2 < h \end{cases} \tag{1}$$

With $R = 1$ and $I^{\text{ext}}(t) = 0$, we have $g(h_0) = \frac{h_0}{J_0}$. J_0 controls the slope of the line.
With $J_0 = 1$: one fixed point, with $J_0 = 3$: three fixed points.

At $J_0 = 2$ we transition from one fixed point to three. $I^{\text{ext}}(t) \neq 0$ controls the bias of the line. Qualitatively, depending on it we may have 0, 1 or 3 fixed points. See next question for the more precise analytical solution.

1.3 In order to give analytical values for h_0 (hence for $f_0 = g(h_0)$) at the fixed point of the dynamics, we have to consider all possible cases.

1. if the slope of the line $\frac{1}{RJ_0}$ is greater than the slope of the transfer function $1/(h_2 - h_1)$ then there is only one fixed point. We have three cases:
   - $f = 0$ is a fixed point if $RI_{\text{ext}} < h_1$
   - $f = 1$ is a fixed point if $RI_{\text{ext}} > h_2 - RJ_0$
   - $f = \frac{h_1 - RI_{\text{ext}}}{RJ_0 - (h_2 - h_1)}$ in between the two previous cases
2. Otherwise if $\frac{1}{RJ_0} < 1/(h_2 - h_1)$, we have 0 or 3 fixed points (we don't do the calculation here).

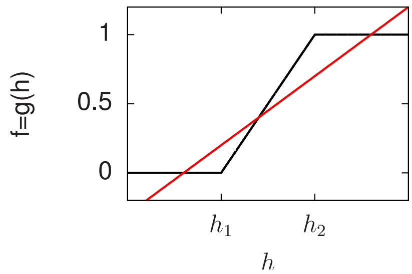

Figure 1: Graphical interpretation of a fixed point

Exercise 2

2.1 Following the steps of the previous exercise for the current, but now we consider the activity over a sub-population of K neurons. For a sub-population we have $A_k(t) = \frac{1}{K}\sum_k \sum_f \delta(t - t_k^f)$. Since the network is homogeneous, the sub-populations will also be homogeneous and we may assume that $A_k(t) \approx A_0$.

$$I_i(t) = \sum_k \sum_f w_{ik}\alpha(t - t_k^f)$$

$$I_i(t) = \sum_k \sum_f w_{ik} \int_0^\infty \alpha(s)\delta(t - t_k^f - s)ds$$

$$I_i(t) = \frac{w_0}{K}\int_0^\infty \alpha(s)\sum_k \sum_f \delta(t - t_k^f - s)ds$$

$$I_i(t) = \frac{w_0}{K}\int_0^\infty \alpha(s)KA_k(t - s)ds$$

$$I_i(t) = \frac{w_0}{K}KA_{k0}\int_0^\infty \alpha(s)ds$$

$$I_i(t) \approx w_0A_0\int_0^\infty \alpha(s)ds$$

To see the approximation from a more intuitive argument, assume a population of, say, 10'000 neurons. Each neuron receives input from K of them. We can say, each neuron draws K samples from the population. Because we have a homogenous network, all of these samples are statistically the same. We expect the sample mean to equal the population mean.

2.2 Nothing. The weights do not scale with N.