1 Emission rate equation

$$|q\rangle = \sum_n a_n |\phi_n\rangle \tag{1}$$

where $a_n = \langle\psi_n|q\rangle$ are the time dependent coefficients of the modal expansion. In particular from Eq.(**??**), and Eq.(**??**) we have $a_0(t) = 1$. The dynamics of the a_n can be determined directly using Eq.(**??**), and Eq.(**??**)

$$\begin{aligned}\dot{a}_n &= \langle\psi_n|\partial_t q\rangle + \langle\partial_t\psi_n|q\rangle \\ &= \langle\psi_n|\mathcal{L}q\rangle + \dot{h}\sum_m a_m\langle\partial_h\psi_n|\phi_m\rangle \\ &= \lambda_n a_n + \dot{h}\sum_m a_m\langle\partial_h\psi_n|\phi_m\rangle\end{aligned} \tag{2}$$

Defining the coupling coefficient as $C_{nm} = \langle\partial_h\psi_n|\phi_m\rangle$ we can rewrite

$$\dot{a}_n = \lambda_n a_n + \dot{h}\sum_m C_{nm}a_m \tag{3}$$

We can finally express the activity $A(t) = q(0,t)$ as

$$A(t) = \sum_n a_n(t)\phi_n(0) \tag{4}$$

Keeping only the first mode, and using the fact that $|\phi_{-n}\rangle = |\bar{\phi}_n\rangle$ and $a_{-n} = \bar{a}_n$, Eq.(4) becomes

$$\begin{aligned}A(t) &= \phi_0(0) + a_1\phi_1(0) + a_{-1}\phi_{-1}(0) \\ &= \phi_0(0) + 2\left(\Re\big[a_1\big]\Re\big[\phi_1(0)\big] - \Im\big[a_1\big]\Im\big[\phi_1(0)\big]\right)\end{aligned} \tag{5}$$

And the dynamics of the a_1 is given by

$$\dot{a}_1 = \lambda_1 a_1 + \dot{h}\big[C_{10} + C_{11}a_1 + C_{1-1}a_{-1}\big]$$

Separating explicitly the real part $X(t)$ and the imaginary part $Y(t)$ of $a_1(t)$

$$a_1(t) = X(t) + iY(t) \tag{6}$$

we derived from Eq.(6) two non linear differential equation

$$\dot{X} =\Re[f]X - \Im[g]Y + \Re[C_{10}]\dot{h} \tag{7}$$

$$\dot{Y} =\Re[g]Y + \Im[f]X + \Im[C_{10}]\dot{h} \tag{8}$$

$$\tag{9}$$

with

$$f =\lambda_1 + \dot{h}(c_{11} + c_{1-1}) \tag{10}$$

$$g =\lambda_1 + \dot{h}(c_{11} - c_{1-1}) \tag{11}$$

We have finally a set of three non linear differential equation $\dot{h}$, $\dot{X}$, $\dot{Y}$