# NOTES ON SIMPLICIAL HOMOTOPY THEORY

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Abstract. Some notes on simplicial homotopy theory

## 1. Introduction

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## 2. Constructive simplicial homotopy theory

- Basics on **SSet** and notation.
- We use $\Delta[n]$, $\partial\Delta[n]$ and $\Lambda^k[n]$ and following Joyal.
- Recall the existence of a weak model structure on **SSet**:
  - Fibrations = Kan fibrations
  - Cofibrations = Levelwise complemented monomorphisms such that degeracies are decidable on the complement of the image.
  - Cofibrant objects are simplicial sets in which degeneracies are decidable.
- It would be good to have an explicit definition of the cofibrant replacement functor.
- Recall the pushout product property.

**Proposition 2.1.** *Cofibrations are closed under pullbacks.*

**Proof.** To be added. $\square$

**Lemma 2.2.** *For $0 \leq k \leq n$, the horn inclusion $i \colon \Lambda^k[n] \to \Delta[n]$ is a retract of the pushout product*

$$i \mathbin{\hat{\times}} \delta^k \colon \big(\Lambda^k[n] \times \Delta[1]\big) \cup (\Delta[n] \times k) \to \Delta[n] \times \Delta[1]\,.$$

**Proposition 2.3.** *A map is a Kan fibration if and only if it has the right lifting property with respect to the pushout products $i \mathbin{\hat{\times}} \delta^k$.*

## 3. Dependent products

We say that a semi-model structure on a category $\mathcal{E}$ has the *restricted Frobenius property* if for every fibration $f\colon B \to A$ with $A$ cofibrant, the pullback functor

$$f^*\colon \mathcal{E}_{/A} \to \mathcal{E}_{/B}$$

preserves trivial cofibrations. When the semi-model structure is cofibrantly generated, it is sufficient [TO CHECK] that the pullback functor sends generating trivial cofibrations to trivial cofibrations. Note that, by adjointness, it follows that the pushforward functor

$$f_*\colon \mathcal{E}_{/B} \to \mathcal{E}_{/A}$$

preserves fibrations. Note, however, that the result of applying $f_*$ may not be a cofibration.

**Lemma 3.1.**

*(i)* $\mathcal{J} \subset \mathsf{Cof} \cap \mathcal{S}$.

*(ii)* $\mathsf{Cof} \cap \mathcal{S} \subseteq \mathsf{TrivCof}$.

**Lemma 3.2.** *For $k \in \{0,1\}$, the pullback of a strong $k$-oriented homotopy equivalence with cofibrant codomain along a fibration is a strong $k$-oriented homotopy equivalence.*

**Theorem 3.3.** *The semi-model structure for Kan complexes on* **SSet** *has the restricted Frobenius condition.*

**Proof.** It suffices to show that, for a pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "j"'] & \Lambda^k[n] \arrow[d, "i"] \\ X \arrow[r, "f"'] & \Delta[n] \end{tikzcd}$$

if $f$ is a Kan fibration then $j$ is a trivial cofibration. First, since $i$ is a trivial cofibration, it is also a cofibration and so its pullback is again a cofibration. Secondly, since $i$ is trivial cofibration with cofibrant codomain, its pullback is a strong homotopy equivalence by Lemma 3.2. But now $j$ is both a cofibration and a strong homotopy equivalence and hence it is a trivial cofibration by Lemma 3.2. □

The plan to interpret $\Pi$-types is as follows. Suppose you have fibrations $q\colon \Gamma.A.B \to \Gamma.A$ and $p\colon \Gamma.A \to \Gamma$, with all objects both fibrant and cofibrant. We begin by applying

$$p_*\colon \mathbf{SSet}_{/\Gamma.A.B} \to \mathbf{SSet}_{/\Gamma.A}$$

to $q$, so as to obtain $p_*(q)\colon \Gamma.(\Pi x\colon A)B(x) \to \Gamma$, which is a fibration by Theorem 3.3. The domain of this map is fibrant but not necessarily cofibrant, so we take a cofibrant replacement [OF WHAT? OF THE OBJECT IN $\mathcal{E}_{/\Gamma}$ OR OF THE MAP?] and obtain a map that we denote as

$$\Gamma.(\widetilde{\Pi} x\colon A)B(x) \to \Gamma\,.$$

This will satisfy the $\beta$-rule but not the $\eta$-rule for $\Pi$-types.

## 4. The universe

Recall that we work in a constructive set theory with two universes $\mathsf{u}_1$ and $\mathsf{u}_2$ and that we refer to elements of $\mathsf{u}_1$ as small sets. We then define a simplicial set $X$ to be *small*

**Definition 4.1.**

(i) We say that a simplicial set $X$ is *small* if $X_n$ is a small set for every $[n] \in \Delta$.

(ii) We say that a map $f \colon Y \to X$ in **SSet** is *small* if for every $x \colon \Delta[n] \to X$ the simplicial set $Y_x$ fitting in the pullback square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\ \Delta[n] \arrow[r, "x"'] & X \end{tikzcd}$$

is small.

By the results in [29] for arbitrary presheaf categories, small maps in **SSet** admit a weak classifier, i.e. a small map $\rho \colon \overline{\mathsf{V}} \to \mathsf{V}$ such that for every small map $f \colon Y \to X$ there exists a pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{V}} \arrow[d, "\rho"] \\ X \arrow[r] & \mathsf{V} \end{tikzcd}$$

Letting $X = \Delta[n]$ in this diagram suggests to define $\mathsf{V}_n$ as the set of all small maps with codomain $\Delta[n]$. In this way, however, one does not obtain a presheaf since the transition functions will satisfy the functorial laws only up to isomorphism rather than equality. To remedy this, the $n$-simplices of $\mathsf{V}$ are defined instead to be the functors $F \colon (\Delta/[n])^{\mathrm{op}} \to \mathbf{Set}$ such that the corresponding map of simplicial sets $\mathsf{El}(F) \to \Delta[n]$ is small. MORE TO BE ADDED.

Following [15, 39], we consider the pullback

$$\begin{tikzcd} \overline{\mathsf{U}} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{V}} \arrow[d, "\rho"] \\ \mathsf{U} \arrow[r] & \mathsf{V} \end{tikzcd}$$

where $\mathsf{U} \subseteq \mathsf{V}$ is defined by letting

$$\mathsf{U}_n = \{ F \in \mathsf{V}_n \mid \mathsf{El}(F) \to \Delta[n] \text{ is a small Kan fibration } \}$$

**Proposition 4.2.**

*(i)* $\pi \colon \overline{\mathsf{U}} \to \mathsf{U}$ *is a small Kan fibration.*

*(ii)* $\pi \colon \overline{\mathsf{U}} \to \mathsf{U}$ *classifies small Kan fibrations, i.e. for every small Kan fibration* $f \colon Y \to X$ *there exists a pullback diagram of the form*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

*(iii) The simplicial set* $\overline{\mathsf{U}}$ *is cofibrant.*

**Proof.** We prove the three claims separately.

(i) Should follow by locality.
(ii) Should be immediate.
(iii) See handwritten notes. Key step is the constructive version of the Eilenberg-Zilber lemma.

□

However, the simplicial set $\mathsf{U}$ does not appear to be cofibrant and hence it does not seem possible to show that $\pi\colon \overline{\mathsf{U}} \to \mathsf{U}$ is a weak classifier for small Kan fibrations with cofibrant codomain. In order to remedy this, we consider the cofibrant replacement $\mathsf{U}_c$ of $\mathsf{U}$, which comes equipped with a trivial fibration $p\colon \mathsf{U}_c \to \mathsf{U}$, and the pullback

$$\begin{tikzcd} \overline{\mathsf{U}}_c \arrow[r] \arrow[d, "\pi_c"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ \mathsf{U}_c \arrow[r, "p"'] & \mathsf{U} \end{tikzcd}$$

We can now prove that $\pi_c\colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ has the desired properties.

**Proposition 4.3.**

*(i)* $\pi_c\colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ *is a small Kan fibration with fibrant codomain.*

*(ii) The map* $\pi_c\colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ *classifies small Kan fibrations with cofibrant domain, i.e. for every small Kan fibration* $f\colon Y \to X$ *with* $X$ *cofibrant there exists a pullback diagram of the form*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \overline{\mathsf{U}}_c \arrow[d, "\pi_c"] \\ X \arrow[r] & \mathsf{U}_c \end{tikzcd}$$

*(iii) The simplicial set* $\overline{\mathsf{U}}_c$ *is cofibrant.*

**Proof.** Part (i) follows from part (i) of Proposition 4.2. For part (ii), let $f\colon Y \to X$ be a small Kan fibration with $X$ cofibrant. Since $f$ is a small Kan fibration, we know from Proposition 4.2 that there is a pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

Since $X$ is cofibrant, we have the lifting diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathsf{U}_c \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & \mathsf{U} \end{tikzcd}$$

which shows that the map $X \to \mathsf{U}$ factors via $\mathsf{U}_c$. We then obtain the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \overline{\mathsf{U}}_c \arrow[r] \arrow[d, "\pi_c"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U}_c \arrow[r, "p"'] & \mathsf{U} \end{tikzcd}$$

Here, the right-hand side square and the rectangle are pullbacks and therefore the left-hand side square is also a pullback, as required. Part (iii) follows from the fact that both $\mathsf{U}_c$ and $\overline{\mathsf{U}}$ are cofibrant, the latter being part (iii) of Proposition 4.2. $\square$

Note that we have not shown yet that $\mathsf{U}_c$ fibrant. This will be done in Section 6, as a consequence of the equivalence extension property for fibrations, which we establish in Section 5.

## 5. **The equivalence extension property**

- Here we follow Kapulkin and Lumsdaine.

## 6. Fibrancy and univalence of the universe

- Fibrancy should follow directly from equivalence extension property, without using 'composition vs filling' but rather retract property for horns (see notes).
- Once we have established fibrancy of $U_c$, then one can prove univalence by showing that $t\colon \mathsf{Weq}(U_c) \to U_c$ is a trivial fibration.
- Question: do we need to know that $\mathsf{Weq}(U_c)$ is a cofibrant object to get univalence?

## 7. Semantics

- This should be essentially straightforward, following Kapulkin and Lumsdaine, but we may need to modify the notion of a $\Pi$-structure to accommodate the cofibrant replacements that we take for $\Pi$.

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