

# NOTES ON SIMPLICIAL HOMOTOPY THEORY

NICOLA GAMBINO AND SIMON HENRY

ABSTRACT. Some notes on simplicial homotopy theory

## 1. Introduction

## 2. Constructive simplicial homotopy theory

- Basics on **SSet** and notation.
- We use  $\Delta[n]$ ,  $\partial\Delta[n]$  and  $\Lambda^k[n]$  and following Joyal.
- Recall the existence of a weak model structure on **SSet**:
  - Fibrations = Kan fibrations
  - Cofibrations = Levelwise complemented monomorphisms such that degeneracies are decidable on the complement of the image.
  - Cofibrant objects are simplicial sets in which degeneracies are decidable.
- It would be good to have an explicit definition of the cofibrant replacement functor.
- Recall the pushout product property.

**Lemma 2.1.** *Cofibrations are closed under pullbacks.*

**Proposition 2.2.** *Cofibrations are closed under pullbacks.*

**Proof.** To be added. □

**Lemma 2.3.** *For  $0 \leq k \leq n$ , the horn inclusion  $i: \Lambda^k[n] \rightarrow \Delta[n]$  is a retract of the pushout product*

$$i \hat{\times} \delta^k: (\Lambda^k[n] \times \Delta[1]) \cup (\Delta[n] \times k) \rightarrow \Delta[n] \times \Delta[1].$$

**Proposition 2.4.** *A map is a Kan fibration if and only if it has the right lifting property with respect to the pushout products  $i \hat{\times} \delta^k$ .*

### 3. Dependent products

The first aim of this section is to show that for every fibration  $p: B \rightarrow A$  with  $A$  cofibrant, the pushforward functor  $\Pi_p: \mathcal{E}_{/B} \rightarrow \mathcal{E}_{/A}$  preserves fibrations. Our second aim is to build on this result to define sufficient structure to interpret Martin-Löf's rules for  $\Pi$ -types.

For our first aim, let  $p: B \rightarrow A$  be a fibration with cofibrant codomain and observe that  $\Pi_p: \mathcal{E}_{/B} \rightarrow \mathcal{E}_{/A}$  preserves fibrations if and only if its left adjoint

$$p^*: \mathcal{E}_{/A} \rightarrow \mathcal{E}_{/B}$$

preserves trivial cofibrations. Furthermore,

**Lemma 3.1.**

- (i)  $\mathcal{J} \subset \text{Cof} \cap \mathcal{S}$ .
- (ii)  $\text{Cof} \cap \mathcal{S} \subseteq \text{TrivCof}$ .

**Lemma 3.2.** *For  $k \in \{0, 1\}$ , the pullback of a strong  $k$ -oriented homotopy equivalence with cofibrant codomain along a fibration is a strong  $k$ -oriented homotopy equivalence.*

**Theorem 3.3.** *The semi-model structure for Kan complexes on  $\mathbf{SSet}$  has the restricted Frobenius condition.*

**Proof.** Since the semi-model structure in which we are working is cofibrantly generated, it is sufficient [TO CHECK] that  $p^*$  sends generating trivial cofibrations to trivial cofibrations. So, let  $p: B \rightarrow \Delta[n]$  be a fibration,  $i: \Lambda^k[n] \rightarrow \Delta[n]$  be a horn inclusion, and define  $j =_{\text{def}} p^*(i)$ , given by the pullback diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \Lambda^k[n] \\ j \downarrow & \lrcorner & \downarrow i \\ B & \xrightarrow[p]{} & \Delta[n] \end{array}$$

We need to show that  $j$  is a trivial cofibration. First, since  $i$  is a trivial cofibration, it is in particular a cofibration and therefore  $j$  is again a cofibration by Lemma 2.1. Secondly, since  $i \in \mathcal{J}$ , by part (i) of Lemma 3.1, it is a cofibration and a strong homotopy equivalence. Since its codomain is cofibrant,  $j$  is a strong homotopy equivalence by Lemma 3.2. But now  $j$  is both a cofibration and a strong homotopy equivalence and hence it is a trivial cofibration, as required, by part (ii) of Lemma 3.2.  $\square$

We now discuss our second aim, namely showing how we can build on Theorem 3.3 to define sufficient structure to interpret  $\Pi$ -types. This is not immediate since the result of applying  $\Pi_p$  to a fibration with cofibrant domain need not have cofibrant domain. Suppose that we have fibrations  $q: B \rightarrow A$  and  $p: A \rightarrow \Gamma$ , with all objects both fibrant and cofibrant. We begin by applying

$$\Pi_p: \mathbf{SSet}_{/A} \rightarrow \mathbf{SSet}_{/\Gamma}$$

to  $q: B \rightarrow A$ , so as to obtain  $\Pi_p(q): \Pi_A(B) \rightarrow \Gamma$ , which is a fibration by Theorem 3.3. The domain of this map is fibrant but not necessarily cofibrant, so we consider its cofibrant replacement

$$\tilde{\Pi}_A(B) =_{\text{def}} \mathbb{L}(\Pi_A(B)),$$

which comes equipped with a trivial fibration  $\varepsilon: \tilde{\Pi}_A(B) \rightarrow \Pi_A(B)$ . Finally, we define the fibration  $\tilde{\Pi}_p(q): \tilde{\Pi}_A(B) \rightarrow \Gamma$  as the composite

$$\begin{array}{ccc} \tilde{\Pi}_A(B) & \xrightarrow{\varepsilon} & \Pi_A(B) \\ & \searrow \tilde{\Pi}_p(q) & \swarrow \Pi_p(q) \\ & \Gamma & \end{array}$$

This will satisfy the  $\beta$ -rule but not the  $\eta$ -rule for  $\Pi$ -types [TO BE ADDED].

#### 4. The universe

Recall that we work in a constructive set theory with two universes  $u_1$  and  $u_2$  and that we refer to elements of  $u_1$  as small sets. We then define a simplicial set  $X$  to be *small*

**Definition 4.1.**

- (i) We say that a simplicial set  $X$  is *small* if  $X_n$  is a small set for every  $[n] \in \Delta$ .
- (ii) We say that a map  $f: Y \rightarrow X$  in **SSet** is *small* if for every  $x: \Delta[n] \rightarrow X$  the simplicial set  $Y_x$  fitting in the pullback square

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ \Delta[n] & \xrightarrow{x} & X \end{array}$$

is small.

By the results in [29] for arbitrary presheaf categories, small maps in **SSet** admit a weak classifier, i.e. a small map  $\rho: \bar{V} \rightarrow V$  such that for every small map  $f: Y \rightarrow X$  there exists a pullback diagram of the form

$$\begin{array}{ccc} Y & \longrightarrow & \bar{V} \\ \downarrow f & \lrcorner & \downarrow \rho \\ X & \longrightarrow & V \end{array}$$

Letting  $X = \Delta[n]$  in this diagram suggests to define  $V_n$  as the set of all small maps with codomain  $\Delta[n]$ . In this way, however, one does not obtain a presheaf since the transition functions will satisfy the functorial laws only up to isomorphism rather than equality. To remedy this, the  $n$ -simplices of  $V$  are defined instead to be the functors  $F: (\Delta/[n])^{\text{op}} \rightarrow \mathbf{Set}$  such that the corresponding map of simplicial sets  $\text{El}(F) \rightarrow \Delta[n]$  is small. MORE TO BE ADDED.

Following [15, 39], we consider the pullback

$$\begin{array}{ccc} \bar{U} & \longrightarrow & \bar{V} \\ \downarrow \pi & \lrcorner & \downarrow \rho \\ U & \longrightarrow & V \end{array}$$

where  $U \subseteq V$  is defined by letting

$$U_n = \{F \in V_n \mid \text{El}(F) \rightarrow \Delta[n] \text{ is a small Kan fibration} \}$$

**Proposition 4.2.**

- (i)  $\pi: \bar{U} \rightarrow U$  is a small Kan fibration.
- (ii)  $\pi: \bar{U} \rightarrow U$  classifies small Kan fibrations, i.e. for every small Kan fibration  $f: Y \rightarrow X$  there exists a pullback diagram of the form

$$\begin{array}{ccc} Y & \longrightarrow & \bar{U} \\ \downarrow f & \lrcorner & \downarrow \pi \\ X & \longrightarrow & U \end{array}$$

- (iii) The simplicial set  $\bar{U}$  is cofibrant.

**Proof.** We prove the three claims separately.

- (i) Should follow by locality.
- (ii) Should be immediate.
- (iii) See handwritten notes. Key step is the constructive version of the Eilenberg-Zilber lemma.  $\square$

However, the simplicial set  $\mathbf{U}$  does not appear to be cofibrant and hence it does not seem possible to show that  $\pi: \overline{\mathbf{U}} \rightarrow \mathbf{U}$  is a weak classifier for small Kan fibrations with cofibrant codomain. In order to remedy this, we consider the cofibrant replacement  $\mathbf{U}_c$  of  $\mathbf{U}$ , which comes equipped with a trivial fibration  $p: \mathbf{U}_c \rightarrow \mathbf{U}$ , and the pullback

$$\begin{array}{ccc} \overline{\mathbf{U}}_c & \longrightarrow & \overline{\mathbf{U}} \\ \pi_c \downarrow & \lrcorner & \downarrow \pi \\ \mathbf{U}_c & \xrightarrow{p} & \mathbf{U} \end{array}$$

We can now prove that  $\pi_c: \overline{\mathbf{U}}_c \rightarrow \mathbf{U}_c$  has the desired properties.

**Proposition 4.3.**

- (i)  $\pi_c: \overline{\mathbf{U}}_c \rightarrow \mathbf{U}_c$  is a small Kan fibration with fibrant codomain.
- (ii) The map  $\pi_c: \overline{\mathbf{U}}_c \rightarrow \mathbf{U}_c$  classifies small Kan fibrations with cofibrant domain, i.e. for every small Kan fibration  $f: Y \rightarrow X$  with  $X$  cofibrant there exists a pullback diagram of the form

$$\begin{array}{ccc} Y & \longrightarrow & \overline{\mathbf{U}}_c \\ f \downarrow & & \downarrow \pi_c \\ X & \longrightarrow & \mathbf{U}_c \end{array}$$

- (iii) The simplicial set  $\overline{\mathbf{U}}_c$  is cofibrant.

**Proof.** Part (i) follows from part (i) of Proposition 4.2. For part (ii), let  $f: Y \rightarrow X$  be a small Kan fibration with  $X$  cofibrant. Since  $f$  is a small Kan fibration, we know from Proposition 4.2 that there is a pullback diagram of the form

$$\begin{array}{ccc} Y & \longrightarrow & \overline{\mathbf{U}} \\ f \downarrow & \lrcorner & \downarrow \pi \\ X & \longrightarrow & \mathbf{U} \end{array}$$

Since  $X$  is cofibrant, we have the lifting diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbf{U}_c \\ \downarrow & \nearrow & \downarrow p \\ X & \longrightarrow & \mathbf{U} \end{array}$$

which shows that the map  $X \rightarrow \mathbf{U}$  factors via  $\mathbf{U}_c$ . We then obtain the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{\mathbf{U}}_c & \longrightarrow & \overline{\mathbf{U}} \\ f \downarrow & & \downarrow \pi_c & \lrcorner & \downarrow \pi \\ X & \longrightarrow & \mathbf{U}_c & \xrightarrow{p} & \mathbf{U} \end{array}$$

Here, the right-hand side square and the rectangle are pullbacks and therefore the left-hand side square is also a pullback, as required. Part (iii) follows from the fact that both  $\mathcal{U}_c$  and  $\bar{\mathcal{U}}$  are cofibrant, the latter being part (iii) of Proposition 4.2.  $\square$

Note that we have not shown yet that  $\mathcal{U}_c$  fibrant. This will be done in Section 6, as a consequence of the equivalence extension property for fibrations, which we establish in Section 5.

## 5. The equivalence extension property

- Here we follow Kapulkin and Lumsdaine.



## 6. Fibrancy and univalence of the universe

- Fibrancy should follow directly from equivalence extension property, without using ‘composition vs filling’ but rather retract property for horns (see notes).
- Once we have established fibrancy of  $U_c$ , then one can prove univalence by showing that  $t: \mathbf{Weq}(U_c) \rightarrow U_c$  is a trivial fibration.
- Question: do we need to know that  $\mathbf{Weq}(U_c)$  is a cofibrant object to get univalence?

## 7. Semantics

- This should be essentially straightforward, following Kapulkin and Lumsdaine, but we may need to modify the notion of a  $\Pi$ -structure to accommodate the cofibrant replacements that we take for  $\Pi$ .

## References

- [1] S. Awodey. Notes on cubical models of type theory. Draft manuscript, March 2015.
- [2] S. Awodey and M. A. Warren. Homotopy theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146:45–55, 2009.
- [3] M. Batanin, D.-C. Cisinski, and M. Weber. Multitensor lifting and strictly unital higher category theory. *Theory and Applications of Categories*, 28:804–856, 2013.
- [4] J. Bénabou and J. Roubaud. Monades et descente. *C. R. Acad. Sc. Paris*, 270, Serie A:96–98, 1970.
- [5] B. van den Berg and R. Garner. Types are weak  $\omega$ -groupoids. *Proceedings of the London Mathematical Society*, 102(3):370–394, 2010.
- [6] B. van den Berg and R. Garner. Topological and simplicial models of identity types. *Transactions of the ACM on Computational Logic*, 13(1):3–44, 2012.
- [7] B. Berger and I. Moerdijk. On an extension of the notion of Reedy category. *Mathematische Zeitschrift*, 269(3-4):977–1004, 2011.
- [8] J. Bergner and C. Rezk. Reedy categories and the  $\Theta$ -construction. *Mathematische Zeitschrift*, 274(1):499–514, 2013.
- [9] M. Bezem, T. Coquand, and S. Huber. A model of type theory in cubical sets. In Ralph Matthes and Aleksey Schubert, editors, *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128. Schloss Dagstuhl — Leibniz-Zentrum für Informatik, 2014.
- [10] M. Bezem, T. Coquand, and E. Parmann. Non-constructivity in Kan simplicial sets. In Thorsten Altenkirch, editor, *13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015)*, volume 38, pages 92–106. Schloss Dagstuhl — Leibniz-Zentrum für Informatik, 2015.
- [11] J. Bourke and R. Garner. Algebraic weak factorisation systems I: accessible AWFS. *Journal of Pure and Applied Algebra*, 220:108–147, 2016.
- [12] J. Bourke and R. Garner. Algebraic weak factorisation systems II: categories of weak maps. *Journal of Pure and Applied Algebra*, 220:148–174, 2016.
- [13] A. K. Bousfield. Constructions of factorisation systems in categories. *Journal of Pure and Applied Algebra*, 9:207–220, 1977.
- [14] D.-C. Cisinski. Les préfaisceaux comme modèles des types d’homotopie. *Astérisque*, 308:xxiv+392, 2006.
- [15] D.-C. Cisinski. Univalent universes for elegant models of homotopy types. arXiv:1406.0058, 2014.
- [16] M. M. Clementino, E. Giuli, and W. Tholen. Topology in a category: compactness. *Portugaliae Mathematica*, 53(4):397–433, 1996.
- [17] C. Cohen, T. Coquand, S. Huber, and A. Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. arXiv:1611.02108. To be published in the post-proceedings of the 21st International Conference on Types for Proofs and Programs, TYPES 2015, 2016.
- [18] J. Emmenegger. A category-theoretic version of the identity type weak factorization system. arXiv:1412.0153, 2014.
- [19] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1967.
- [20] N. Gambino and R. Garner. The identity type weak factorisation system. *Theoretical Computer Science*, 409:94–109, 2008.
- [21] N. Gambino and J. Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154(1):153–192, 2013.
- [22] R. Garner. A homotopy-theoretic universal property of Leinster’s operad for weak  $\omega$ -categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 147:615–628, 2009.
- [23] R. Garner. Understanding the small object argument. *Applied Categorical Structures*, 17(3):247–285, 2009.
- [24] R. Garner. Homomorphisms of higher categories. *Advances in Mathematics*, 224(6):2269–2311, 2010.
- [25] R. Garner and S. Lack. On the axioms for adhesive and quasiadhesive categories. *Theory and Applications of Categories*, 27(3):27–46, 2012.
- [26] P. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Birkhäuser, 1999.
- [27] M. Grandis and W. Tholen. Natural weak factorisation systems. *Archivum Mathematicum*, 42:397–408, 2006.
- [28] P. Hirschhorn. *Model categories and their localizations*. American Mathematical Society, 2003.

- [29] M. Hofmann and T. Streicher. Lifting grothendieck universes. Available from the second-named author's web page, 1997.
- [30] M. Hovey. *Model categories*. American Mathematical Society, 1999.
- [31] S. Huber. A model of type theory in cubical sets. Licentiate of philosophy thesis, University of Gothenburg, 2015.
- [32] J. M. E. Hyland. First steps in synthetic domain theory. In M.-C. Pedicchio A. Carboni and G. Rosolini, editors, *Category Theory*, volume 1488 of *Lecture Notes in Mathematics*, pages 280–301. Springer, 1991.
- [33] P. T. Johnstone. *Sketches of an elephant: a Topos theory compendium*. Oxford Logic Guides. Oxford University Press, New York, NY, 2002.
- [34] A. Joyal. The theory of quasi-categories and its applications. Quaderns 45, Centre de Recerca Matemàtica, 2008.
- [35] A. Joyal and M. Tierney. An introduction to simplicial homotopy theory. Available from <http://hopf.math.purdue.edu/Joyal-Tierney/JT-chap-01.pdf>, 1999.
- [36] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. American Mathematical Society, 2007.
- [37] A. Joyal and M. Tierney. Notes on simplicial homotopy theory. Lecture notes, available at <http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf>, 2008.
- [38] K. Kamps and T. Porter. *Abstract homotopy and simple homotopy theory*. World Scientific Publishing Co., 1997.
- [39] C. Kapulkin and P. LeFanu Lumsdaine. The simplicial model of Univalent Foundations (after Voevodsky). arXiv:1211.2851v4, 2016.
- [40] G. M. Kelly and R. Street. Review of the elements of 2-categories. In *Category Seminar*, volume 420 of *Lecture Notes in Mathematics*. Springer, 1974.
- [41] S. Mac Lane. *Categories for the working mathematician*. Springer, second edition, 1998.
- [42] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.
- [43] F. W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In *Proceedings of the American Mathematical Society Symposium on Pure Mathematics XVII*, pages 1–14, 1970.
- [44] J. Lurie. *Higher topos theory*. Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009.
- [45] I. Moerdijk and J. Nuiten. Minimal fibrations of dendroidal sets. arXiv:1509.01073, 2015.
- [46] B. Nordström, K. Petersson, and J. Smith. Martin-löf type theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, Oxford Logic Guides, chapter V, pages 1–37. Oxford University Press, 2001.
- [47] I. Orton and A. M. Pitts. Axioms for modelling cubical type theory in a topos. In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 24:1–24:19, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.
- [48] A. M. Pitts. Nominal presentation of cubical sets models of type theory. In H. Herbelin, P. Letouzey, and M. Sozeau, editors, *20th International Conference on Types for Proofs and Programs (TYPES 2014)*. Schloss Dagstuhl — Leibniz-Zentrum für Informatik, 2015.
- [49] D. G. Quillen. *Homotopical algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer, 1967.
- [50] E. Riehl. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra*, 217:1069–1104, 2013.
- [51] E. Riehl and D. Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29(9):256–301, 2014.
- [52] E. Riehl. Algebraic model structures. *New York Journal of Mathematics*, 17:173–231, 2011.
- [53] E. Riehl. *Categorical homotopy theory*. Cambridge University Press, 2014.
- [54] J. Rosický and W. Tholen. Factorization, fibration and torsion. *Journal of Homotopy and Related Structures*, 2(295–314), 2007.
- [55] G. Rosolini. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986.
- [56] M. Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25:1203–1277, 2015.
- [57] M. Shulman. The univalence axiom for elegant Reedy presheaves. *Homology, Homotopy and Applications*, To appear.
- [58] A. Swan. An algebraic weak factorisation system on 01-substitution sets: a constructive proof. arXiv:1409.1829, 2014.
- [59] V. Voevodsky. Univalent foundations project. [http://www.math.ias.edu/vladimir/files/univalent\\_foundations\\_project.pdf](http://www.math.ias.edu/vladimir/files/univalent_foundations_project.pdf), 2010.
- [60] V. Voevodsky. The equivalence axiom and univalent models of type theory. (talk at cmu on february 4, 2010). arXiv:1402.5556v2, 2014.

- [61] M. A. Warren. *Homotopy theoretic aspects of constructive type theory*. PhD thesis, Carnegie Mellon University, 2008.