NOTES ON SIMPLICIAL HOMOTOPY THEORY

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Abstract. Some notes on simplicial homotopy theory

1. Introduction

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2. Review of the semi-model structure on simplicial sets

Let $\mathcal{E}$ be a category with an initial object 0 and a terminal object 1. Recall that a semi-model structure on $\mathcal{E}$ consists of three classes of maps Weq, Fib and Cof in $\mathcal{E}$ satisfying some axioms, which we now recall. When stating these axioms, and in the following, we refer to elements of Weq, Fib and Cof as weak equivalences, fibrations and cofibrations, respectively, call *fibrant* the objects X for which the unique map $\top_X : X \to 1$ is a fibration, and *cofibrant* the objects Y for which the unique map $\perp_Y : 0 \to Y$ is a cofibration.

The axioms for a semi-model structure can then be stated as follows:

(SM1) A

(SM2)

It is important to note the similarities and differences between semi-model structures and model structures.

- Basics on **SSet** and notation.
- We use Δ^n, $\partial\Delta^n$ and Λ^n_k.
- Recall the semi model structure on **SSet**:
  - Generating cofibrations are the boundary inclusions,
  $$\mathcal{I} =_{\mathrm{def}} \{i^n \colon \partial\Delta^n \to \Delta^n \mid n \geq 0\}$$
  - Trivial Kan fibrations = maps with the right lifting property with respect to $\mathcal{I}$.
  - Cofibrations = Levelwise complemented monomorphisms such that degeracies are decidable on the complement of the image.
  - Cofibrant objects are simplicial sets in which degeneracies are decidable.
  - Generating trivial cofibrations are the horn inclusions:
  $$\mathcal{J} =_{\mathrm{def}} \{h^k_n \colon \Lambda^n_k \to \Delta^n \mid 0 \leq k \leq n\}\,.$$
  - Kan fibrations = maps with the right lifting property with respect to $\mathcal{J}$.
  - Fibrant objects are Kan complexes.
  - Interval object is Δ^1. Endpoint inclusions $\delta^k \colon \{k\} \to \Delta^1$ are defined by $\delta^k =_{\mathrm{def}} h^1_k$.
- It would be good to have an explicit definition of the cofibrant replacement functor.
- Recall the pushout product property.

Lemma 2.1. *If B and C are cofibrants, and A is any simplicial sets with maps $B \to A$, $C \to A$ then the pullback $B \times_A C$ is cofibrant.*

Proof. Sketche: we check it first for the product $B \times C$, and then observe that $B \times_A C \subset B \times C$ and a subobject of a cofibrant object is cofibrant. □

Proposition 2.2. *If A, B, C are cofibrant simplicial sets, $A \hookrightarrow C$ is a cofibrations and $f : B \to C$ is any morphism, then the natural projection map $A \times_C B \to B$ is a cofibration.*

Proof. To be added. □

Lemma 2.3. *For $0 \leq k \leq n$, the horn inclusion $h^k_n \colon \Lambda^n_k \to \Delta^n$ is a retract of the pushout product*
$$h^k_1 \mathbin{\hat{\times}} i^n \colon (\Delta^1 \times \partial\Delta^n) \cup (\{k\} \times \Delta^n) \to \Delta^1 \times \Delta^n$$

Proposition 2.4. *A map is a Kan fibration if and only if it has the right lifting property with respect to the pushout products $i \mathbin{\hat{\times}} \delta^k$.*

3. Dependent products

The first aim of this section is to show that for every fibration $p\colon B \to A$ with A cofibrant, the pushforward functor $\Pi_p\colon \mathcal{E}_{/B} \to \mathcal{E}_{/A}$ preserves fibrations. Our second aim is to build on this result to define sufficient structure to interpret Martin-Löf's rules for Π-types.

For our first aim, let $p\colon B \to A$ be a fibration with cofibrant codomain and observe that $\Pi_p\colon \mathcal{E}_{/B} \to \mathcal{E}_{/A}$ preserves fibrations if and only if its left adjoint

$$p^*\colon \mathcal{E}_{/A} \to \mathcal{E}_{/B}$$

preserves trivial cofibrations. Furthermore,

Lemma 3.1.

(i) $\mathcal{J} \subset \mathsf{Cof} \cap \mathcal{S}$.
(ii) $\mathsf{Cof} \cap \mathcal{S} \subseteq \mathsf{TrivCof}$.

Lemma 3.2. *For $k \in \{0,1\}$, the pullback of a strong k-oriented homotopy equivalence with cofibrant codomain along a fibration is a strong k-oriented homotopy equivalence.*

Theorem 3.3. *The semi-model structure for Kan complexes on* **SSet** *has the restricted Frobenius condition.*

Proof. Since the semi-model structure in which we are working is cofibrantly generated, it is sufficient [TO CHECK] that p^* sends generating trivial cofibrations to trivial cofibrations. So, let $p\colon B \to \Delta[n]$ be a fibration, $i\colon \Lambda^k[n] \to \Delta[n]$ be a horn inclusion, and define $j =_{\mathrm{def}} p^*(i)$, given by the pullback diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^k[n] \arrow[d, "i"] \\ B \arrow[r, "p"'] & \Delta[n] \end{tikzcd}$$

We need to show that j is a trivial cofibration. First, since i is a trivial cofibration, it is in particular a cofibration and therefore j is again a cofibration by Proposition 2.2. Secondly, since $i \in \mathcal{J}$, by part (i) of Lemma 3.1, it is a cofibration and a strong homotopy equivalence. Since its codomain is cofibrant, j is a strong homotopy equivalence by Lemma 3.2. But now j is both a cofibration and a strong homotopy equivalence and hence it is a trivial cofibration, as required, by part (ii) of Lemma 3.2. □

We now discuss our second aim, namely showing how we can build on Theorem 3.3 to define sufficient structure to interpret Π-types. This is not immediate since the result of applying Π_p to a fibration with cofibrant domain need not have cofibrant domain. Suppose that we have fibrations $q\colon B \to A$ and $p\colon A \to \Gamma$, with all objects both fibrant and cofibrant. We begin by applying

$$\Pi_p\colon \mathbf{SSet}_{/A} \to \mathbf{SSet}_{/\Gamma}$$

to $q\colon B \to A$, so as to obtain $\Pi_p(q)\colon \Pi_A(B) \to \Gamma$, which is a fibration by Theorem 3.3. The domain of this map is fibrant but not necessarily cofibrant, so we consider its cofibrant replacement

$$\widetilde{\Pi}_A(B) =_{\mathrm{def}} \mathbb{L}\big(\Pi_A(B)\big)\,,$$

which comes equipped with a trivial fibration $\varepsilon\colon \widetilde{\Pi}_A(B) \to \Pi_A(B)$. Finally, we define the fibration $\widetilde{\Pi}_p(q)\colon \widetilde{\Pi}_A(B) \to \Gamma$ as the composite

$$\begin{tikzcd} \widetilde{\Pi}_A(B) \arrow[rr, "\varepsilon"] \arrow[dr, "\widetilde{\Pi}_p(q)"'] & & \Pi_A(B) \arrow[dl, "\Pi_p(q)"] \\ & \Gamma & \end{tikzcd}$$

This will satisfy the β-rule but not the η-rule for Π-types [TO BE ADDED].

4. The universe

Recall that we work in a constructive set theory with two universes u_1 and u_2 and that we refer to elements of u_1 as small sets. We then define a simplicial set X to be *small*

Definition 4.1.

(i) We say that a simplicial set X is *small* if X_n is a small set for every $[n] \in \Delta$.

(ii) We say that a map $f\colon Y \to X$ in **SSet** is *small* if for every $x\colon \Delta[n] \to X$ the simplicial set Y_x fitting in the pullback square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \Delta[n] \arrow[r, "x"'] & X \end{tikzcd}$$

is small.

By the results in [29] for arbitrary presheaf categories, small maps in **SSet** admit a weak classifier, i.e. a small map $\rho\colon \overline{\mathsf{V}} \to \mathsf{V}$ such that for every small map $f\colon Y \to X$ there exists a pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{V}} \arrow[d, "\rho"] \\ X \arrow[r] & \mathsf{V} \end{tikzcd}$$

Letting $X = \Delta[n]$ in this diagram suggests to define V_n as the set of all small maps with codomain $\Delta[n]$. In this way, however, one does not obtain a presheaf since the transition functions will satisfy the functorial laws only up to isomorphism rather than equality. To remedy this, the n-simplices of V are defined instead to be the functors $F\colon (\Delta/[n])^{\mathrm{op}} \to \mathbf{Set}$ such that the corresponding map of simplicial sets $\mathsf{El}(F) \to \Delta[n]$ is small. MORE TO BE ADDED.

Following [15, 39], we consider the pullback

$$\begin{tikzcd} \overline{\mathsf{U}} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{V}} \arrow[d, "\rho"] \\ \mathsf{U} \arrow[r] & \mathsf{V} \end{tikzcd}$$

where $\mathsf{U} \subseteq \mathsf{V}$ is defined by letting

$$\mathsf{U}_n = \{ F \in \mathsf{V}_n \mid \mathsf{El}(F) \to \Delta[n] \text{ is a small Kan fibration } \}$$

Proposition 4.2.

(i) $\pi\colon \overline{\mathsf{U}} \to \mathsf{U}$ *is a small Kan fibration.*

(ii) $\pi\colon \overline{\mathsf{U}} \to \mathsf{U}$ *classifies small Kan fibrations, i.e. for every small Kan fibration* $f\colon Y \to X$ *there exists a pullback diagram of the form*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

(iii) The simplicial set $\overline{\mathsf{U}}$ *is cofibrant.*

Proof. We prove the three claims separately.

(i) Should follow by locality.
(ii) Should be immediate.
(iii) See handwritten notes. Key step is the constructive version of the Eilenberg-Zilber lemma.

□

However, the simplicial set U does not appear to be cofibrant and hence it does not seem possible to show that $\pi \colon \overline{\mathsf{U}} \to \mathsf{U}$ is a weak classifier for small Kan fibrations with cofibrant codomain. In order to remedy this, we consider the cofibrant replacement U_c of U, which comes equipped with a trivial fibration $p \colon \mathsf{U}_c \to \mathsf{U}$, and the pullback

$$\begin{tikzcd} \overline{\mathsf{U}}_c \arrow[r] \arrow[d, "\pi_c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ \mathsf{U}_c \arrow[r, "p"'] & \mathsf{U} \end{tikzcd}$$

We can now prove that $\pi_c \colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ has the desired properties.

Proposition 4.3.

(i) $\pi_c \colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ *is a small Kan fibration with fibrant codomain.*
(ii) The map $\pi_c \colon \overline{\mathsf{U}}_c \to \mathsf{U}_c$ *classifies small Kan fibrations with cofibrant domain, i.e. for every small Kan fibration* $f \colon Y \to X$ *with* X *cofibrant there exists a pullback diagram of the form*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \overline{\mathsf{U}}_c \arrow[d, "\pi_c"] \\ X \arrow[r] & \mathsf{U}_c \end{tikzcd}$$

(iii) The simplicial set $\overline{\mathsf{U}}_c$ *is cofibrant.*

Proof. Part (i) follows from part (i) of Proposition 4.2. For part (ii), let $f \colon Y \to X$ be a small Kan fibration with X cofibrant. Since f is a small Kan fibration, we know from Proposition 4.2 that there is a pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

Since X is cofibrant, we have the lifting diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathsf{U}_c \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & \mathsf{U} \end{tikzcd}$$

which shows that the map $X \to \mathsf{U}$ factors via U_c. We then obtain the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \overline{\mathsf{U}}_c \arrow[r] \arrow[d, "\pi_c"] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{\mathsf{U}} \arrow[d, "\pi"] \\ X \arrow[r] & \mathsf{U}_c \arrow[r, "p"'] & \mathsf{U} \end{tikzcd}$$

Here, the right-hand side square and the rectangle are pullbacks and therefore the left-hand side square is also a pullback, as required. Part (iii) follows from the fact that both U_c and $\overline{\mathsf{U}}$ are cofibrant, the latter being part (iii) of Proposition 4.2. $\square$

Note that we have not shown yet that U_c fibrant. This will be done in Section 6, as a consequence of the equivalence extension property for fibrations, which we establish in Section 5.

5. The equivalence extension property

- Here we follow Kapulkin and Lumsdaine.

6. Fibrancy and univalence of the universe

- Fibrancy should follow directly from equivalence extension property, without using 'composition vs filling' but rather retract property for horns (see notes).
- Once we have established fibrancy of U_c, then one can prove univalence by showing that $t\colon \mathsf{Weq}(U_c) \to U_c$ is a trivial fibration.
- Question: do we need to know that $\mathsf{Weq}(U_c)$ is a cofibrant object to get univalence?

7. Semantics

- This should be essentially straightforward, following Kapulkin and Lumsdaine, but we may need to modify the notion of a Π-structure to accommodate the cofibrant replacements that we take for Π.

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