MATH255: Mathematics for Computing

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Definition

*The **universal quantifier** is denoted by $\forall$ and indicates "for all".*

*The **existential quantifier** is denoted by $\exists$ and indicates "there exists".*

Definition

*A **predicate** is an expression that becomes T or F when values are substituted for the variables.*

*The **domain** of a predicate is the set of all possible values it can take on.*

*The **truth set** of a predicate is the subset of the domain that makes the predicate true.*

Example

What is the truth set of $p(x)$: x is an integer less than 5, $\operatorname{dom} p = \mathbb{Z}$?

Answer. Truth set: $\{\ldots, -2, -1, 0, 1, 2, 3, 4\}$

Example

What is the truth set of $q(x)$: $x^2 > x, \operatorname{dom} q = \mathbb{R}$?

Answer. Truth set: $(-\infty, 0) \cup (1, \infty)$

Definition

*The general form of a **universal statement** is as follows.*

$$\forall\, x \in D, p(x)$$

"For all x in the domain of predicate p, $p(x)$."

- All humans are mortal.
- Every real number has a nonnegative square: $\forall\, x \in \mathbb{R}, x^2 \geq 0$.

To prove a universal statement, you must show that the statement is true for every element in the domain. To disprove a universal statement, you need to find one counterexample, i.e. a member of the domain that makes the statement false.

Example

Prove or disprove: $\forall\, x \in \mathbb{R}, x^2 > x$.

This is false; there are many counterexamples. For instance, $x = \frac{1}{2}$ gives $\left(\frac{1}{2}\right)^2 > \frac{1}{2} \rightarrow \frac{1}{4} > \frac{1}{2}$, which is false.

Example

Prove or disprove: every integer greater than zero has a prime factor.

False: let $x = 1$. Then x does not have a prime factor, since the primes start at 2.

Example

Let $D = \{1, 2, 3, 4, 5\}$. Show that

(a) $\forall\ x \in D, x^2 \geq x$ *is TRUE, and*

(b) $\forall\ x \in D, \frac{1}{x^2} < \frac{1}{x}$ *is FALSE.*

(a) $1^2 \geq 1 \leftrightarrow 1 \geq 1$ T
$2^2 \geq 2 \leftrightarrow 4 \geq 2$ T
$3^2 \geq 3 \leftrightarrow 9 \geq 3$ T
$4^2 \geq 4 \leftrightarrow 16 \geq 4$ T
$5^2 \geq 5 \leftrightarrow 25 \geq 5$ T: therefore, $\forall\ x \in D, x^2 \geq x$ is TRUE.

(b) $\frac{1}{1^2} < 1 \leftrightarrow 1 < 1$ F: therefore, $\forall\ x \in D, \frac{1}{x^2} < \frac{1}{x}$ is FALSE.

Definition

*The general form of an **existential statement** is as follows.*

$$\exists\, x \in D \text{ such that } p(x)$$

"There exists x in the domain of predicate p such that $p(x)$."

- There is a cat in my house.
- There exist integers m, n such that $m + n = mn$.

To prove an existential statement, you must find one element of the domain that makes the statement true. To disprove an existential statement, you need to prove the statement false for every element of the domain.

Example

Show that the statement $\exists\ m \in \mathbb{Z}\ \texttt{s.t.}\ m^2 = m$ *is TRUE.*

$1^2 = 1$, or $0^2 = 0$: therefore, $\exists\ m \in \mathbb{Z}\ \texttt{s.t.}\ m^2 = m$ is TRUE.

Example

Let $E = \{5, 6, 7, 8, 9, 10\}$*. Show that the statement* $\exists\ m \in E\ \texttt{s.t.}\ m^2 = m$ *is FALSE.*

$5^2 = 25 \neq 5$
$6^2 = 36 \neq 6$
$7^2 = 49 \neq 7$
$8^2 = 64 \neq 8$
$9^2 = 81 \neq 9$
$10^2 = 100 \neq 10$: therefore, $\exists\ m \in E\ \texttt{s.t.}\ m^2 = m$ is FALSE.

Negation of Universal Statement

What is the negation of, "all mathematicians wear glasses"?

Negation of Universal Statement

What is the negation of, "all mathematicians wear glasses"?

"There exists a mathematician who does not wear glasses."

Negation of Universal Statement

What is the negation of, "all mathematicians wear glasses"?

"There exists a mathematician who does not wear glasses."

Definition

The negation of the universal statement

$$\forall\, x \in D, p(x)$$

is the existential statement

$$\exists\, x \in D \text{ s.t. } \sim p(x).$$

Negation of Universal Statement

Example

No computer hacker is over 40 years old.

There is a computer hacker over 40 years old.

All prime numbers are odd.

There exists an even prime number.

Every blonde person has blue eyes.

There is a blonde person who does not have blue eyes.

$\forall\, x \in \mathbb{R}, \frac{1}{x} > 1$

$\exists\, x \in \mathbb{R}$ s.t. $x \leq 1$

What is the negation of, "some fish breathe air"?

Negation of Existential Statement

What is the negation of, "some fish breathe air"?

"No fish breathes air."

What is the negation of, "some fish breathe air"?

"No fish breathes air."

Definition

The negation of the existential statement

$$\exists\, x \in D \text{ s.t. } p(x)$$

is the universal statement

$$\forall\, x \in D, \sim p(x).$$

Example

There is a triangle whose sum of angles is 200 degrees.

All triangles have sum of angles not equal to 200 degrees.

There is a 120-year-old woman in Australia.

All women in Australia are not 120 years old.

$\exists\, x \in \mathbb{R}$ s.t. $x^2 = -1$

$\forall\, x \in \mathbb{R}, x^2 \neq -1$

Proof of Valid Argument

Definition

*An **argument** is a sequence of statements, all of which are assumptions except the last one, which is the conclusion. If the conclusion is true whenever all assumptions are true, then the argument is valid.*

Example

If x is a pig, then x is pink. (assumption)

Peppa is a pig. (assumption)

Therefore, Peppa is pink. (conclusion)

Proof of Valid Argument

Definition

*A **proof** is a valid argument used to establish a result.*

Example

Prove that if $x \in \mathbb{R}$ and n is even, then $x^n \geq 0$.

Proof.

Since n is even, we know that $n = 2m$ for some $m \in \mathbb{Z}$. Then

$$x^n = x^{2m} = (x^m)^2 \geq 0$$

$\square$

To test an argument for validity, use a truth table. Identify all the rows in which all assumptions are true and verify that the conclusion is also true in those rows.

Example

$p \rightarrow (q \vee \sim r)$
$q \rightarrow (p \wedge r)$
$\therefore p \rightarrow r$

p	q	r	$q \vee \sim r$	$p \wedge r$	$p \rightarrow (q \vee \sim r)$	$q \rightarrow (p \wedge r)$	$p \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	T	F	T	T
T	F	F	T	F	T	T	F
F	T	T	T	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	F	F	T	T	T
F	F	F	T	F	T	T	T

Since there is a row that has true assumptions and false conclusion, the argument is **invalid.**

Proof of Valid Argument

Homework. Test the validity.

(a) $p \vee (q \vee r)$
$\sim r$
$\therefore p \vee r$

(b) $p \rightarrow q$
p
$\therefore q$

Definition

An argument that has two assumptions is called a ***syllogism****.*
The most basic form of syllogism is the ***modus ponens*** *form:*

$p \to q$
p
$\therefore q$

Definition

The ***law of syllogism*** *is as follows.*

$$(p \to q) \wedge (q \to r) \to (p \to r)$$

Homework. Prove it, either by quick method or by truth table.

Proof of Existential Statement

To prove a statement of the form

$$\exists\, x \in D \text{ s.t. } p(x),$$

one must find at least one $x \in D$ that makes $p(x)$ true.

Example

Prove there is an even number that can be written in two different ways as the sum of two primes.

Proof.

$$\begin{aligned} 2 &= 2 \\ 4 &= 2+2 \\ 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 5+5 = 3+7 \end{aligned}$$

□

Proof of Existential Statement

Example

Prove that there exist $m, n \in \mathbb{N}$ whose sum of reciprocals is an integer: $\frac{1}{m} + \frac{1}{n} \in \mathbb{Z}$.

Proof.

Let $m = n = 1$. Then

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{1} + \frac{1}{1} = 1 + 1 = 2.$$

$\square$

Proof of Universal Statement

To prove a statement of the form

$$\forall\, x \in D, p(x),$$

one must prove that $p(x)$ is true for every possible choice of $x \in D$. If the domain is reasonably small, the method of exhaustion involves substituting each $x \in D$ into p and verifying the $p(x)$ is true for each one.

Proof of Universal Statement

Example

Prove that every number that is even between 4 and 18 is the sum of two prime numbers.

Proof.

$$\begin{aligned}
4 &= 2 + 2 \\
6 &= 3 + 3 \\
8 &= 3 + 5 \\
10 &= 3 + 7 \\
12 &= 5 + 7 \\
14 &= 7 + 7 \\
16 &= 5 + 11
\end{aligned}$$

□

Proof of Universal Statement

Homework. Investigate proofs (not by exhaustion!) that every even number can be written as the sum of two prime numbers.

Another proving method is called the generalised proof, in which arbitrary (unspecified) elements of the domain are used to shoe validity for all elements.

Example

Prove that if $a, b \in \mathbb{Z}$, then $10a + 8b$ is even.

Proof.

Let $a, b \in \mathbb{Z}$. Then

$10a + 8b = 2(5a + 4b)$.
Since $a, b \in \mathbb{Z}$, $5a$ and $4b$ are also in $\mathbb{Z}$, as is the sum $5a + 4b$.
Therefore,

$2(5a + 4b)$ is even. $\square$

Disproof of Existential Statement

To disprove a statement, prove its negation. In the case of an existential statement, the negation is universal.

$$\sim (\exists\, x \in D \text{ s.t. } p(x)) \equiv \ \forall\, x \in D, \sim p(x)$$

Example

Disprove: there exists an even prime number greater than 2.

Proof.

The negation is: all prime numbers greater than 2 are odd. Let $x > 2$ be prime. Suppose that x is even. Then $x = 2m, m \in \mathbb{Z}$ and

$$\frac{x}{2} = \frac{2m}{2} = m \in \mathbb{Z},$$

which means that x is divisible by 2, thus not a prime number. This is a contradiction, since we started out with, 'let x be prime'. So our supposition that x is even is false, therefore, x is odd. $\square$

In the case of a universal statement, the negation is existential.

$$\sim (\forall\, x \in D, p(x)) \equiv\ \exists\, x \in D \texttt{ s.t. } \sim p(x)$$

Example

Disprove: $\forall\, x \in \mathbb{R}, x < 0\ \vee\ x > 0.$

Proof.

The negation is $\exists\, x \in \mathbb{R} \texttt{ s.t. } x \geq 0\ \wedge\ x \leq 0$. Let $x = 0$. Then $x \geq 0$ and $x \leq 0$. $\square$

Example

Disprove: $\forall\ a, b \in \mathbb{R}, a^2 = b^2 \to a = b.$

Proof.

Let $a = -1, b = 1$. Then $a^2 = (-1)^2 = 1$ and $b^2 = 1^2 = 1$, so $a^2 = b^2$, but $a \neq b$. $\square$

Homework. Prove or disprove: $\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R}\ \texttt{s.t.}\ x + y = 0.$

Direct Proof

A **direct proof** of a statement begins with supposing that the assumptions are true and using them to prove that the conclusion must also be true.

Example

Prove that if $3x - 9 = 15$, *then* $x = 8$.

Proof.

$$
\begin{aligned}
3x - 9 &= 15 \\
3x - 9 + 9 &= 15 + 9 \\
3x &= 24 \\
\frac{3x}{3} &= \frac{24}{3} \\
x &= 8
\end{aligned}
$$

$\square$

Example

Prove that the sum of any two even numbers is even.

Proof.

Let a, b be even numbers. Then $\exists\ c, d \in \mathbb{Z}$ s.t. $a = 2c, b = 2d$.

$$\begin{aligned} a + b &= 2c + 2d \\ &= 2(c + d) \end{aligned}$$

Since $c, d \in \mathbb{Z}$, $c + d \in \mathbb{Z}$. So $a + b$ is an integer times two, therefore, it's an even number. $\square$

Example

Prove that if a, b are perfect squares, then ab is a perfect square.

Proof.

We have $a = c^2, b = d^2$ for some $c, d \in \mathbb{Z}$.

$$\begin{aligned} ab &= c^2 d^2 \\ &= (cd)^2 \end{aligned}$$

Since $c, d \in \mathbb{Z}$, $cd \in \mathbb{Z}$. Therefore, ab is a perfect square. $\square$

Example

Prove that $\forall\ x \in \mathbb{R}, -x^2 + 2x + 1 \leq 2.$

Proof.

Let $x \in \mathbb{R}$.

$$\begin{aligned} -x^2 + 2x + 1 \leq 2 &\Leftrightarrow -x^2 + 2x + 1 - 2 \leq 0 \\ &\Leftrightarrow (-1)(-x^2 + 2x - 1) \geq (-1)0 \\ &\Leftrightarrow x^2 - 2x + 1 \geq 0 \\ &\Leftrightarrow (x-1)^2 \geq 0 \ \ \text{TRUE} \end{aligned}$$

$\square$

Proof by Contradiction

To prove a statement **by contradiction**, suppose that the conclusion is false and show that an assumption also must be false. This means that if all assumptions are true, the conclusion must be true.

Homework. Prove by truth table that $p \to q \equiv \sim q \to \sim p$.

Example

Prove that $\forall\ n \in \mathbb{N}$, *if* n^2 *is even, then* n *is even.*

Proof.

Suppose that n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} n^2 &= (2k+1)^2 = 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 = \quad \text{ODD} \end{aligned}$$

Therefore, if n^2 is even, then n is even. $\square$

Proof by Contradiction

Example

Prove that $y \in \mathbb{R} \setminus \mathbb{Q} \to y + 7 \in \mathbb{R} \setminus \mathbb{Q}$.

Proof.

Suppose $y + 7 \in \mathbb{Q}$. Then $\exists\ a, b \in \mathbb{Z}$ $\texttt{s.t.}$ $y + 7 = \frac{a}{b}$.

$$\begin{aligned} y + 7a = \frac{a}{b} &\Leftrightarrow y = \frac{a}{b} - 7 \\ &\Leftrightarrow y = \frac{a - 7b}{b} \in \mathbb{Q} \end{aligned}$$

Therefore, $y \in \mathbb{R} \setminus \mathbb{Q} \to y + 7 \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

Example

Prove that there is an infinite number of prime numbers.

Proof.

Suppose there is a finite number of prime numbers: $\{p_1, p_2, \ldots, p_n\}$. Define $p = p_1 p_2 \cdots p_n + 1$ and note that $p > p_n$. Now for any prime number p_i, we have

$$\begin{aligned} \frac{p}{p_i} &= \frac{p_1 p_2 \cdots p_n + 1}{p_i} = \frac{p_1 p_2 \cdots p_n}{p_i} + \frac{1}{p_i} \\ &= p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n + \frac{1}{p_i} \notin \mathbb{Z} \end{aligned}$$

Since p is not divisible by any prime number p_i, p is prime. This is a contradiction, since $\{p_1, \ldots, p_n\}$ is the list of all primes. Therefore, there is an infinite number of prime numbers. $\square$