6.7900: Machine Learning
Lecture 5

Lecture start: Tues/Thurs 2:35pm
Who's speaking today? Prof. Tamara Broderick
Course website: gradml.mit.edu
Questions? Ask here or on piazza.com/mit/fall2024/67900/
Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Times

I. MLE, ERM, Bayes
II. Linear regression
    A. Motivation
    B. MLE & ERM

Upcoming

I. Challenges with MLE/ERM for linear regression
II. Bayesian linear regression
    A. Bayes with Gaussians
    B. Posterior, predictive

Recap

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Proposition from Lecture 1

- **Proposition**. Consider regression with $\mathcal{X} = \mathbb{R}^D, \mathcal{Y} = \mathbb{R}$
  - Assume *X*,*Y* density *p*(*x*,*y*) & square loss $L(a, g) = (a - g)^2$
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  - What if those conditions don't hold?

Visualizing the full-rank case

- We want to minimize $\mathrm{RSS}(\theta) = (X\theta - Y)^\top (X\theta - Y)$
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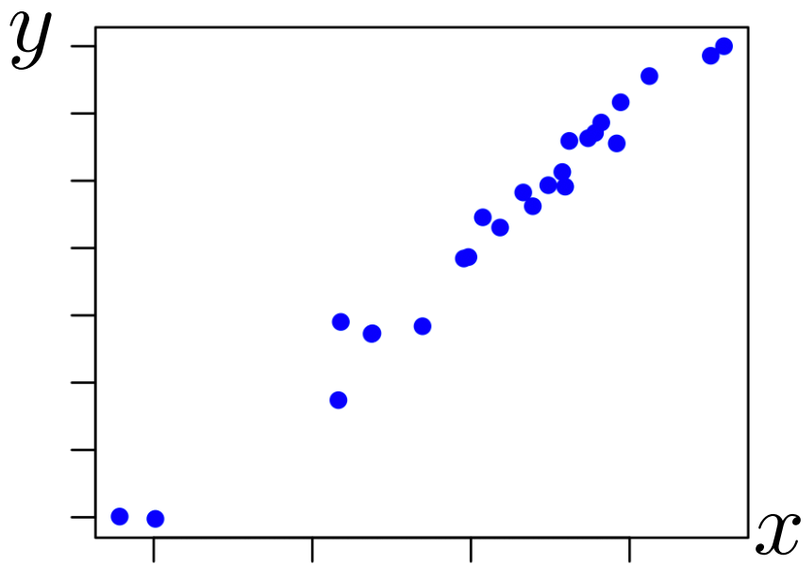

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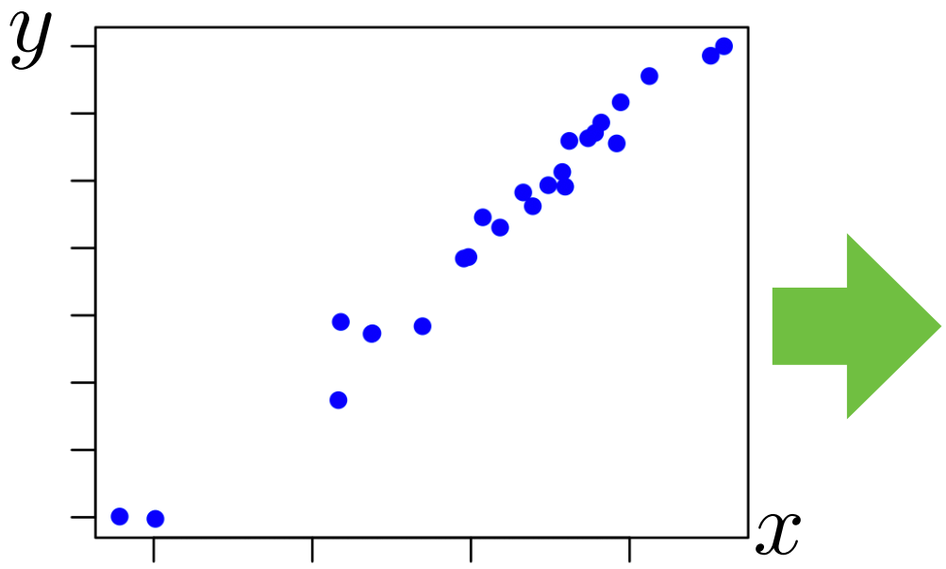

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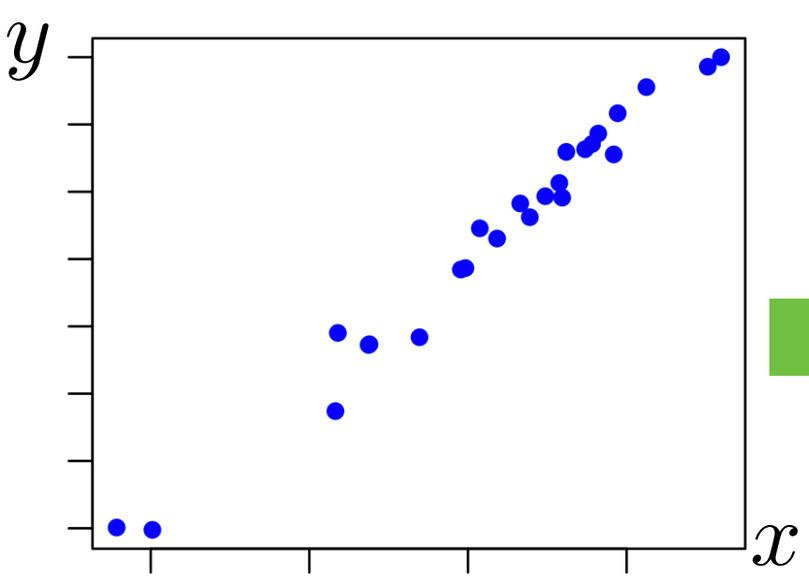

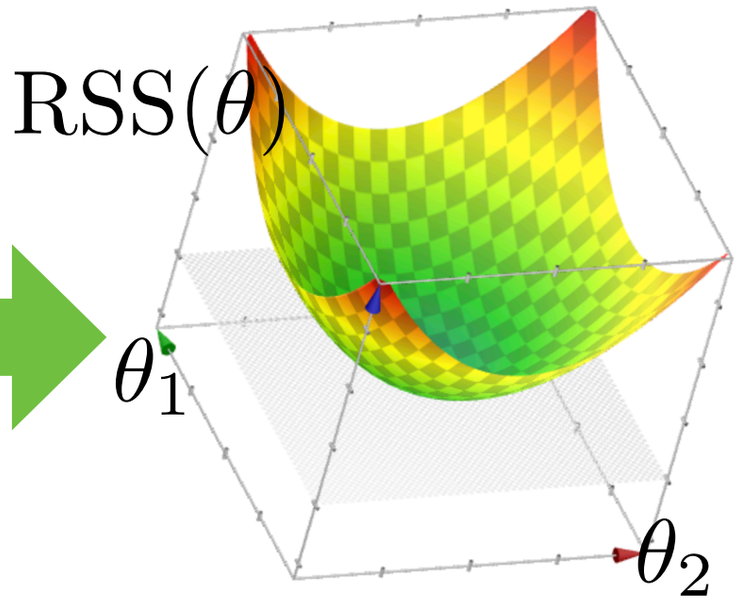

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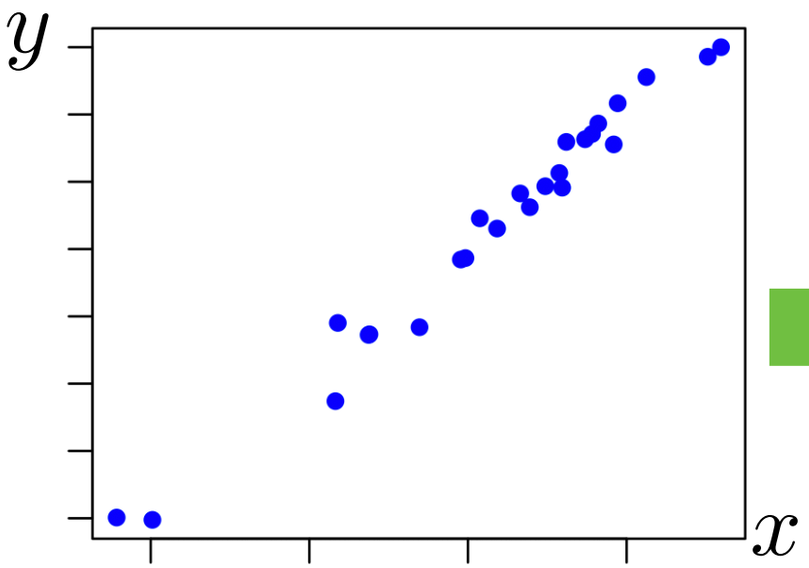

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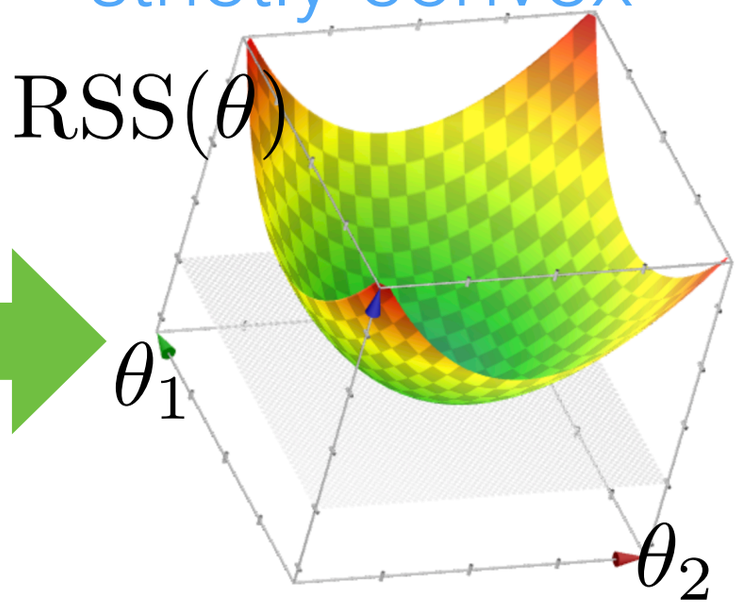

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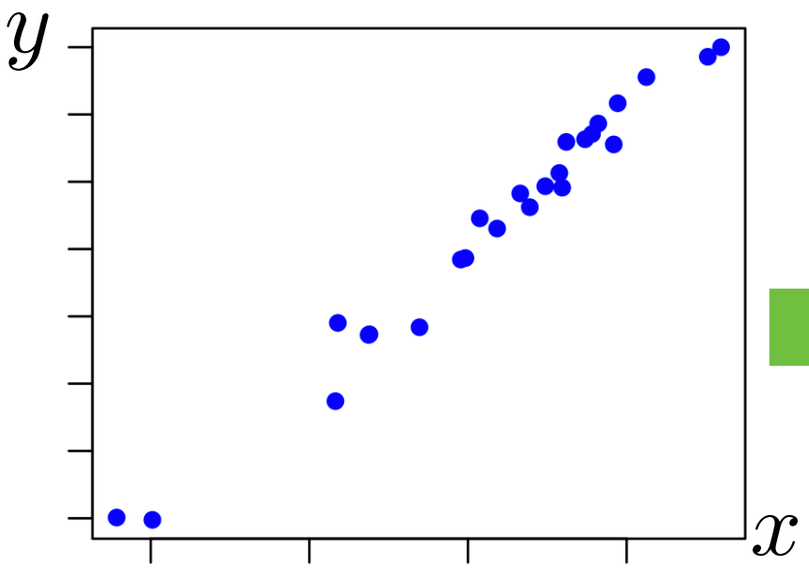

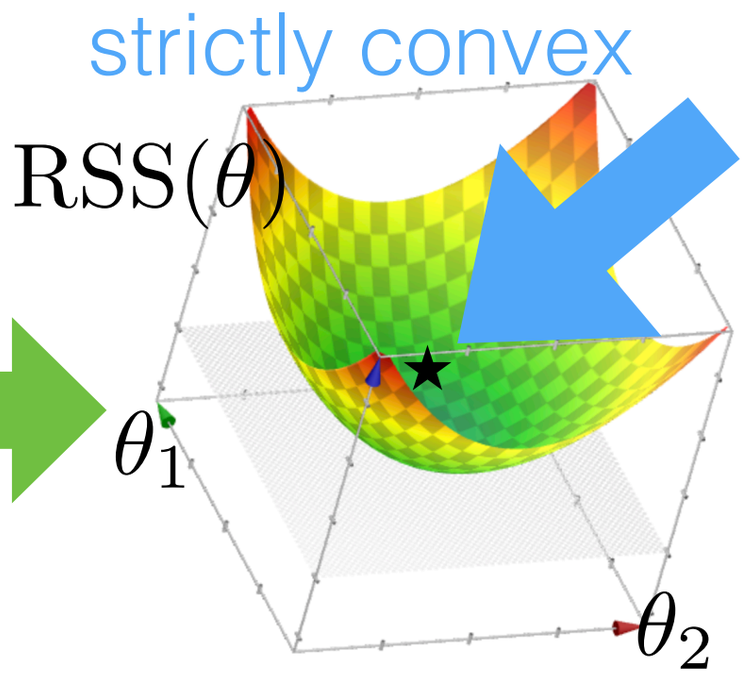

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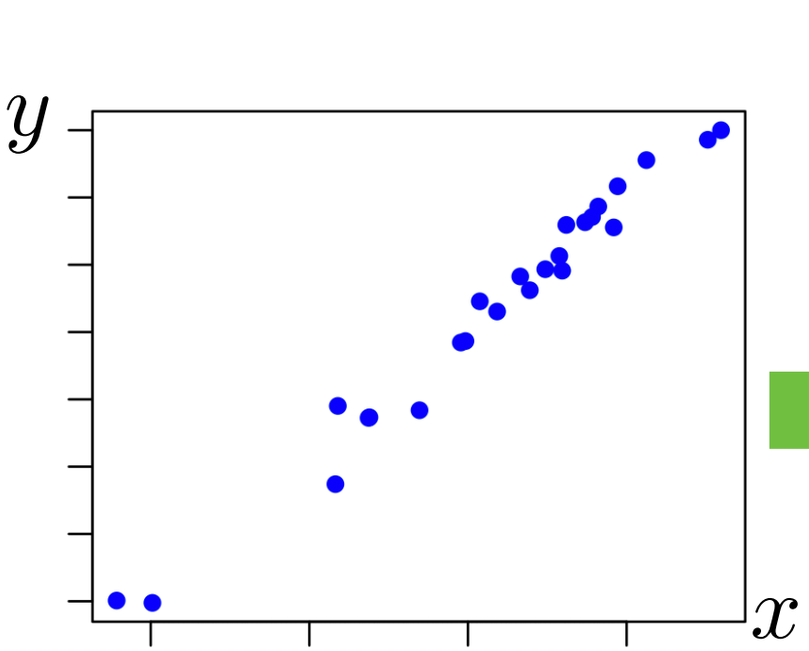

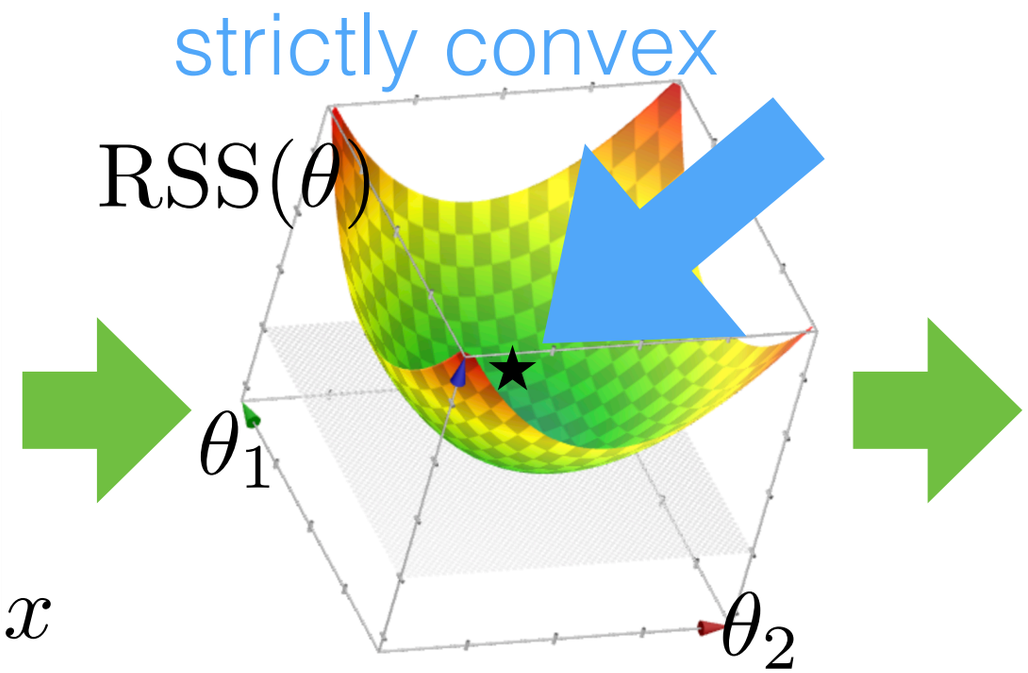

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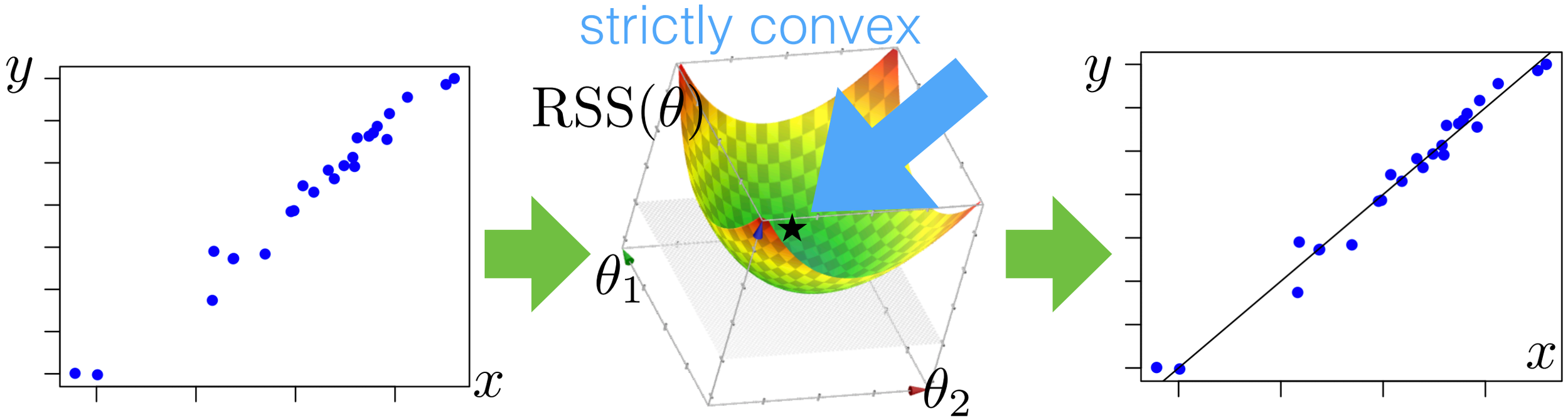

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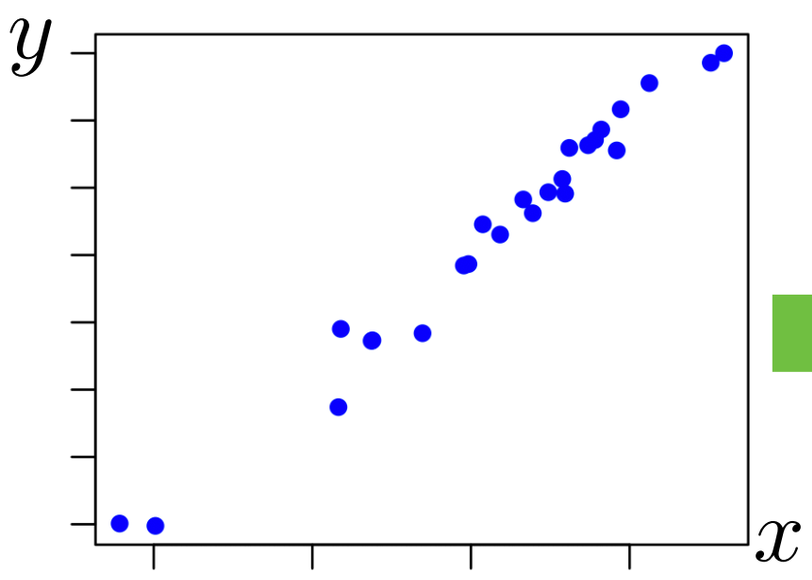

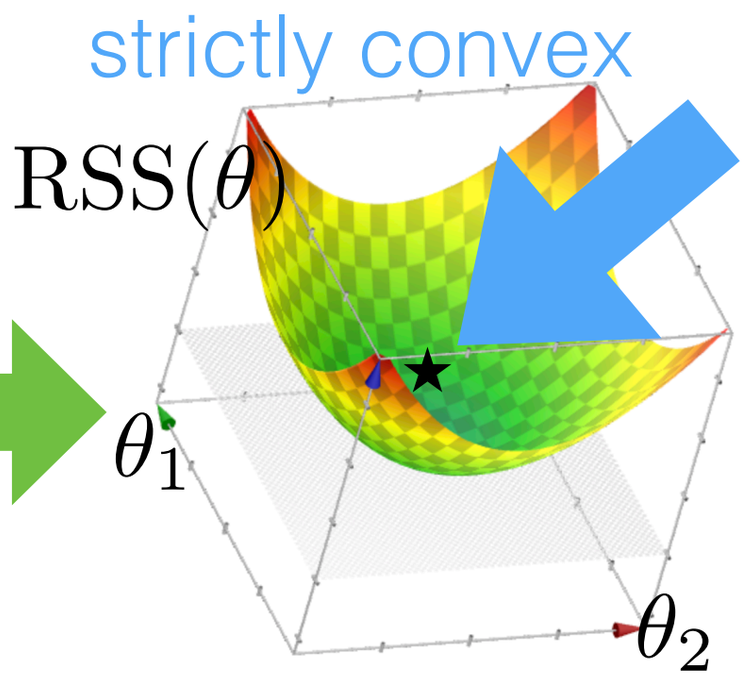

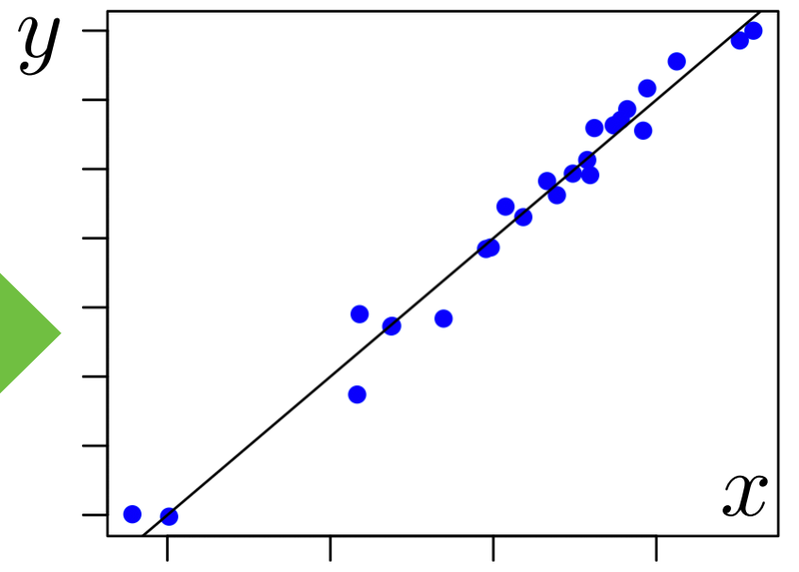

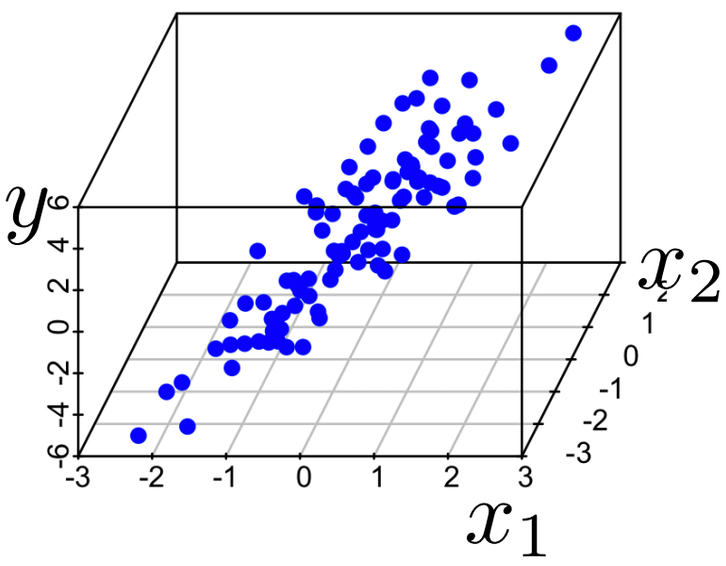

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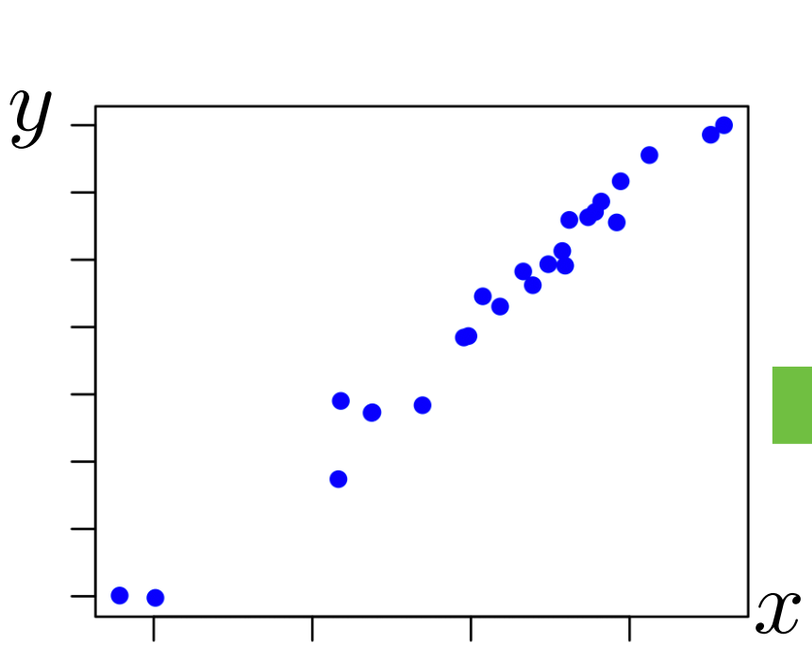

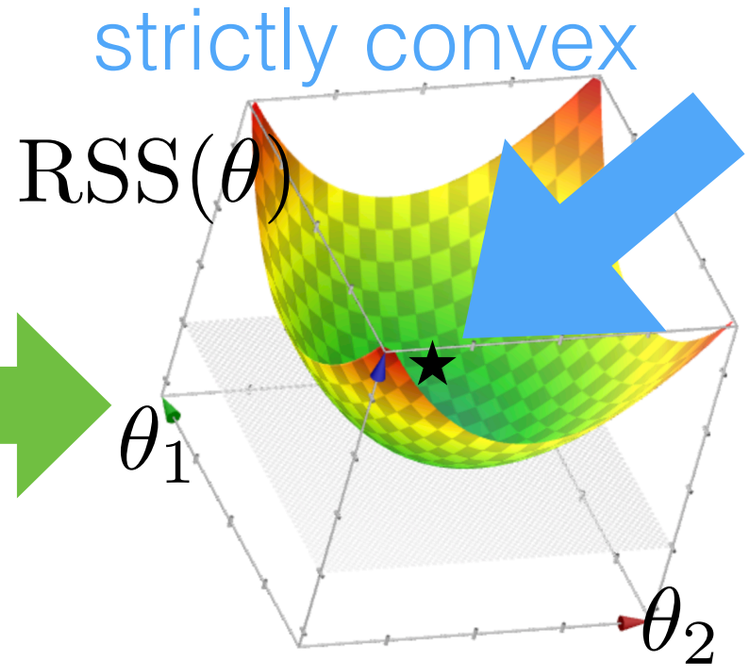

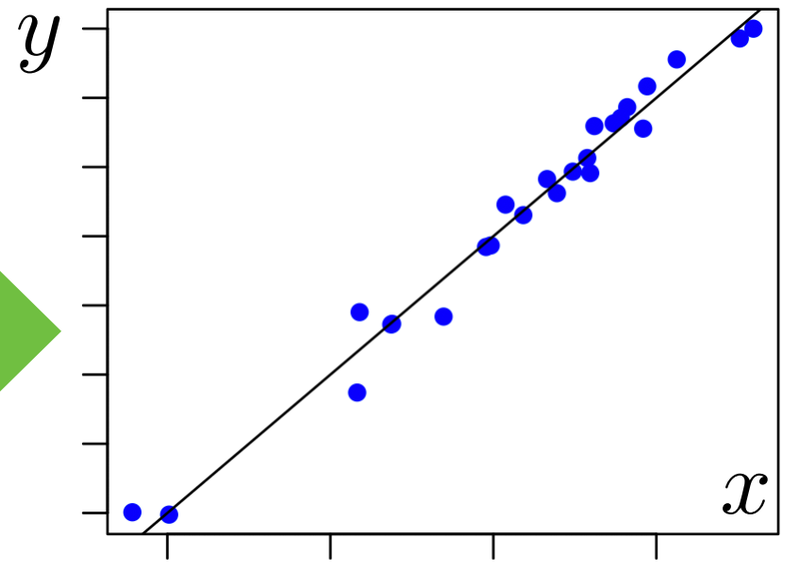

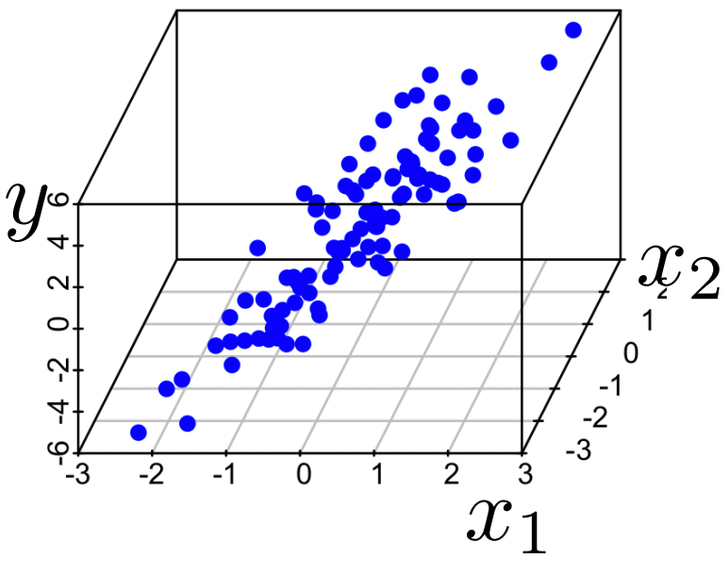

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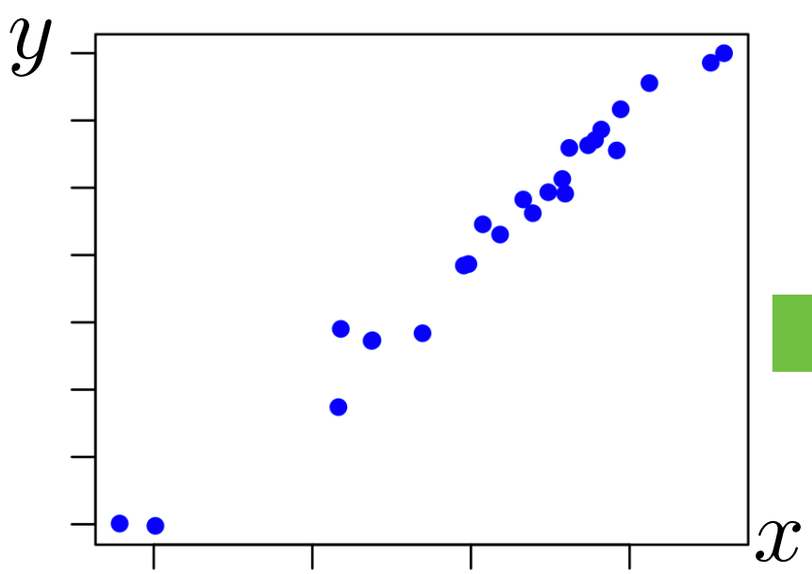

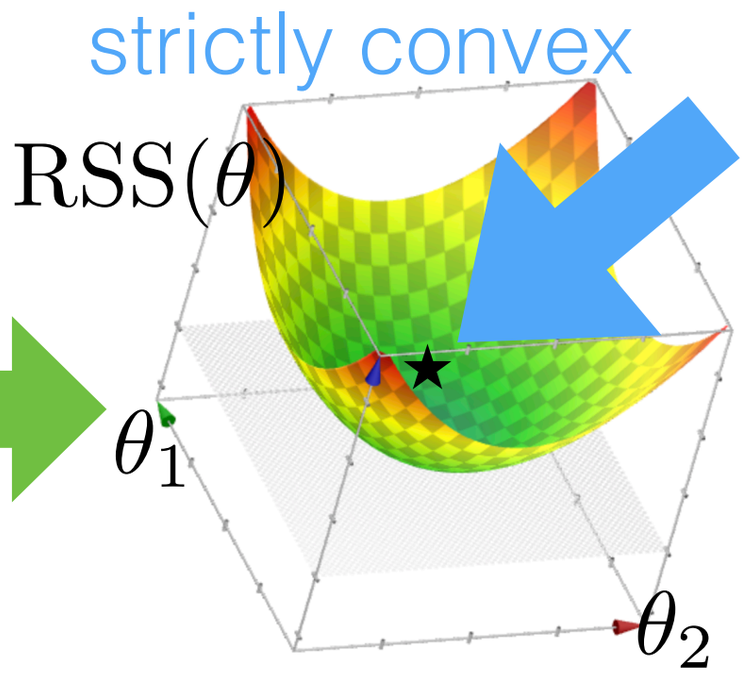

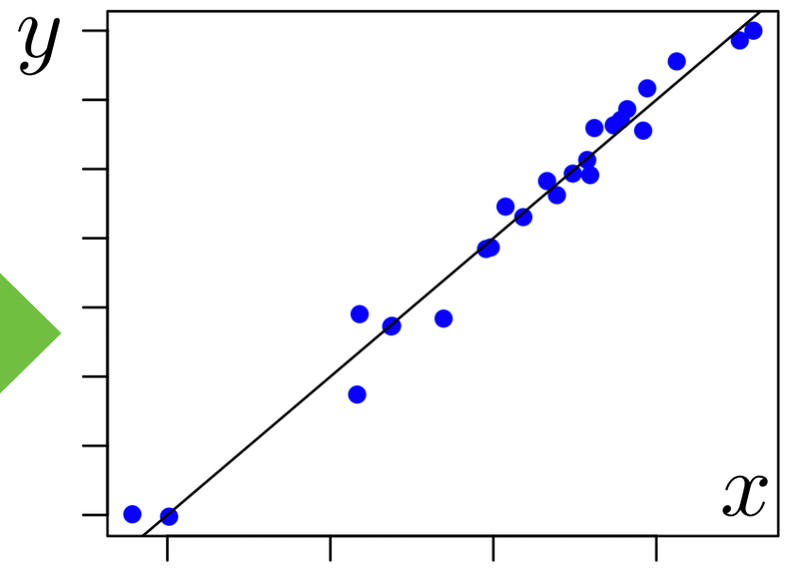

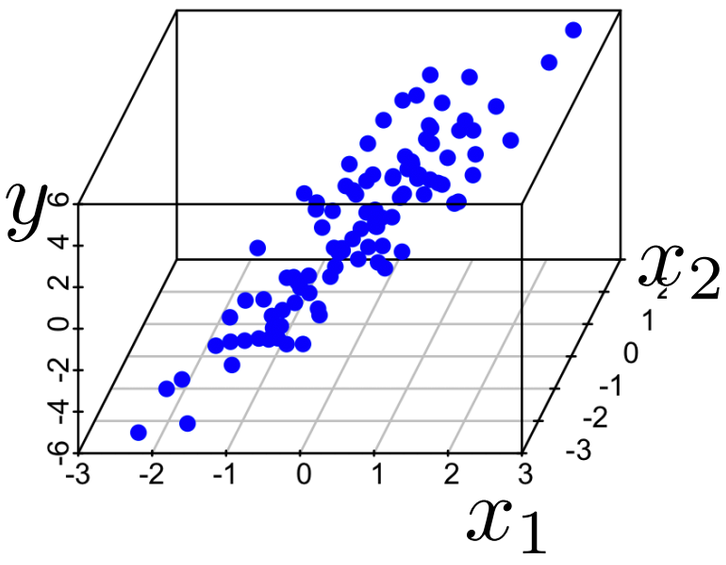

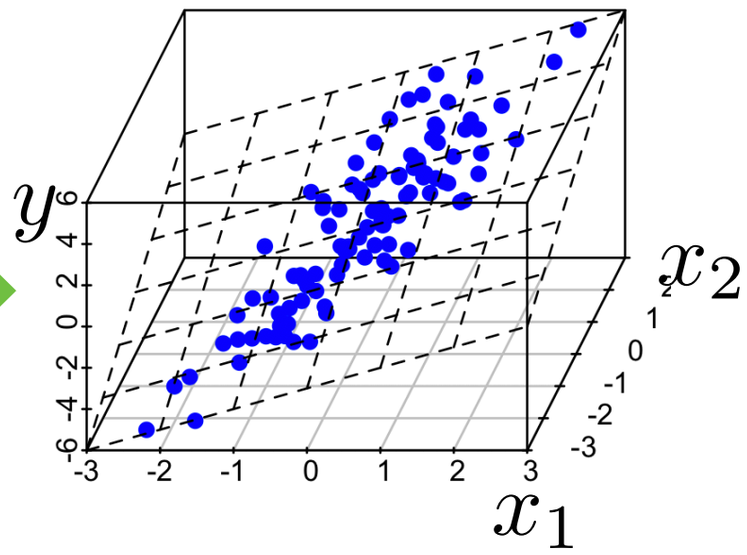

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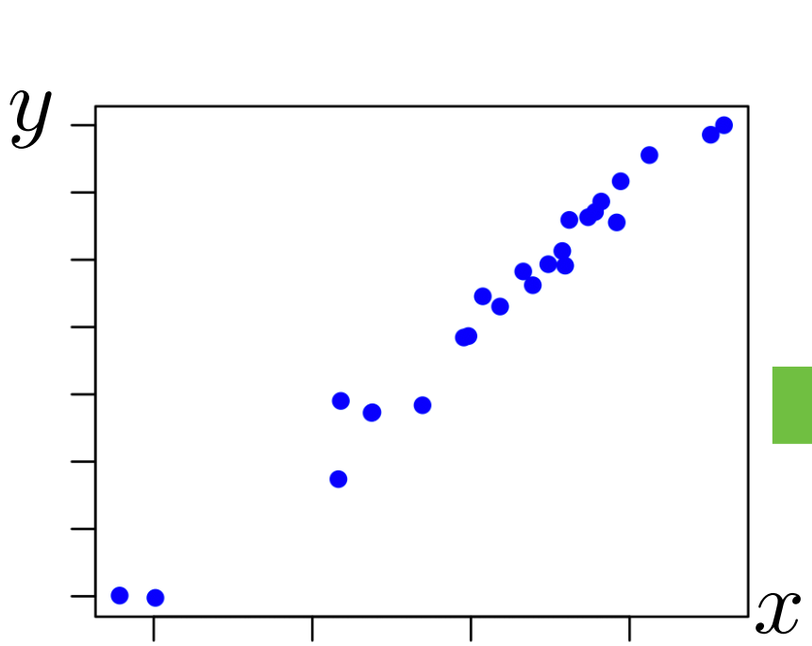

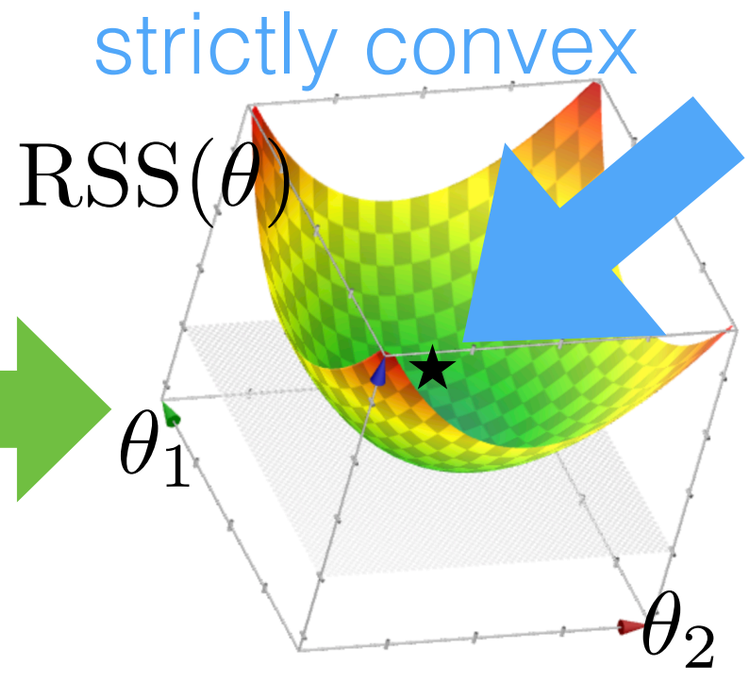

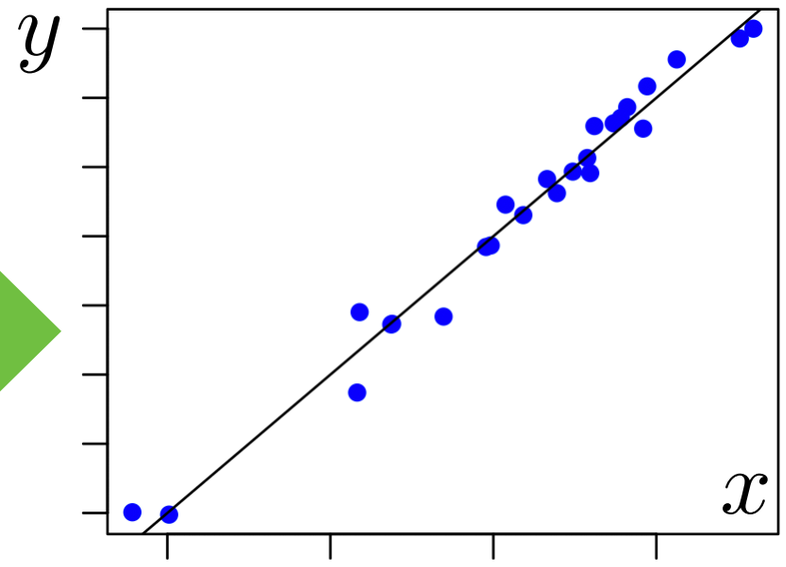

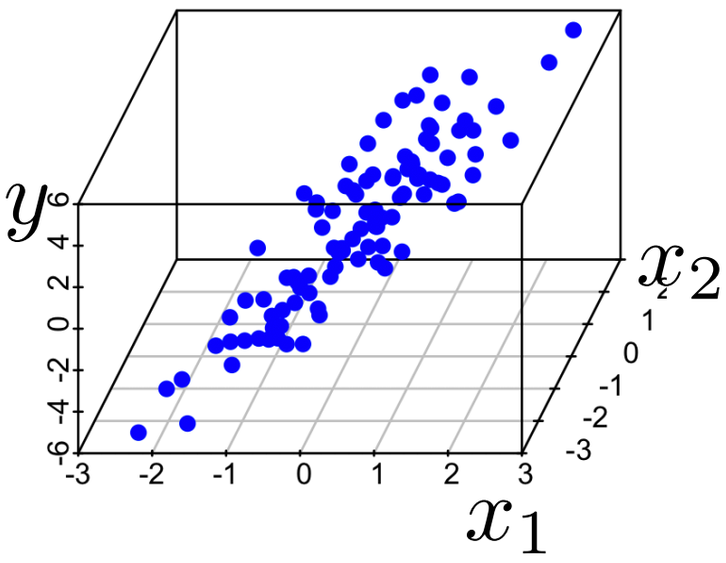

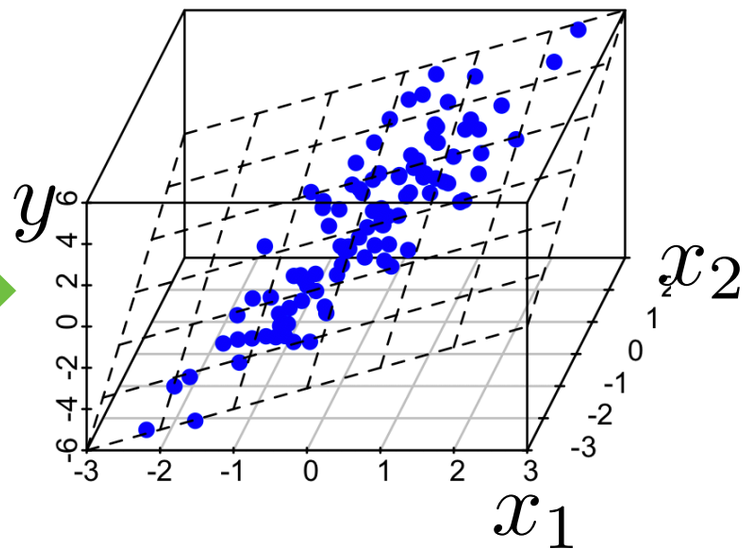

- What could go wrong?

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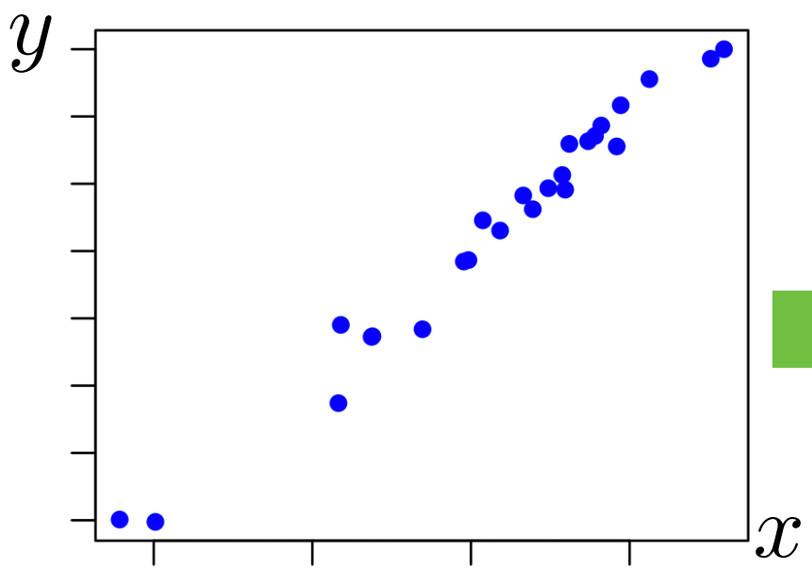

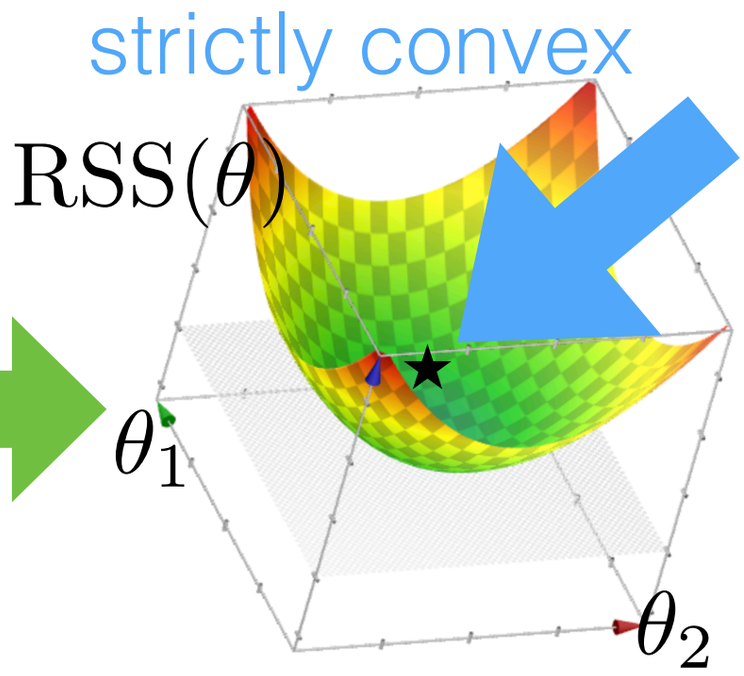

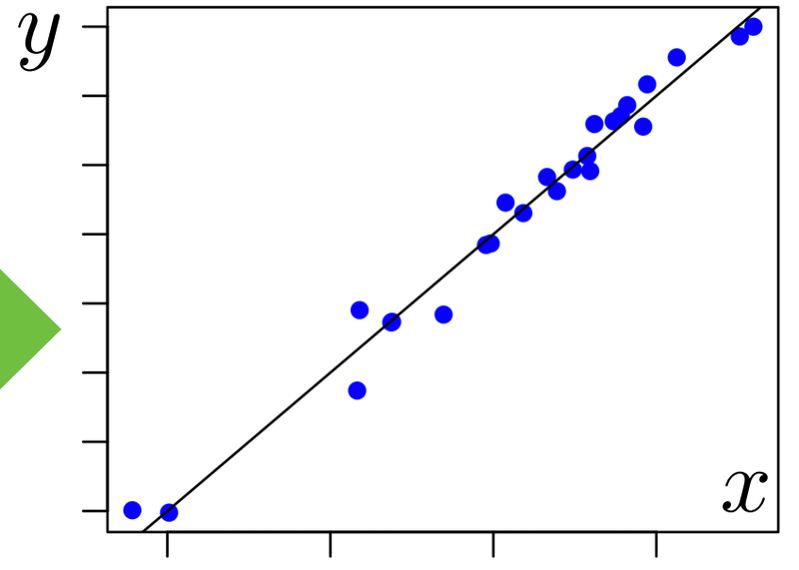

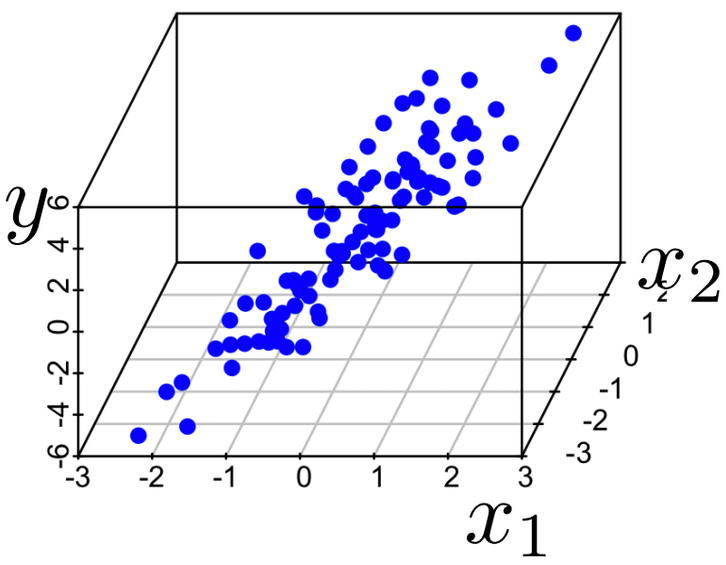

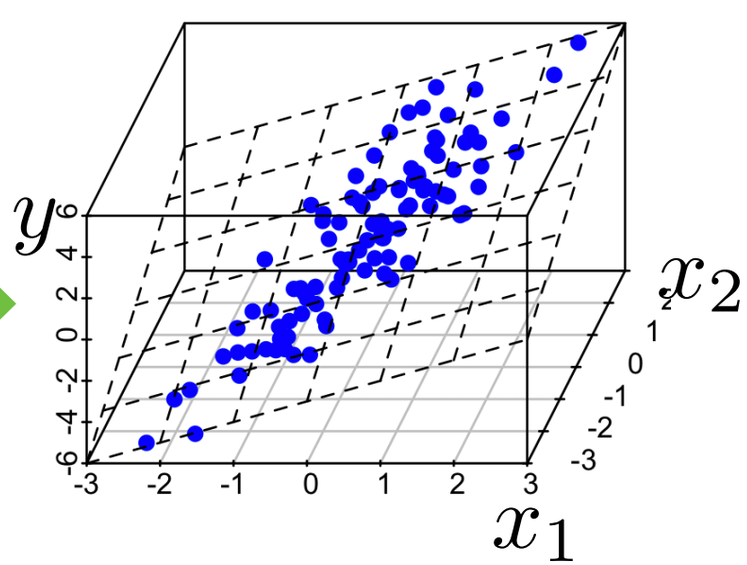

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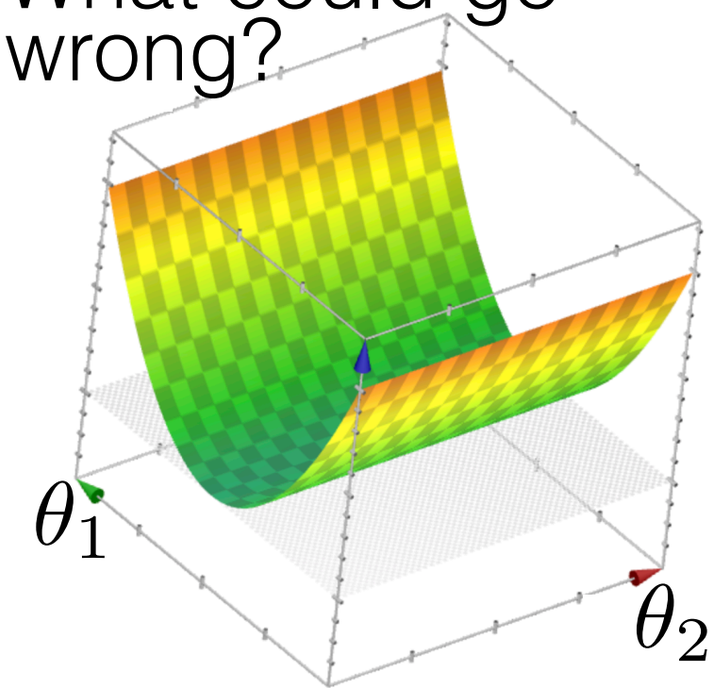

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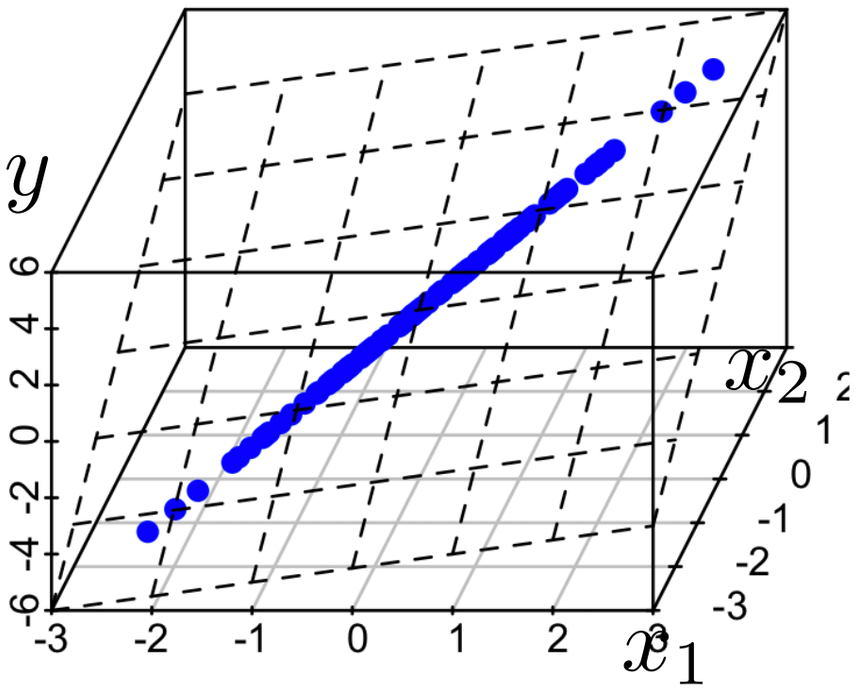

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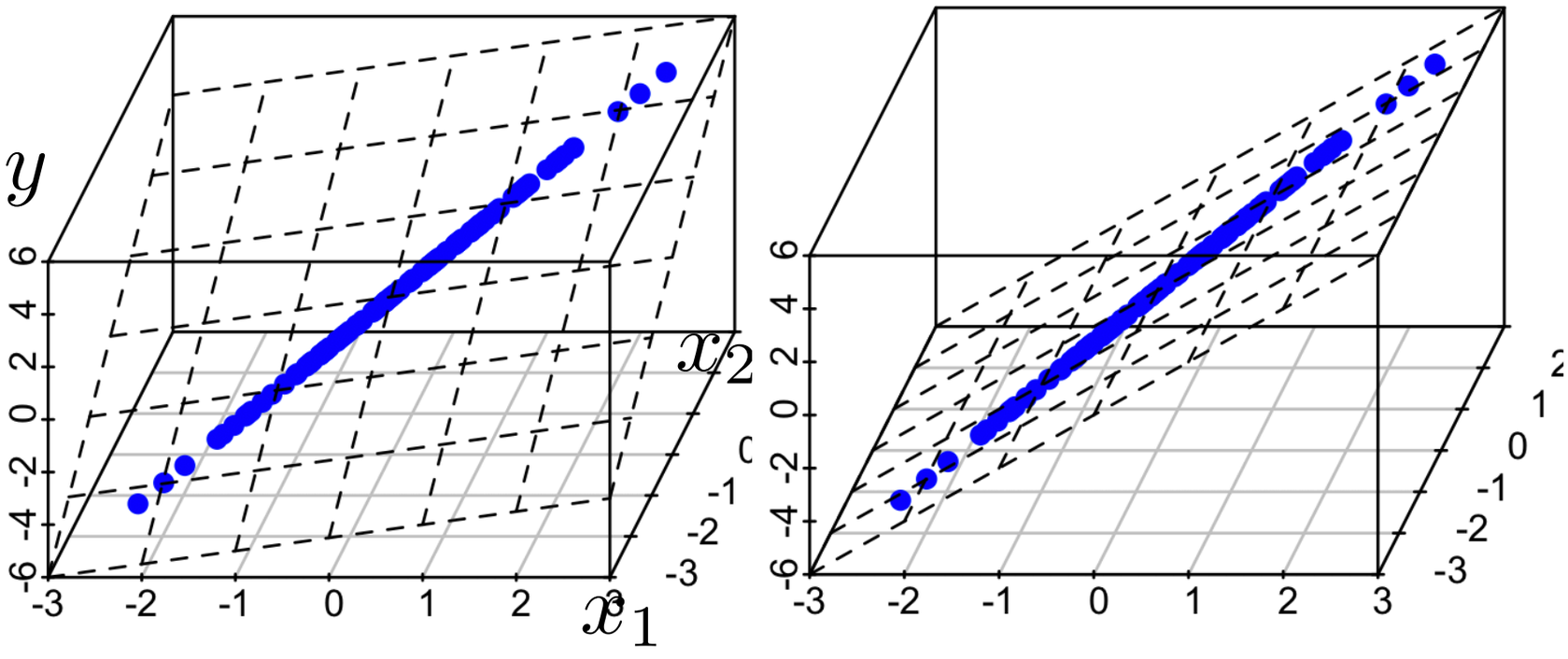

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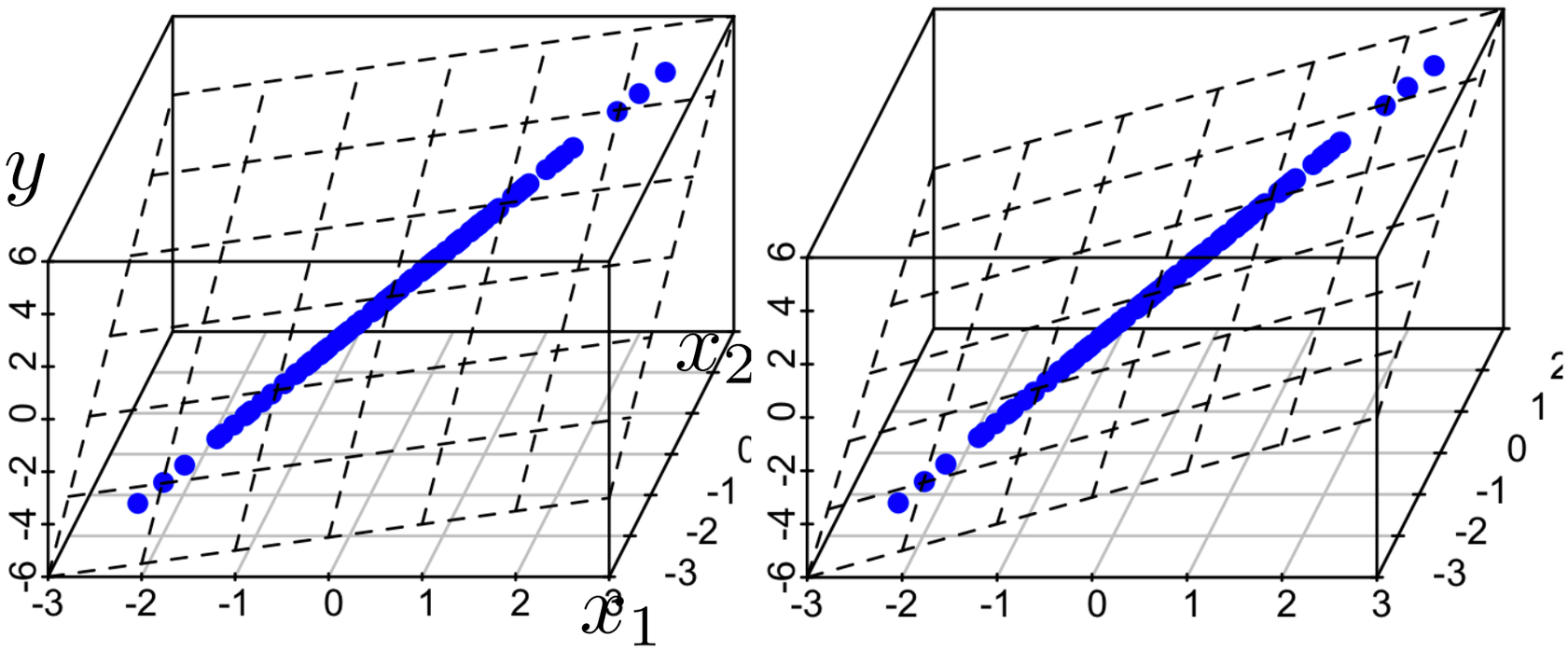

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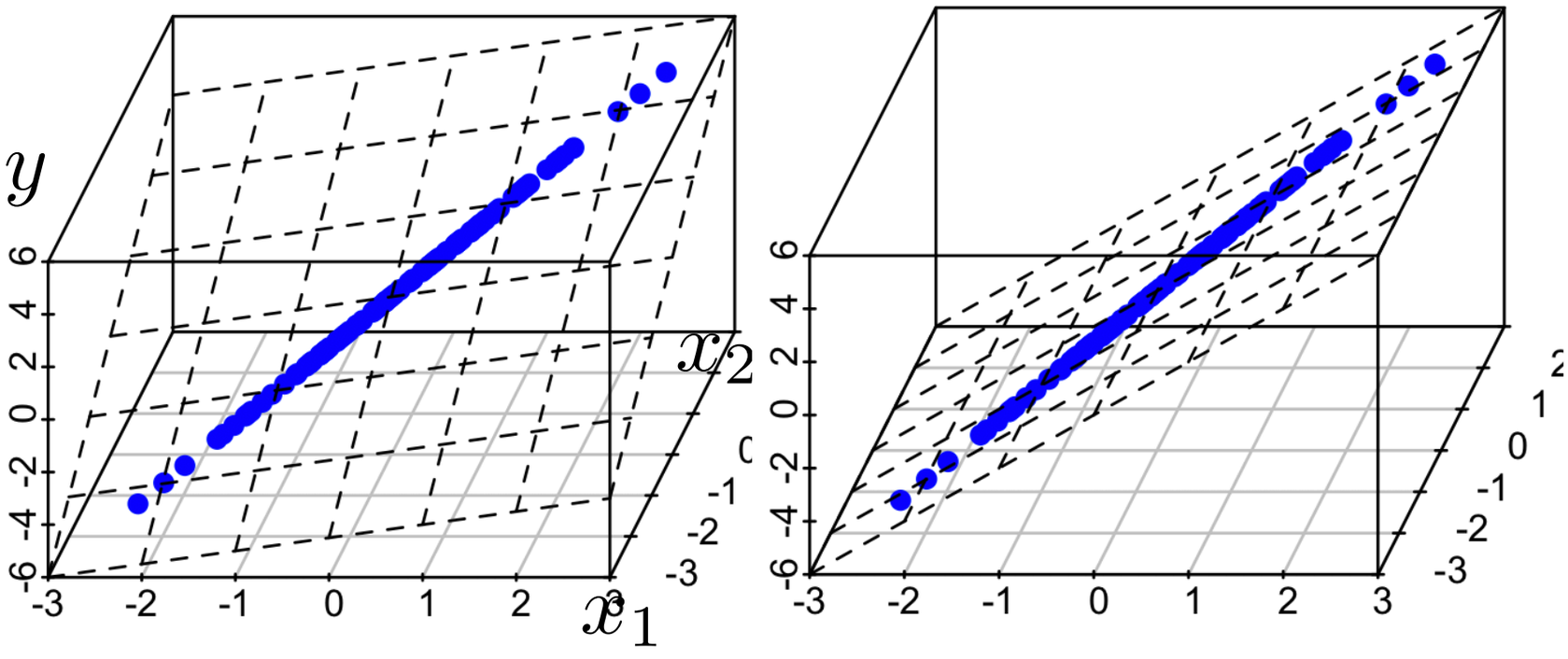

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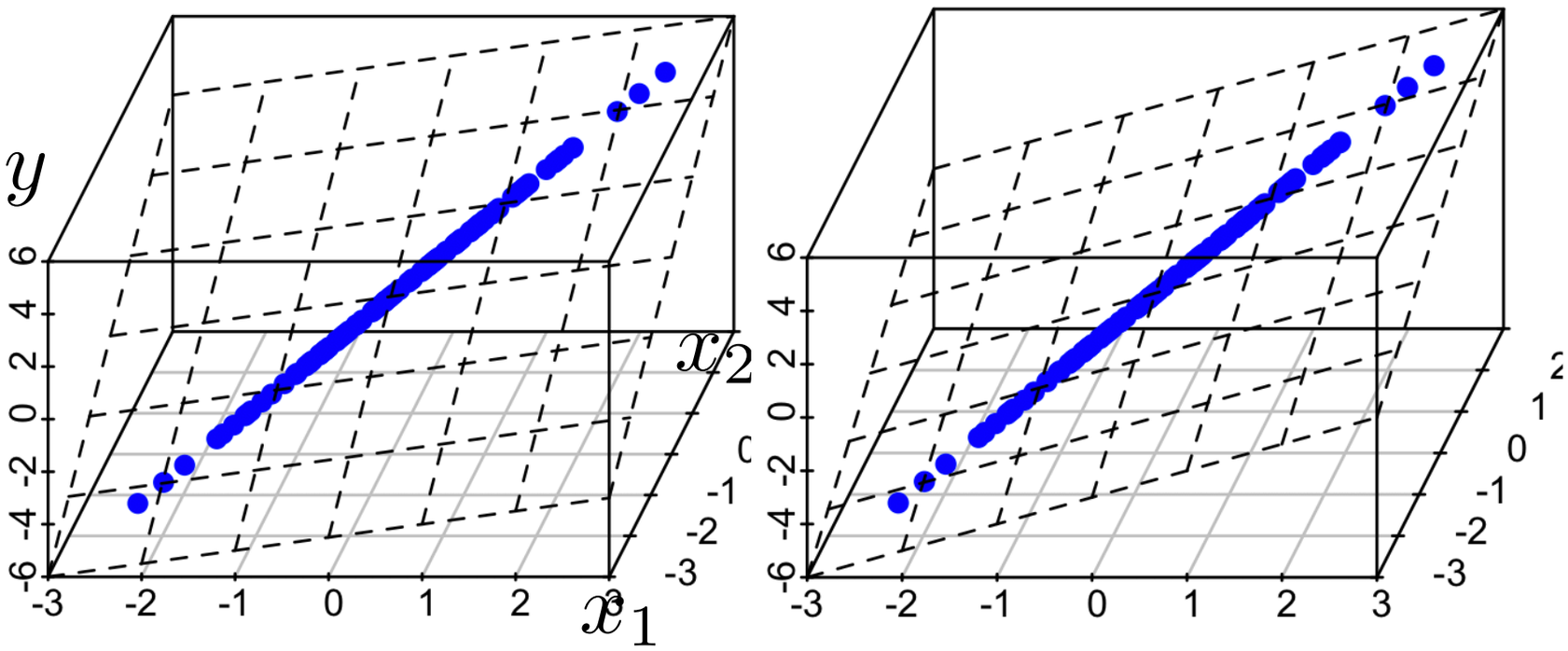

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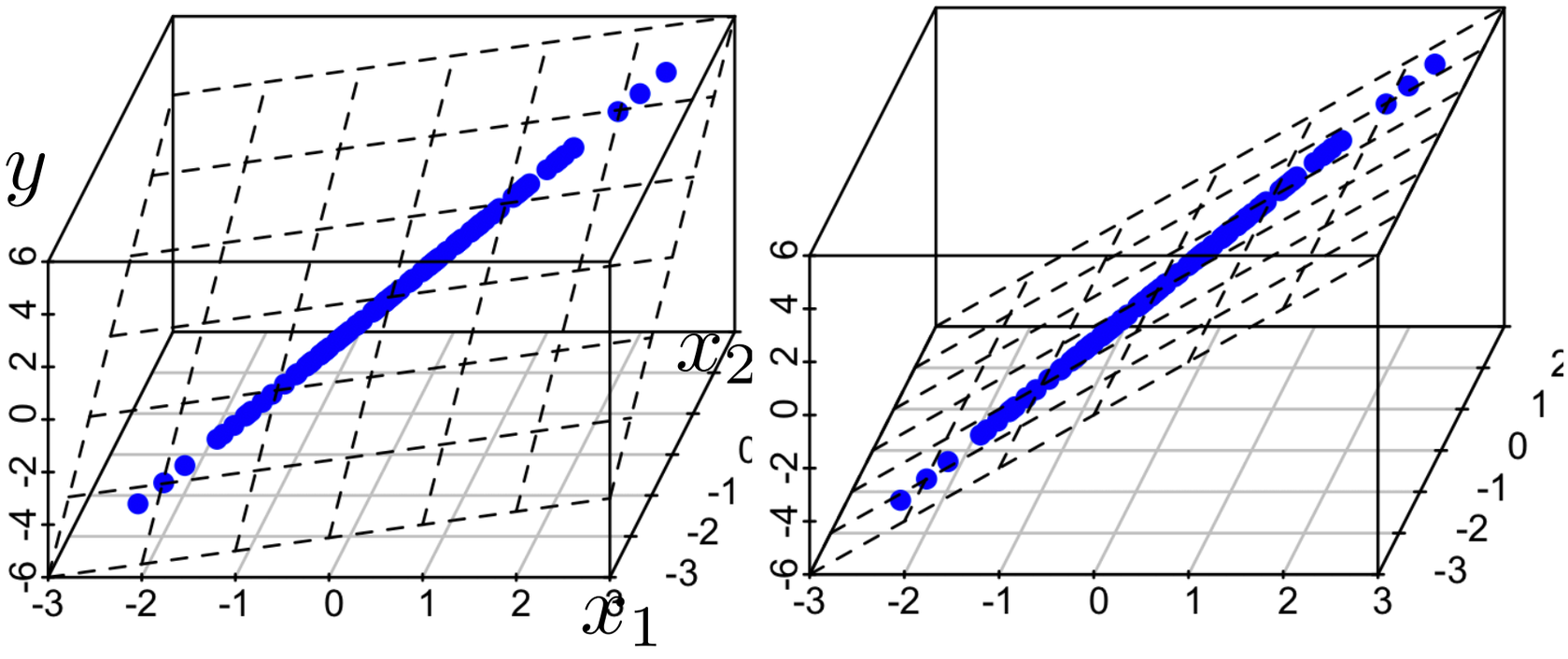

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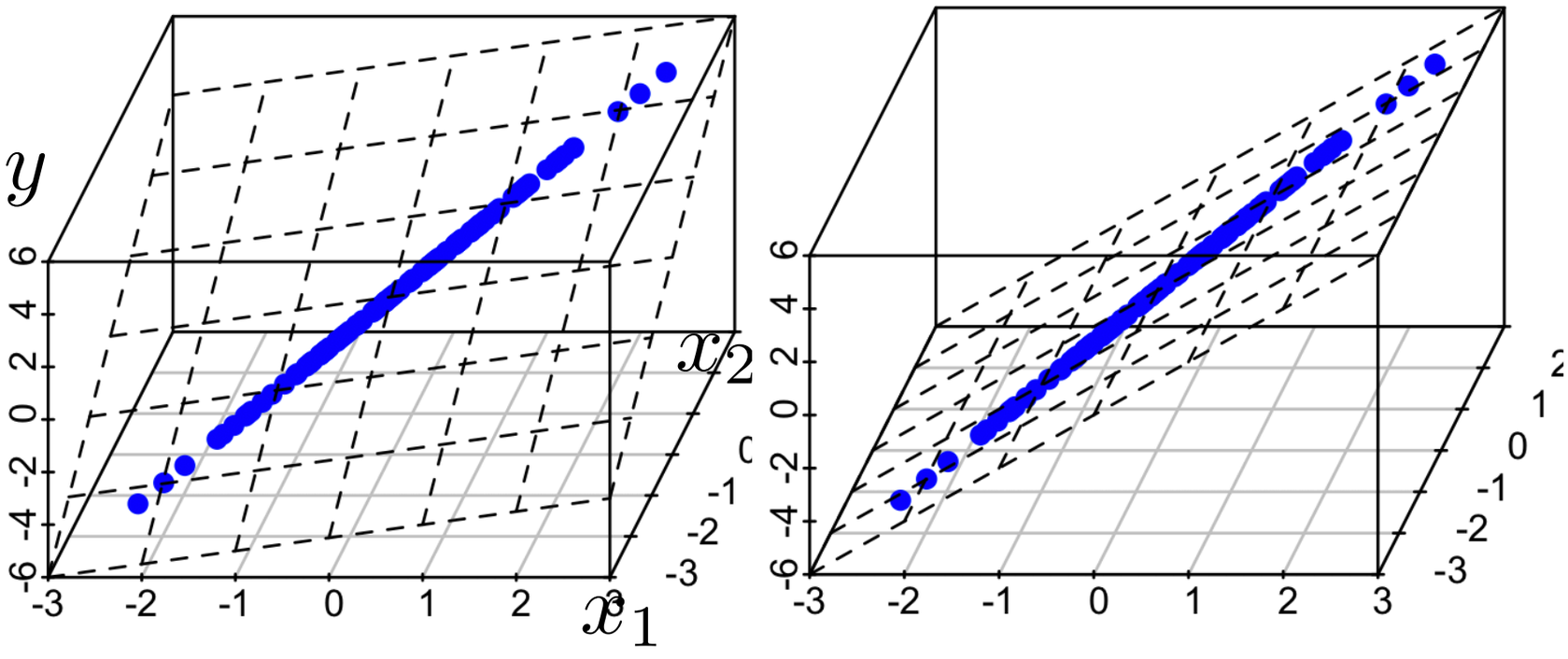

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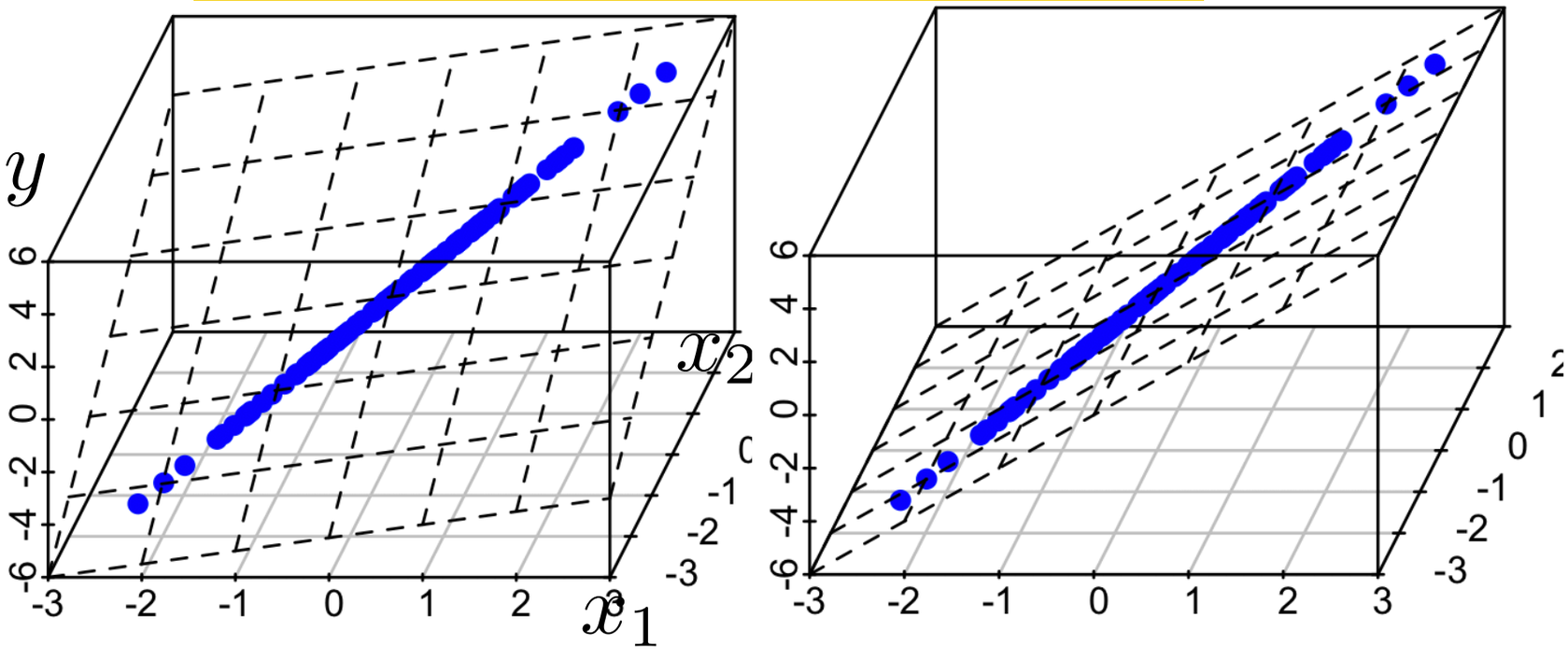

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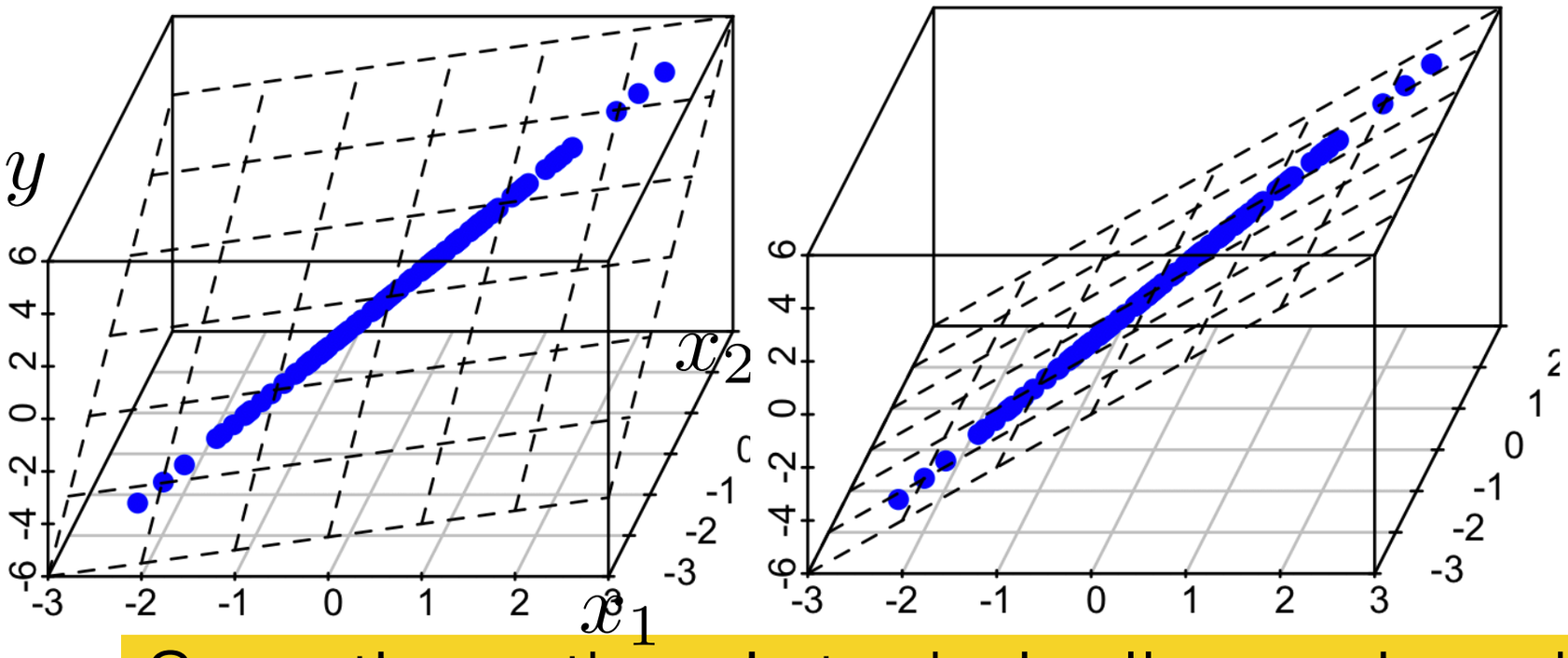

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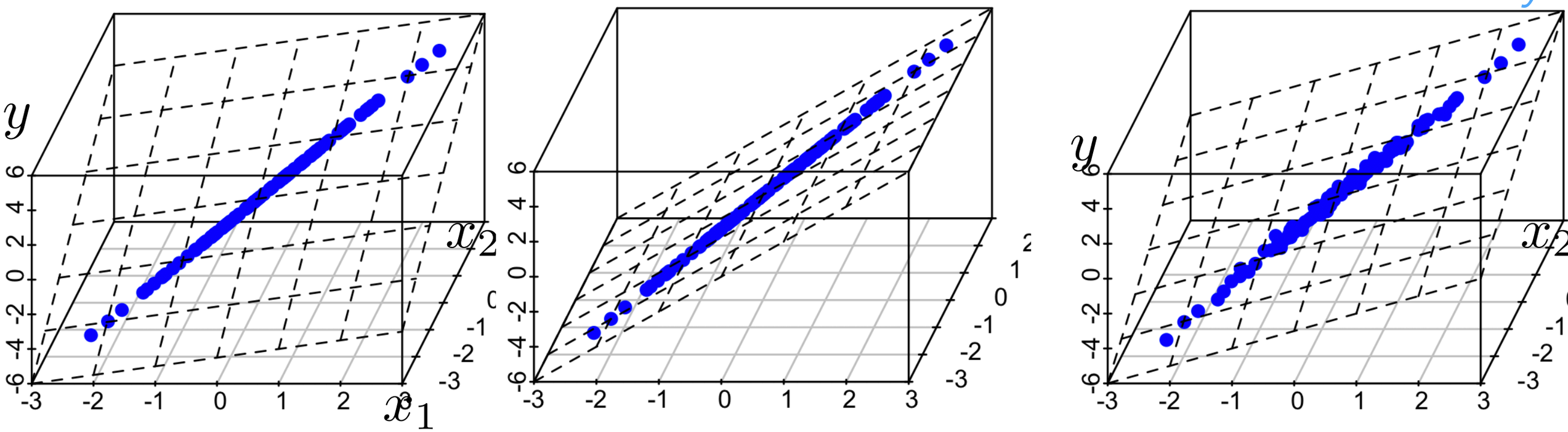

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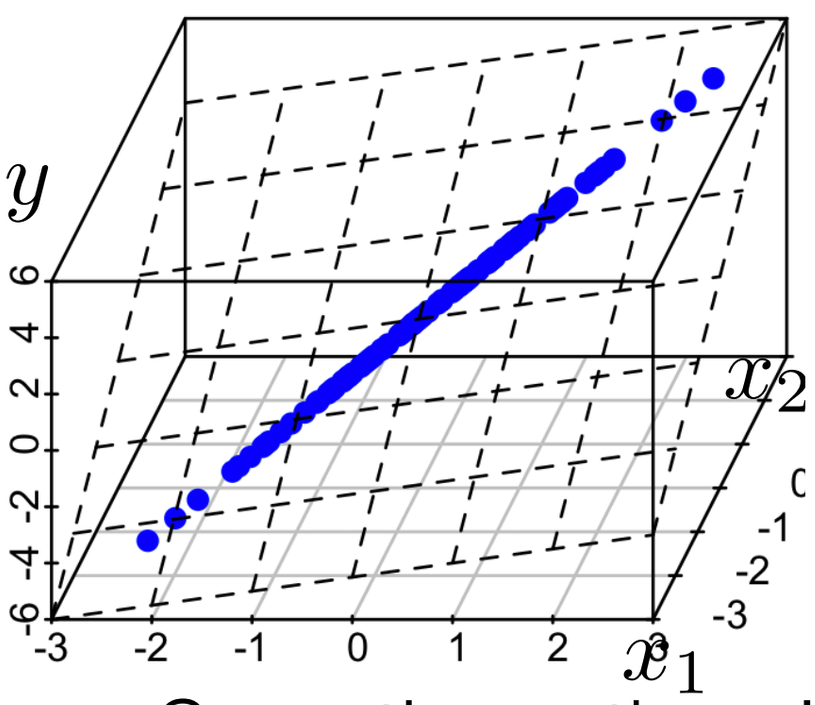

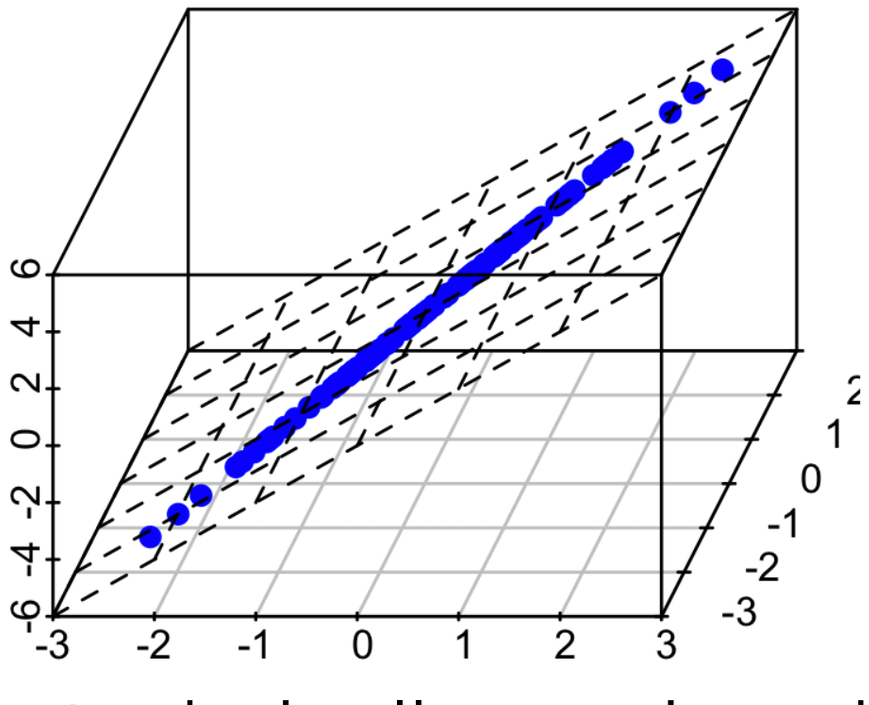

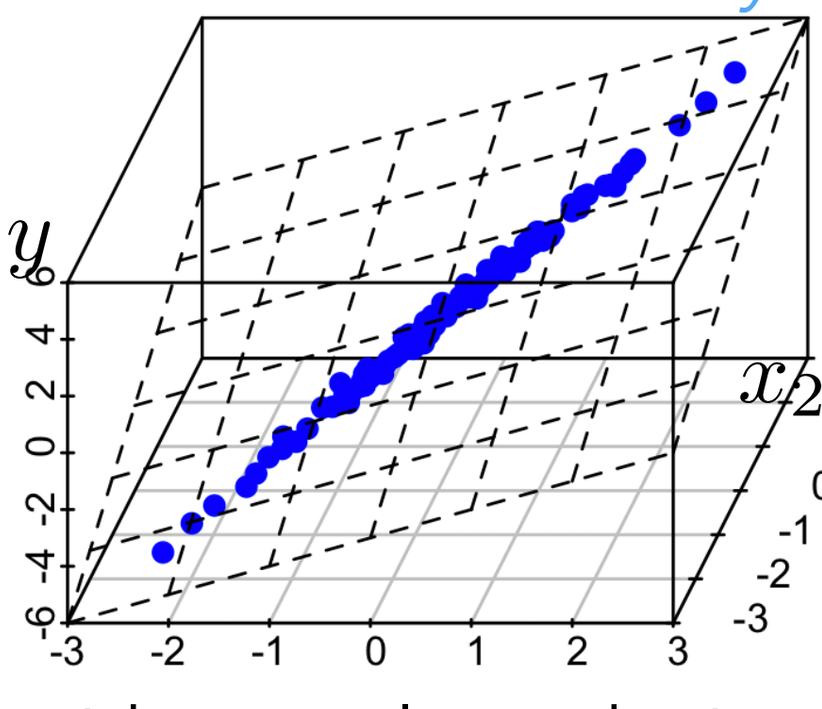

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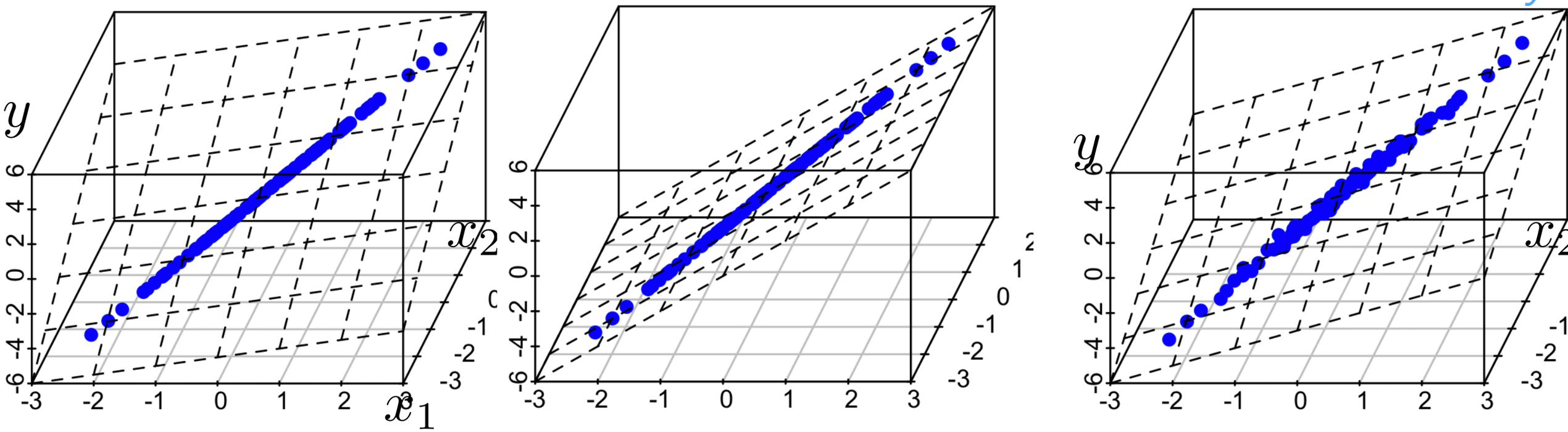

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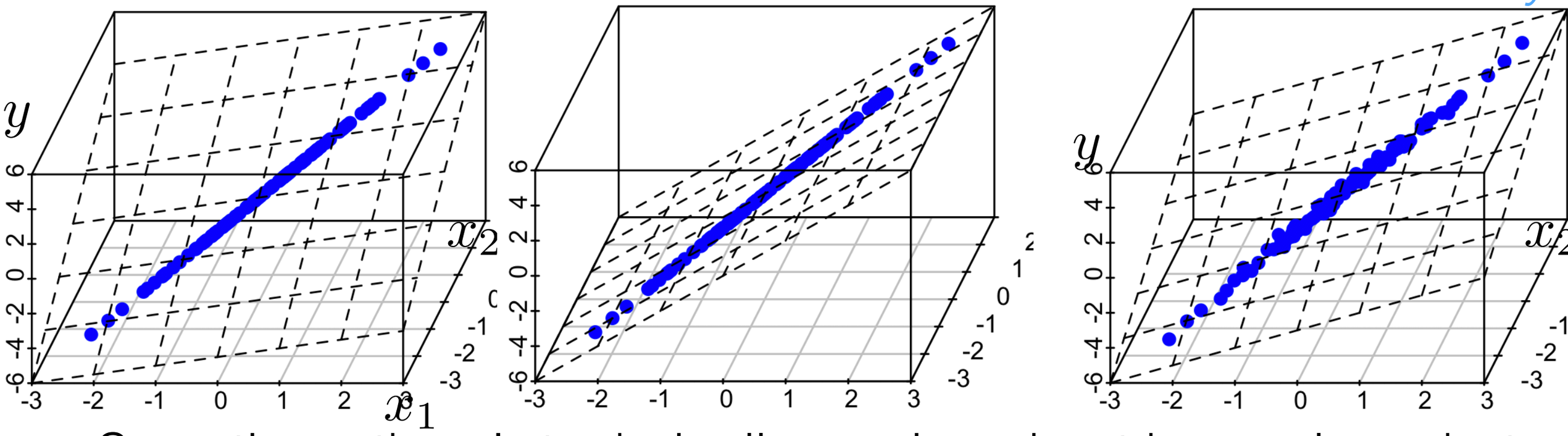

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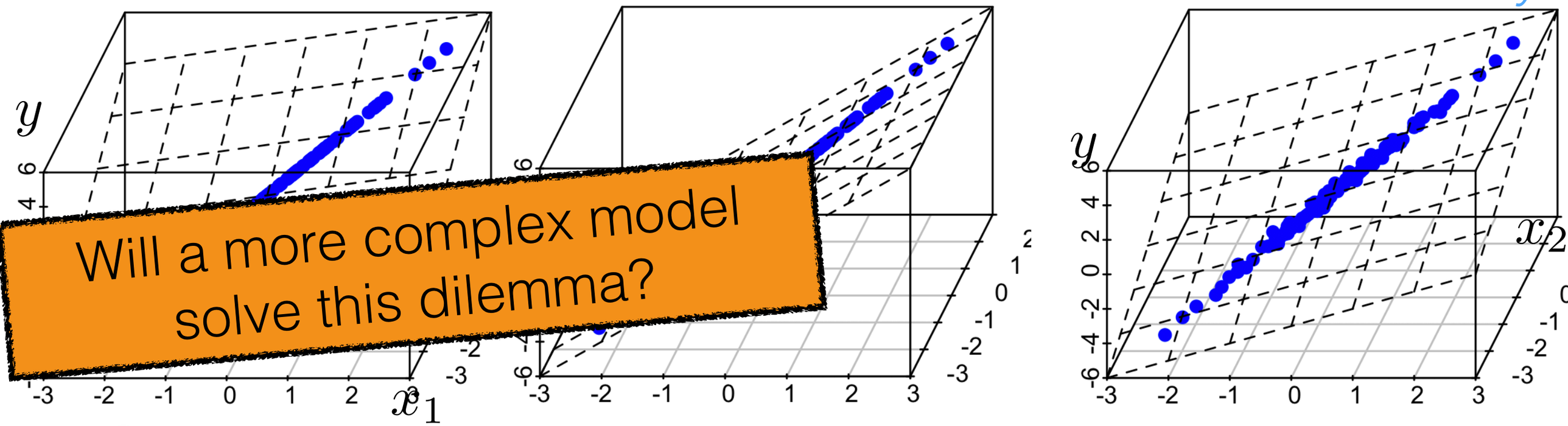

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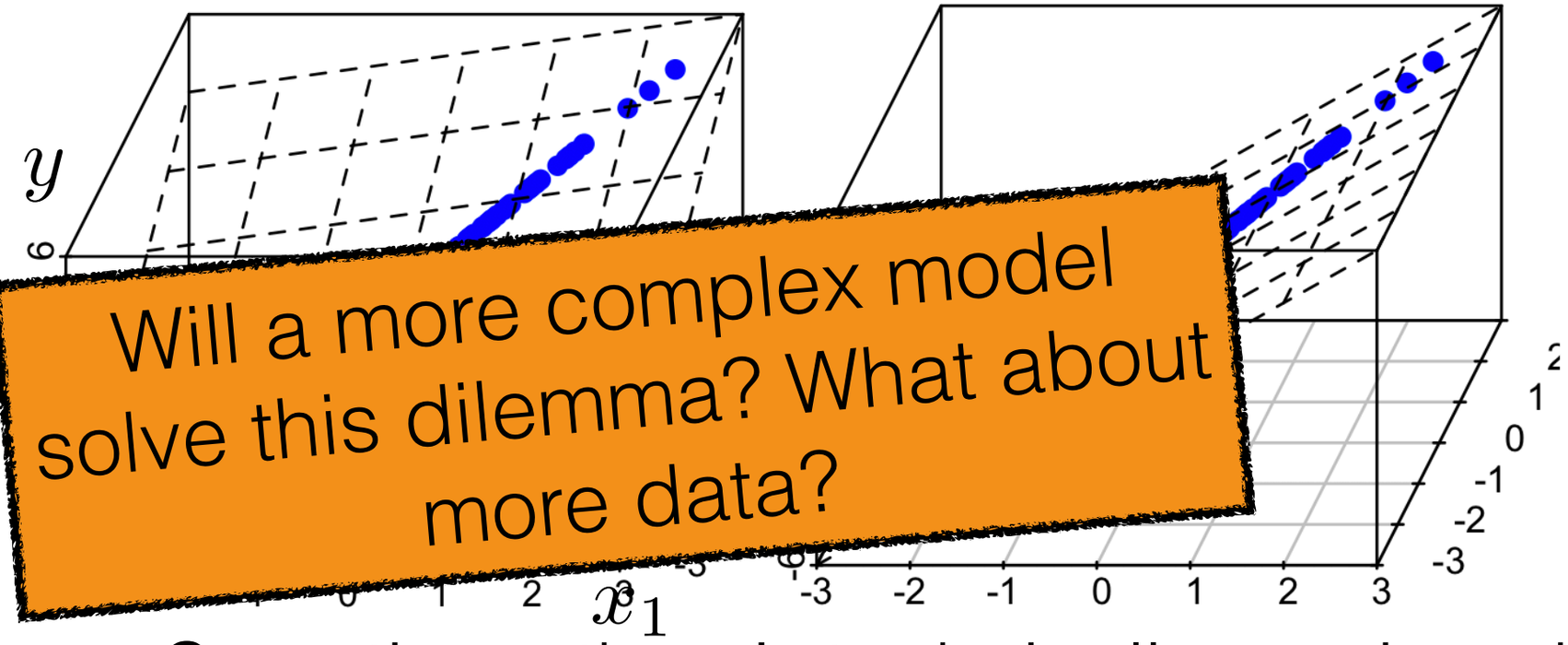
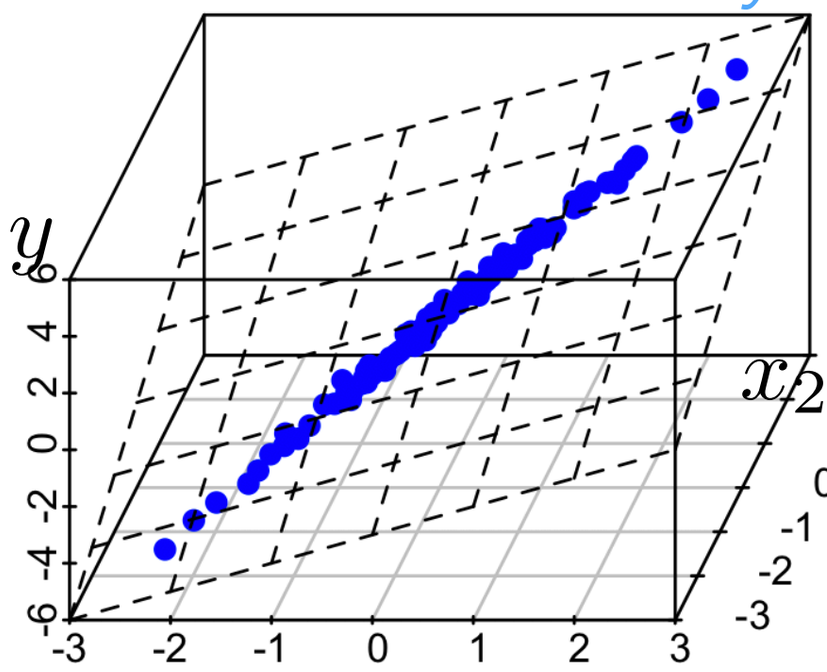


Will a more complex model solve this dilemma? What about more data?



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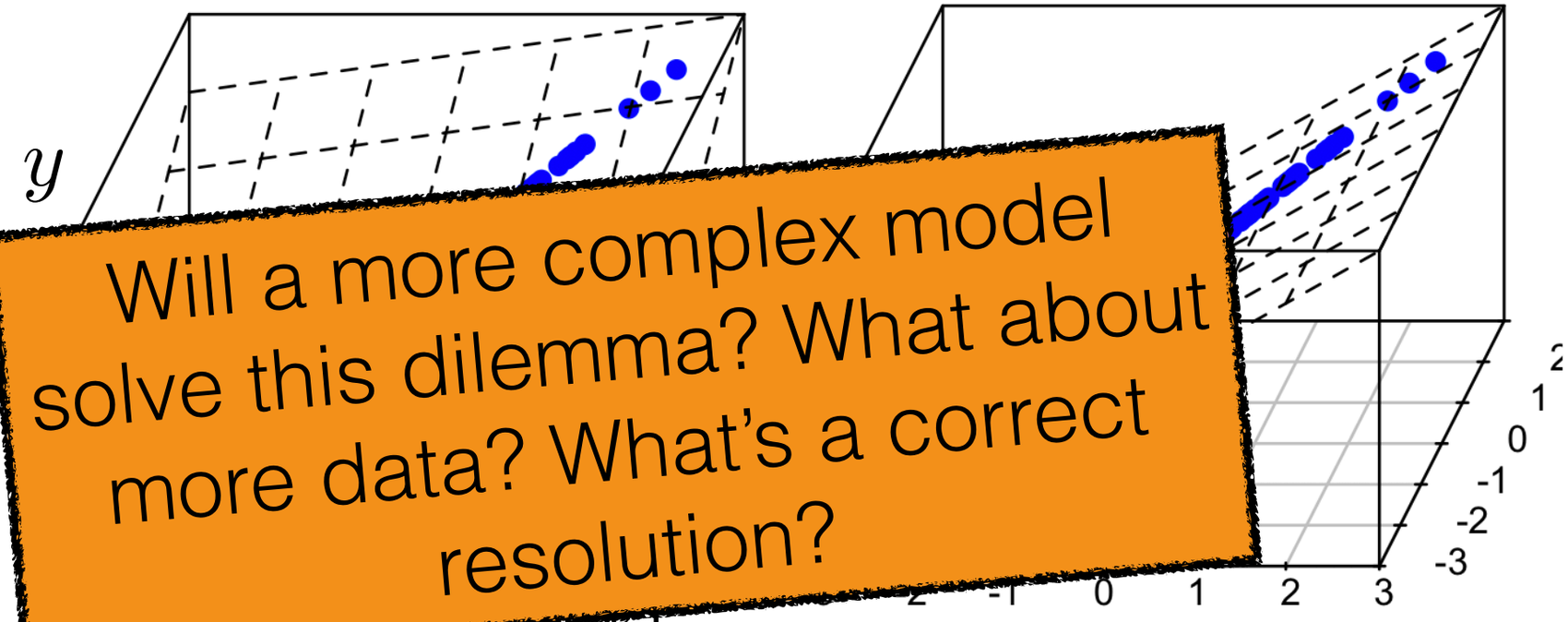

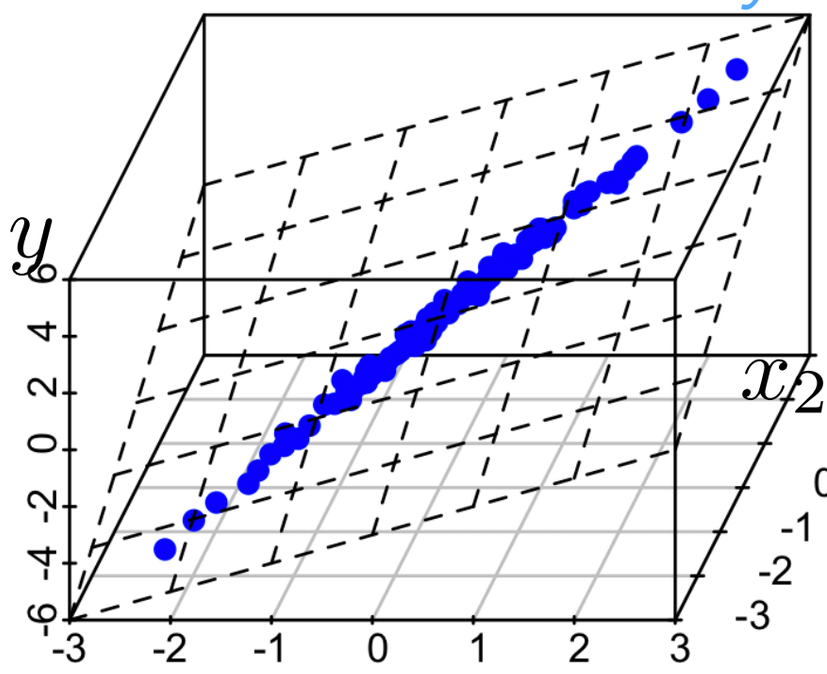

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[Stigler 1990, "A Galtonian perspective on shrinkage estimators"]

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- Sense check: one label (D_y=1)

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  check: *N* = 0, *N* = really large

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Note: new uses of notation here

μ_n, Σ_n

exercise: check

not bothering to write x since conditioning on it everywhere

Bayesian linear regression

- Likelihood: $\mathcal{N}(y^{(n)}|\theta^\top x^{(n)}, \sigma^2)$; we'll assume $\sigma^2 > 0$ is known

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DxD DxD 1x1 Dx1 1xD

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recall: nth row of *X* matrix is nth data point's features

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check: is pos def

$$= \Sigma_0^{-1} + (\sigma^2)^{-1} \underbrace{X^\top}_{\text{DxN}} \underbrace{X}_{\text{NxD}}$$

recall: nth row of *X* matrix is nth data point's features

Bayesian linear regression

- Likelihood: $\mathcal{N}(y^{(n)}|\theta^\top x^{(n)}, \sigma^2)$; we'll assume $\sigma^2 > 0$ is known

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1x1

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DxD DxD 1x1 Dx1

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