MIT EECS
ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

6.7900: Machine Learning
Lecture 16

Lecture start: Tues/Thurs 2:35pm

Who's speaking today? Prof. Tamara Broderick

Course website: gradml.mit.edu

Questions? Ask here or on piazza.com/mit/fall2024/67900/

Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Times

I. Various supervised learning methods: linear, polynomial, trees, forests, neural networks

Today

I. A common motif in spatiotemporal (and similar) data

II. Gaussian processes: model and inference

Sub-type of regression

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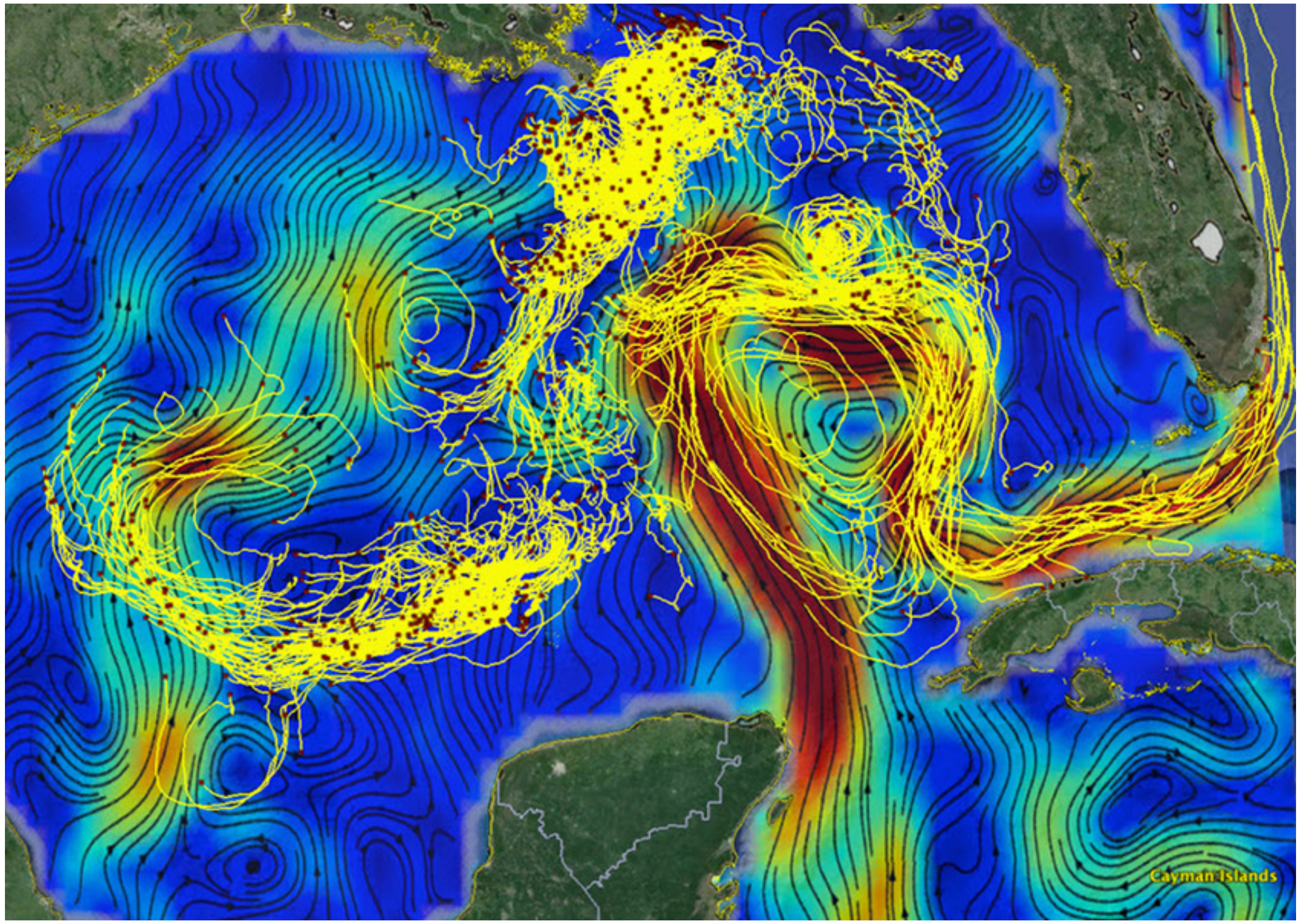

Example:

[Ryan, Özgökmen 2023; Zewe 2023; Gonçalves et al 2019; Lodise et al 2020; Berlinghieri et al 2023]

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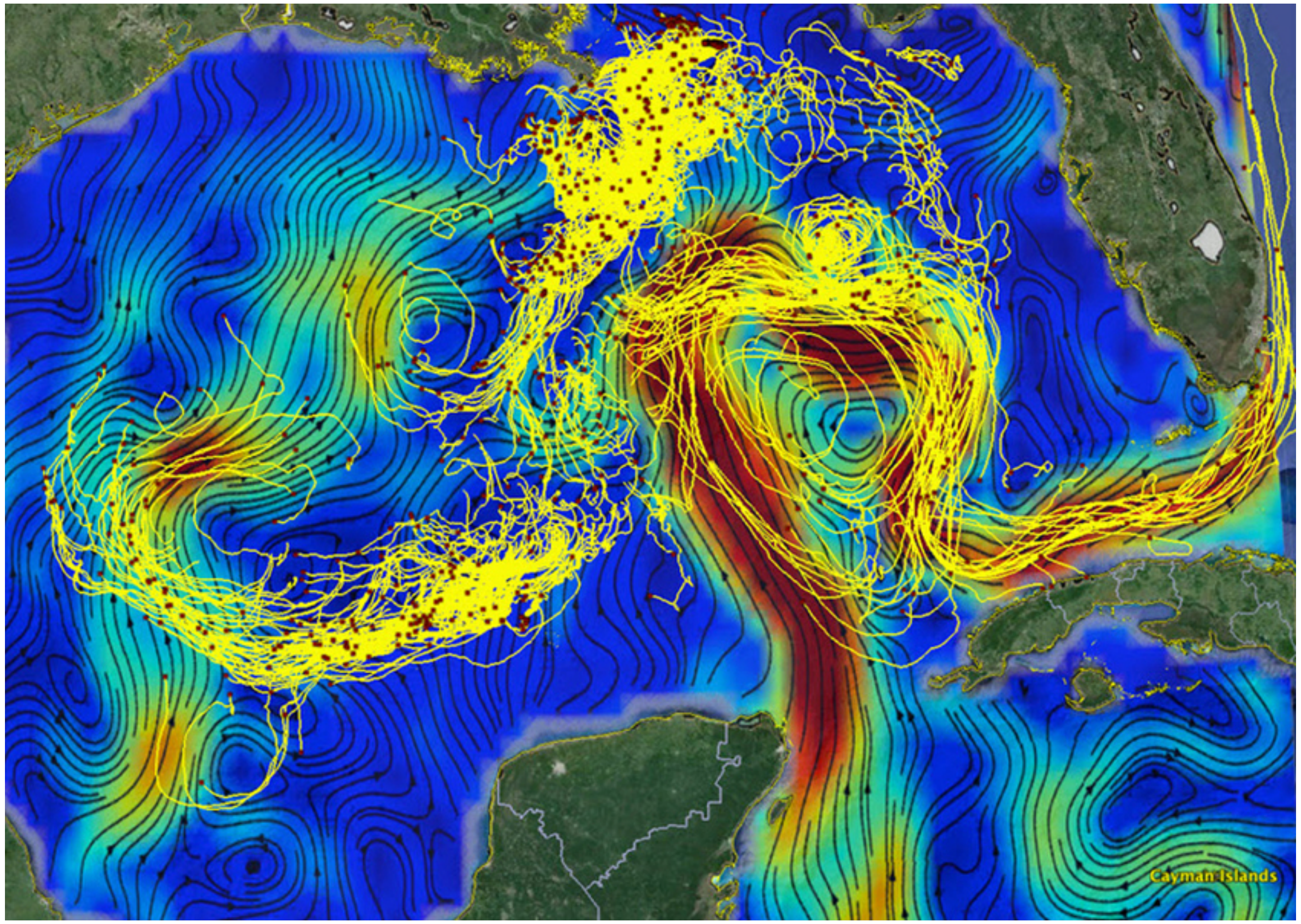

Example:

- The ocean current (velocity vector field) varies by space & time

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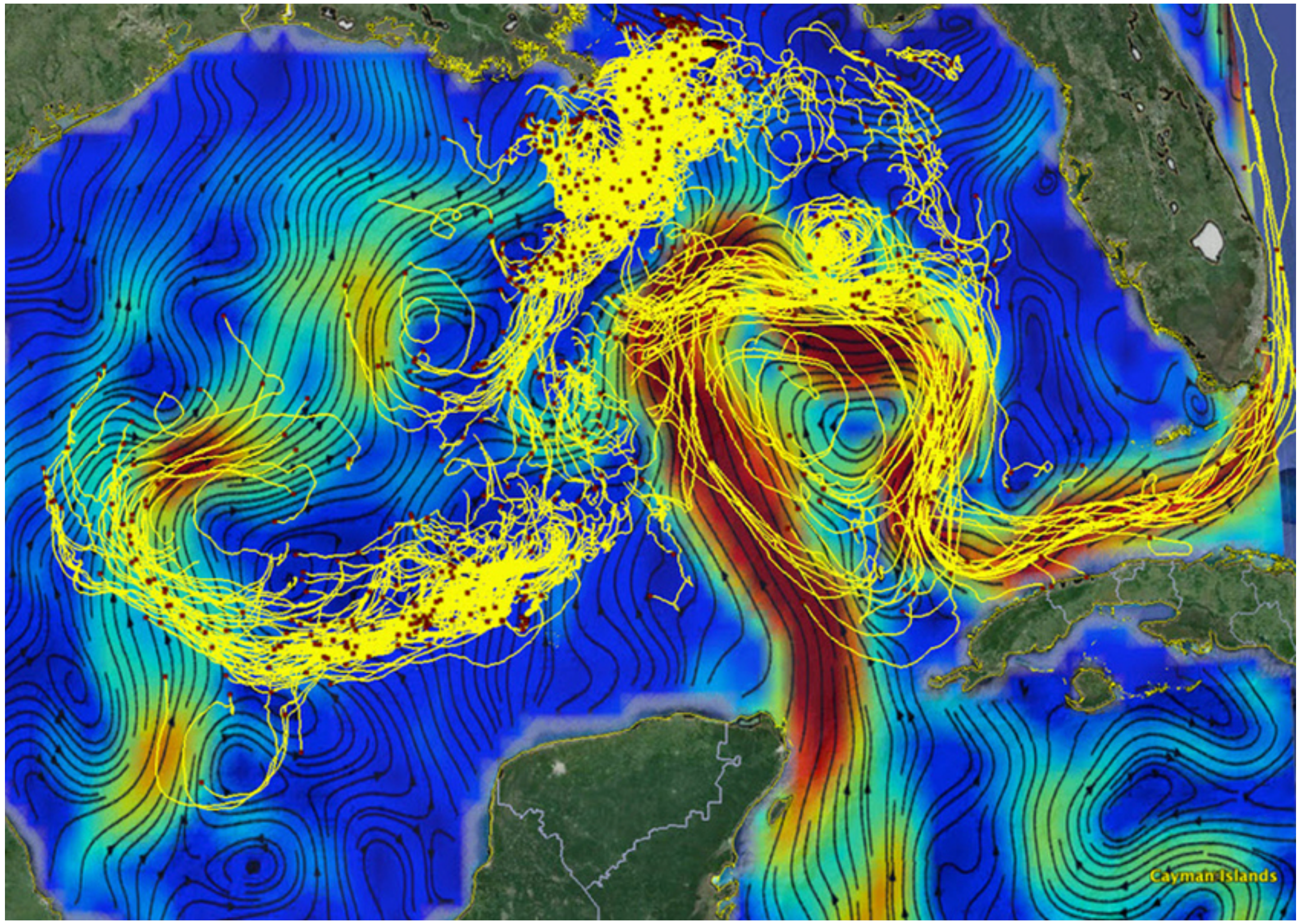

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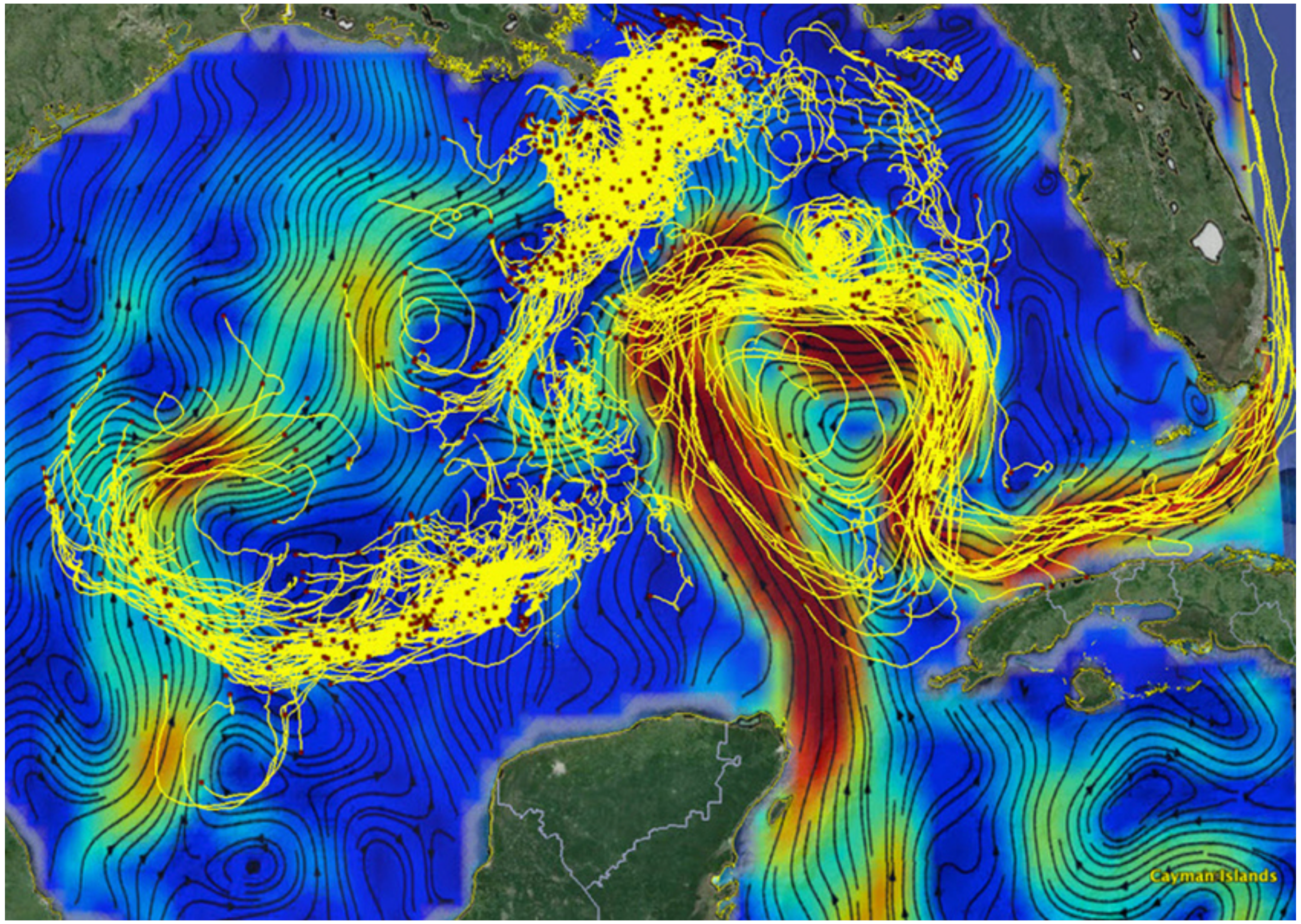

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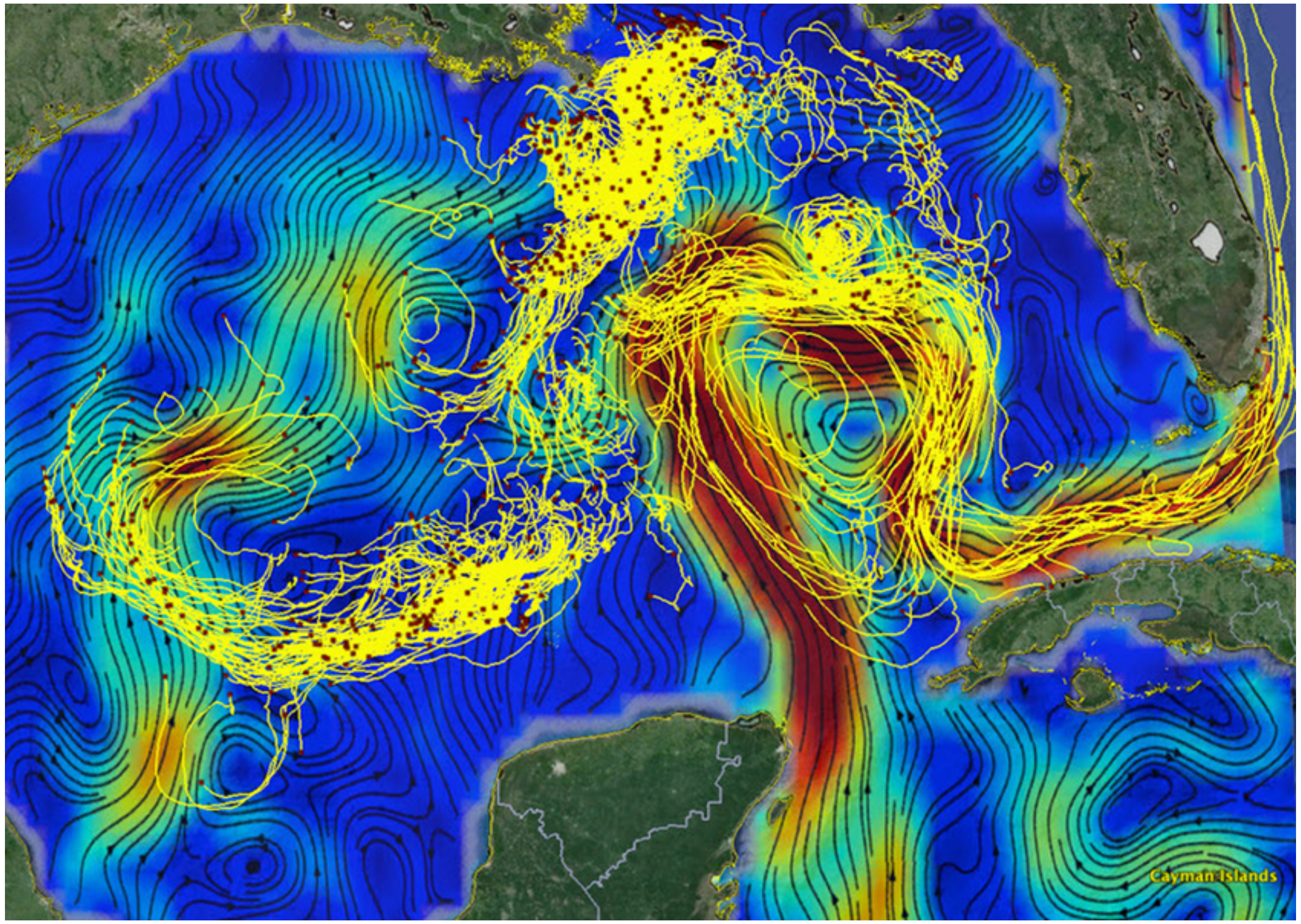

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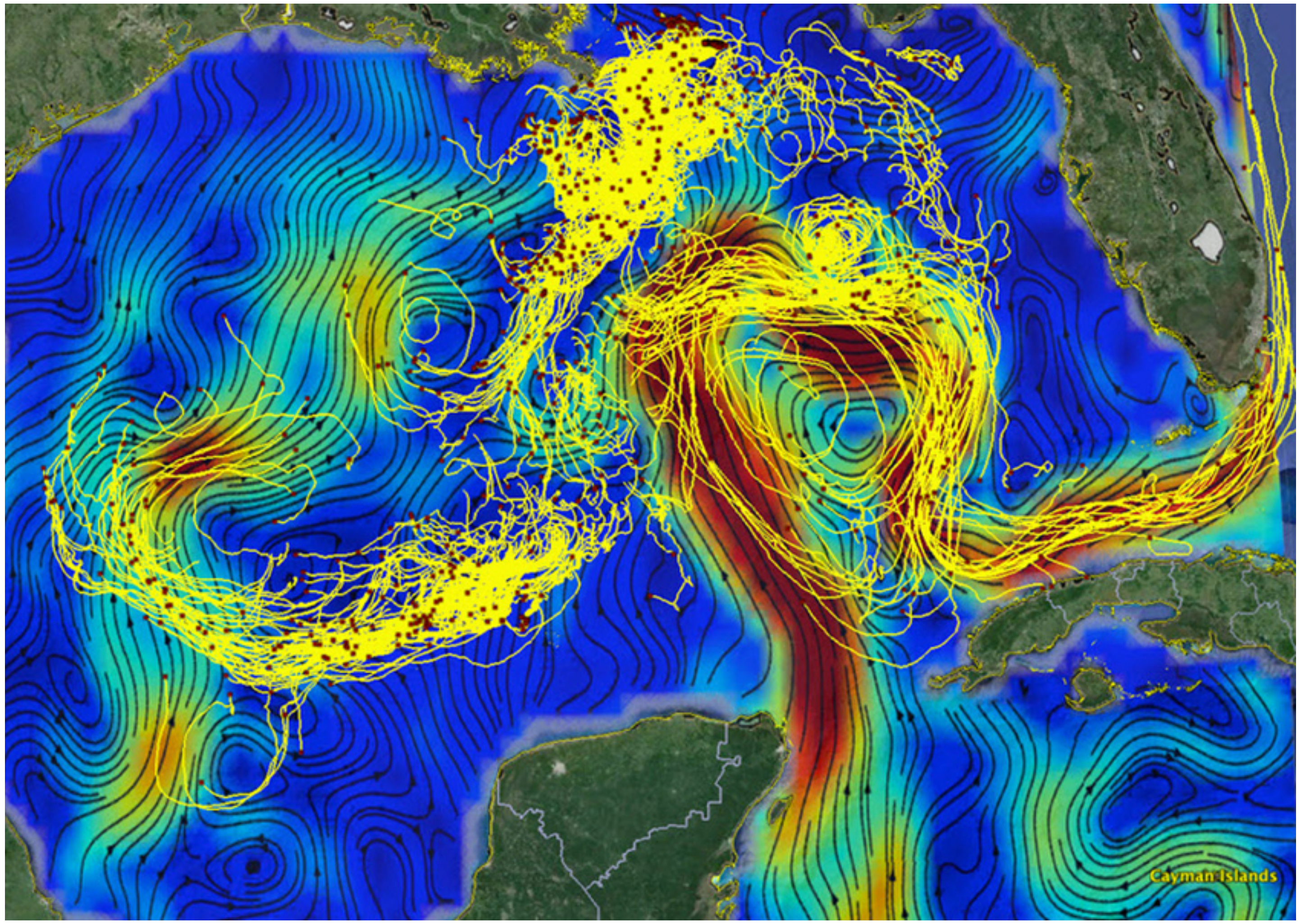

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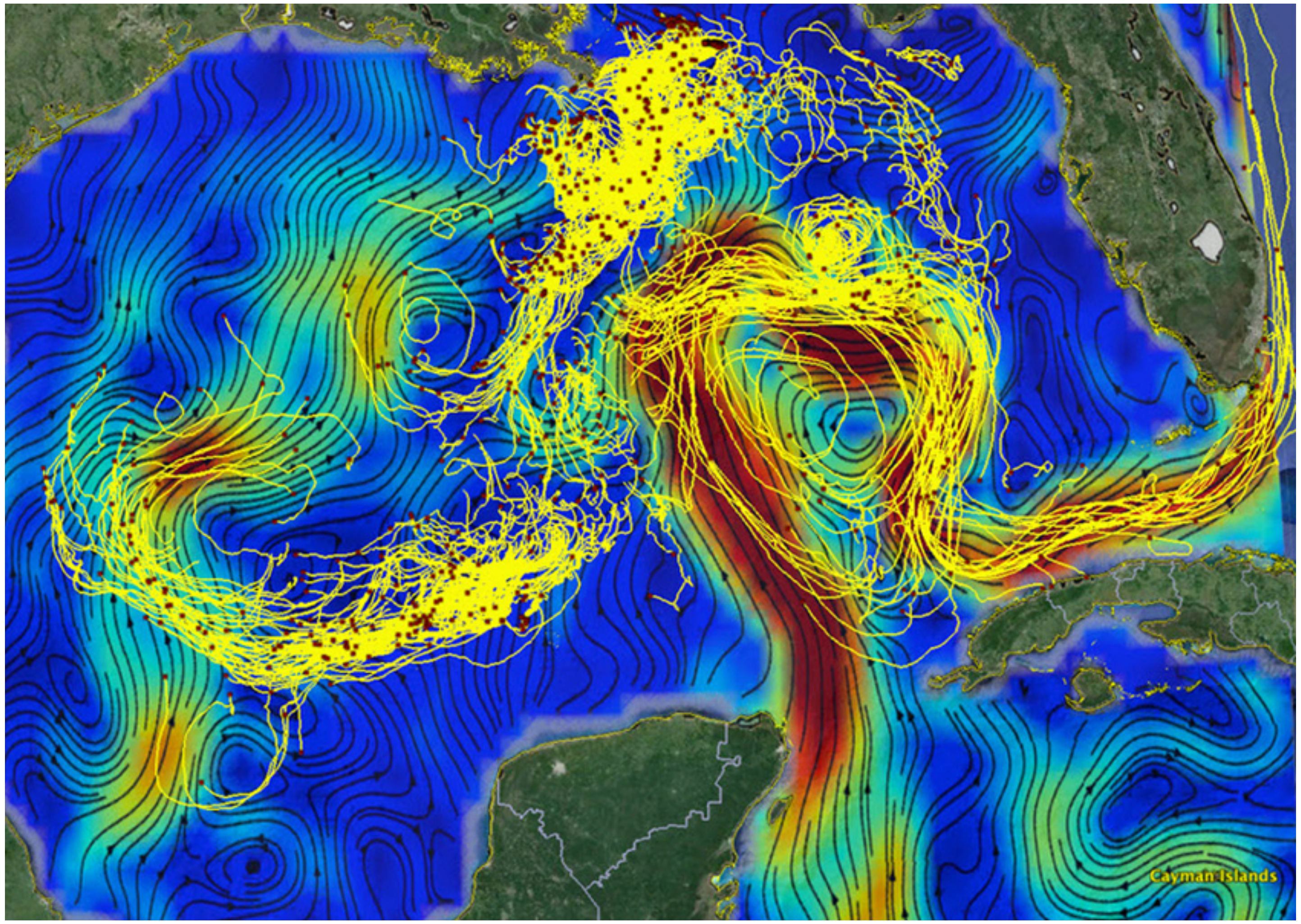

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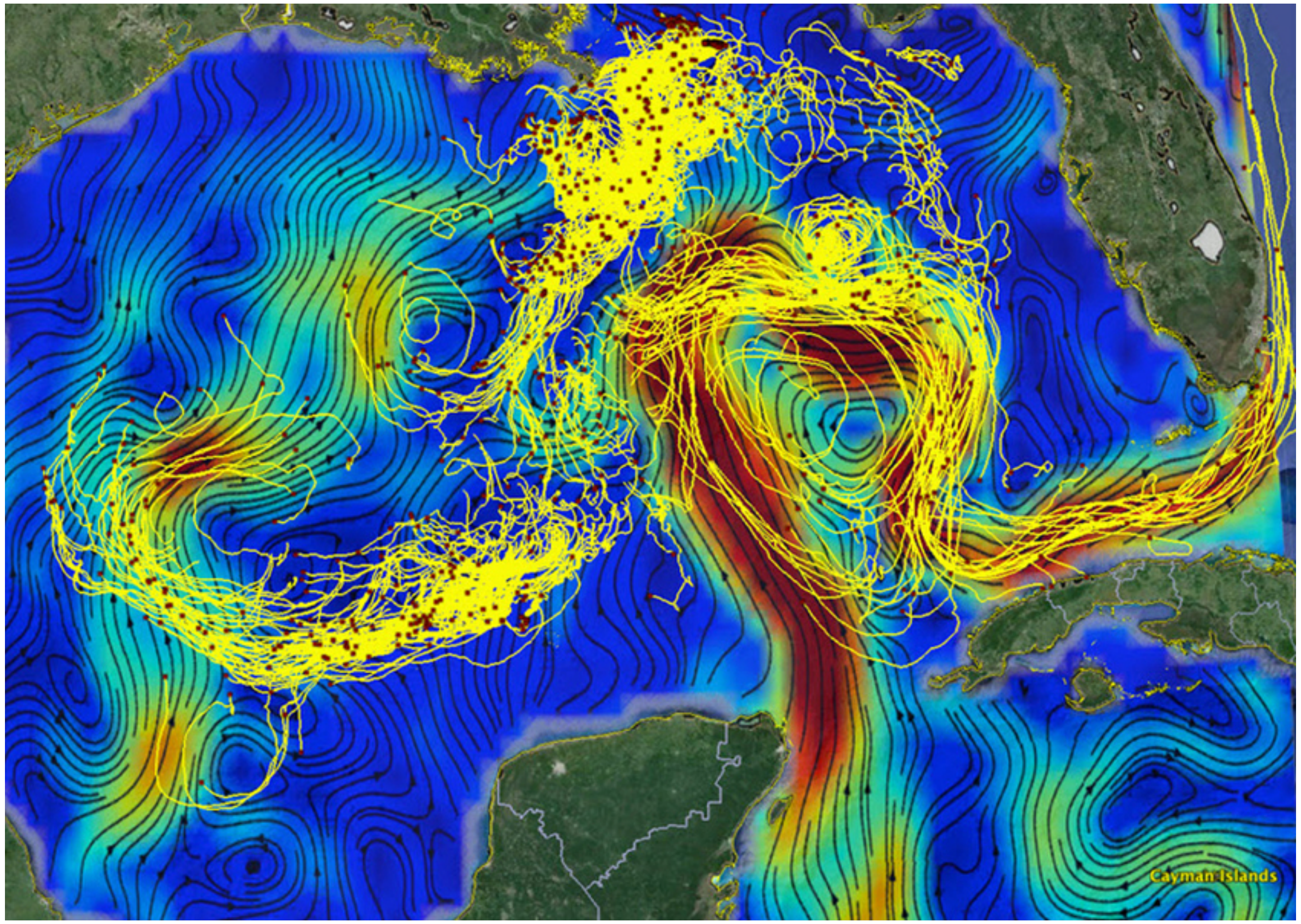

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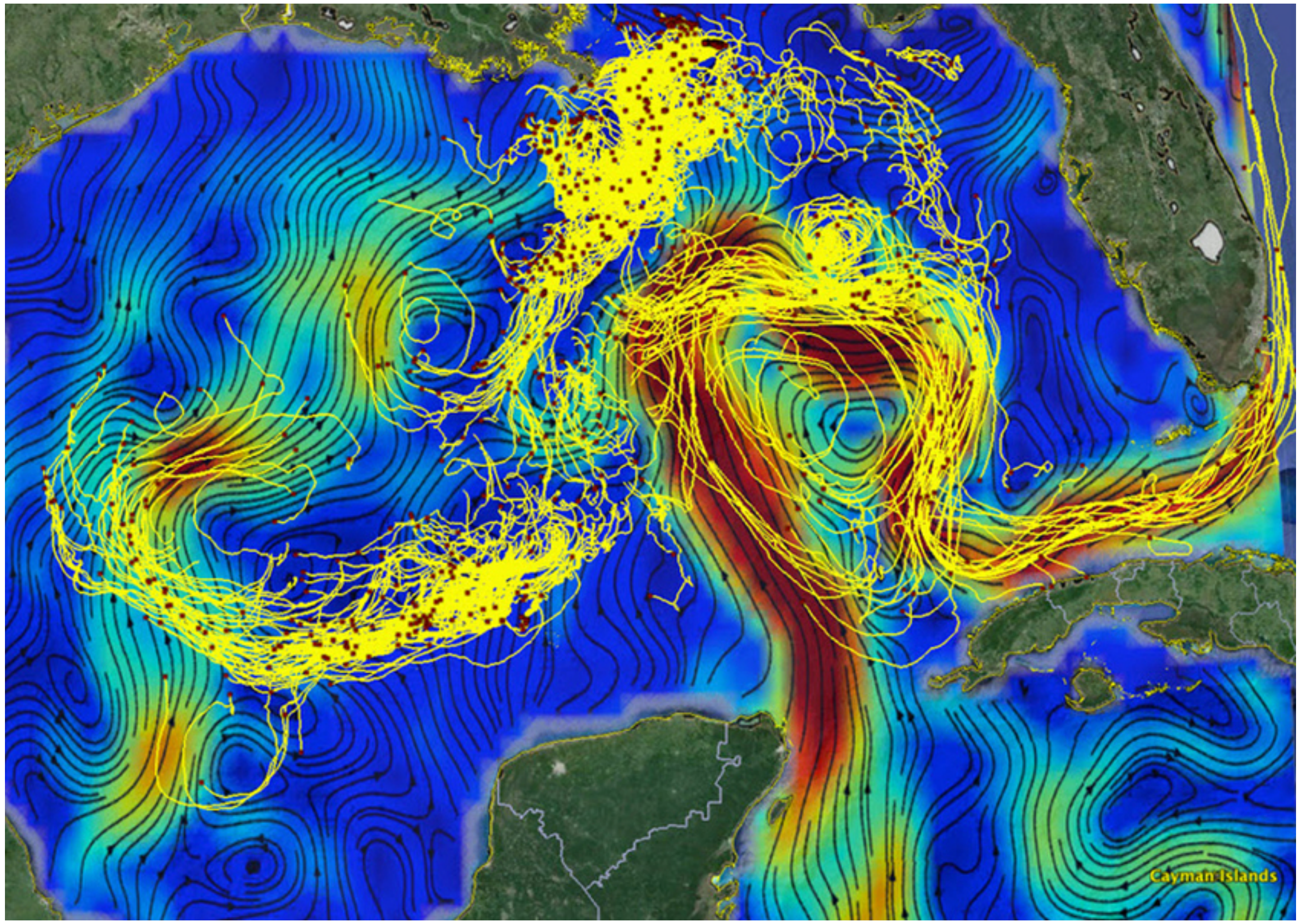

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Many similar problems: e.g. predicting groundwater quality from well measurements, etc. [Belkhiri et al 2020]

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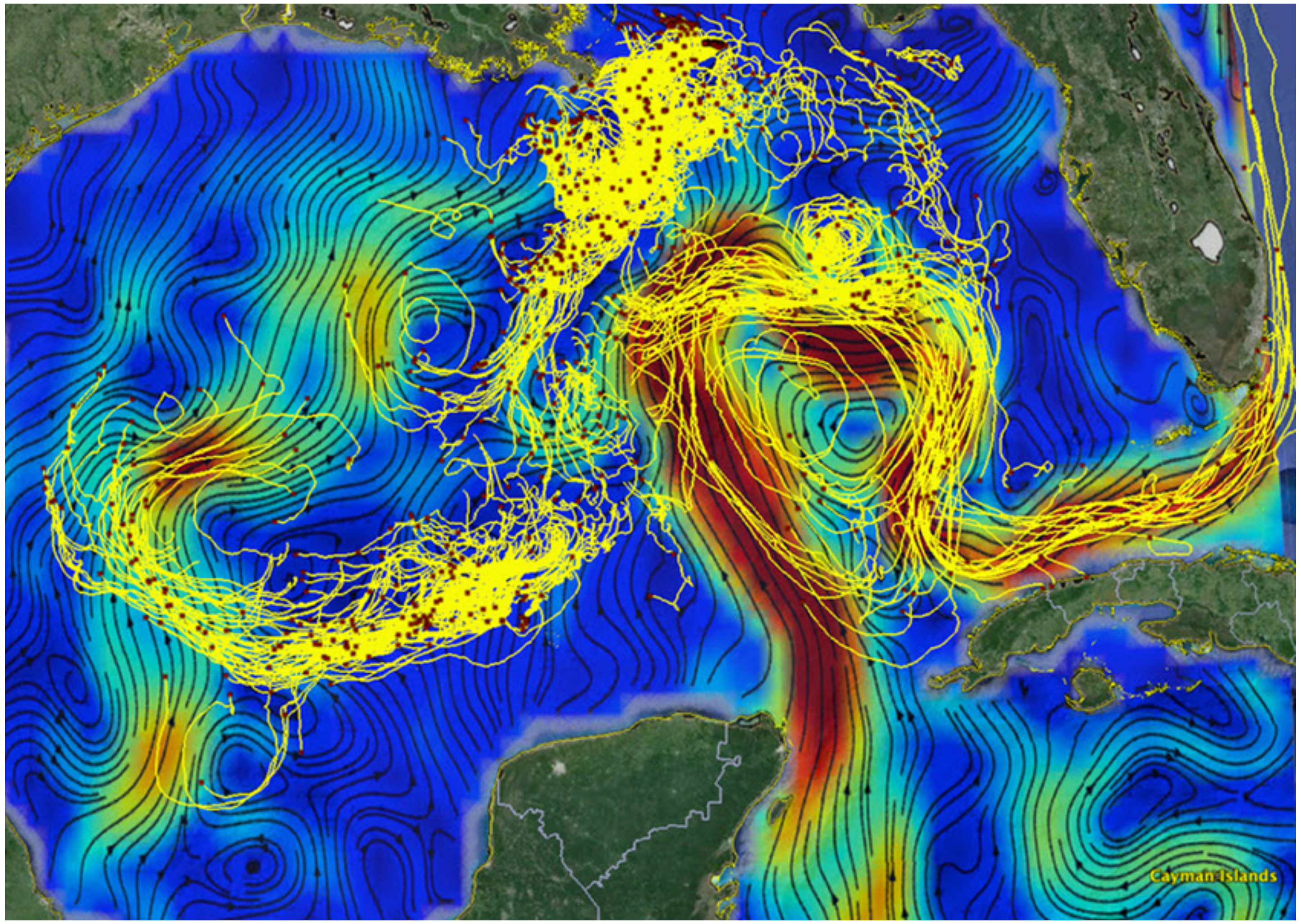

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Challenges:
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Many similar (low dimensional) problems: e.g. predicting groundwater quality from well measurements, etc. [Belkhiri et al 2020]

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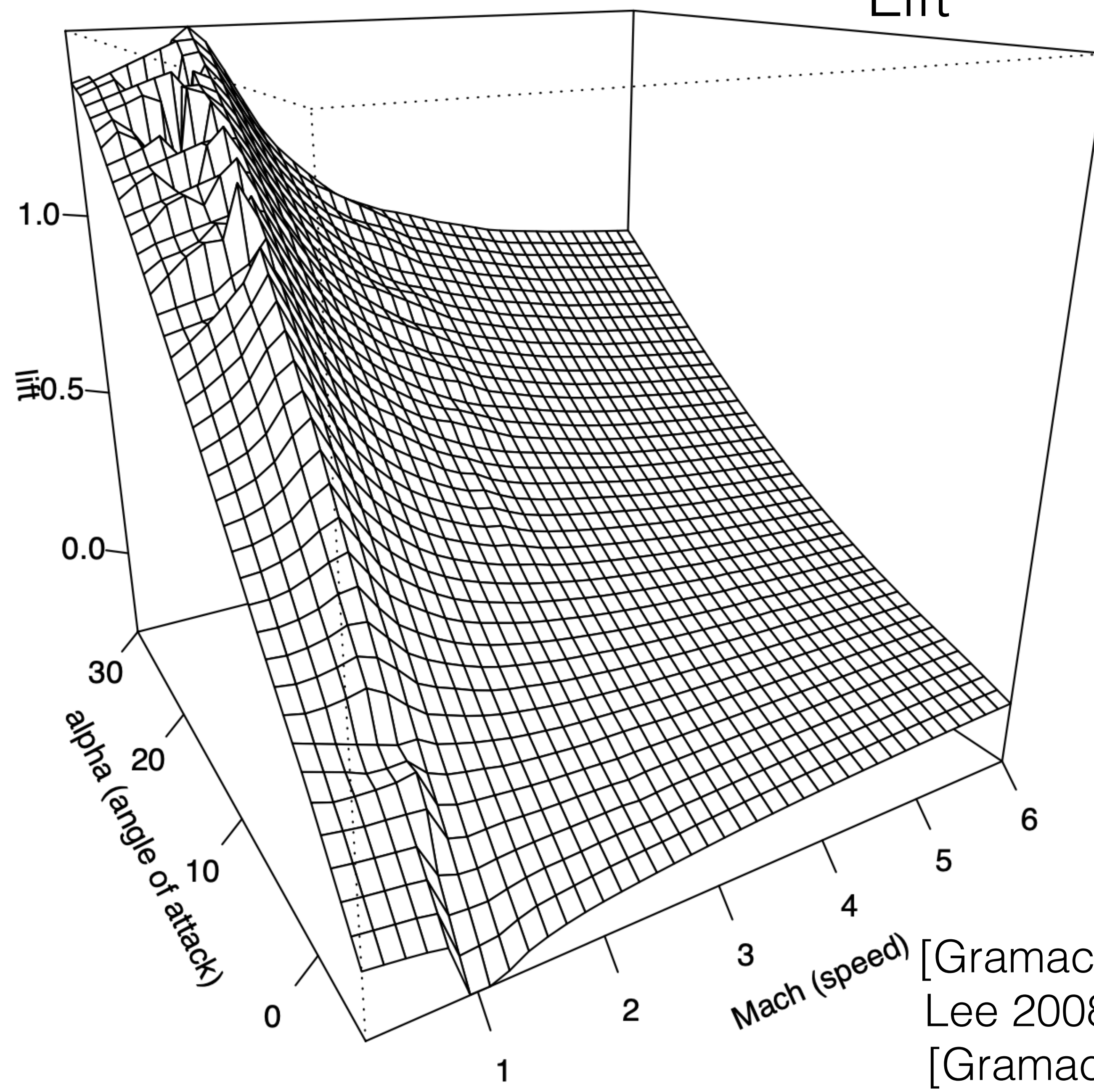

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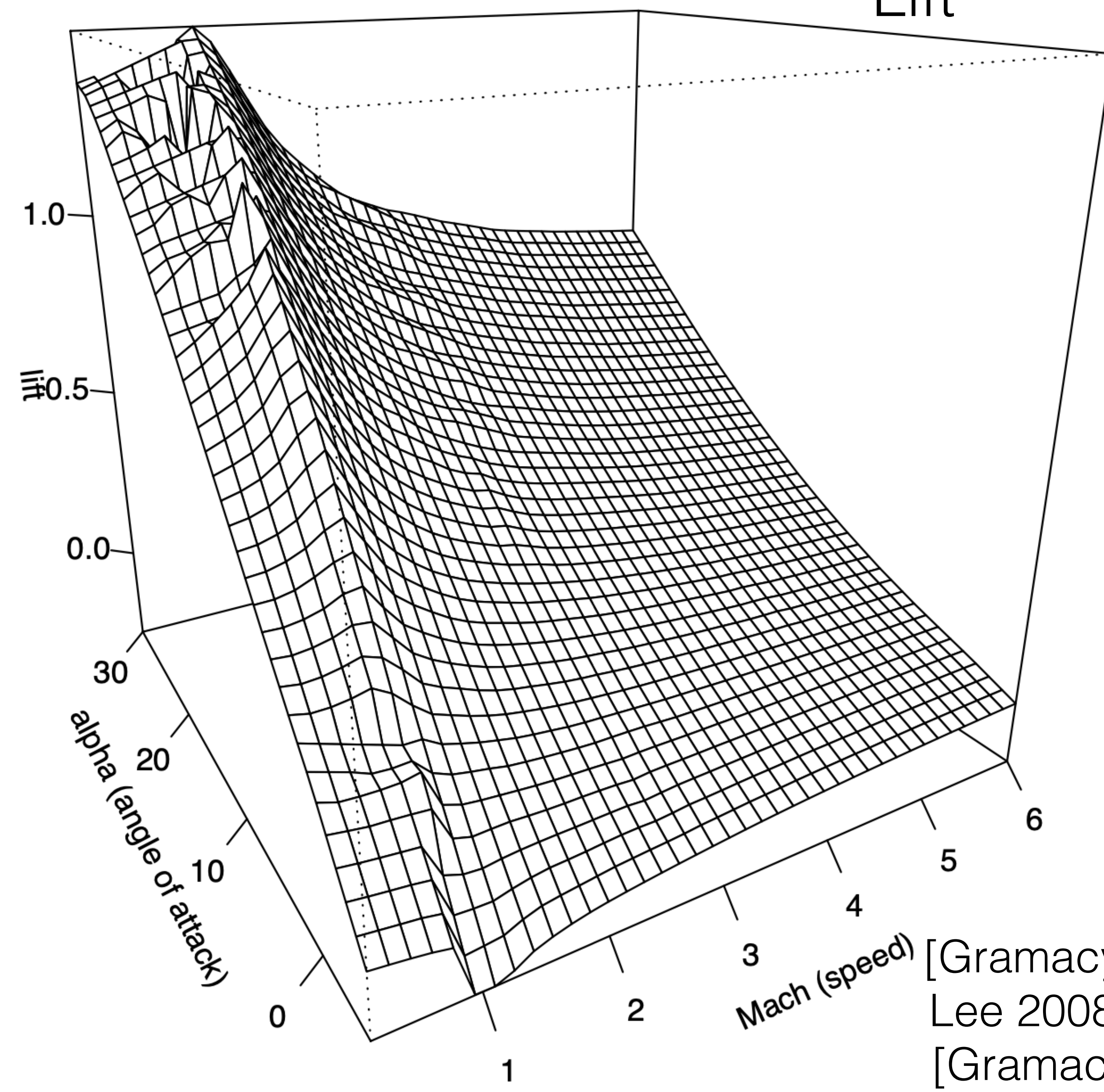

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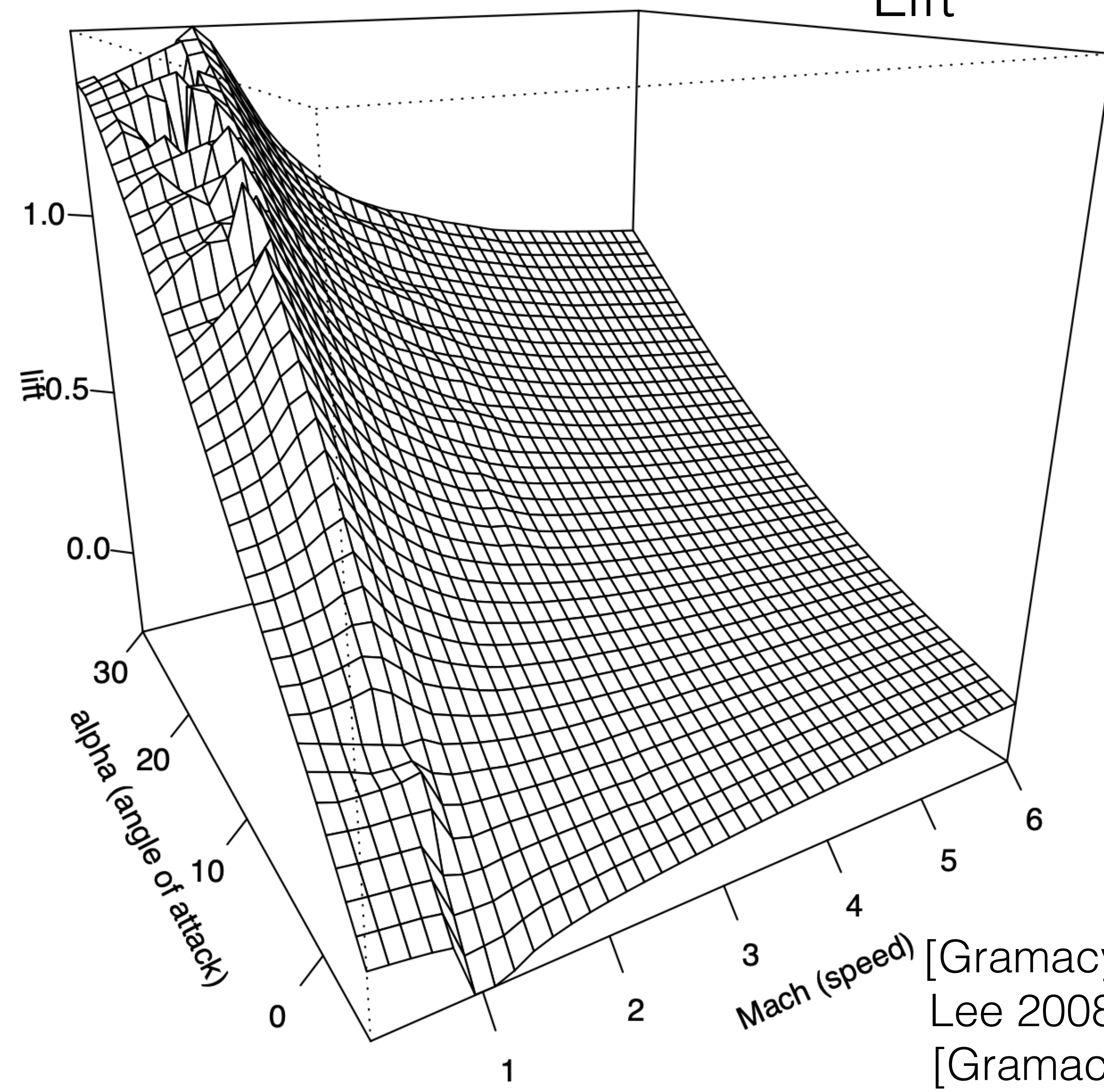

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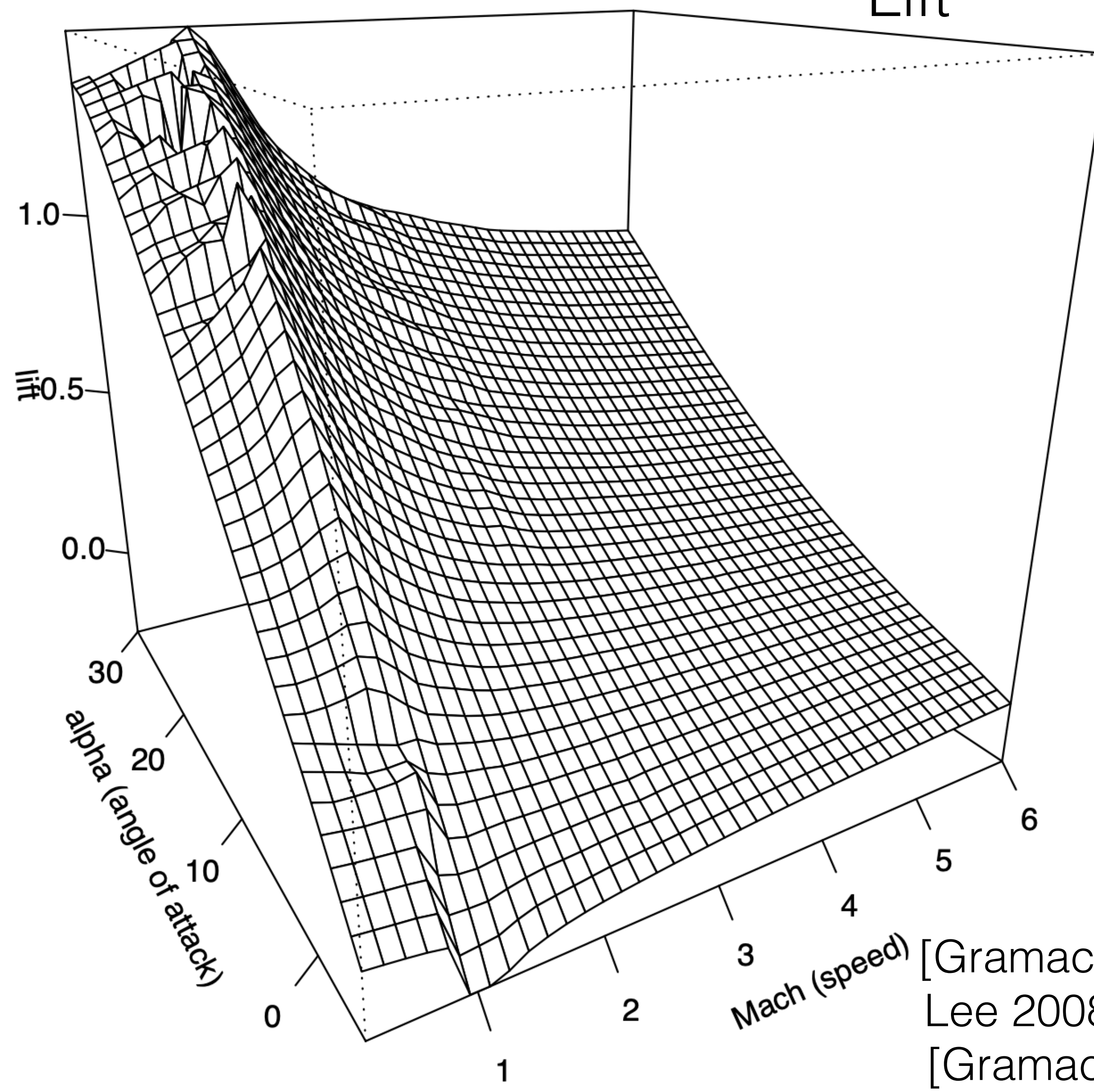

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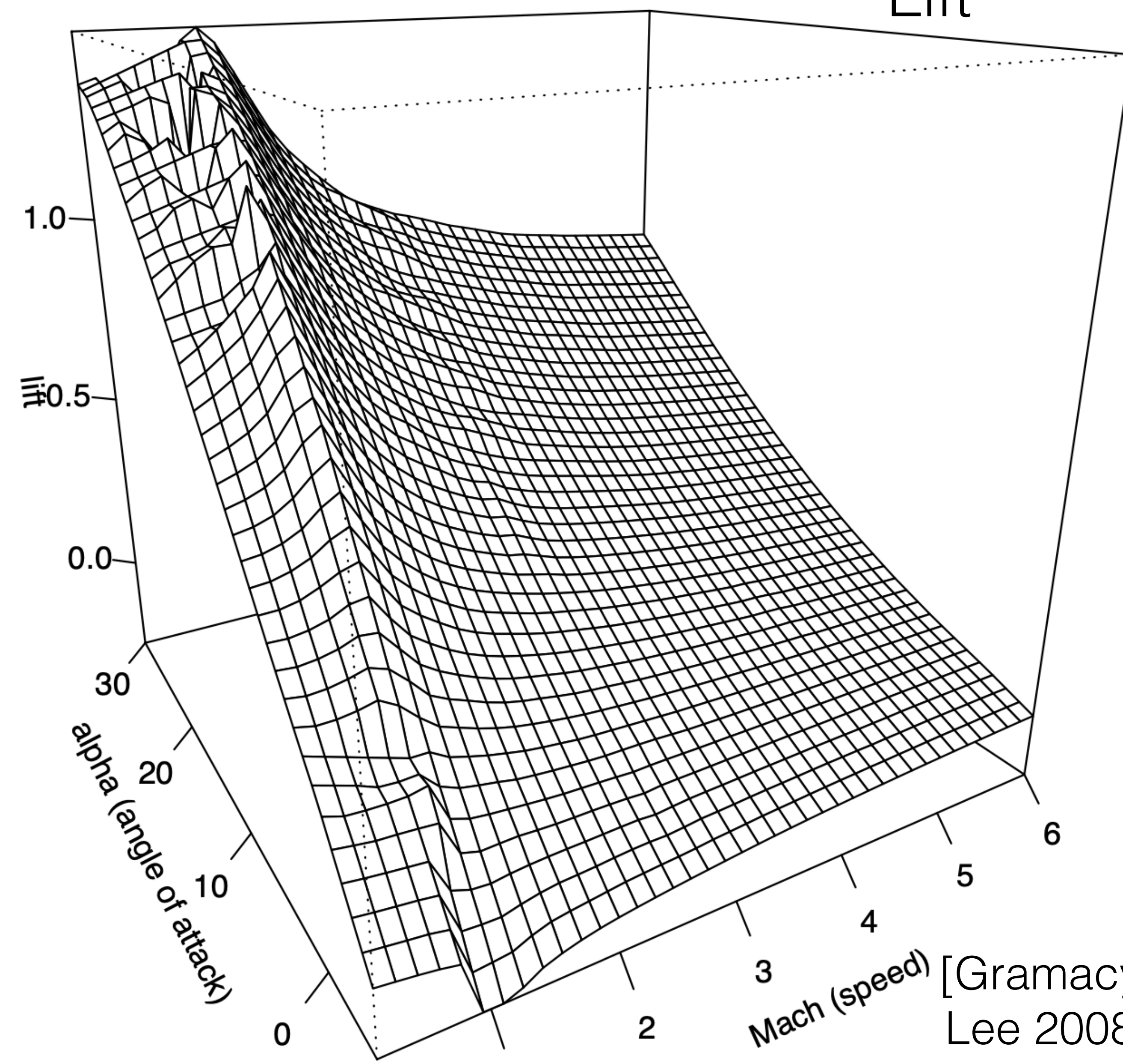

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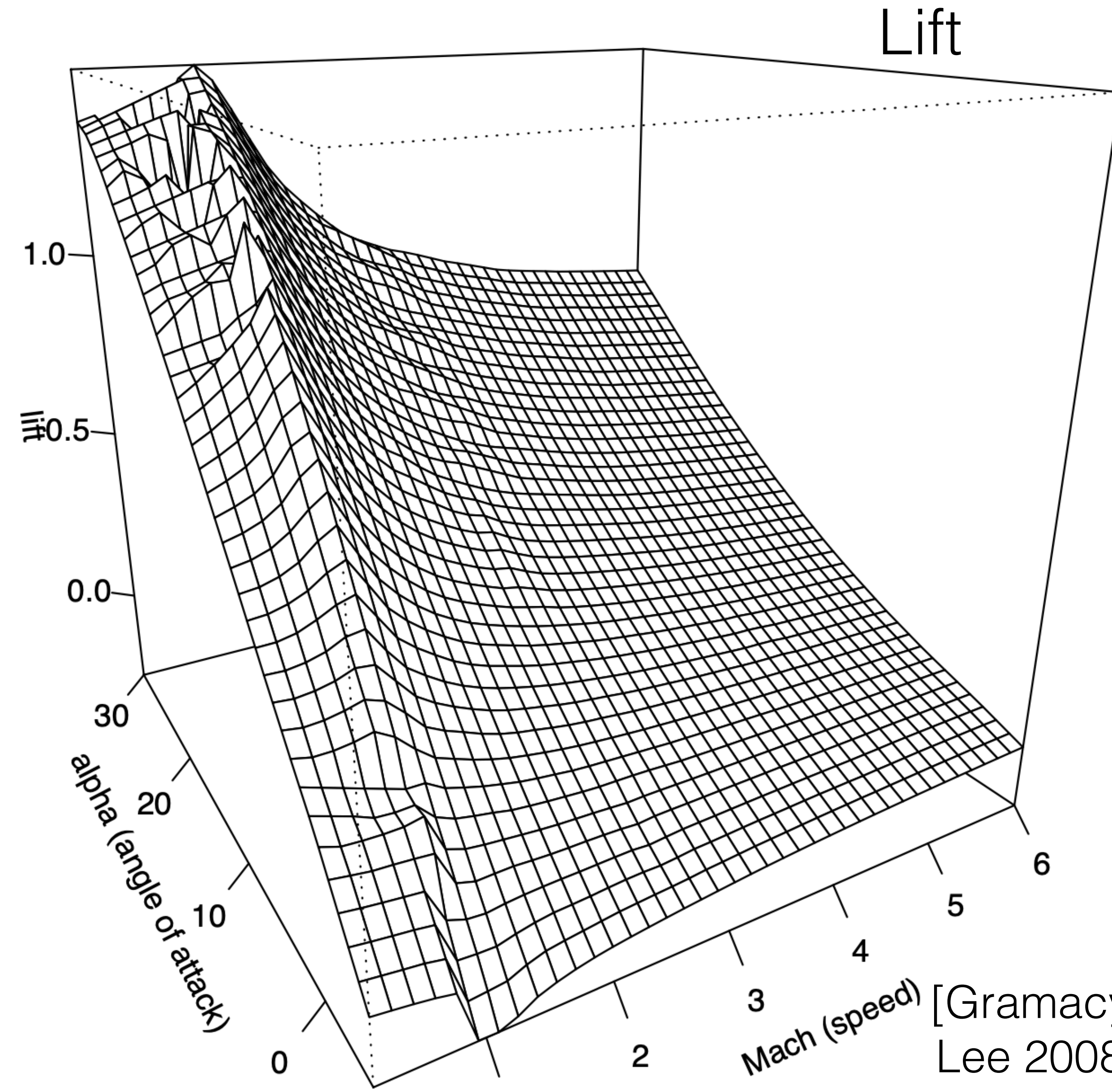

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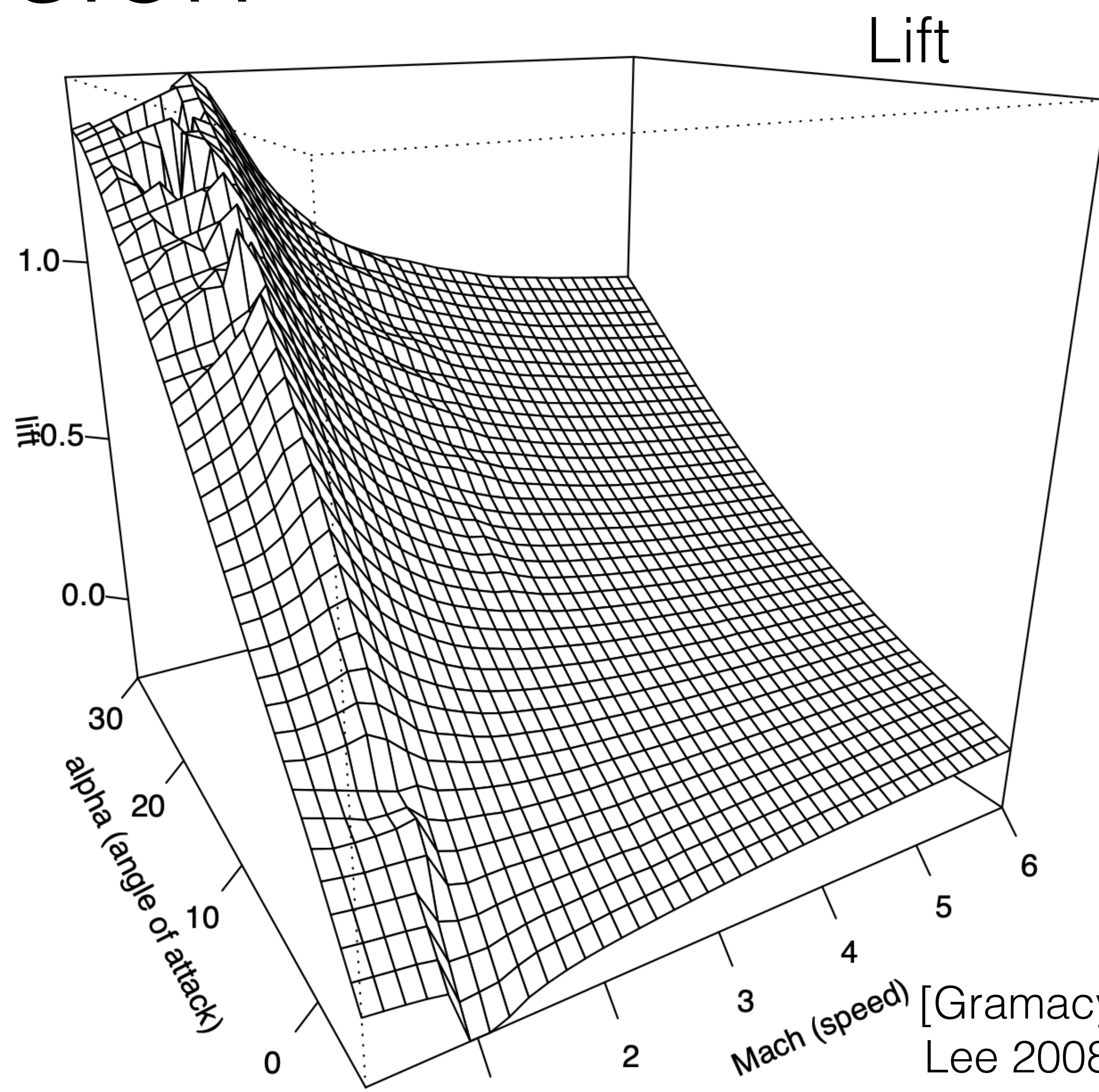

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Common similar problem: learn (& optimize) performance in ML as a function of scalar hyperparameters [Snoek et al 2012, 2015; Garnett 2023]

- E.g. tune momentum, learning rate, dropout rates, etc.

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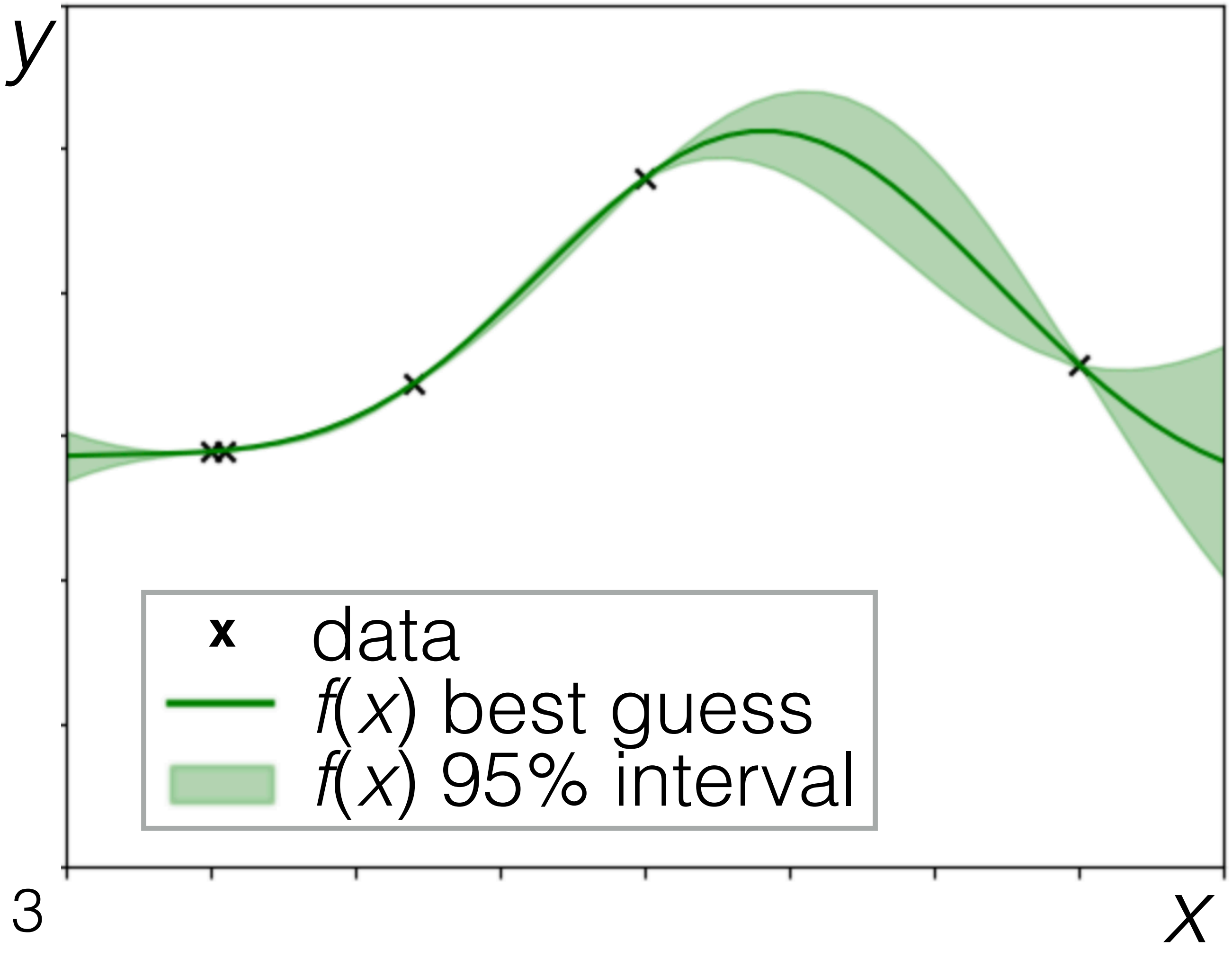

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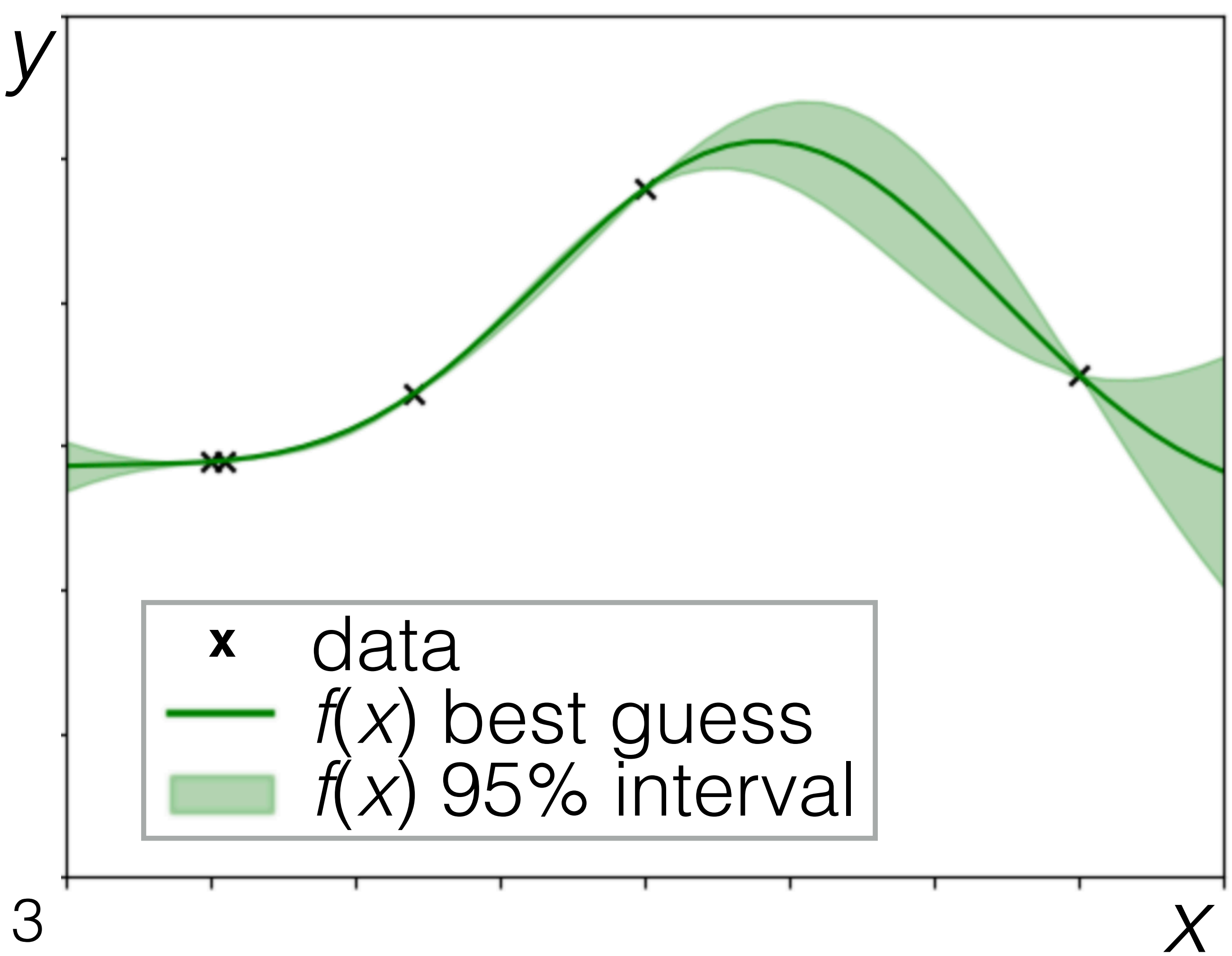

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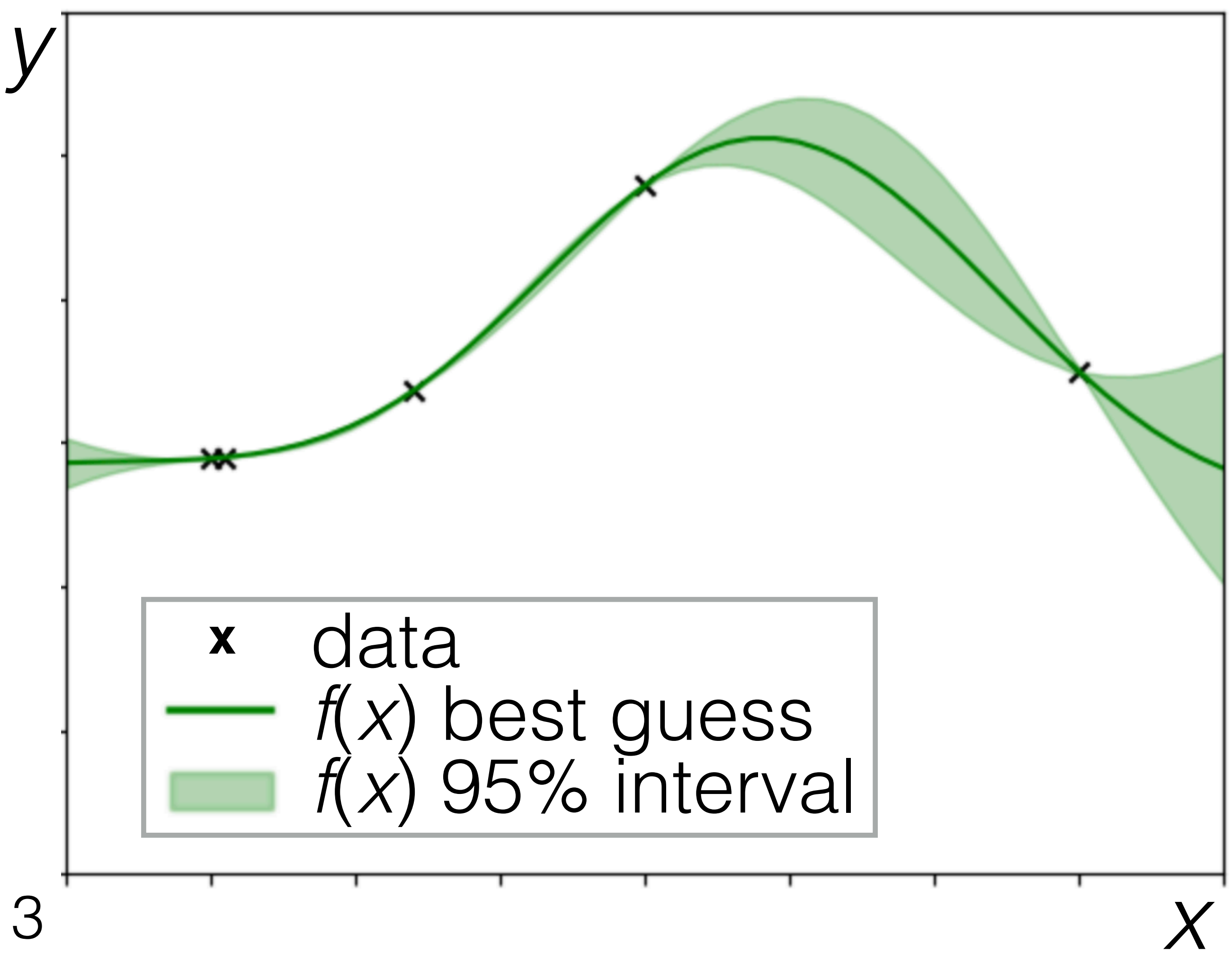

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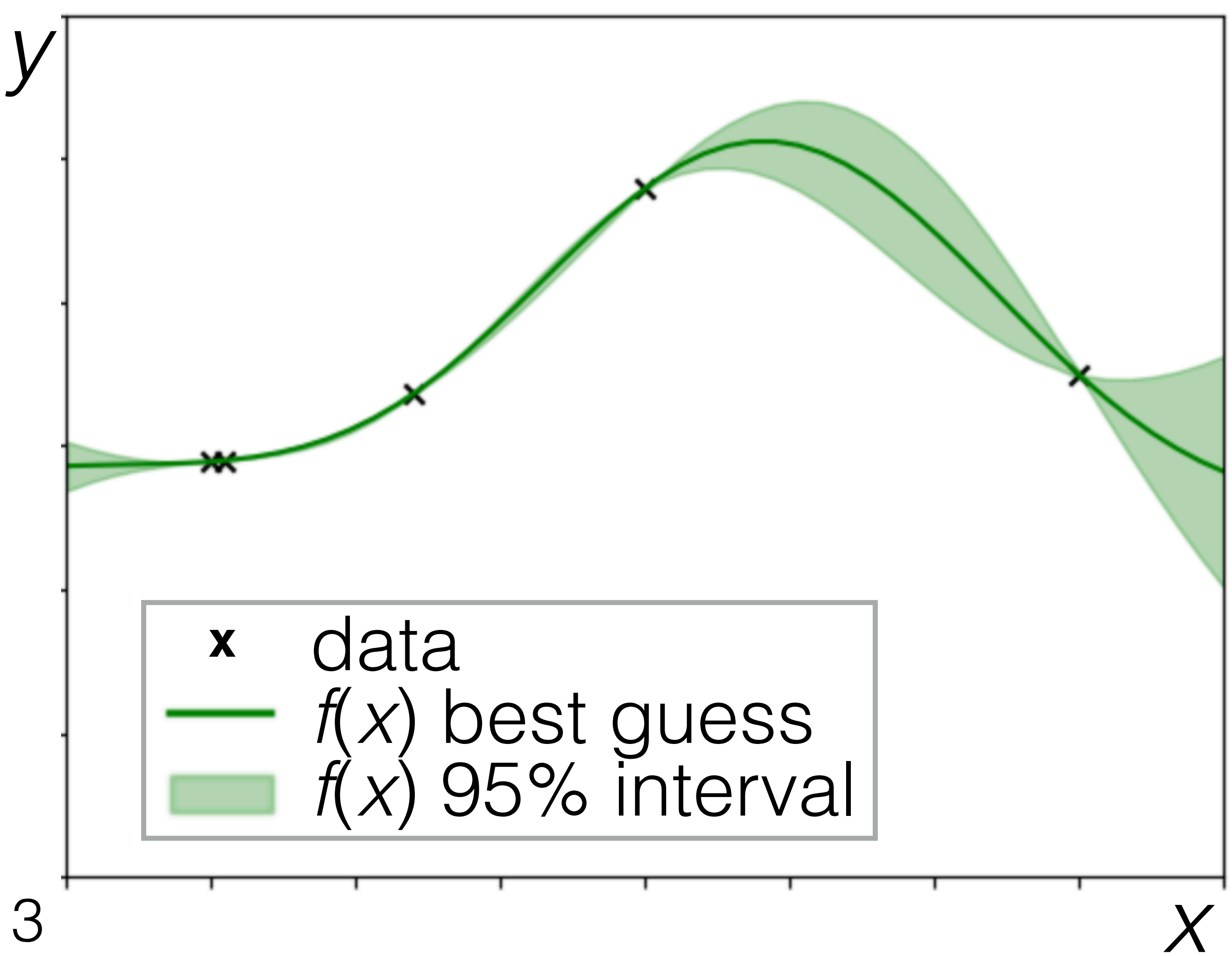

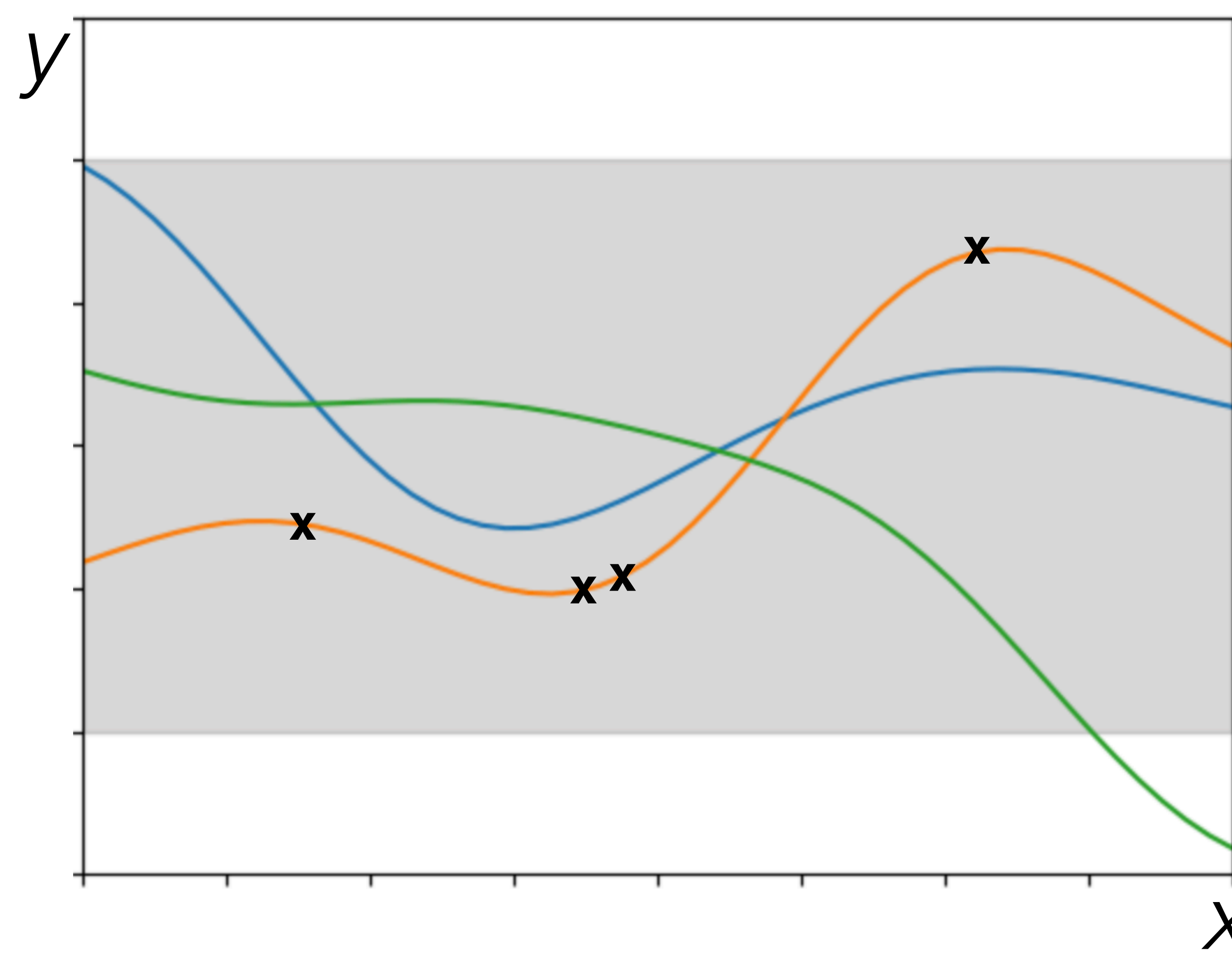

Multivariate Gaussian using locations

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- *M*=2 dimensions: $[y^{(1)}, y^{(2)}]^\top \sim \mathcal{N}(\mu, K)$
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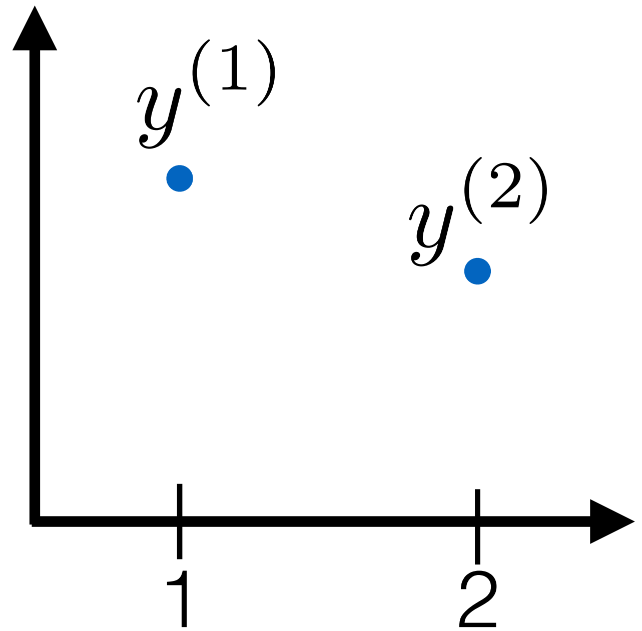

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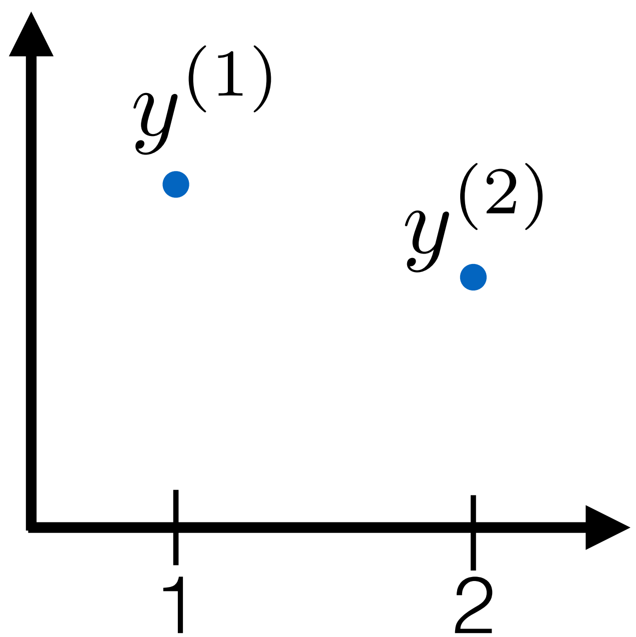

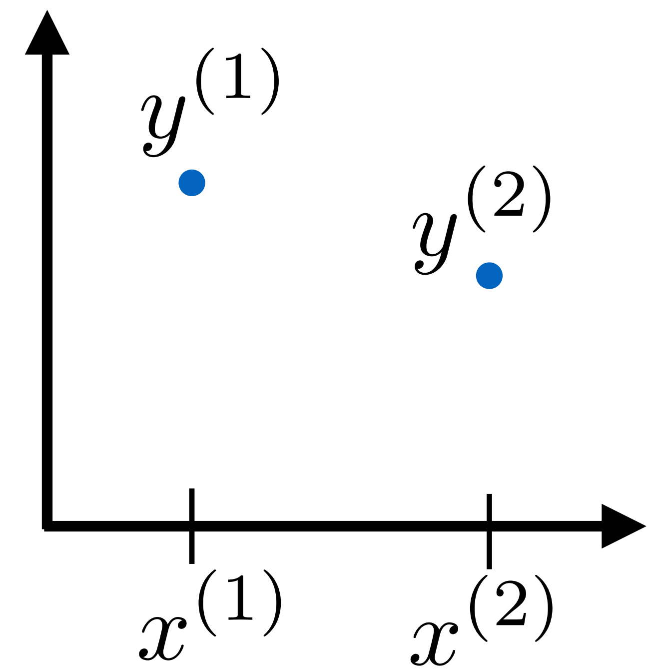

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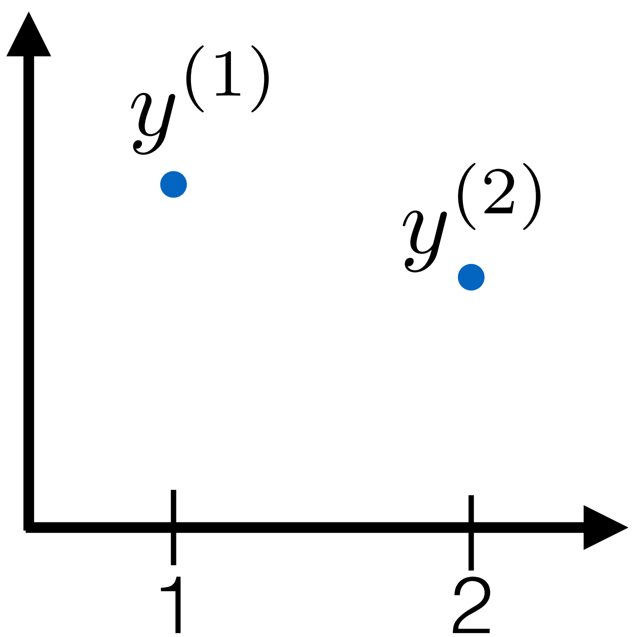

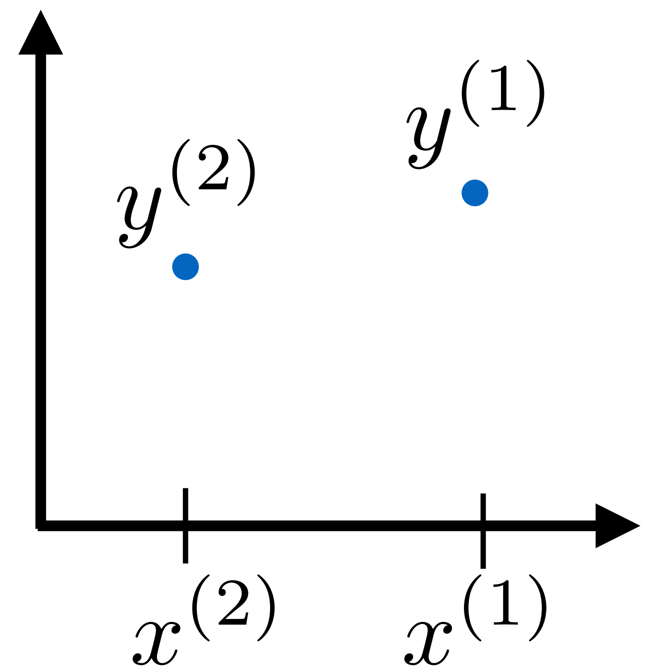

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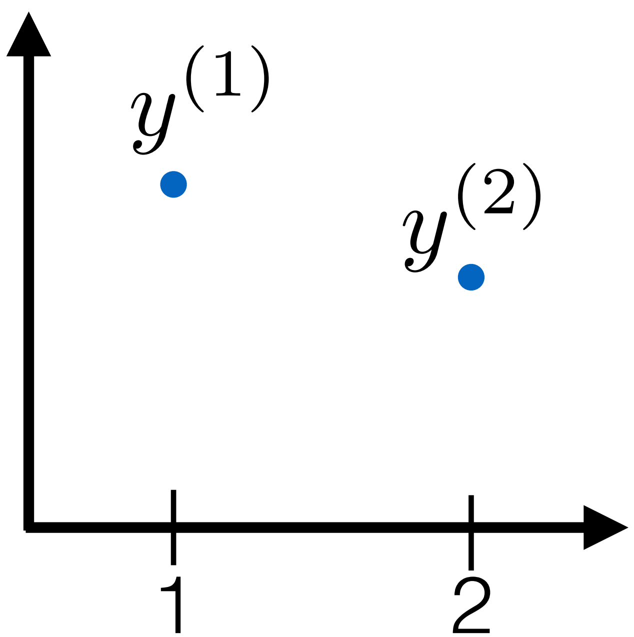

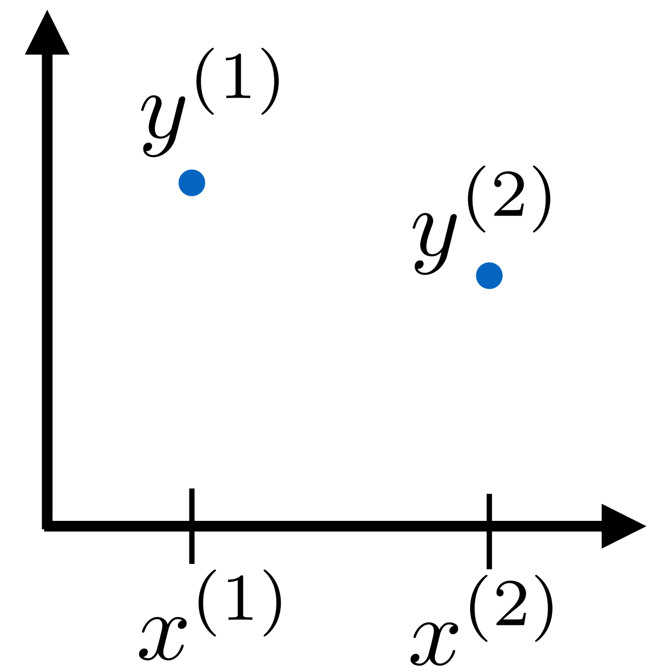

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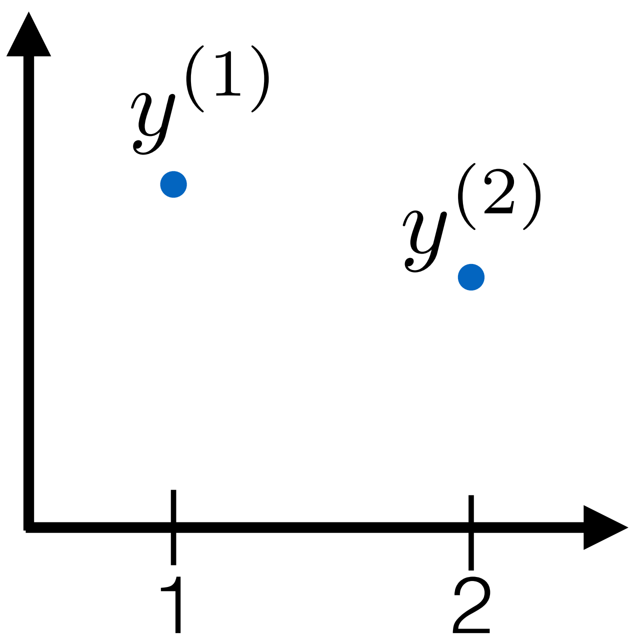

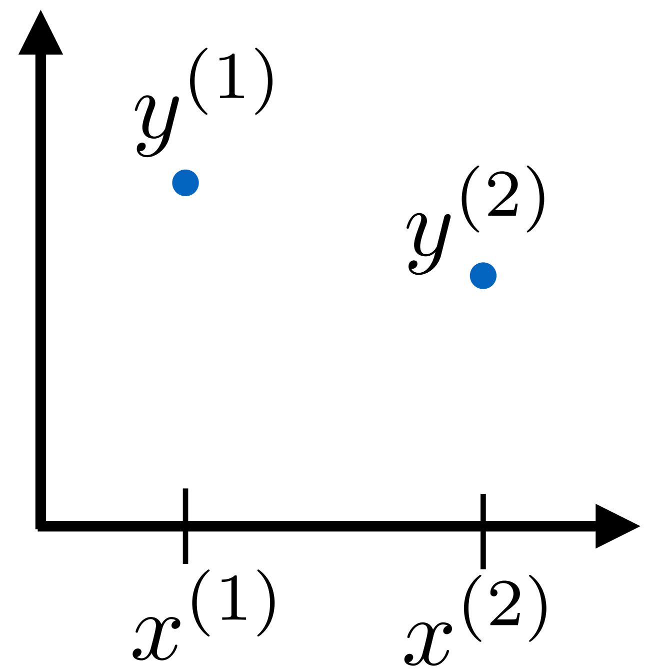

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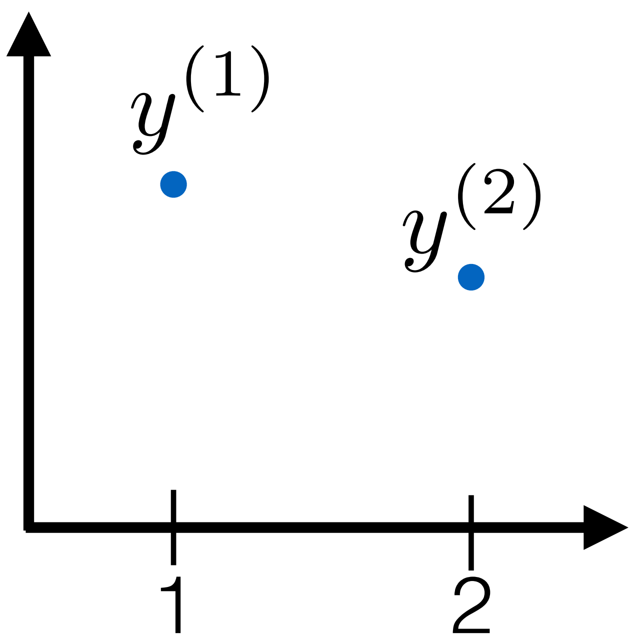

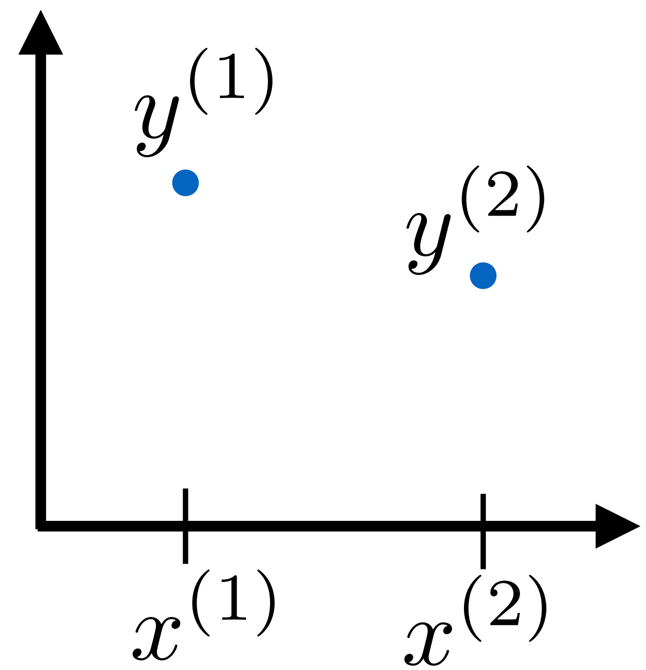

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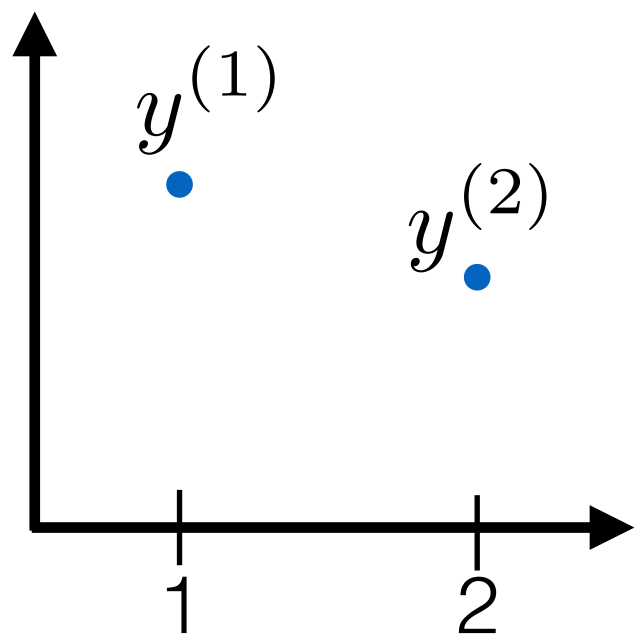

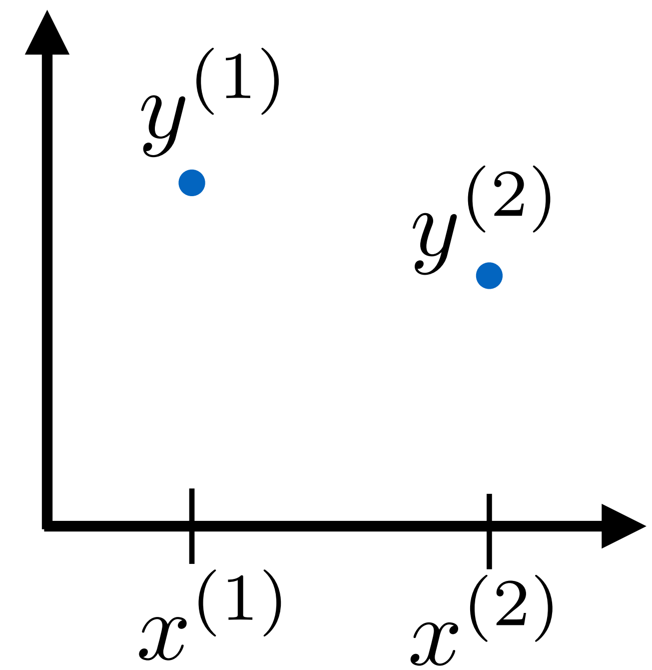

- What if we let the correlation depend on the x's?
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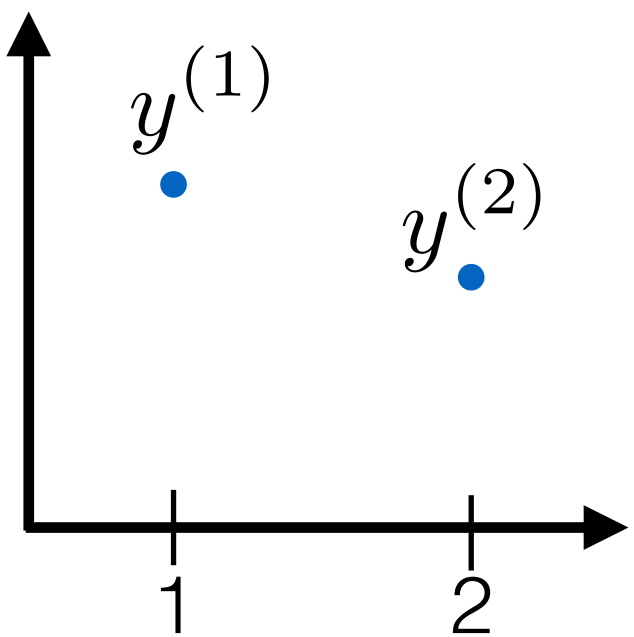

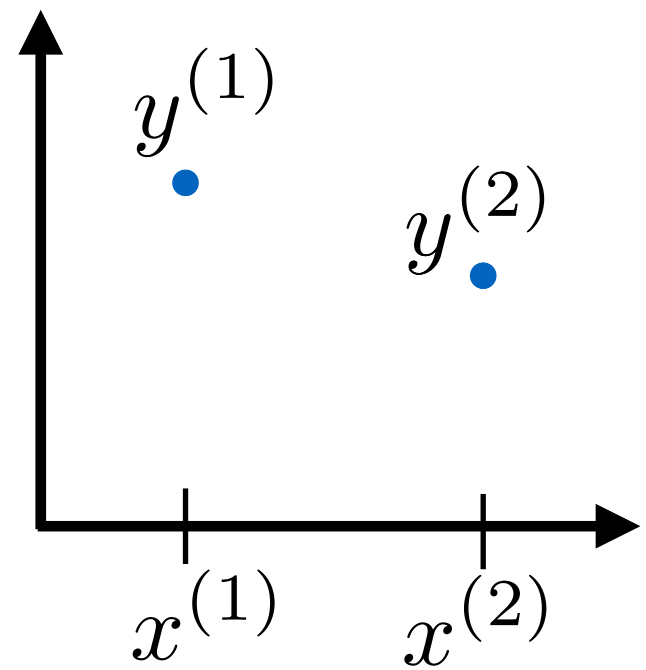

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  - Let $\rho = \rho(|x^{(1)} - x^{(2)}|)$
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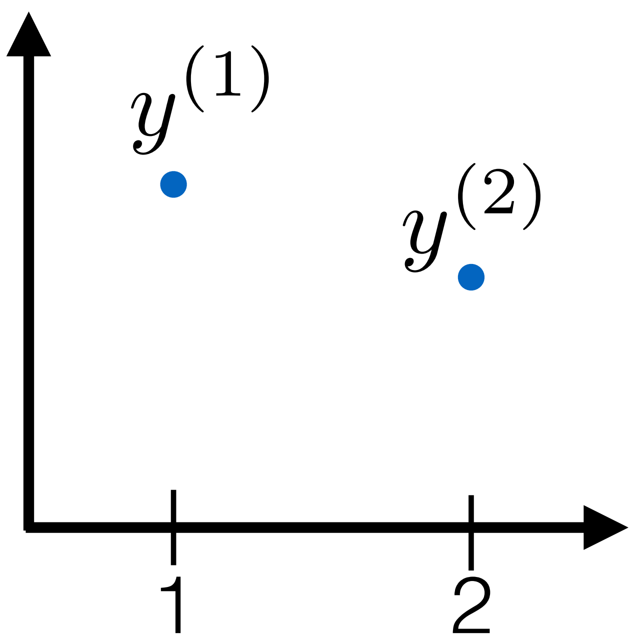

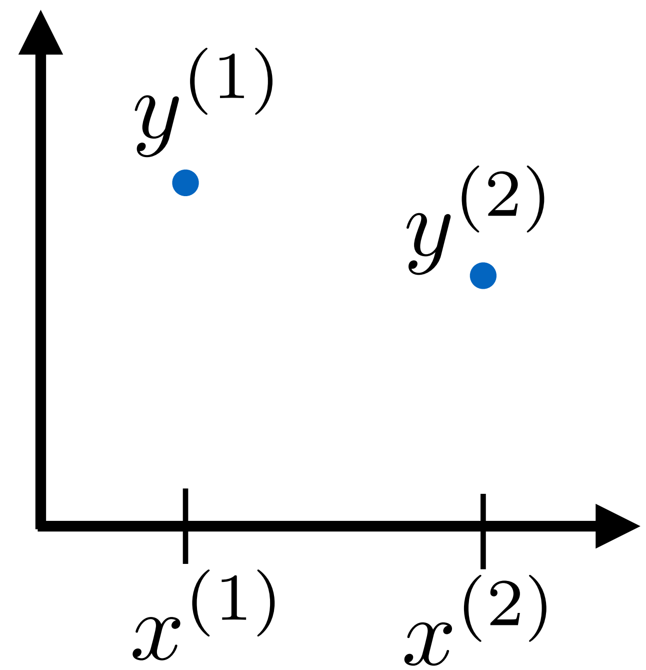

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[demo]

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We just drew random functions from a type of "Gaussian process"!

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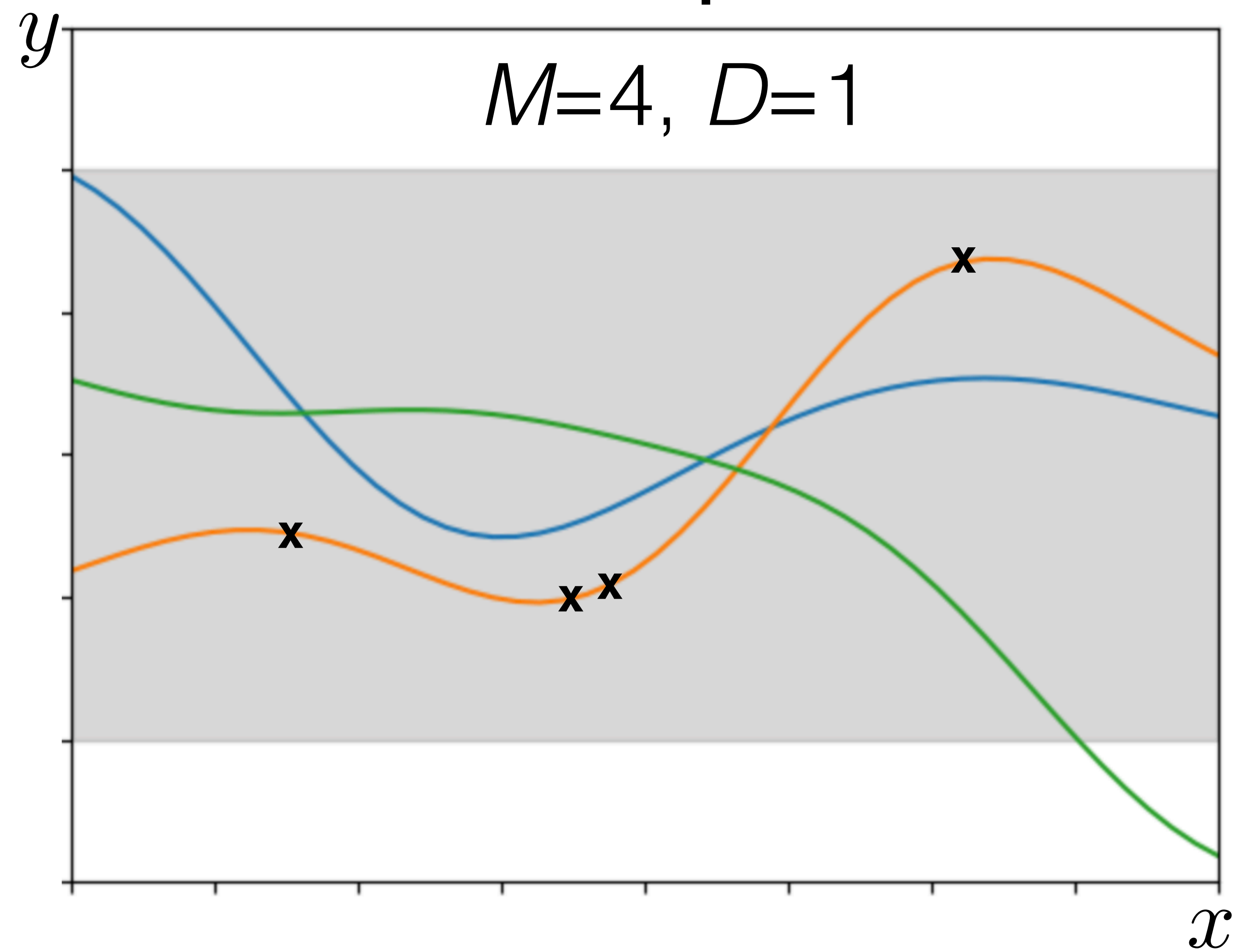

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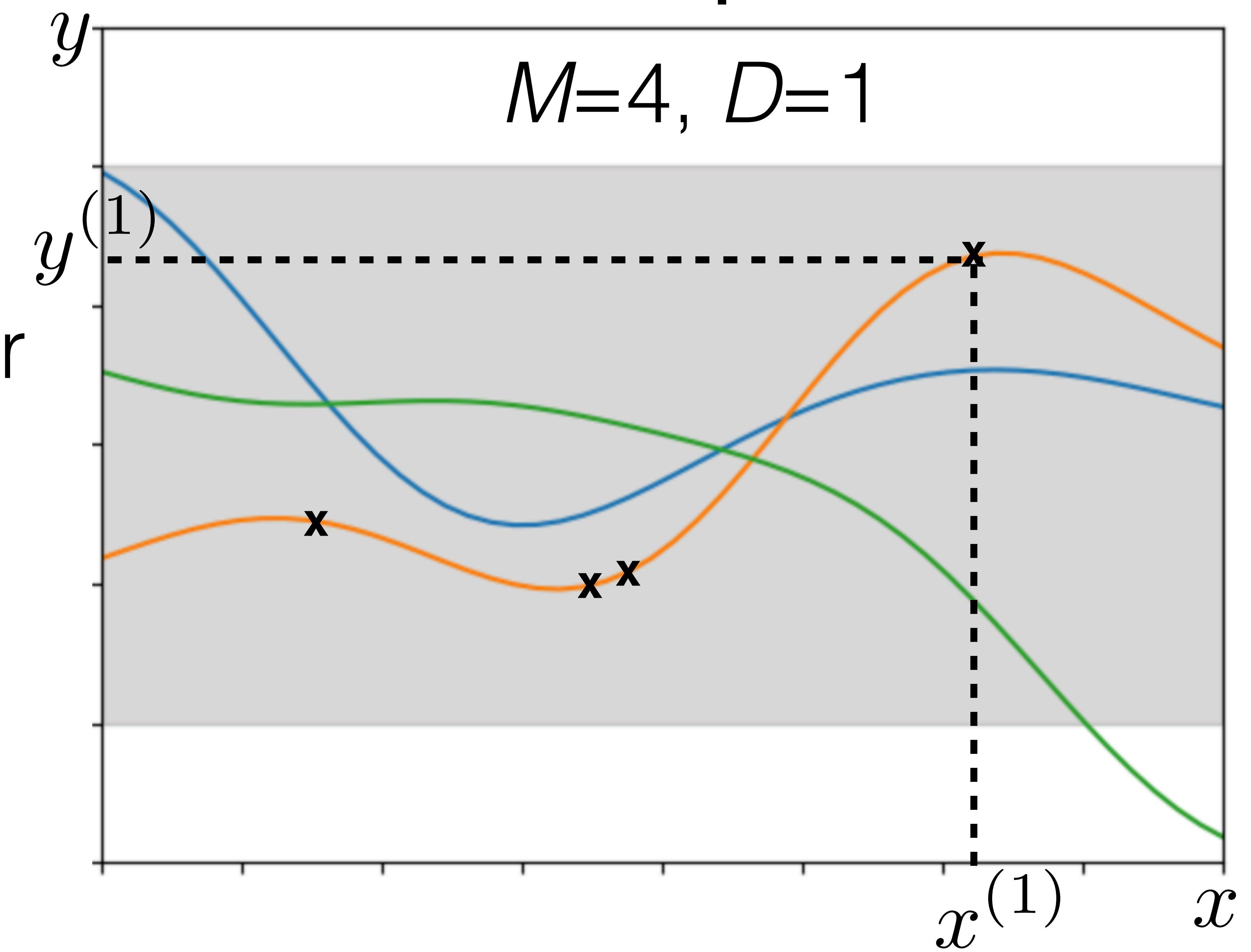

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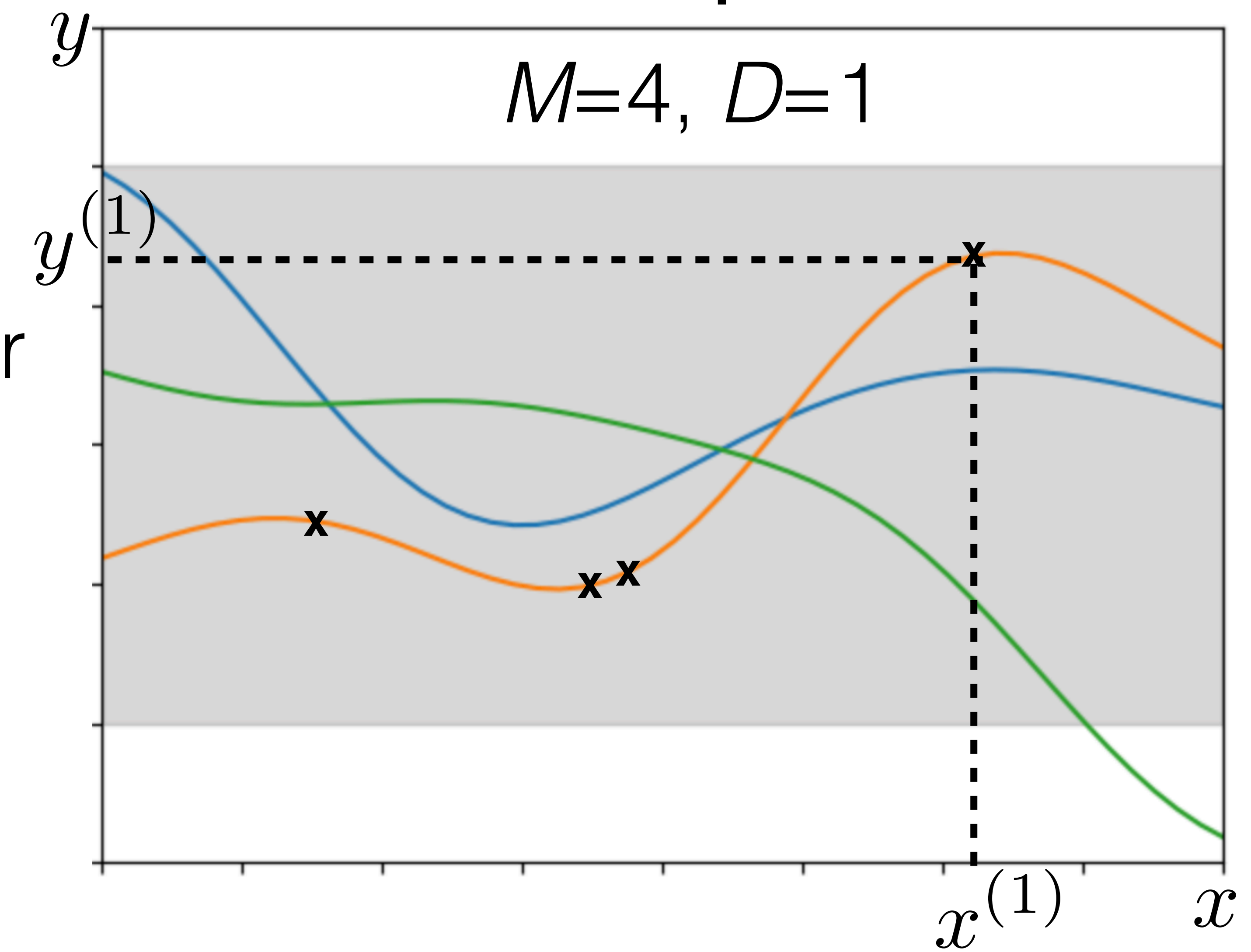

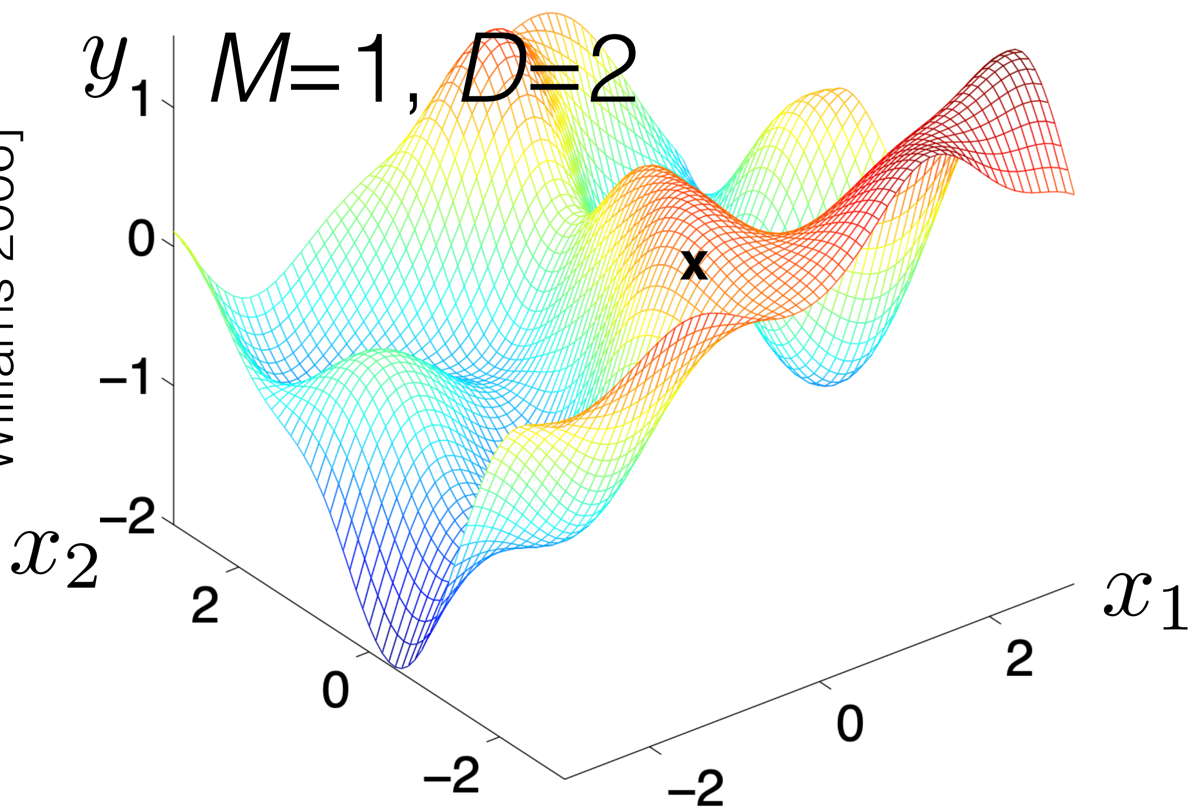

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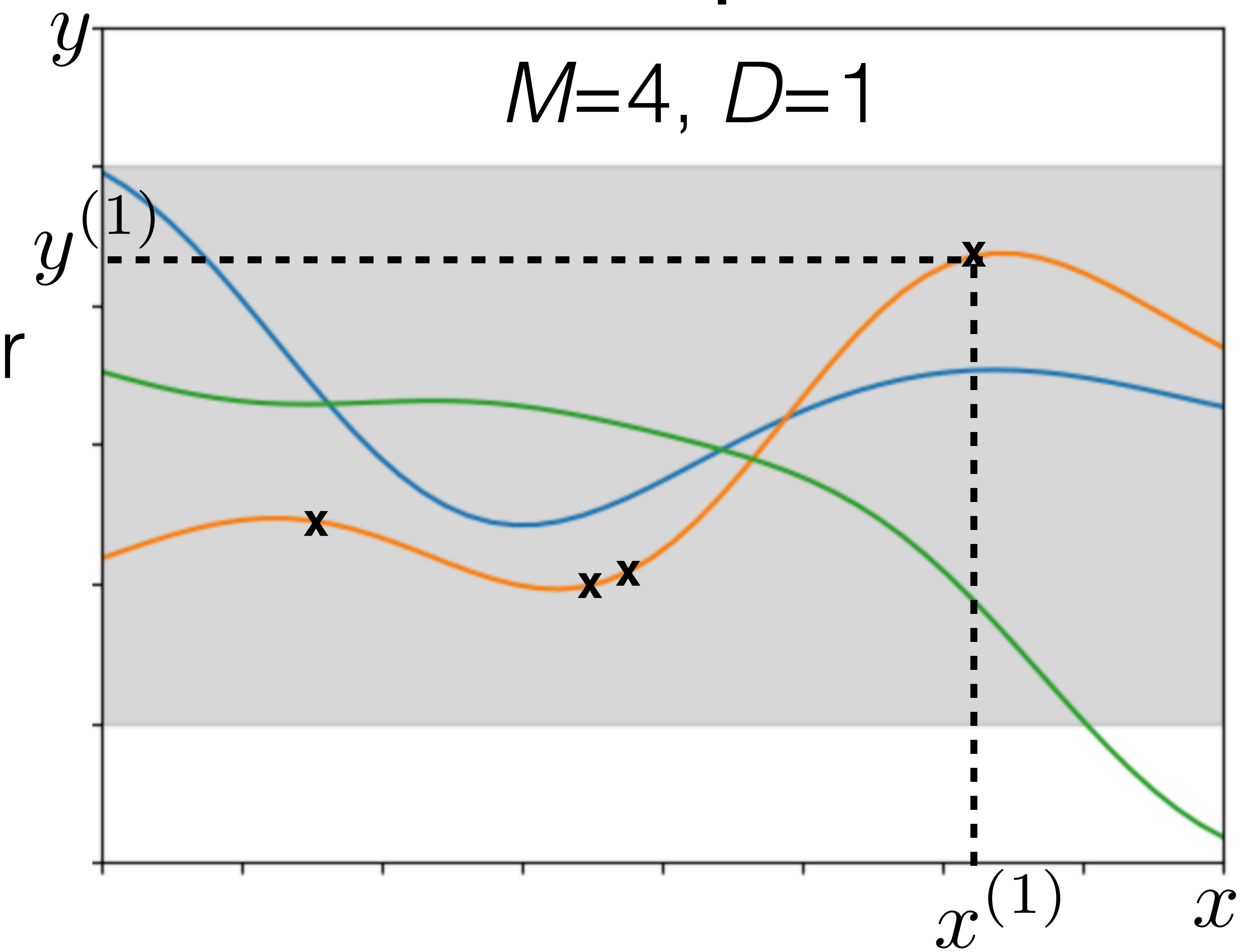

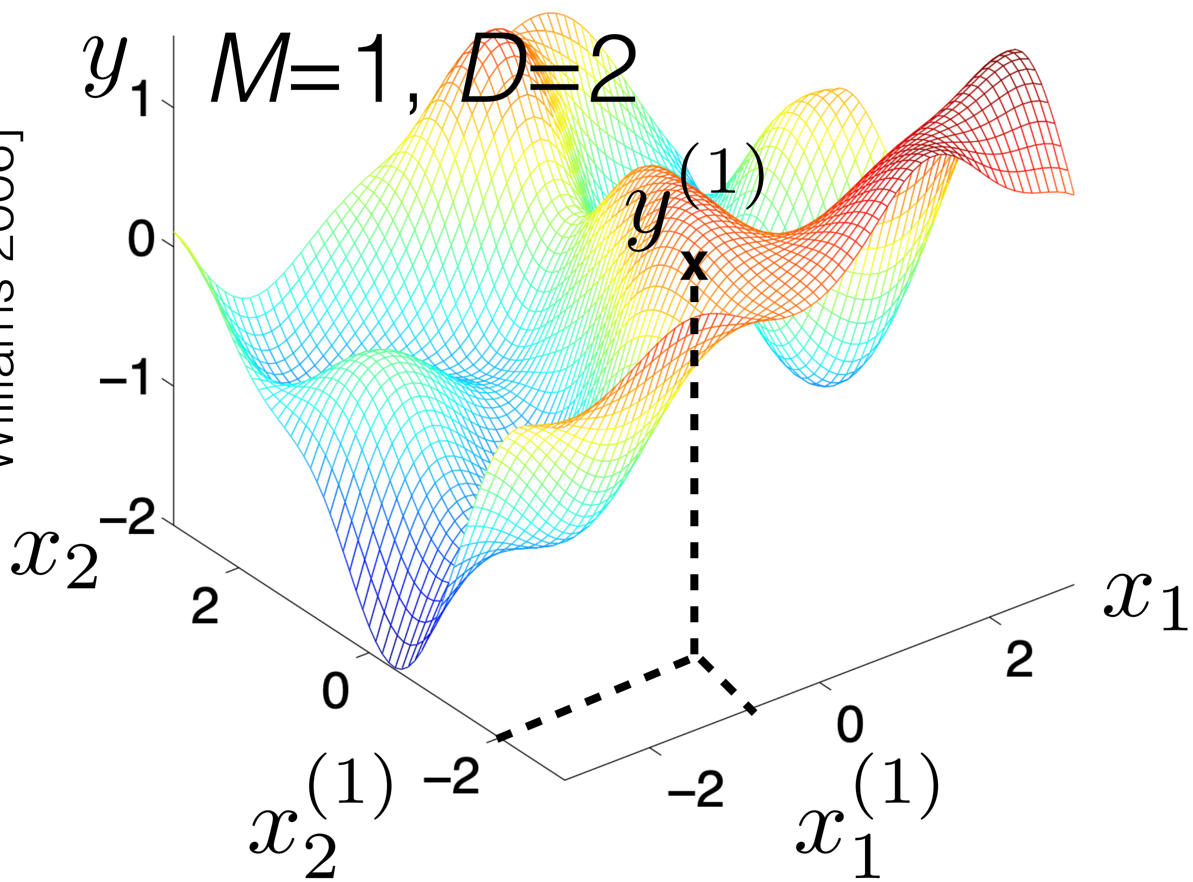

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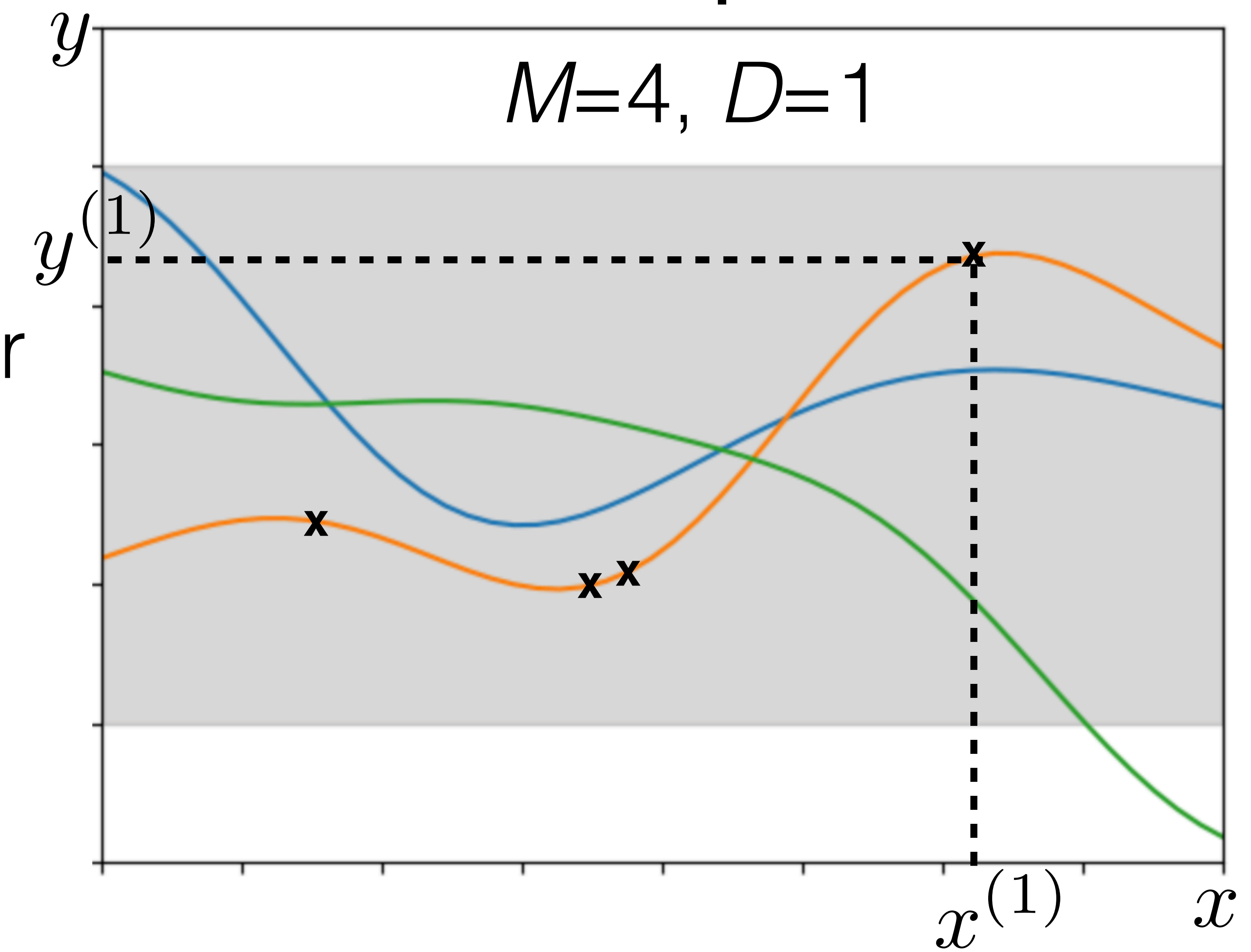

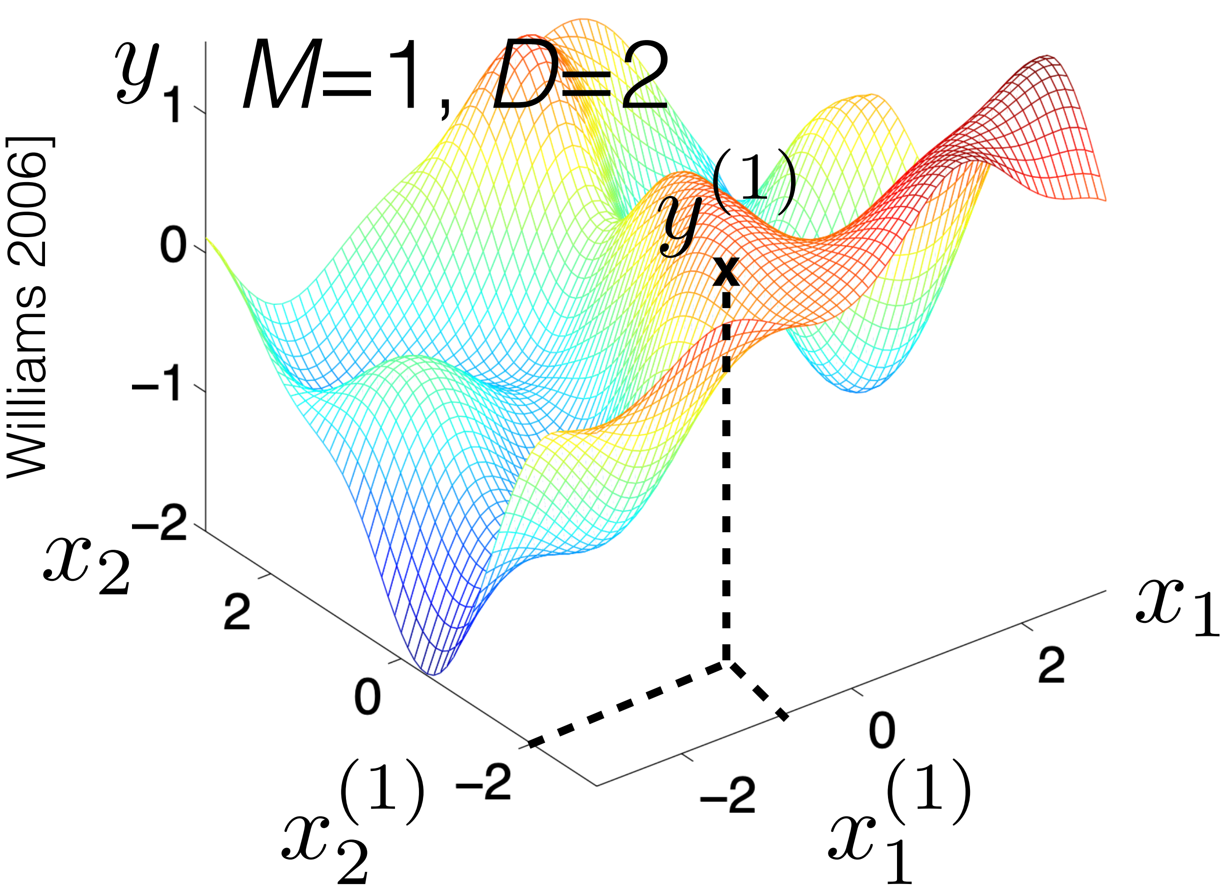

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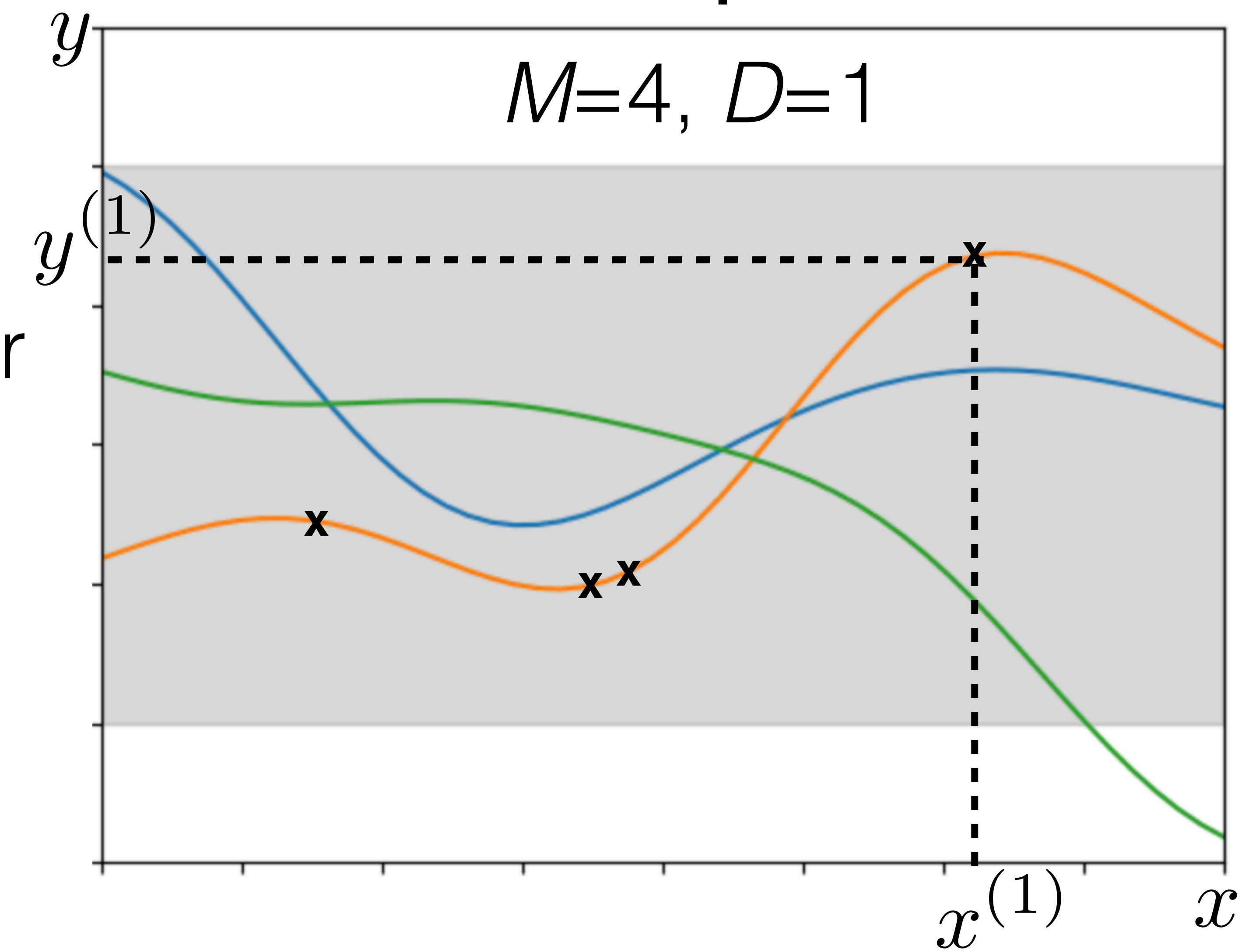

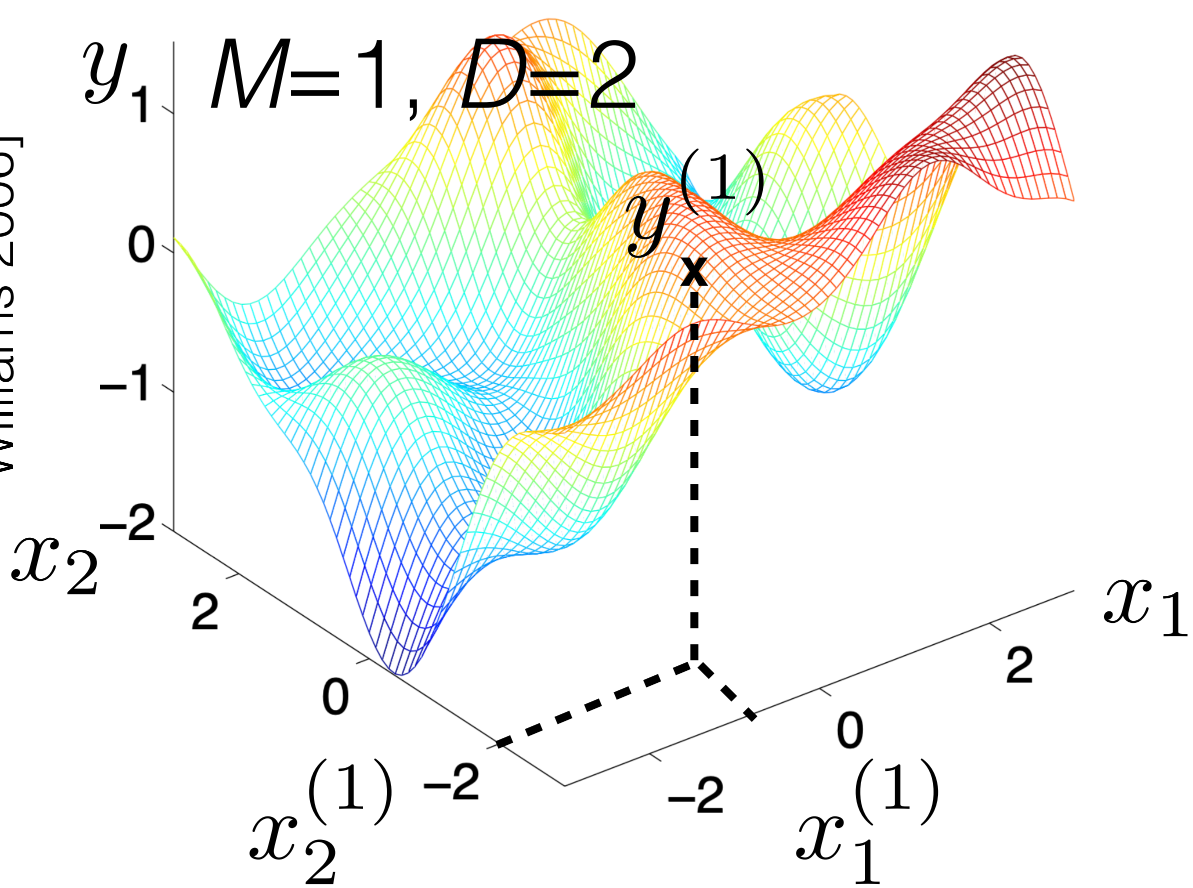

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Inference about unknowns given data

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$$k(x, x') = \sigma^2 \exp(-\tfrac{1}{2}(x - x')^2), \sigma = 1$$

Inference about unknowns given data

- Let X collect the N "training" data points (indexed 1 to N)
- Let X' collect the M "test" data points

  X: NxD
  - Where we want to evaluate the function
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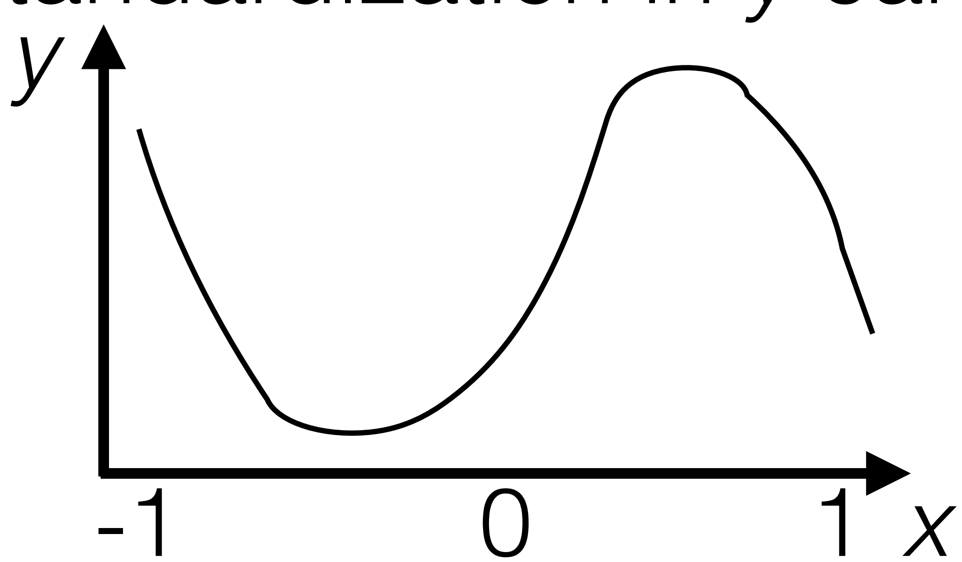

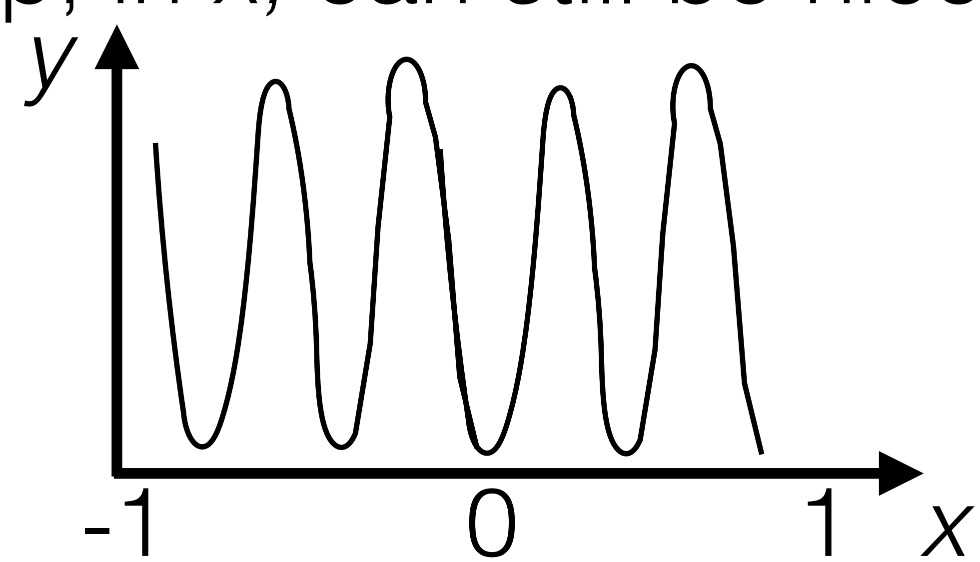

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  - Given those values, now compute and report the mean and uncertainty intervals.

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