6.7900: Machine Learning
Lecture 3

Lecture start: Tues/Thurs 2:35pm
Who's speaking today? Prof. Tamara Broderick
Course website: gradml.mit.edu
Questions? Ask here or on piazza.com/mit/fall2024/67900/
Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Time

I. Empirical risk minimization
II. Modeling
III. Maximum likelihood estimate (MLE)
IV. Pros & cons of the MLE

Today's Plan

I. A Bayesian approach
II. Prior
III. Posterior
IV. Predictive

Recap

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we'll get back to having features in the next lecture

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  - Often have domain knowledge
- A **Bayesian approach** tries to address overfitting, quantify uncertainty, and incorporate domain knowledge

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  - We have a likelihood
  - A Bayesian model needs both a likelihood and a prior

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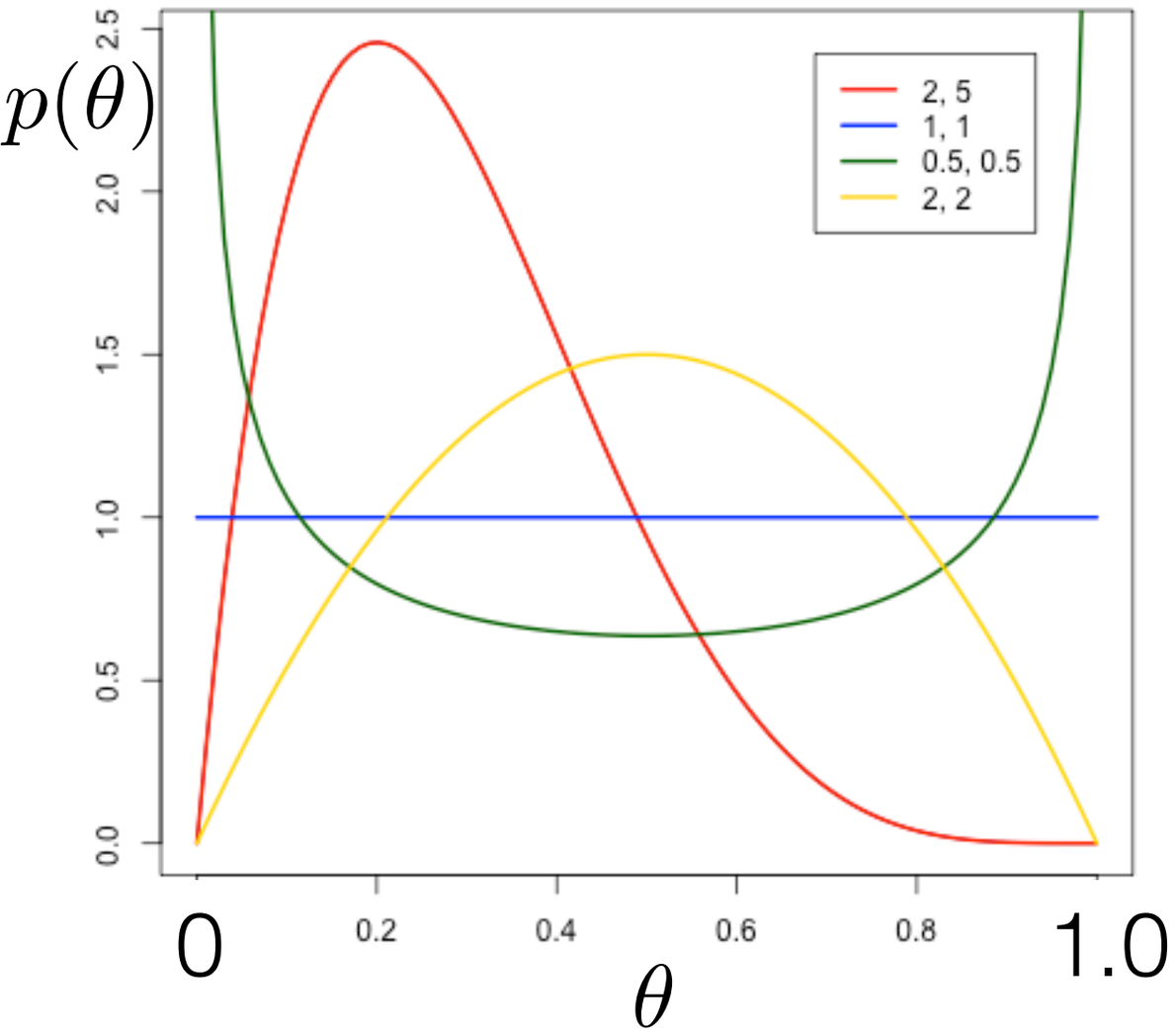

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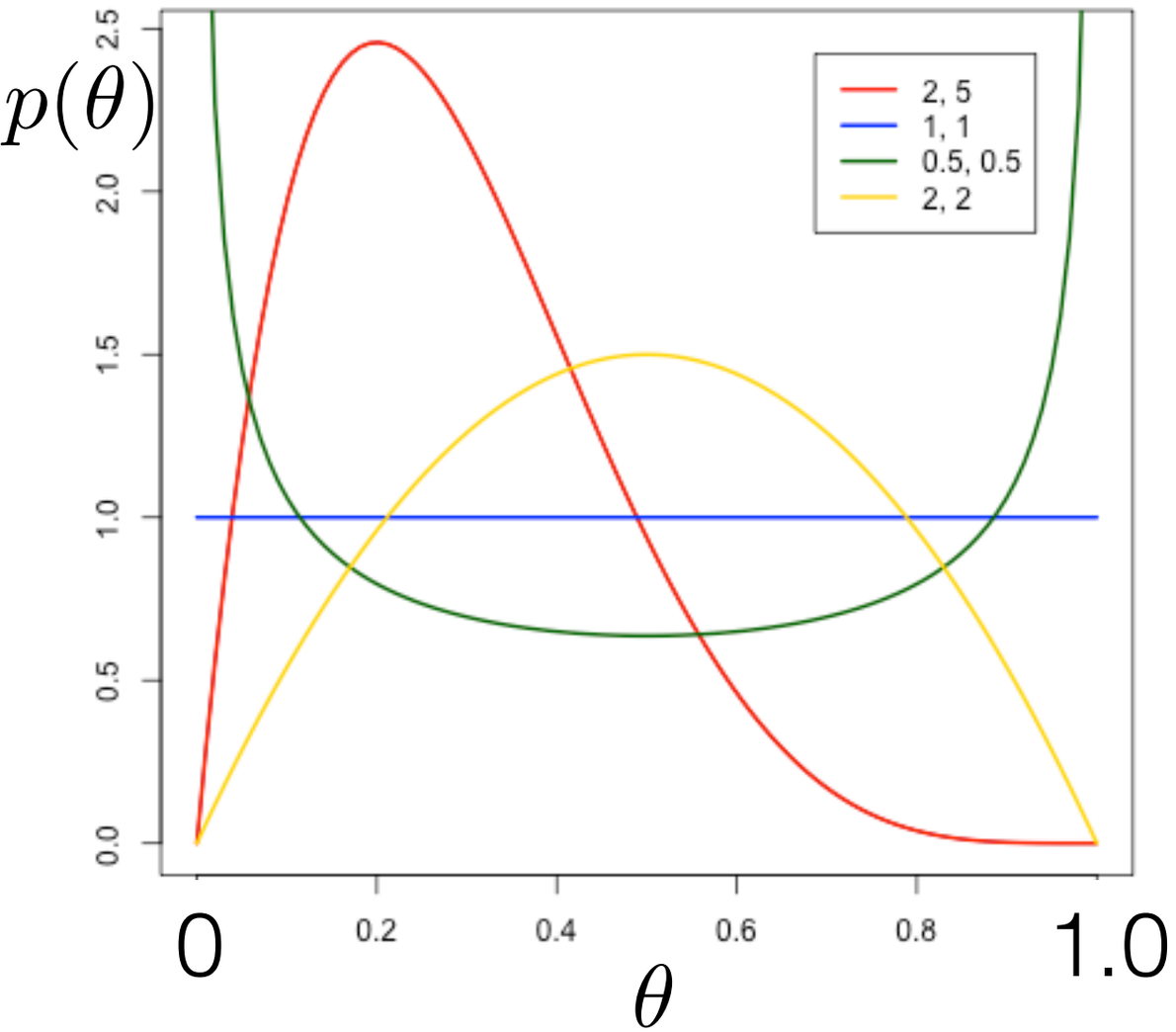

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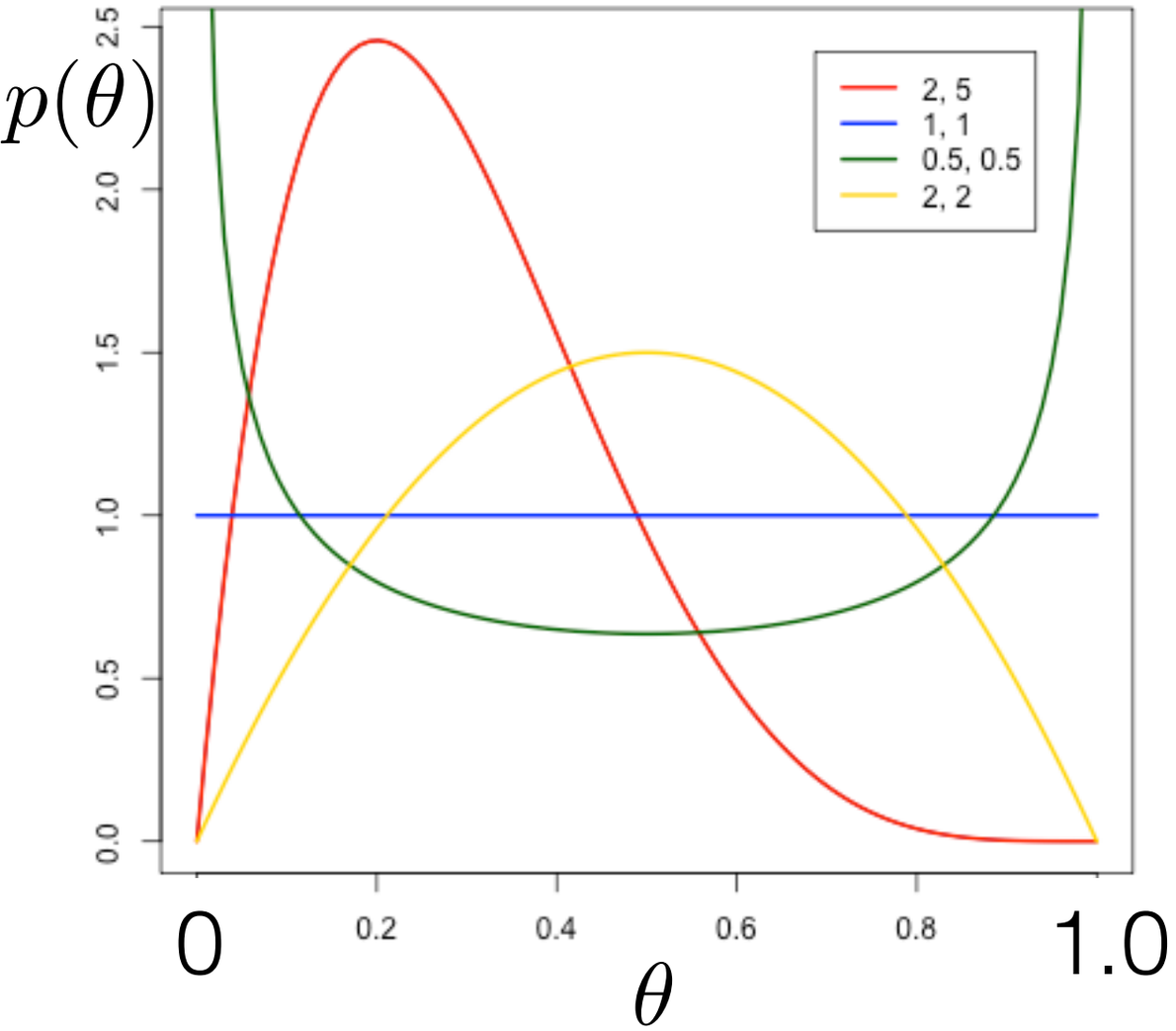

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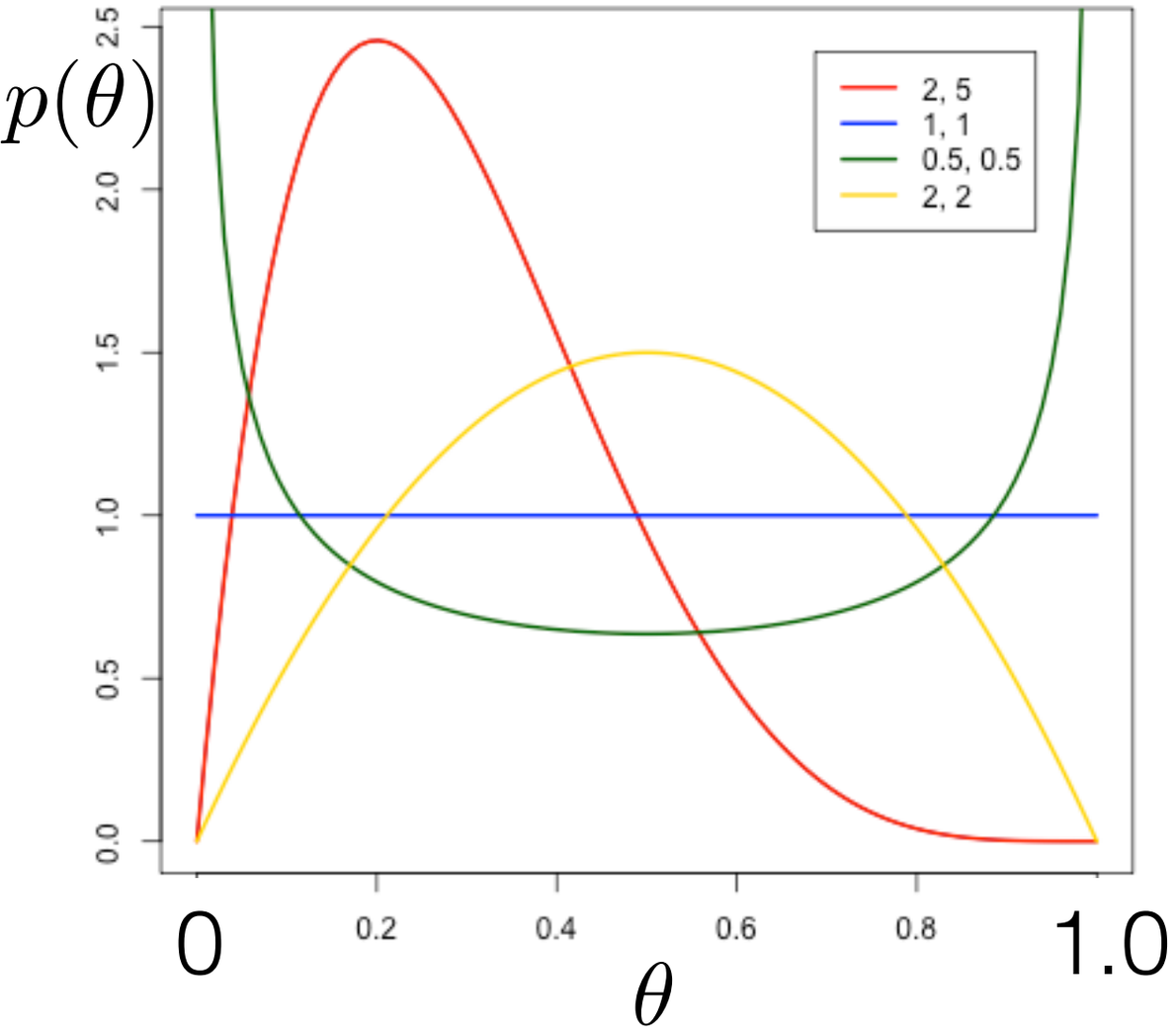

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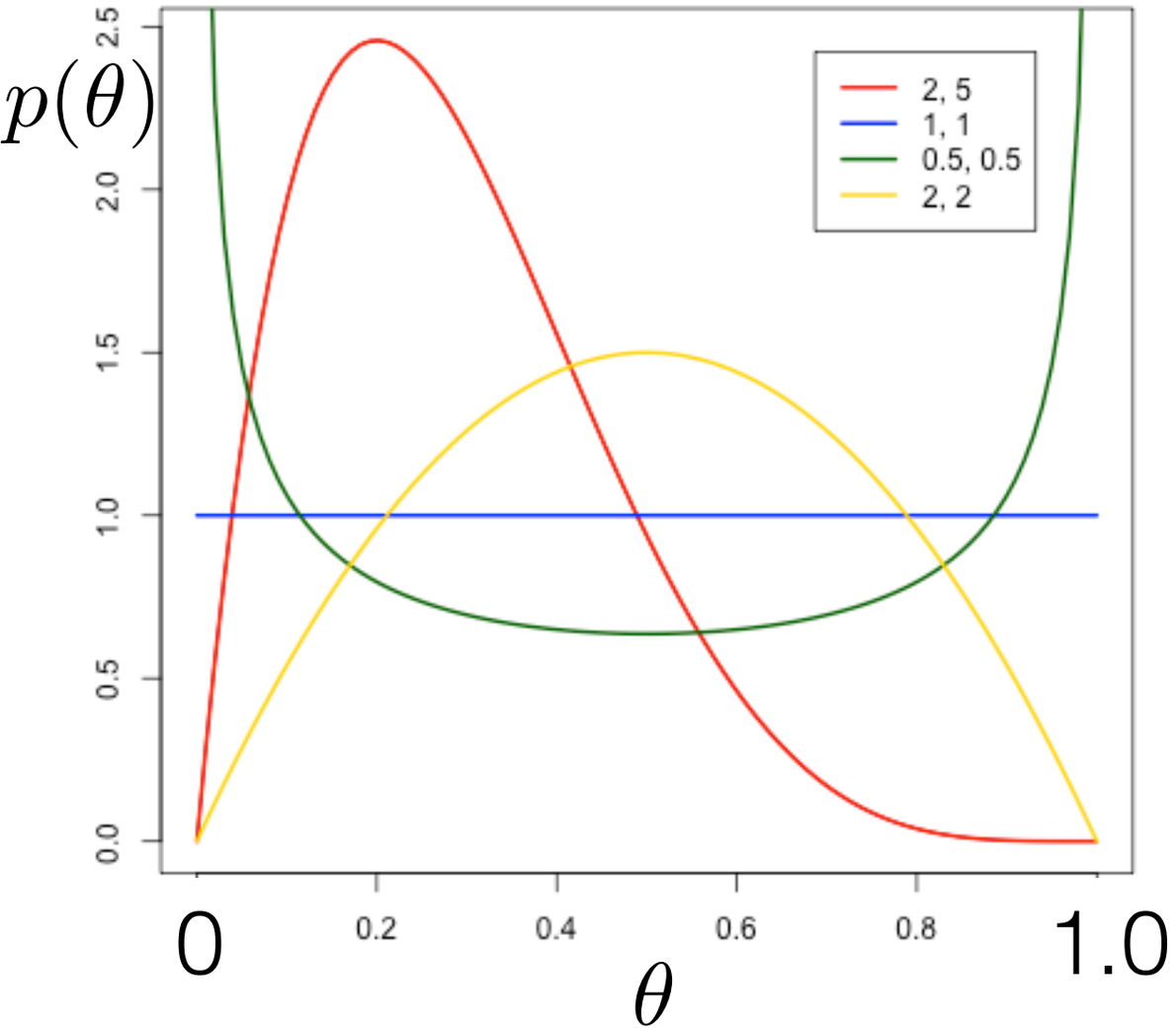

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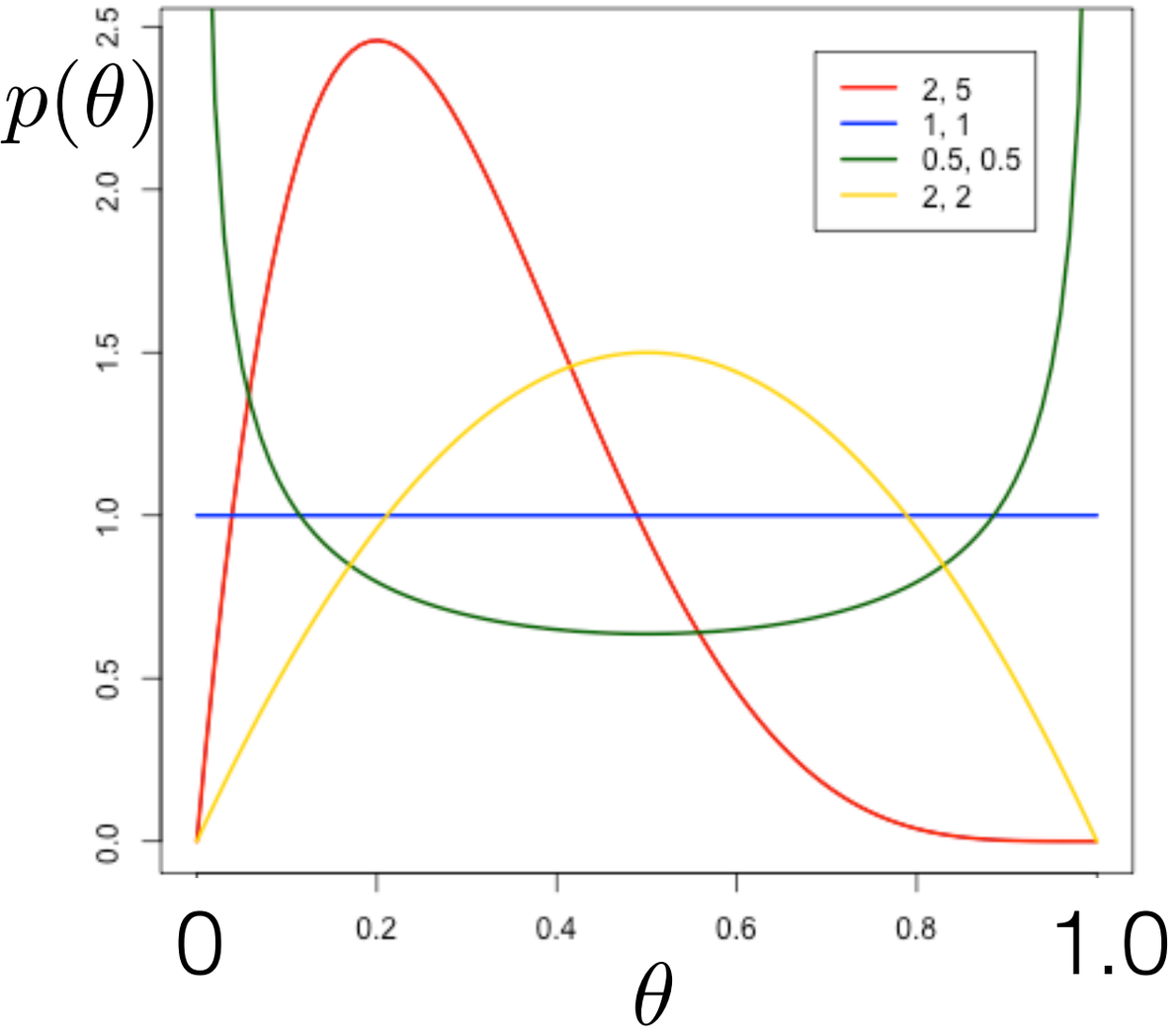

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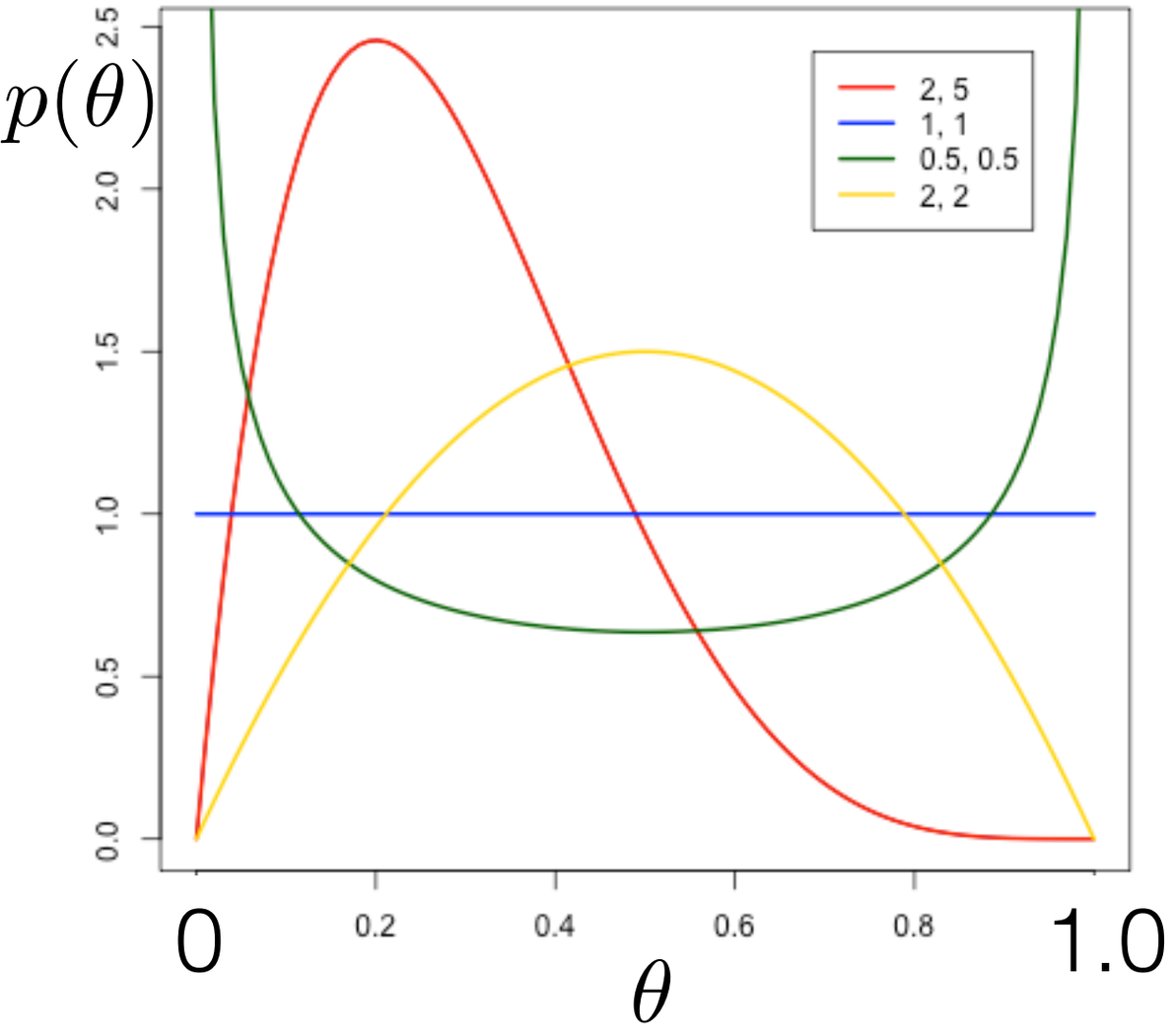

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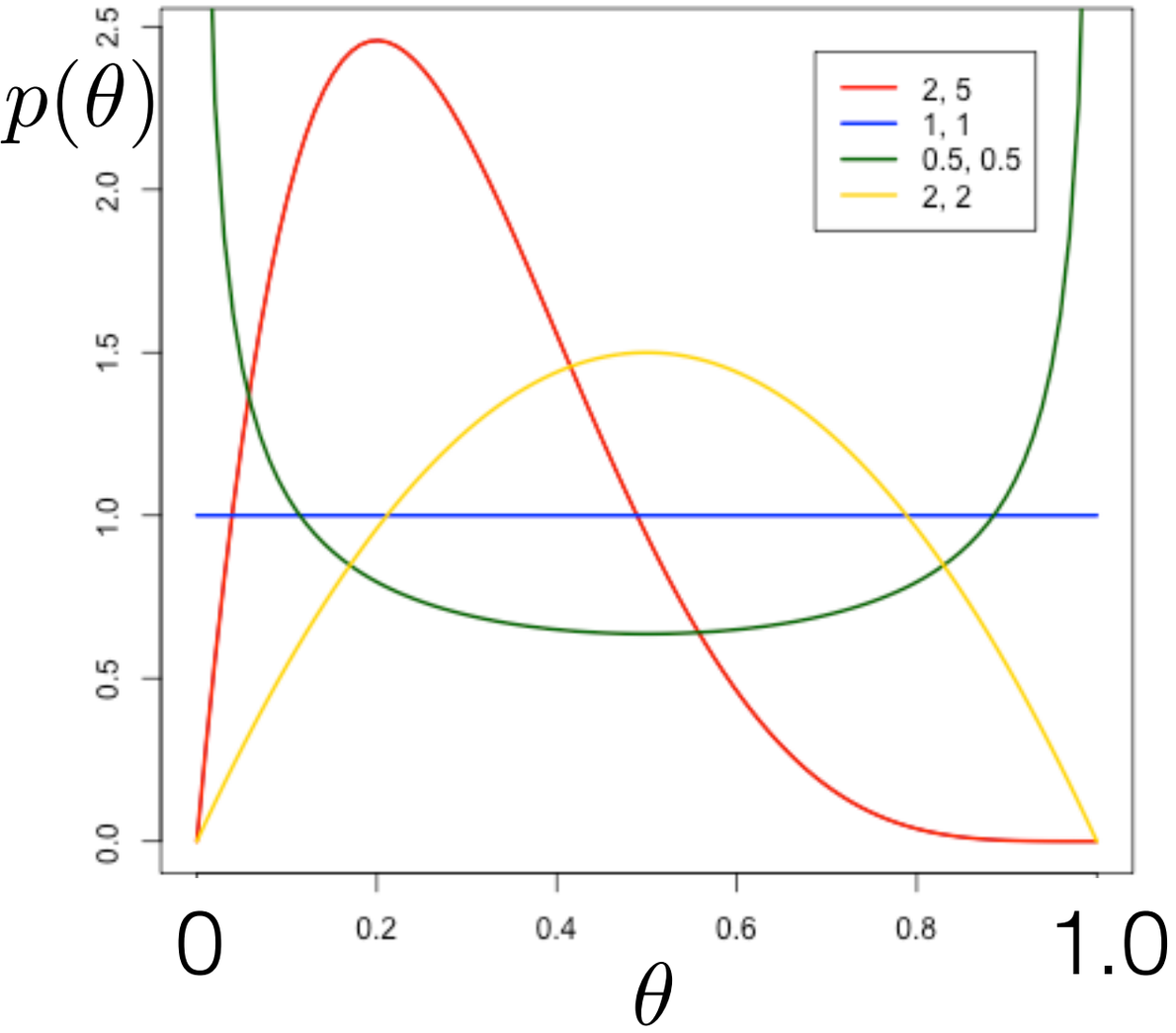

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revisit questions above

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- Predictive: $p(y|\hat{\theta}_{\mathrm{MAP}})$ vs. $p(y|\hat{\theta}_{\mathrm{MLE}})$ vs. $p(y^{(n+1)} = y|\mathcal{D})$