6.7900: Machine Learning
Lecture 18

Lecture start: Tues/Thurs 2:35pm

Who's speaking today? Prof. Tamara Broderick

Course website: gradml.mit.edu

Questions? Ask here or on piazza.com/mit/fall2024/67900/

Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Times

I. GPs for regression: model and inference

II. Examples and common types of missing data

Today

I. How to proceed when data is missing

II. Dimensionality reduction

III. Principal components analysis (PCA)

Recap: missing data

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- Some examples:

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  - Missing not at random (MNAR or NMAR)
    - Missingness can depend on something besides the observed data

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  - More generally, can introduce a “missingness” feature

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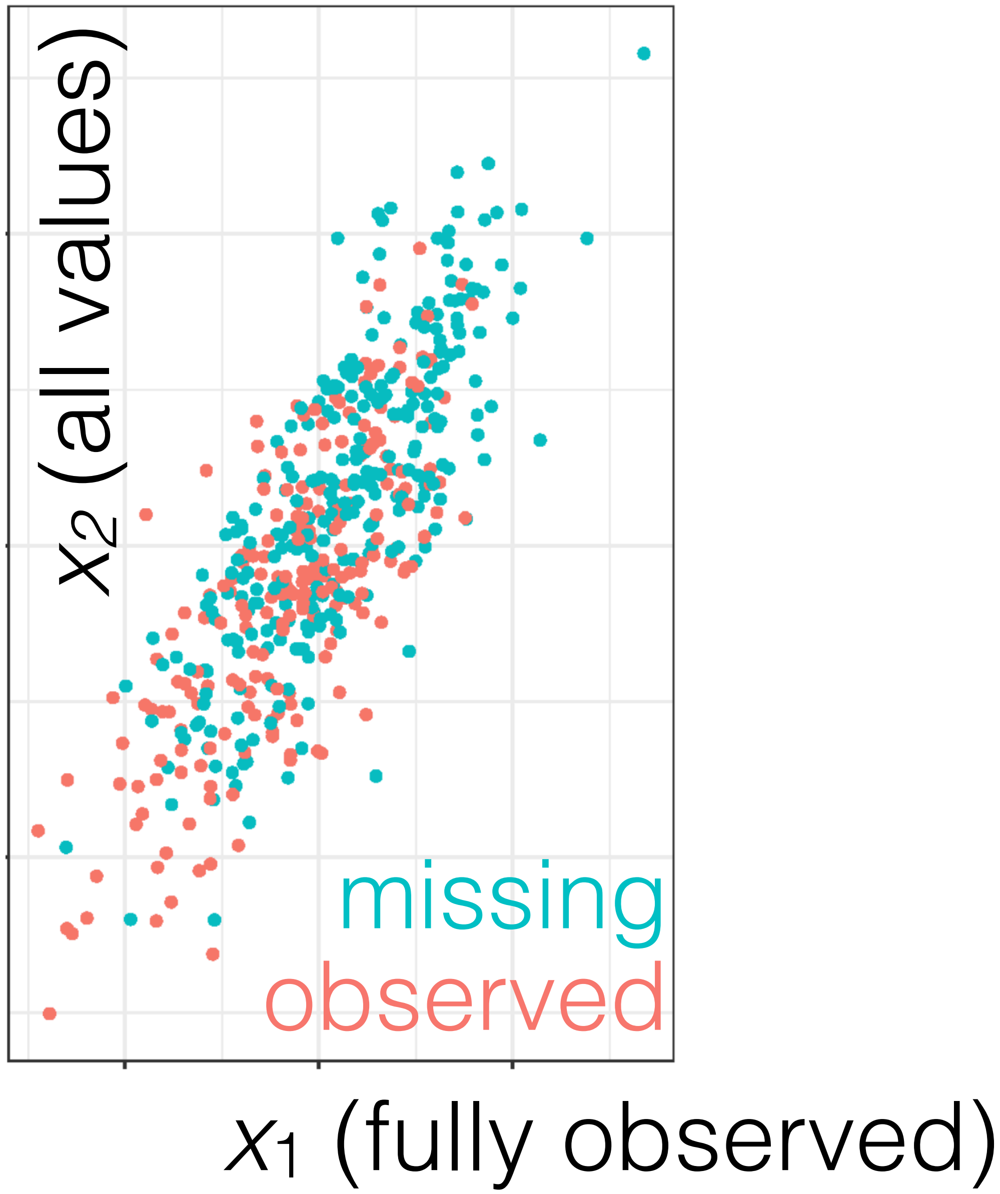

[adapted from Petito 2023]

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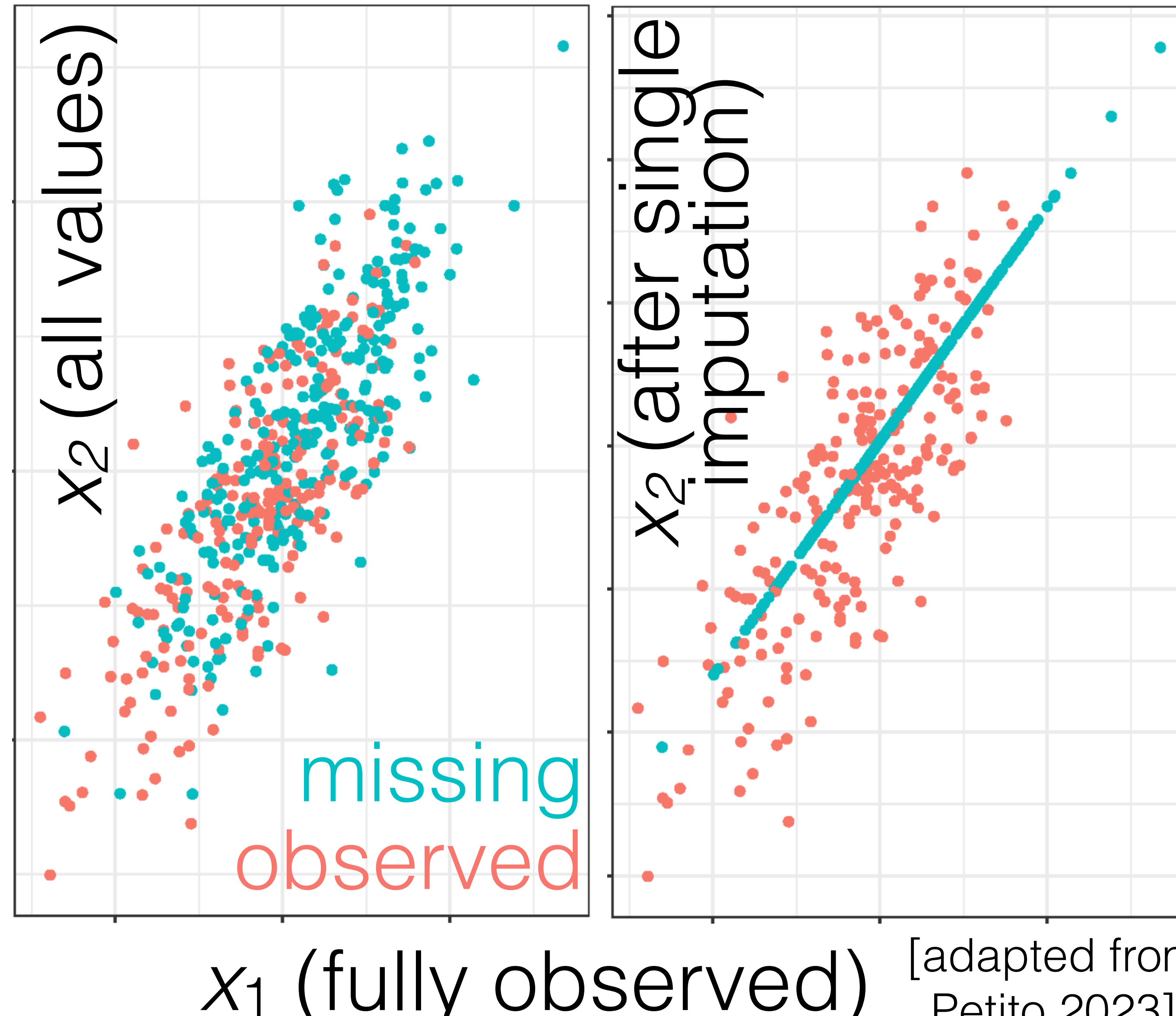

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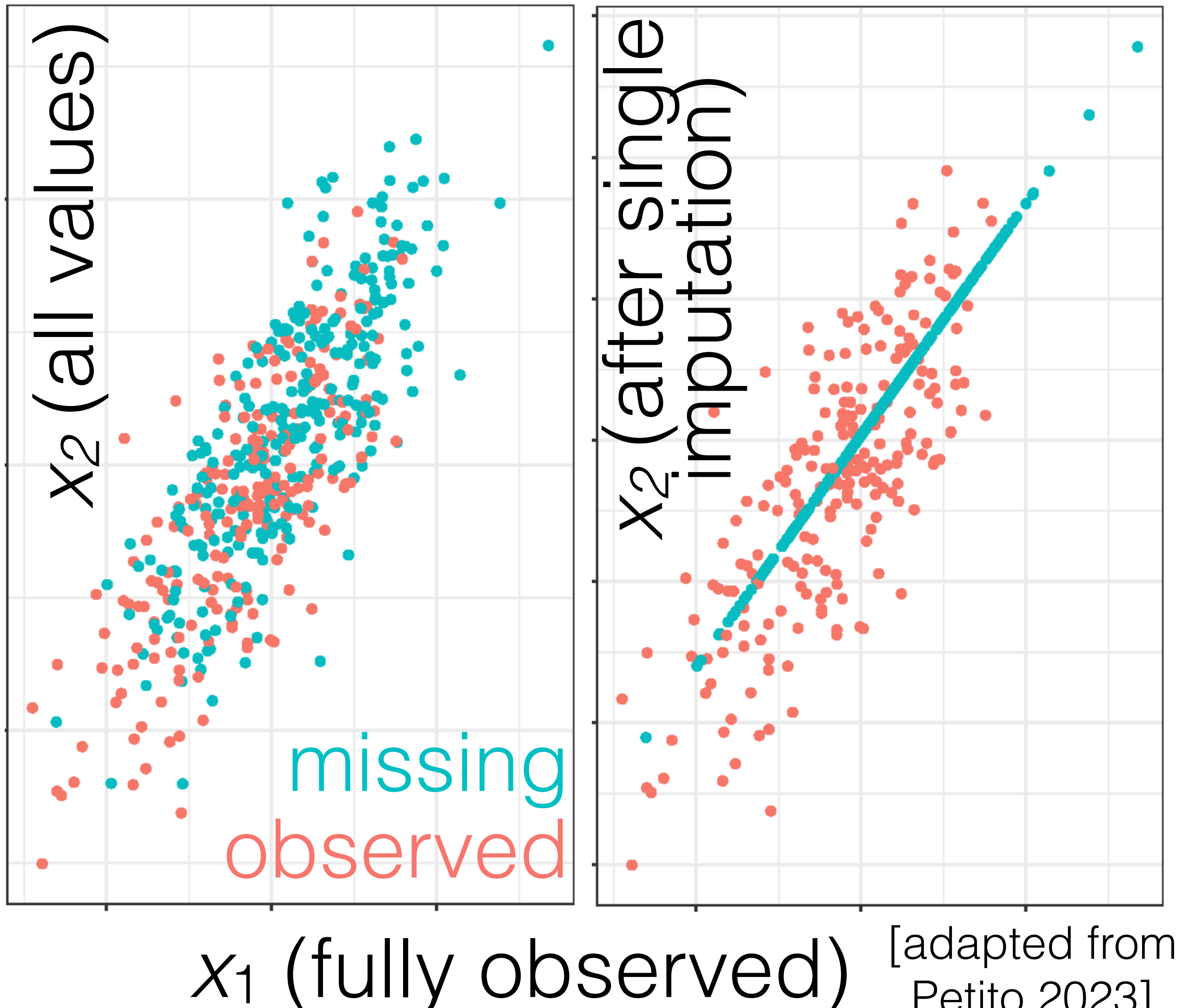

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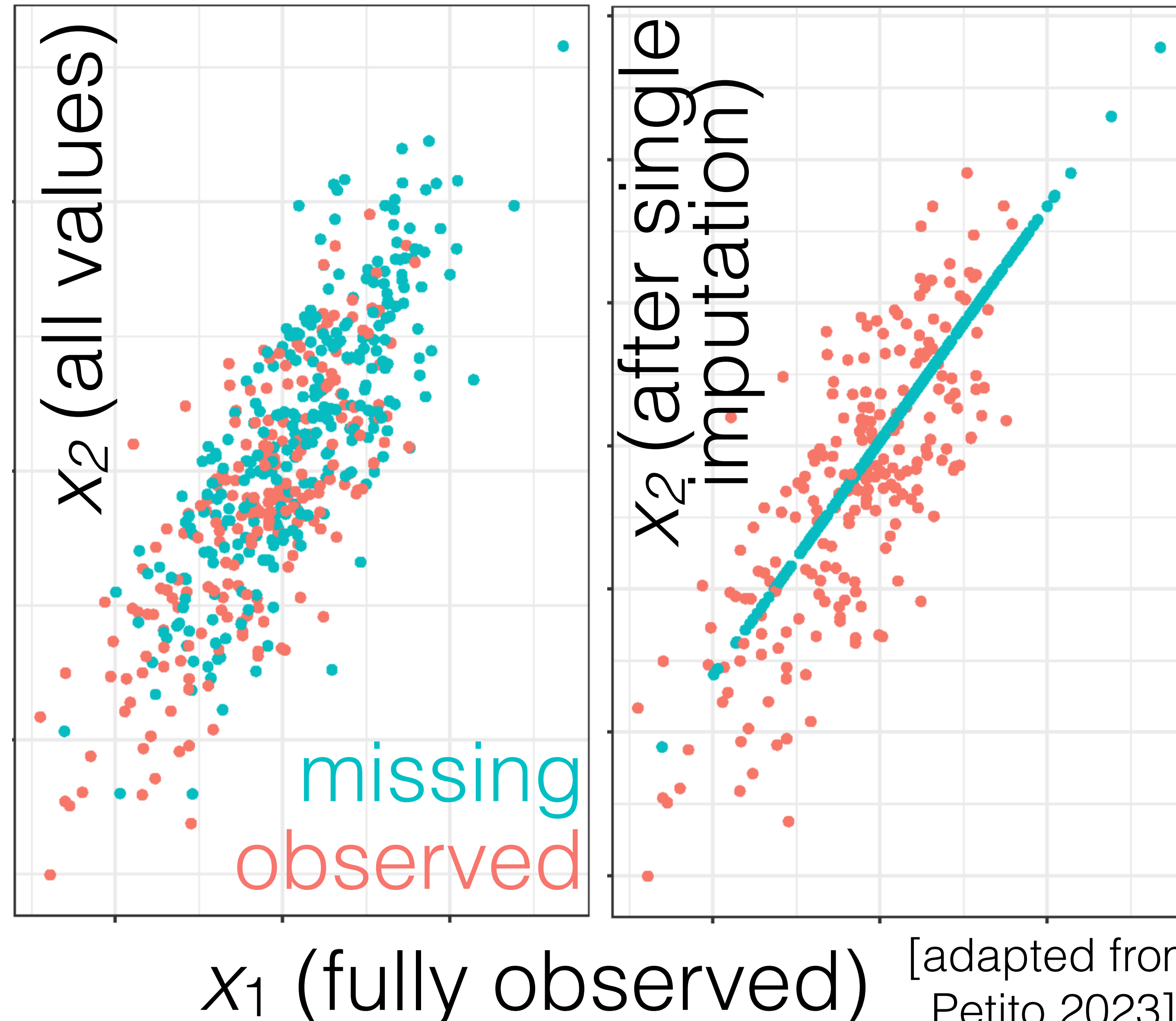

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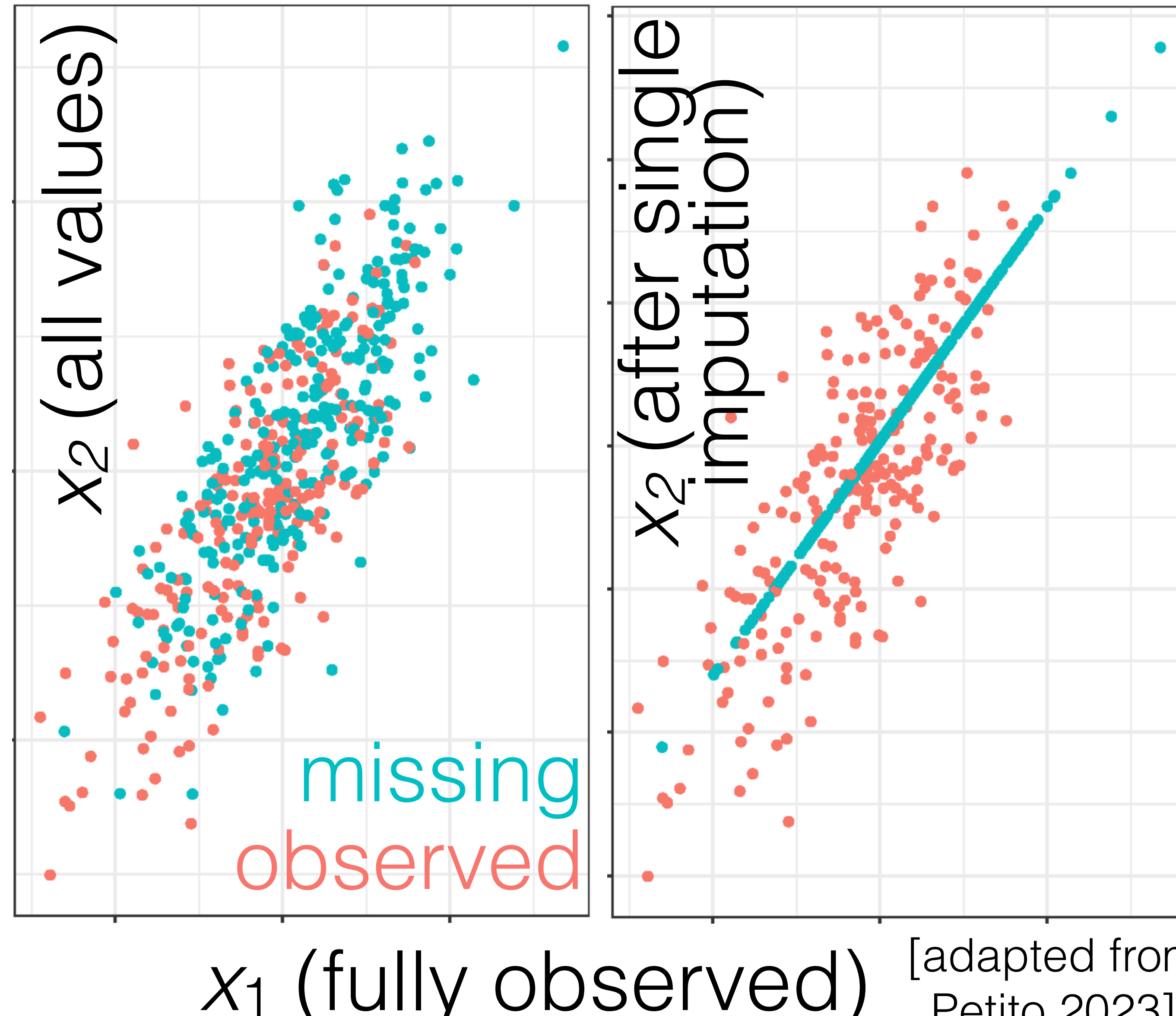

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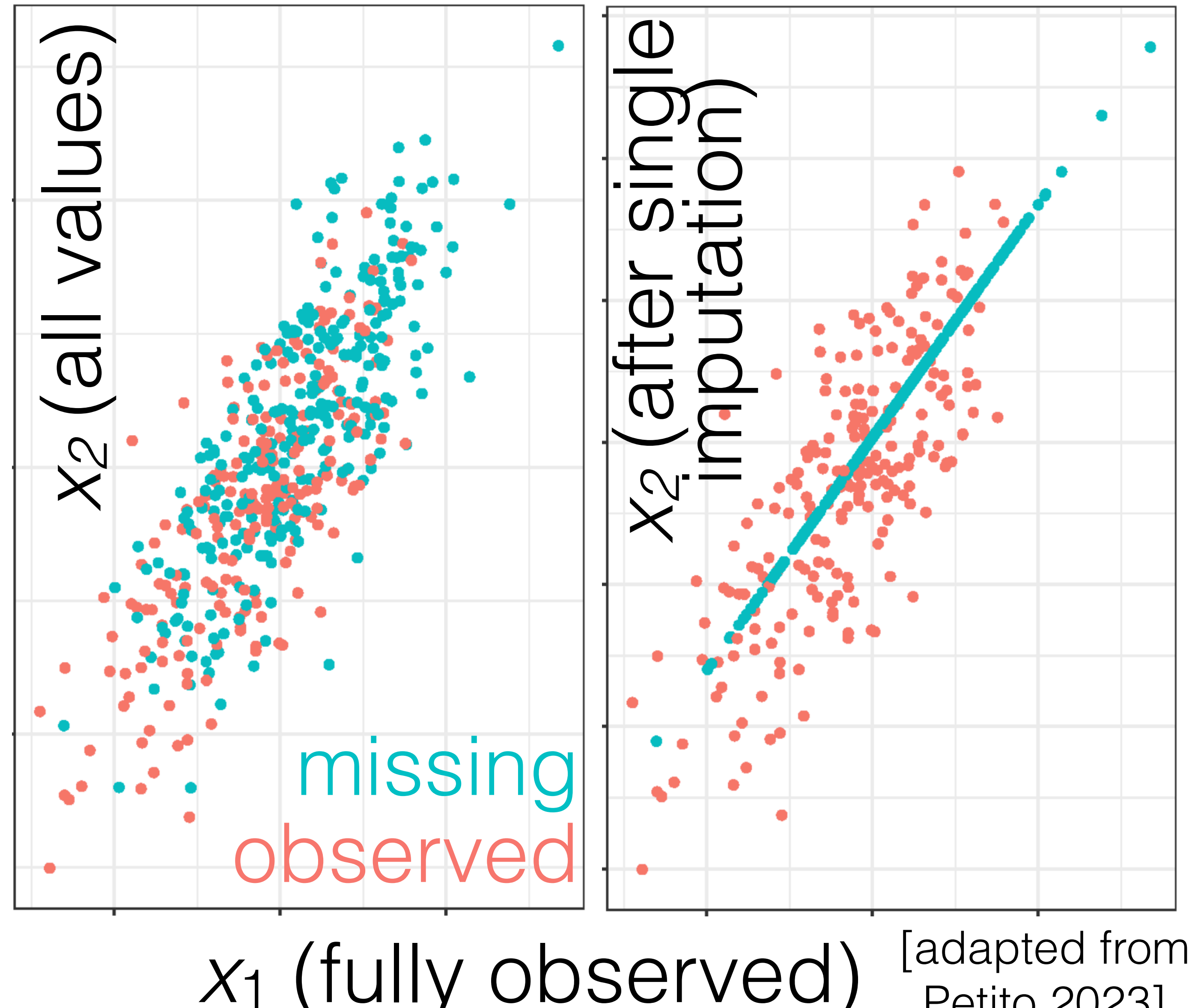

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- Option: full Bayesian model, including over missing values
- Con of methods above: more work for the data analyst

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[Elhaik 2022; not clear where these numbers come from though]

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[Example thanks to Schlens 2014]

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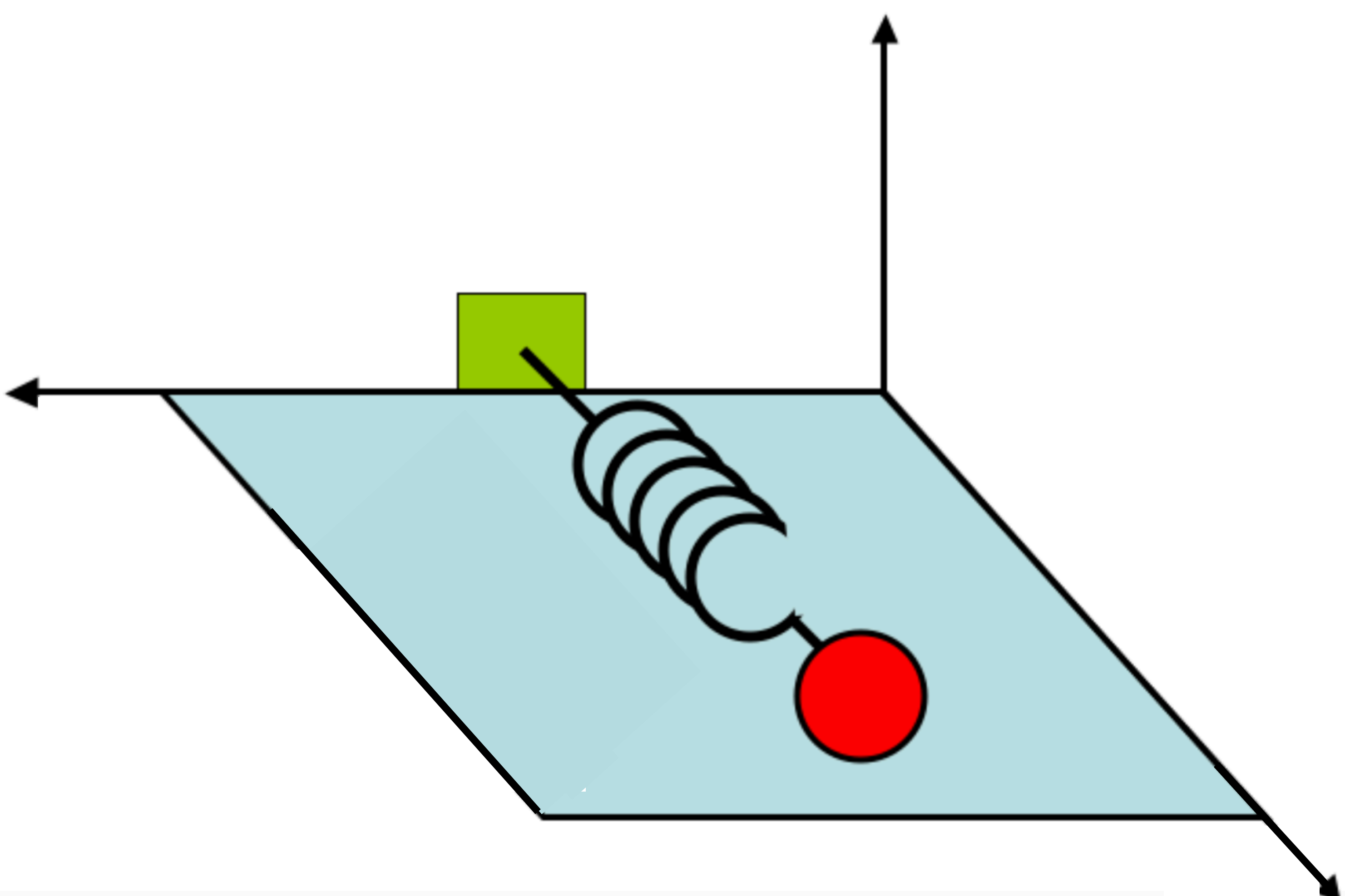

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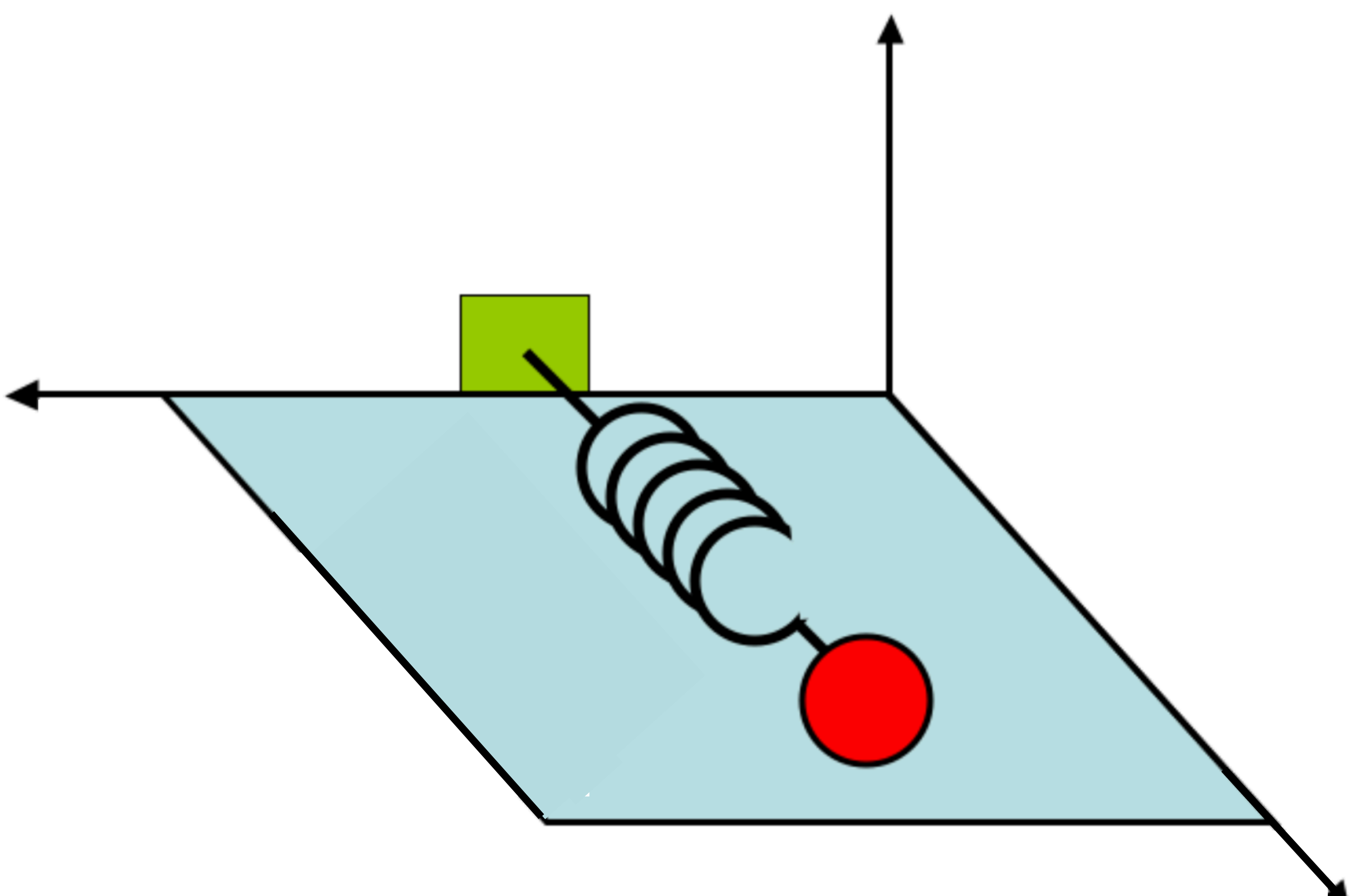

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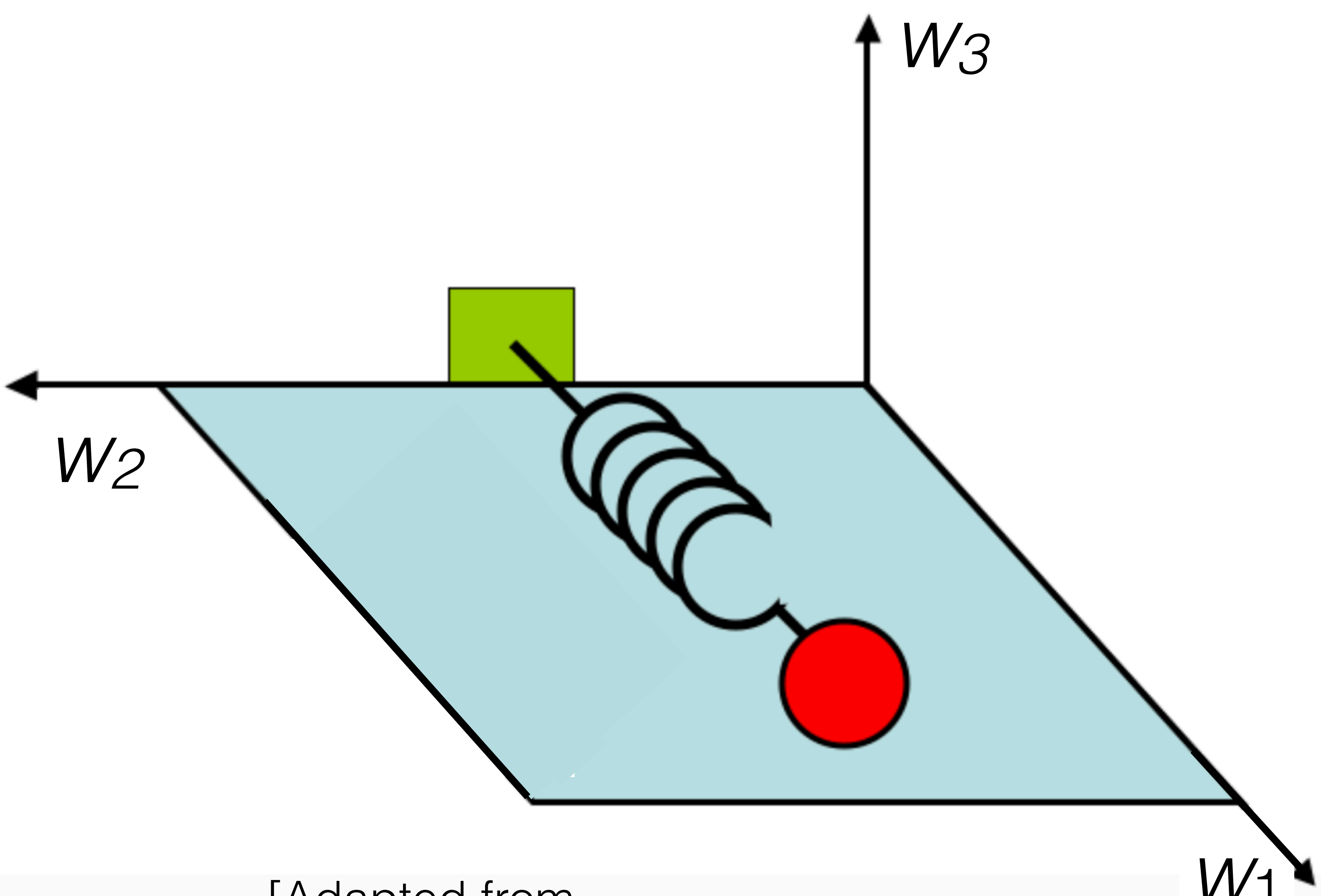

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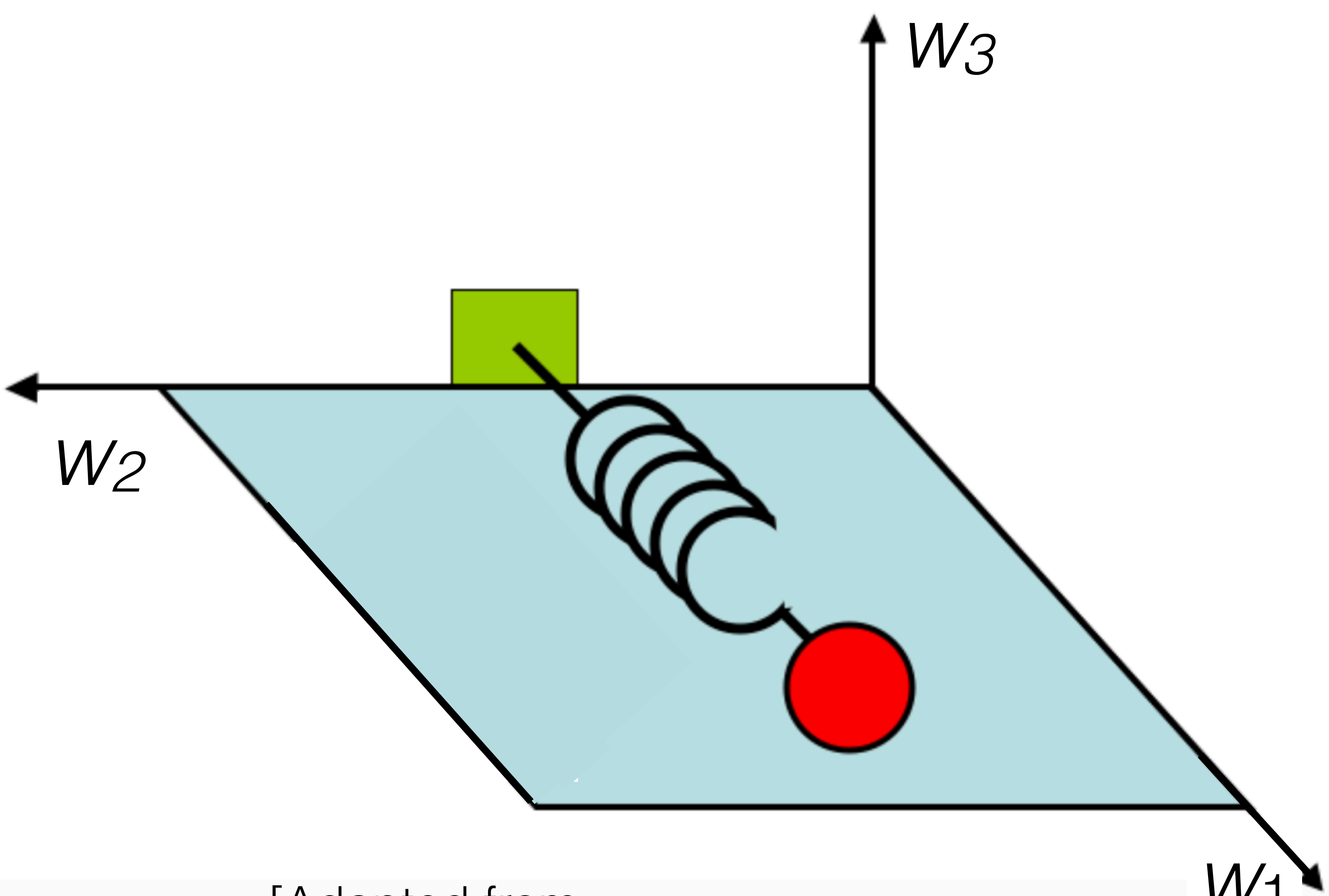

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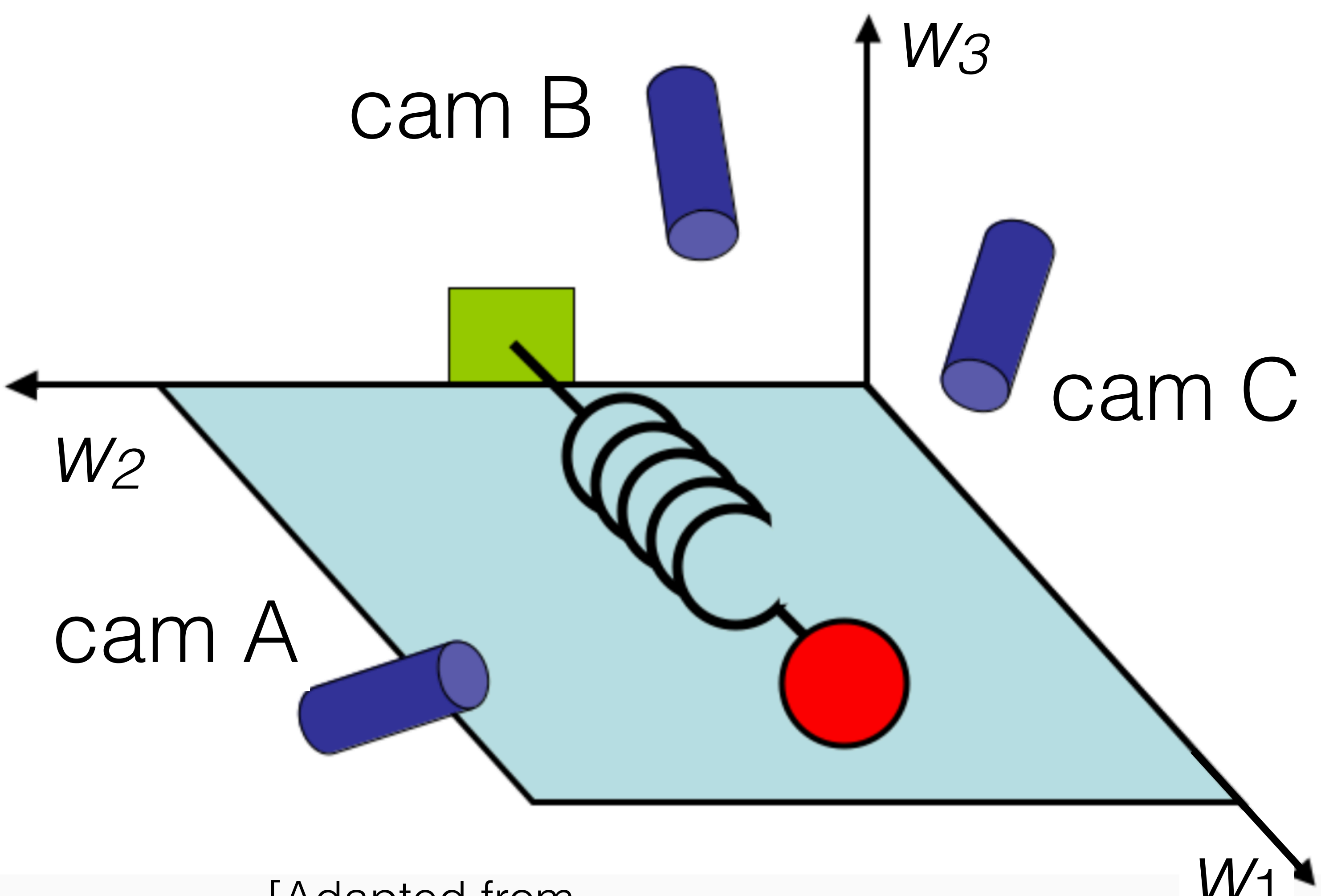

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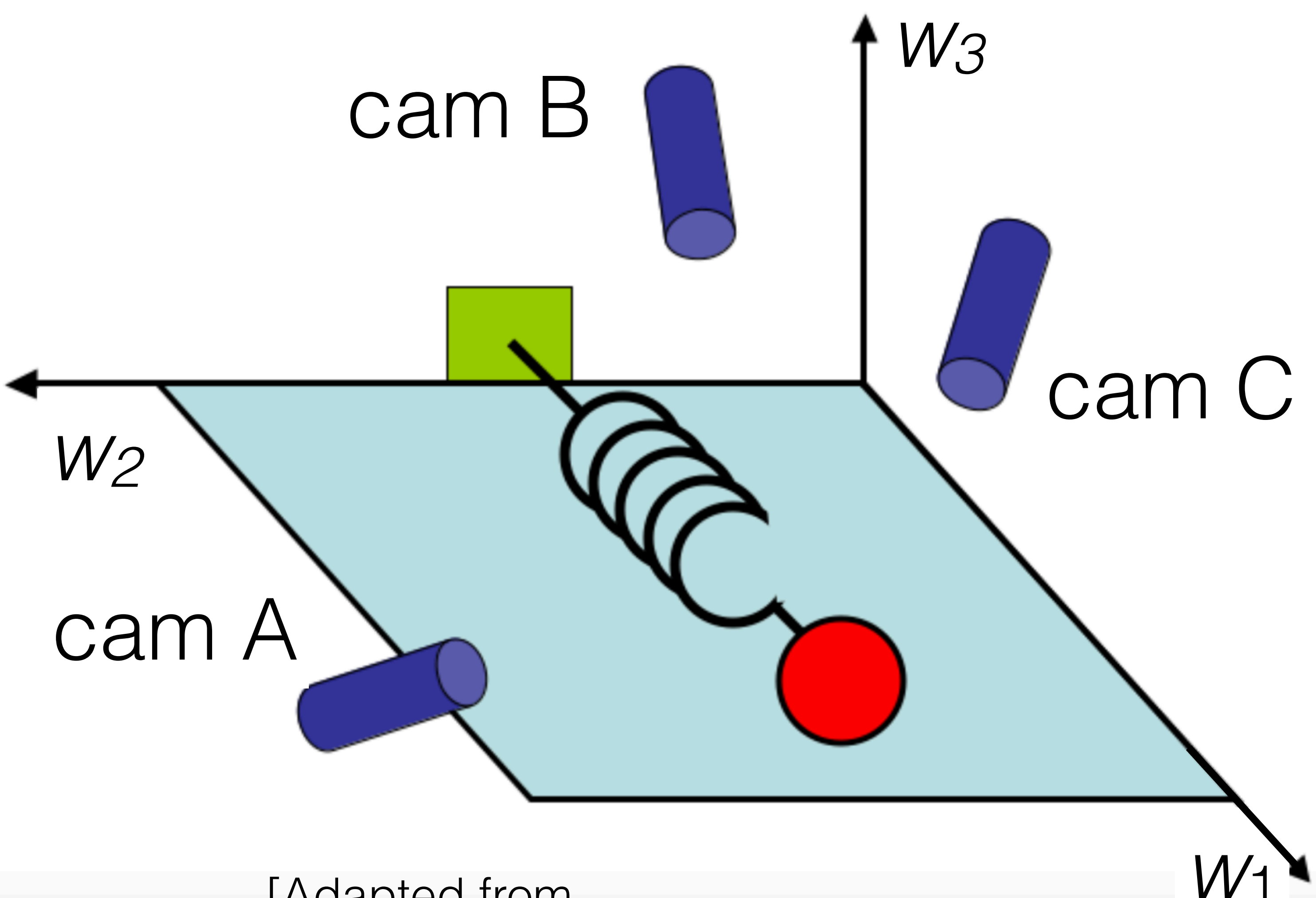

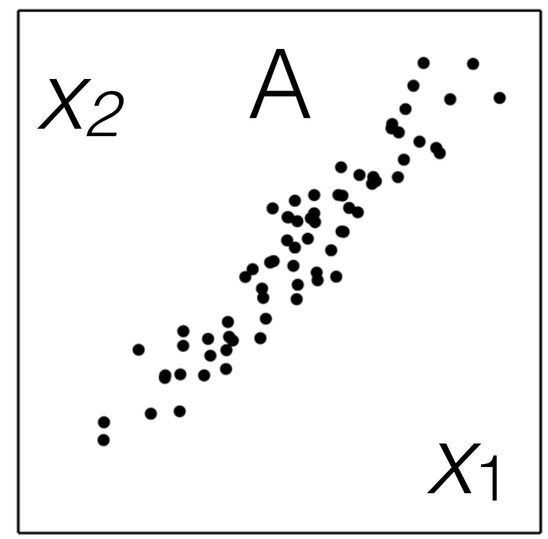

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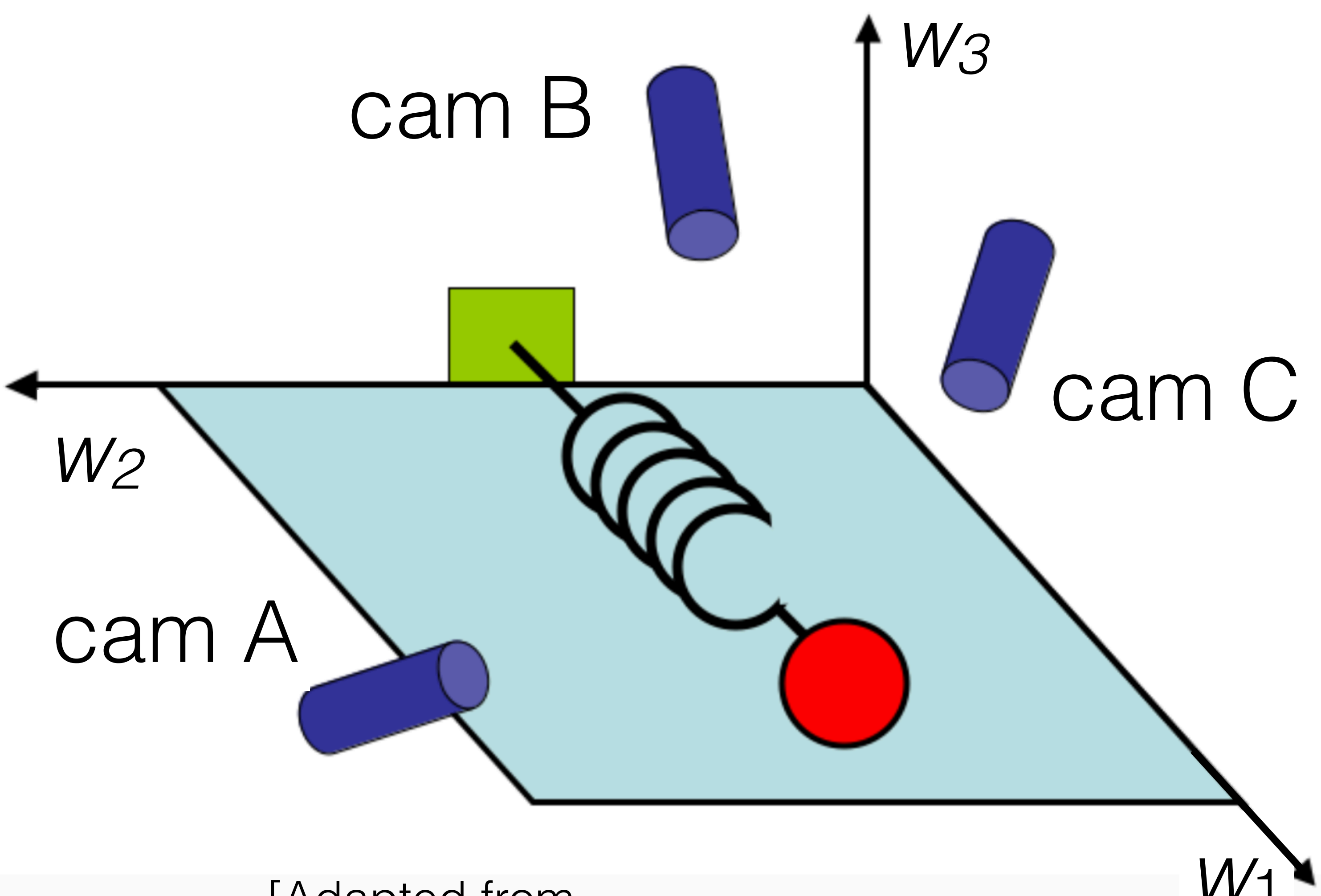

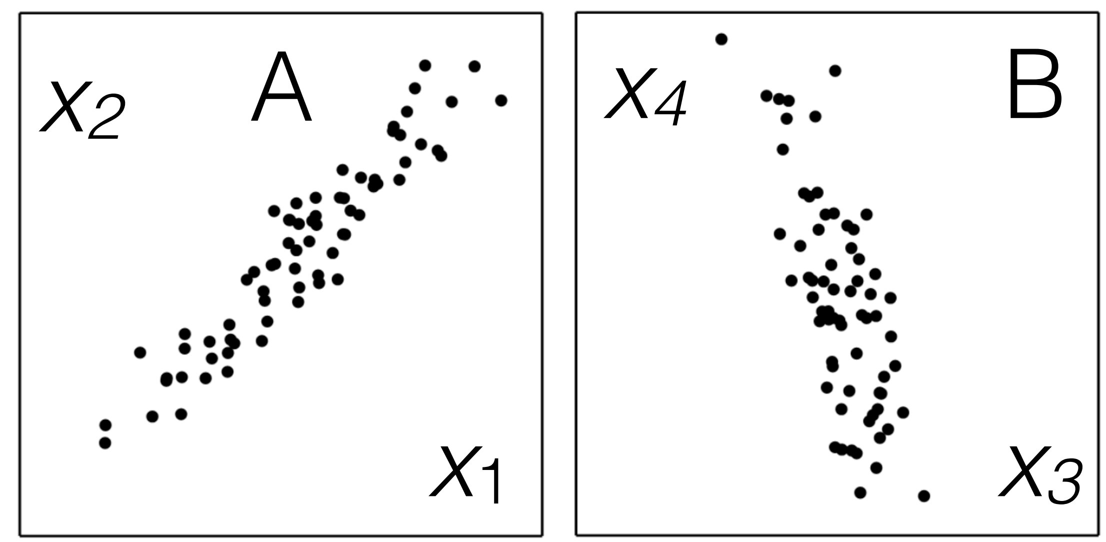

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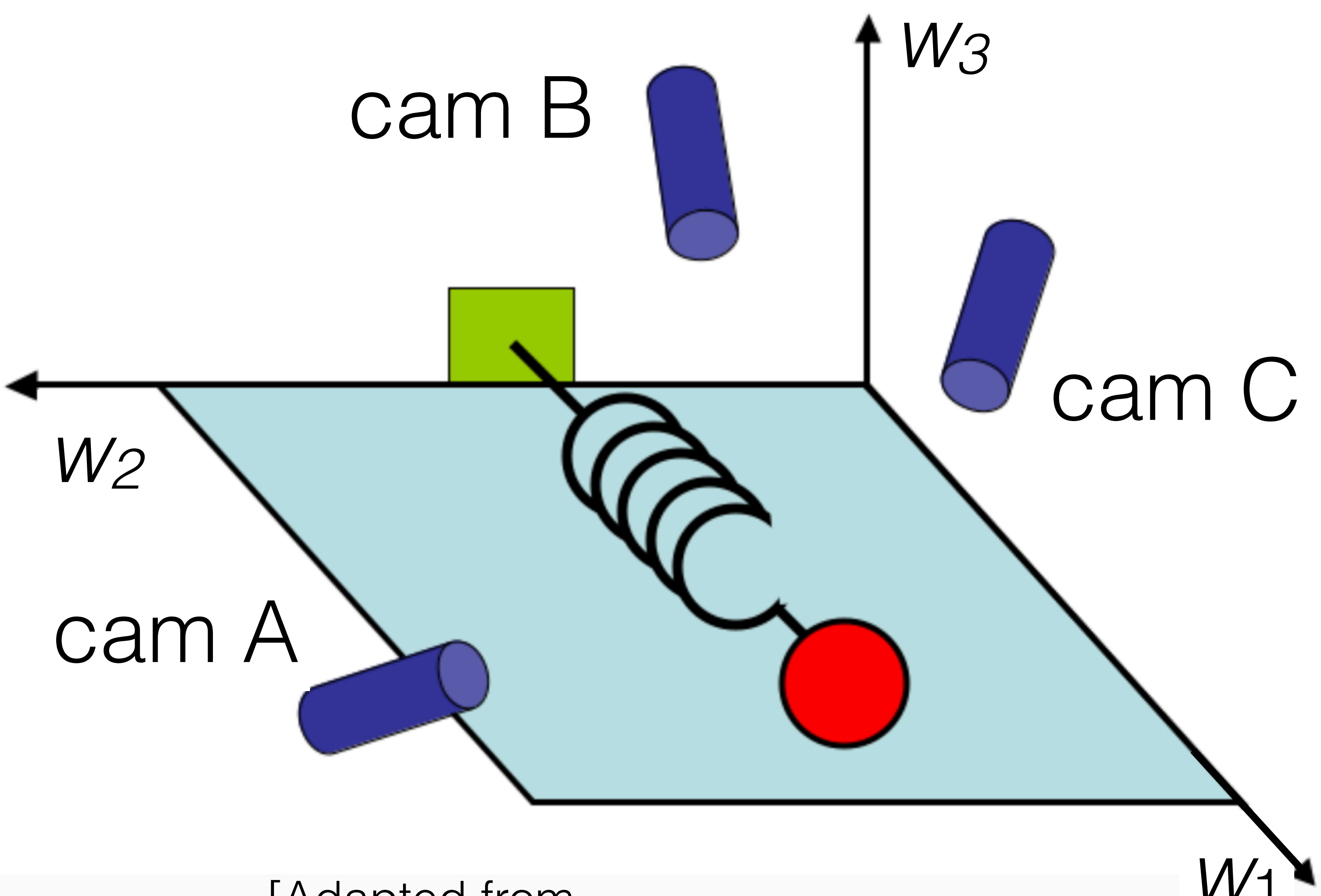
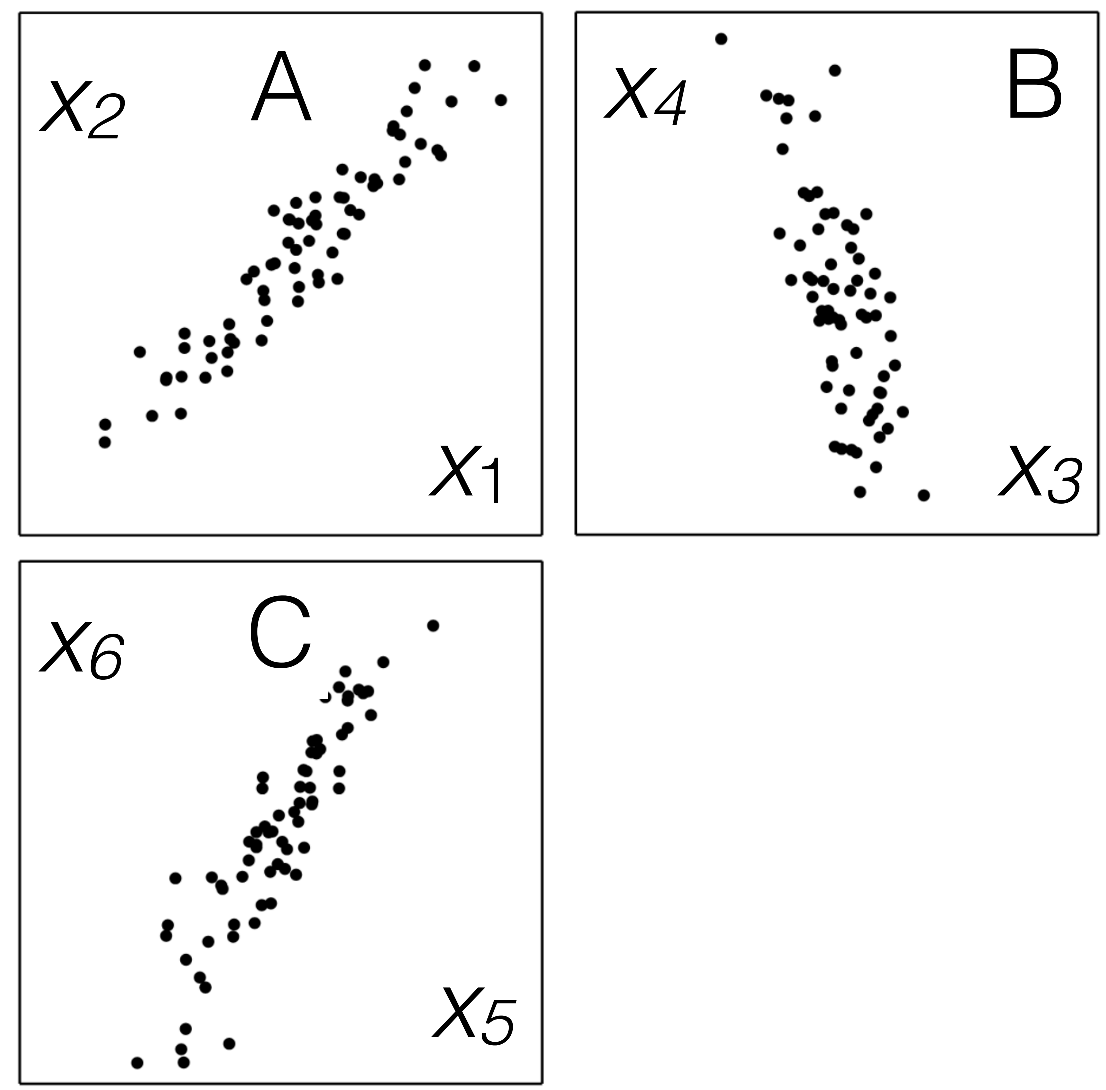


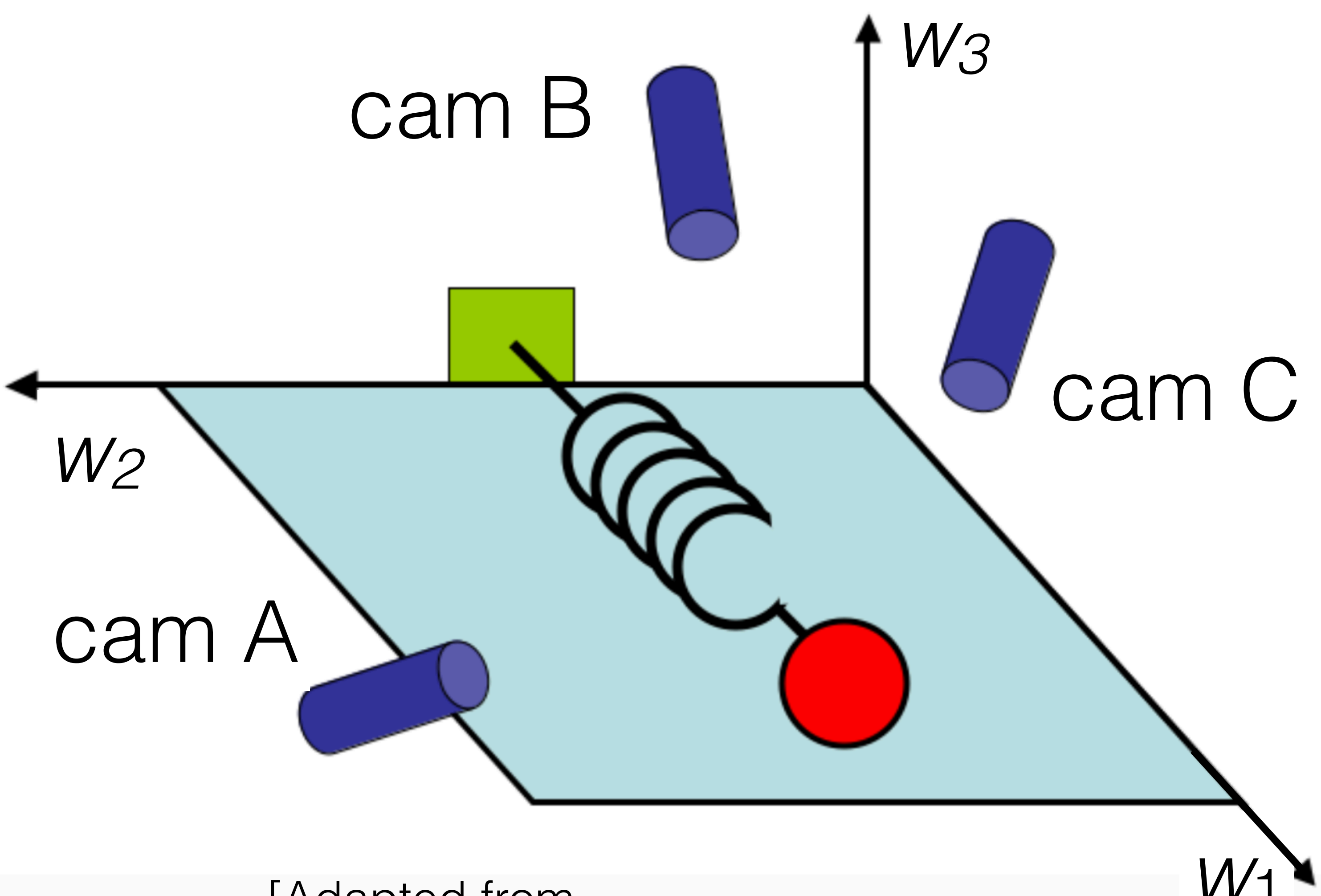


[Adapted from Schlens 2014]

Motivating example for Principal Components Analysis (PCA)

[Example thanks to Schlens 2014]

- We often make lots of noisy, possibly-redundant measurements to try to understand some phenomenon
- Suppose we attach a ball to a spring.
  - There are some true, unknown axes w_1, w_2, w_3 such that the ball and spring movement are very largely in w_1.
  - We set up three cameras, take occasional snapshots.



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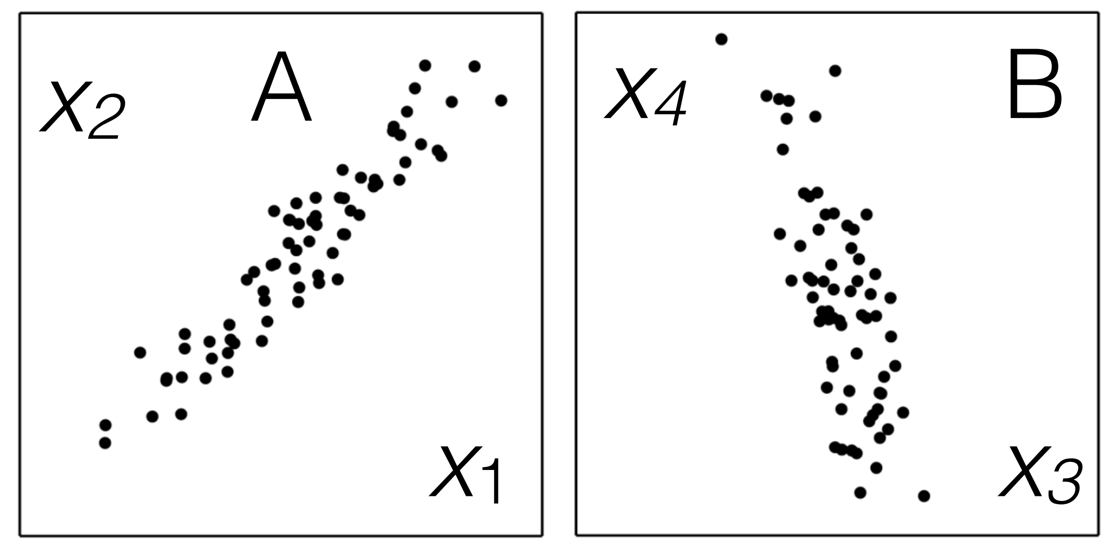
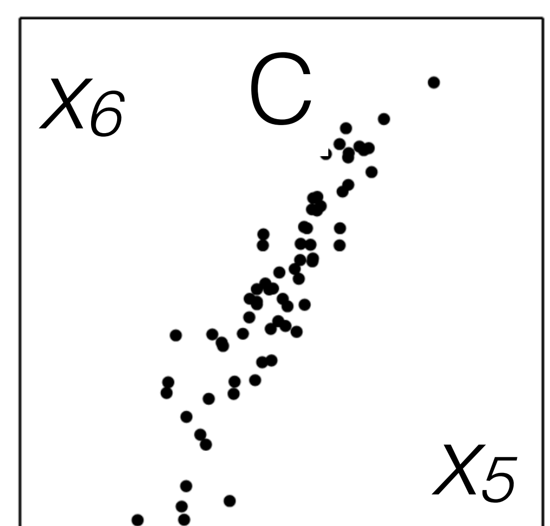
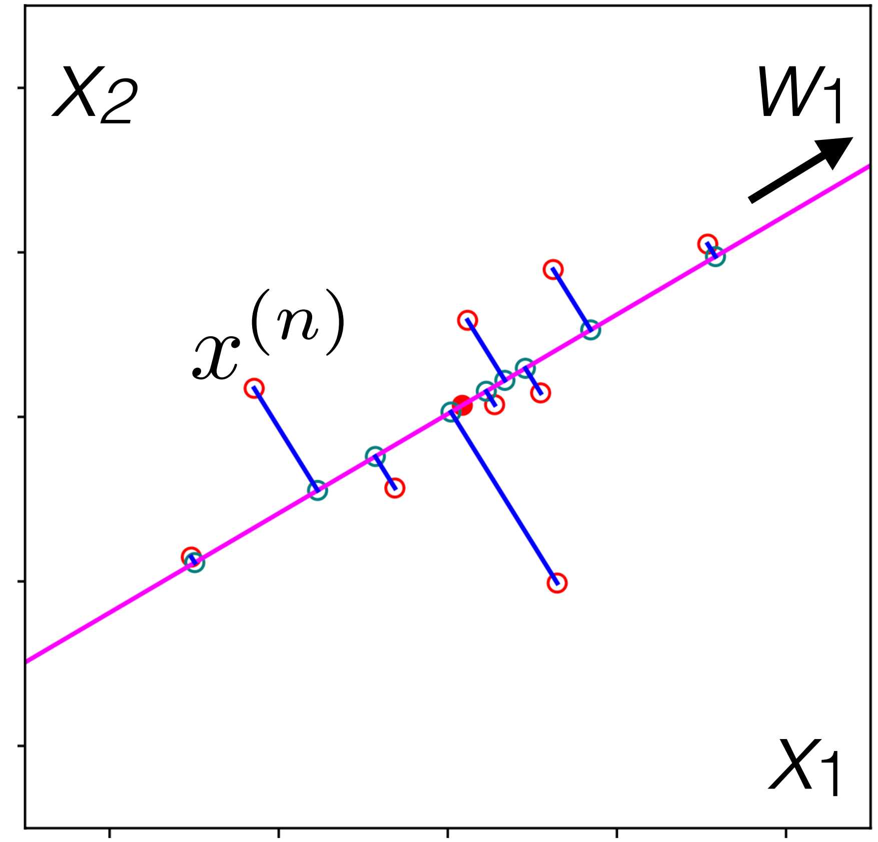




- We have 6 features, but need only 1

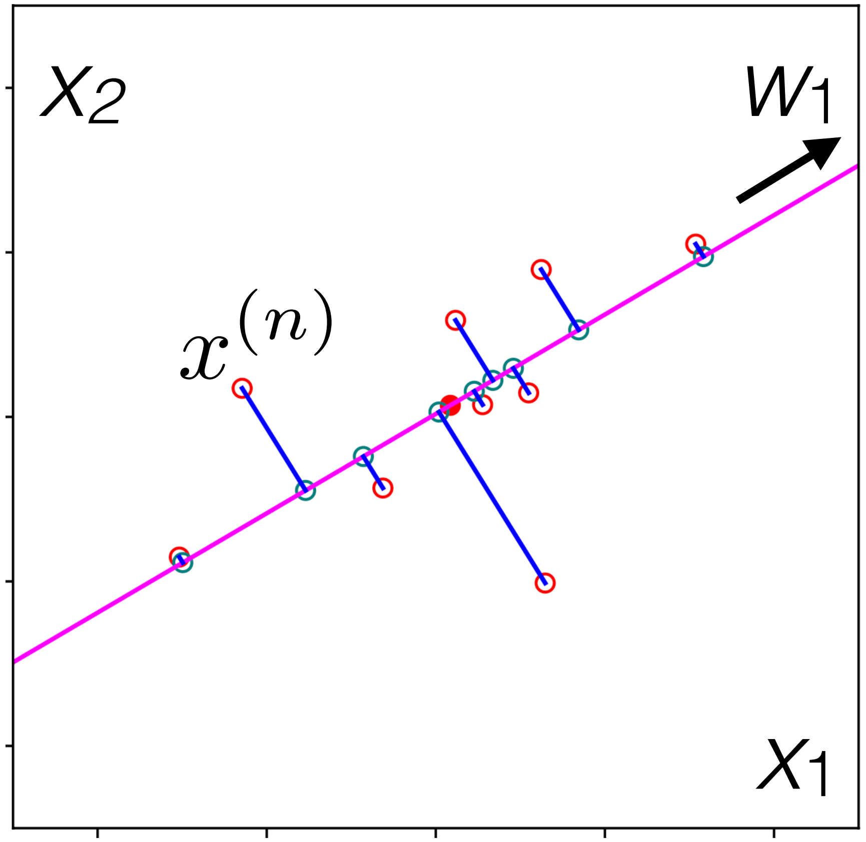
Problem setup for PCA



[adapted from Murphy 2022, Fig 20.1]

Problem setup for PCA

- As usual $x^{(n)}$ is a Dx1 vector



[adapted from Murphy 2022, Fig 20.1]

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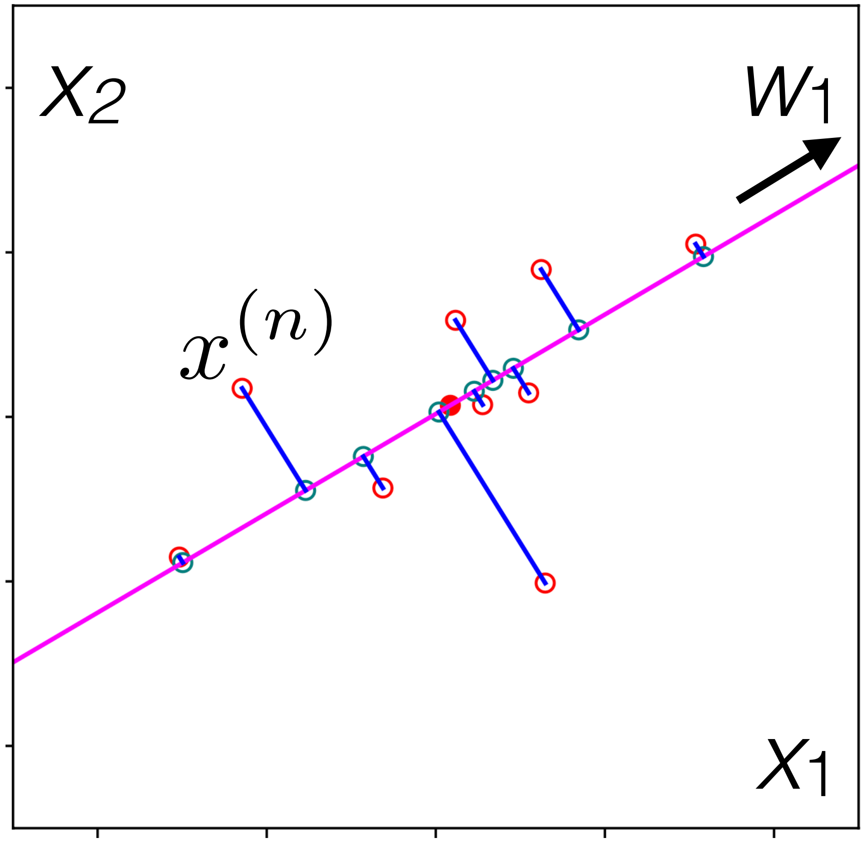
- Pre-process so $\frac{1}{N}\sum_{n=1}^{N} x^{(n)} = 0_D$



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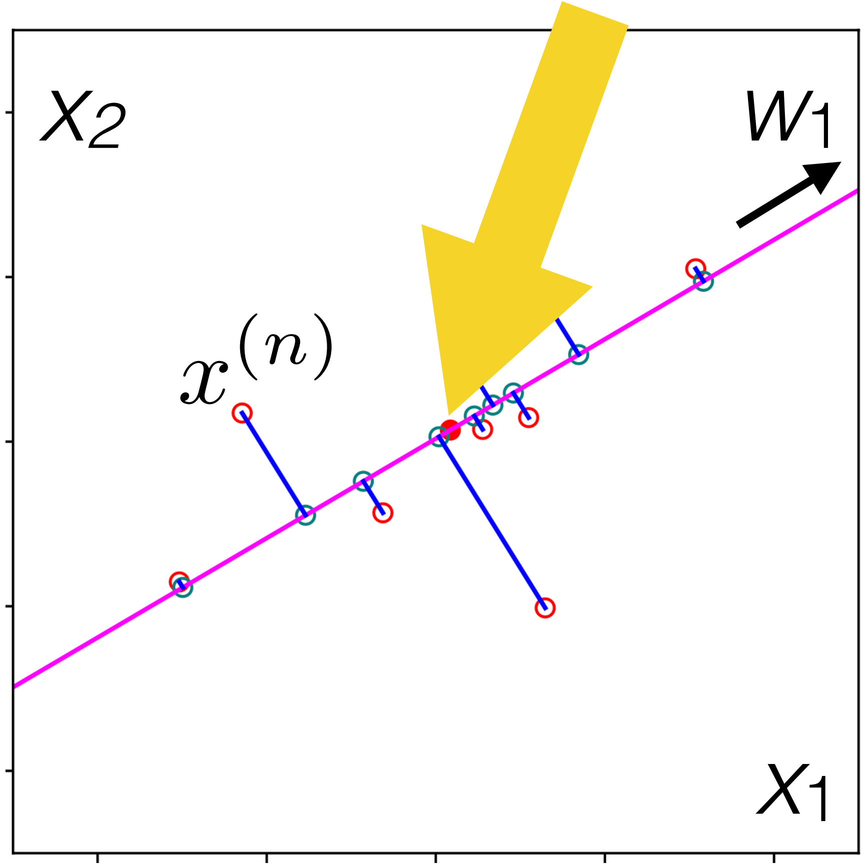

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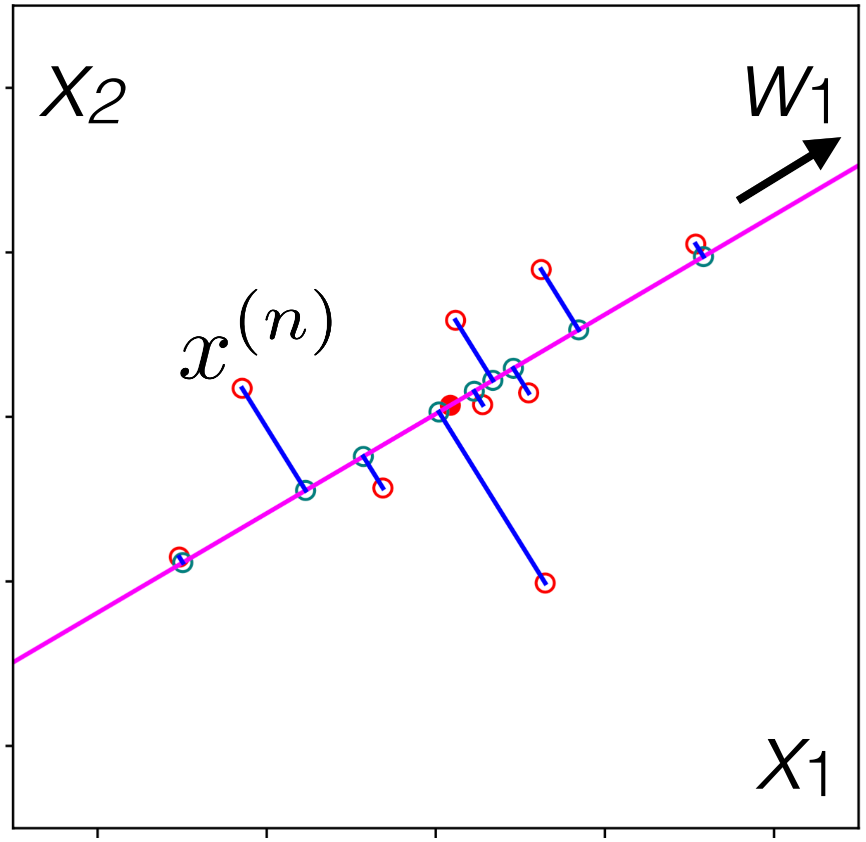

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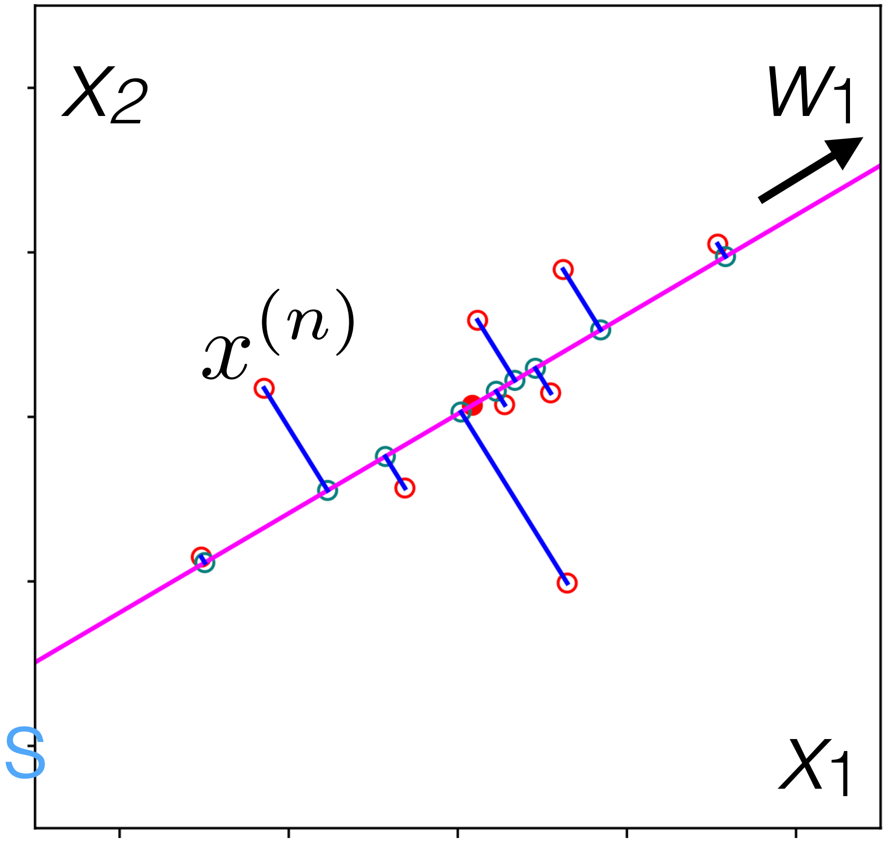

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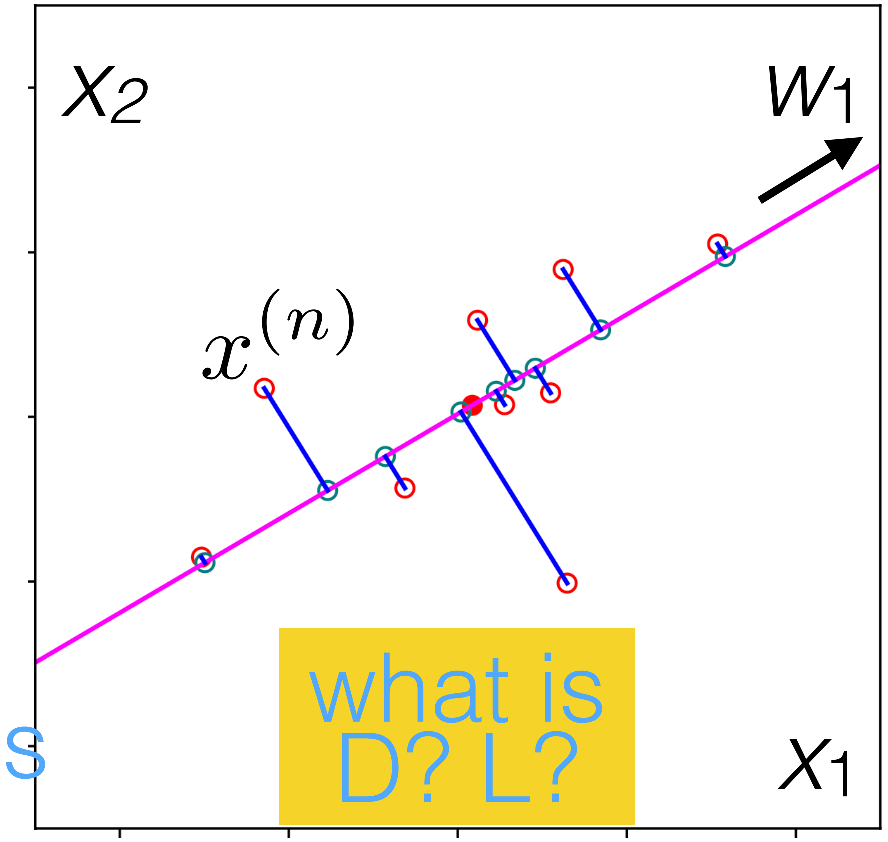

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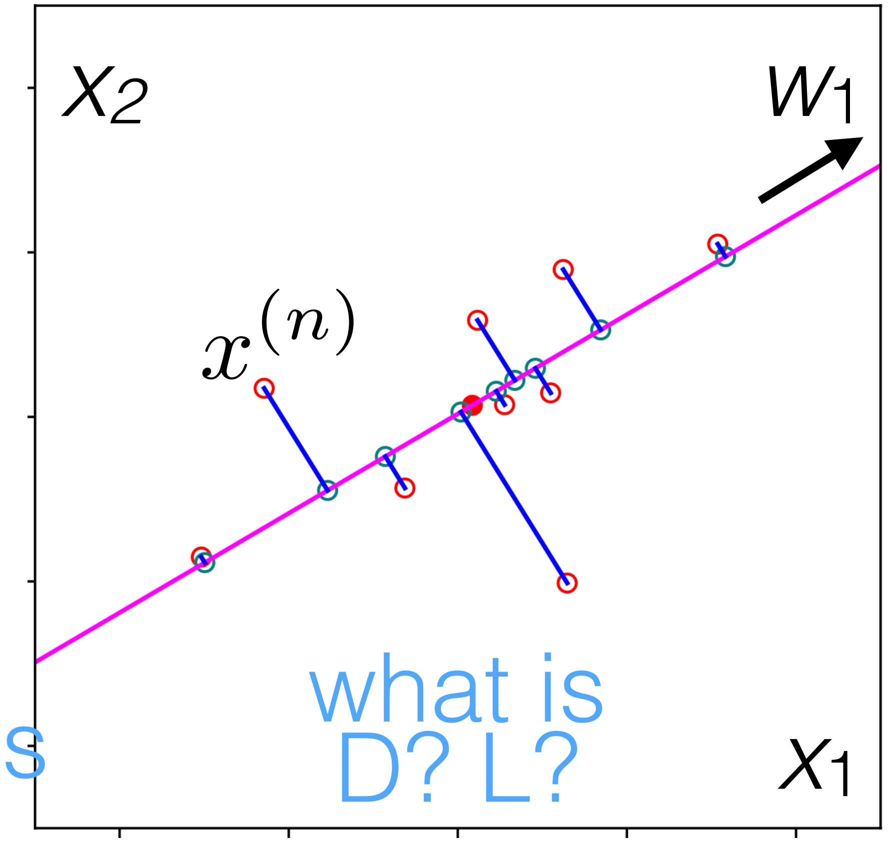
$x^{(n)}$
Dx1



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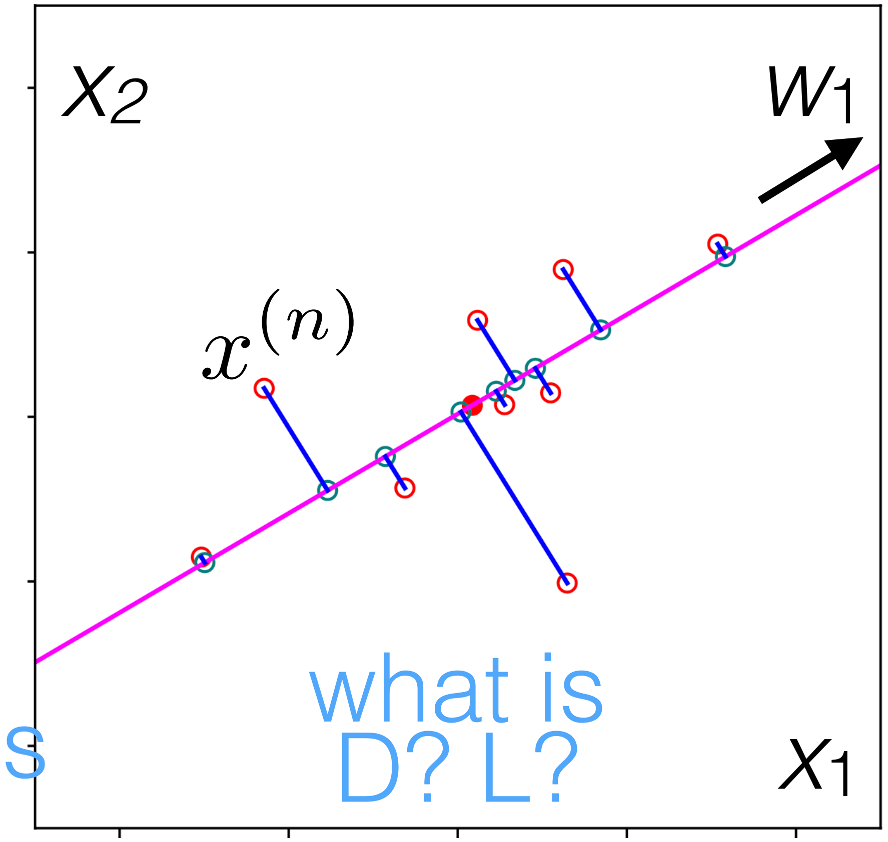

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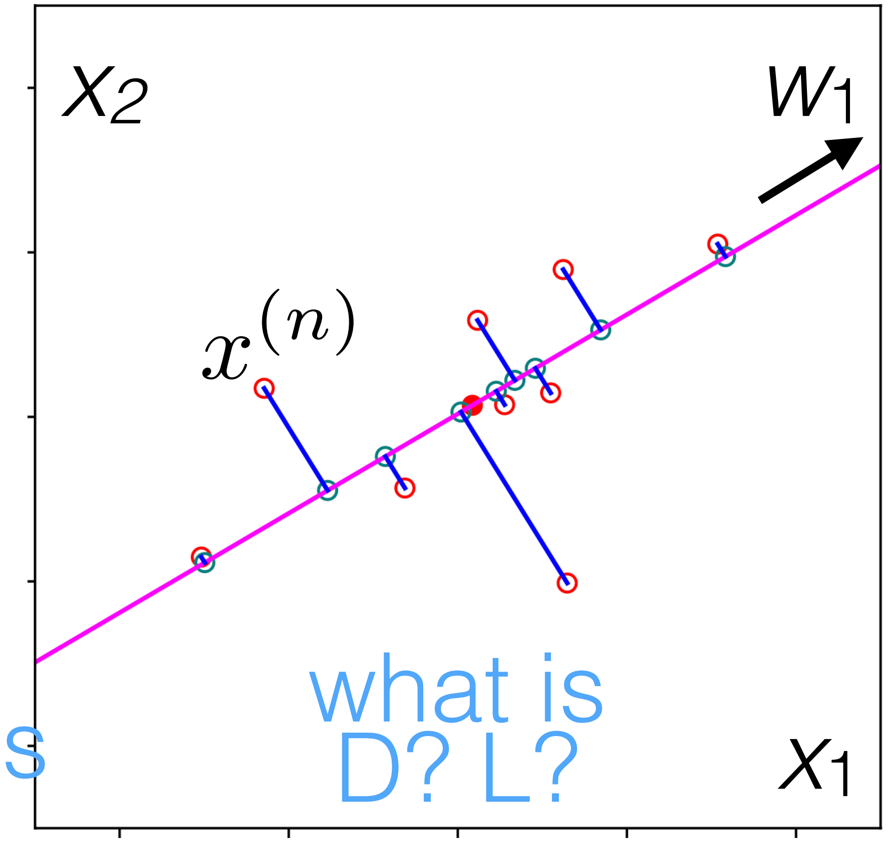

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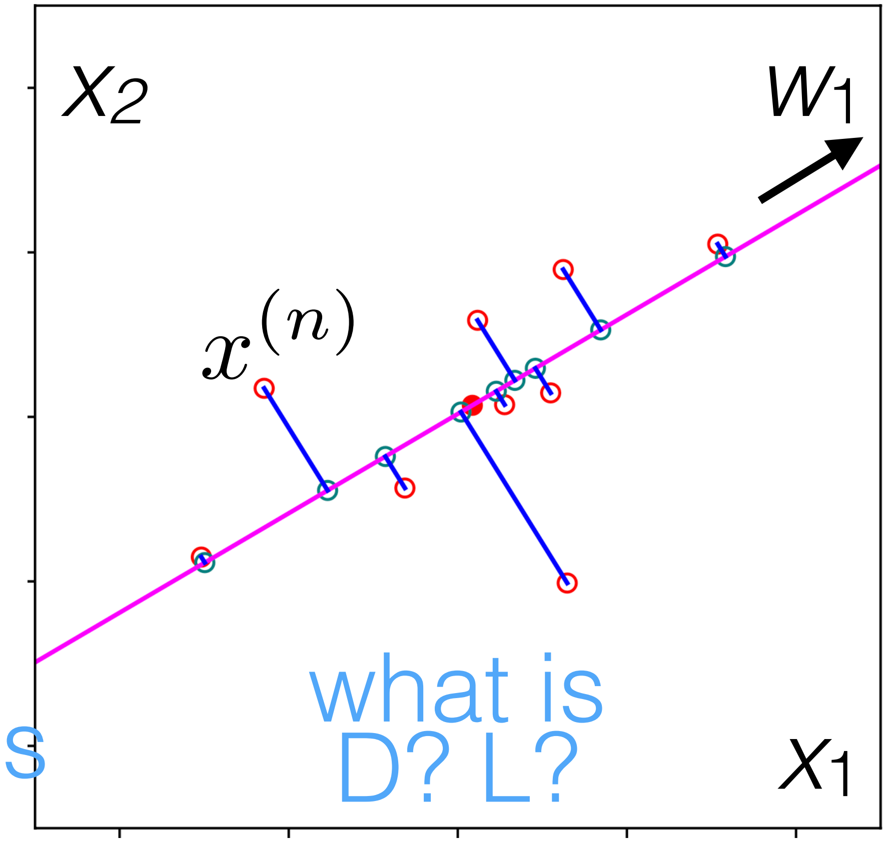

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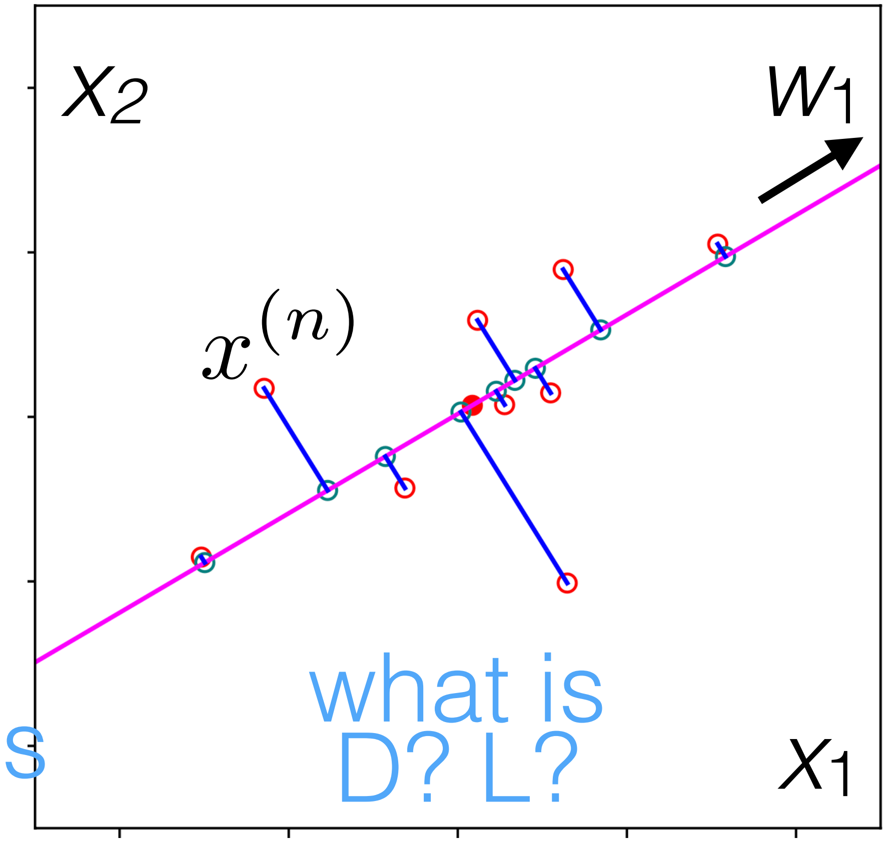

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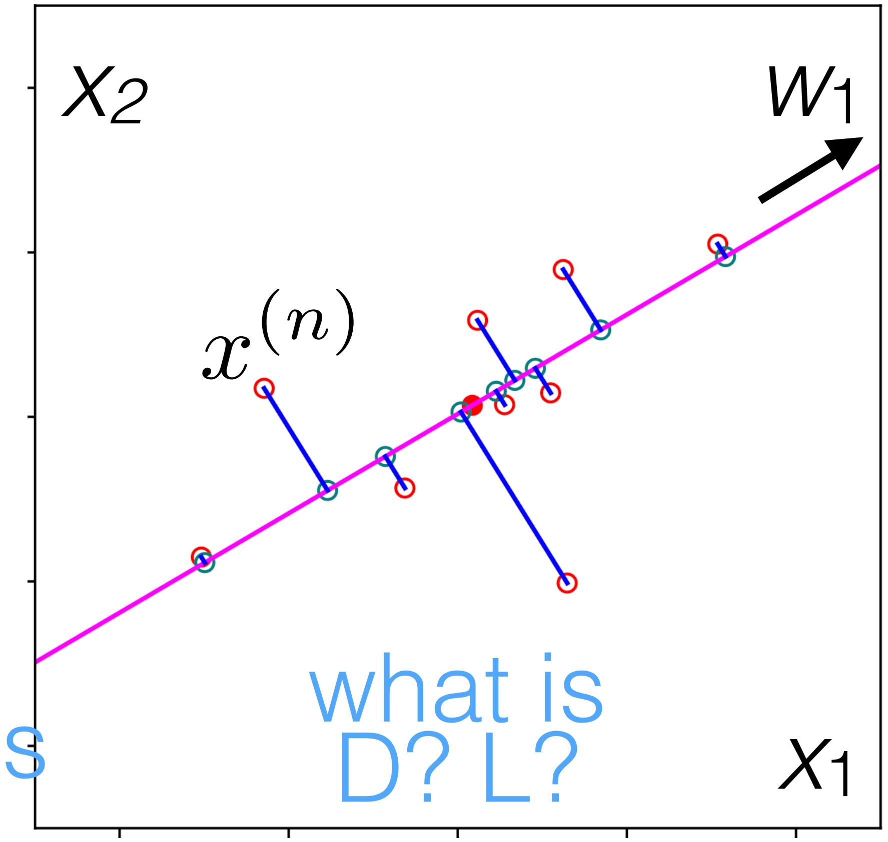

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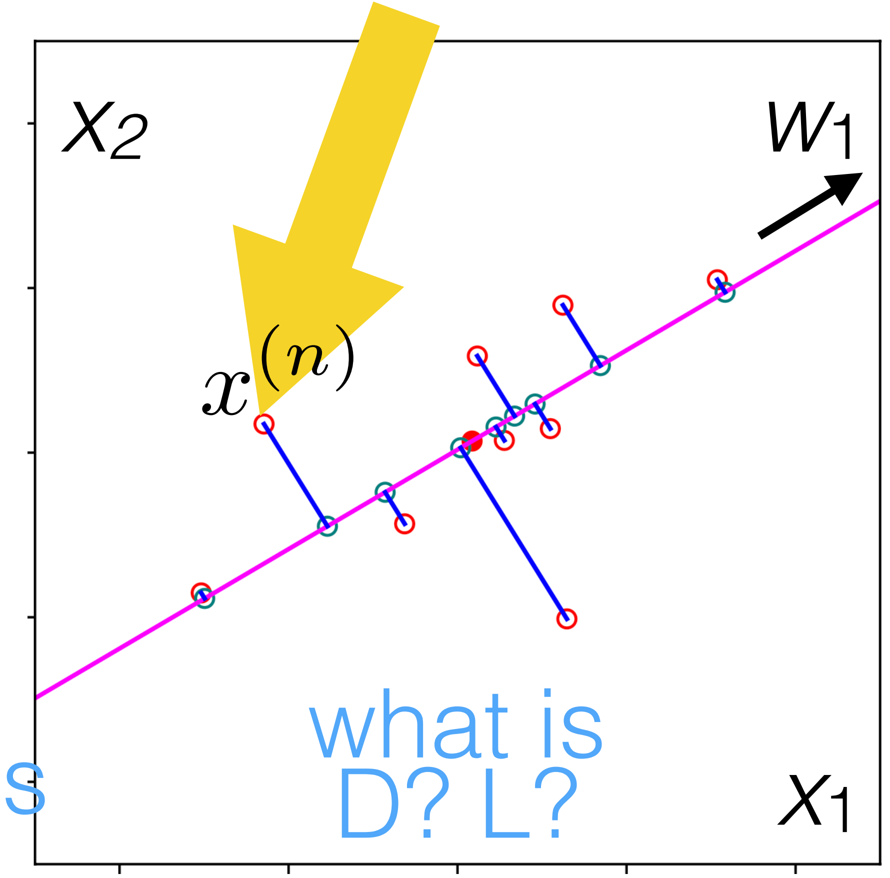

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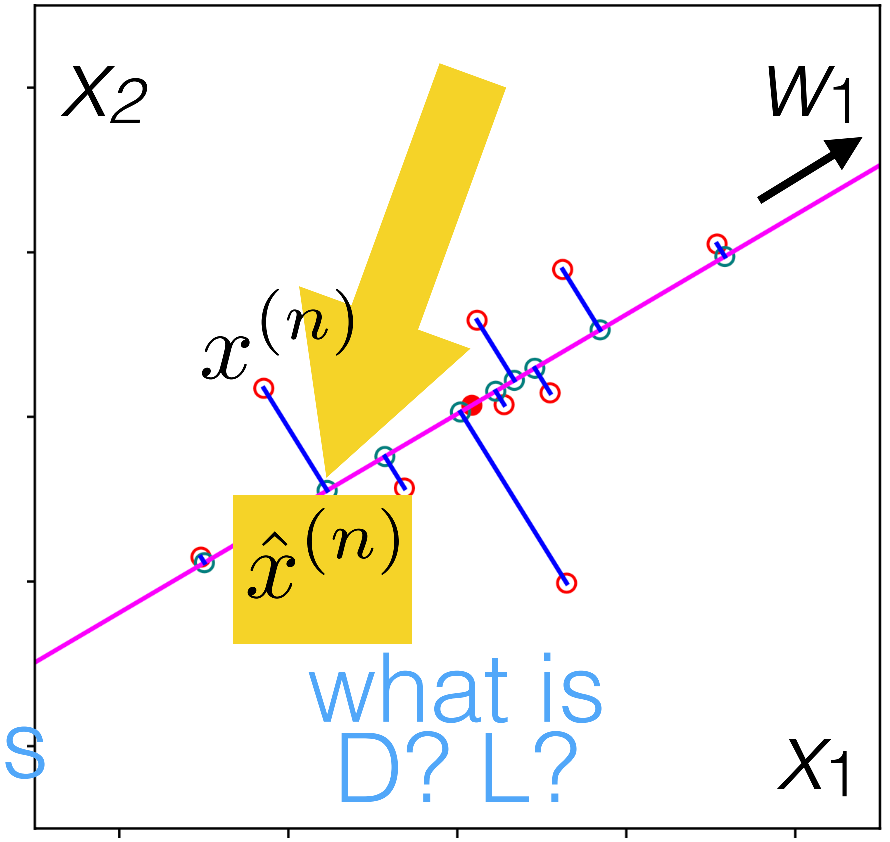

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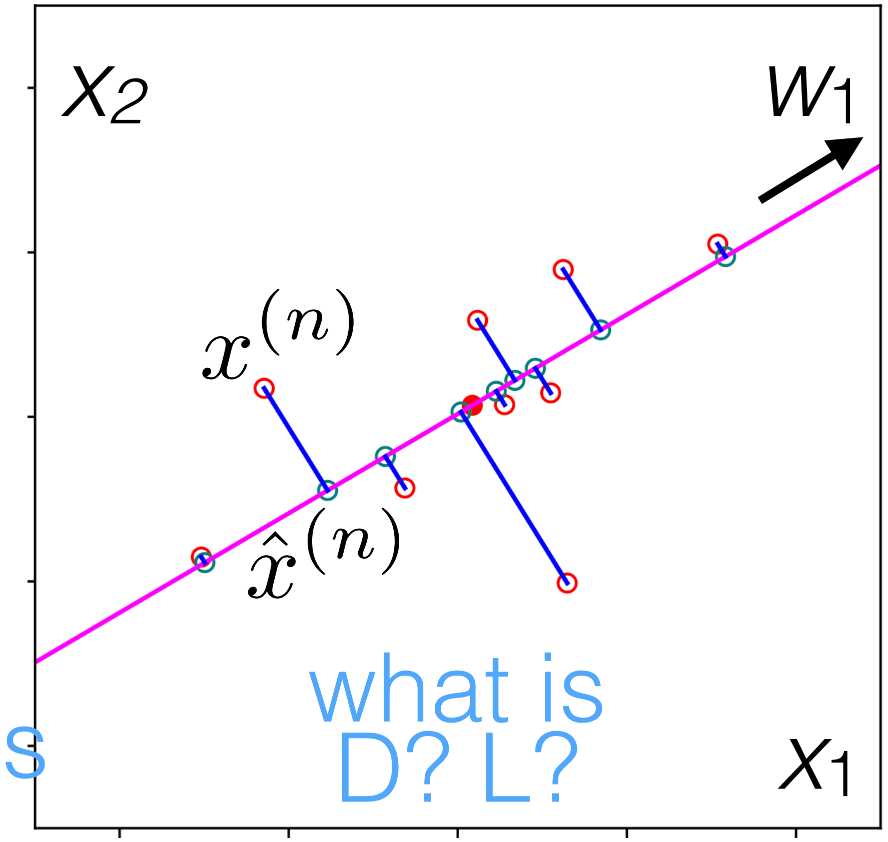

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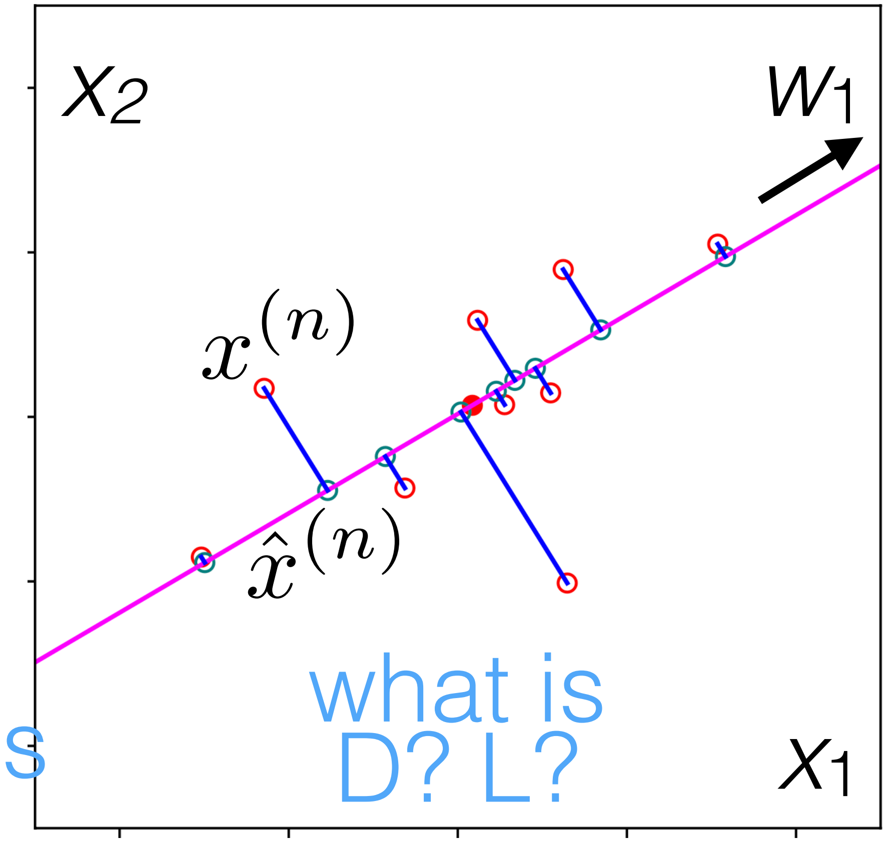

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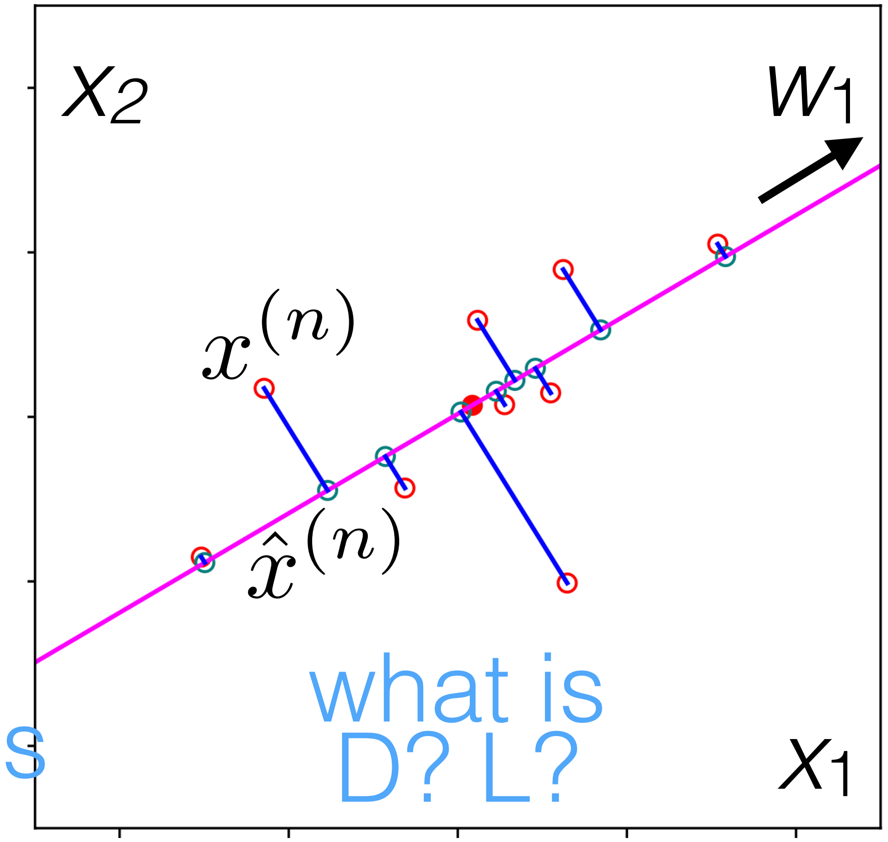

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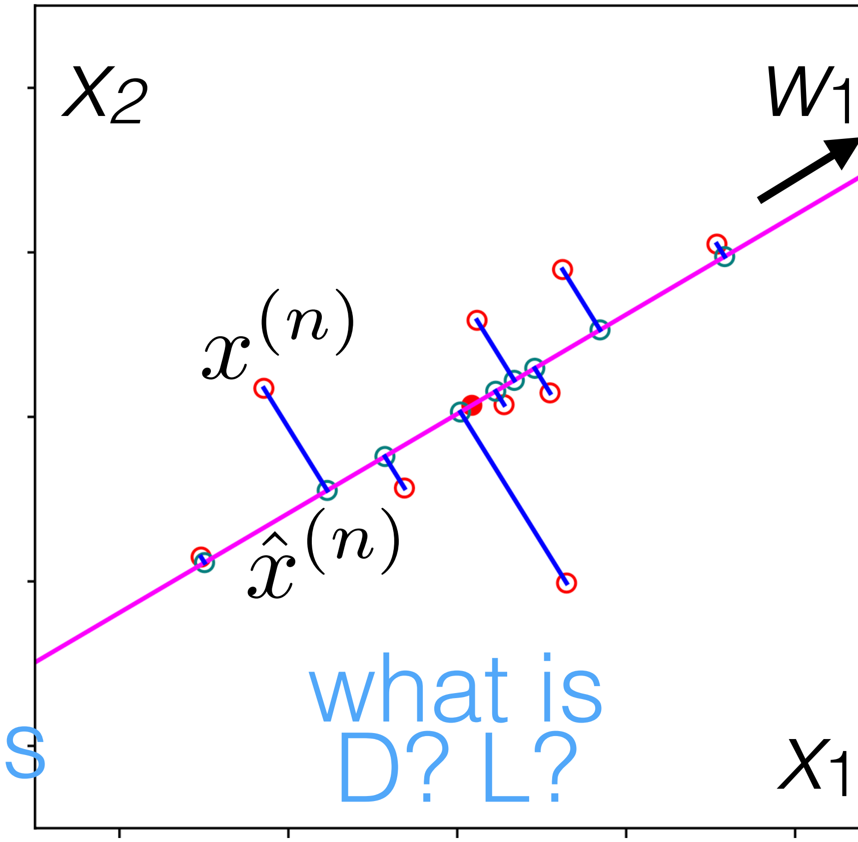

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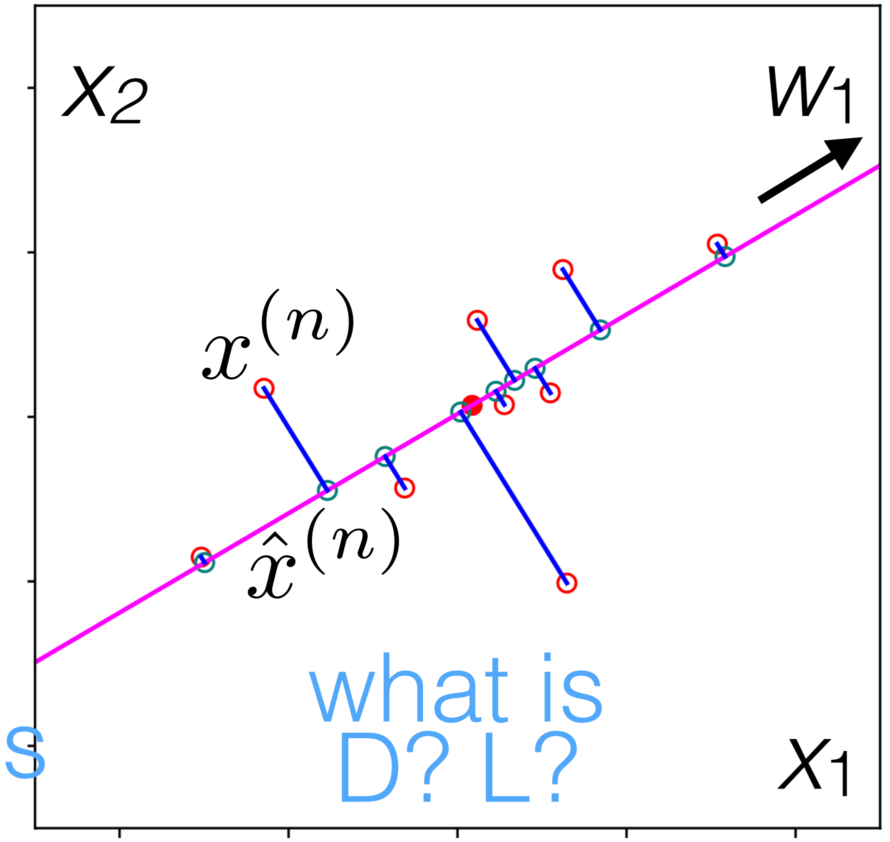

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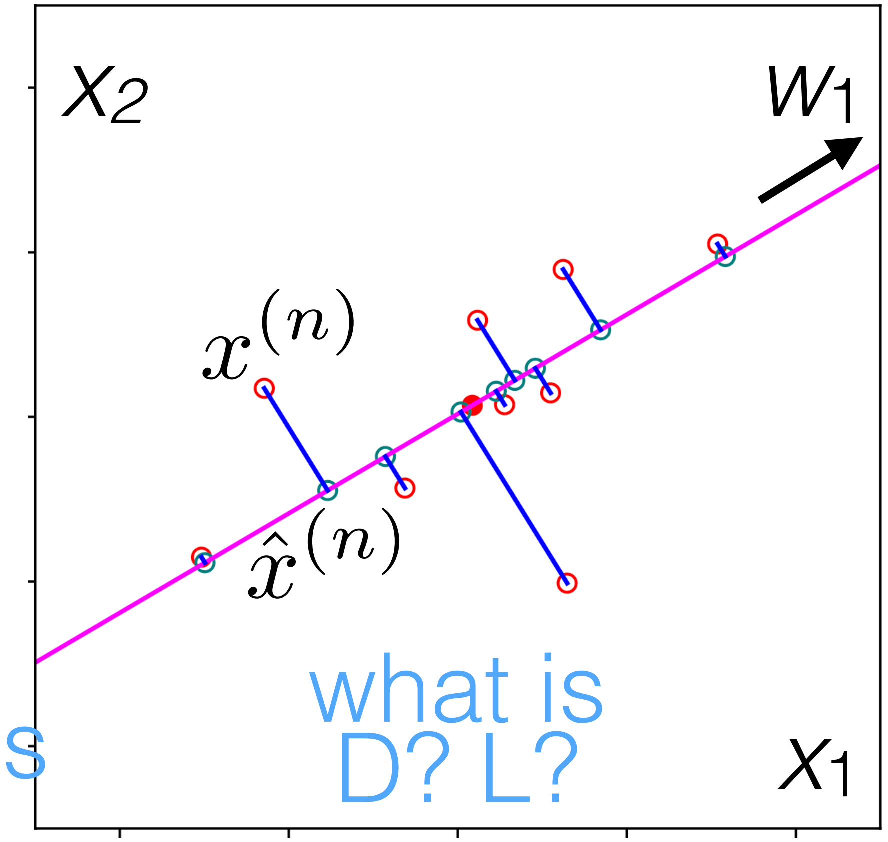

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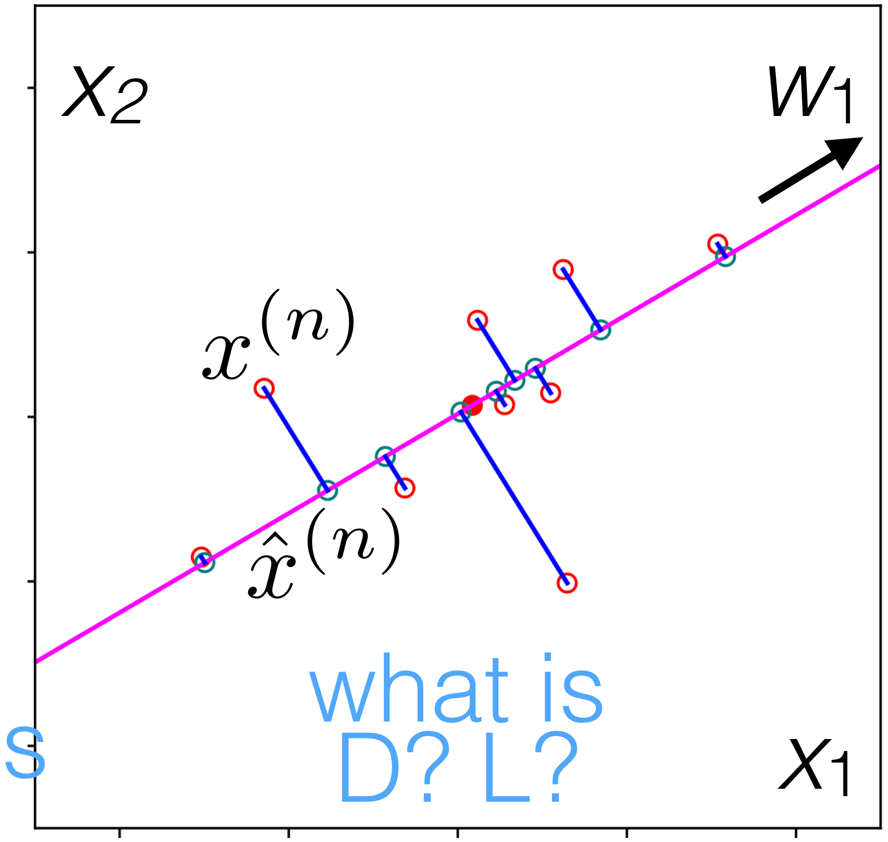

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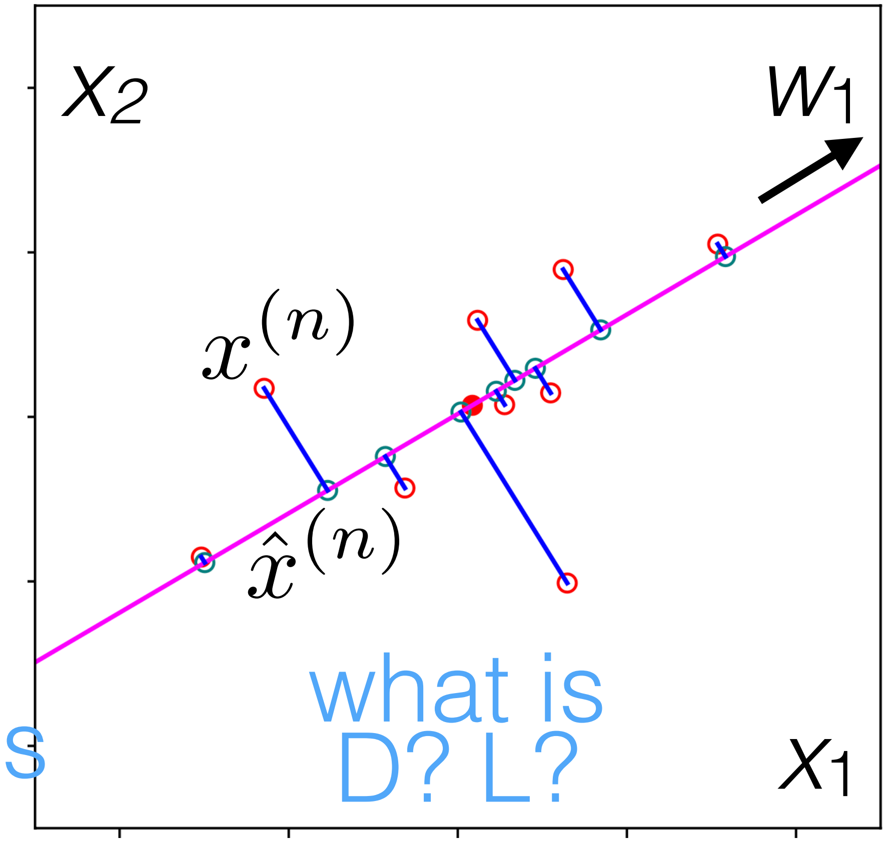

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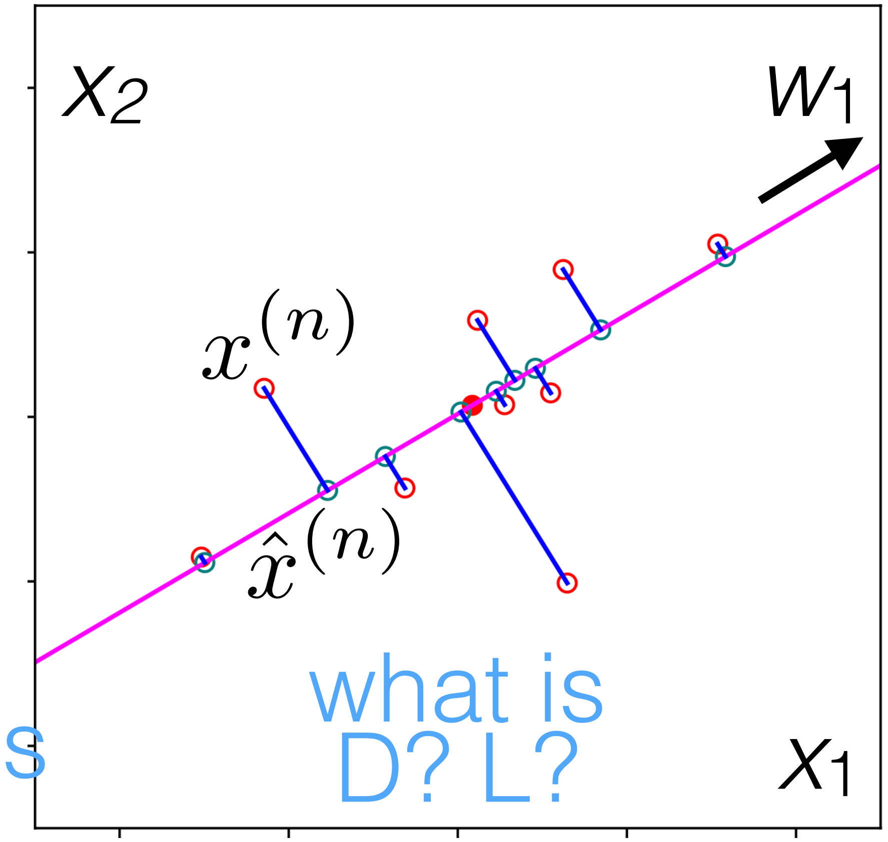

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Frobenius norm (square is sum of square entries)

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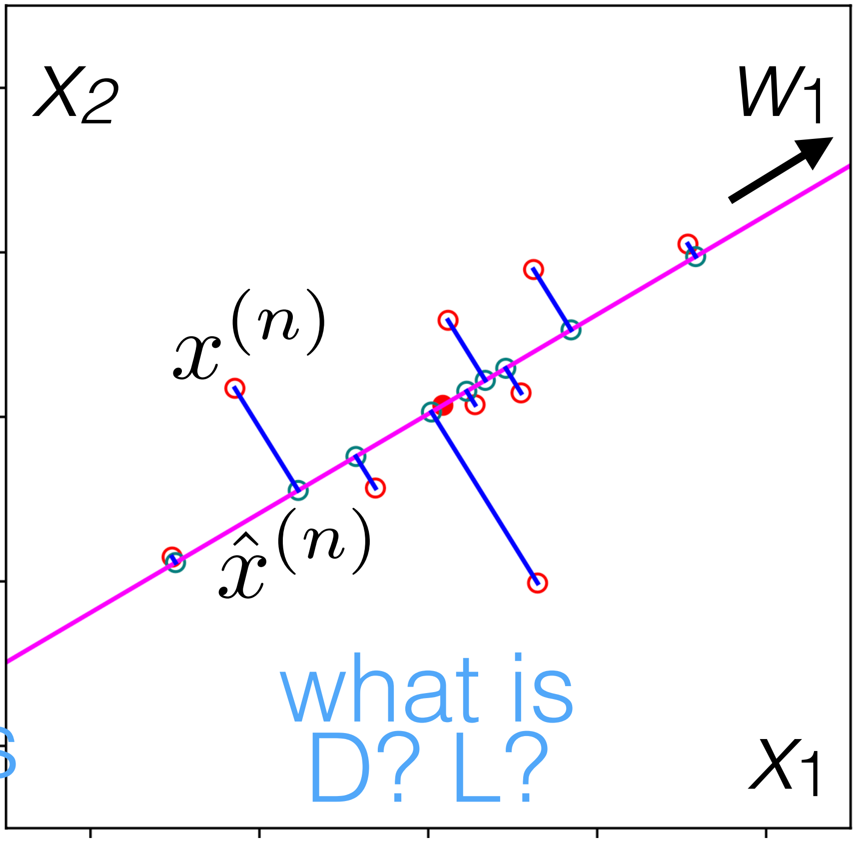

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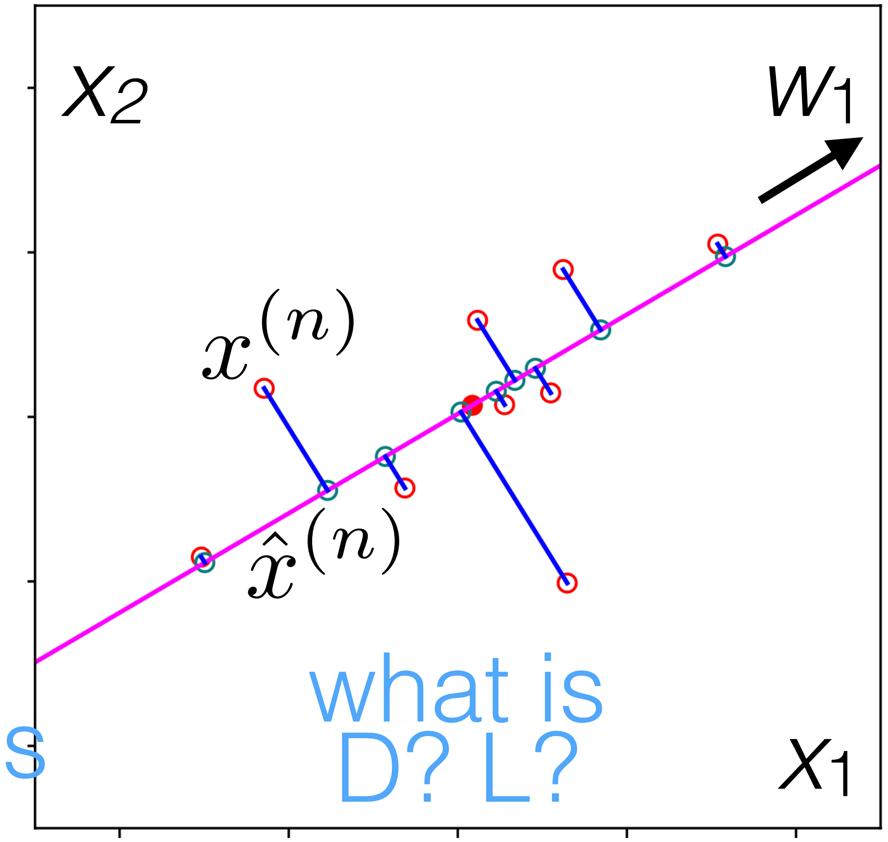

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DxN DxL LxN NxD NxL LxD

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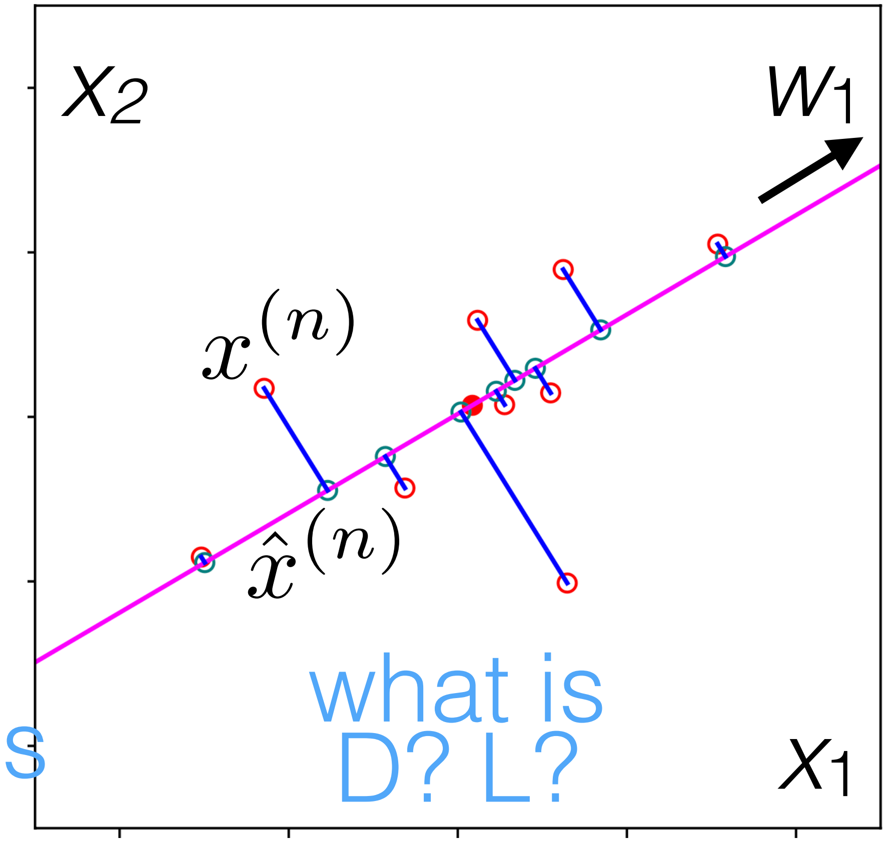


[adapted from Murphy 2022, Fig 20.1]

- As usual $x^{(n)}$ is a *D*x1 vector
- Pre-process so $\frac{1}{N}\sum_{n=1}^{N} x^{(n)} = 0_D$
- Assume: we'd like to approximate the data with its projection onto a low-dimensional subspace, with orthonormal basis $w_1, \ldots, w_L$ principal components

$$x^{(n)} \approx \sum_{\ell=1}^{L} z_\ell^{(n)} w_\ell = W z^{(n)} =: \hat{x}^{(n)}$$

Dx1 weights:1x1 Dx1 DxL Lx1 no offset (just rotating basis)

- Goal: projection is "close" to original data (square loss)

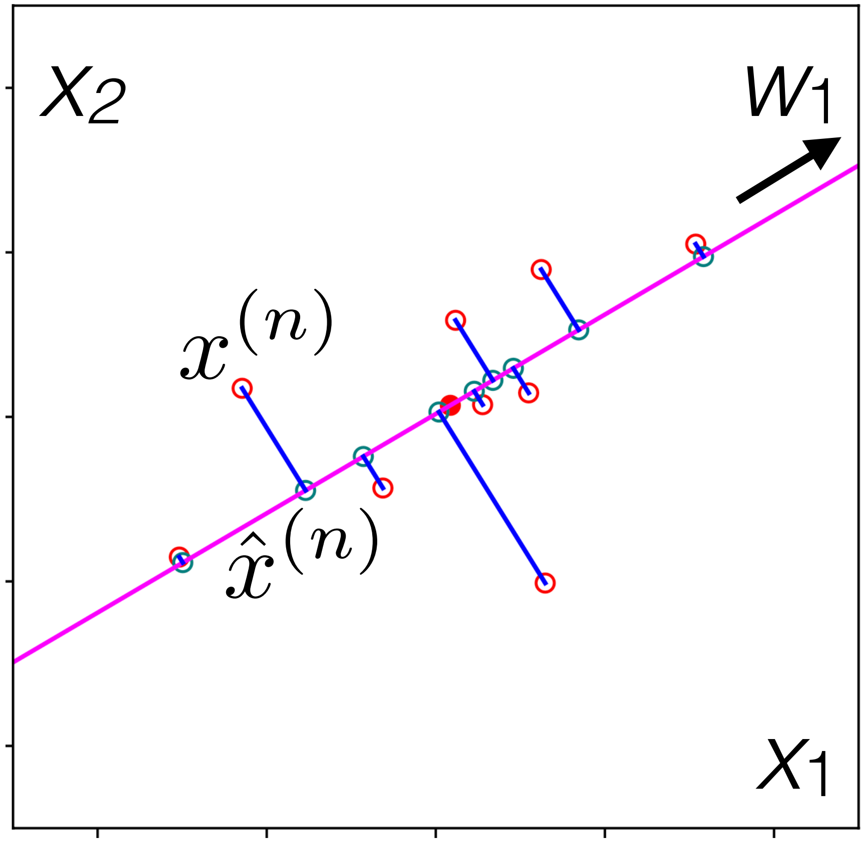
$$\min \sum_{n=1}^{N} \|x^{(n)} - \hat{x}^{(n)}\|^2 = \|X^\top - W Z^\top\|_F^2 = \|X - Z W^\top\|_F^2$$

DxN DxL LxN NxD NxL LxD

Frobenius norm (square is sum of square entries)

  - optimizing over *W* (DxL) and *Z* (NxL)
  - constraint: *W* represents an orthonormal basis
  - Observe: if we find a best basis, could instead use -1 times any basis vector, -1 times the corresponding *z* values, and get the same result

Solving when $L=1$



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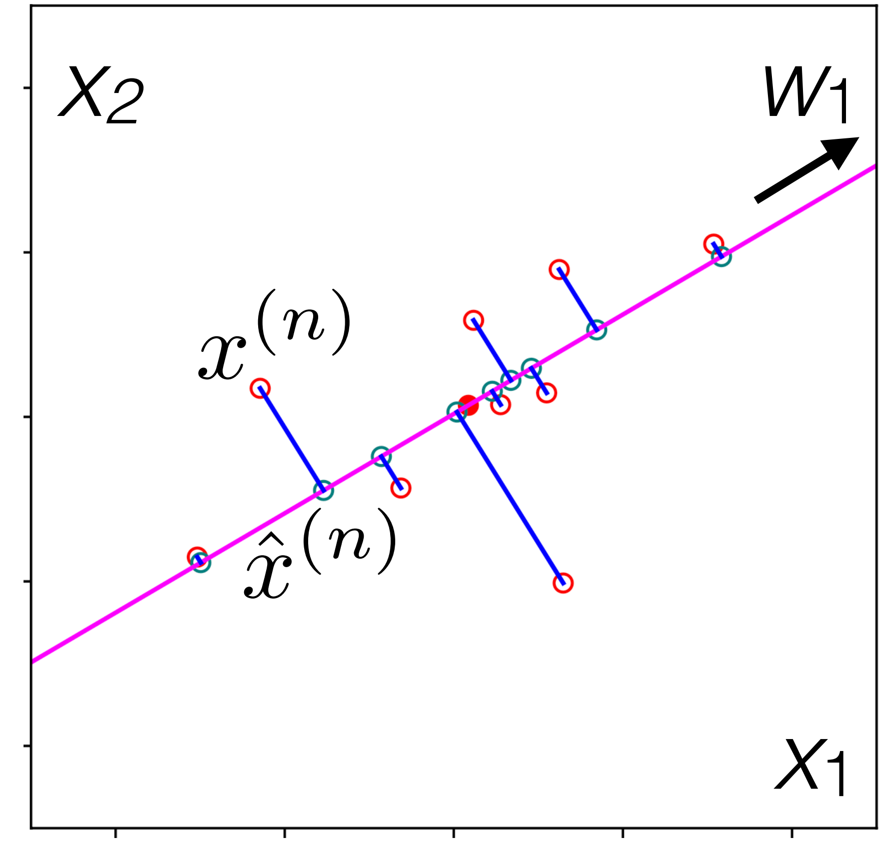

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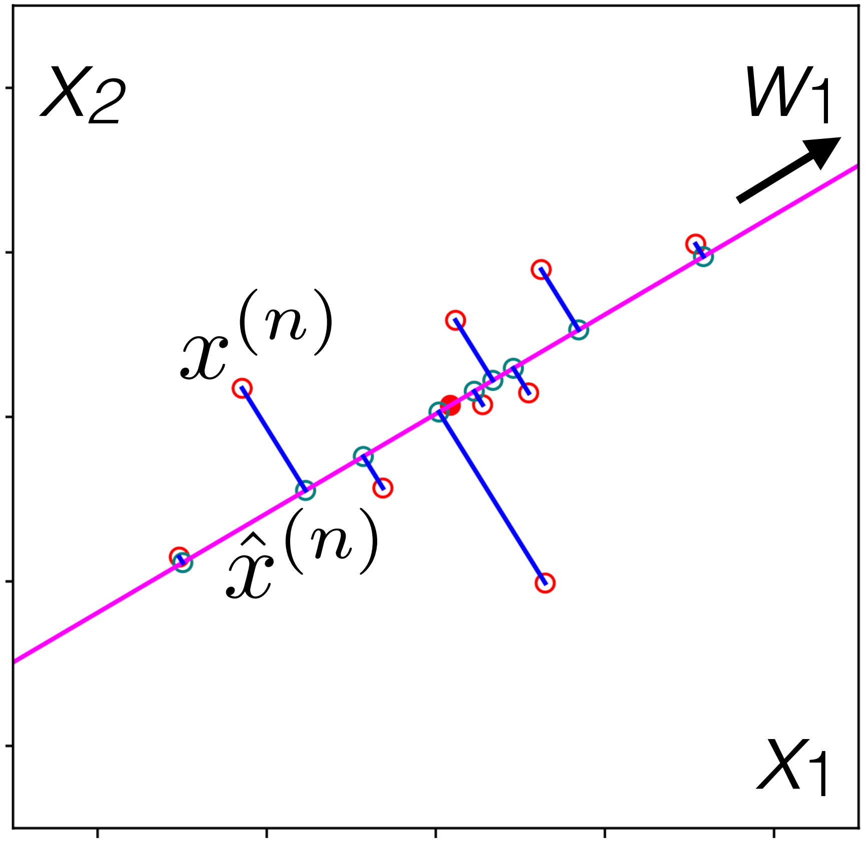

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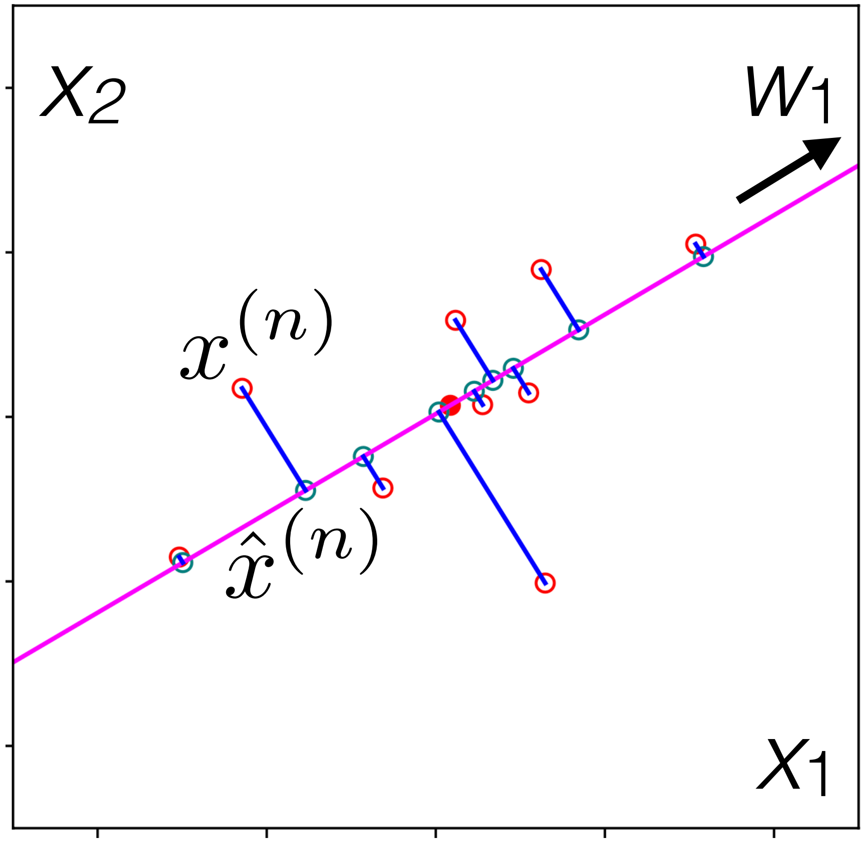

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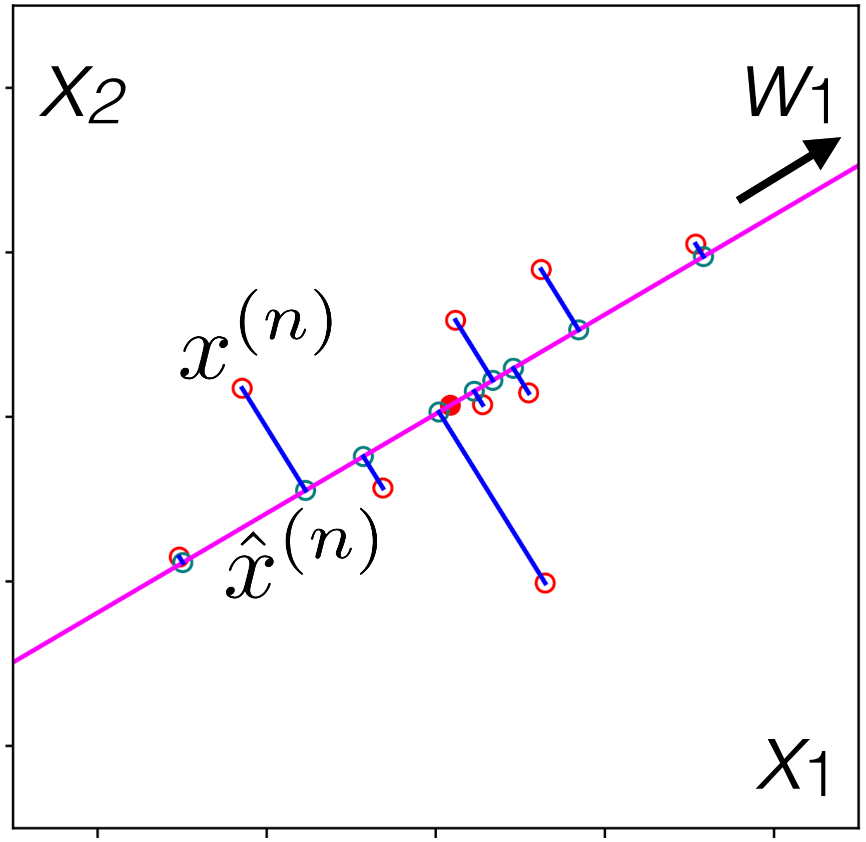

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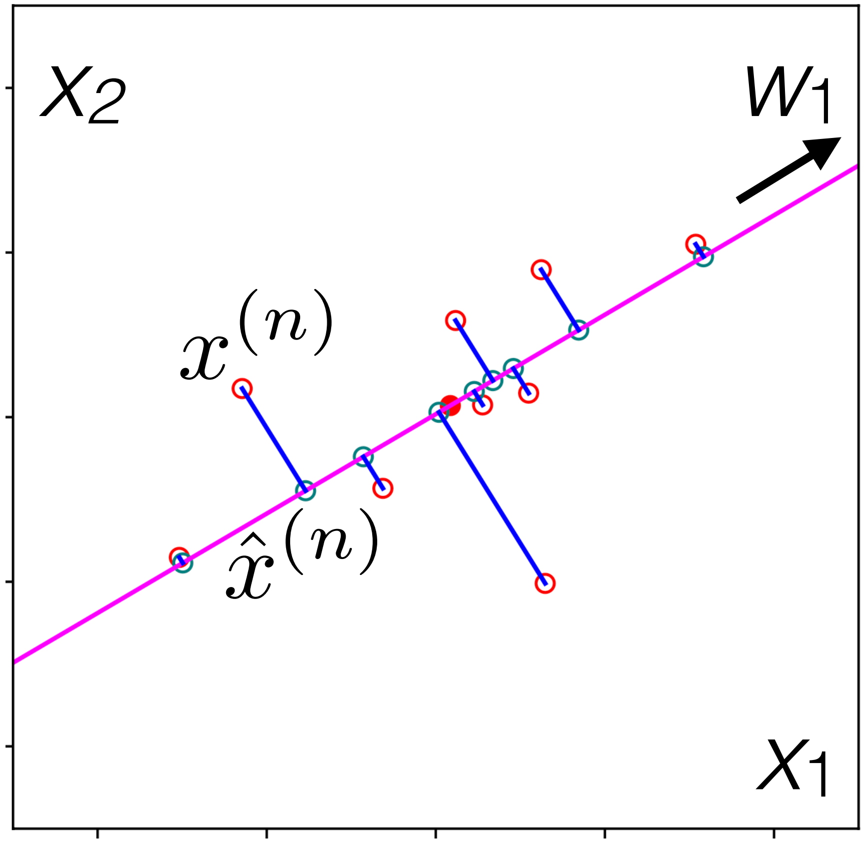

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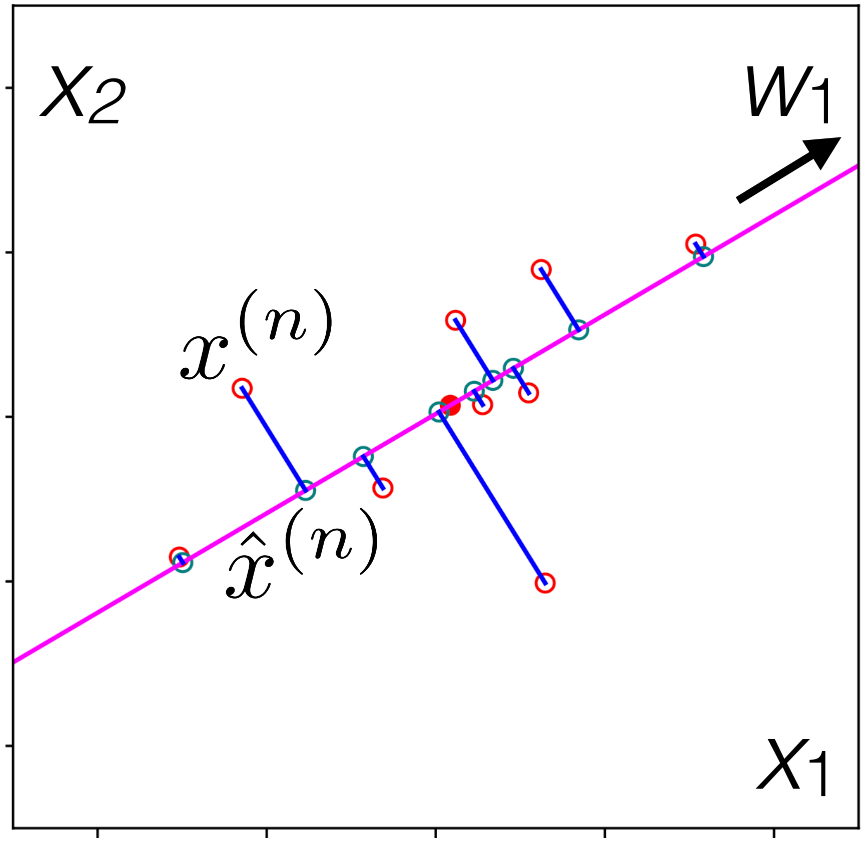

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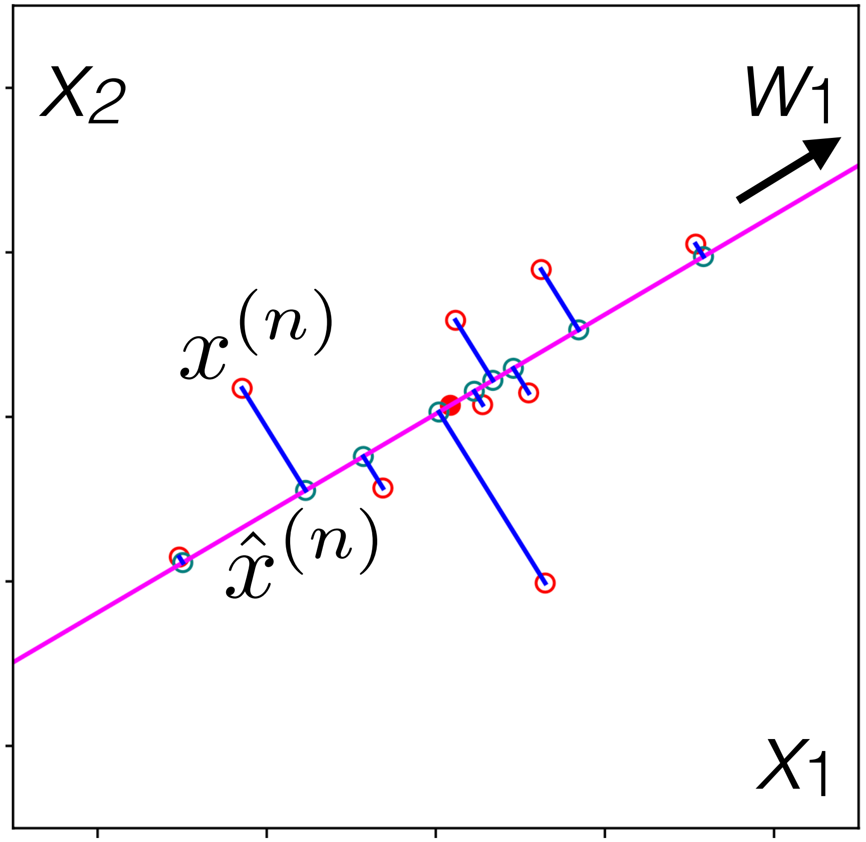

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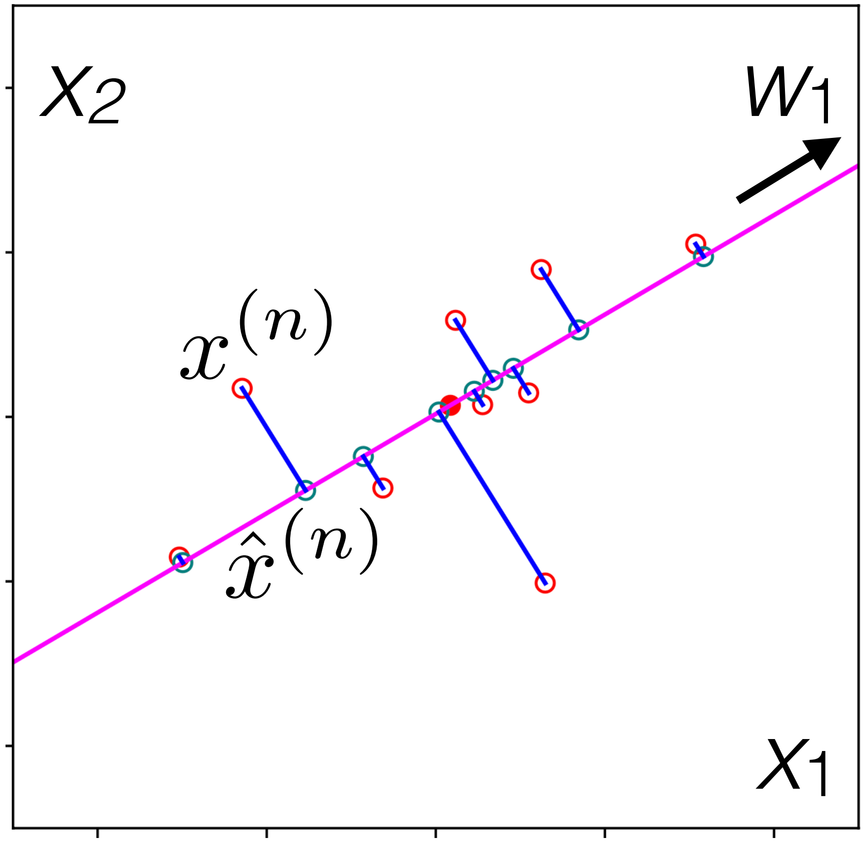

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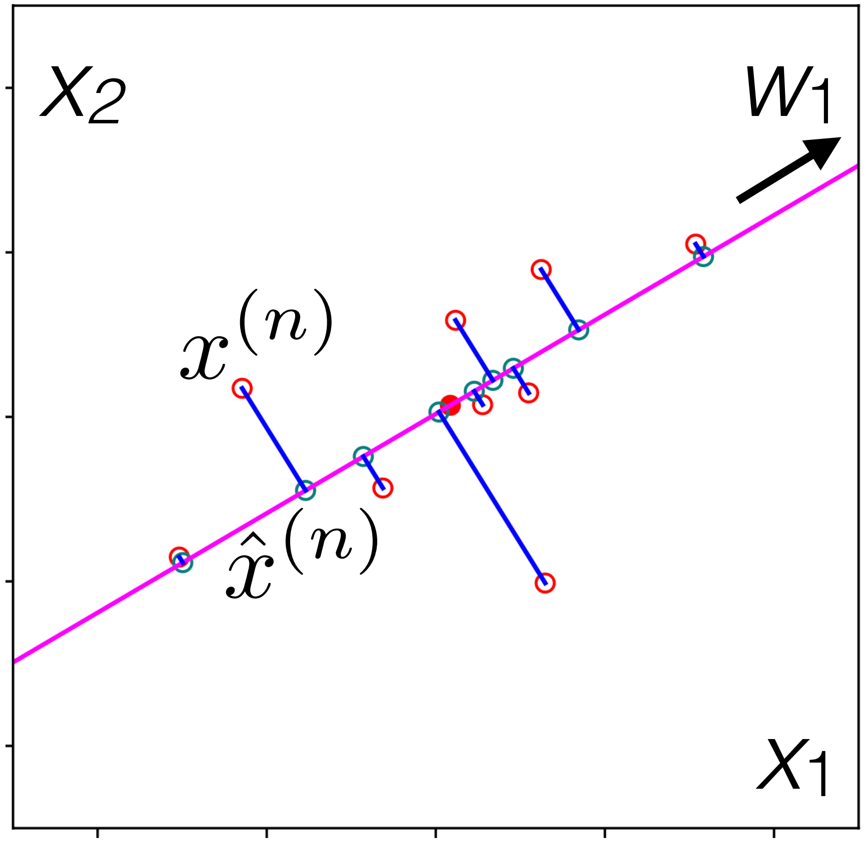

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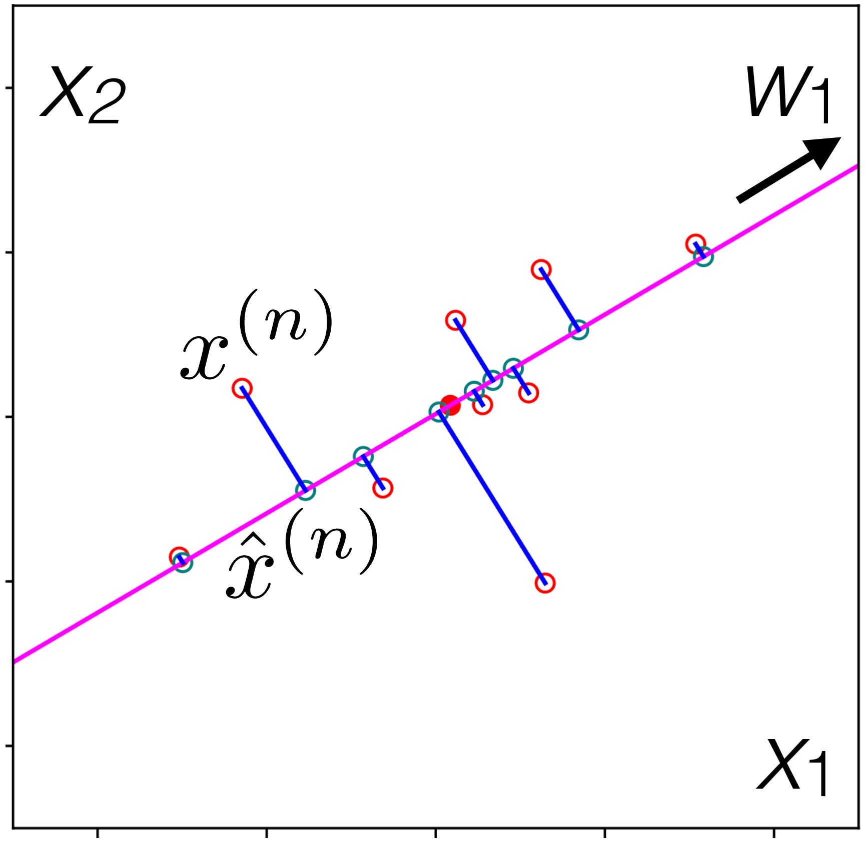

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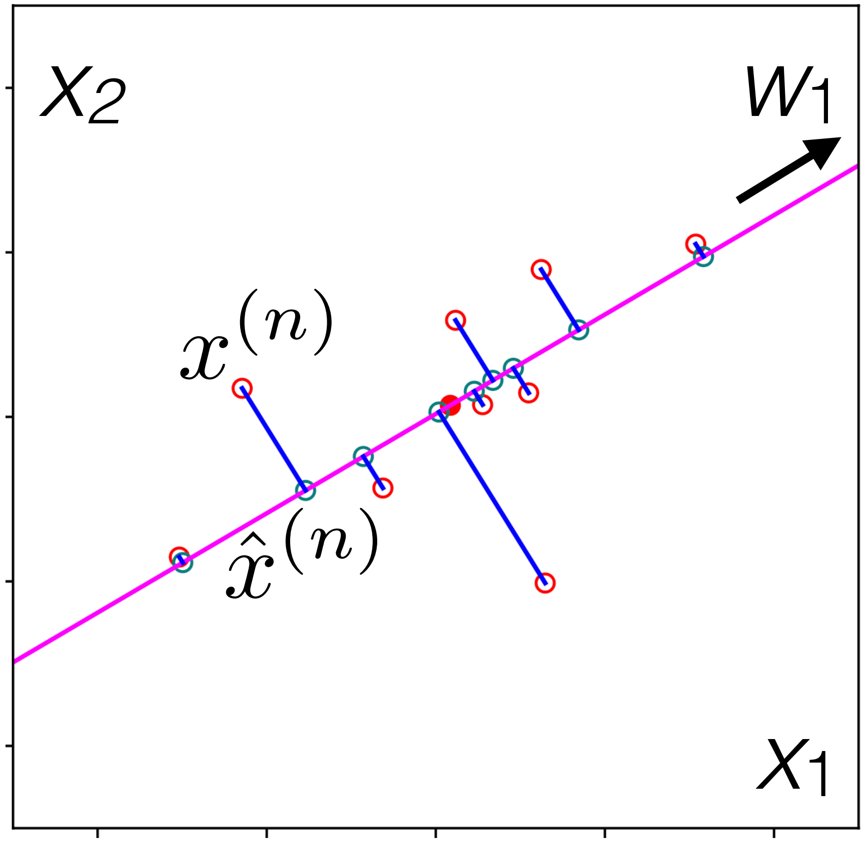

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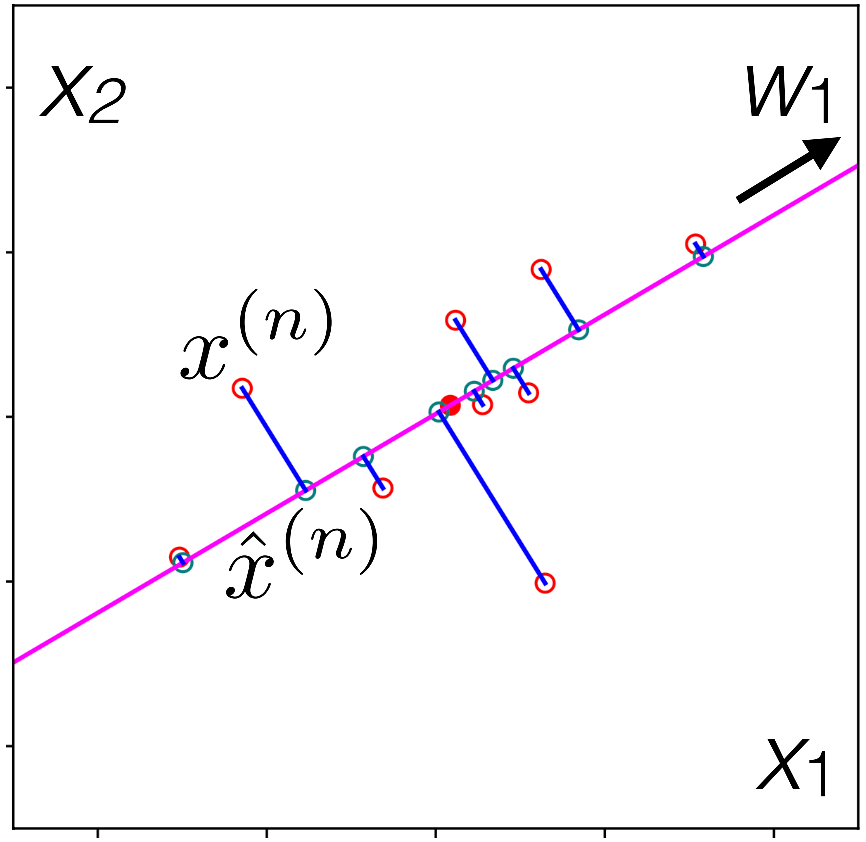

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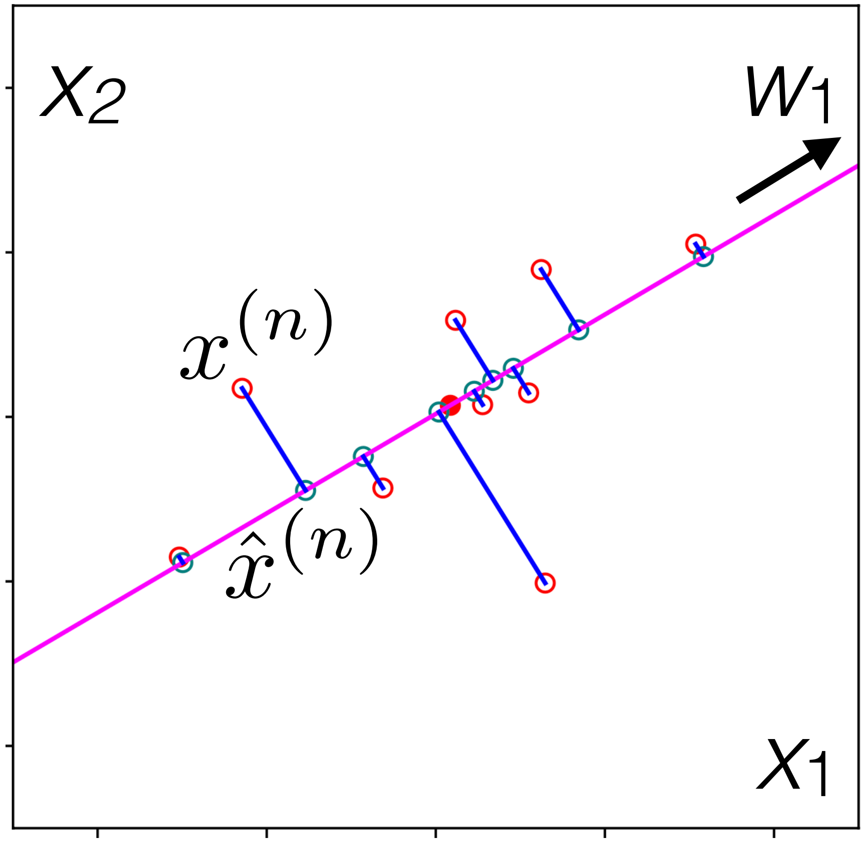

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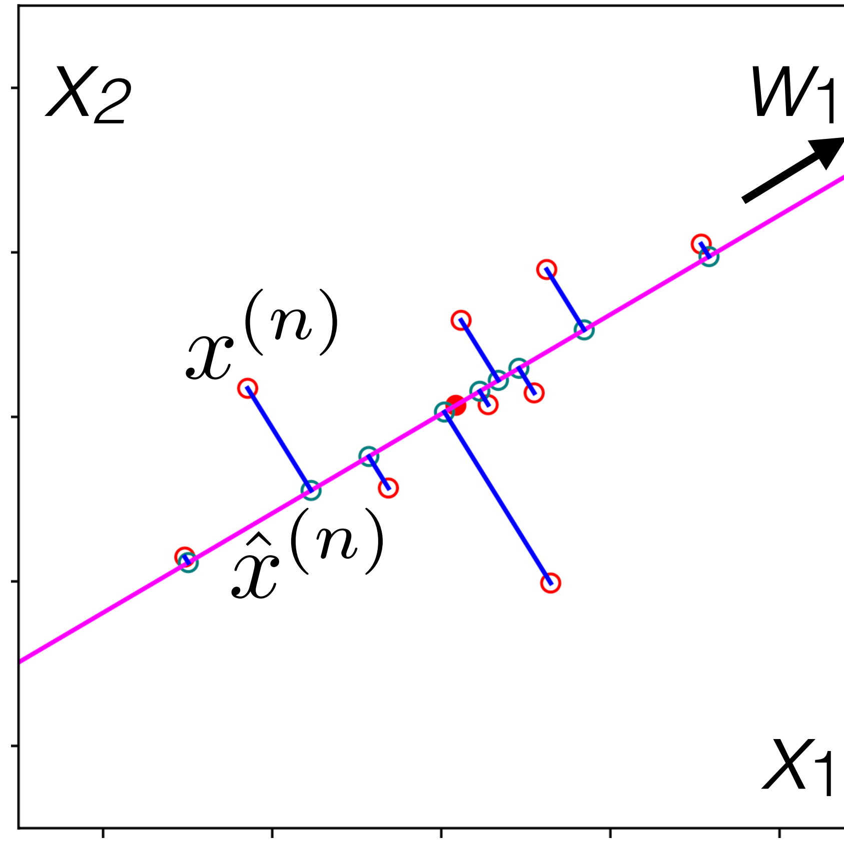

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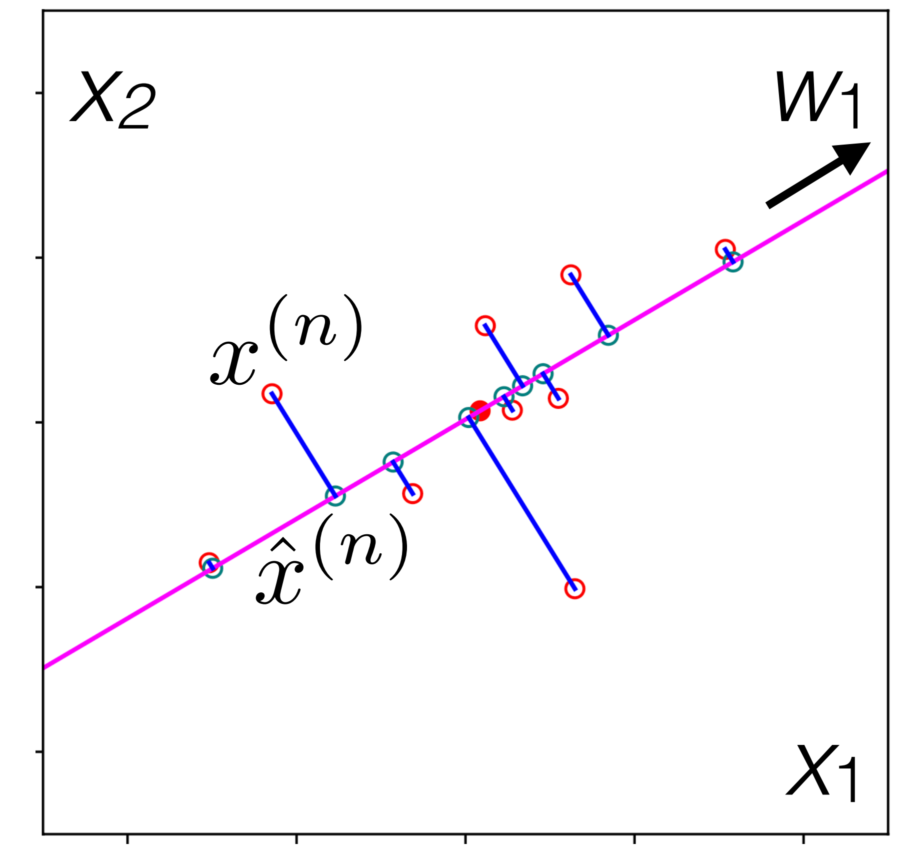

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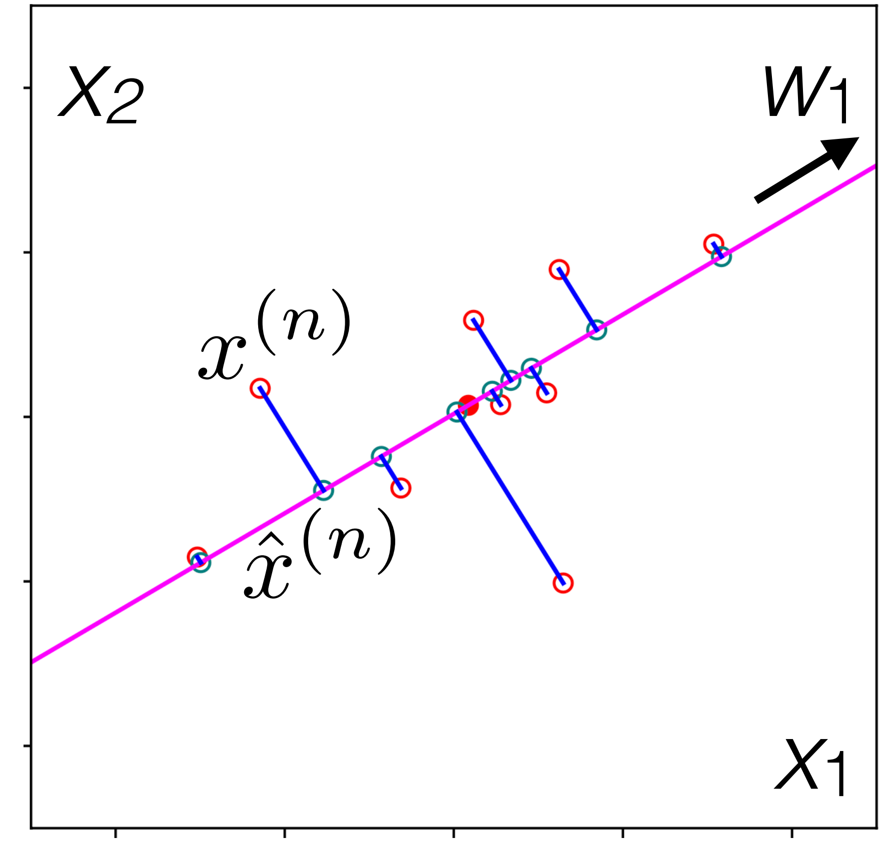

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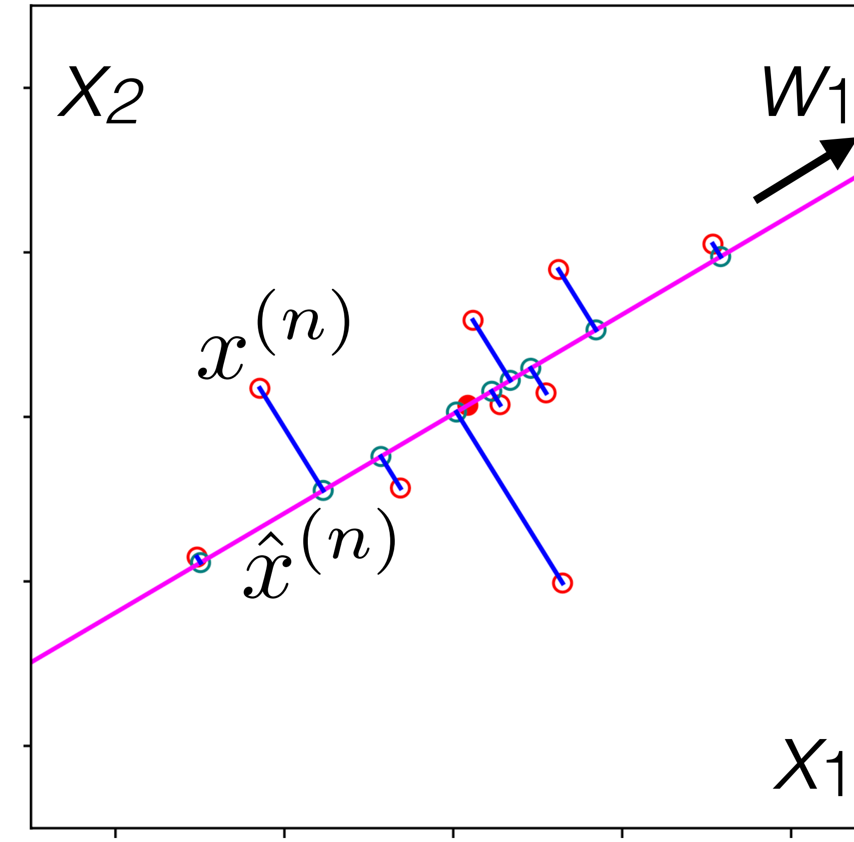

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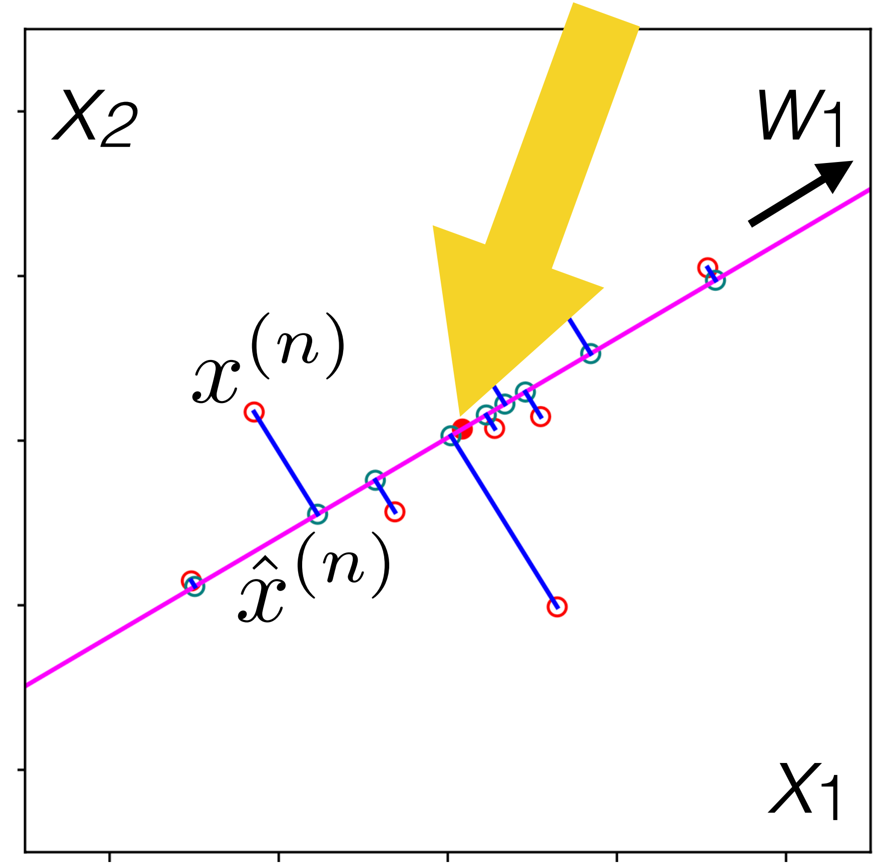

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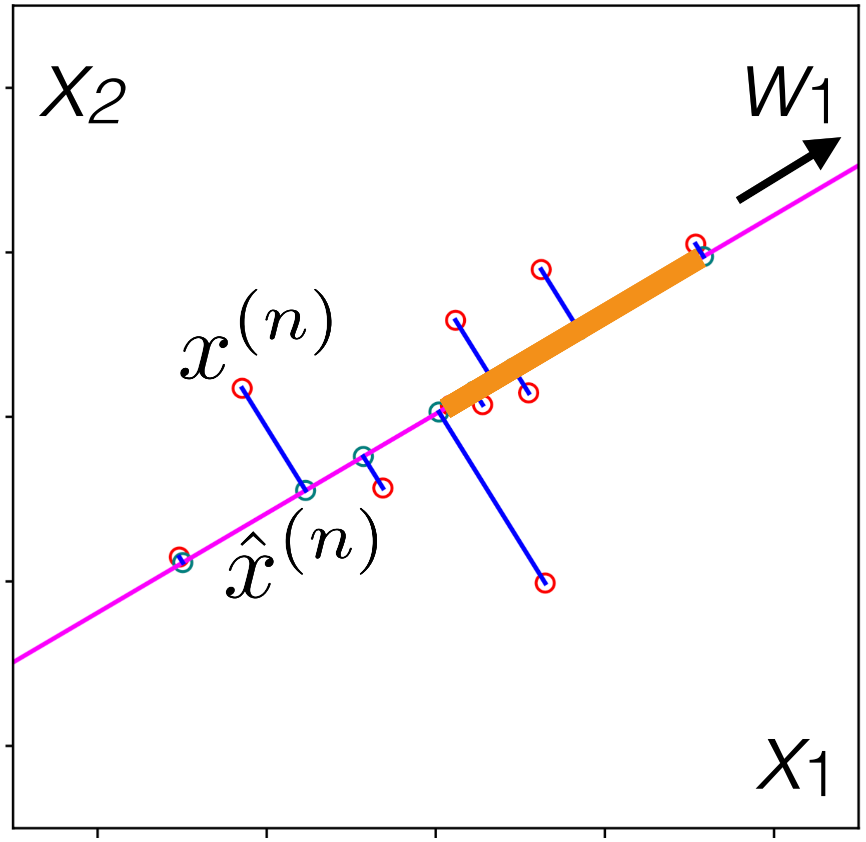

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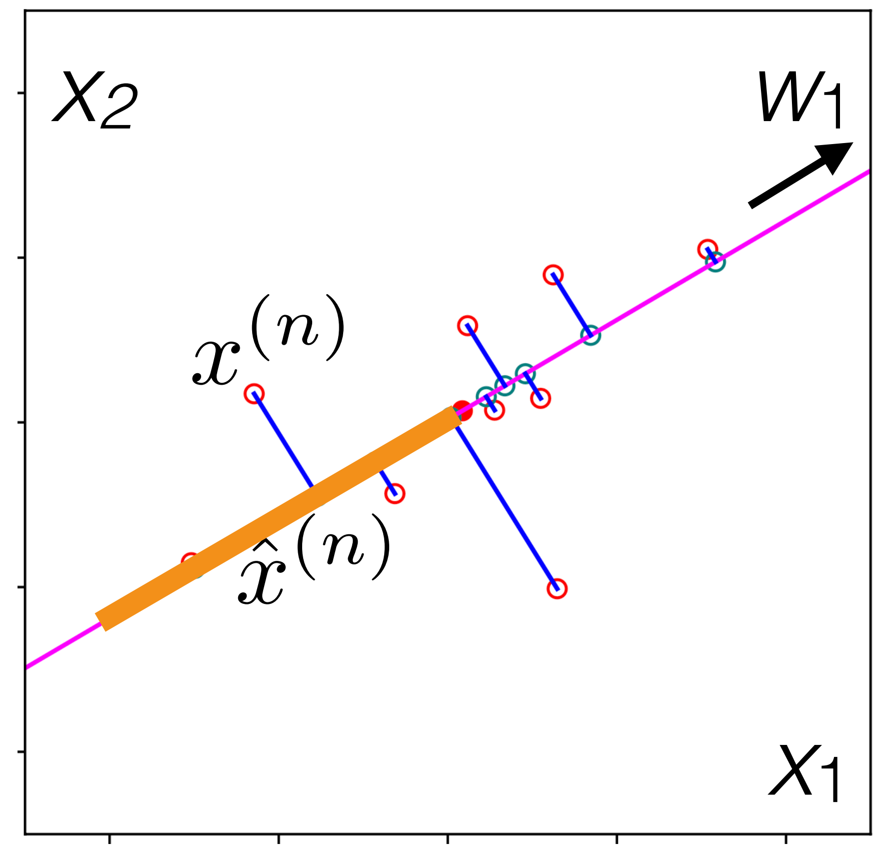

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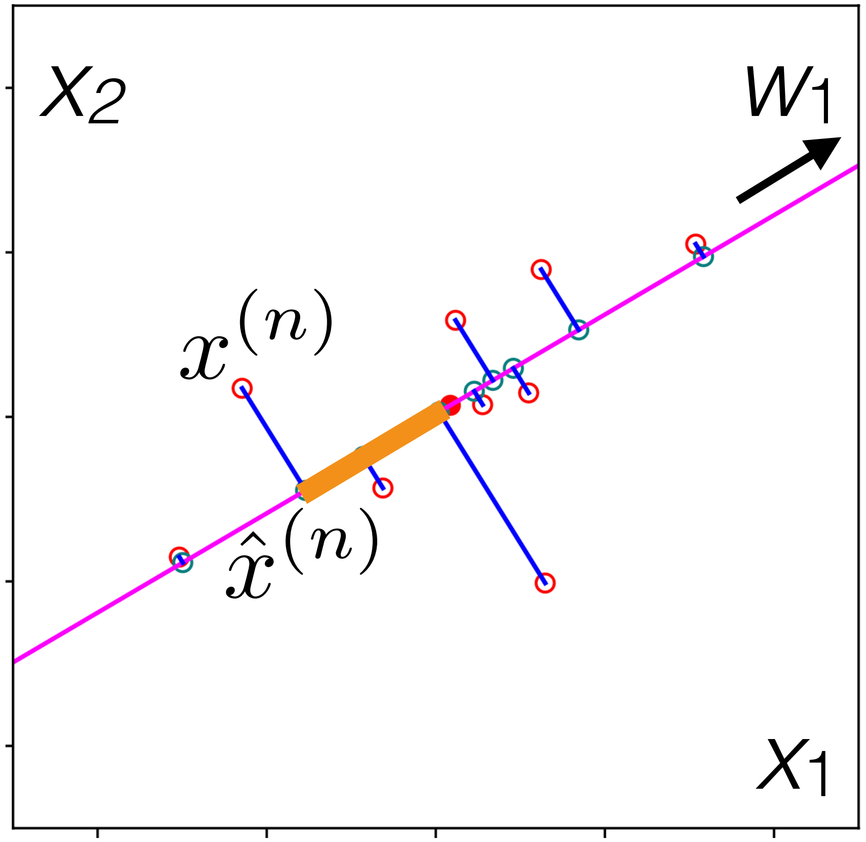

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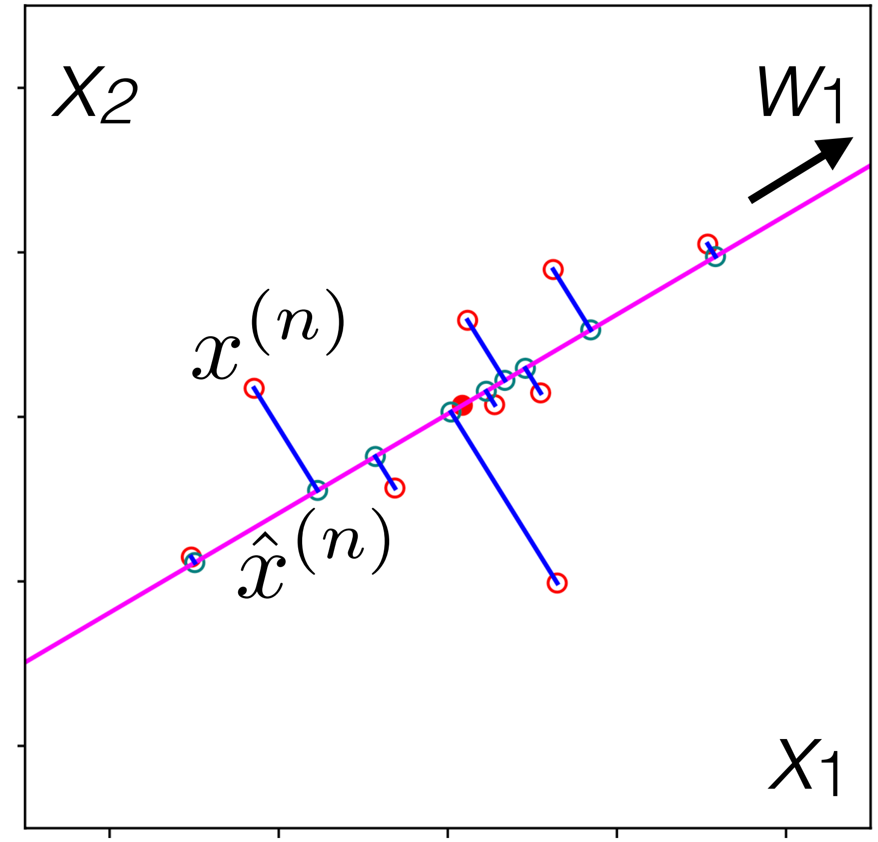

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- Plug back in to the objective

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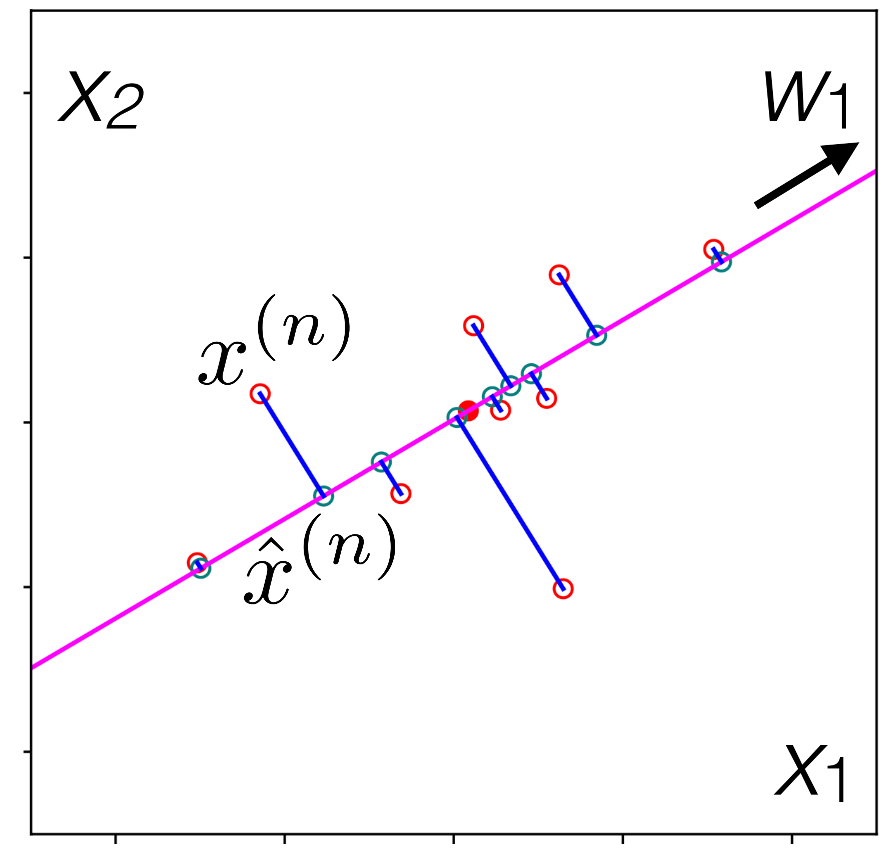

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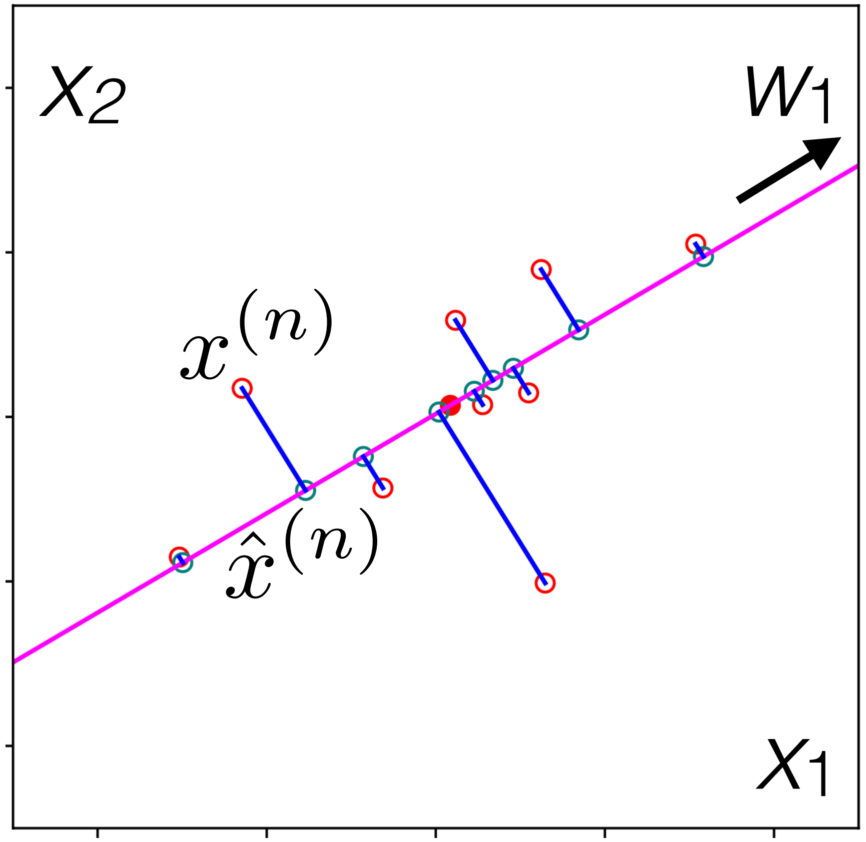

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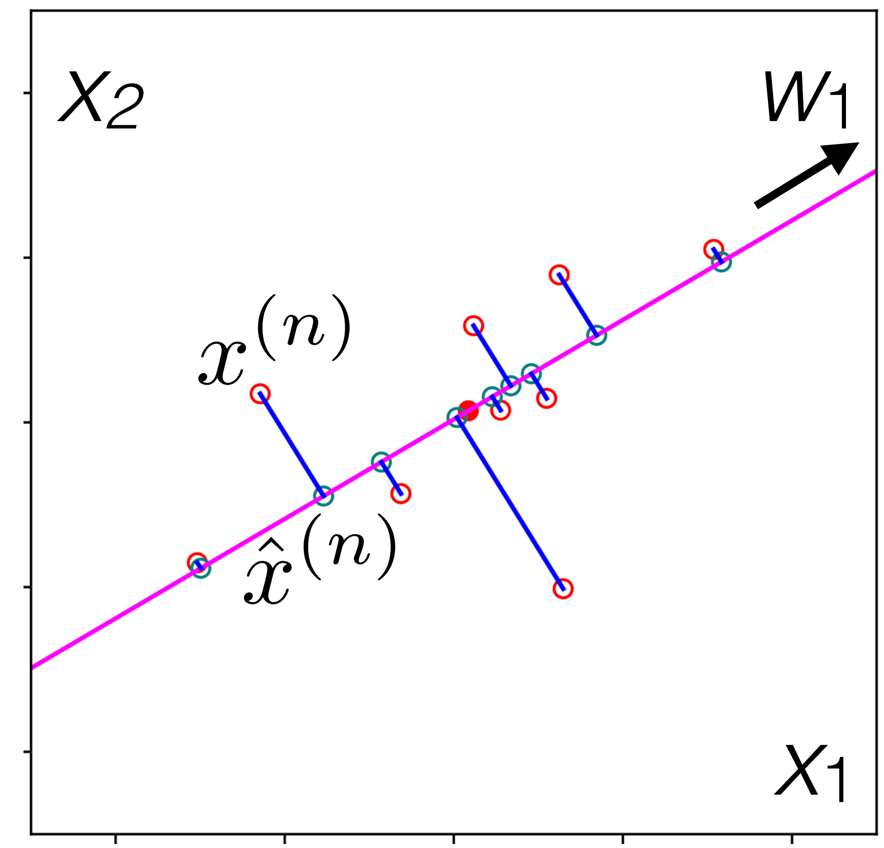

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corrected from live lecture!

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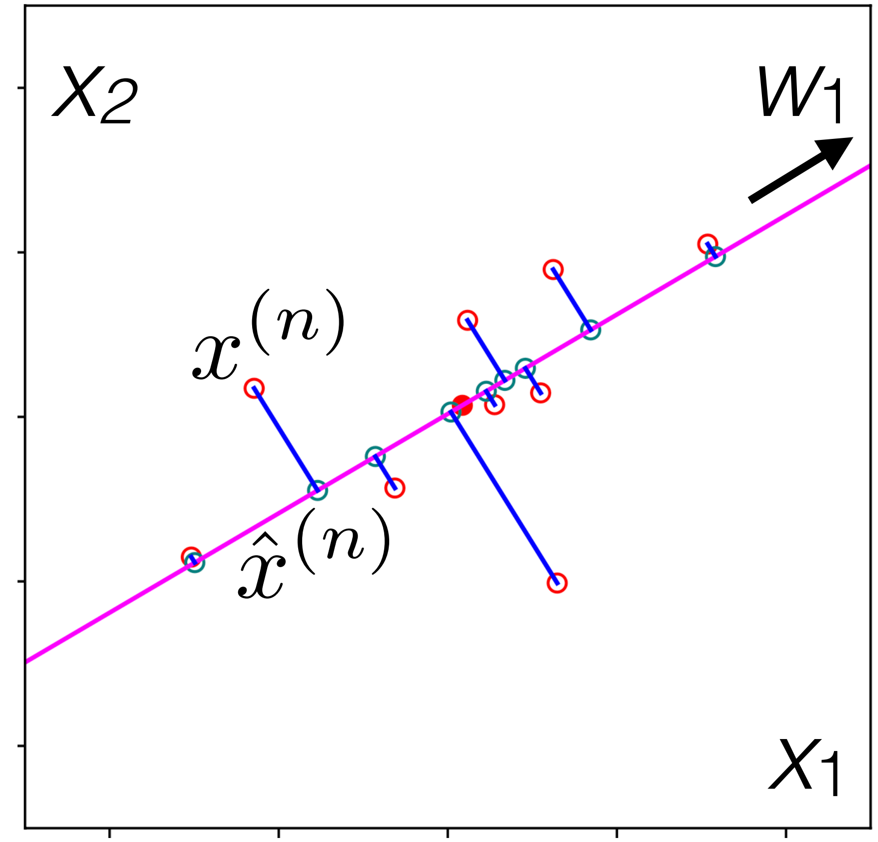

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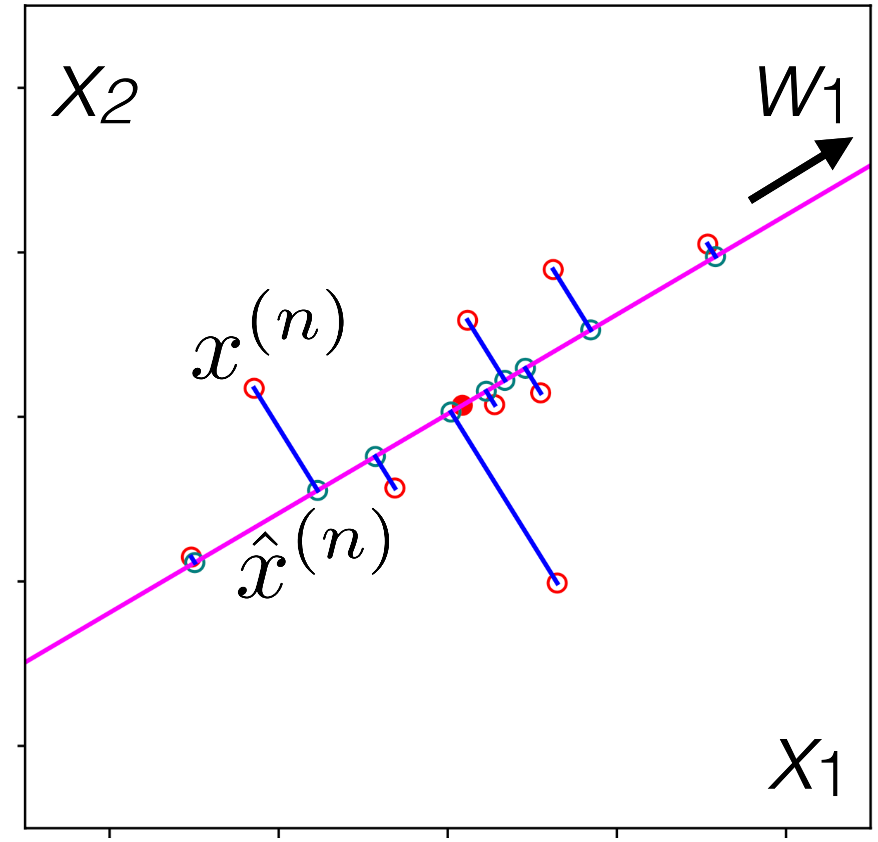

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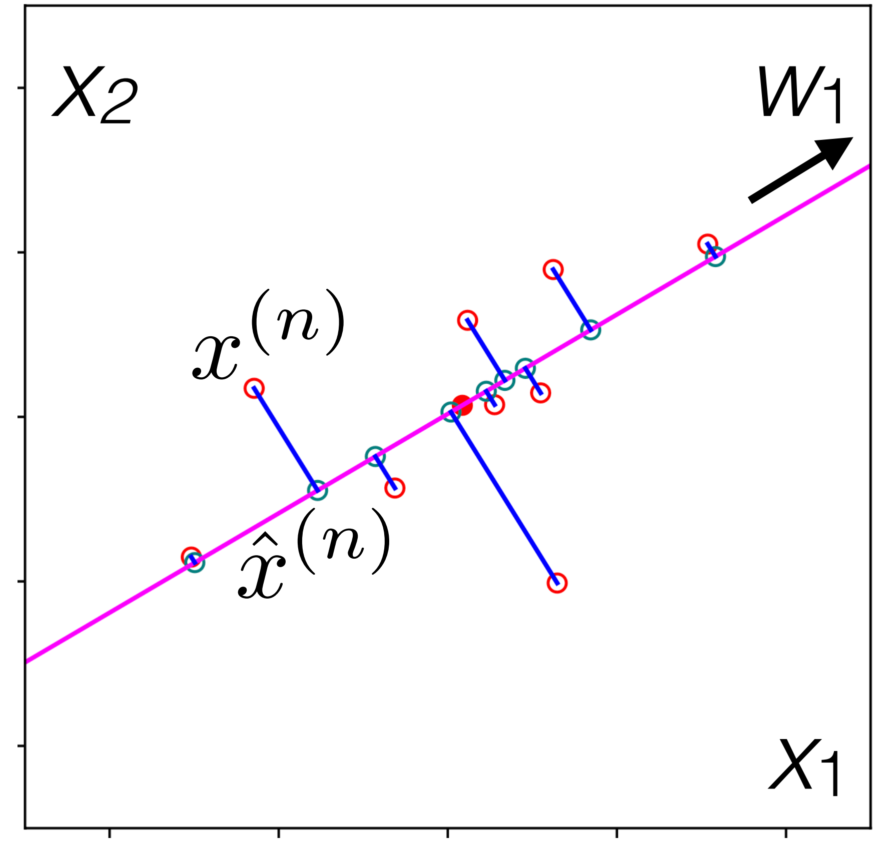

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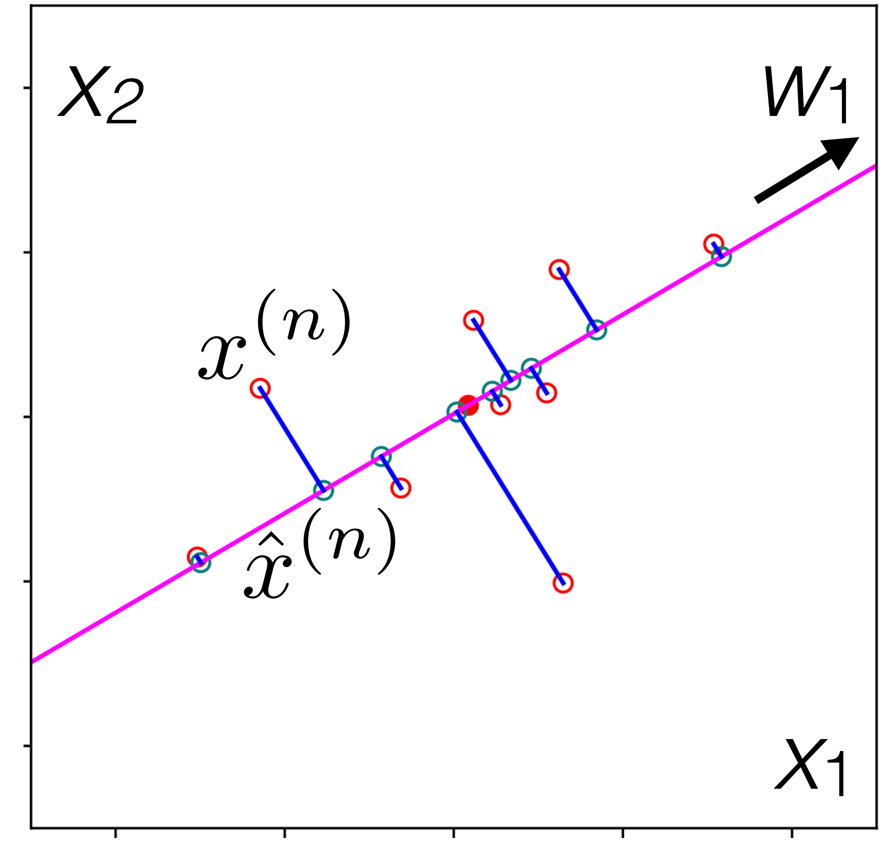

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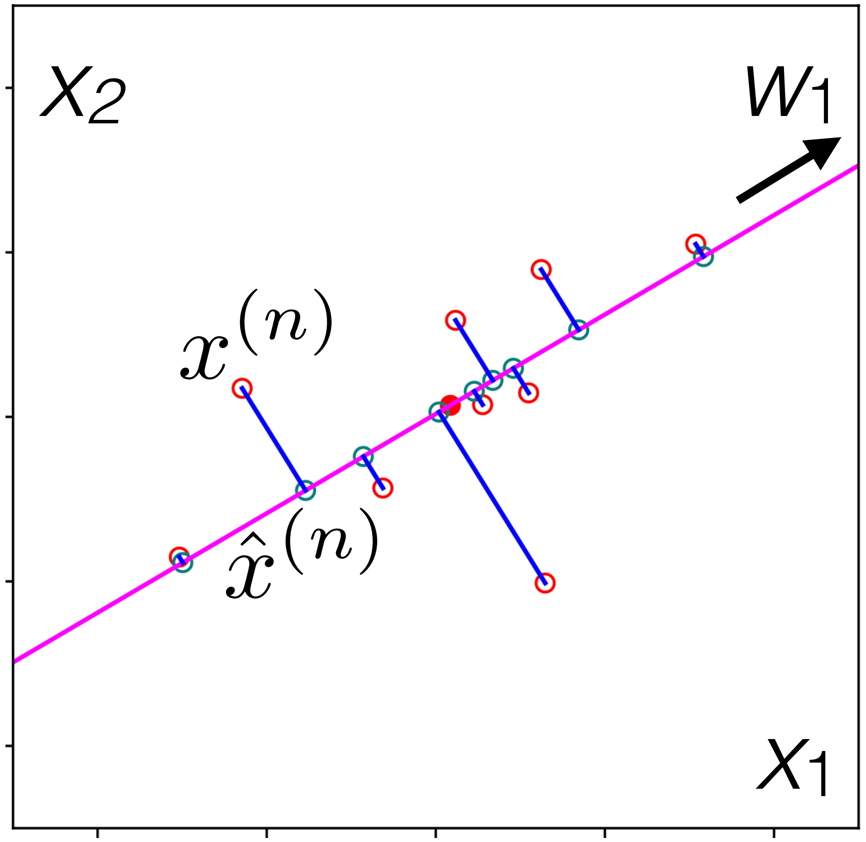

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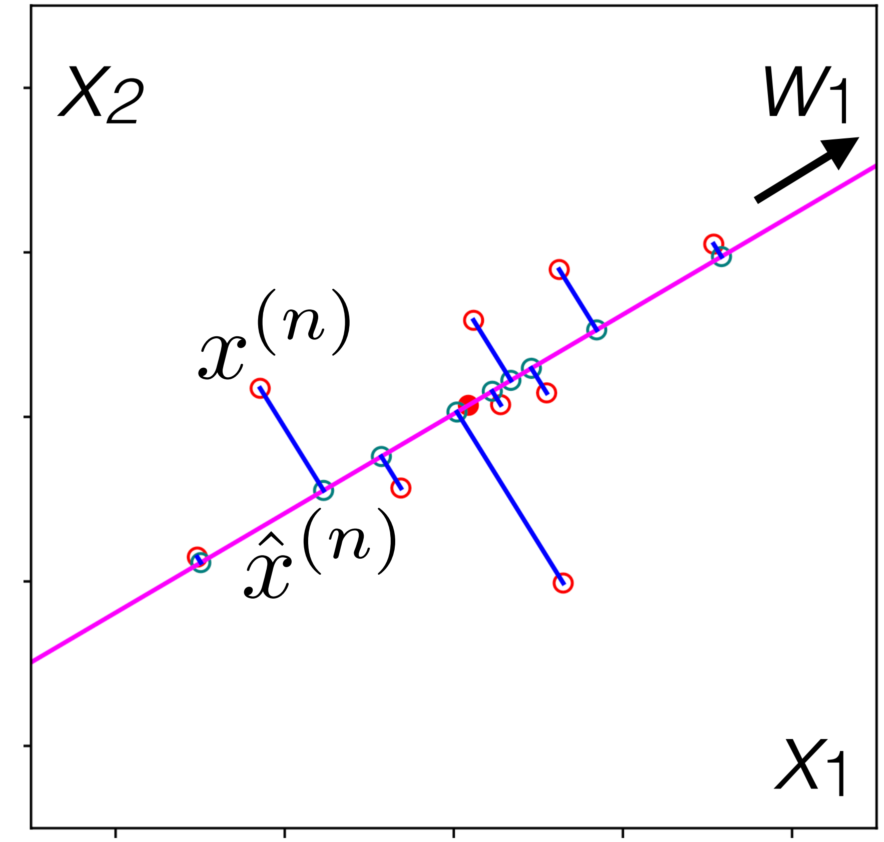

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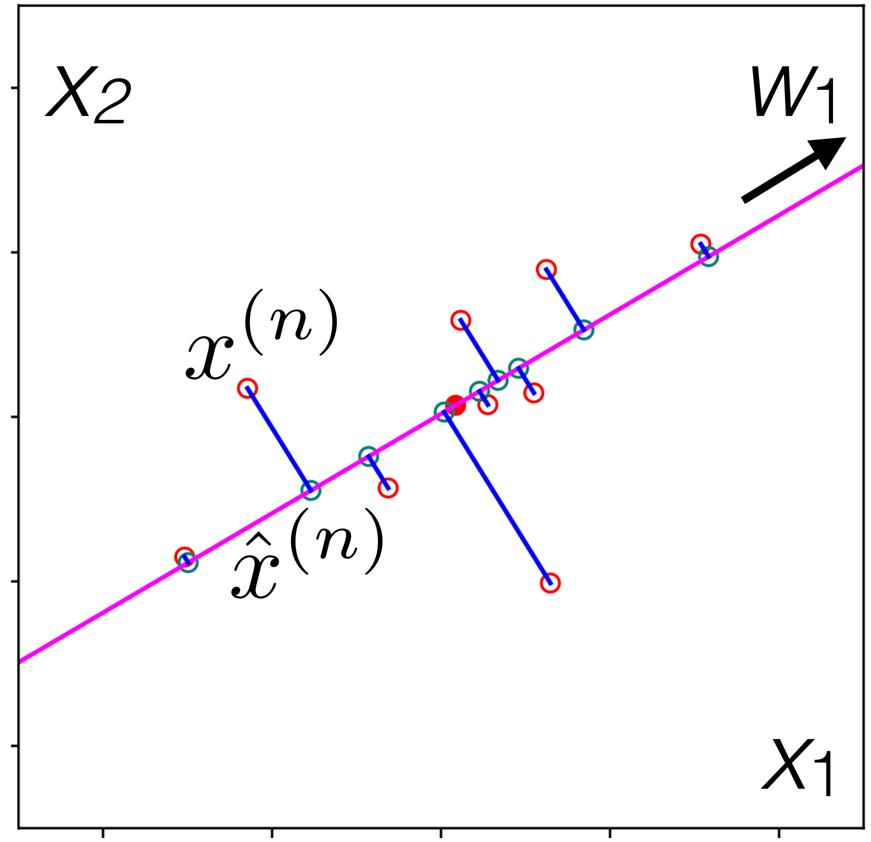

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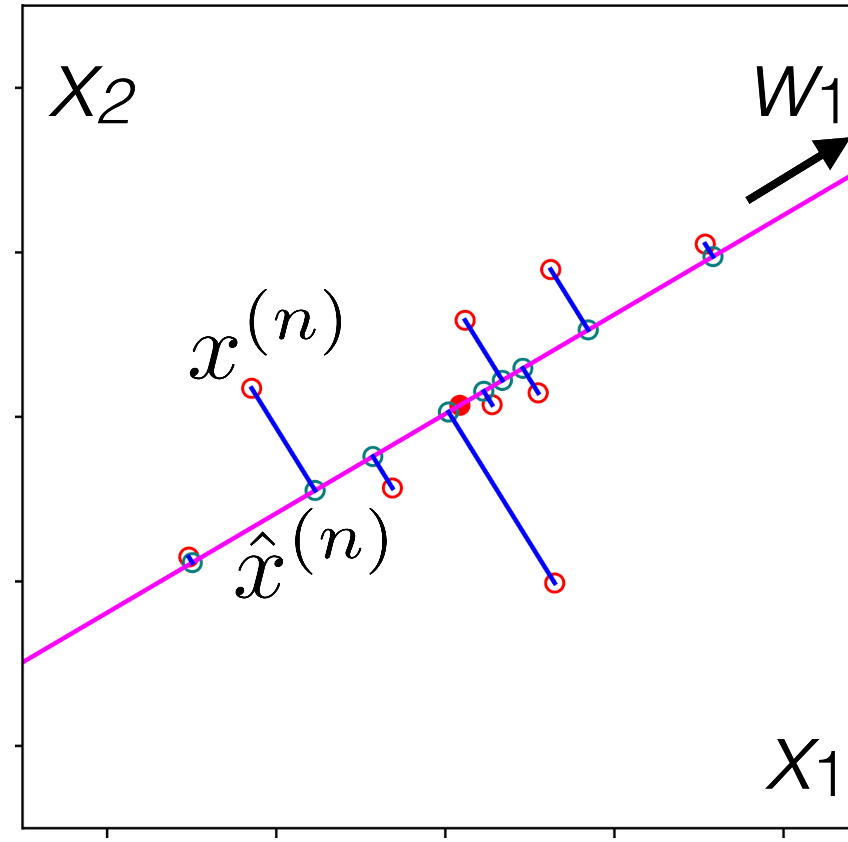

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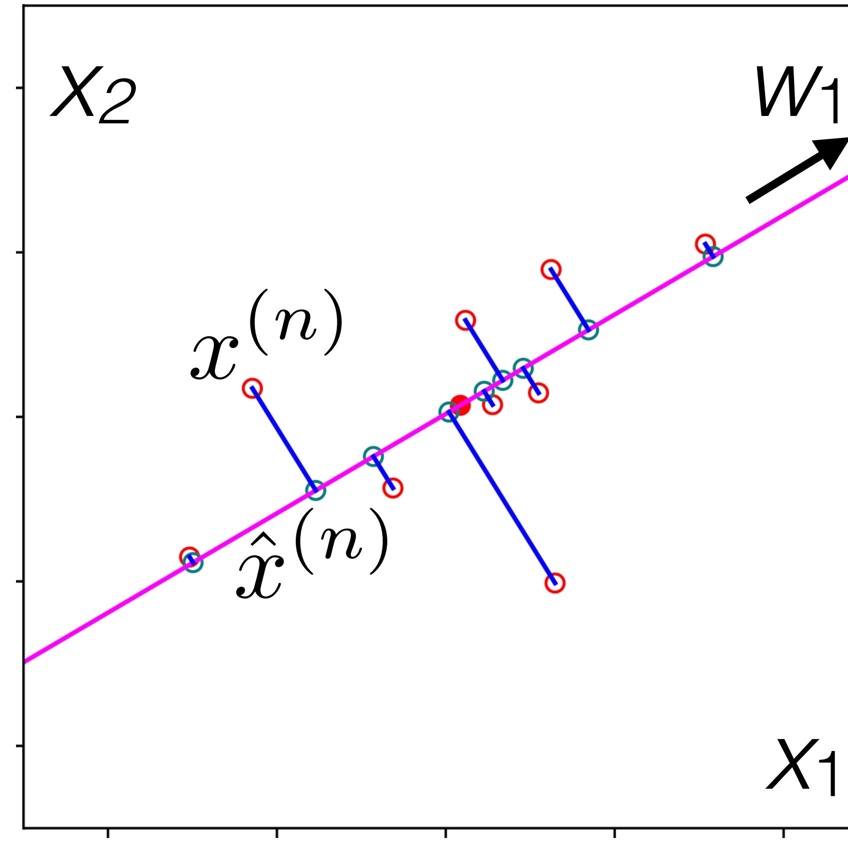

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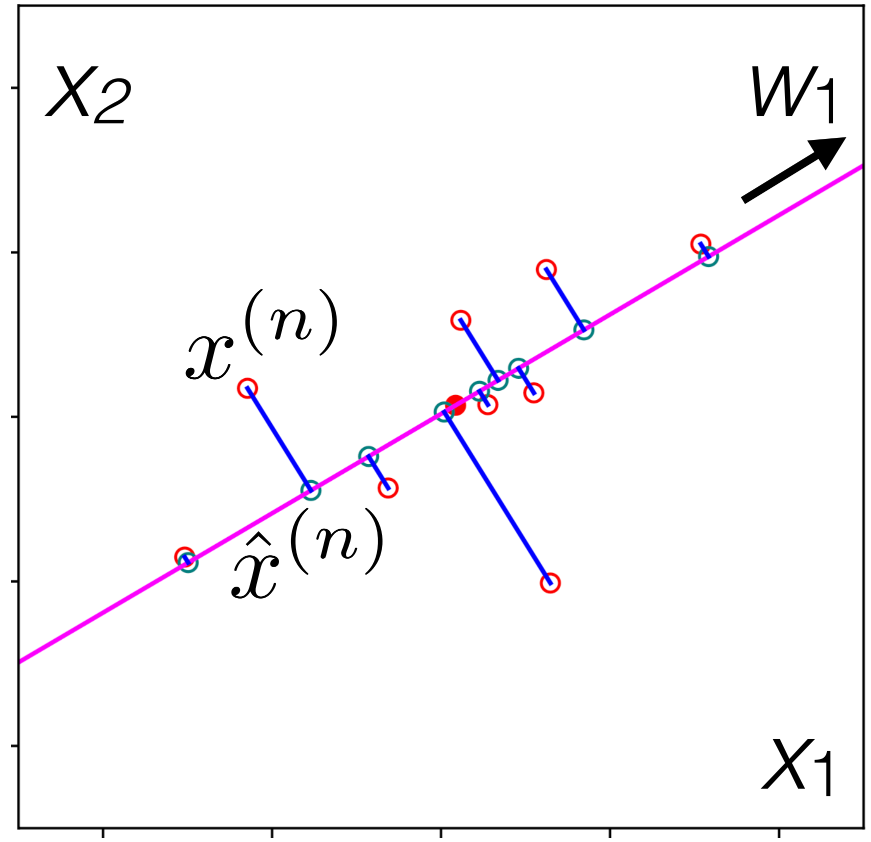

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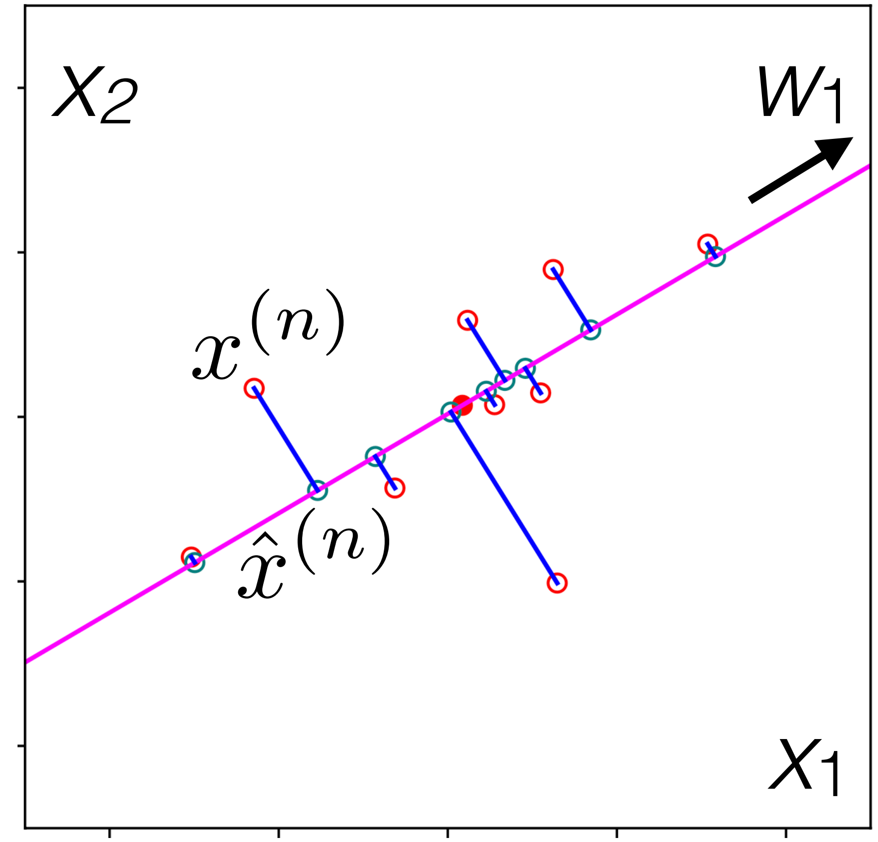

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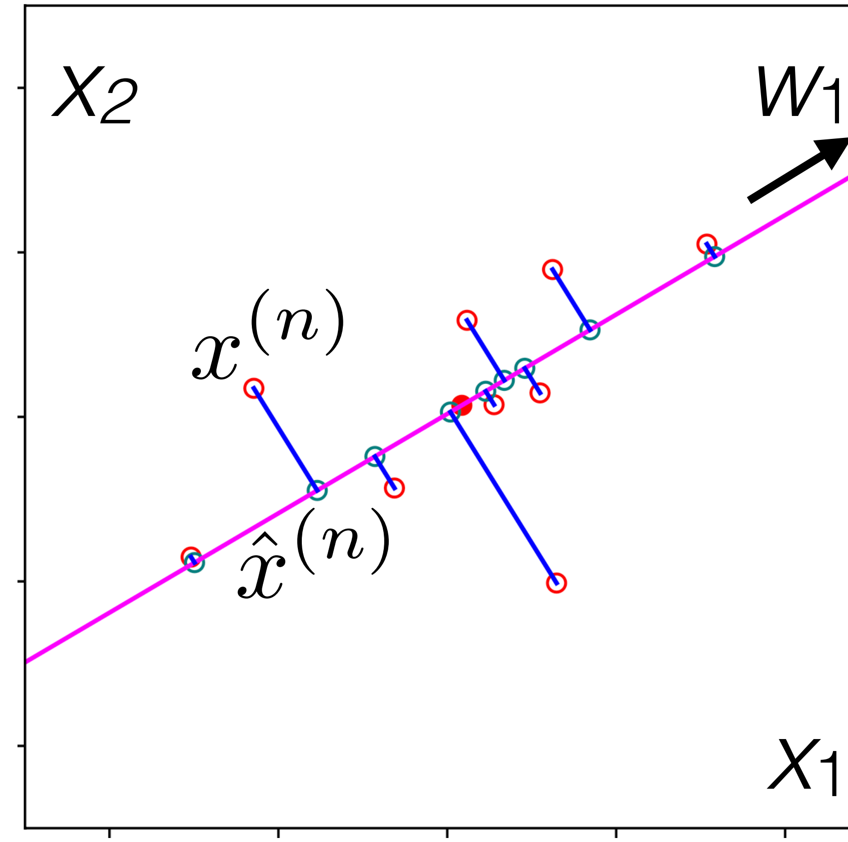

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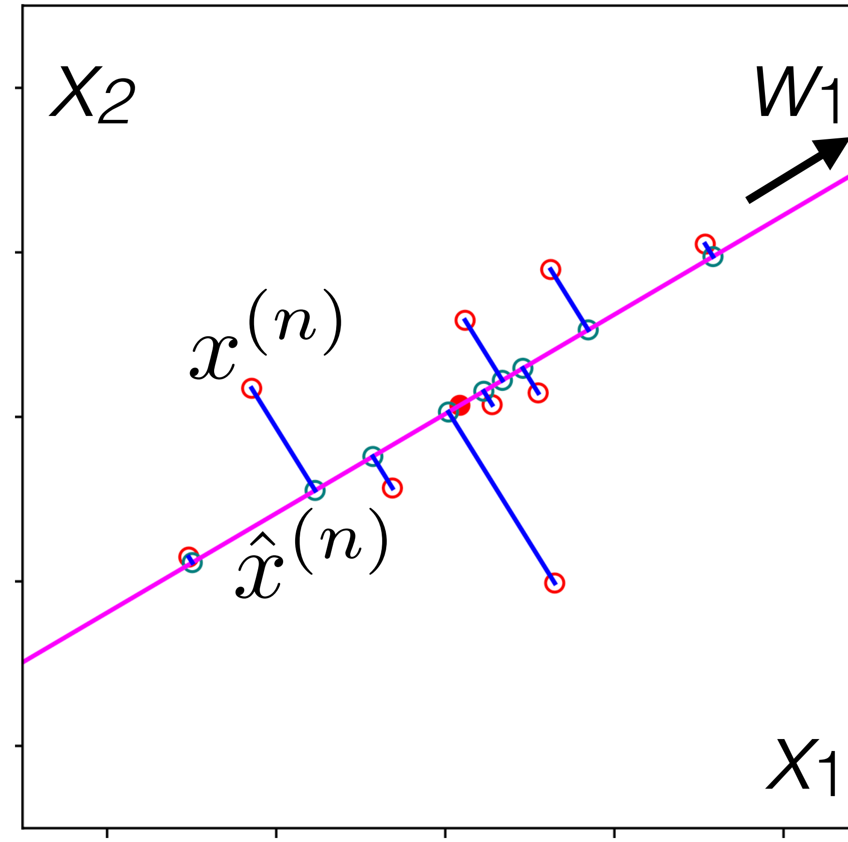

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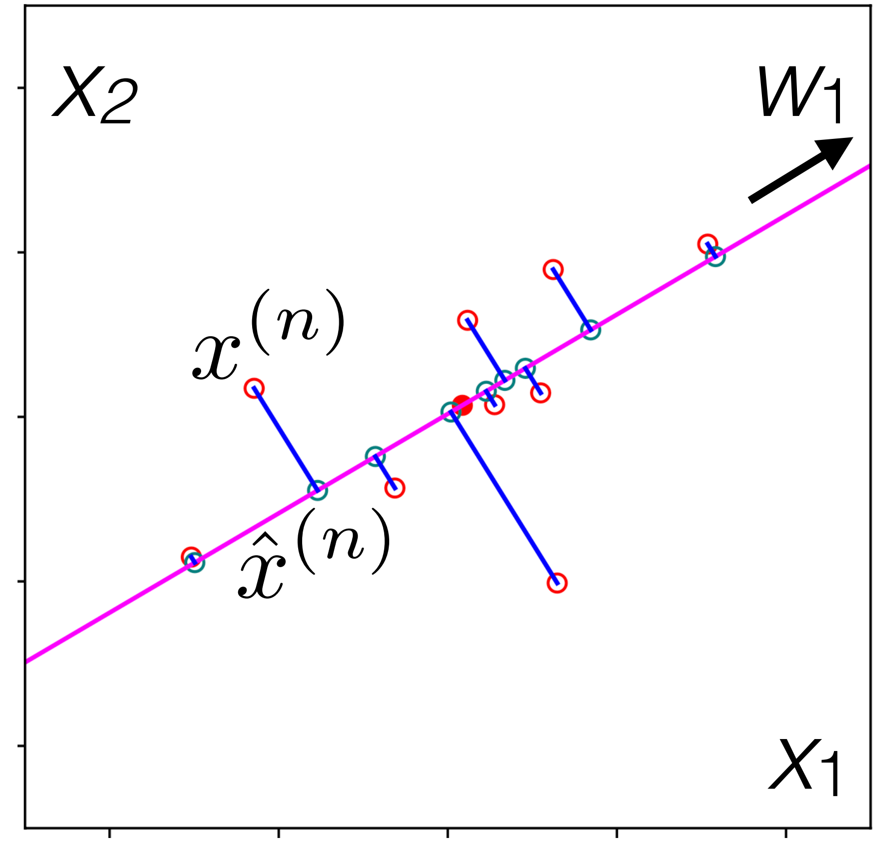

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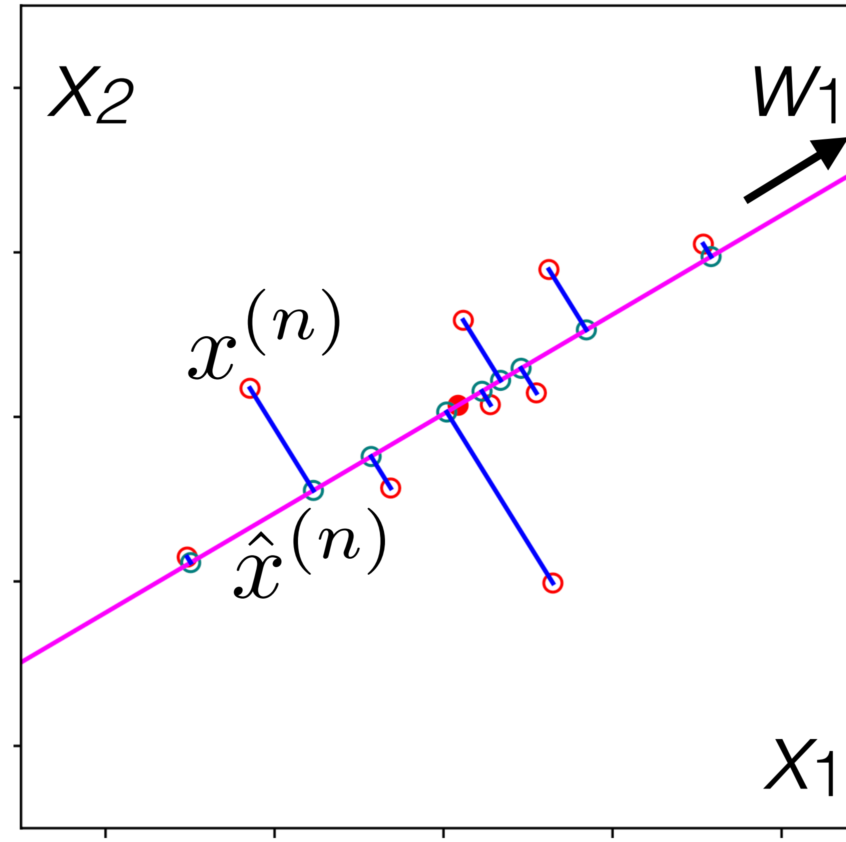

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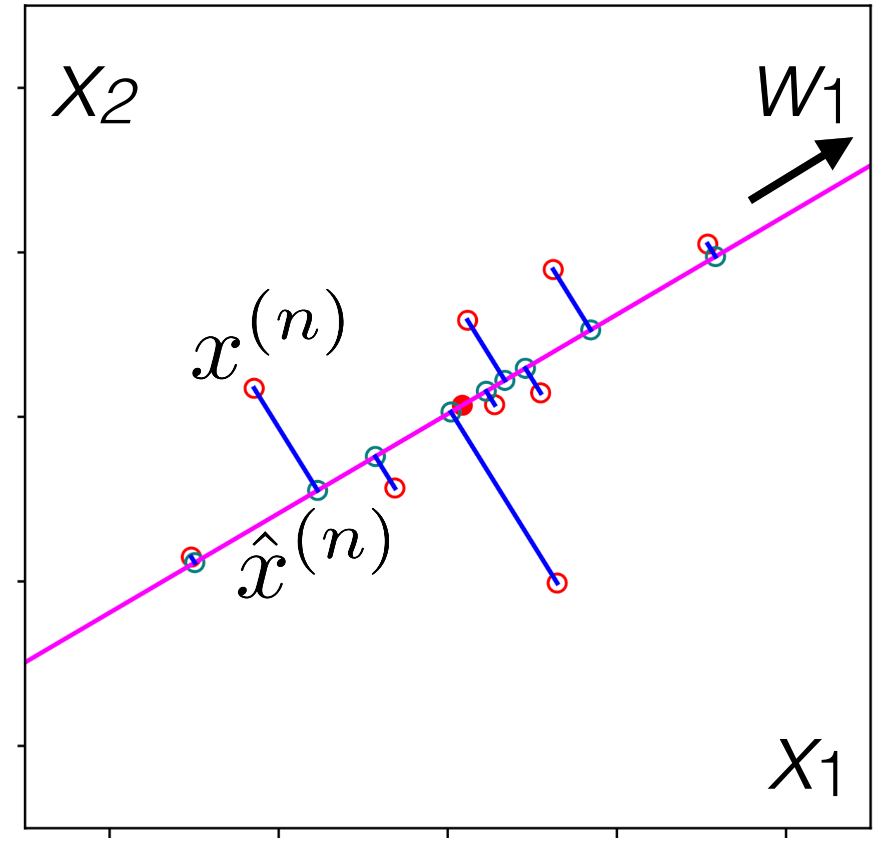

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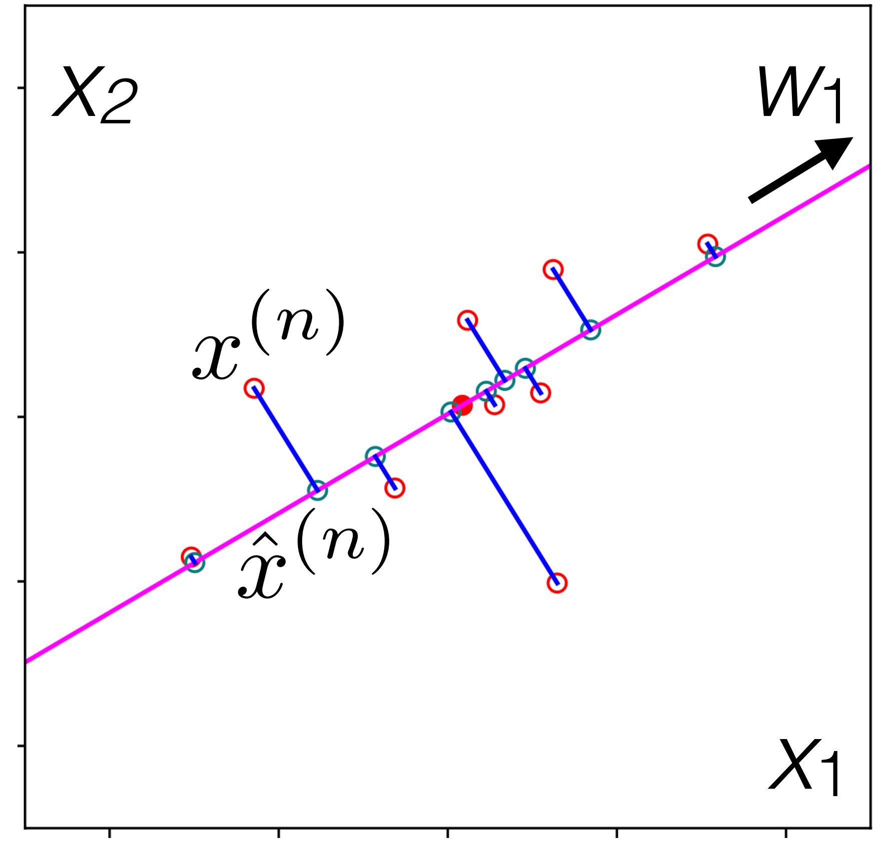

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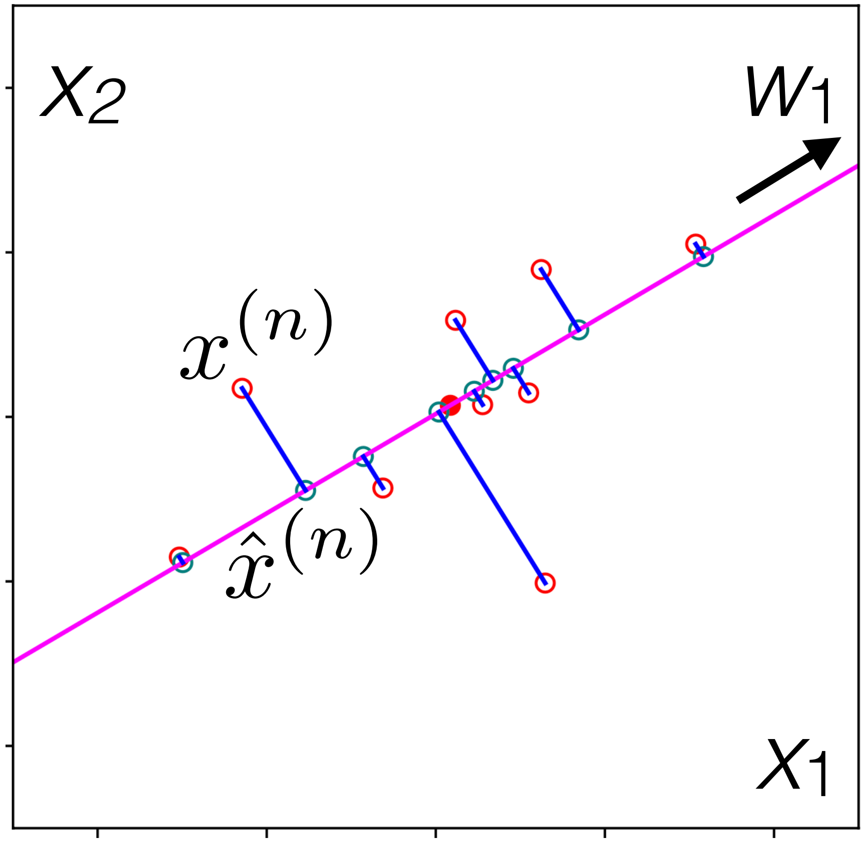

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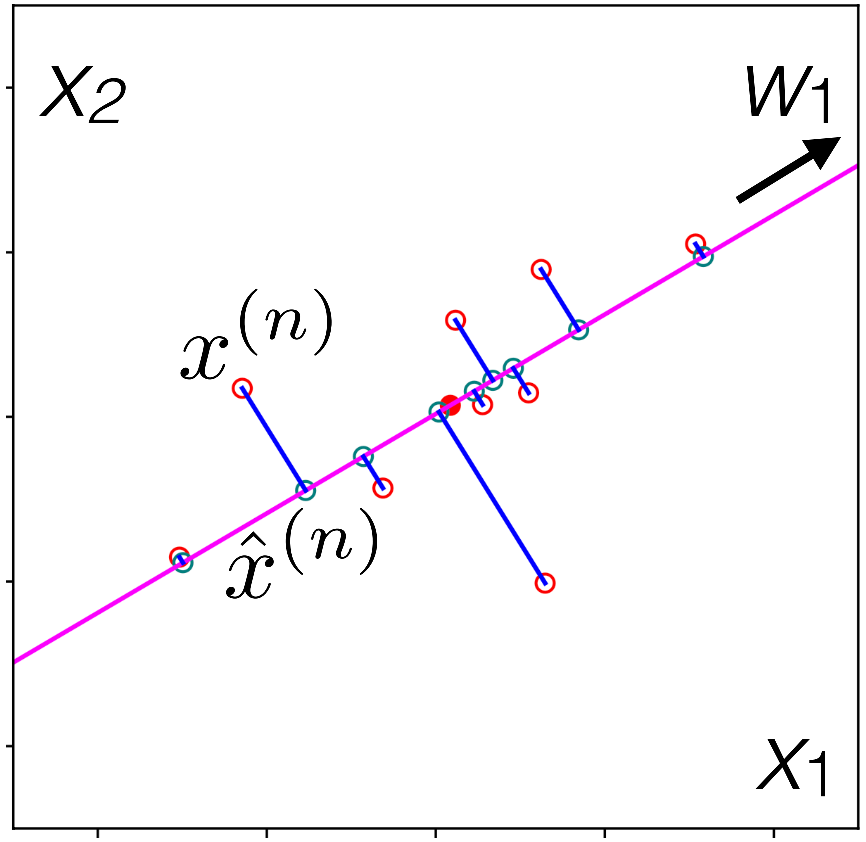

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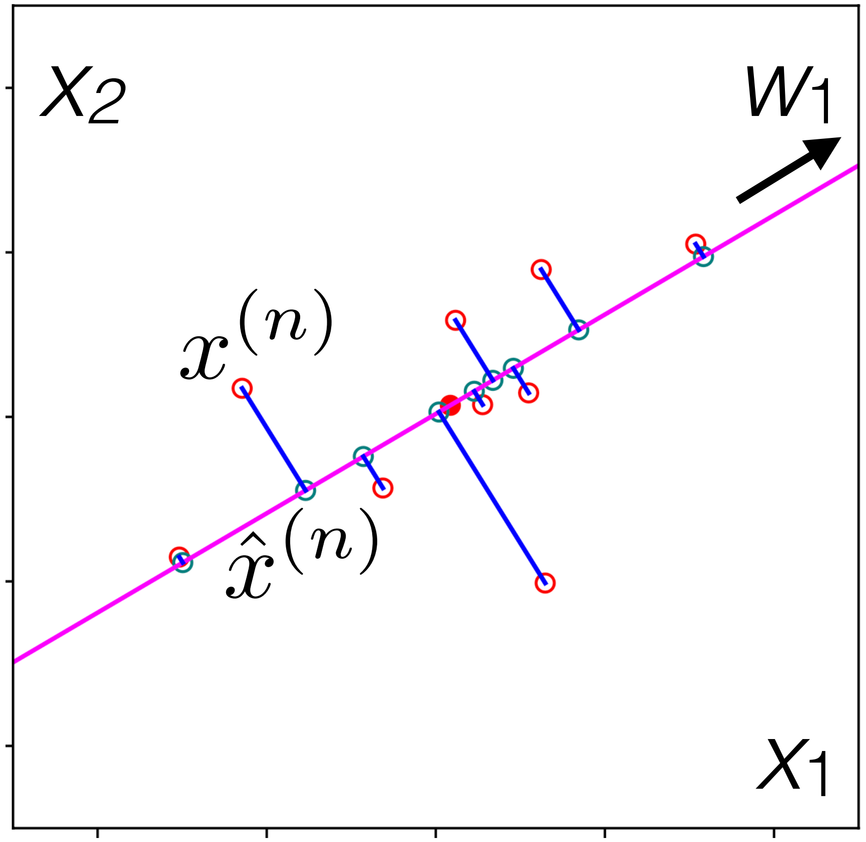

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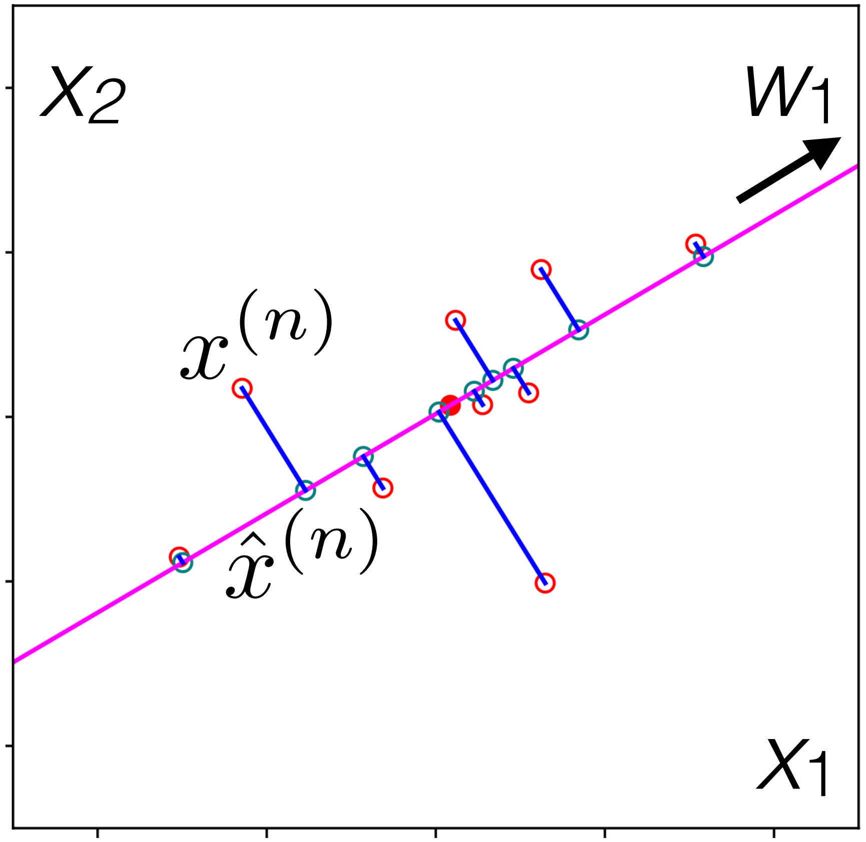

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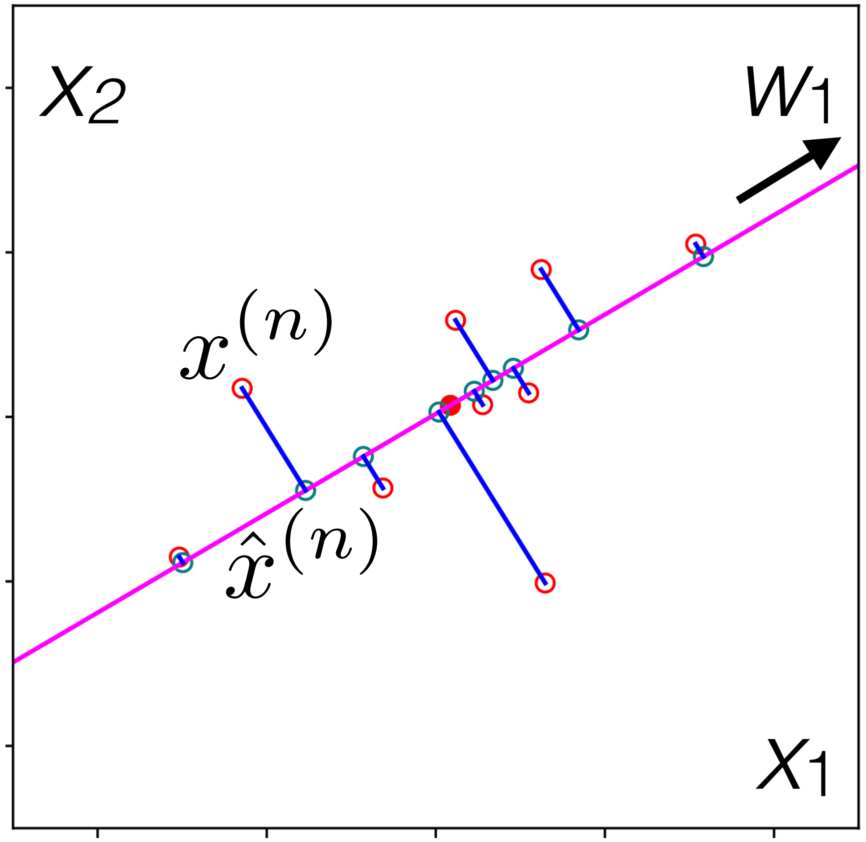

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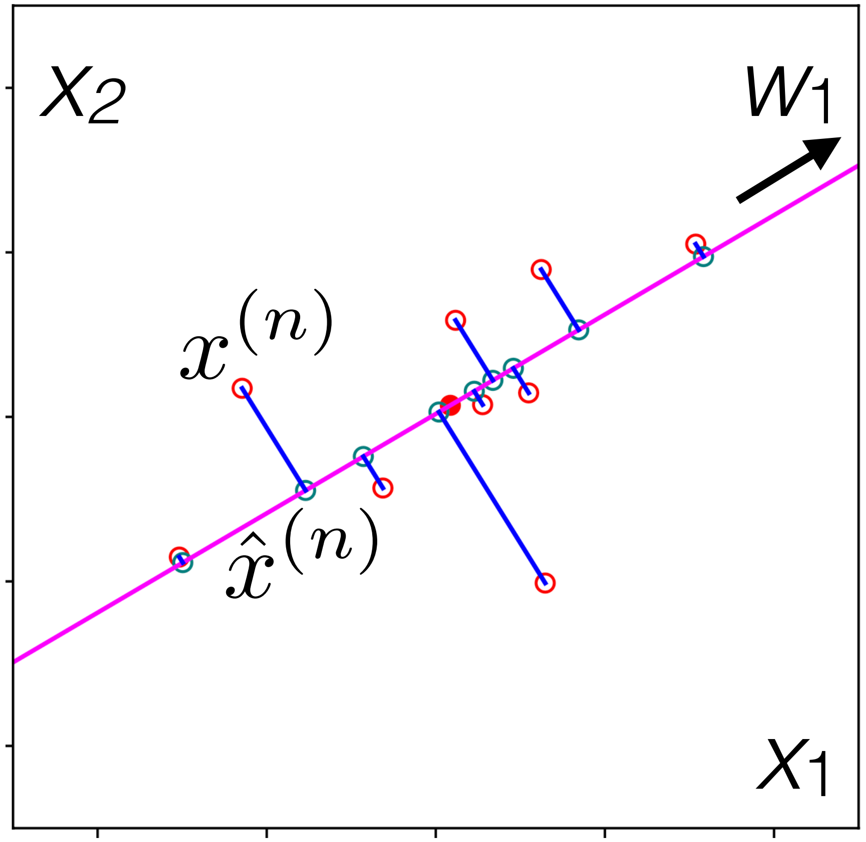

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$$w_\ell^\top w_\ell = 1 \quad \& \quad \forall k \in \{1, \ldots, \ell - 1\}, w_\ell^\top w_k = 0$$

References (1/1)

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