6.7900: Machine Learning
Lecture 7

Lecture start: Tues/Thurs 2:35pm

Who's speaking today? Prof. Tamara Broderick

Course website: gradml.mit.edu

Questions? Ask here or on piazza.com/mit/fall2024/67900/

Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Time

I. Visualizing regression
II. Uncertainty
III. Ridge regression
IV. More flexible/complex features

Today

I. Evaluation for supervised learning
II. Choosing hyperparams
III. Validation data and empirical risk

More complex features

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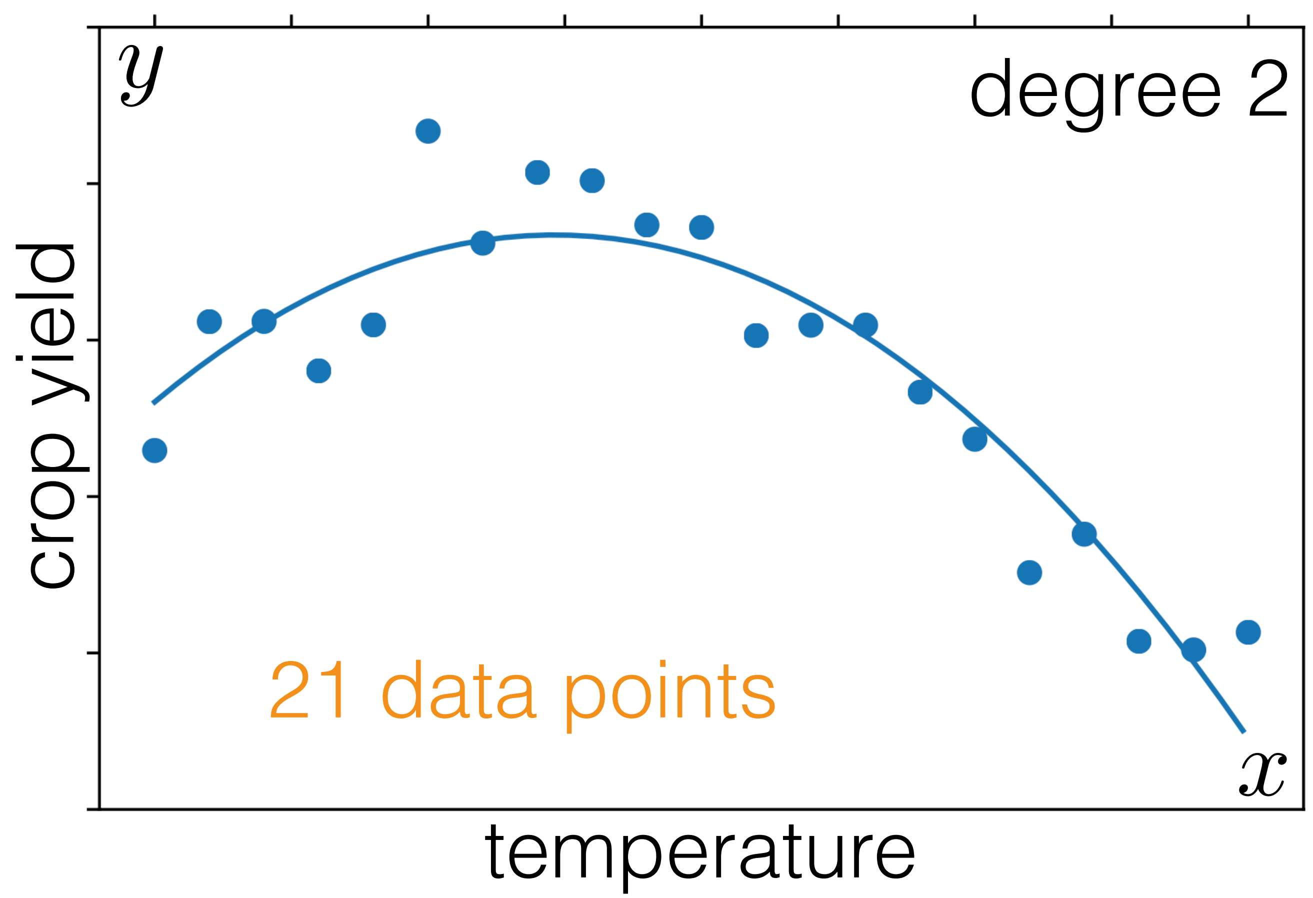

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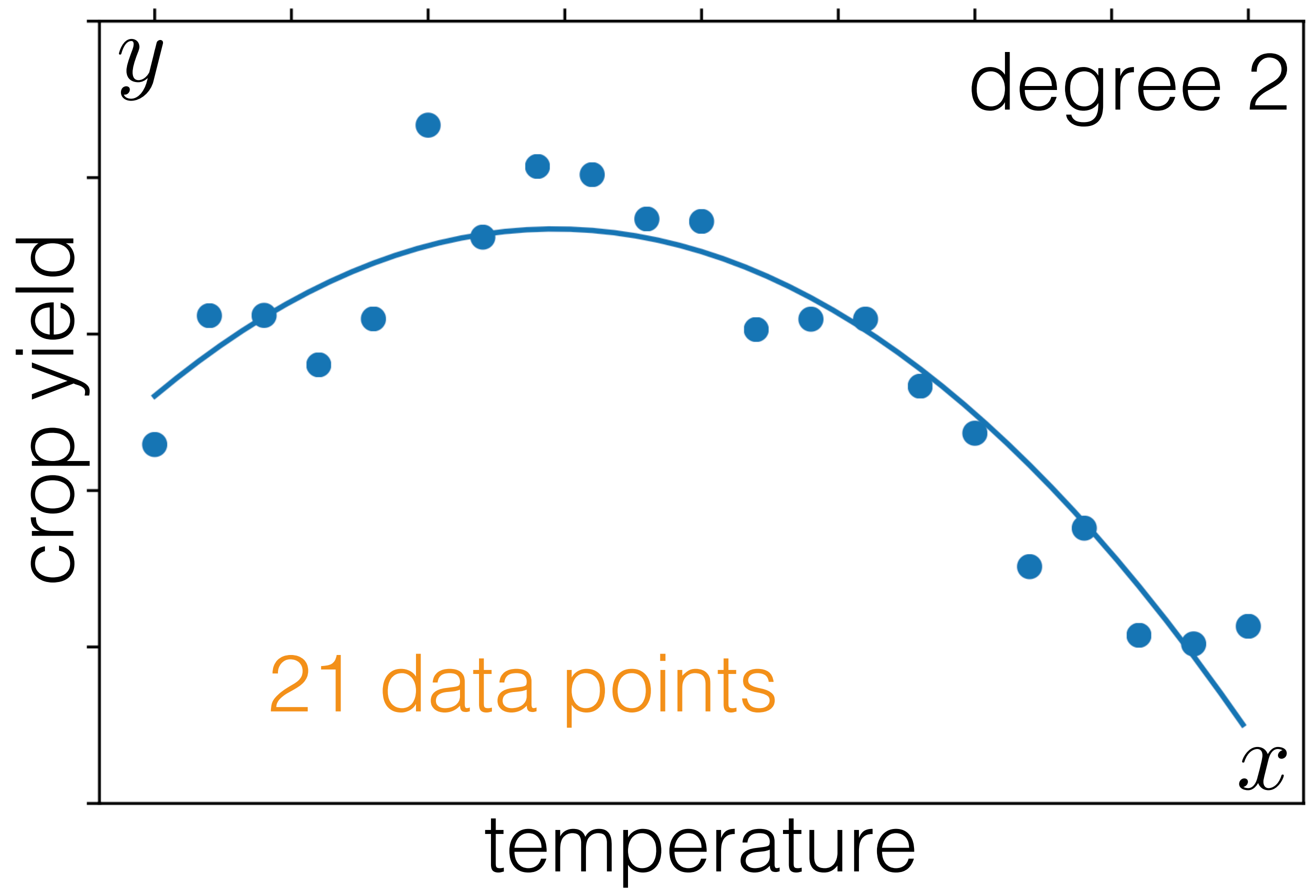

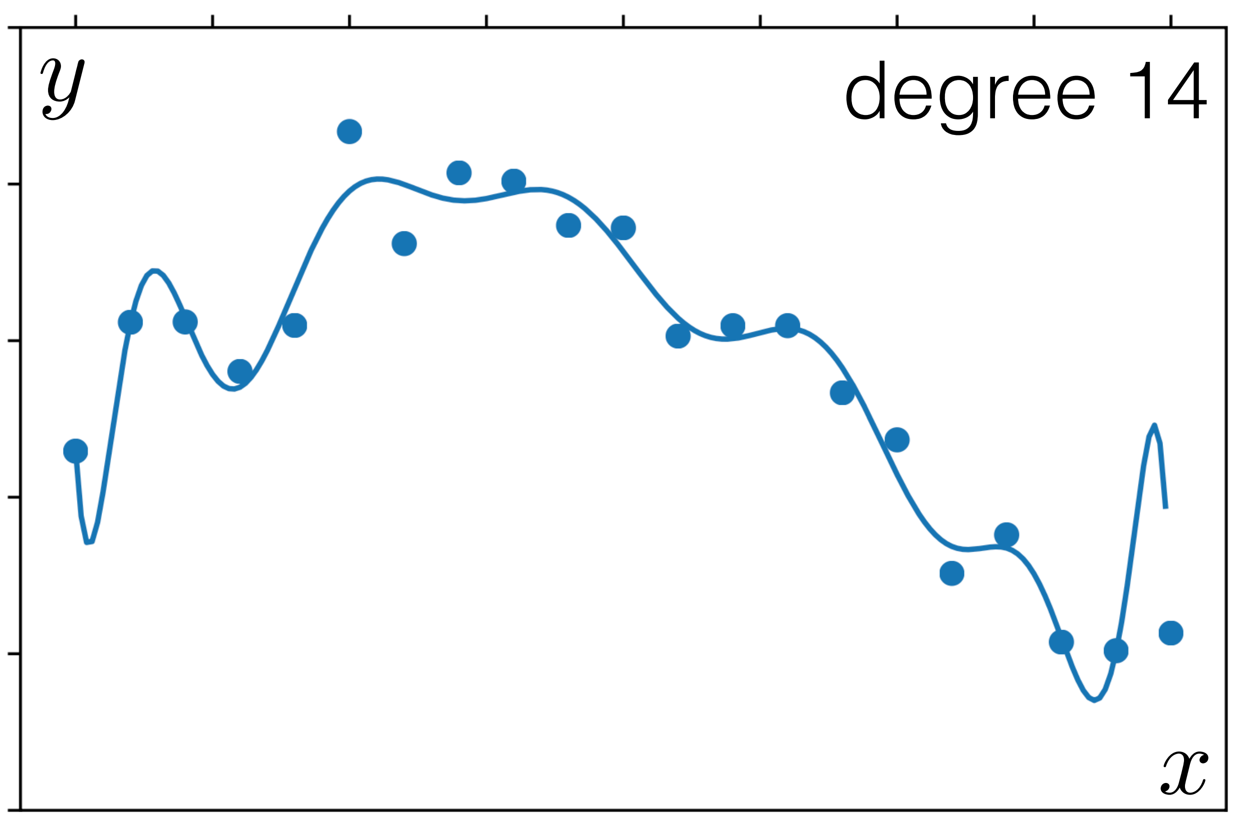

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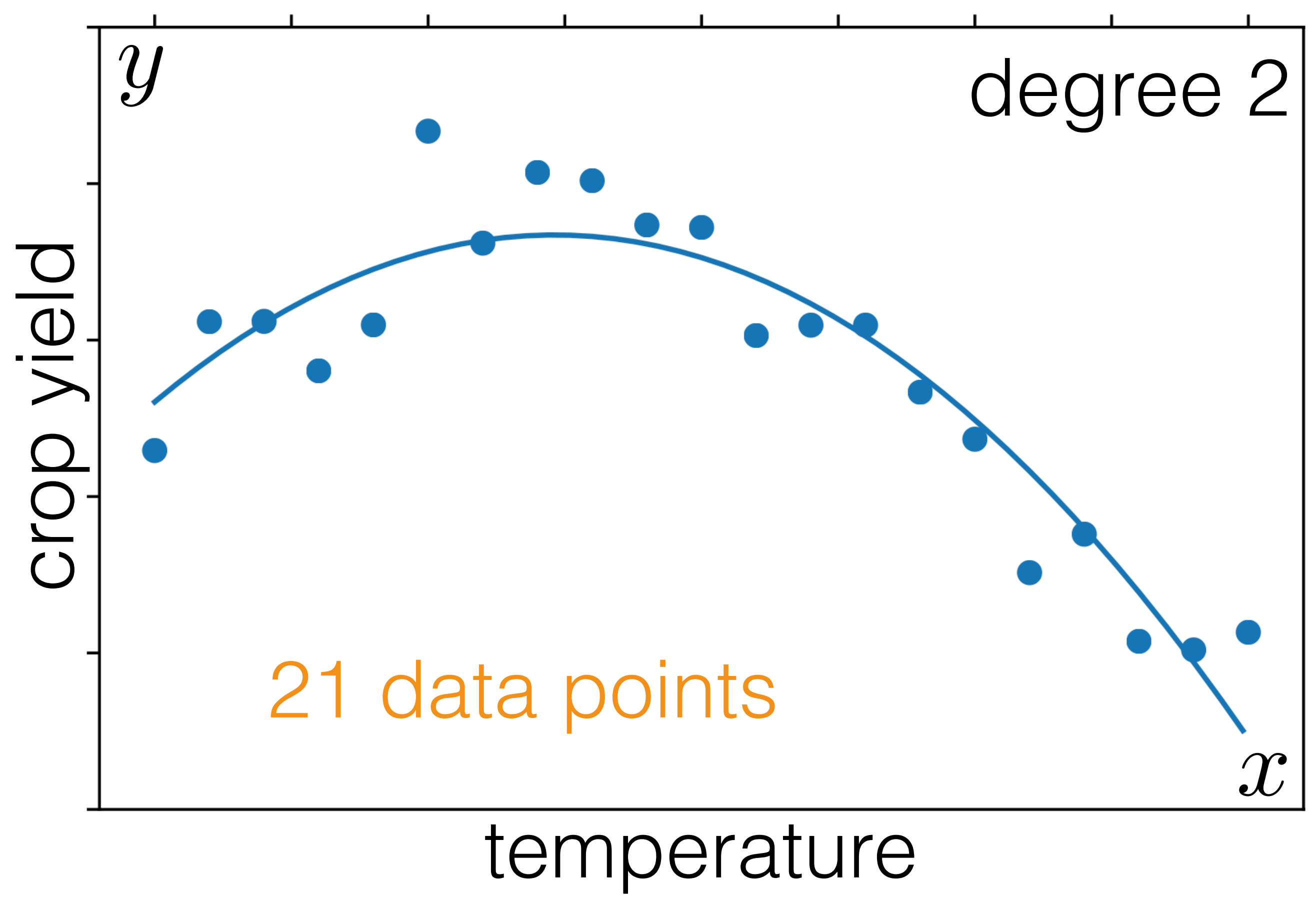

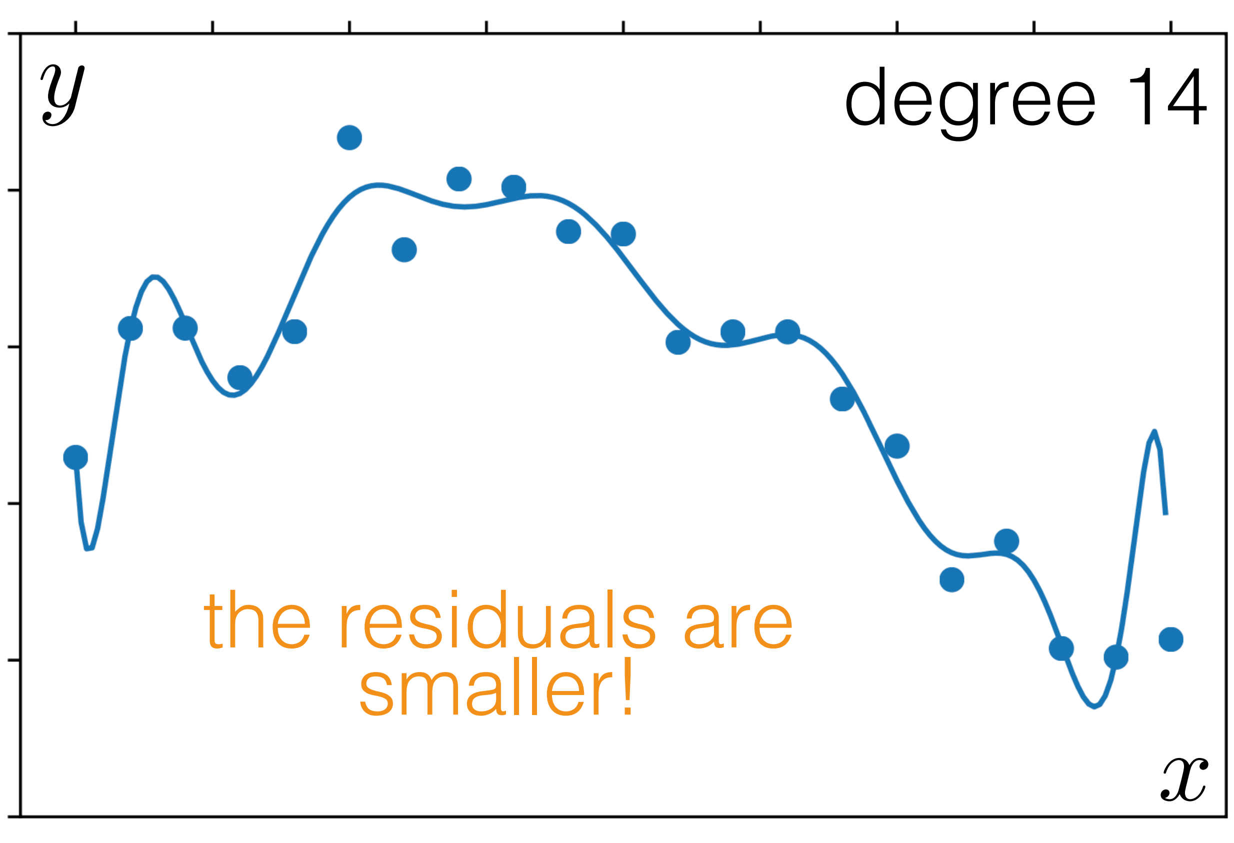

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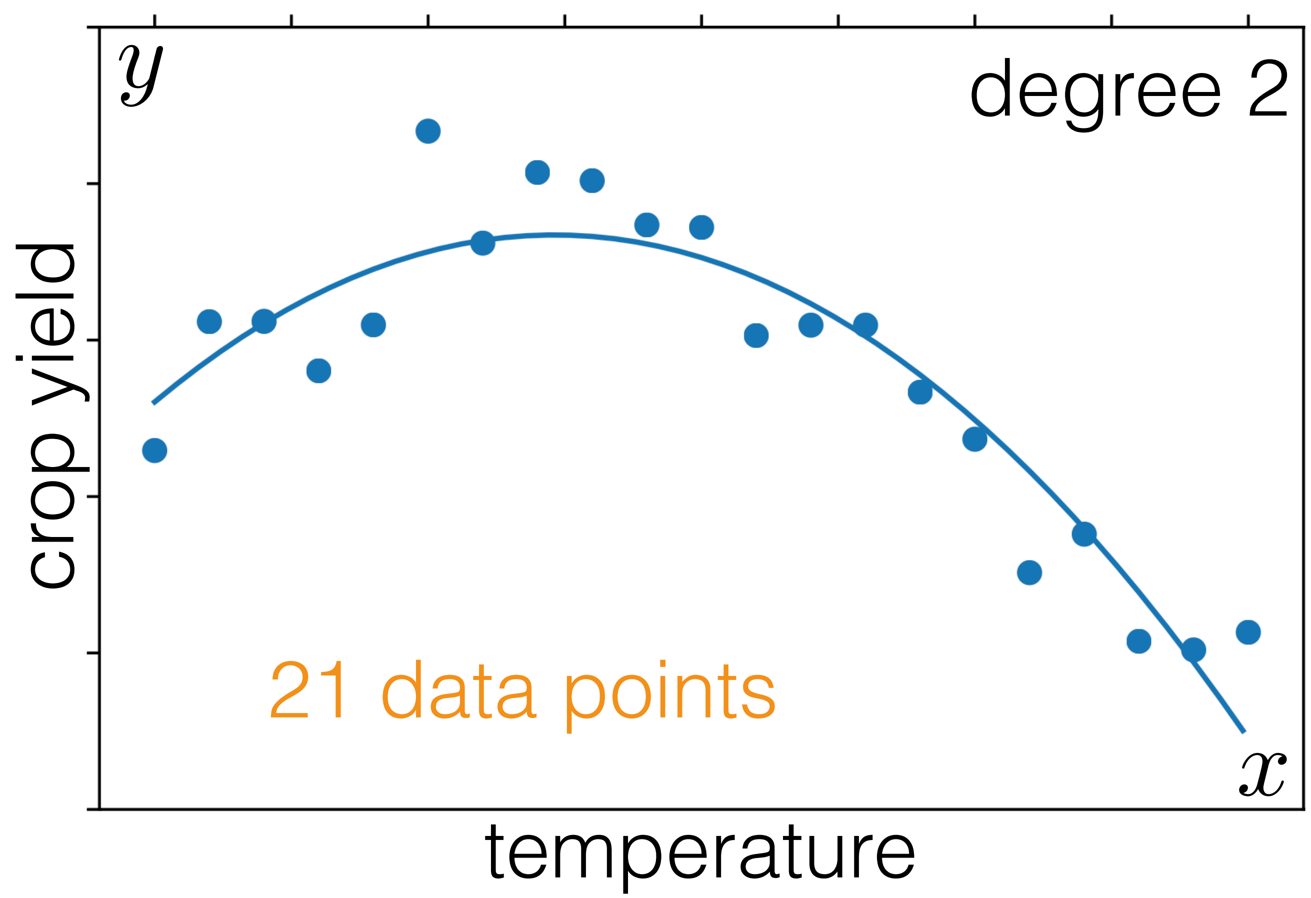

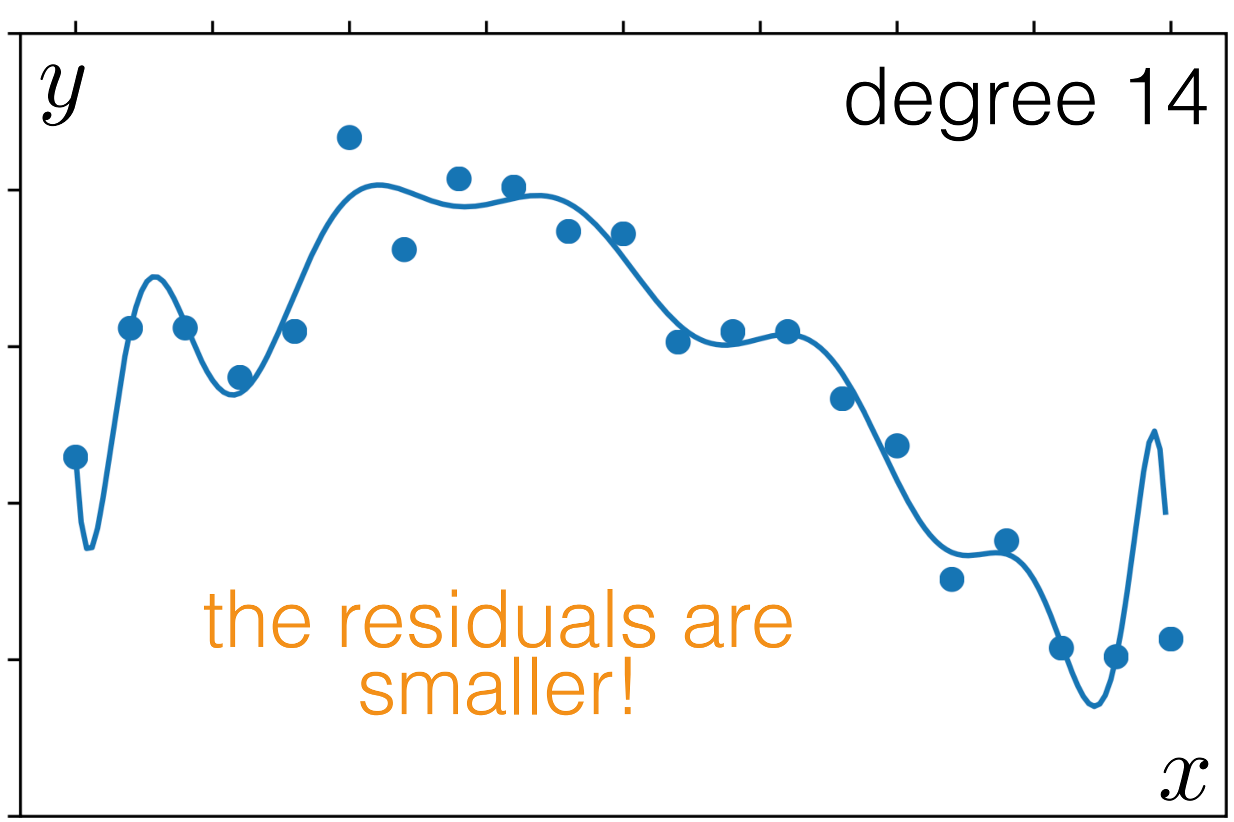

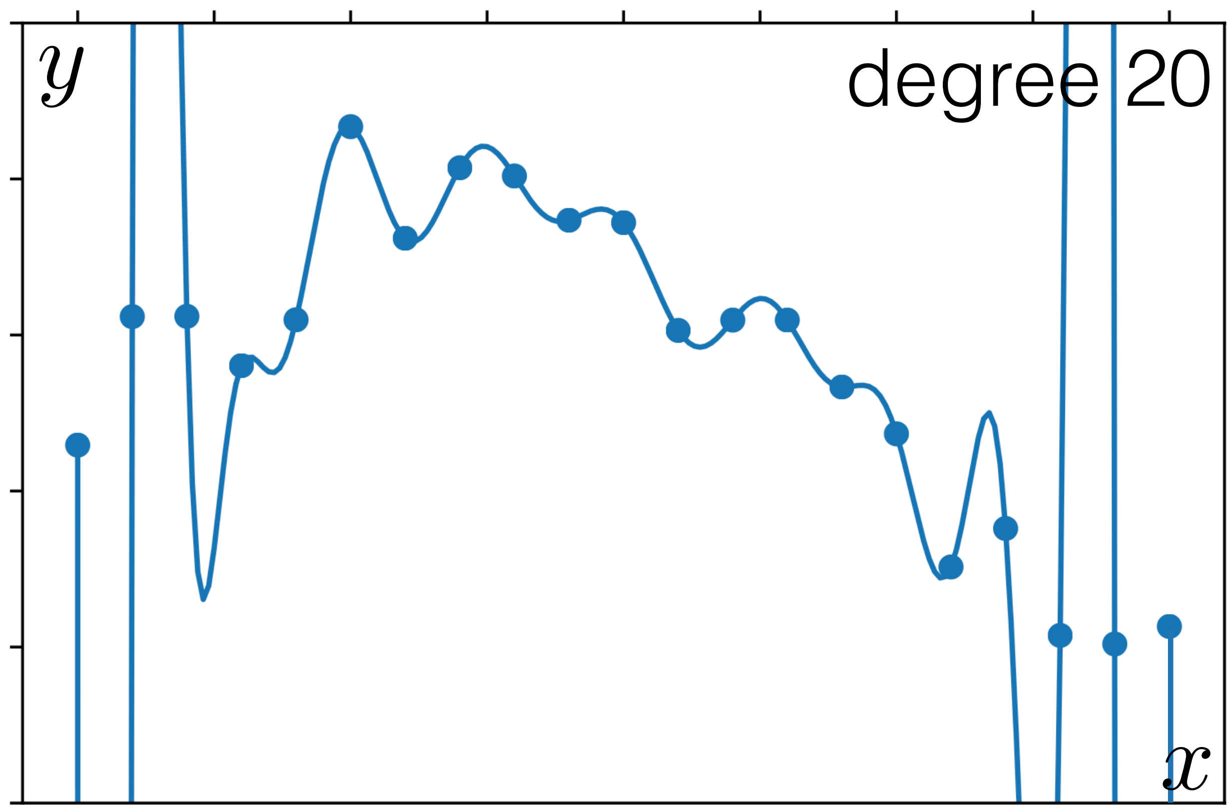

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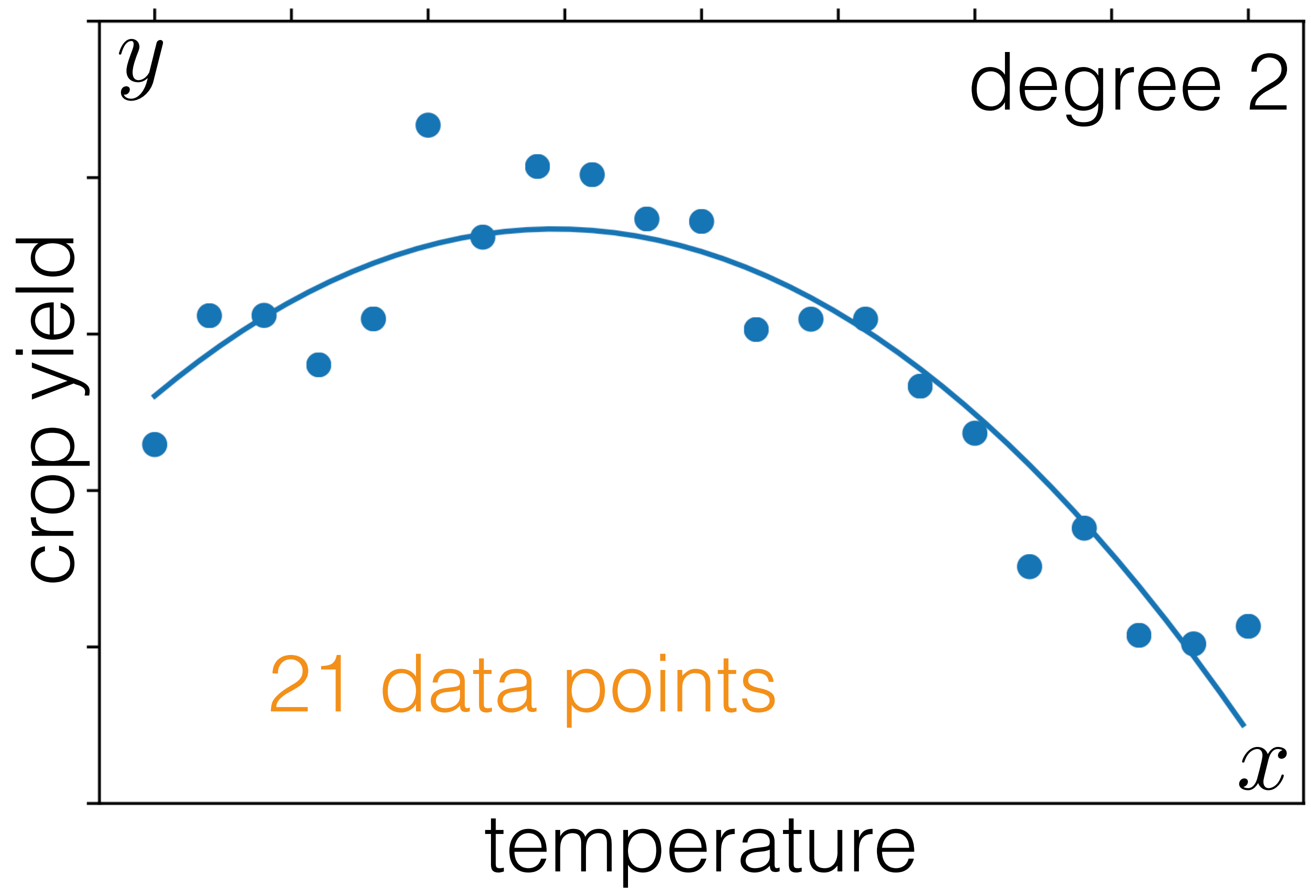

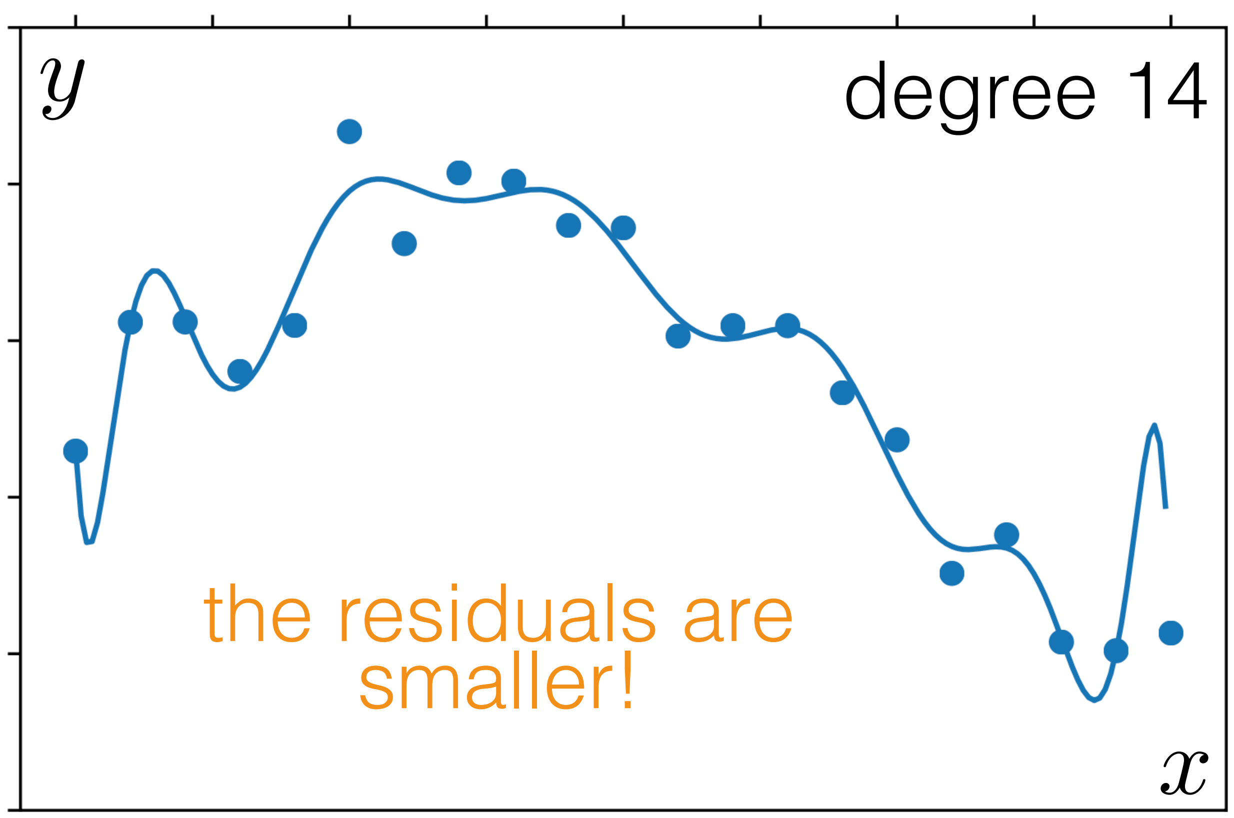

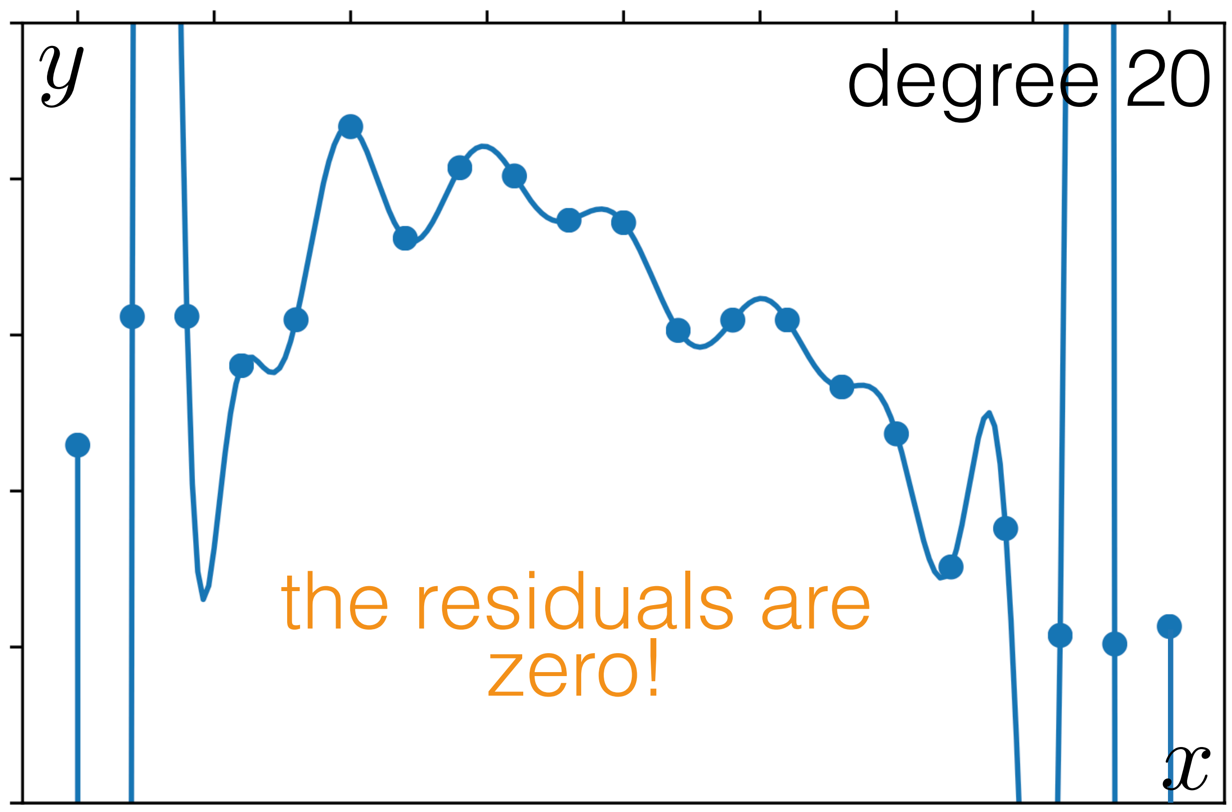

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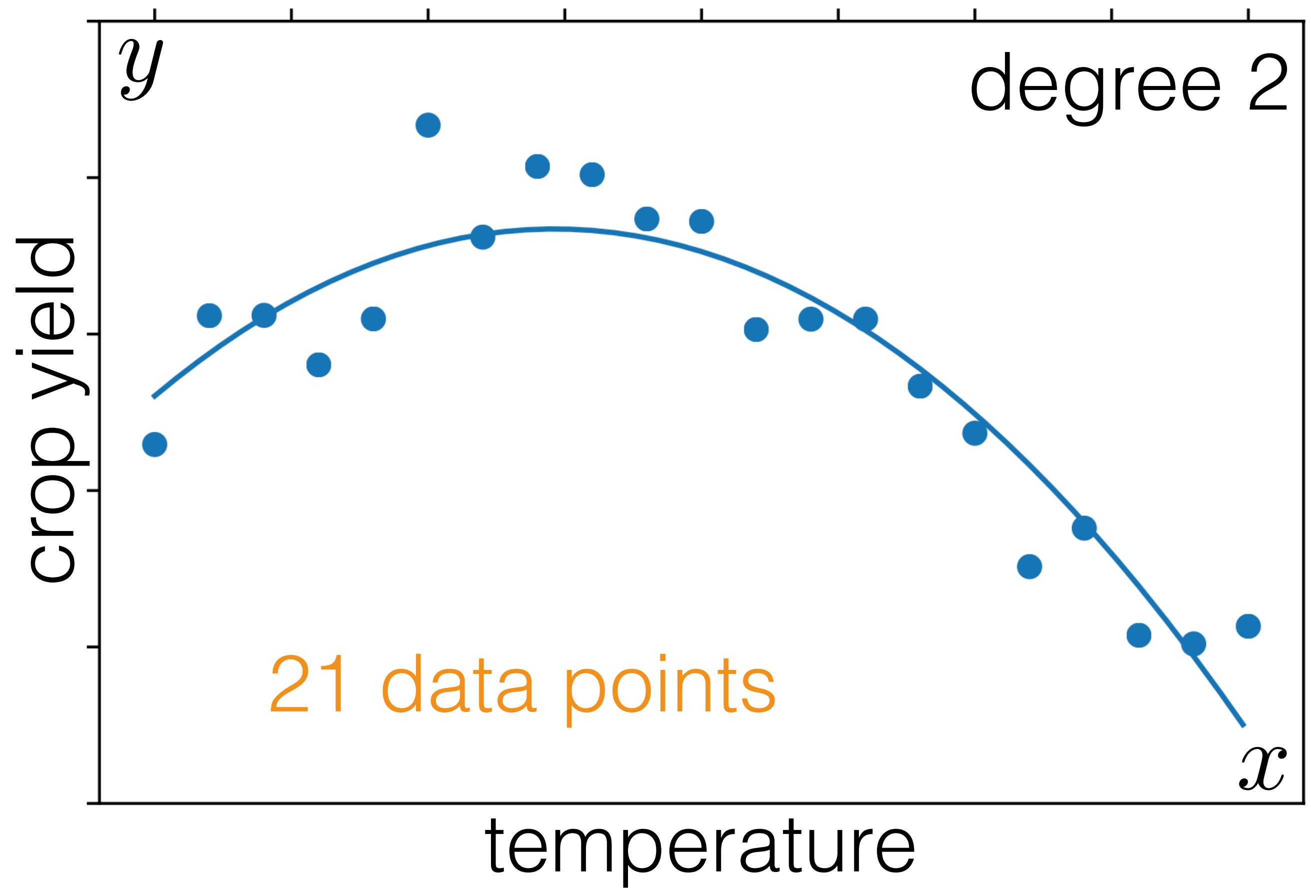
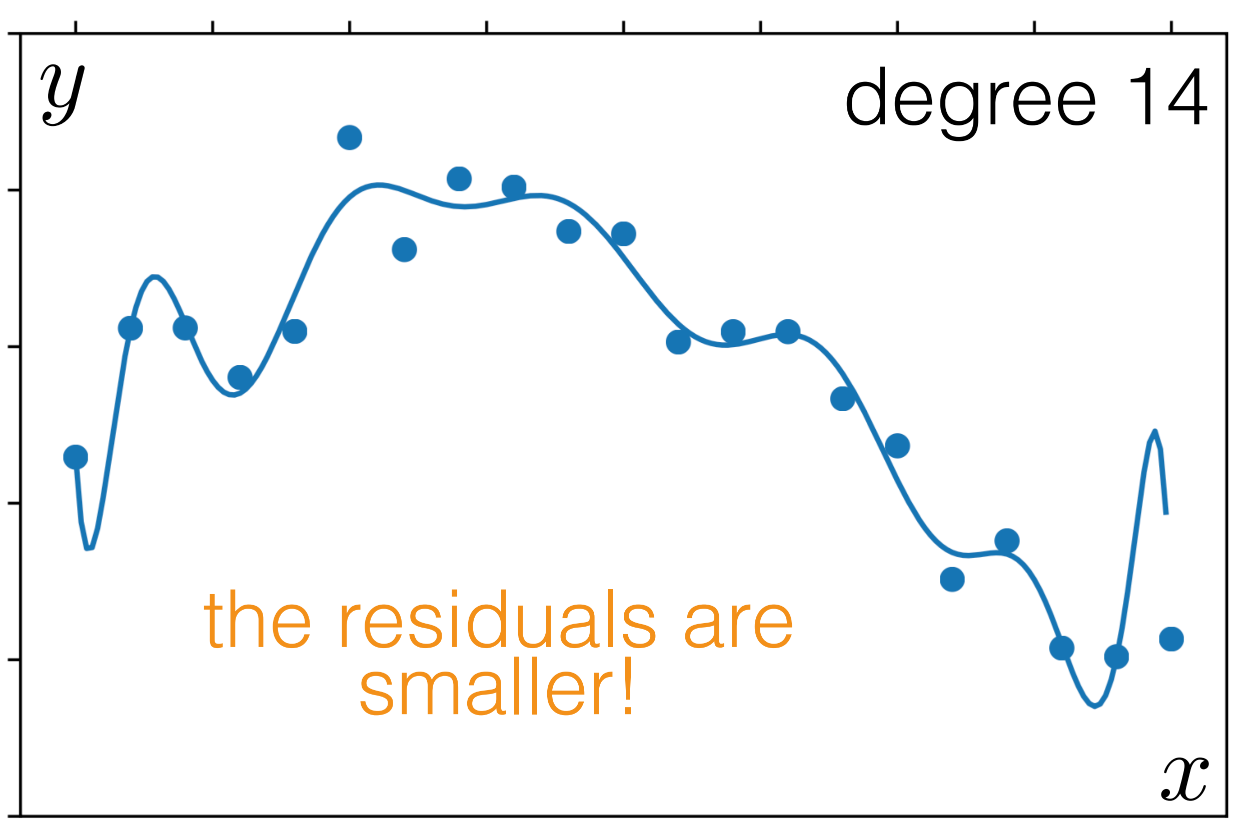
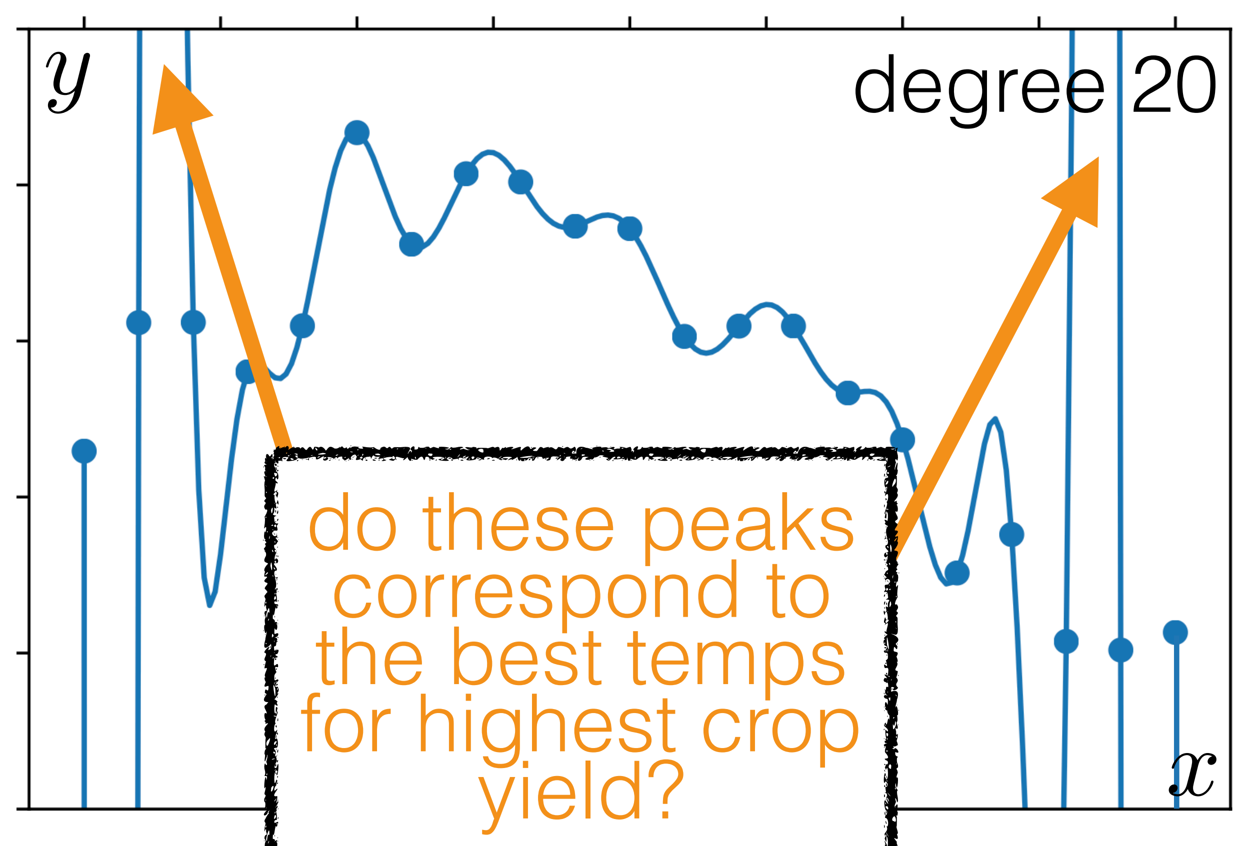
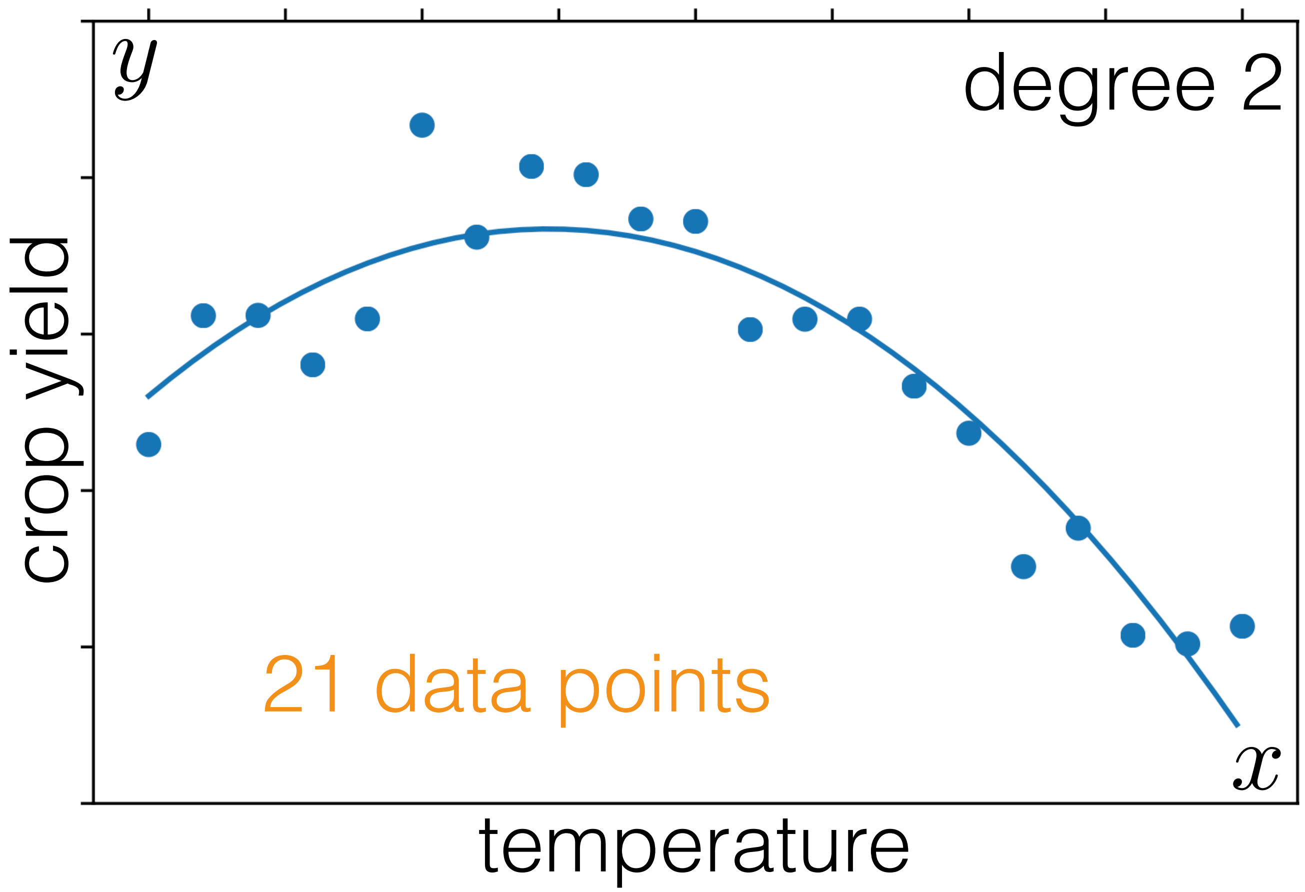


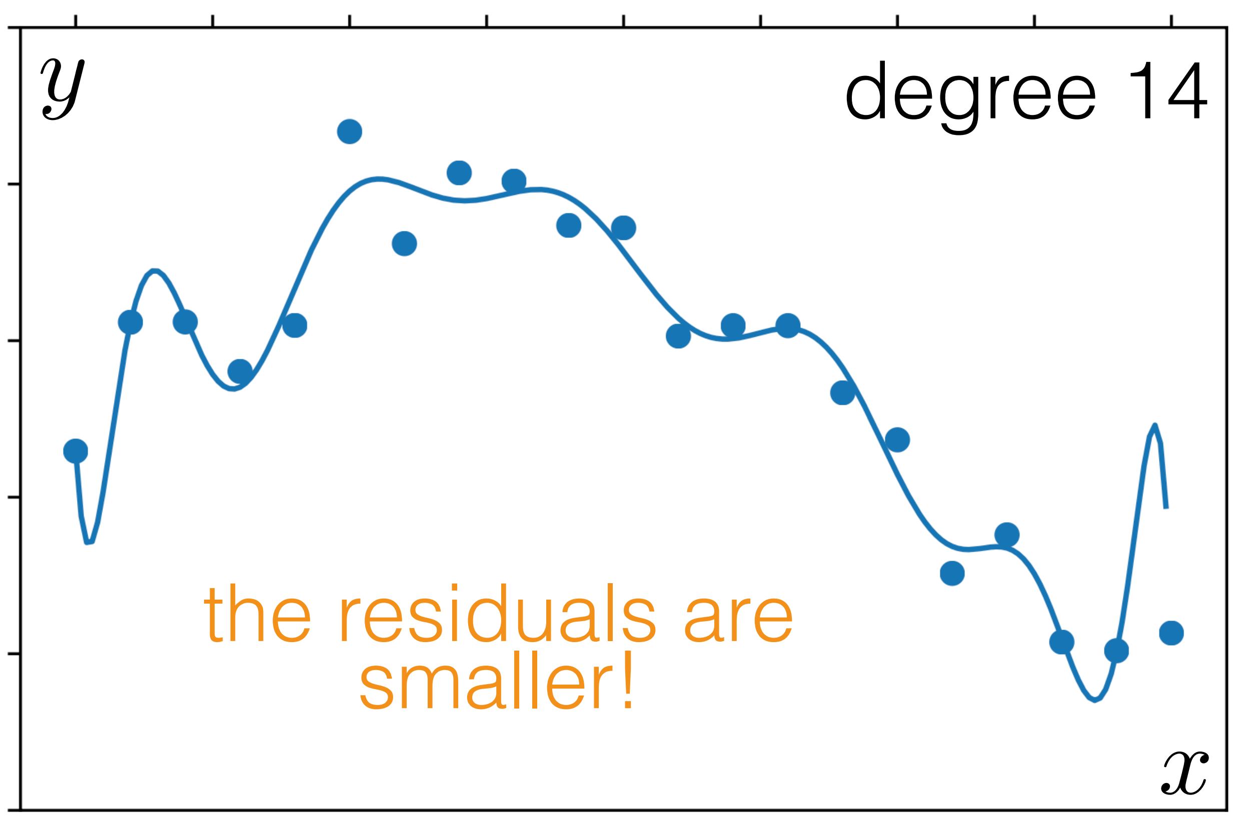


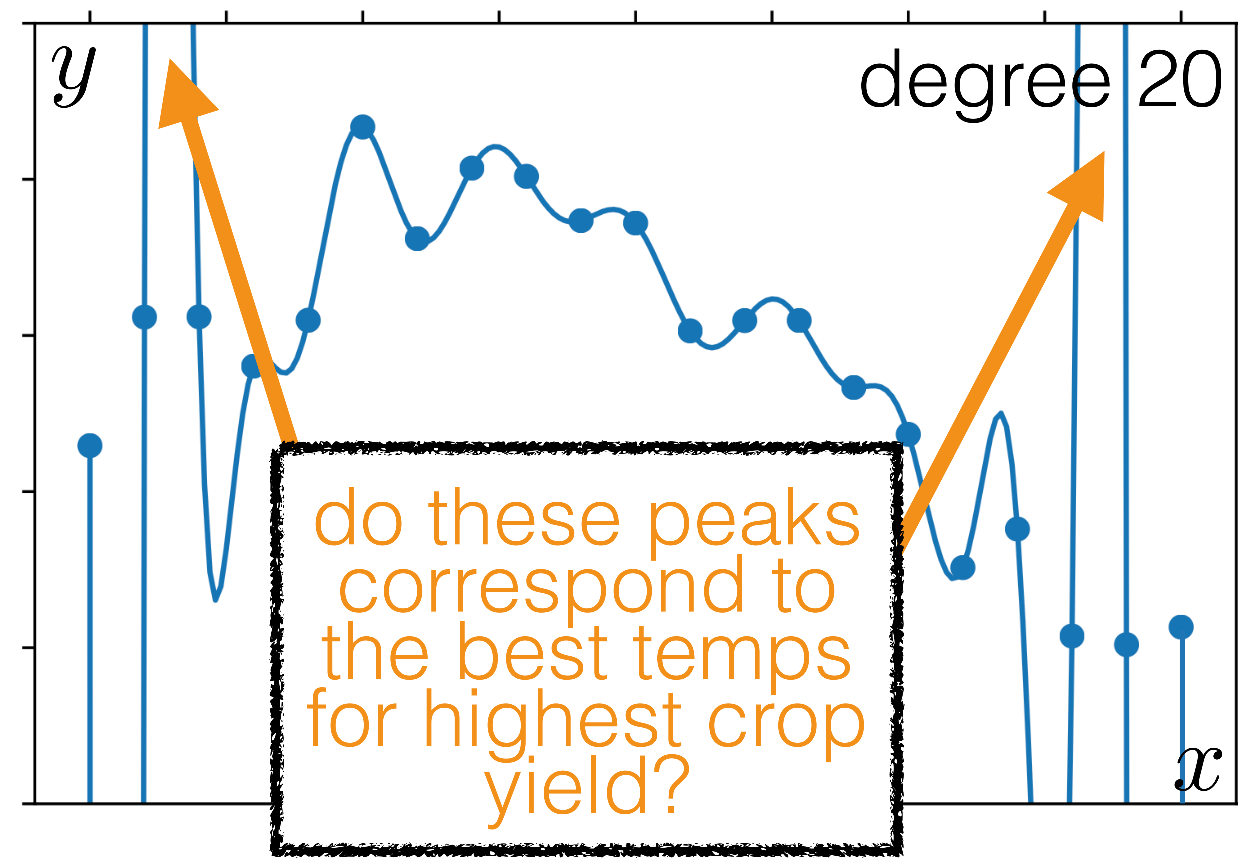


More complex features

- Is it always better to use more complex/flexible feature sets? OLS with polynomial features up to specified degree
- E.g. deg 2: $\phi(x) = [1, x, x^2]^\top$
- Always plot your data!
- Harder in higher dimensions







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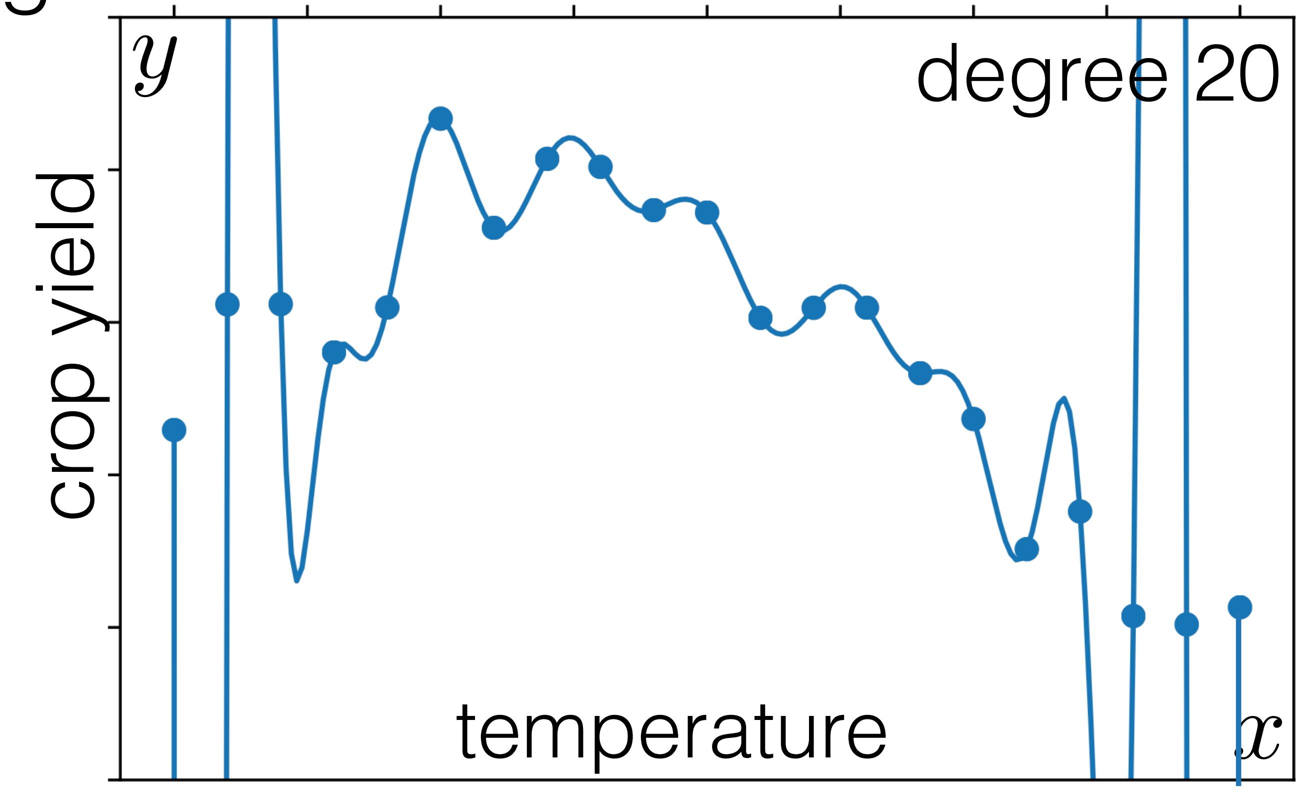

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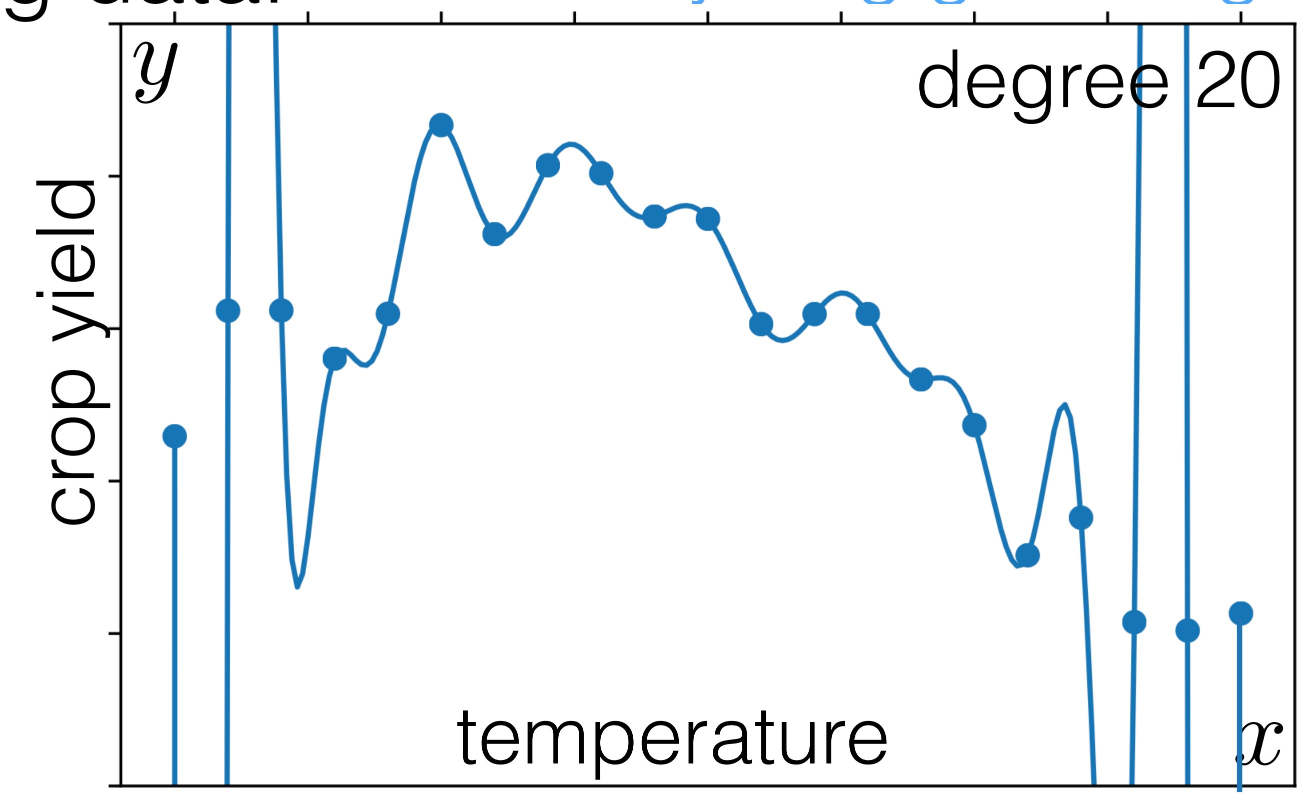

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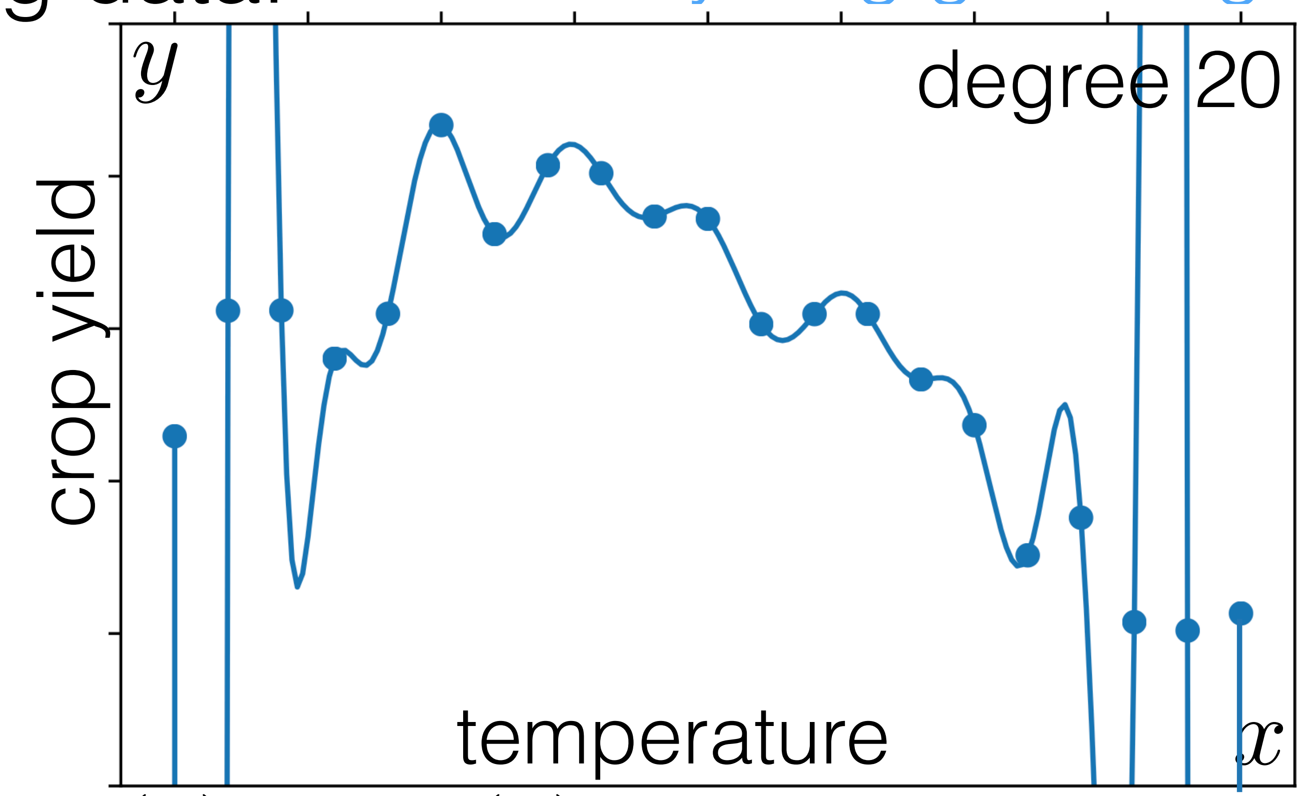

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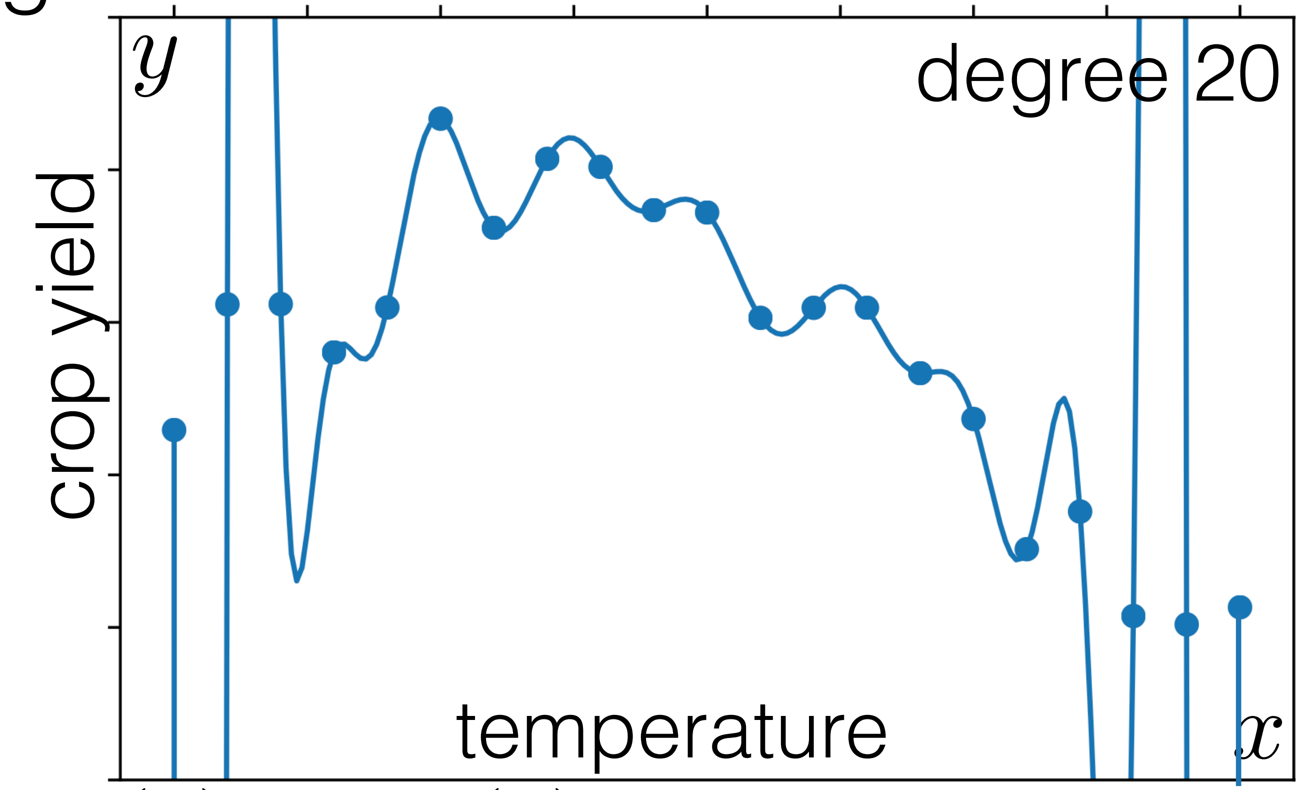

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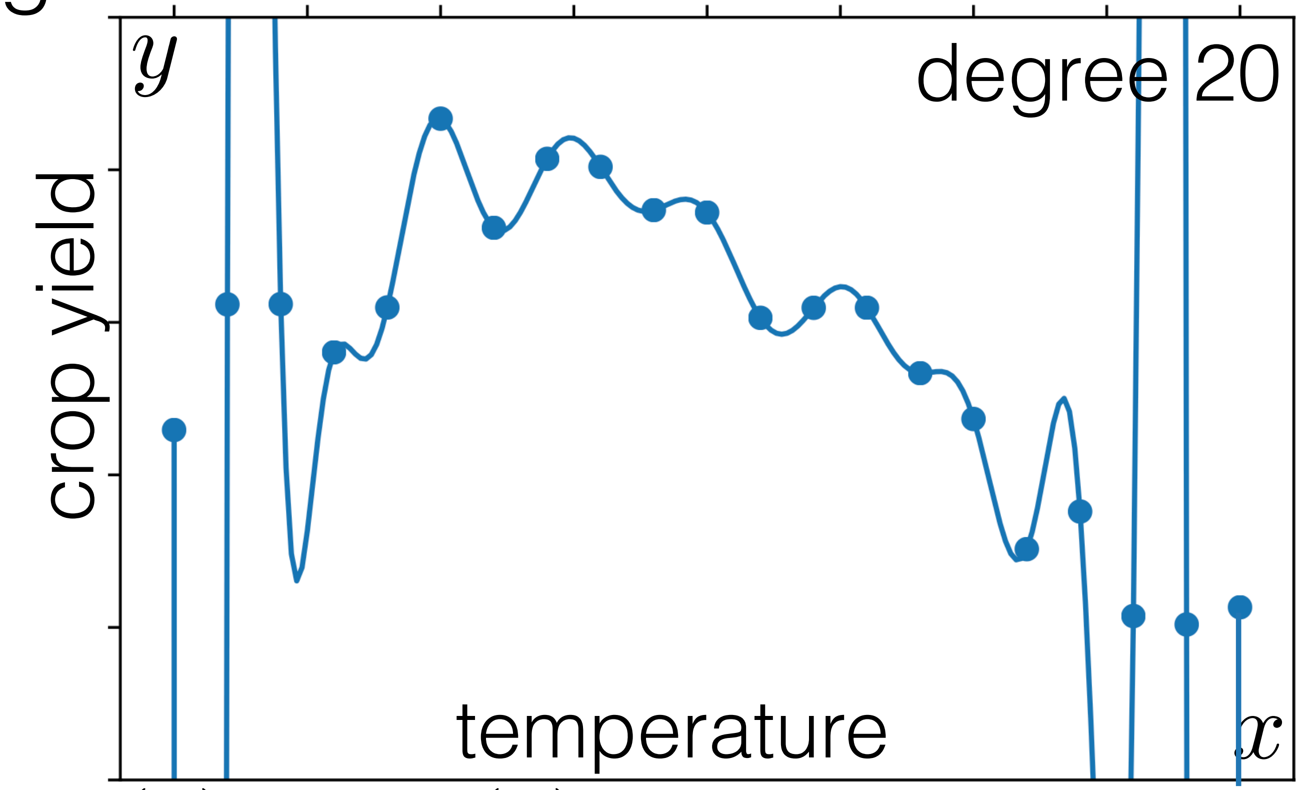

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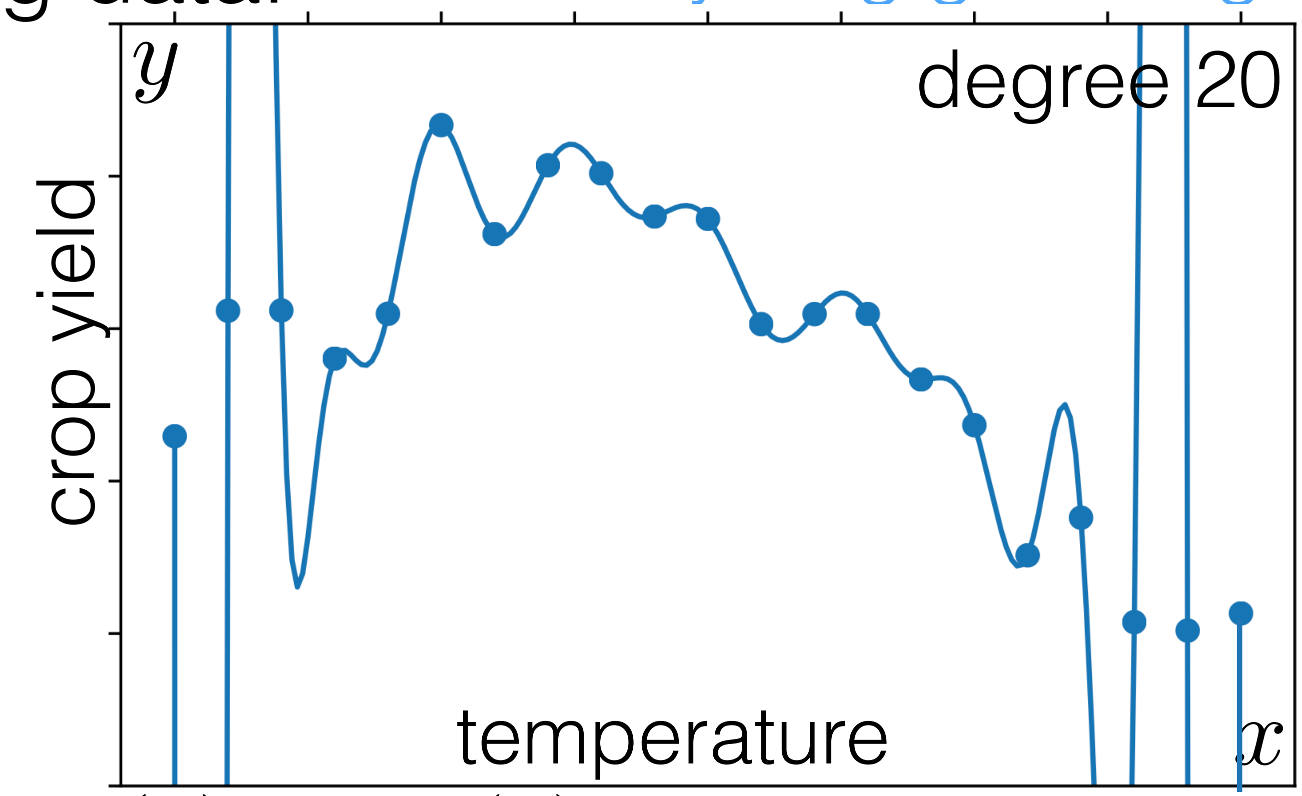

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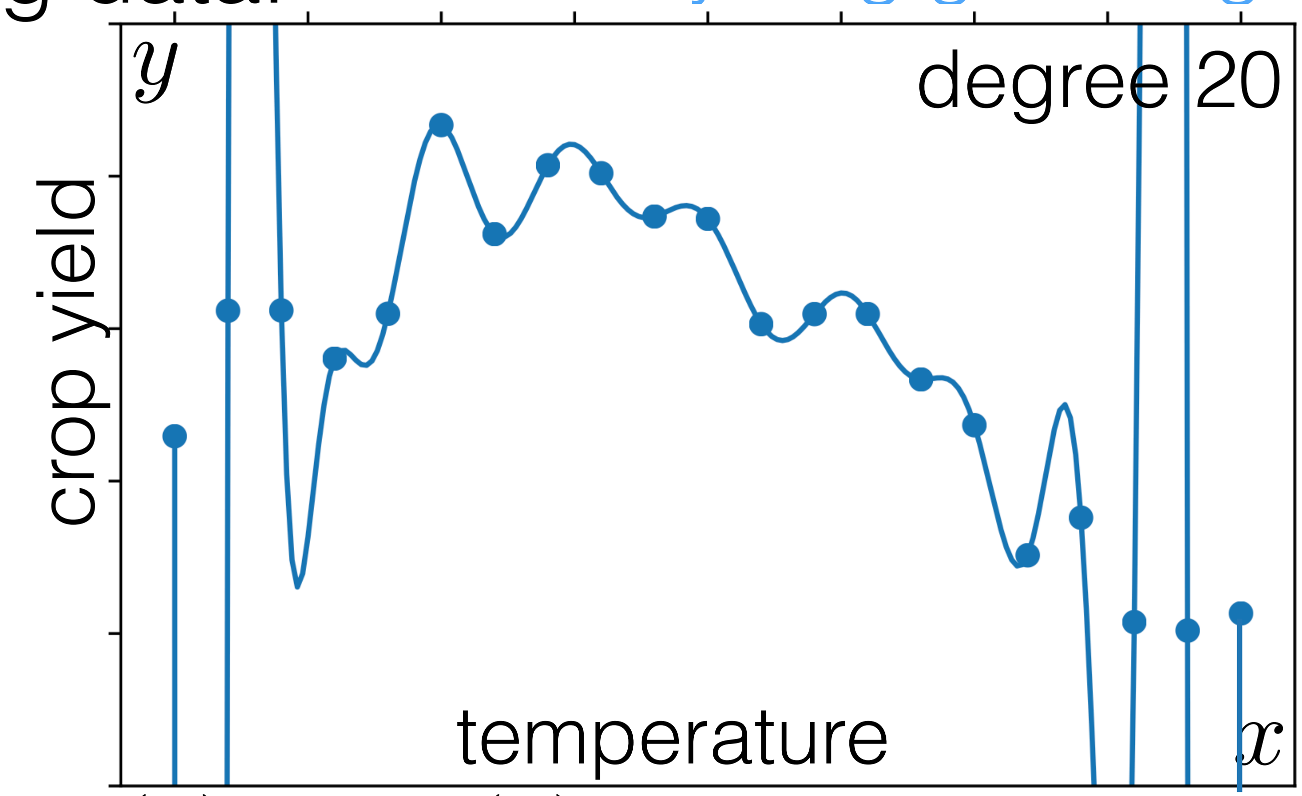

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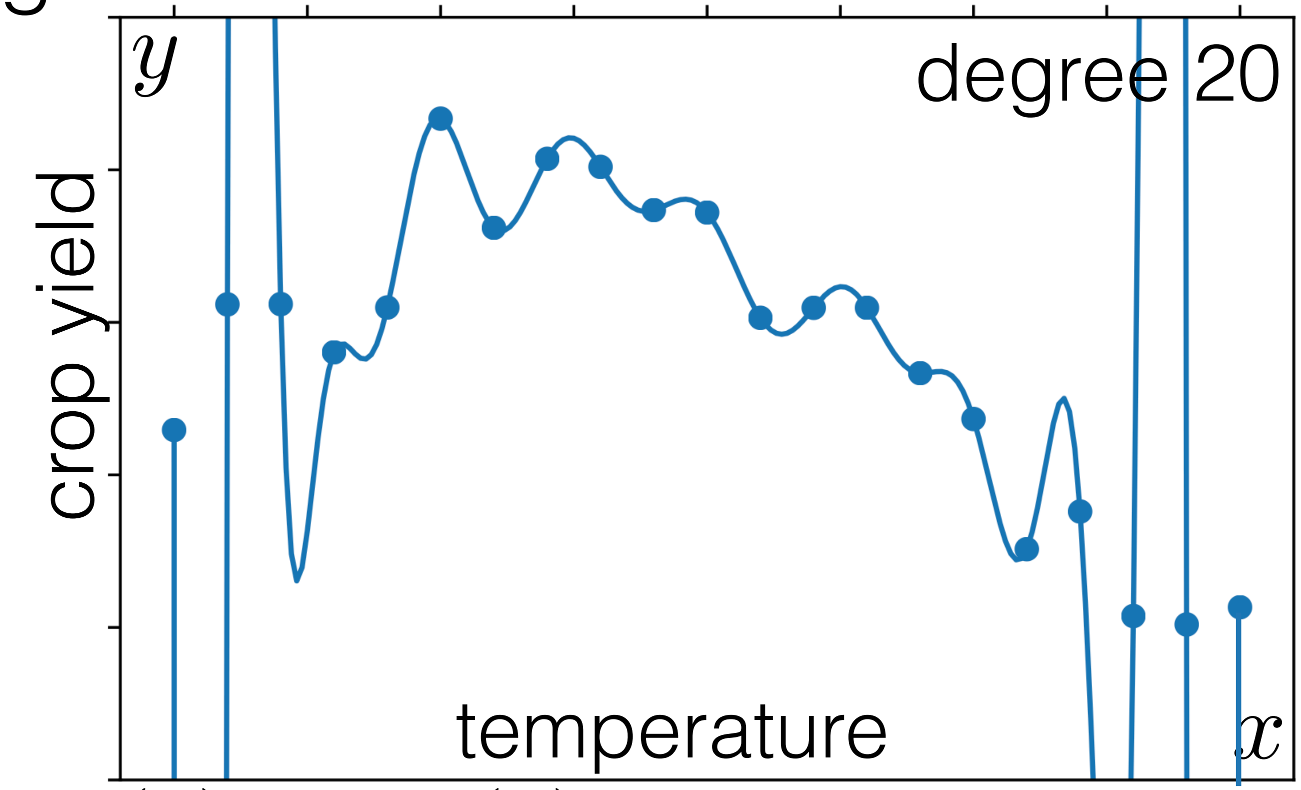

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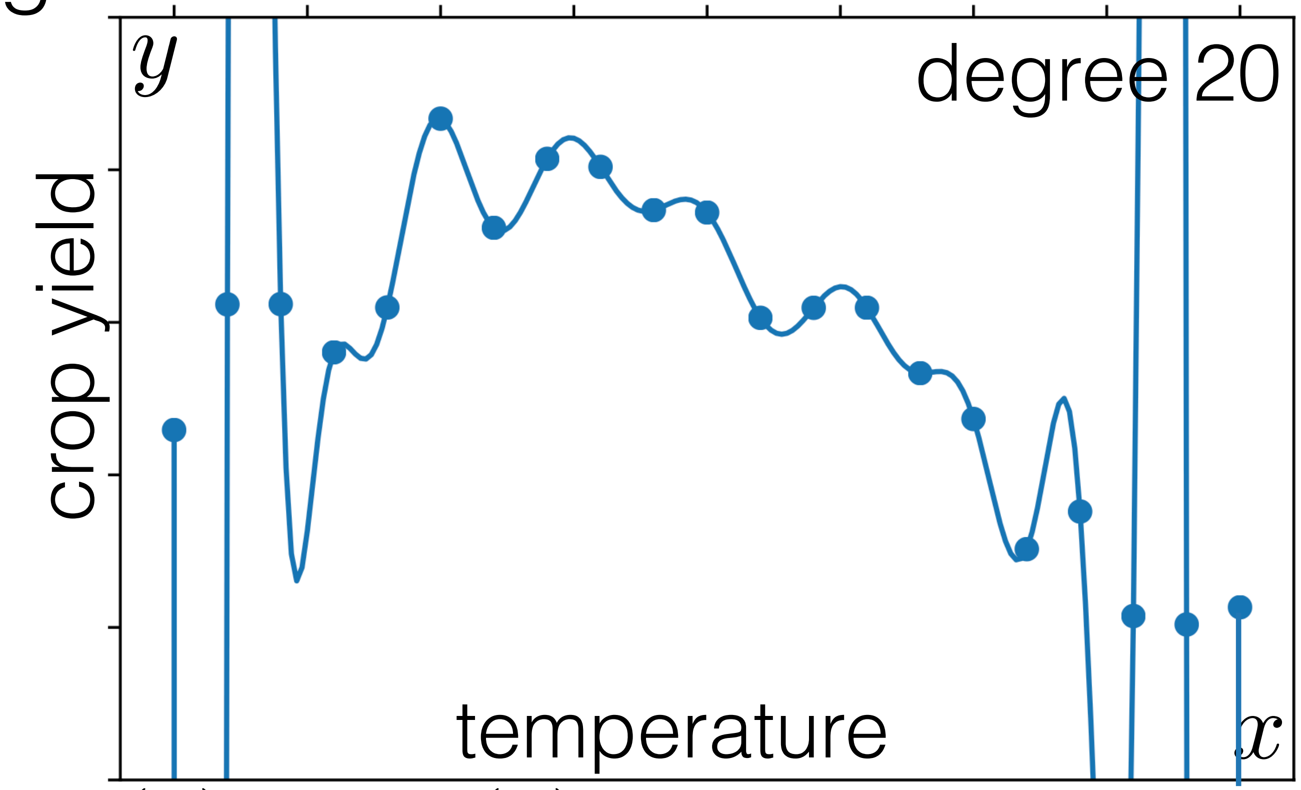

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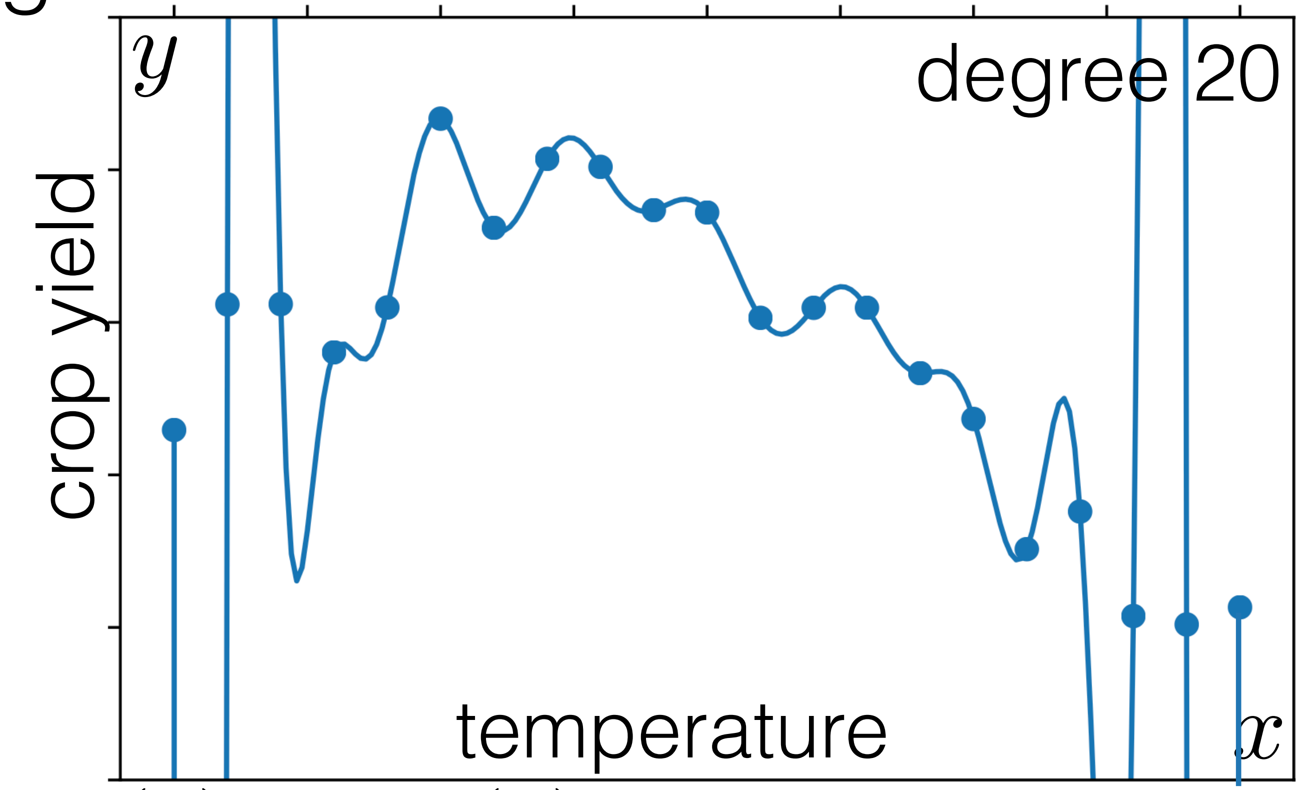

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  - For any training point, let $h_{\mathcal{D}}(X^{(n)}) = Y^{(n)}$; else $h_{\mathcal{D}}(x) = 0$
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- Which assumption isn't true? The terms in the sum aren't iid

$$L(Y^{(n)}, h_{\mathcal{D}}(X^{(n)})) = f(X^{(n)}, Y^{(n)}, \mathcal{D} = \{(X^{(n')}, Y^{(n')})\}_{n'=1}^{N})$$

Empirical risk over the training data

- Proposal: Let's estimate $\mathbb{E}[L(Y, h_{\mathcal{D}}(X))]$ with the empirical average of loss over the training data: could anything go wrong?
  $\frac{1}{N}\sum_{n=1}^{N} L(Y^{(n)}, h_{\mathcal{D}}(X^{(n)}))$
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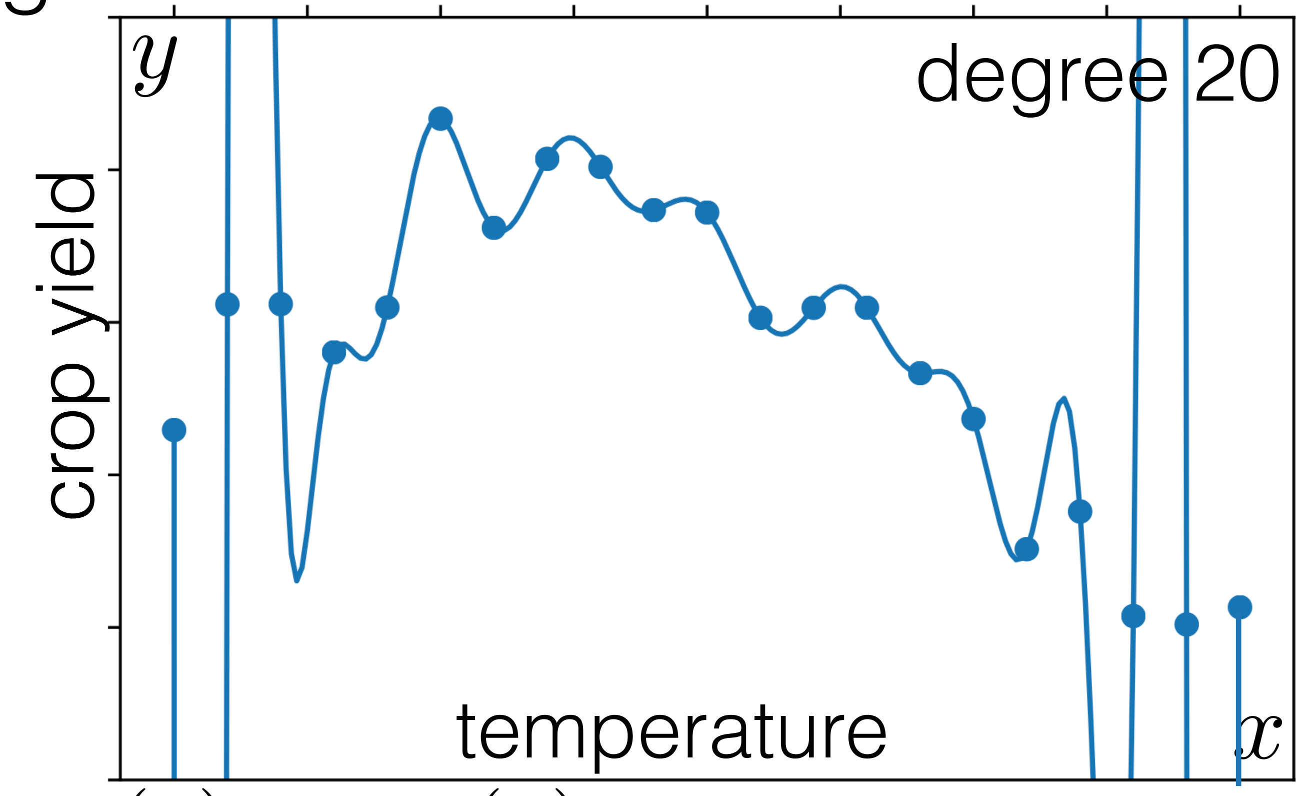

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=? (the Law of Large Numbers doesn't tell us)

Empirical risk over validation data

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- Note: can use validation data to estimate risk at a new data point even if the decision rule didn't arise from training data

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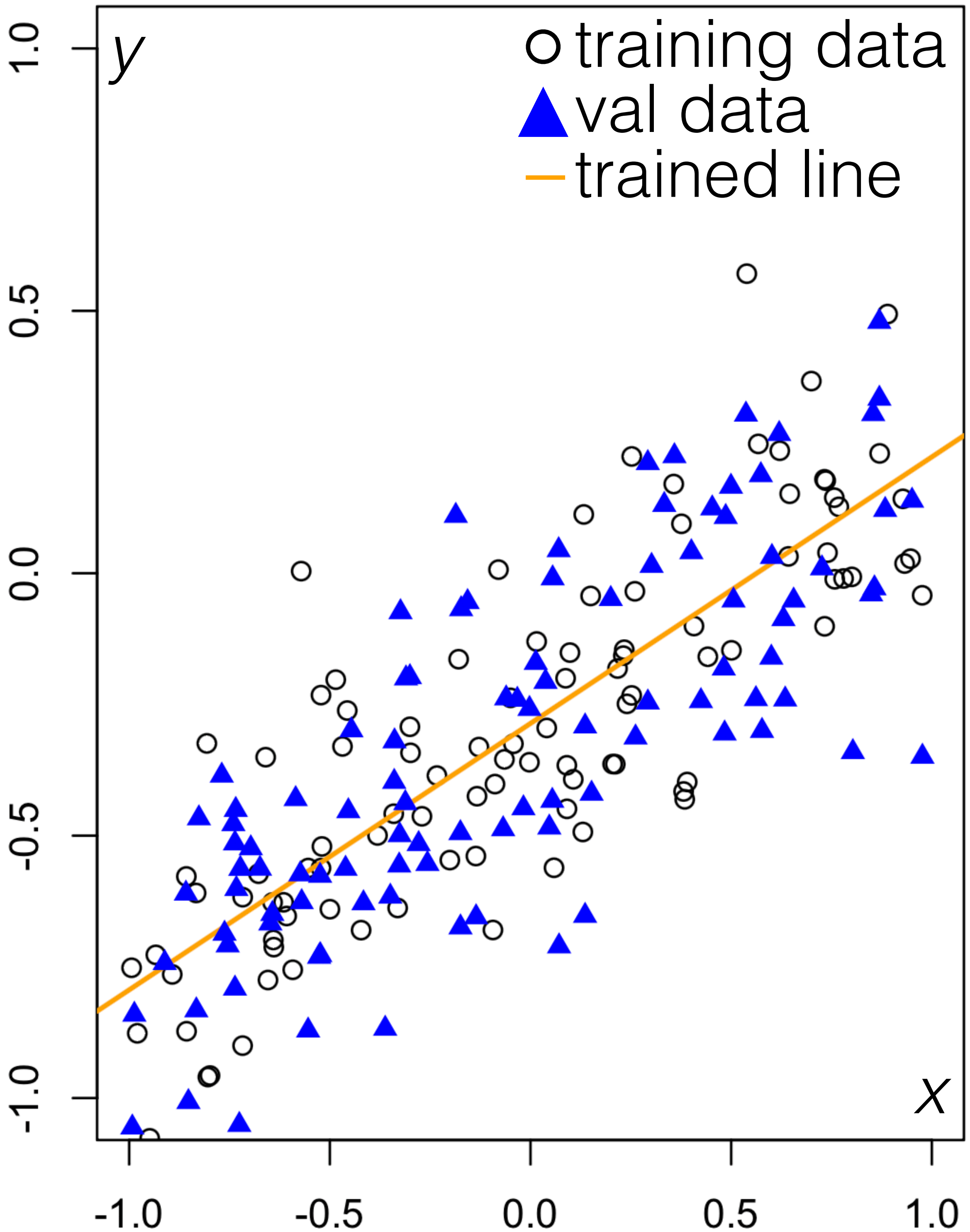

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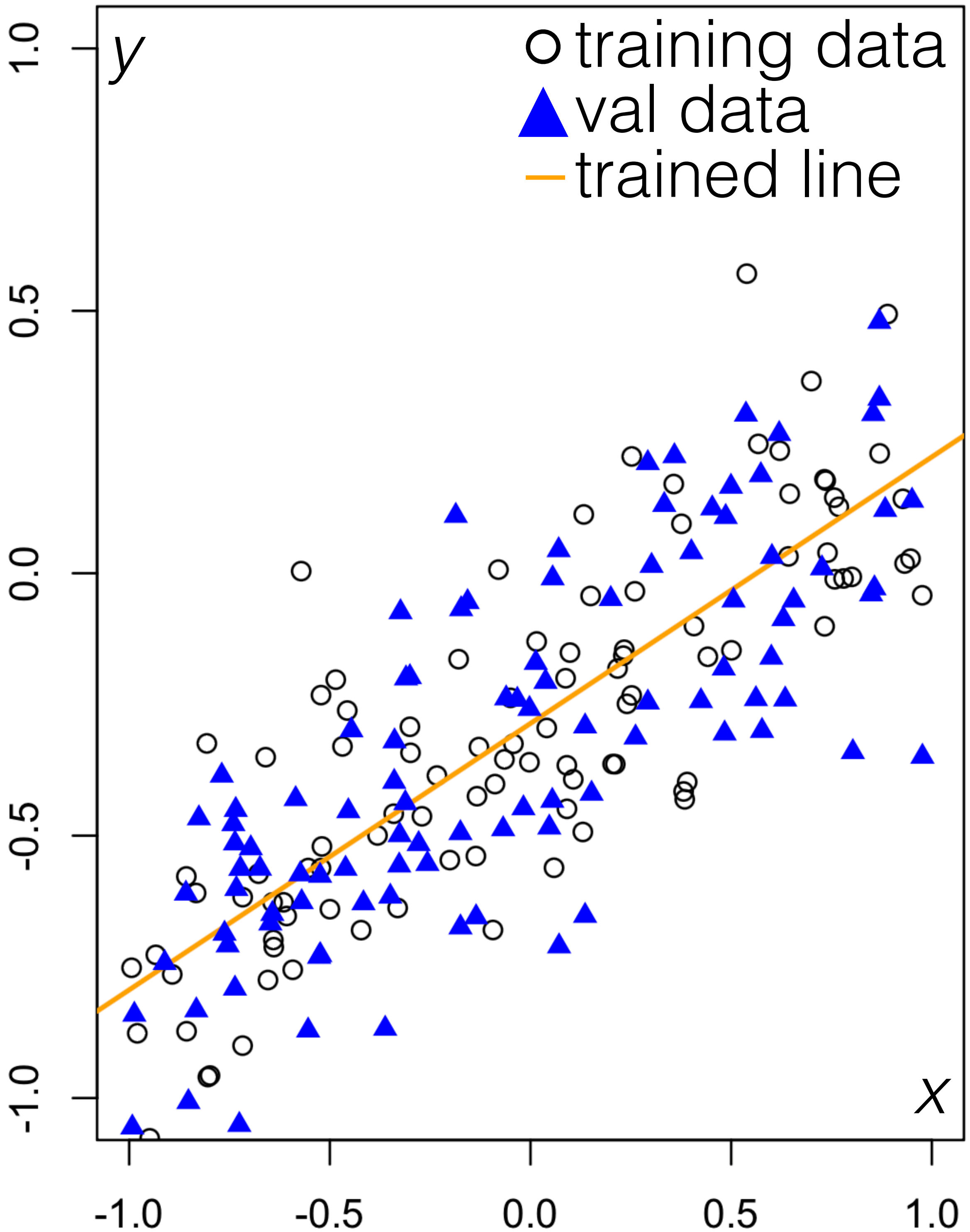

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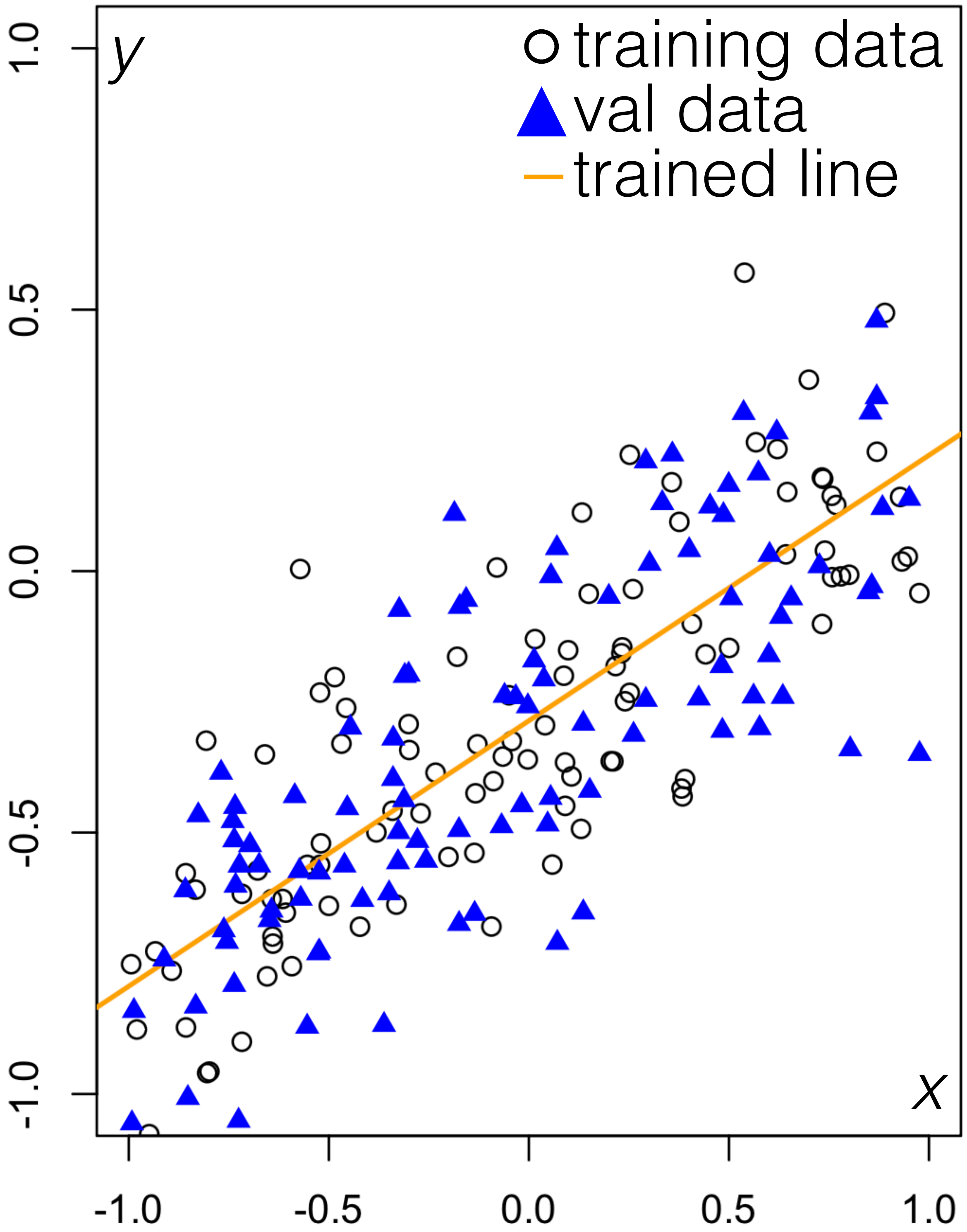

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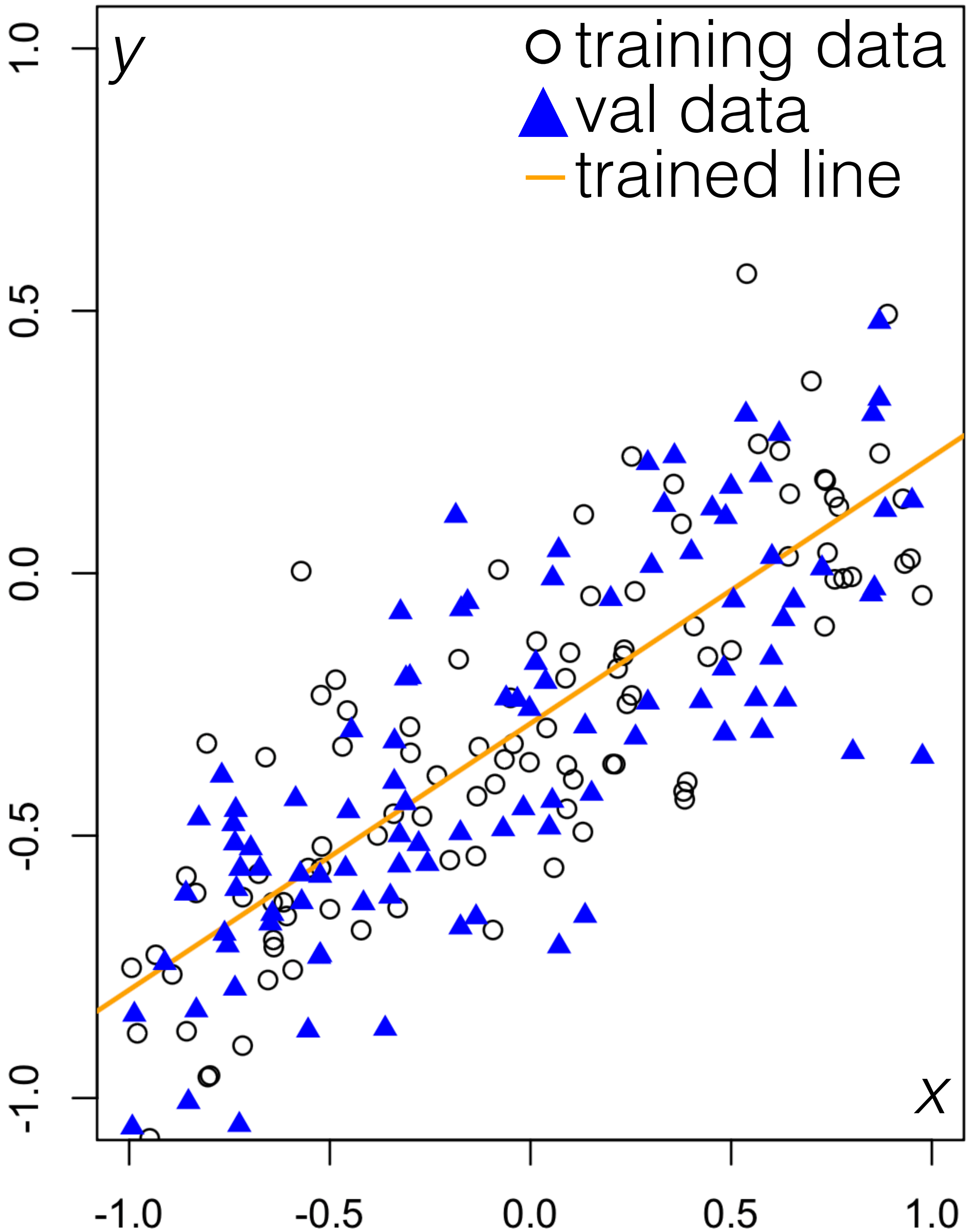

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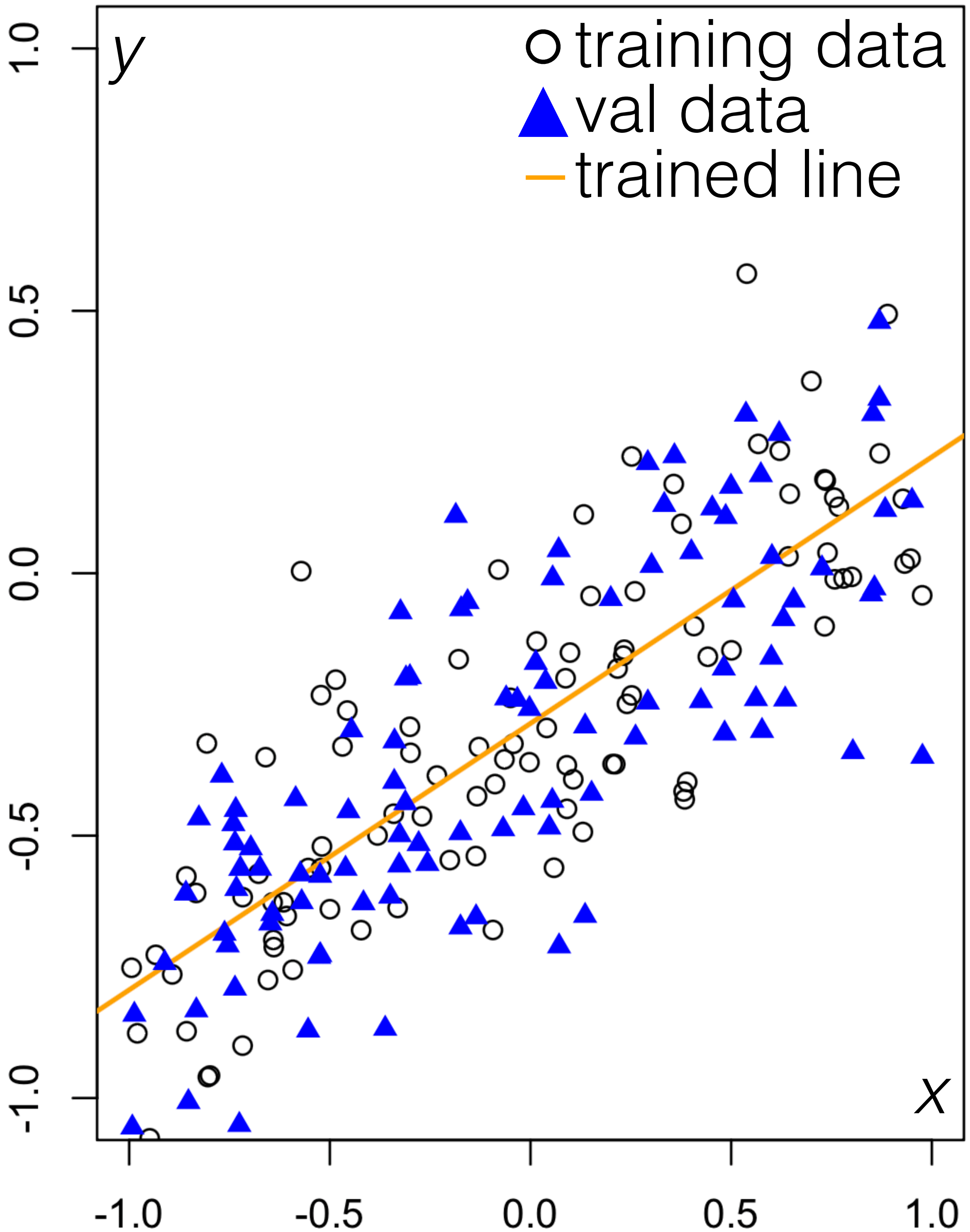

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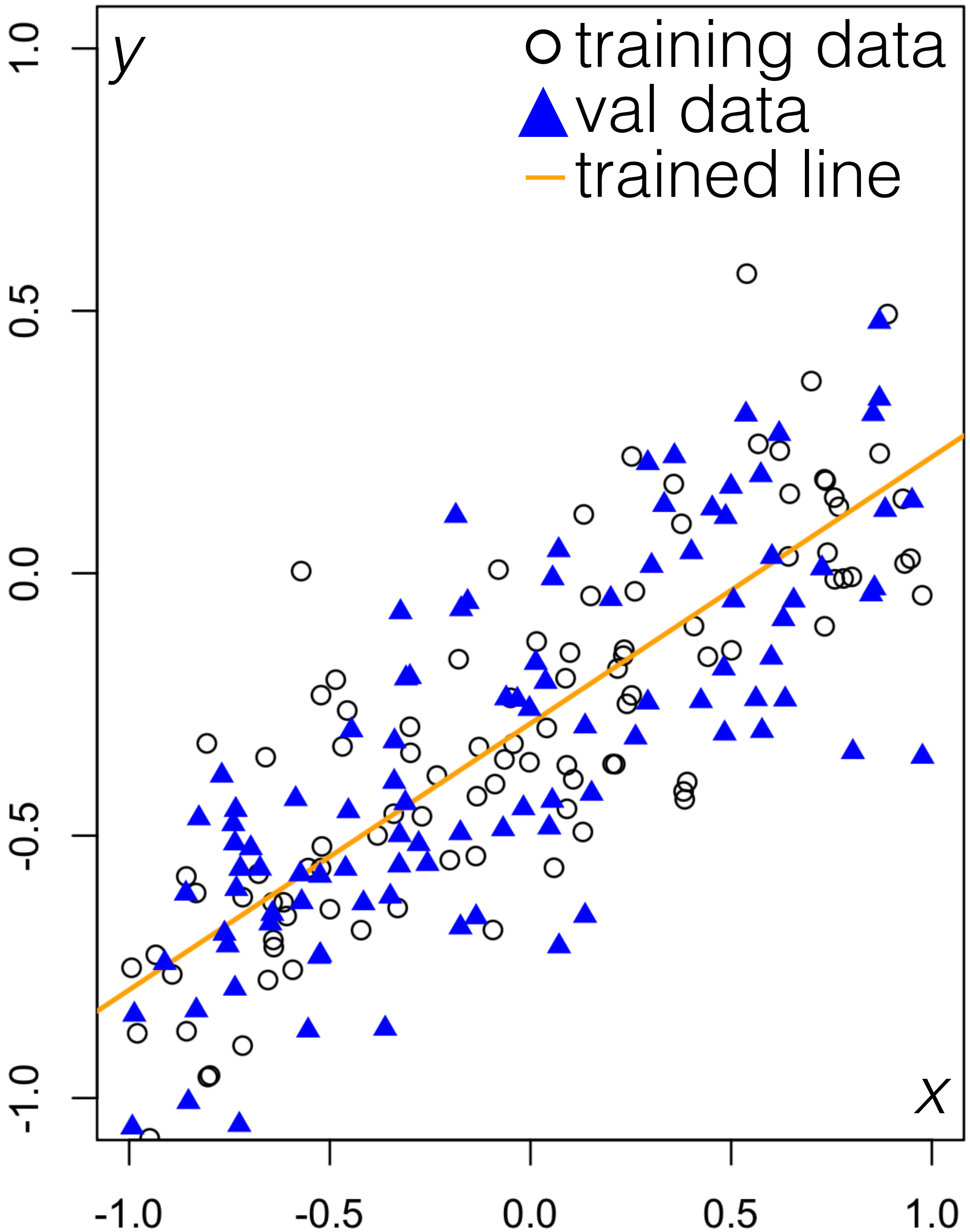

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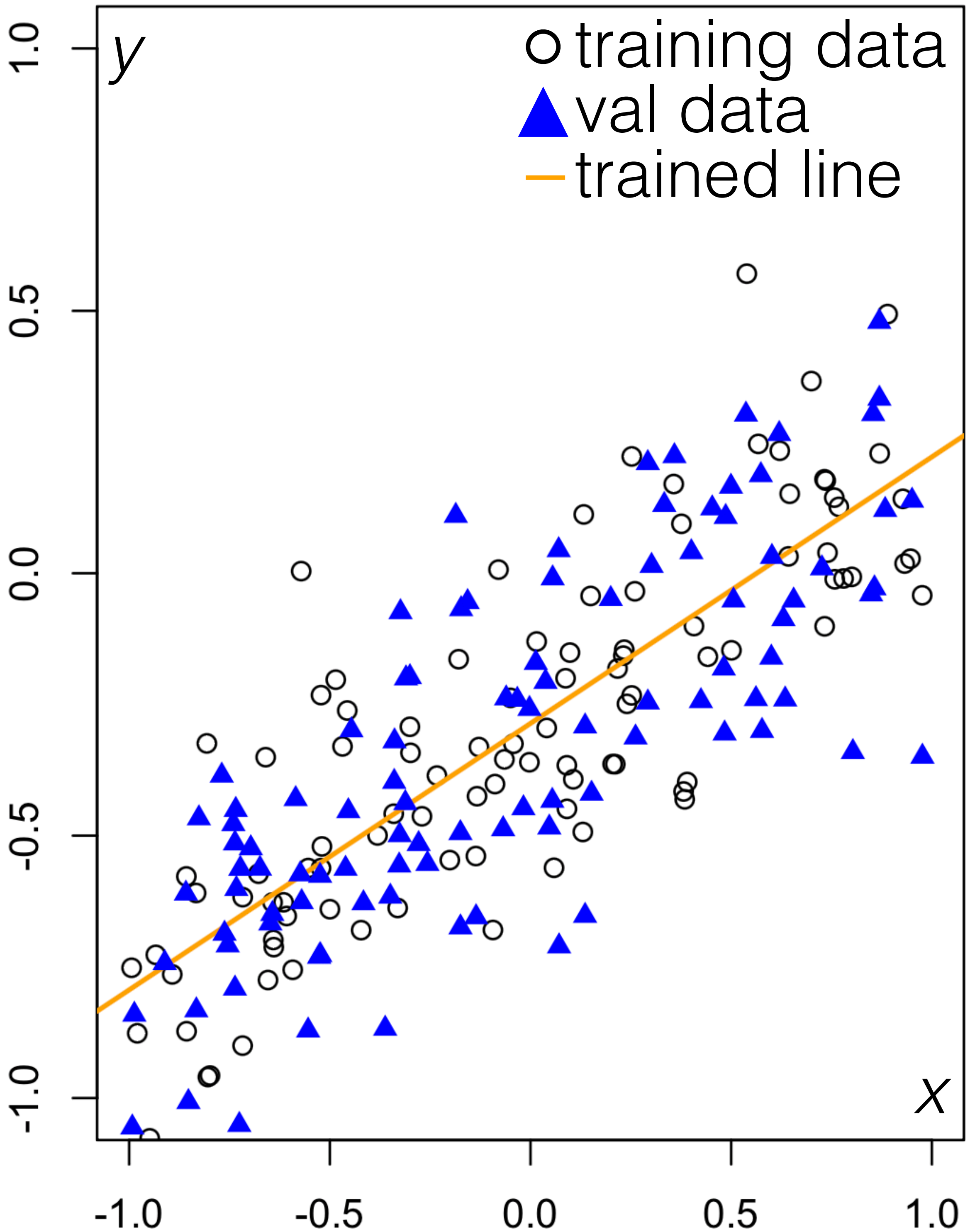

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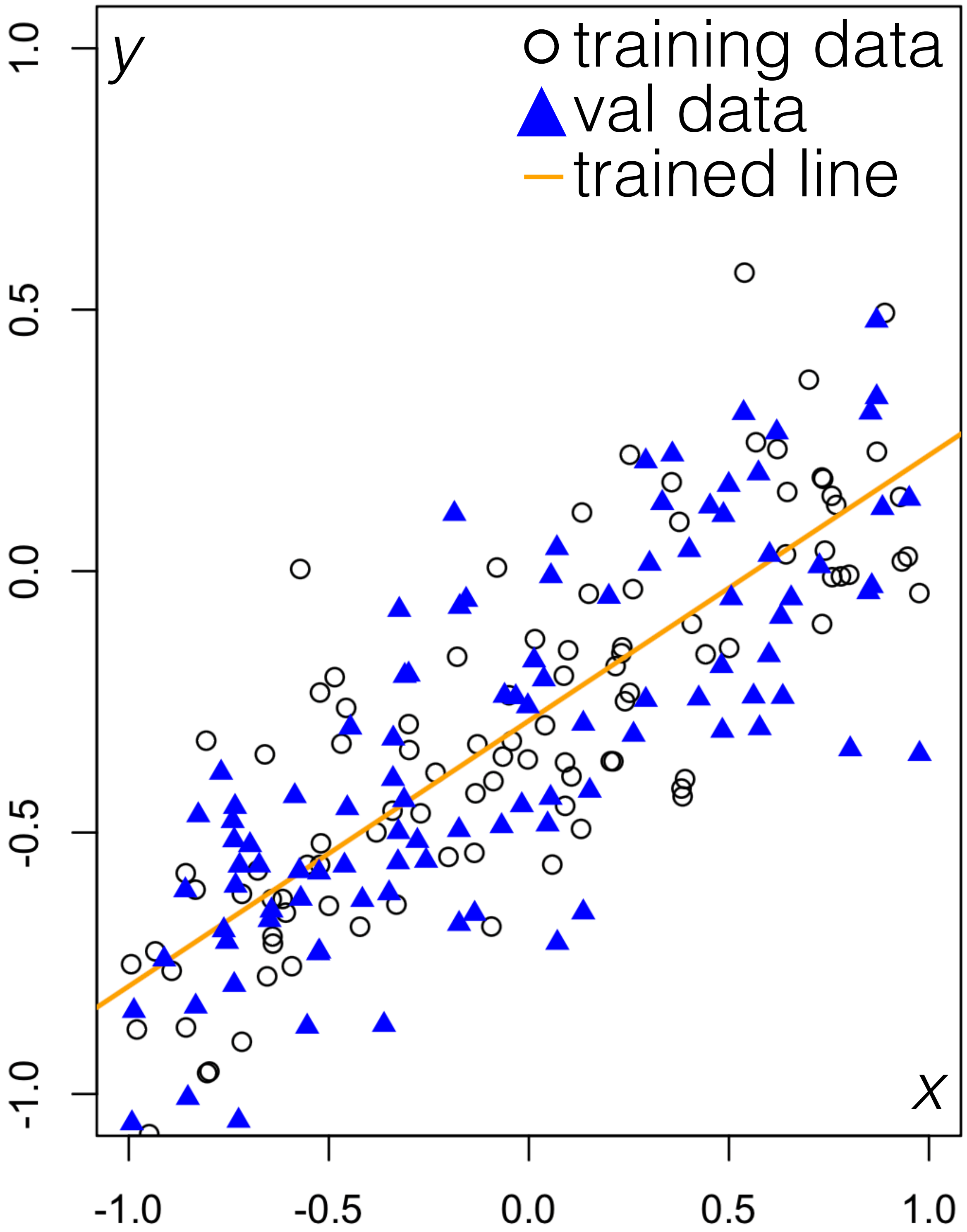

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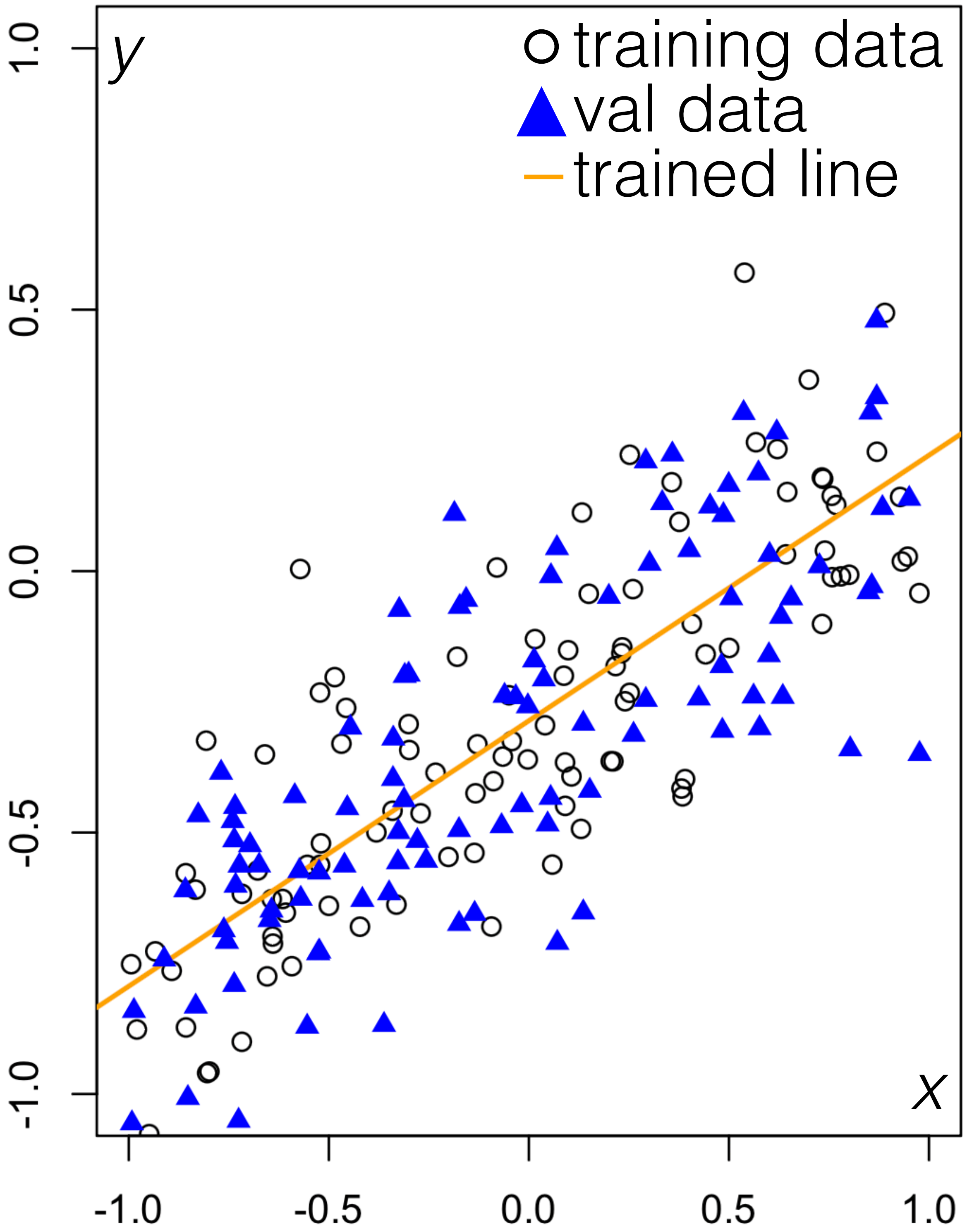

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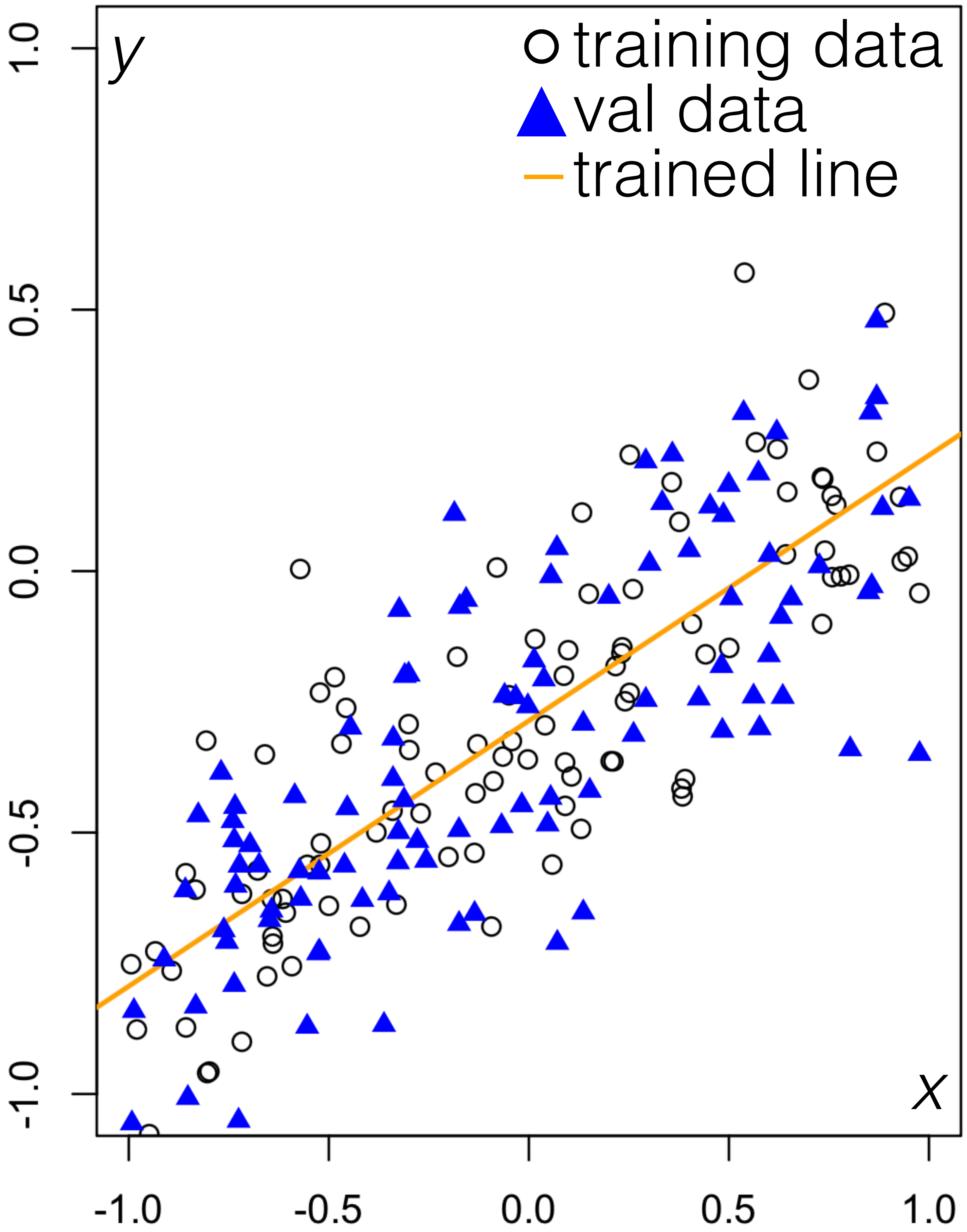

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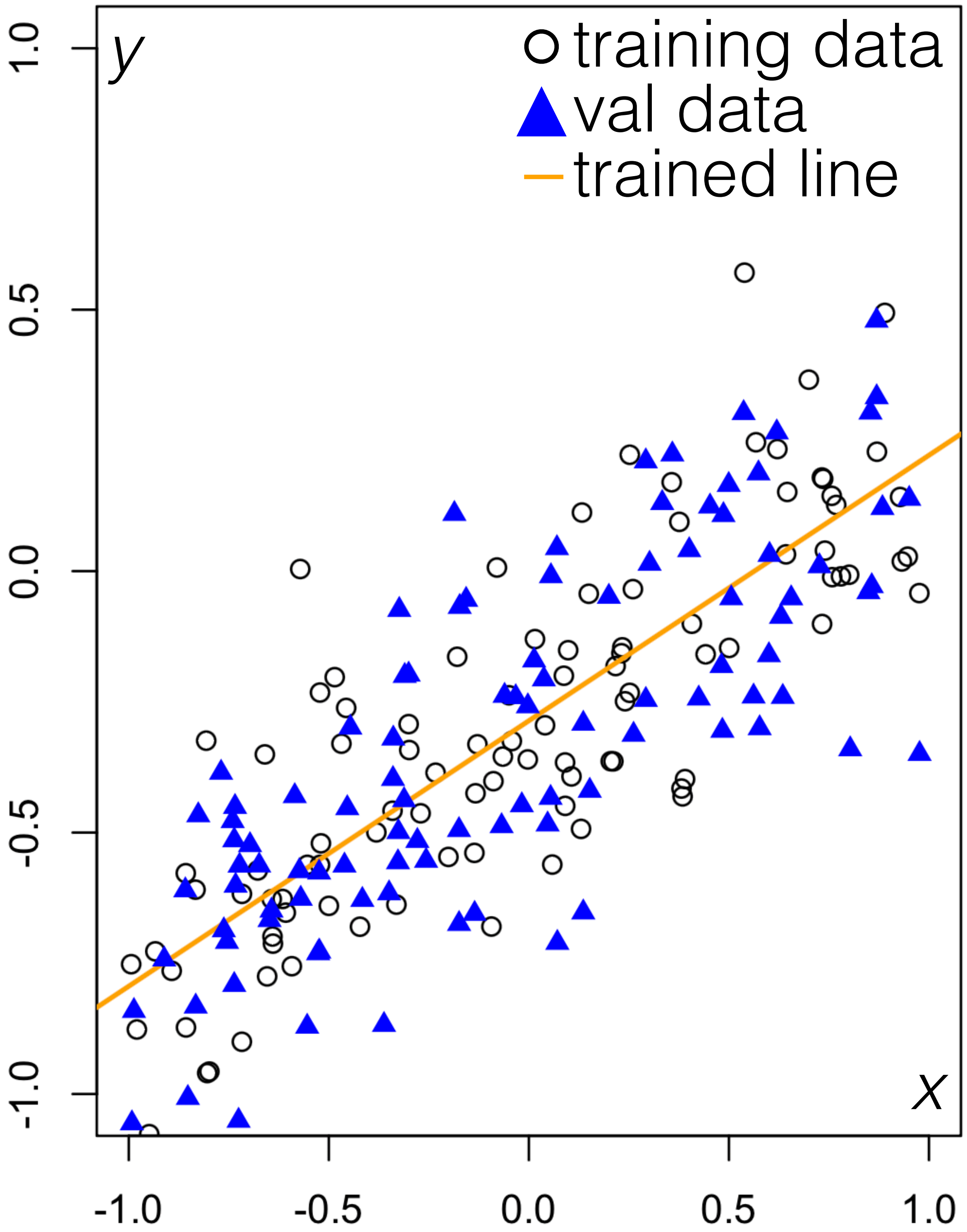

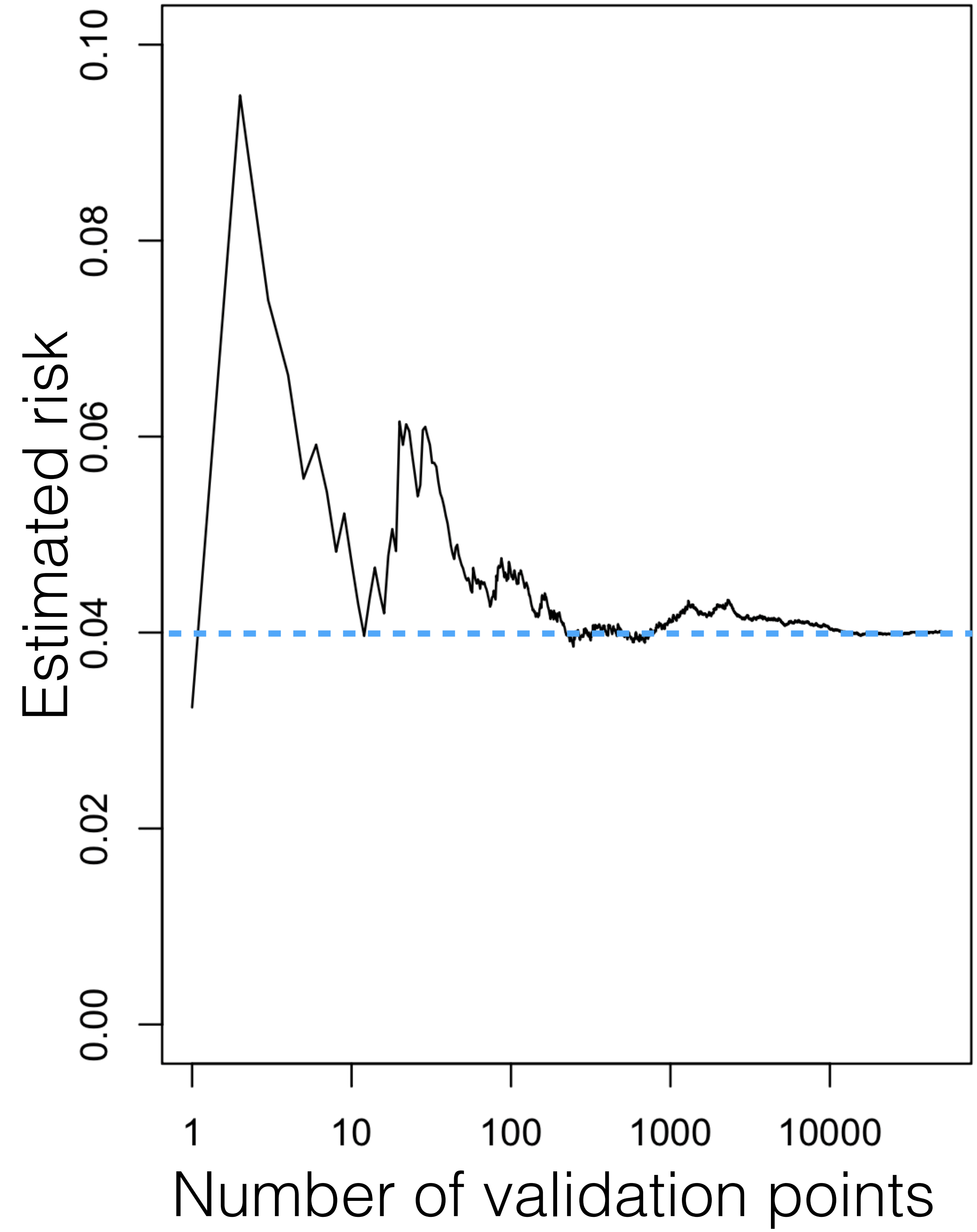

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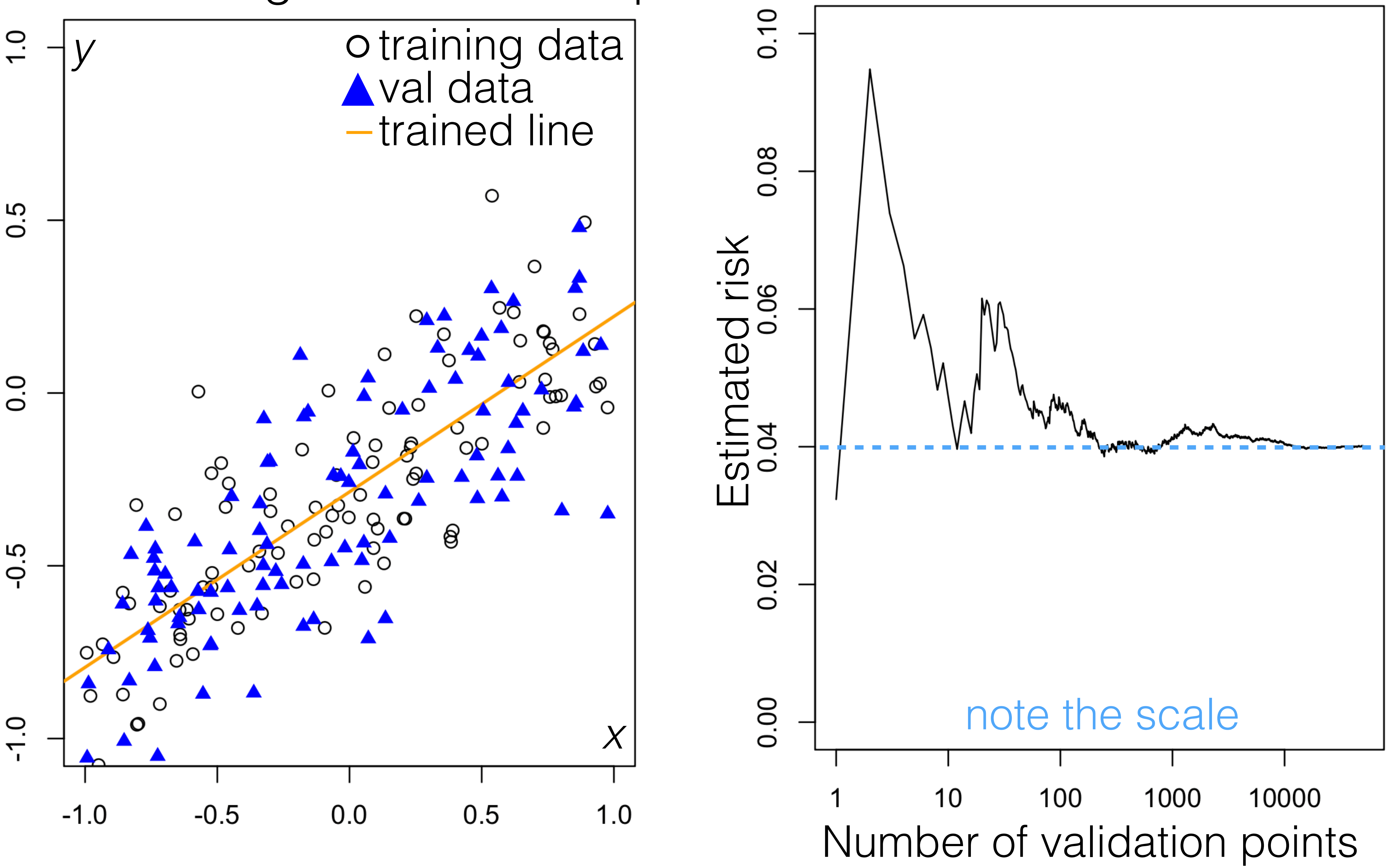

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Is this a good way to evaluate my predictor?

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- I care about the risk (expected loss) for a new person.

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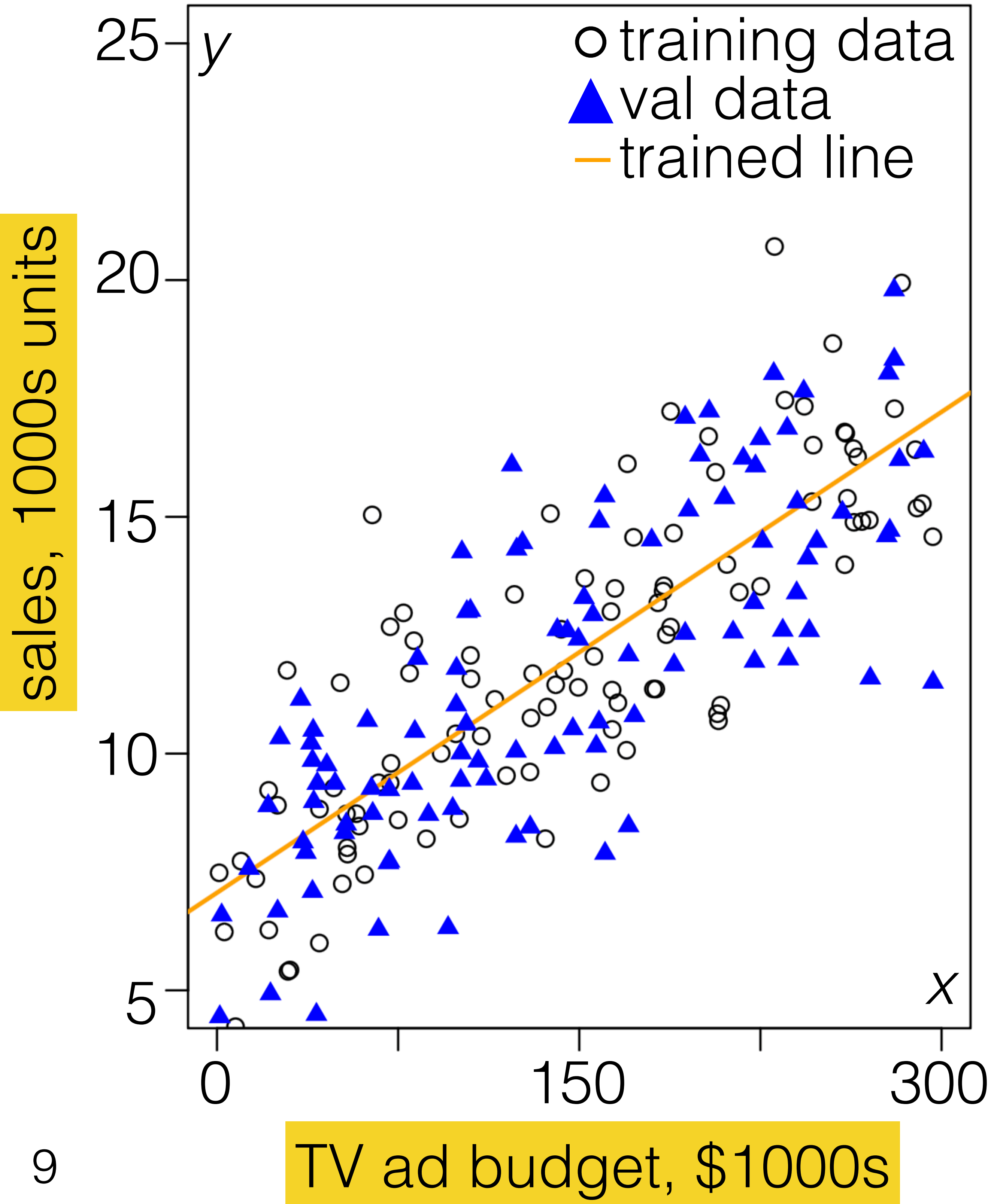

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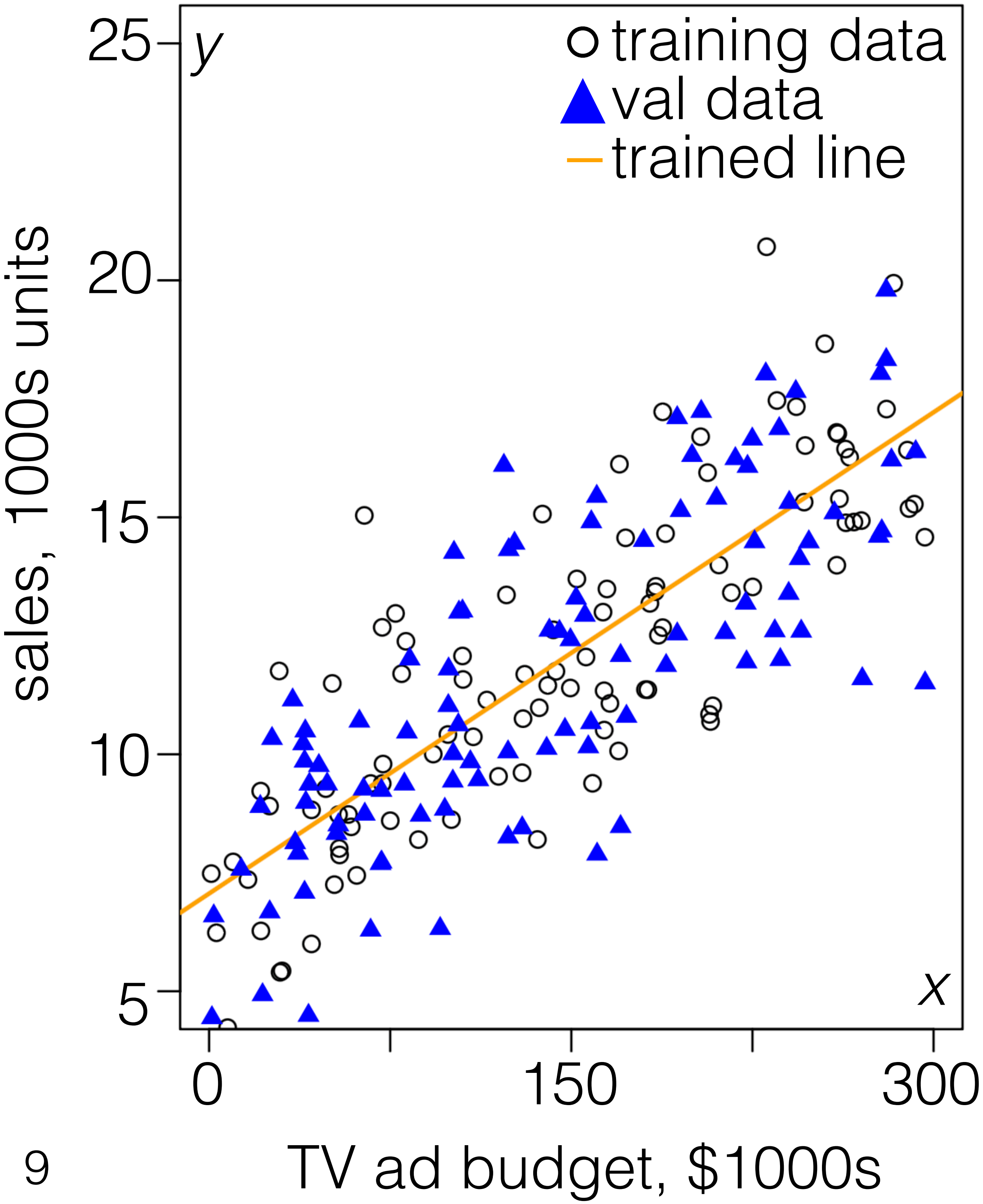

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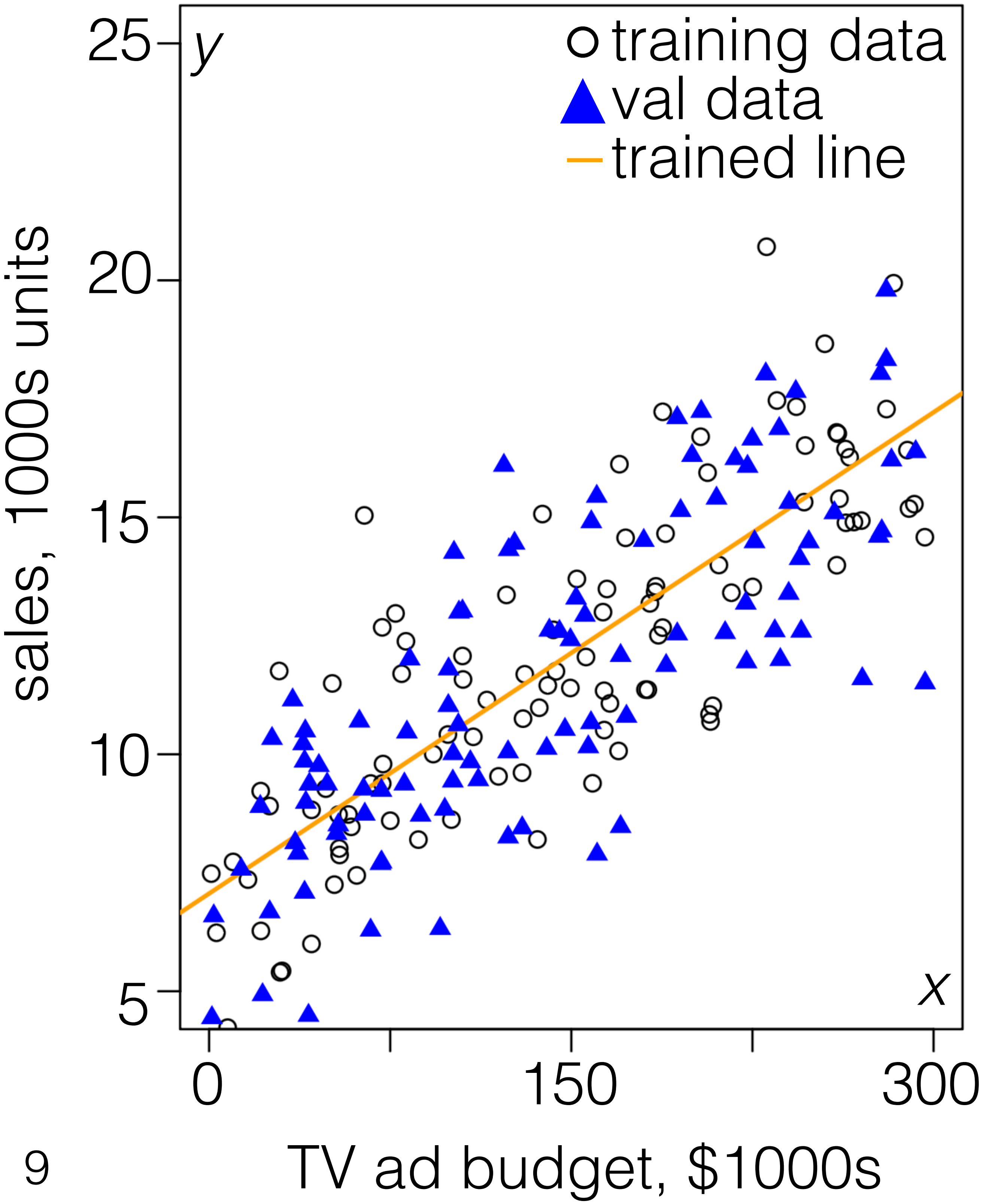

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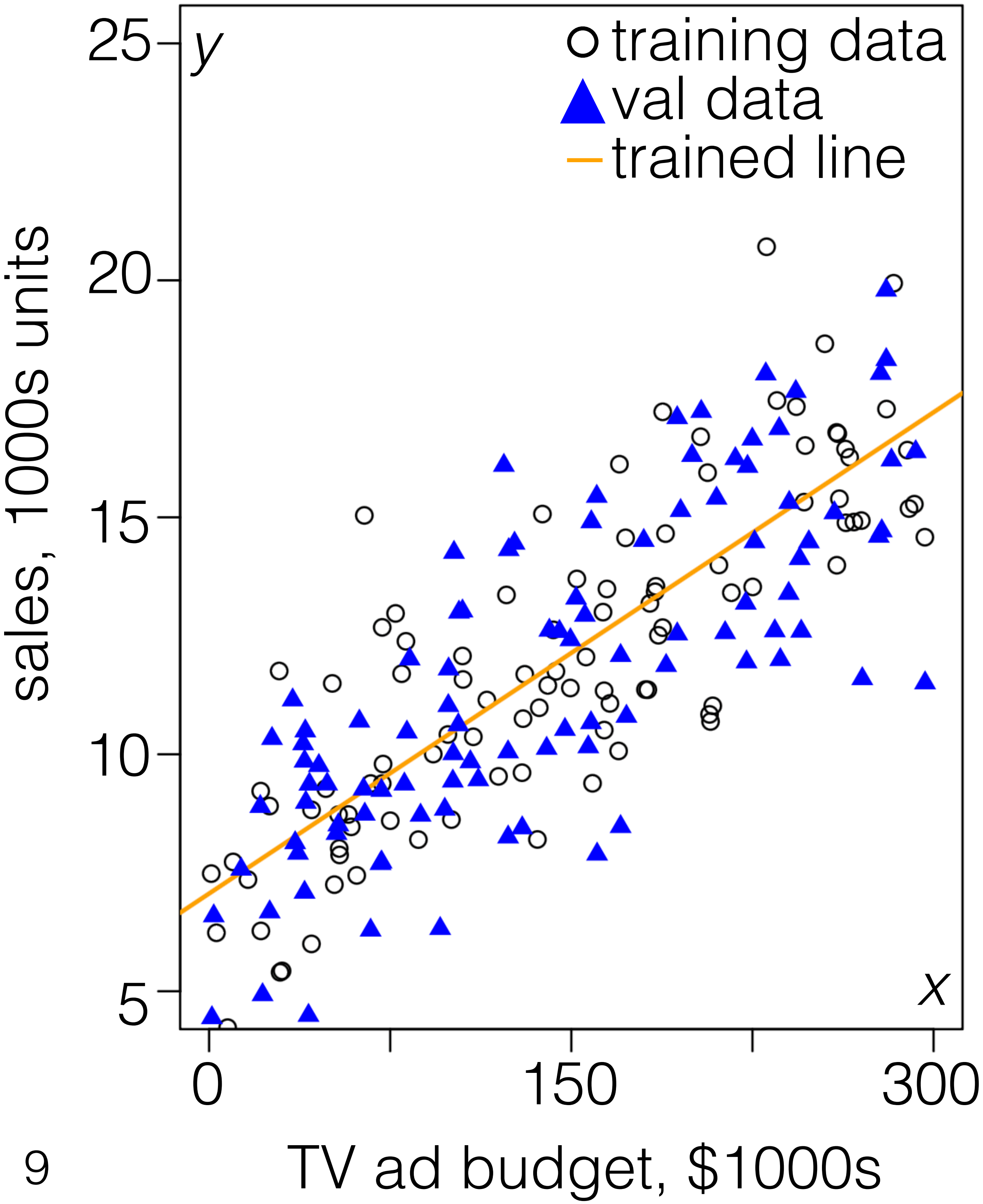

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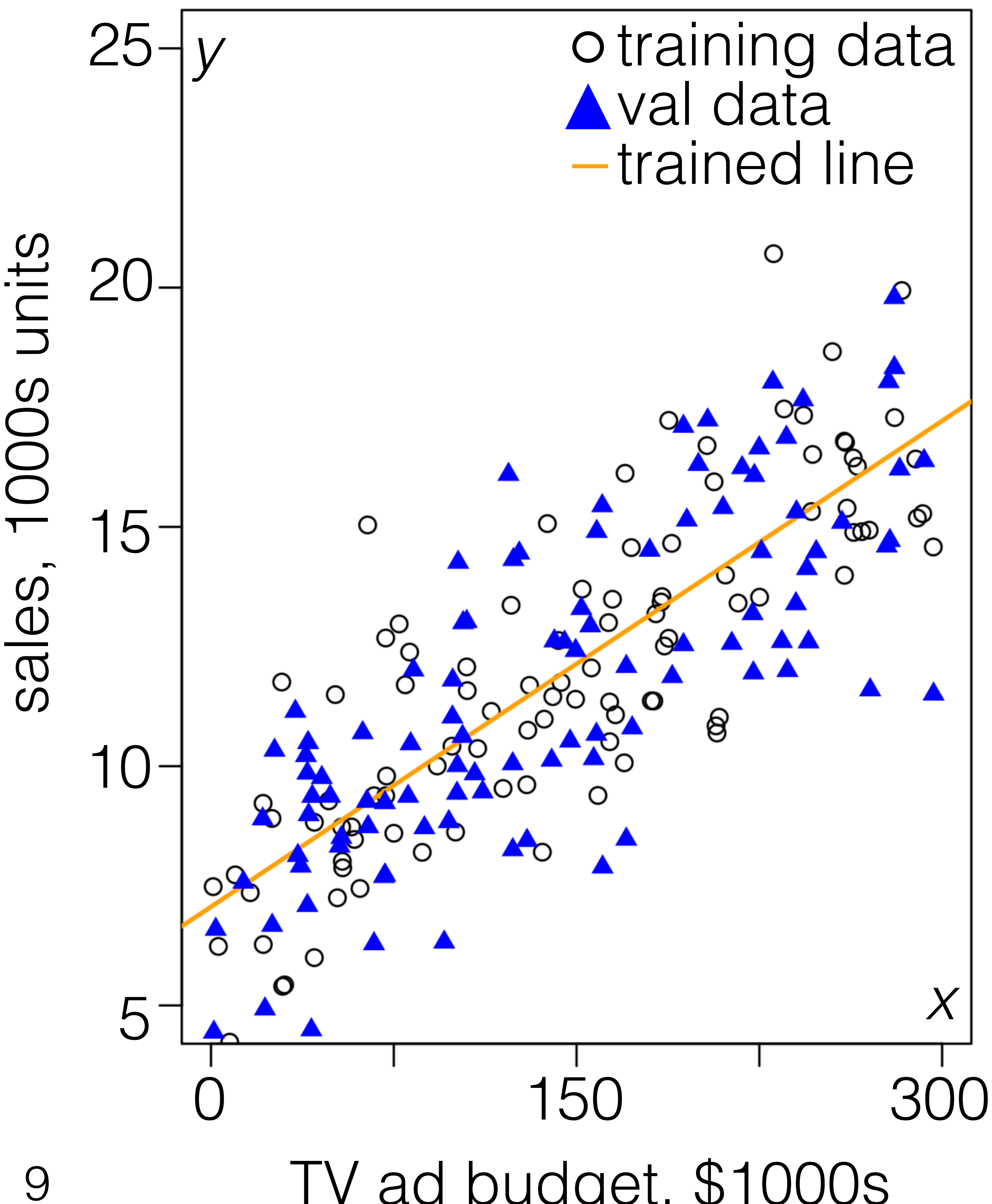

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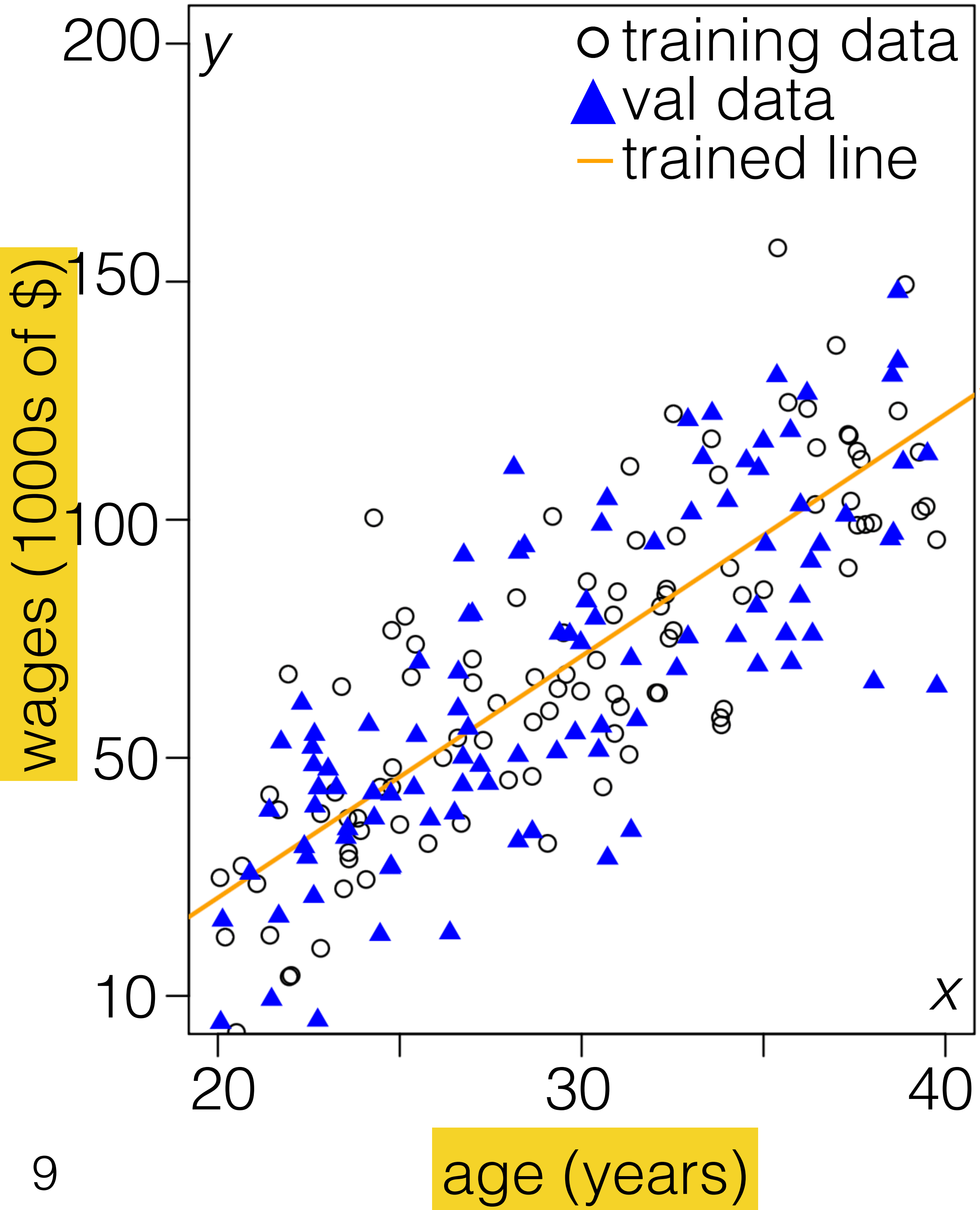

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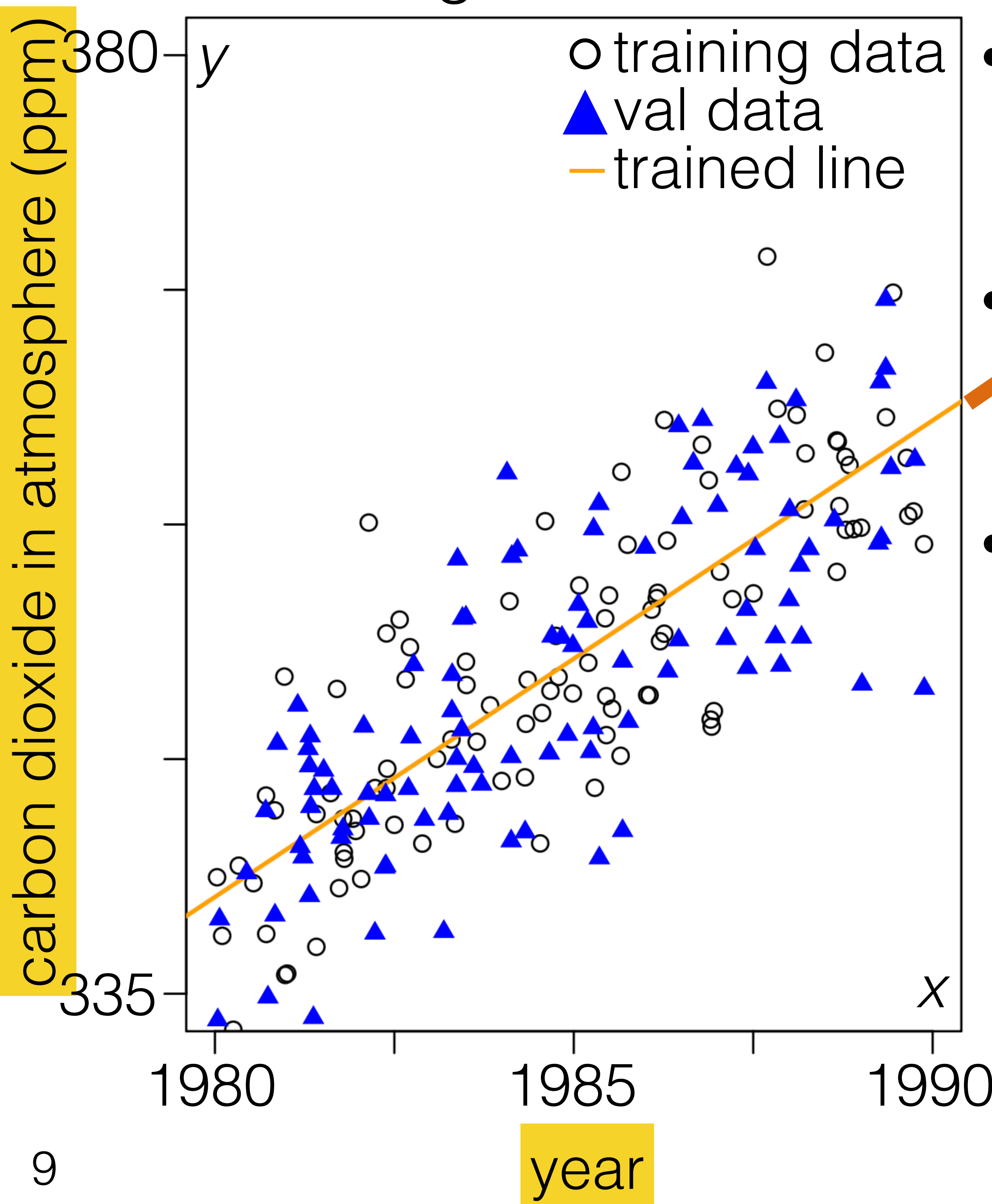

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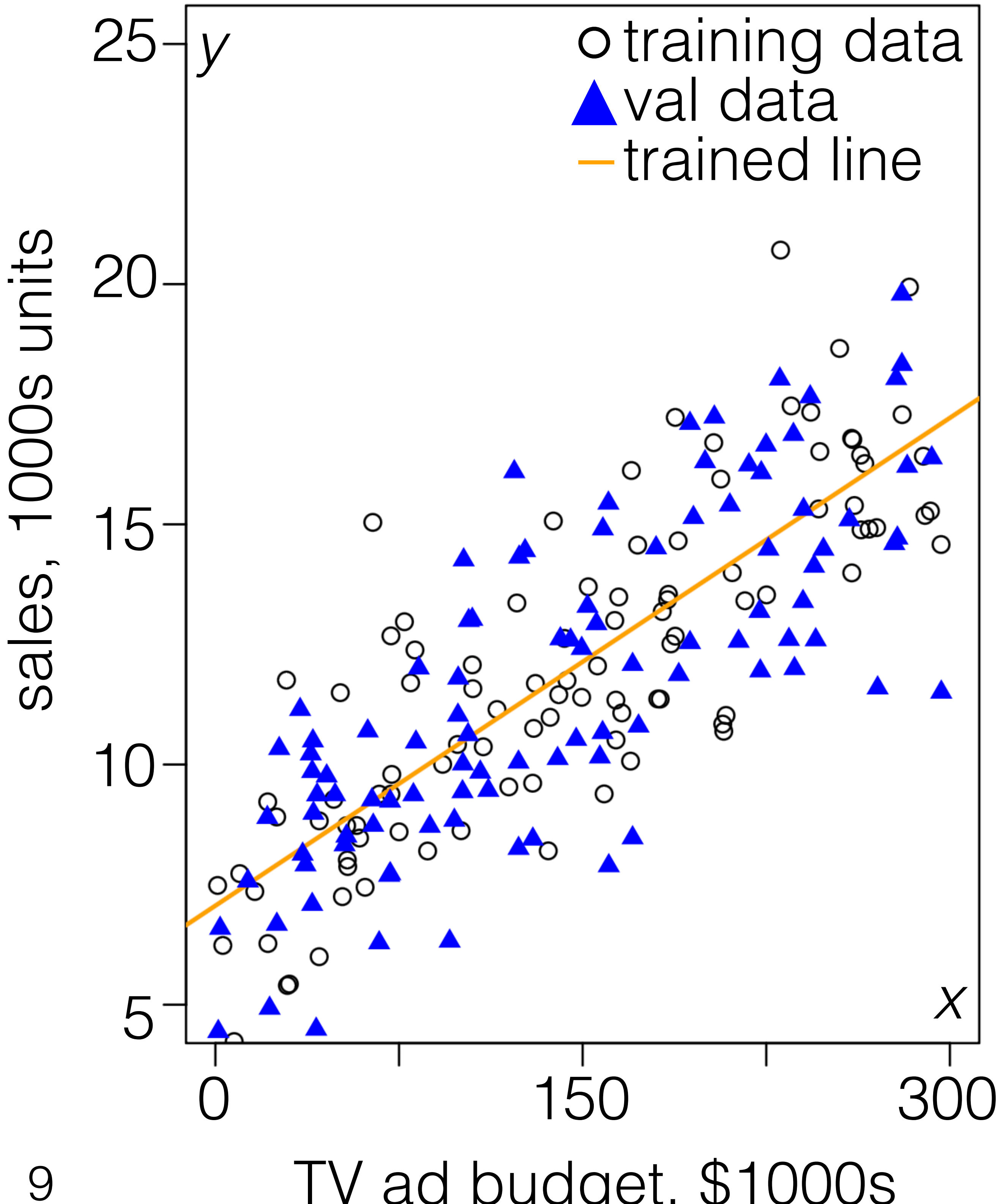

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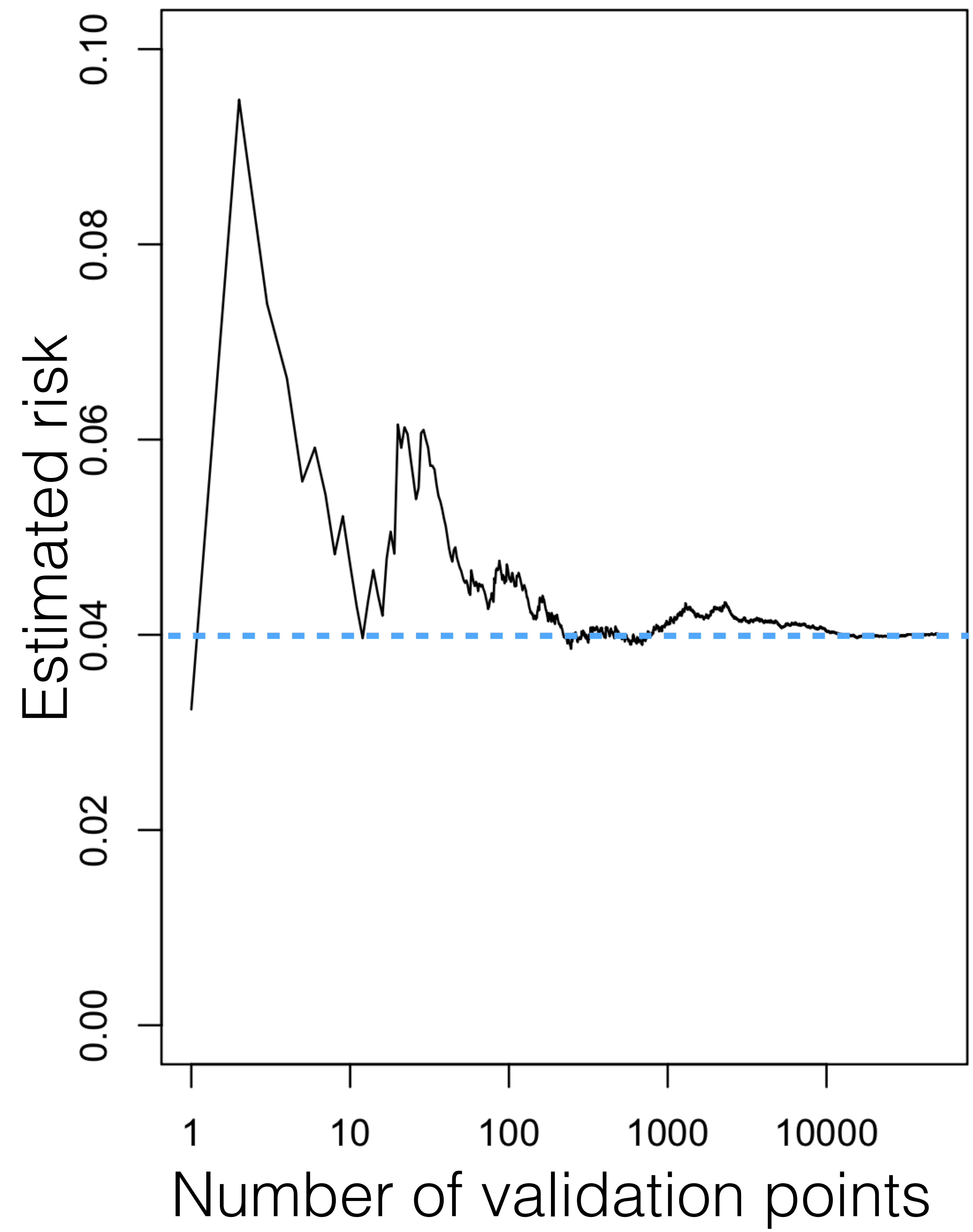
  - People make this kind of prediction all the time. Ask: What assumptions are they making?

Did I use enough validation points?



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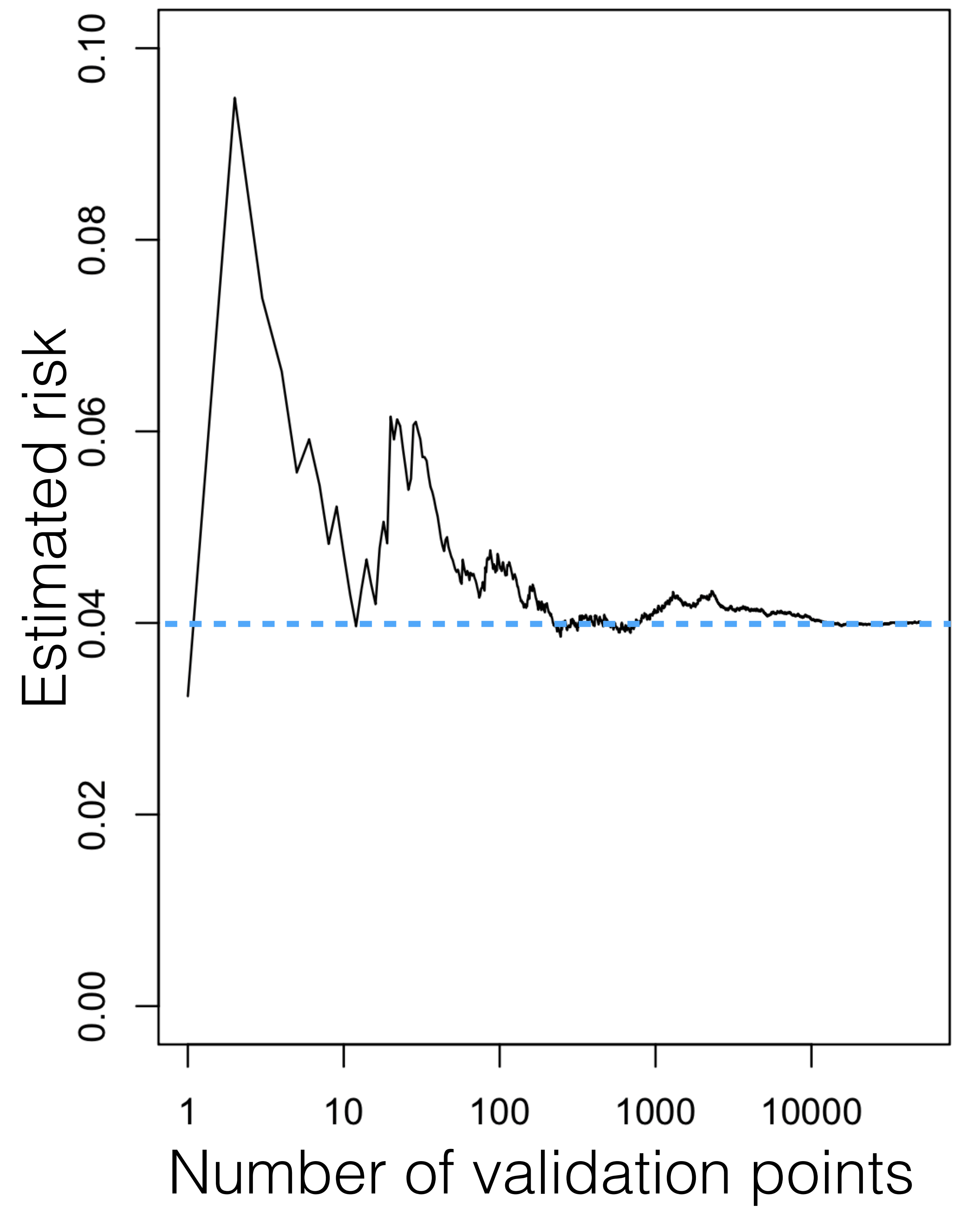

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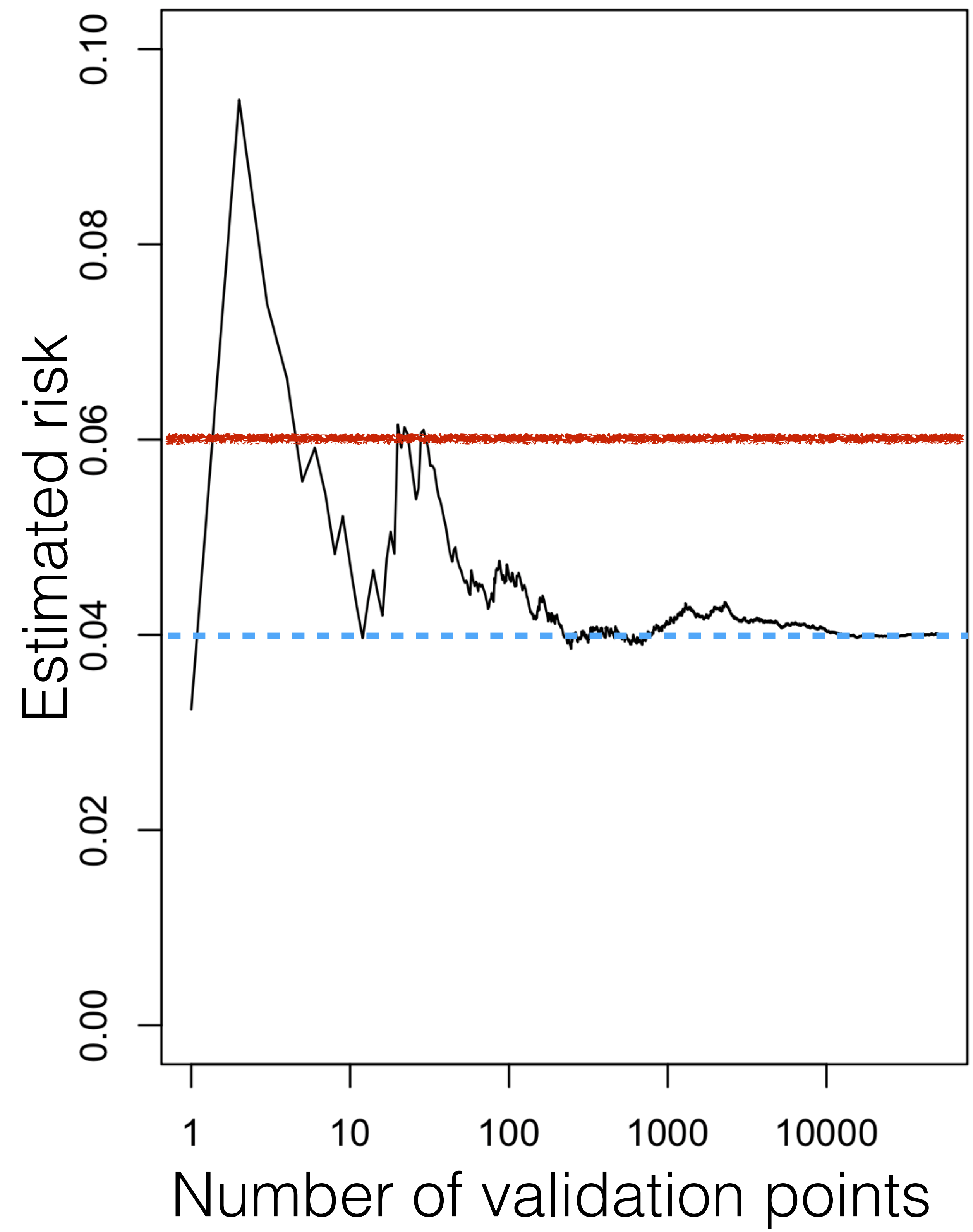

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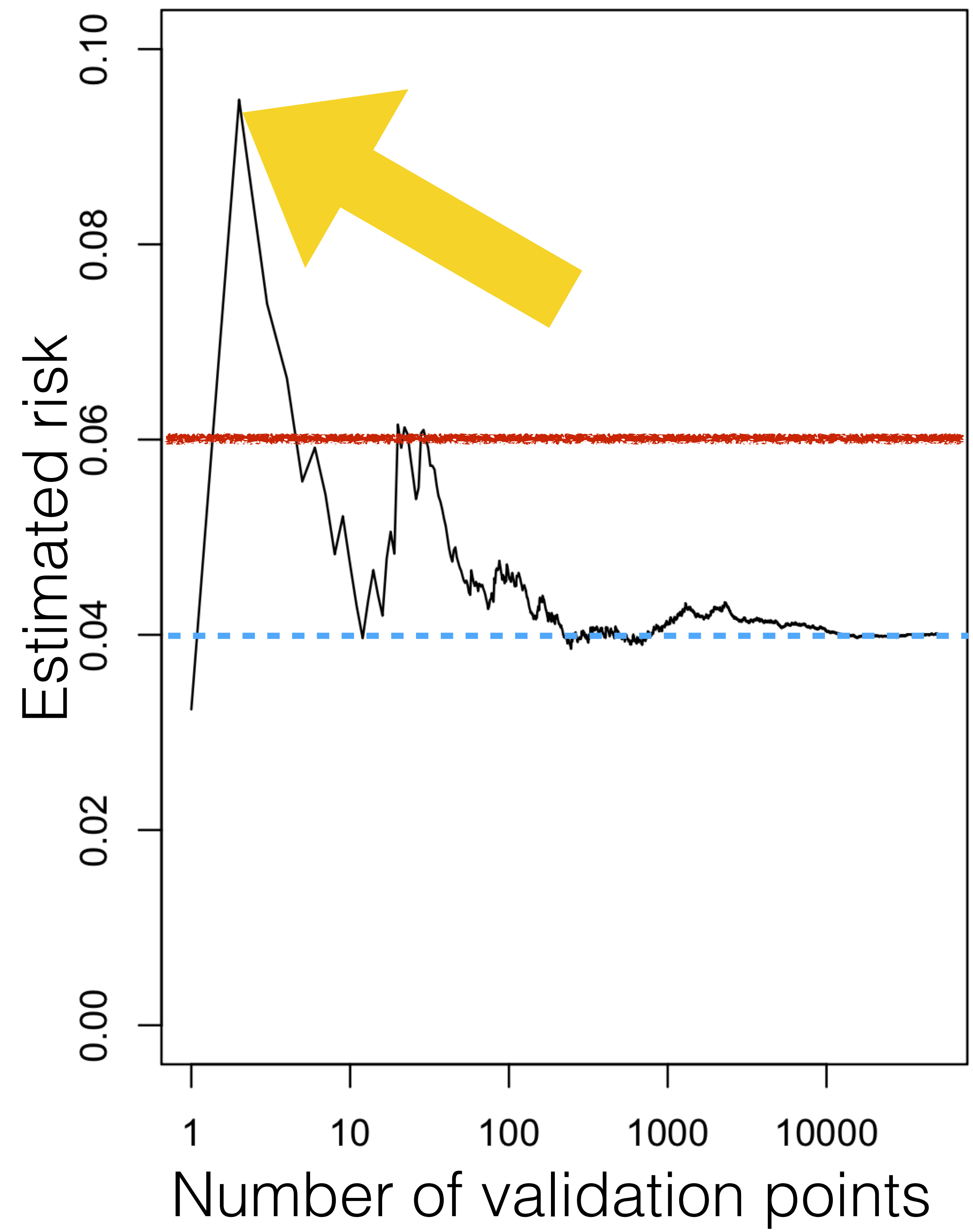

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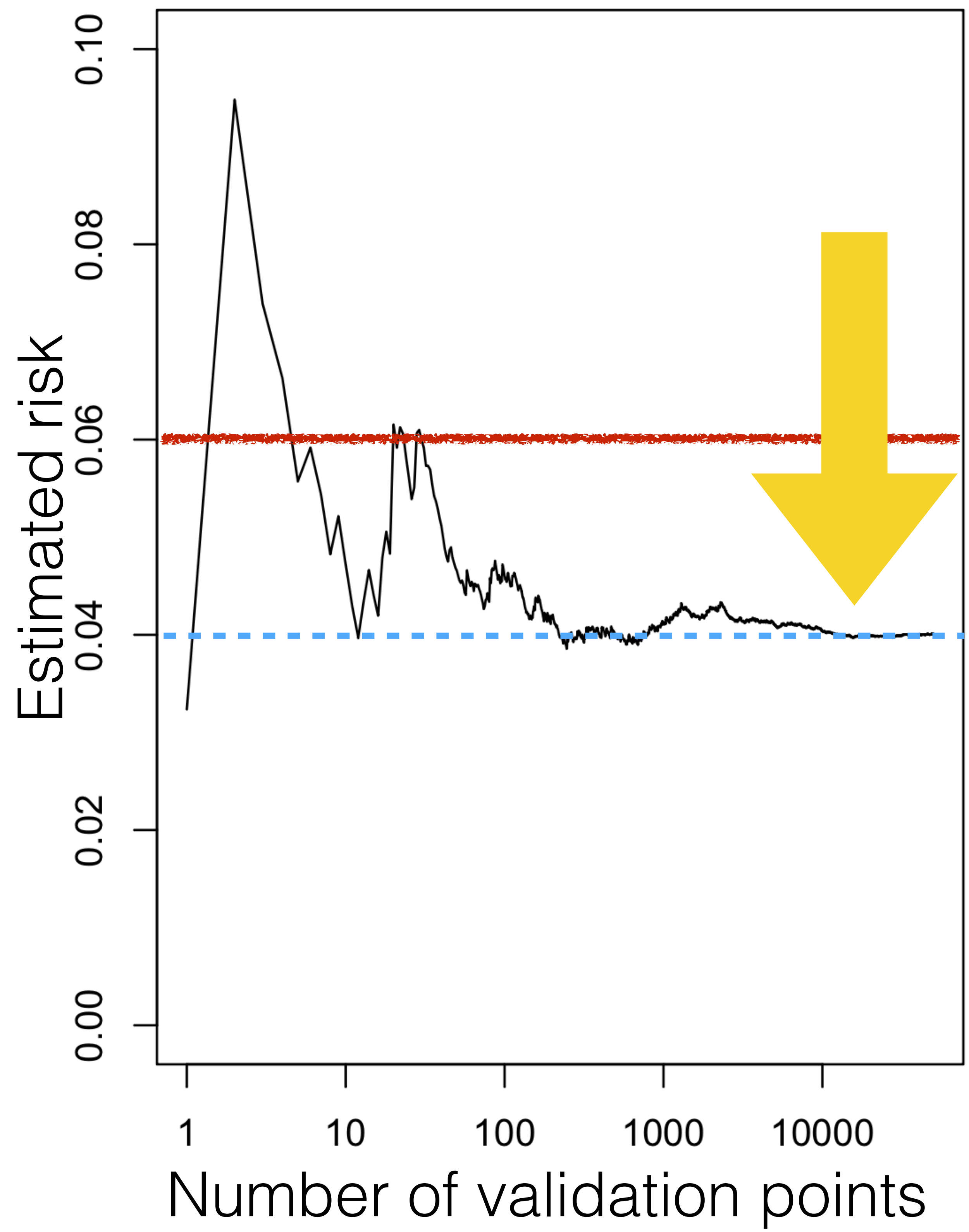

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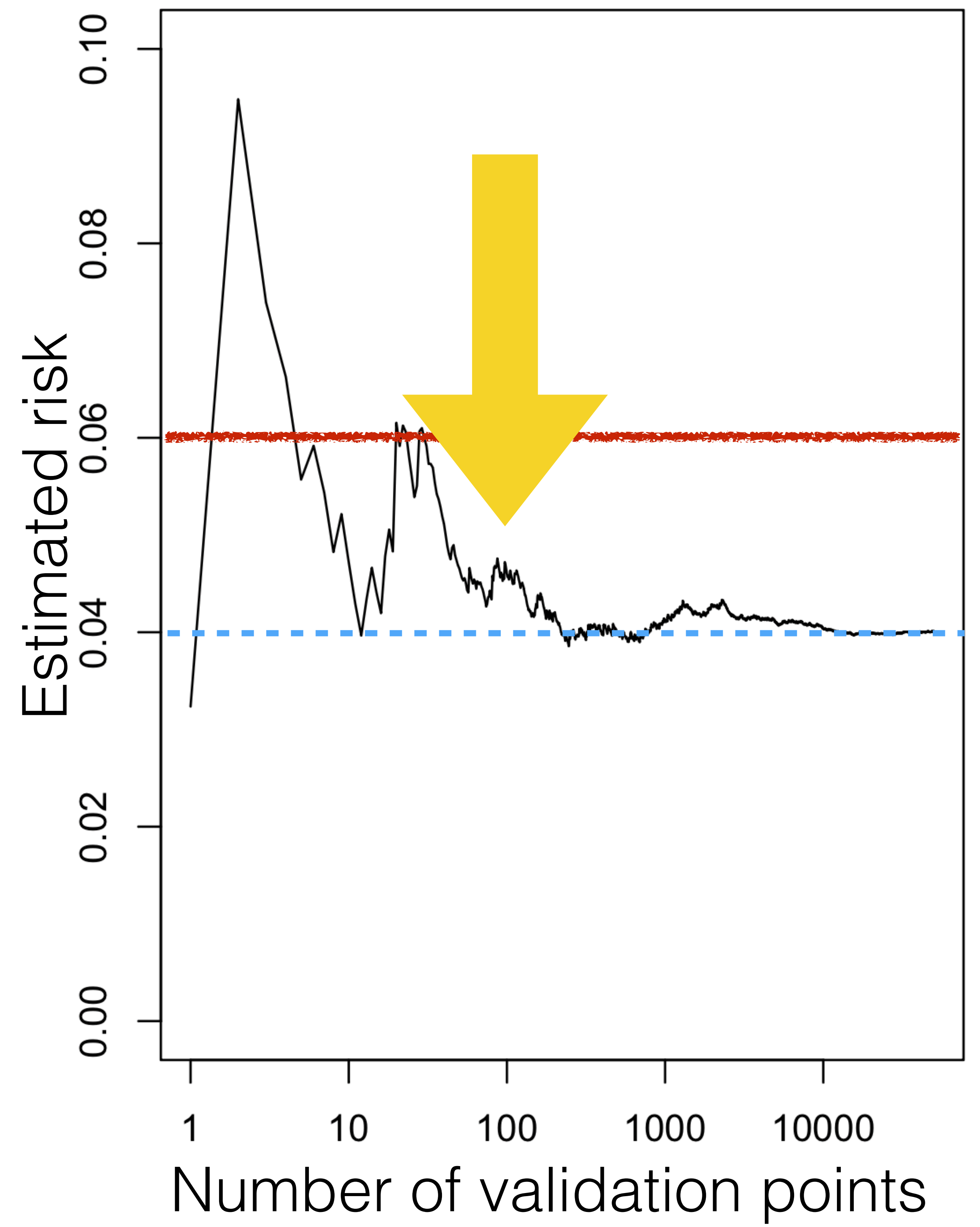

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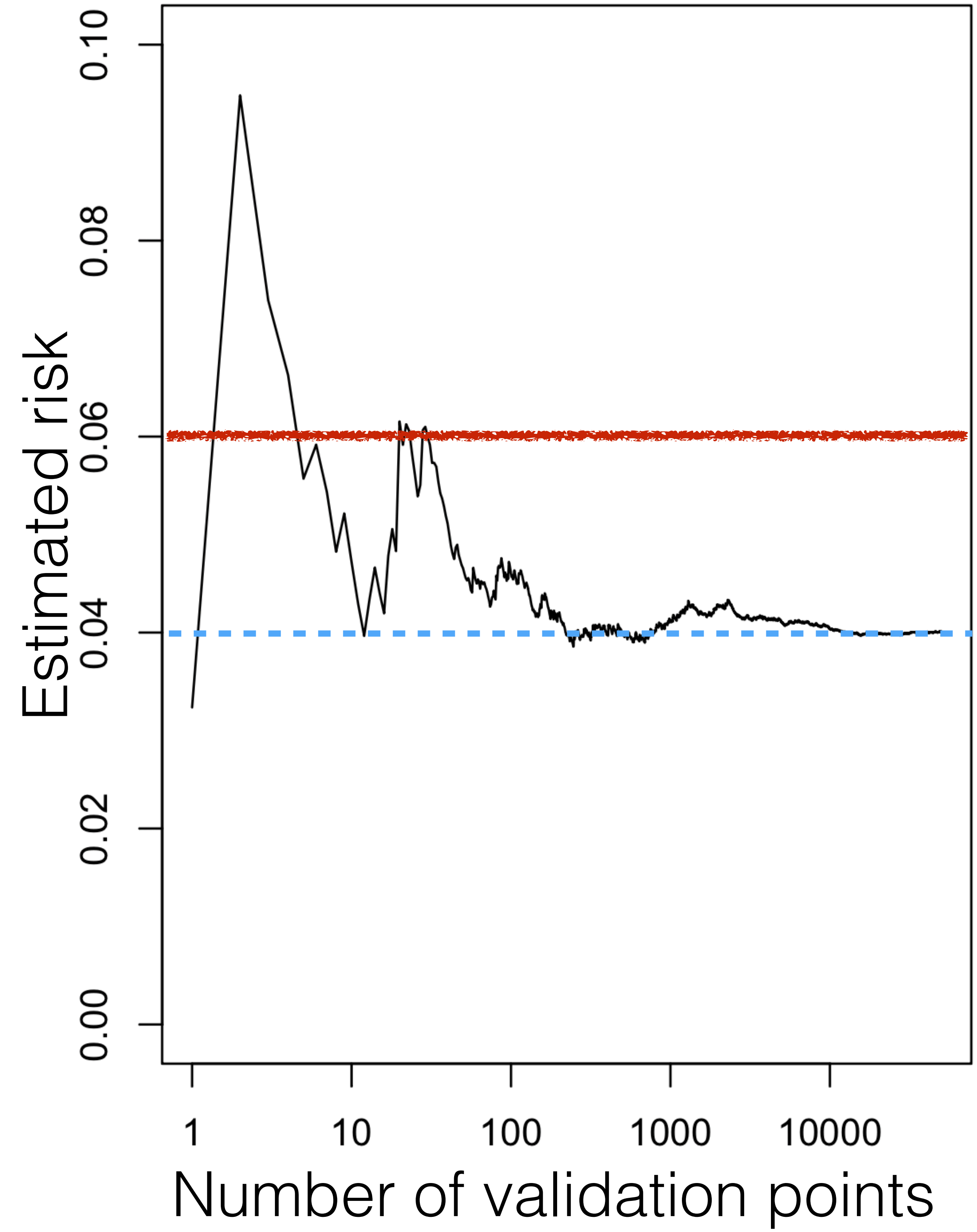

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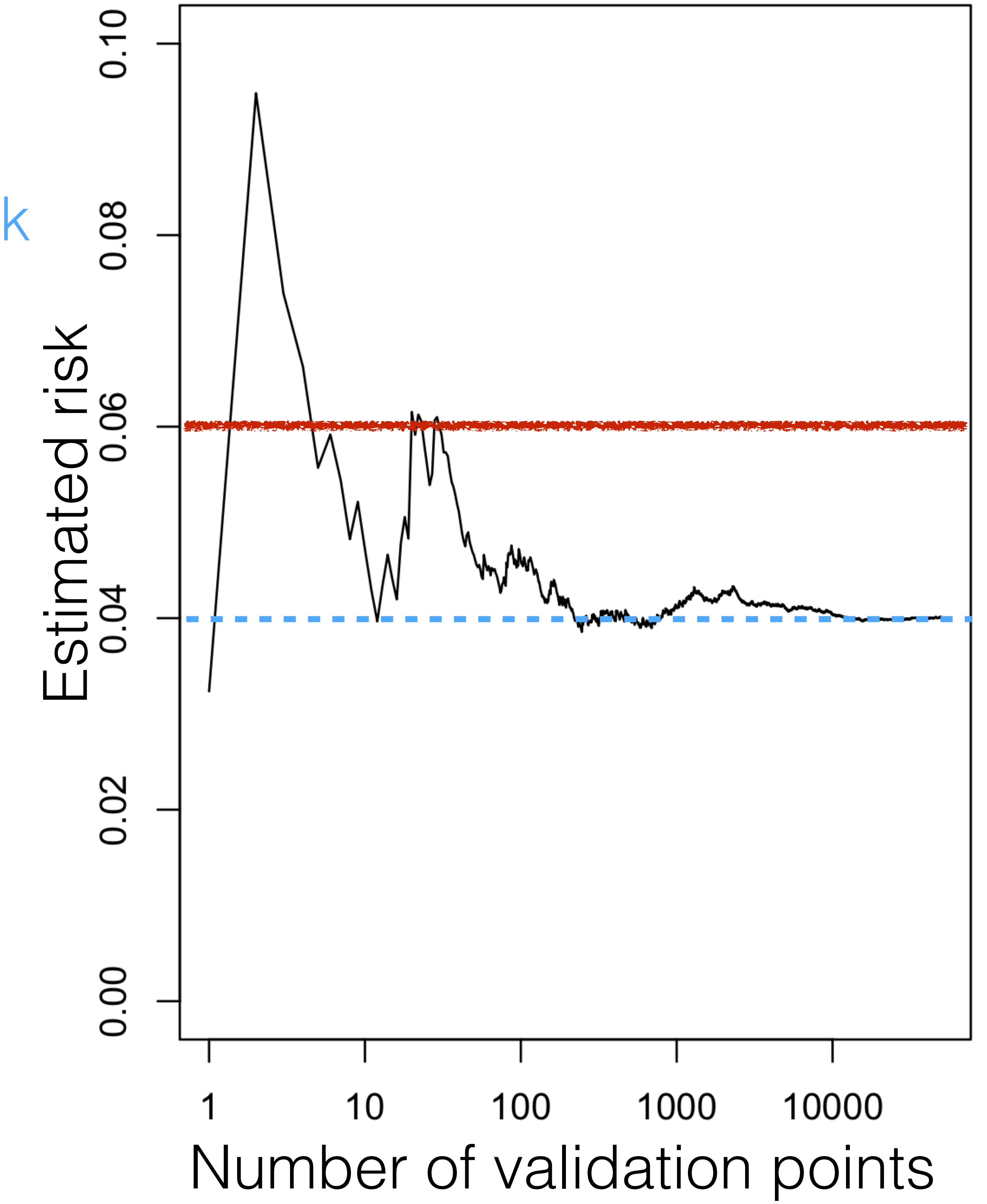

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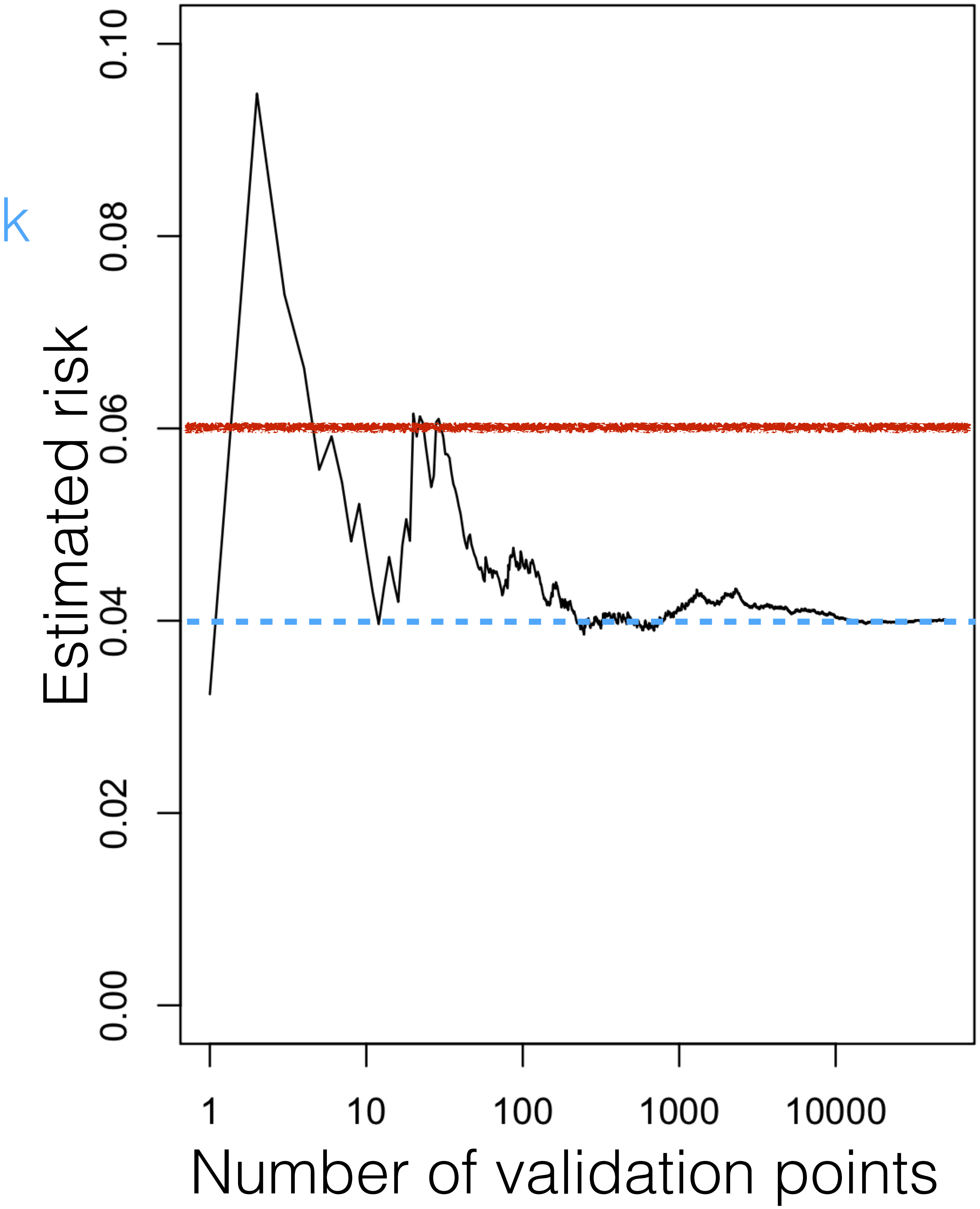

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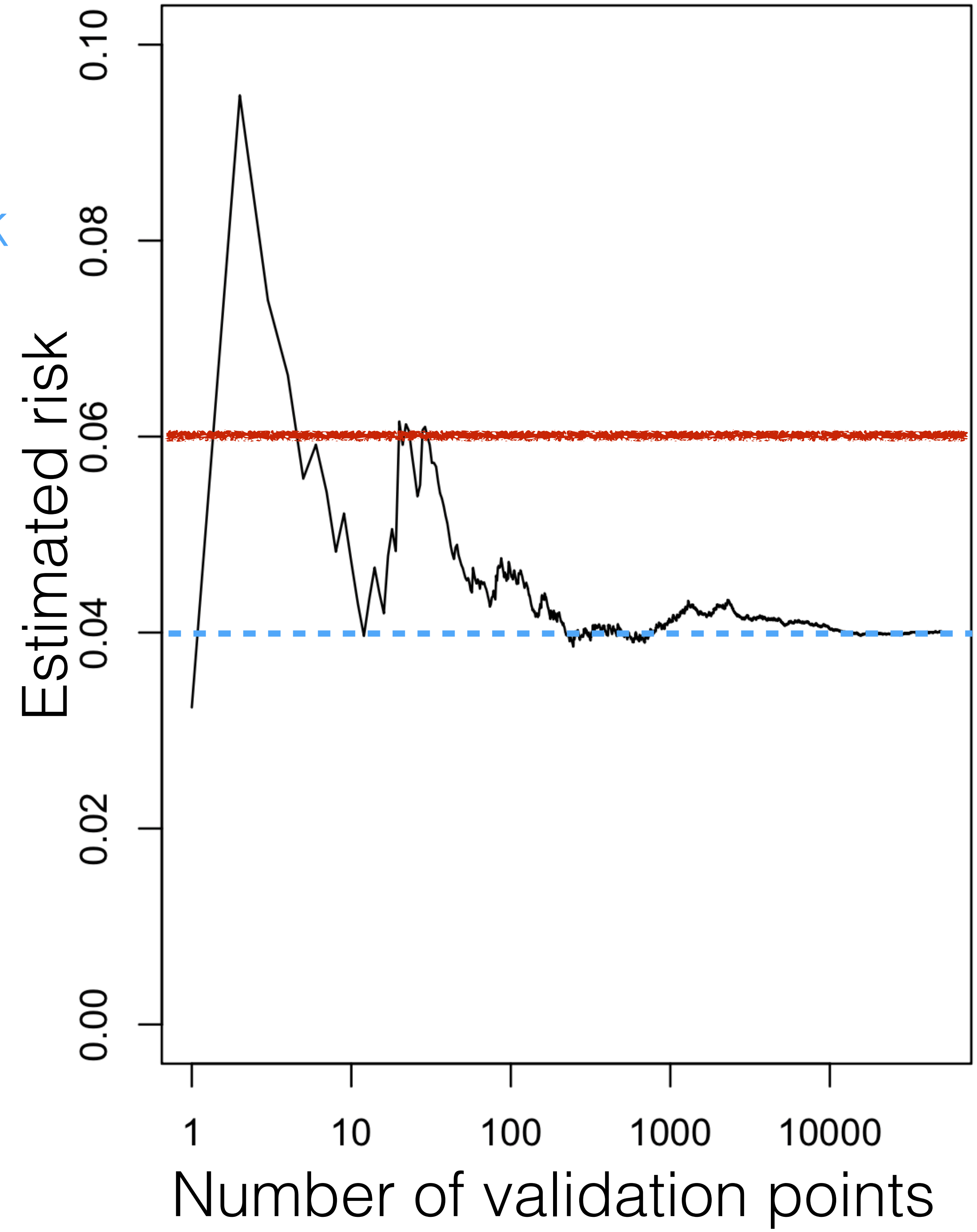

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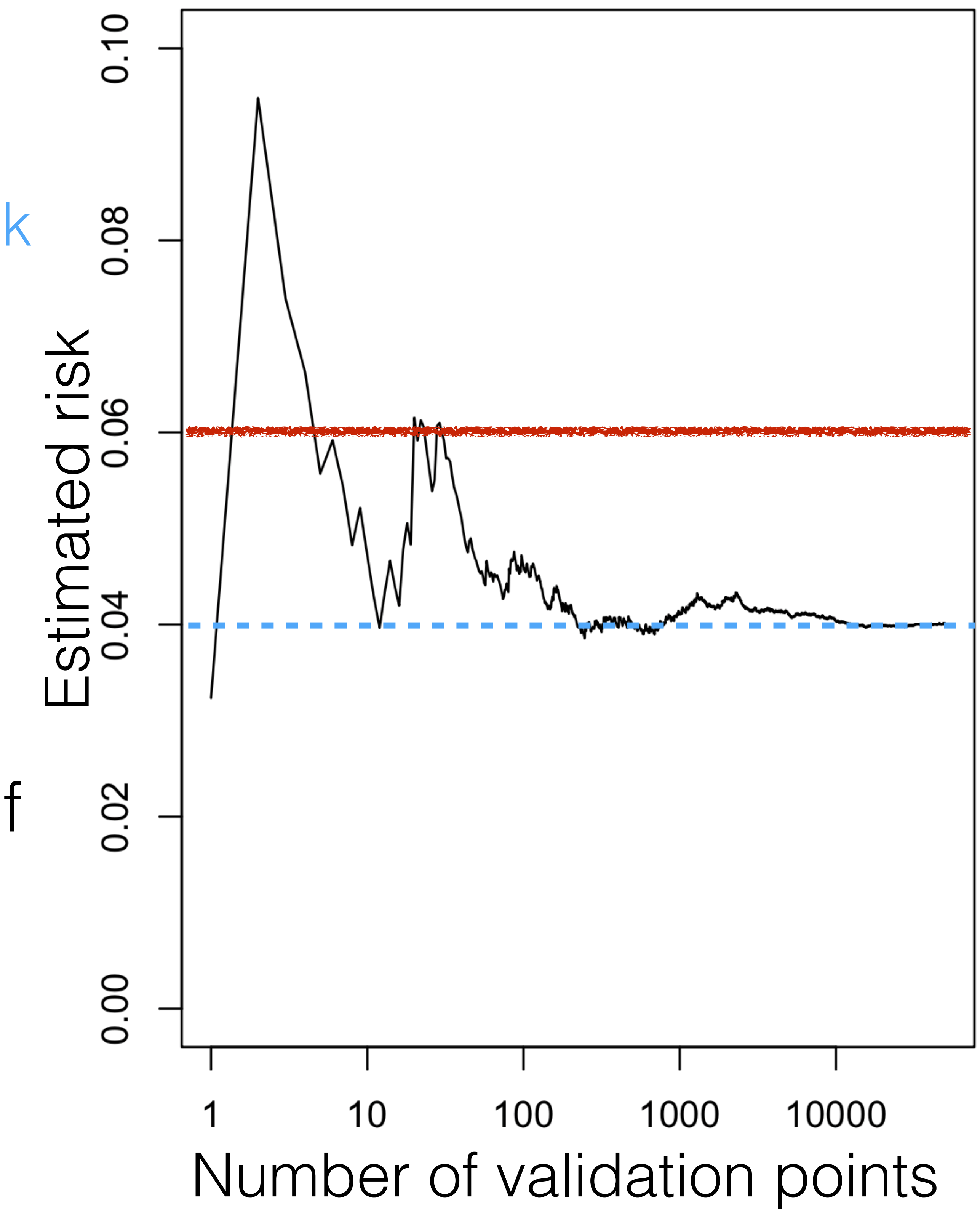

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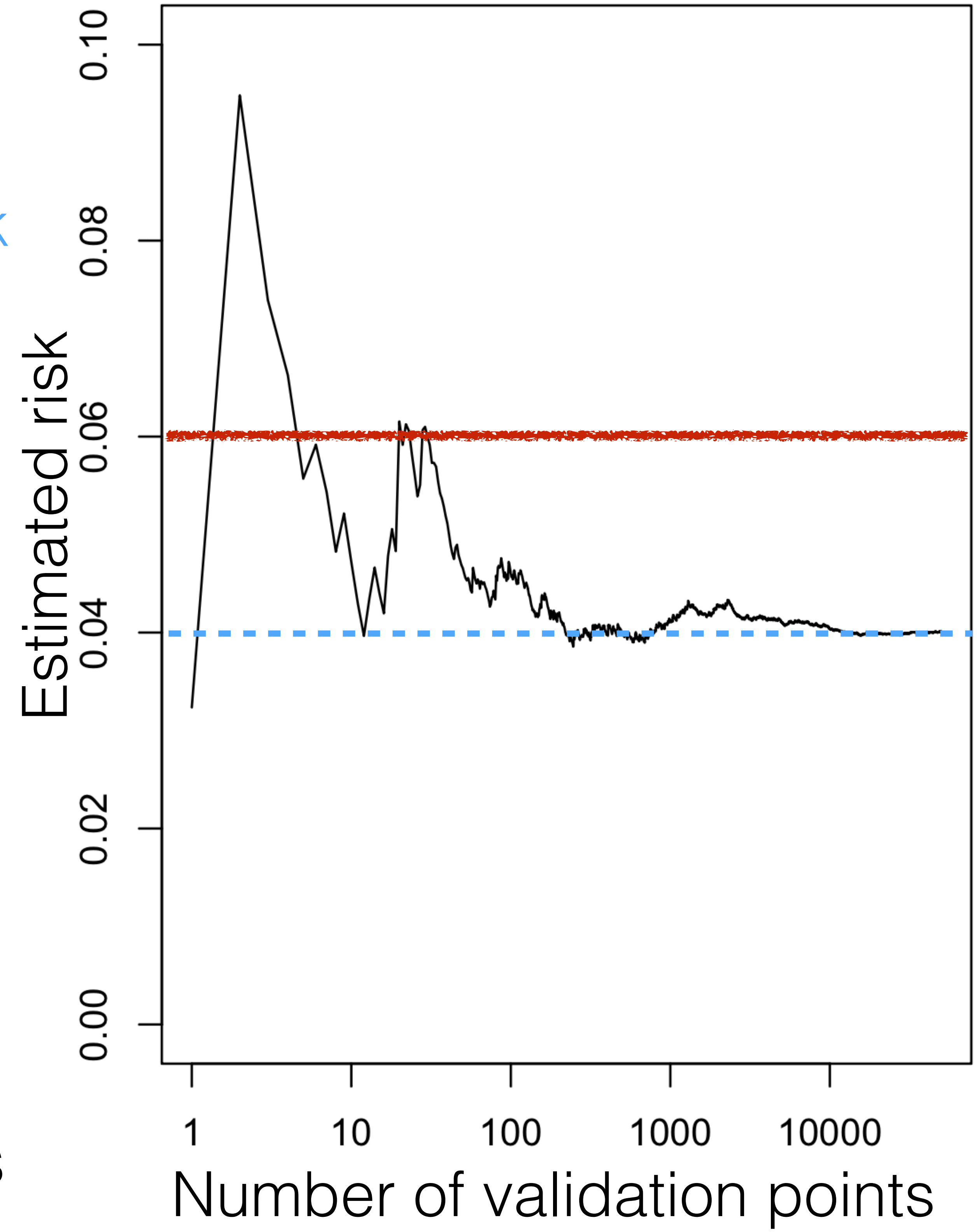

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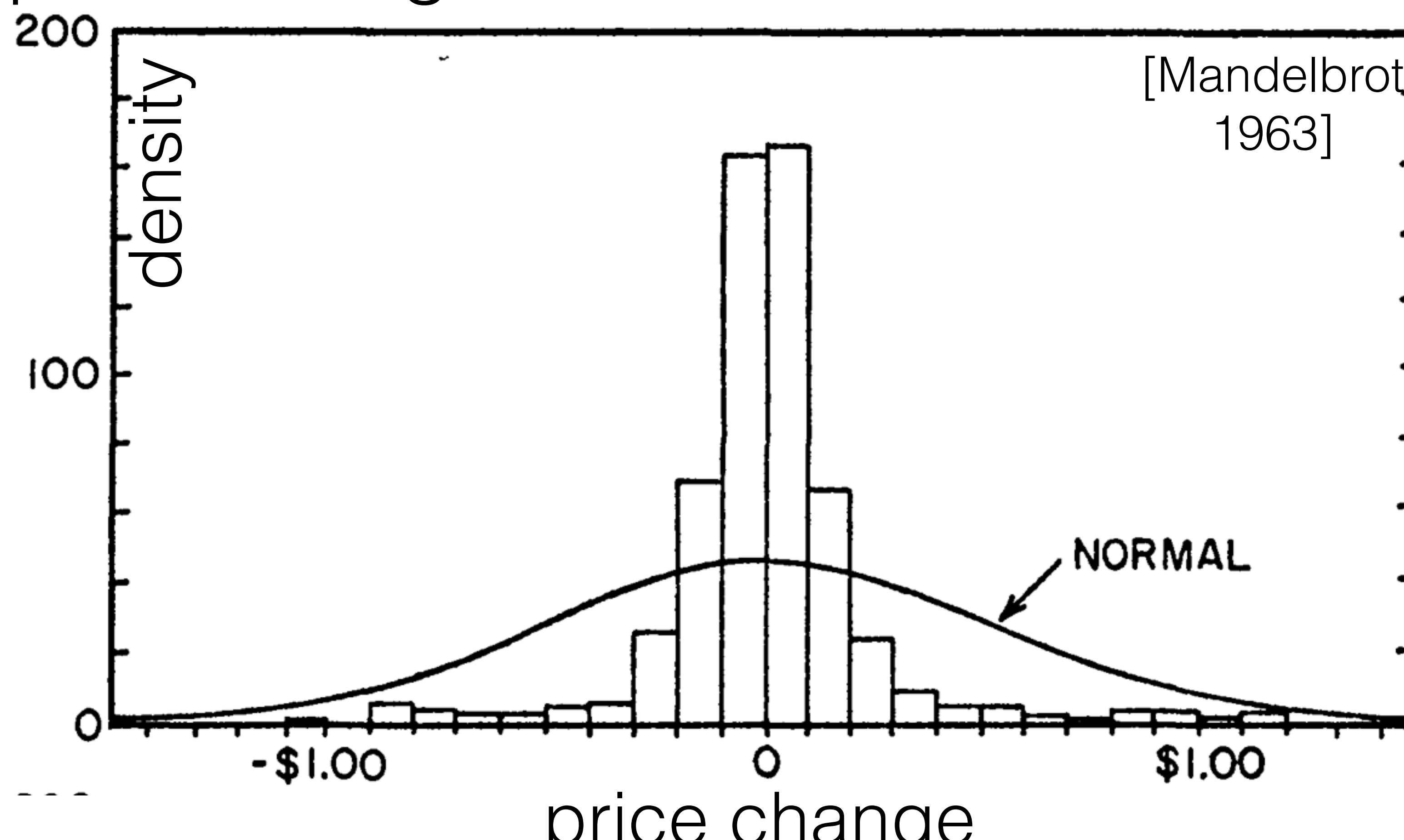

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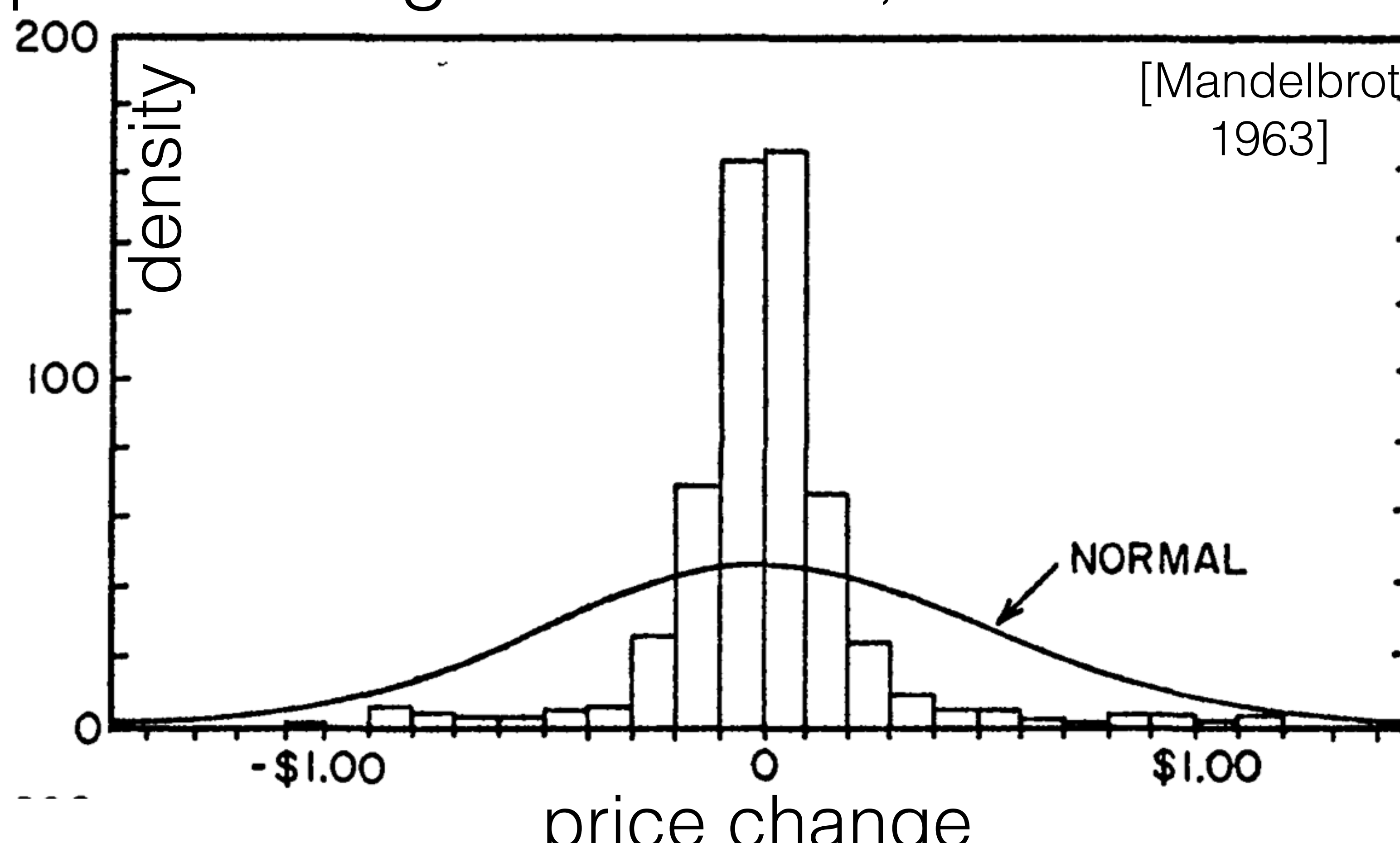

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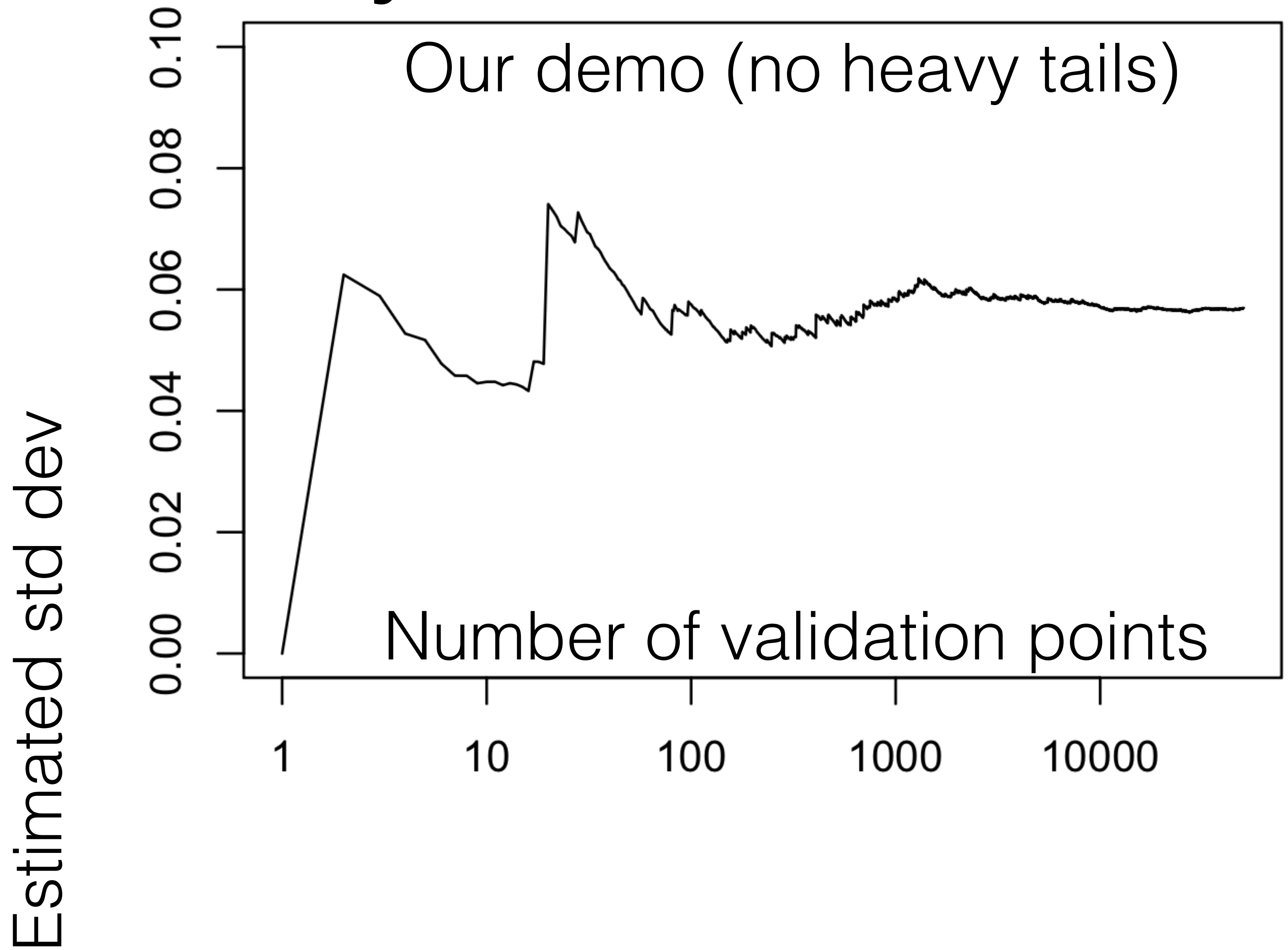

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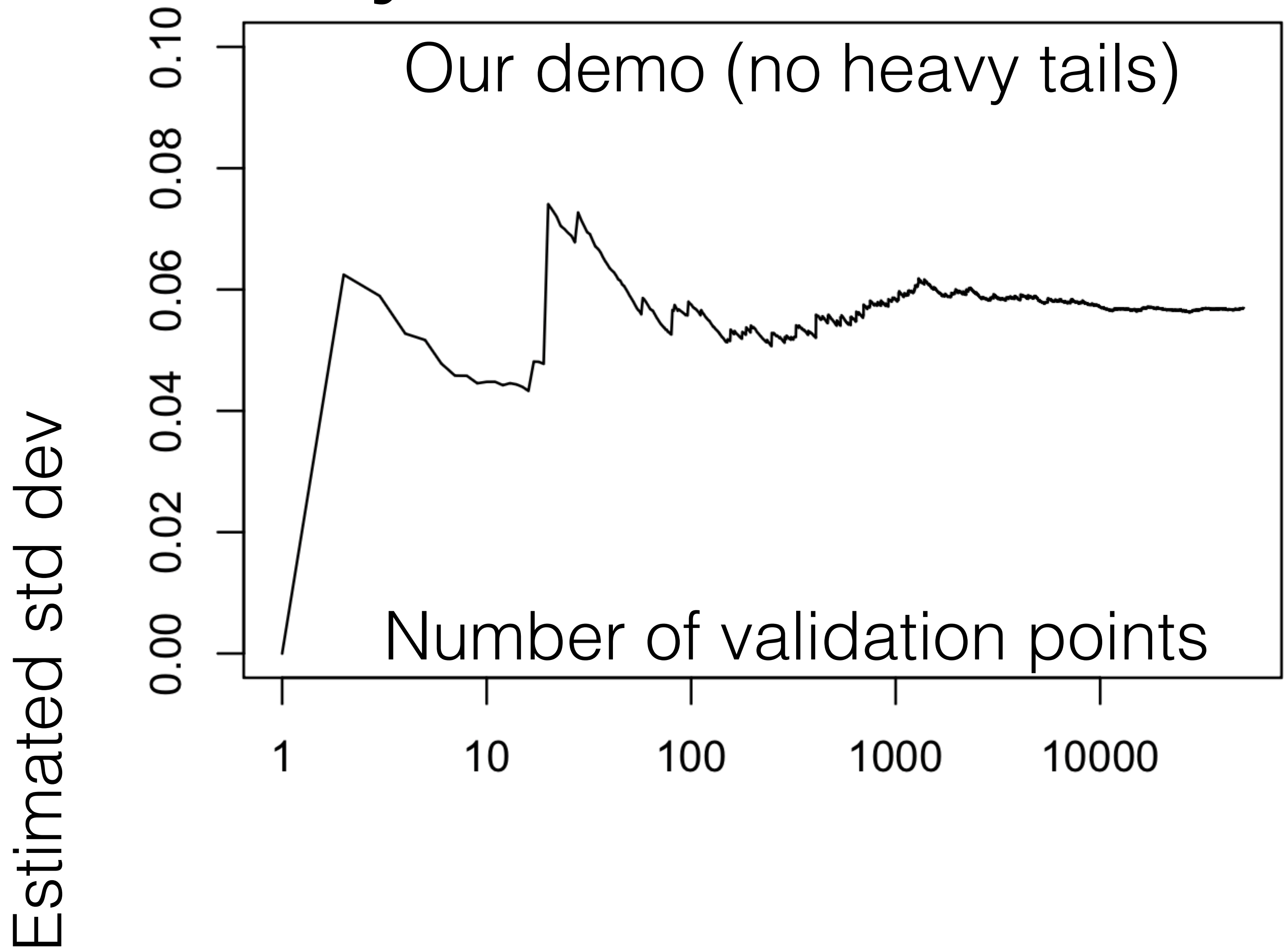

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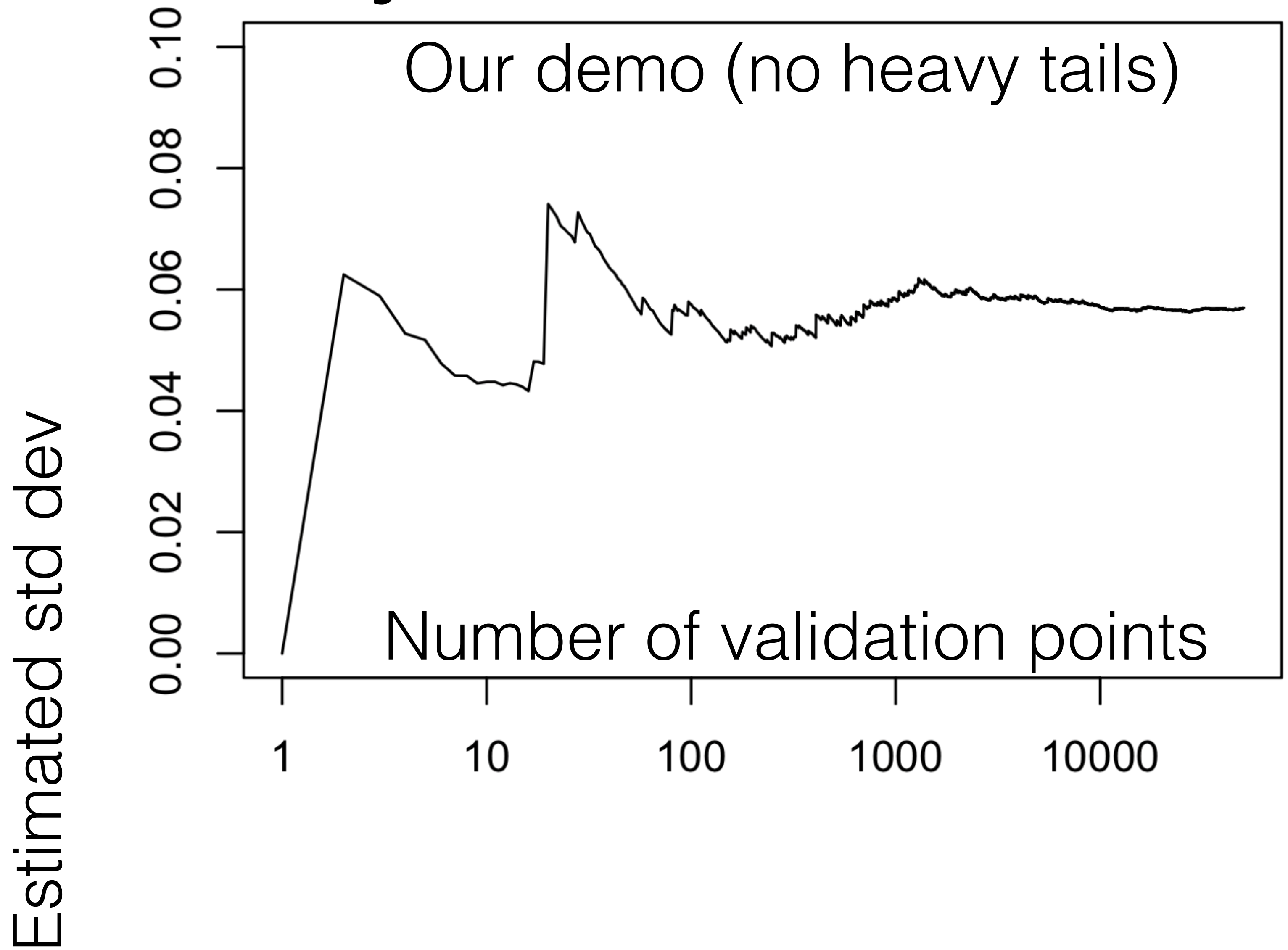

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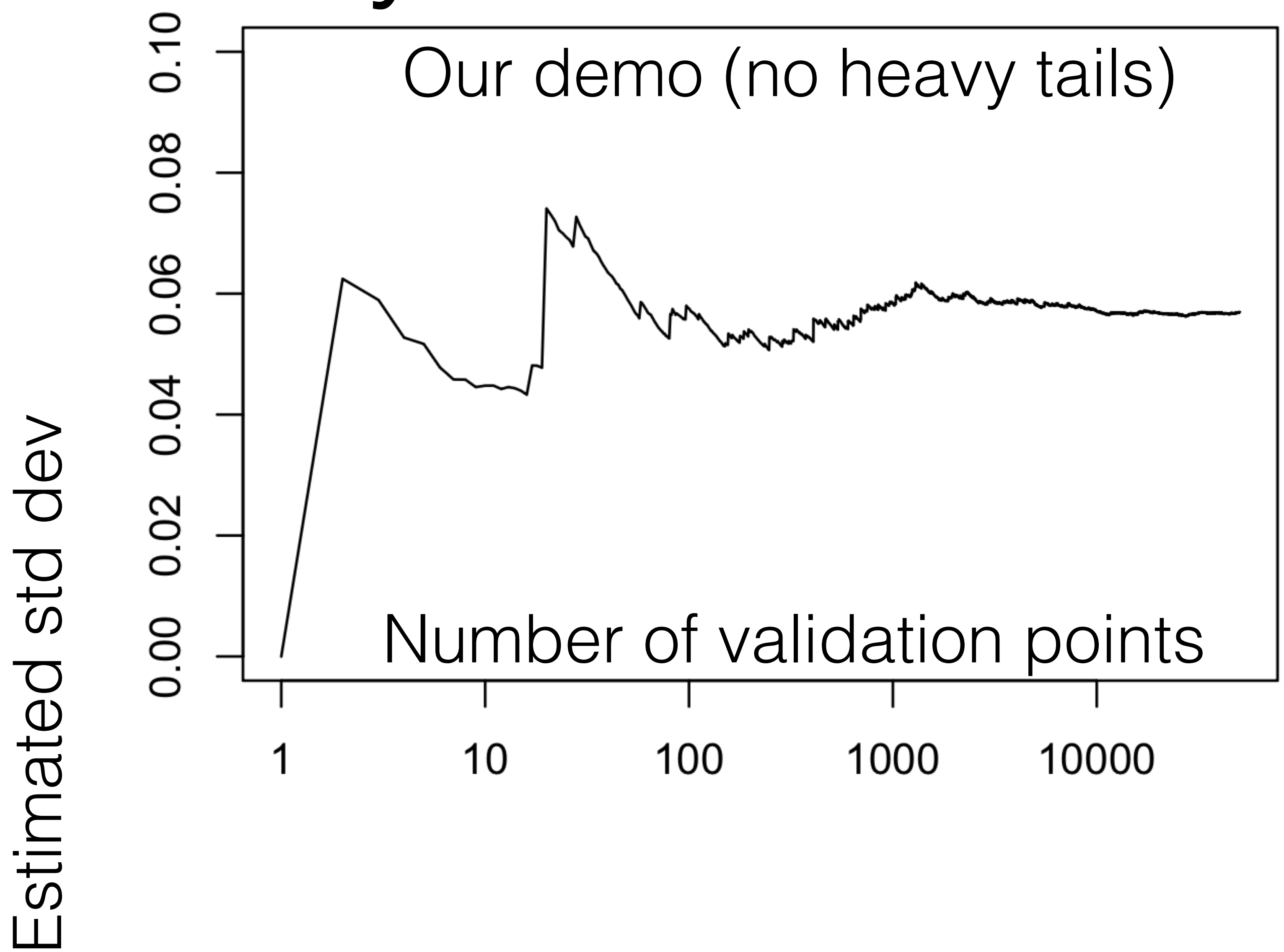

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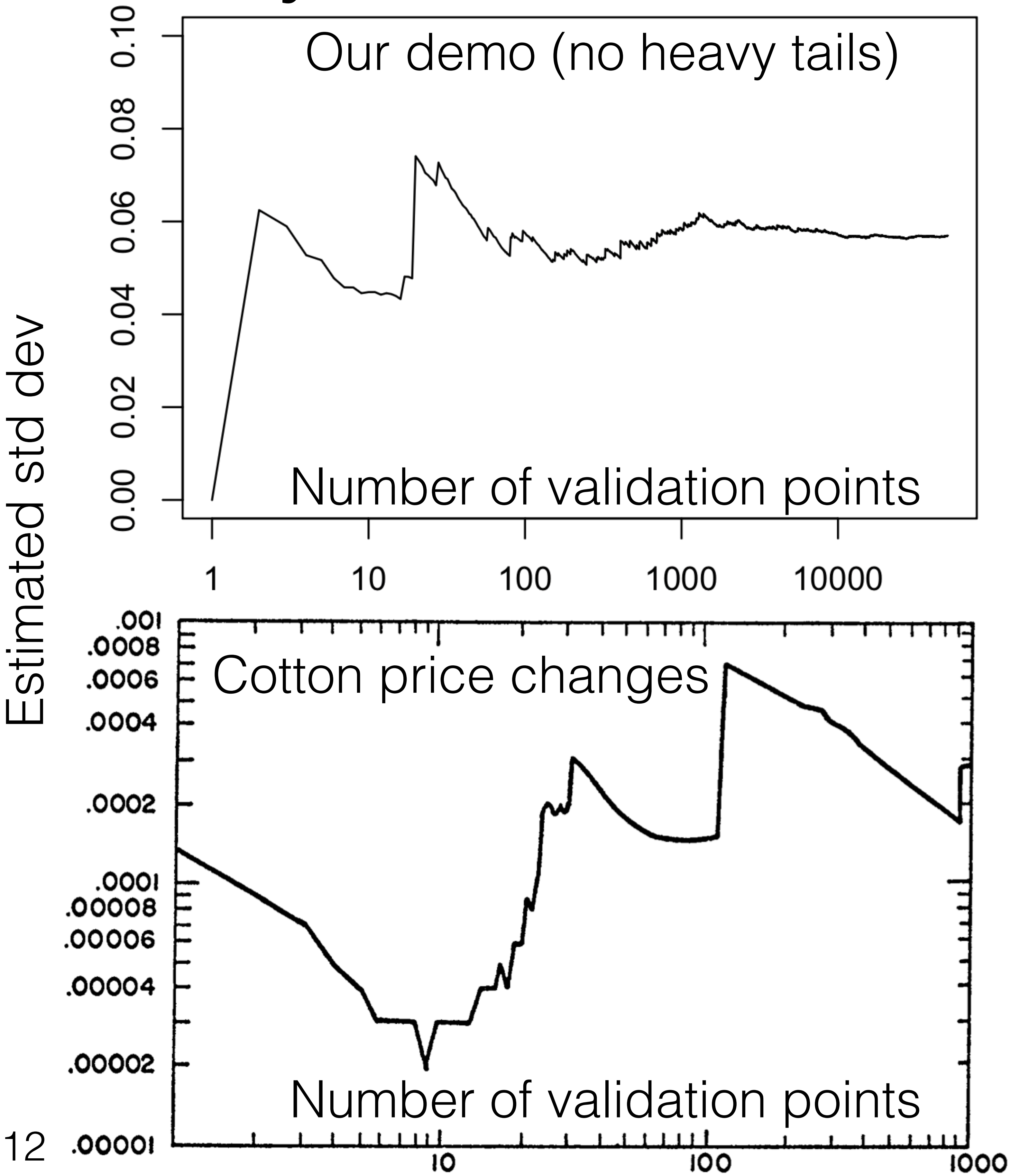

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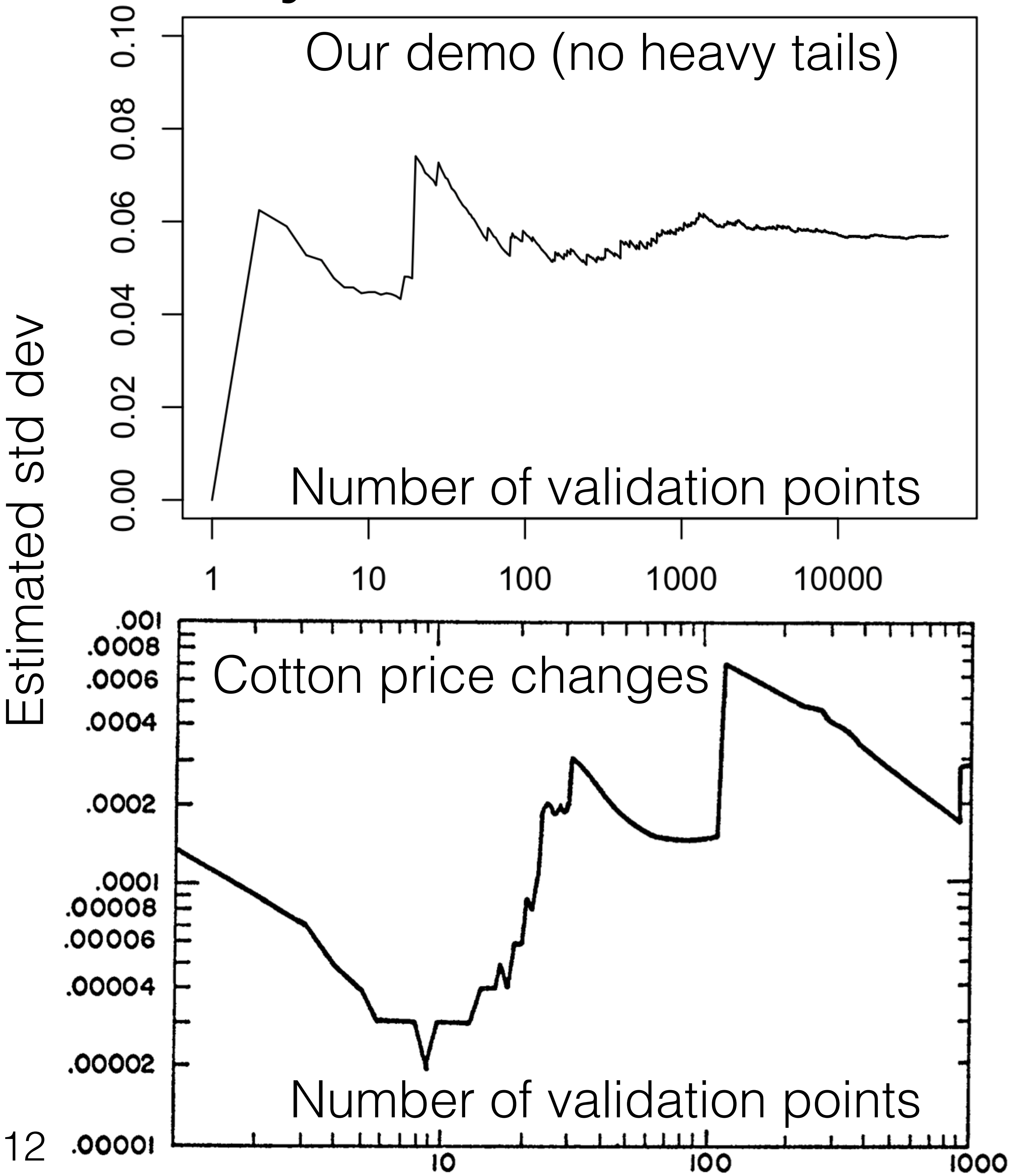

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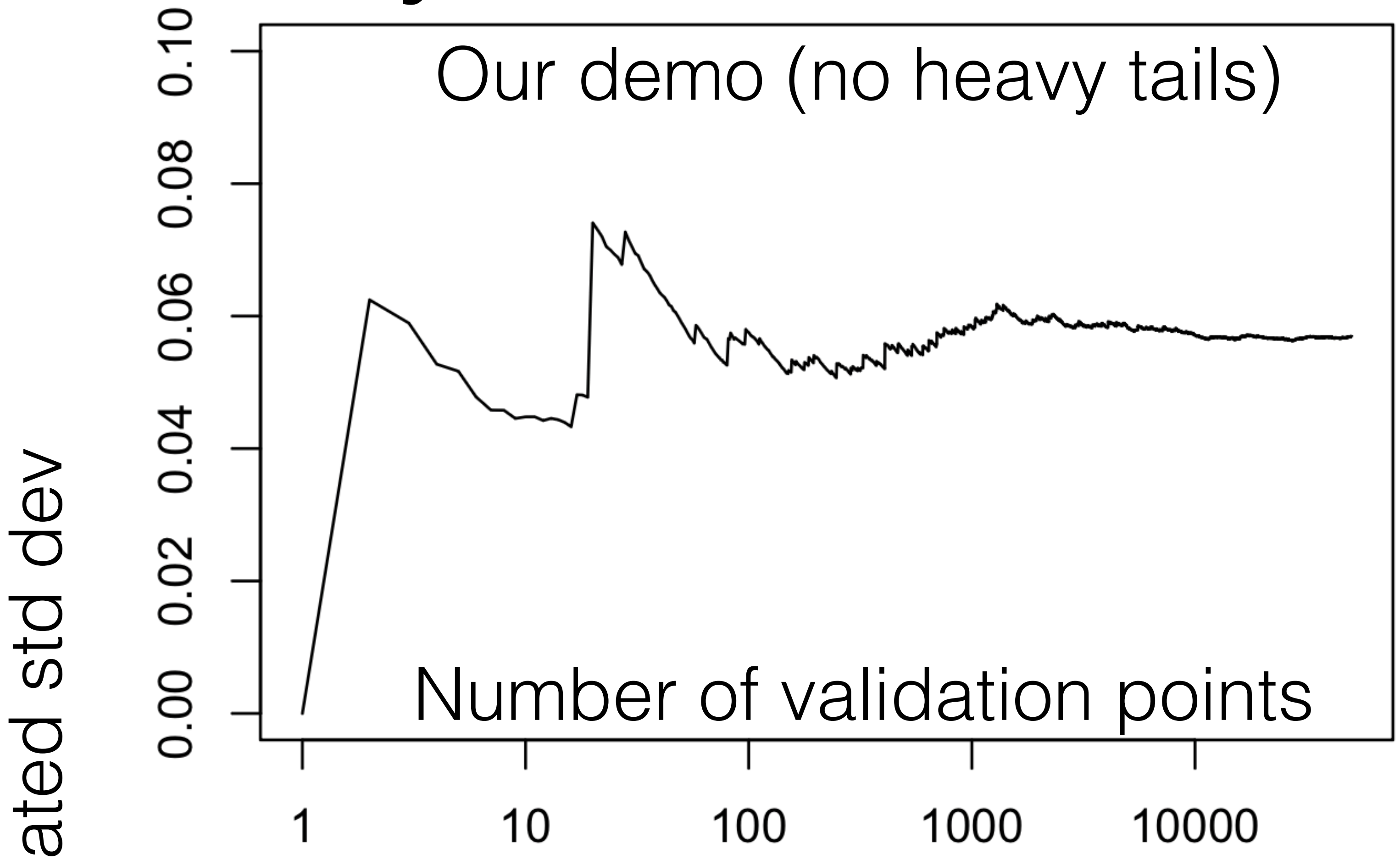

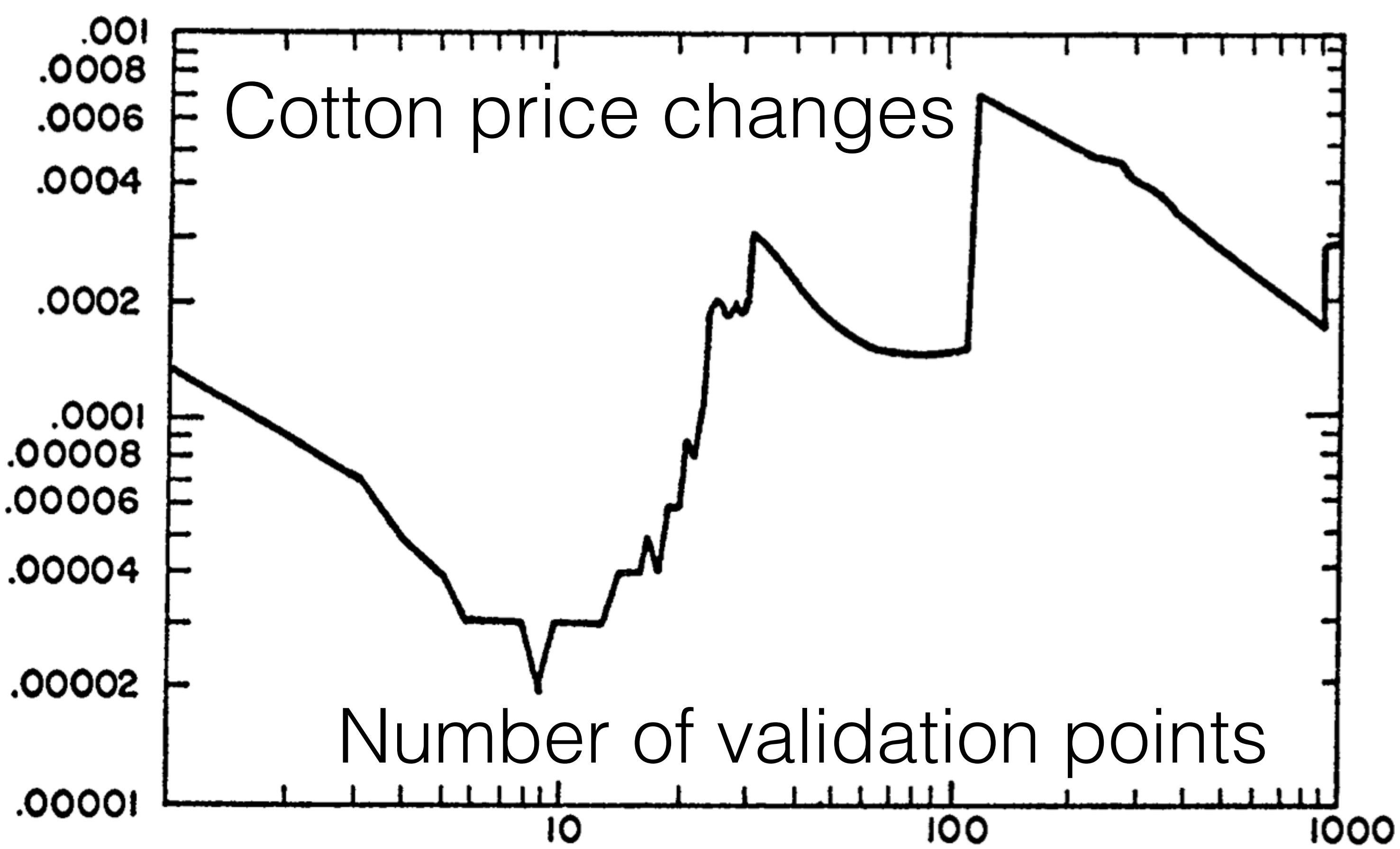

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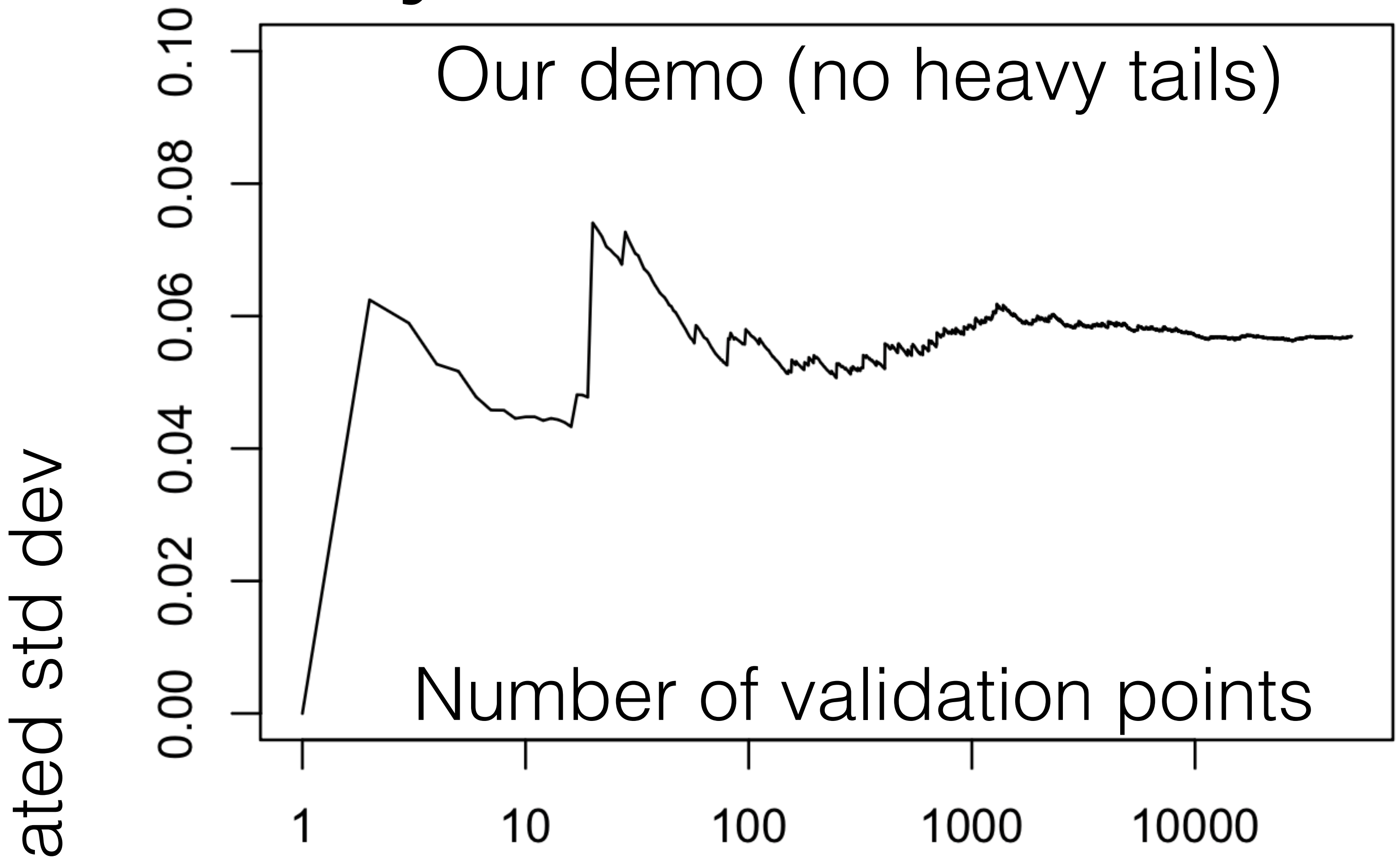

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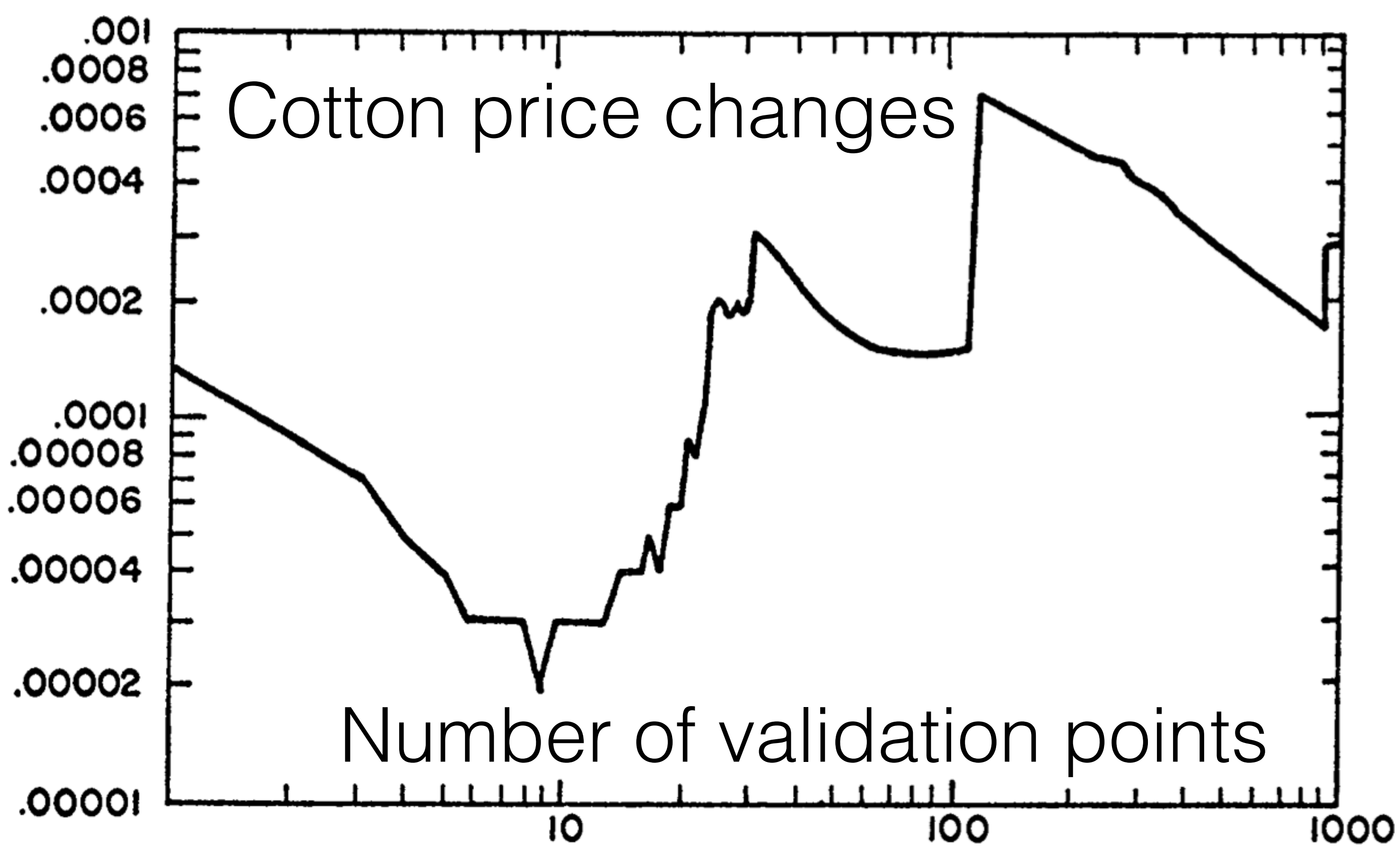

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- Variance doesn't exist
- Estimate jumps around with more data, does not converge

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References (1/1)

Meerschaert, M. M. and Hans-Peter Scheffler. Nonparametric methods for heavy tailed vector data: A survey with applications from finance and hydrology.

Mandelbrot, Benoit B. “The variation of certain speculative prices.” The Journal of Business, 1963.