6.7900: Machine Learning
Lecture 17

Lecture start: Tues/Thurs 2:35pm
Who's speaking today? Prof. Tamara Broderick
Course website: gradml.mit.edu
Questions? Ask here or on piazza.com/mit/fall2024/67900/
Materials: Slides, video, etc linked from gradml.mit.edu after the lecture (but there is no livestream)

Last Times

I. A common motif in spatiotemporal (and similar) data
II. GPs for regression: model and inference

Today

I. Gaussian processes: fitting hyperparameters, observation noise, high-dimensional inputs
II. Missing data

Recap: Gaussian processes (GPs)

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signal variance lengthscales

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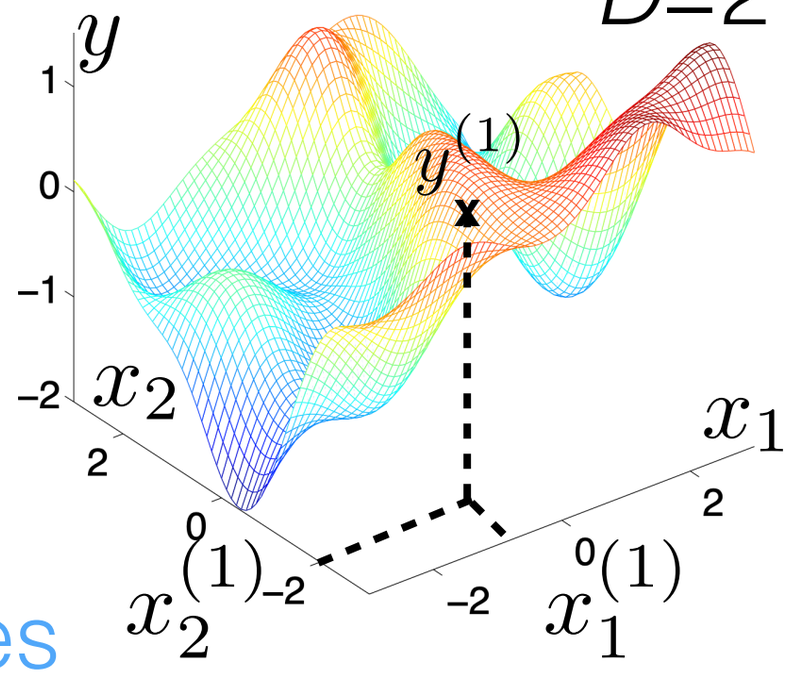

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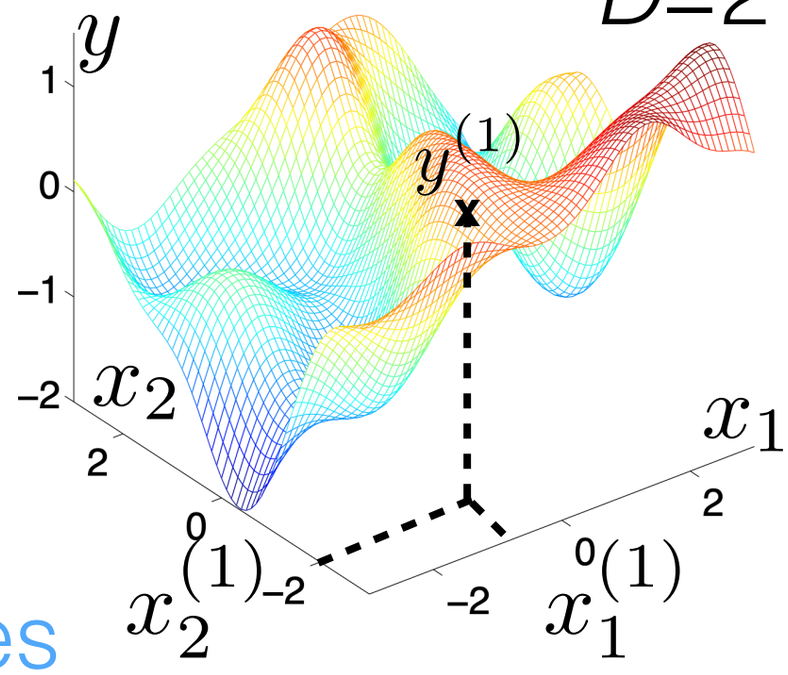

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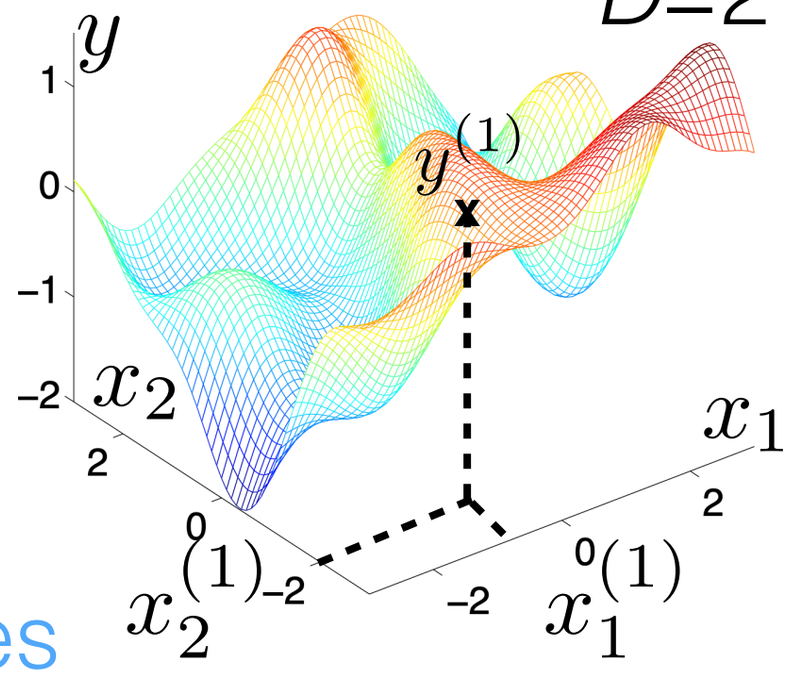

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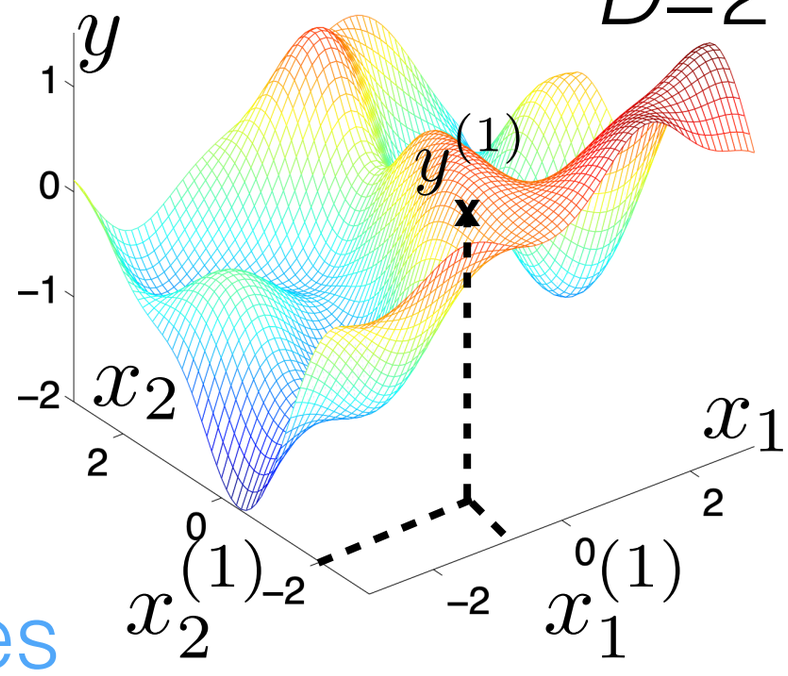

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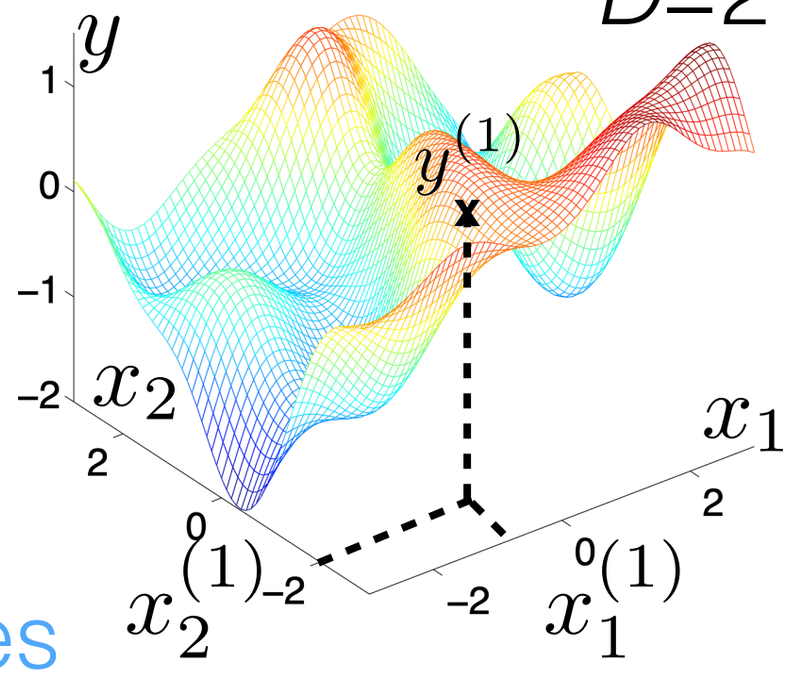

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- Automation / ease of use

[demo1,2,3]

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What if we put *y* here instead?

Observation noise

- So far we've been assuming that we observed *f*(*x*) directly
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  - We observe $\{(x^{(n)}, y^{(n)})\}_{n=1}^{N}$ and want to learn the latent *f*
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What if we put *y* here instead? [demo2, demo3]

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Can you state a non-trivial lower bound on the marginal variance of a test $y^{(m)}$? [demo2, demo3]

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Even when observations are "perfect," use a (very small) *nugget* for numerical reasons

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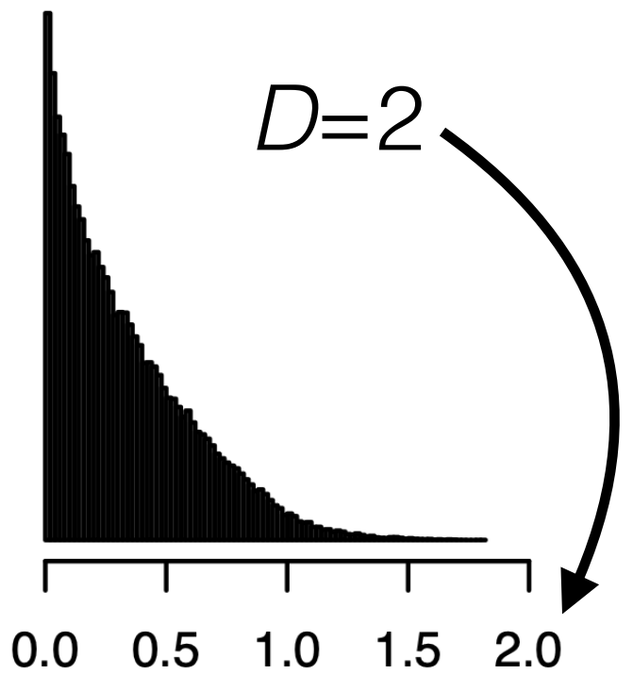

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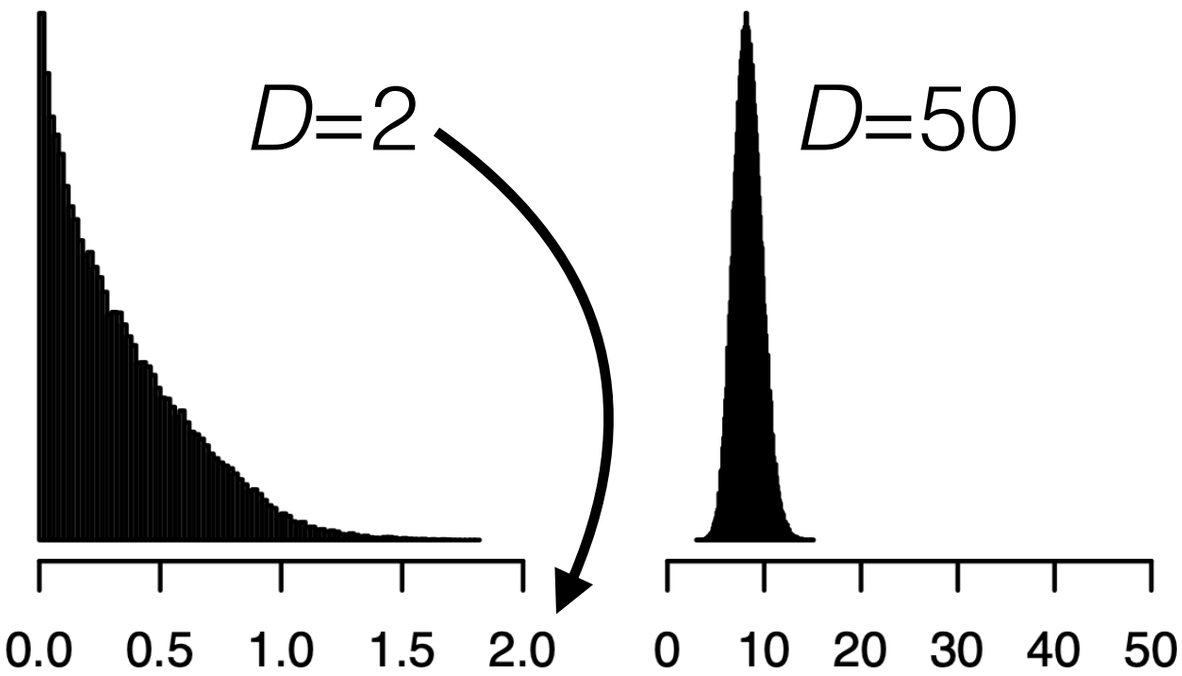

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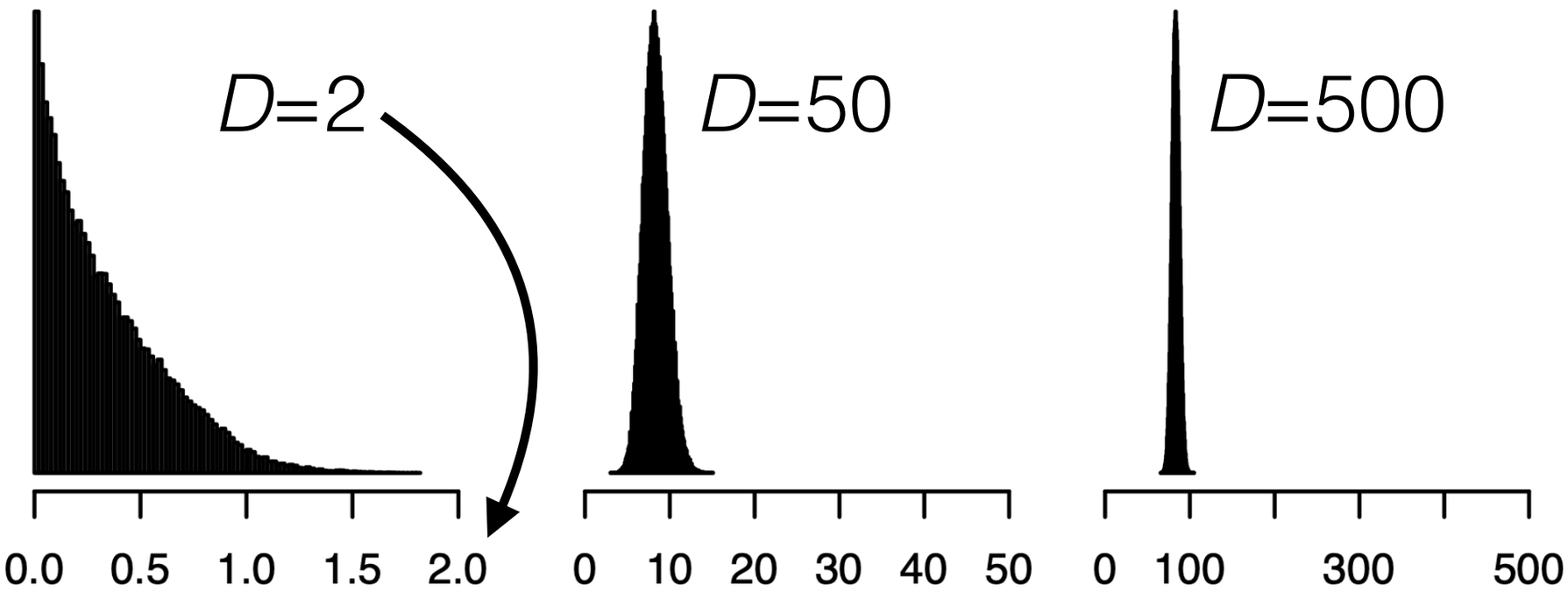

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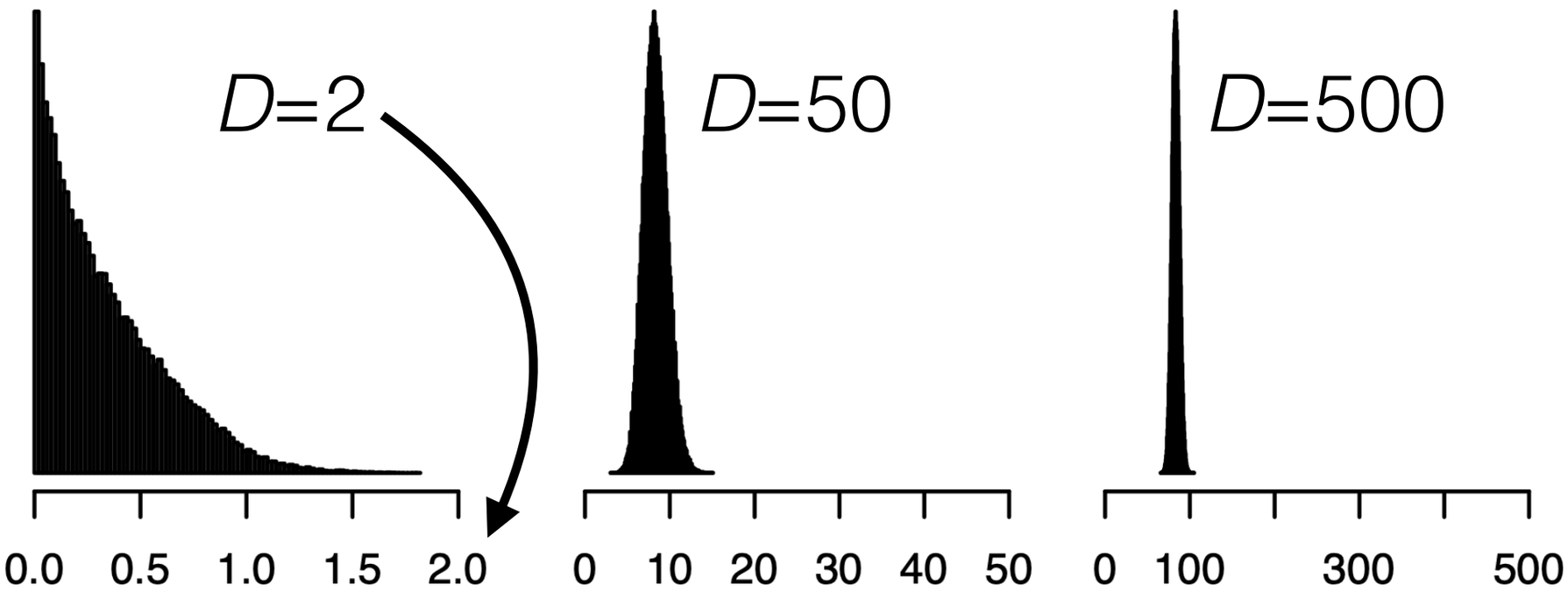

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- Recall: points "far" from data default to the prior mean and variance

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- What are our options?
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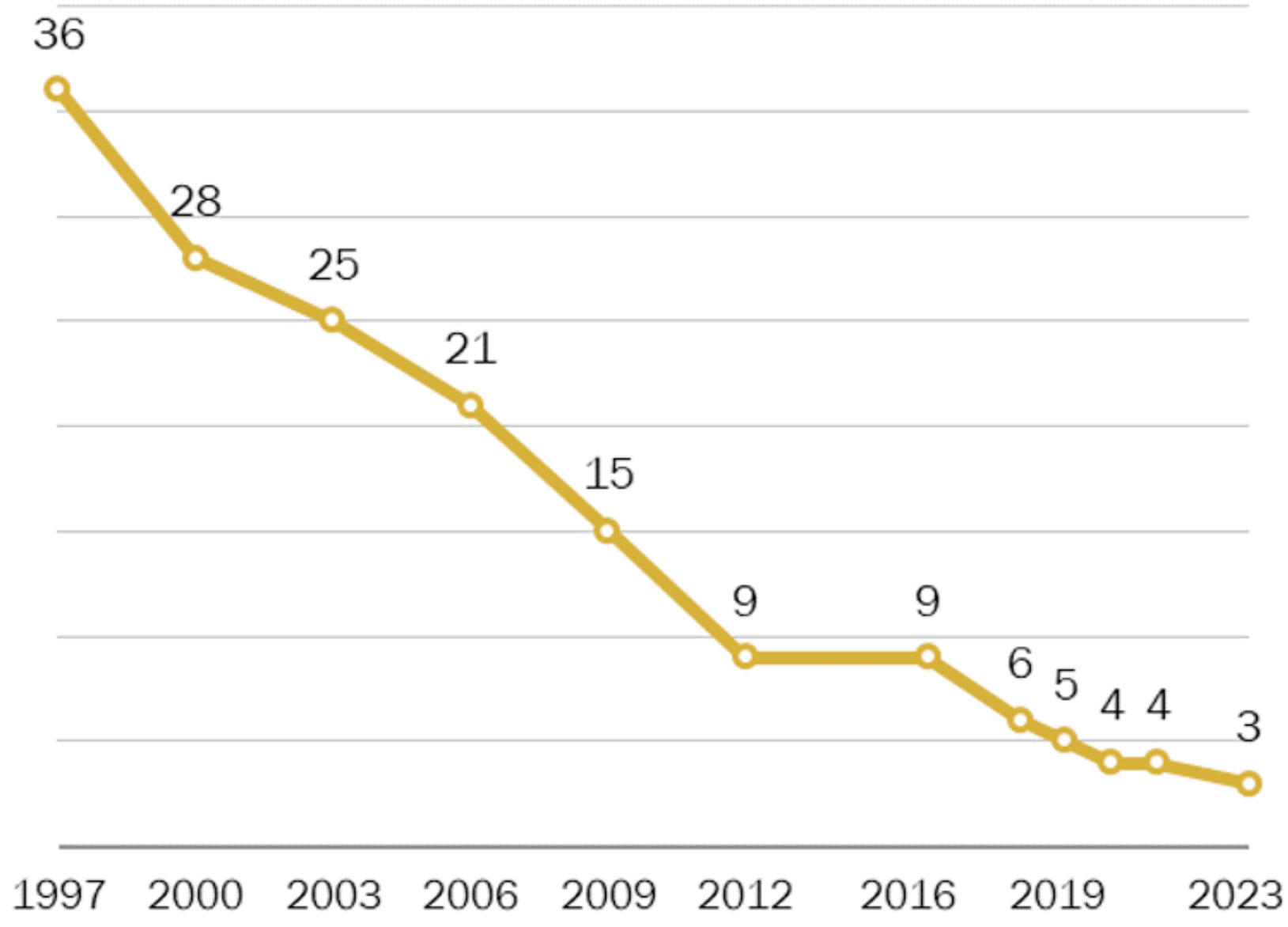
Telephone survey response rates, 1997-2023

% responding, based on AAPOR RR3



Annual averages of Pew Research Center telephone survey response rates

PEW RESEARCH CENTER

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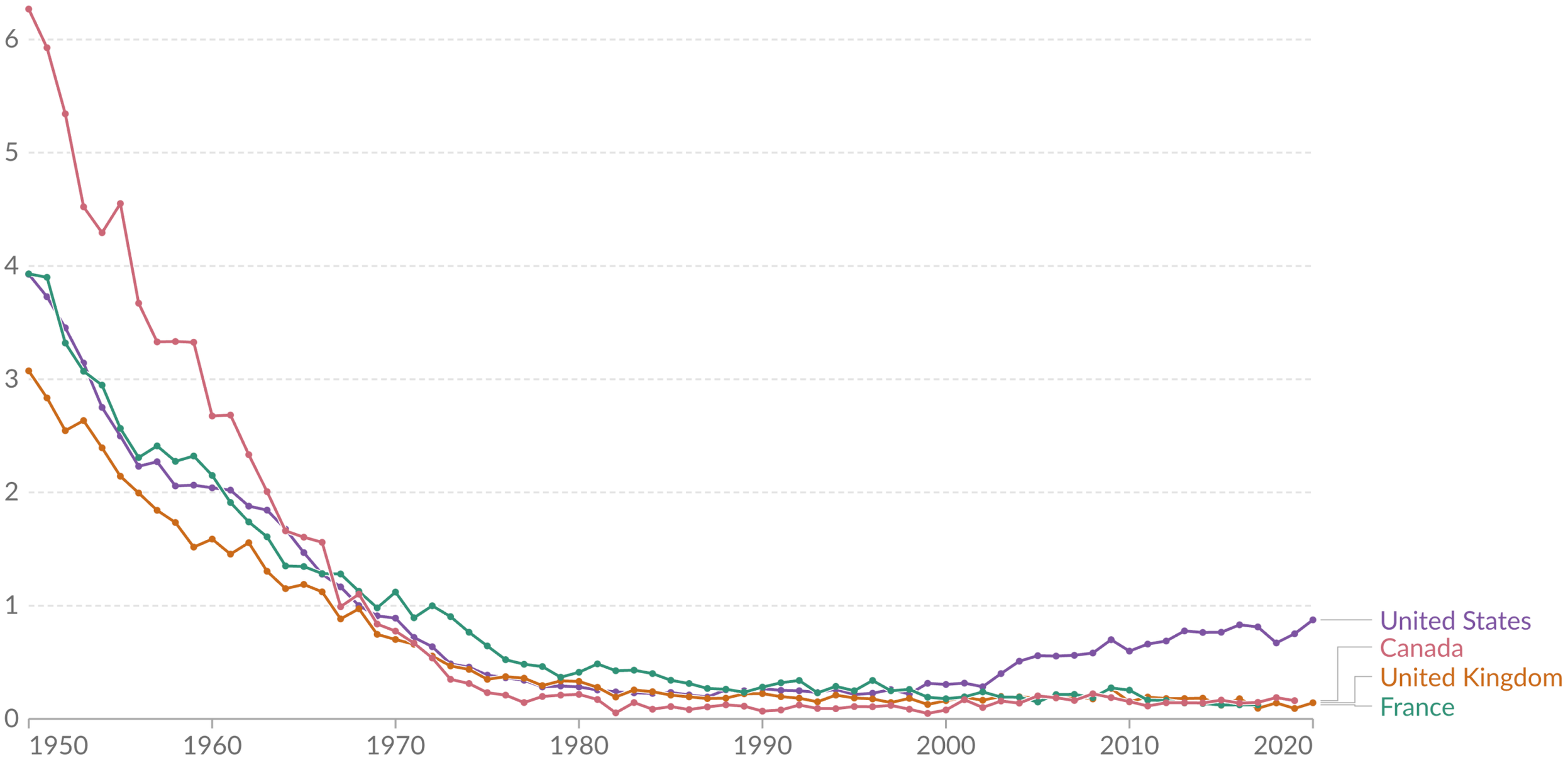
Reported maternal mortality rate

Our World in Data

Reported annual death rate from maternal conditions per 100,000 women and girls, based on official statistics from each country. This includes late maternal deaths that occur up to 1 year after the end of pregnancy. Due to limited reporting, figures are lower than the true number of maternal deaths.



Data source: WHO Mortality Database (2022)

OurWorldInData.org/causes-of-death | CC BY

Note: To allow for comparisons between countries and over time, this metric is age-standardized. All deaths in a country may not have been registered with a cause of death.

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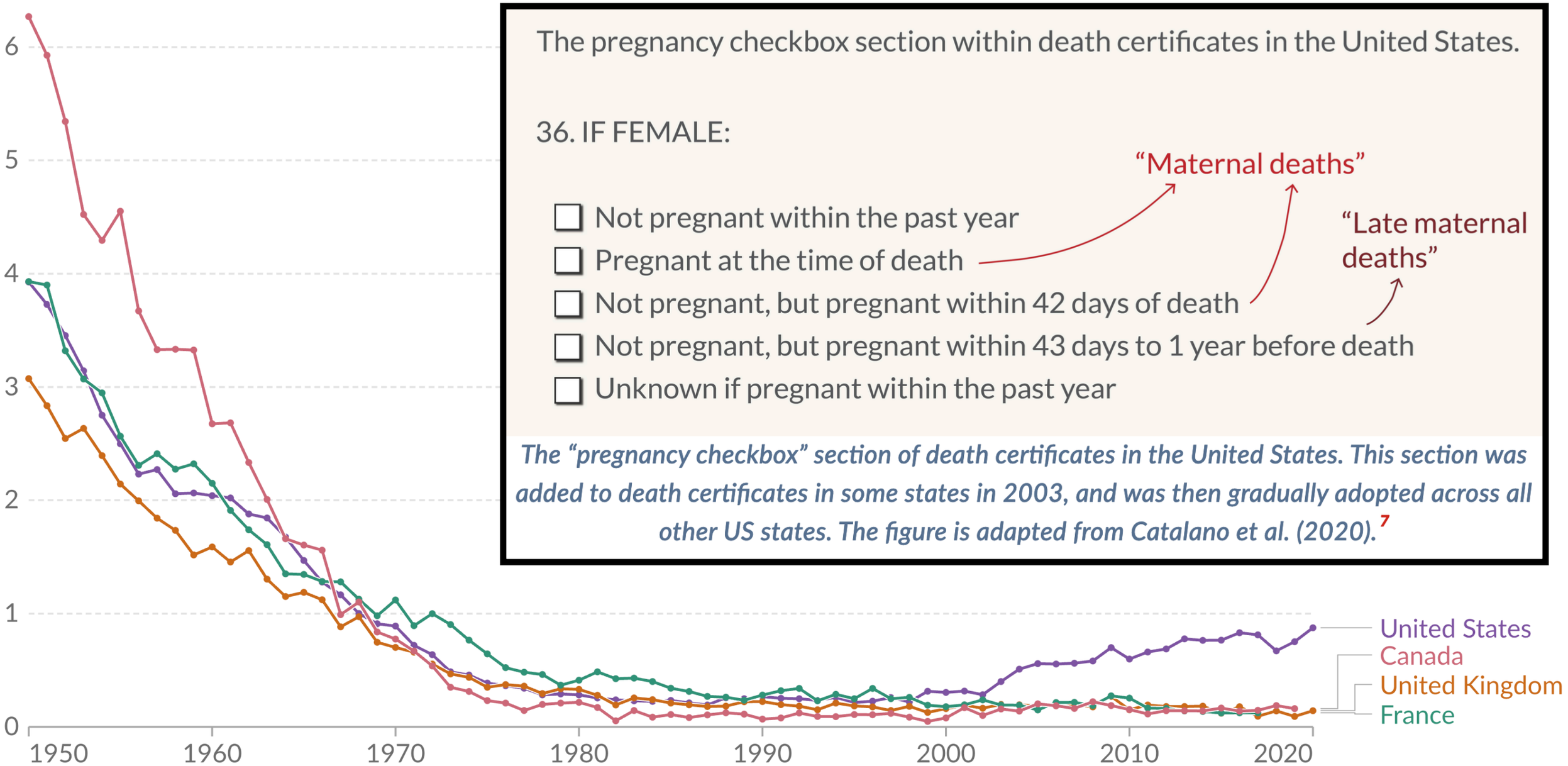

The pregnancy checkbox section within death certificates in the United States.

36. IF FEMALE:

"Maternal deaths"

"Late maternal deaths"

- ☐ Not pregnant within the past year
- ☐ Pregnant at the time of death
- ☐ Not pregnant, but pregnant within 42 days of death
- ☐ Not pregnant, but pregnant within 43 days to 1 year before death
- ☐ Unknown if pregnant within the past year

The "pregnancy checkbox" section of death certificates in the United States. This section was added to death certificates in some states in 2003, and was then gradually adopted across all other US states. The figure is adapted from Catalano et al. (2020).[7]

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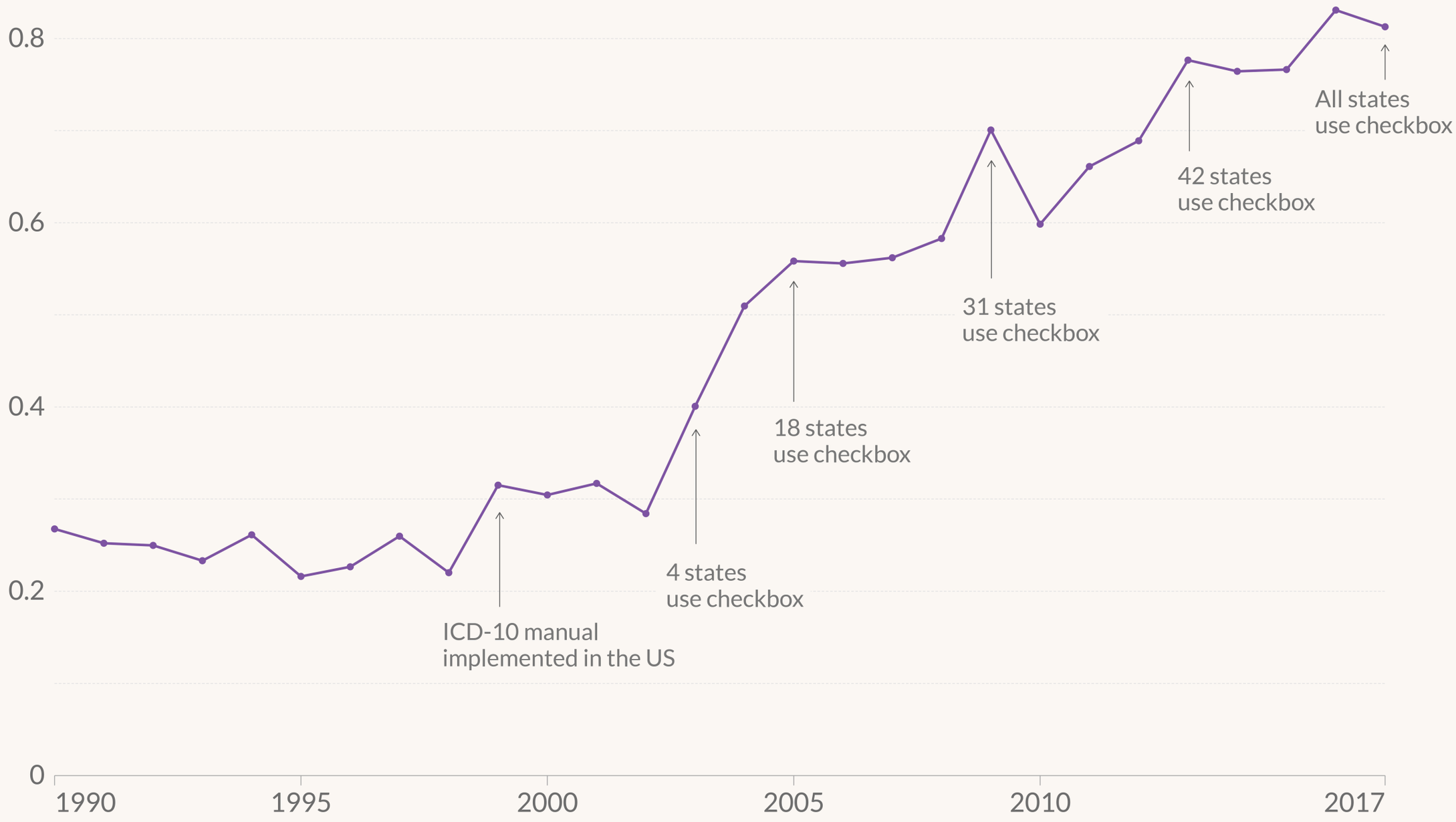
The US maternal mortality rate rose as more states adopted the “pregnancy checkbox”

Our World in Data

As more states in the US adopted the “pregnancy checkbox” on death certificates — which asked if the deceased had been pregnant or recently pregnant — the reported maternal mortality rate rose.

Maternal mortality rate, per 100,000 females



Source: WHO Mortality Database (2022). Adapted from KS Joseph et al. (2021) Maternal mortality in the United States.
Data includes “late maternal deaths”, which occur up to 1 year after the end of pregnancy.

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  - Aside: worth digging into data provenance

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  - Still feasible that we might take practical actions

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    - Cf. our discussion on extrapolation

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