1 First-Order Logic

Important: In this section, assume that w, x, y, z are variables; A, B, C, D are constants; and $f(\cdot), g(\cdot), h(\cdot)$ are functions; and $P(\cdot), Q(\cdot), R(\cdot), S(\cdot), \ldots$ are predicates.

1.1 Normal forms

Problem 1 (12 pts): Convert to prenex normal form (4 points each):

$\exists_y \neg P = \neg \forall_y P$

1. $\forall x \neg (\exists y \neg P(x,y))$ $\neg\neg \forall_y P = \forall_y P$ $\boxed{\forall_x \forall_y P(x,y)}$

$\neg\forall_x(P \vee \neg\forall_y Q) = \neg\forall_x P \vee \neg\forall_x \neg\forall_y Q = \exists_x \exists_y \neg(P \vee Q)$

2. $\neg \forall x \left(P(x) \vee \neg(\exists y \neg Q(x,y))\right)$ $\boxed{\exists_x \exists_y \neg(P(x) \vee Q(x,y))}$

3. $\neg \forall x \left(\exists y Q(x,y) \rightarrow \neg P(x)\right)$ $\boxed{\exists_x \forall_y (Q \wedge P)}$

$\exists_x \neg(\neg\exists_y Q \vee \neg P)$

$\exists_x \neg\exists_y (\neg Q \vee \neg P)$

$\exists_x \forall_y \neg(\neg Q \vee \neg P)$

Problem 2 (15 pts): Skolemize the expressions (3 points each):

$\exists_x \leftarrow a = h()$

1. $\exists x P(x)$ $P(a)$

$\exists_y \leftarrow h(x)$

2. $\forall x \exists y P(x,y)$ $P(x, h(x))$

$a = h()$ $b = g()$ z

3. $\exists x \exists y \forall z (P(x,y) \wedge Q(y,z))$ $P(a,b) \wedge Q(b,z)$

x $h(x)$ $g(x)$

4. $\forall x \exists y \exists z (P(x,y) \wedge Q(y,z))$ $P(x, h(x)) \wedge Q(h(x), g(x))$

x y $h(x,y)$

5. $\forall x \forall y \exists z (P(x,y) \wedge Q(y,z))$ $P(x,y) \wedge Q(y, h(x,y))$

Problem 3 (12 pts): Convert the following into a canonical form (convert to prenex normal form, then CNF, then skolemize):

$$\forall x \left[(\forall y(\neg P(x,y) \rightarrow Q(y))) \rightarrow \neg(\exists z R(x,z))\right]$$

$\forall_x [\forall_y (P \vee Q) \rightarrow \neg\exists_z (R)]$

$\forall_x [\neg\forall_y (P \vee Q) \vee \neg\exists_z (R)]$

$\forall_x [\exists_y \neg(P \vee Q) \vee \forall_z \neg(R)]$

pre NF $\forall_x \exists_y \forall_z [\neg(P \vee Q) \vee \neg(R)]$

$\forall_x \exists_y \forall_z [(\neg P \wedge \neg Q) \vee \neg R]$

CNF $\forall_x \exists_y \forall_z [(\neg P \vee \neg R) \wedge (\neg Q \vee \neg R)]$

x $h(x)$ z

skol $(\neg P(x, h(x)) \vee \neg R(x,z)) \wedge (\neg Q(h(x)) \vee \neg R(x,z))$

1.2 Substitution and Unification

Problem 4 (12 pts): Apply the following substitutions to the expressions (4 point each);

1. Apply $\{x/f(A)\}$ to $P(x,y) \vee Q(x)$. $P(f(A),y) \vee Q(f(A))$
2. Apply $\{x/A, y/f(z)\}$ to $P(x,y) \vee Q(x)$. $P(A, f(z)) \vee Q(A)$
3. Apply $\{x/f(y), y/B\}$ to $P(x,y) \vee Q(x)$. $P(f(B), B) \vee Q(f(B))$

Problem 5 (12 pts): For each of the following, (1) find the unifier, and (2) show the unified expression. For example, given $P(A)$ and $P(x)$, the unifier would be $\{x/A\}$, and the unified expression $P(A)$. If the pair of expressions is not unifiable, indicate so. (3 points each). Note: Show all steps, including the disagreement set D_k for each iteration, and the σ_k.

1. $P(x, f(B))$ and $P(A, f(y))$ $\{x/A, y/B\}$ $P(A, f(B))$
2. $P(x, f(A))$ and $P(y, y)$ $\{x/f(A), y/f(A)\}$ $P(f(A), f(A))$
3. $P(x, f(y), y)$ and $P(A, f(g(w)), g(A))$ $\{x/A, y/g(A), w/A\}$ $P(A, f(g(A)), g(A))$
4. $P(A, f(y), y, A)$ and $P(x, f(g(x)), g(B), w)$ invalid. $x/A, y/g(x) = y/g(A), y/g(B) \rightarrow A \neq B$

1.
$D_0 = \{x, A\}$
$\sigma_1 = x/A$
$W_1 = P(A, f(B)),$
$P(A, f(y))$
$D_1 = \{y/B\}$
$\sigma_2 = y/B$
$W_2 = P(A, f(B))$

2.
$D_0 = \{x, y\}$
$\sigma_1 = x/y$
$W_1 = P(y, f(A)),$
$P(y, y)$
$D_1 = \{f(A), y\}$
$\sigma_2 = y/f(A)$
$W_2 = P(f(A), f(A))$

3.
$D_0 = \{x, A\}$
$\sigma_1 = x/A$
$W_1 = P(A, f(y), y),$
$P(A, f(g(w)), g(A))$
$D_1 = \{y, g(w)\}$
$\sigma_2 = y/g(w)$
$W_2 = P(A, f(g(w)), g(w)),$
$P(A, f(g(w)), g(A))$
$D_2 = \{w, A\}$
$\sigma_3 = w/A$
$W = P(A, f(g(A)), g(A))$

4.
$D_0 = \{A, x\}$
$\sigma_1 = x/A$
$W_1 = P(A, f(y), y, A),$
$P(A, f(g(x)), g(B), w)$
$D_1 = \{y, g(x)\}$
$\sigma_2 = y/g(x)$
$W_2 = P(A, f(g(x)), g(x), A),$
$P(A, f(g(x)), g(B), w)$
$D_2 = \{x, B\}$
$\sigma_3 = x/B$
invalid

1.3 Proof by resolution

Problem 6 (15 pts):

(1) Translate the following into first-order logic, (2) convert the resulting formulas into a canonical form, and (3) prove the theorem using resolution.

Given: (1)

1. Every kid likes Garfield. $\forall x\,(Kid(x) \rightarrow Likes(x, Garfield))$
2. Everyone who likes Garfield likes all cats. $\forall x \forall y\,(Likes(x, Garfield) \wedge Cat(y) \rightarrow Likes(x, y))$
3. Cheshier Cat is a cat, and Cheshier Cat is magical. $Cat(CheshierCat) \wedge Magical(CheshierCat)$
4. Every magical being can time travel or it can become invisible. $\forall x\,(Magical(x) \rightarrow Time(x) \vee Invisible(x))$
5. No cat can time travel. $\forall x\,(Cat(x) \rightarrow \neg Time(x))$
6. Jon does not like all things that is invisible. $\forall x\,(Invisible(x) \rightarrow \neg Likes(Jon, x))$

Show that the following is a logical consequence:

7. (conclusion) Jon is not a kid. $\neg Kid(Jon)$

(2)

1. $\neg Kid(x) \vee Likes(x, Garfield)$
2. $\neg Likes(x, Garfield) \vee \neg Cat(y) \vee Likes(x, y)$

3a. $Cat(CheshierCat)$

3b. $Magical(CheshierCat)$

4. $\neg Magical(x) \vee Time(x) \vee Invisible(x)$
5. $\neg Cat(x) \vee \neg Time(x)$
6. $\neg Invisible(x) \vee \neg Likes(Jon, x)$

resolution

7. $Kid(Jon)$
8. (1,7) $Likes(Jon, Garfield)$
9. (2,8) $\neg Cat(y) \vee Likes(Jon, y)$
10. (3a,9) $Likes(Jon, CheshierCat)$
11. (6,10) $\neg Invisible(CheshierCat)$
12. (3b,4) $Time(CheshierCat) \vee Invisible(CheshierCat)$
13. (3a,5) $\neg Time(CheshierCat)$
14. (12,13) $Invisible(CheshierCat)$
15. (11,14) False. Proved by resolution