First problems in integer Ramsey theory

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1. Notation

Denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and the set of positive integers, respectively.

2. Partition Ramsey theory

Definition 2.1. Let $A \subseteq \mathbb{N}$.

- The set A is *syndetic* if there exists a finite set $F \subseteq \mathbb{N} \cup \{0\}$ such that

$$\mathbb{N} \subseteq \bigcup_{n \in F} (A - n). \tag{2.1}$$

- The set A is *thick* if it contains arbitrarily long intervals.
- The set A is *piecewise syndetic* if there exists a finite set $F \subseteq \mathbb{N} \cup \{0\}$ such that

$$\bigcup_{n \in F} (A - n) \text{ is thick.} \tag{2.2}$$

Problem 1. *Suppose $A = \{a_1 < a_2 < a_3 < \cdots\}$. Show that A is syndetic if and only if $\sup_n(a_{n+1} - a_n) < \infty$. (We say that A has "bounded gaps".)*

Proof. We want to prove that if $\sup_n(a_{n+1} - a_n) < \infty$ then A is syndetic.There exists M such that max$(a_{n+1} - a_n)$=M.We choose $F = \{0, 1, ..., M - 1\}$.Then $A \cup .. \cup (A - (M - 1)) \supseteq N$.

We want to prove that if A is syndetic then $\sup_n(a_{n+1} - a_n) < \infty$.There exists F=$\{n_1, .., n_k\}$ (n_1 <...< n_k) such that $N \subseteq \bigcup_{n \in F}(A - n)$. To satisfy this, n_k+1 should be greater than or equal to sup$(a_{n+1} - a_n)$.Thus sup$(a_{n+1} - a_n) \leq n_k$+1, therefore,sup$(a_{n+1} - a_n) < \infty$. □

Problem 2. *Show that A is syndetic if and only if for all thick sets T, the intersection $A \cap T$ is non-empty. Similarly, the set A is thick if and only if for all syndetic sets S, the intersection $A \cap S$ is non-empty. Conclude that A is syndetic if and only if $\mathbb{N} \setminus A$ is not thick. (We say that the family of syndetic sets and the family of thick sets are* dual.*)*

Proof. if A is syndetic then sup$(a_{n+1} - a_n)$=M. Since every thick set contains arbitrarily long intervals, there exist intervals of length greater than M in T , $T \cap A$ is non-empty.If A is not syndetic, sup$(a_{n+1} - a_n)$=∞.$\mathbb{N} \setminus A$ contains arbitrarily long intervals, and $(\mathbb{N} \setminus A) \cap A$ is empty. If A is thick then it contains arbitrarily long intervals and sup$(a_{n+1} - a_n)$=M for S, therefore,$A \cap S$ is non-empty. If A is not thick , then it contains bounded intervals,so $\mathbb{N} \setminus A$ contains bounded gaps.$N \cap A$ is syndetic and $(\mathbb{N} \setminus A) \cap A$ is empty. If A is syndetic, sup$(a_{n+1} - a_n)$ <∞ for A and elements of $\mathbb{N} \setminus A$ are gaps in A and they are bounded,so $\mathbb{N} \setminus A$ is not thick..If A is thick then it contains arbitrarily long intervals and elements of A are gaps in $\mathbb{N} \setminus A$,they are not bounded therefore, $N \cap A$ is not syndetic. □

Problem 3. *Show that A is piecewise syndetic if and only if it is equal to the intersection of a syndetic set and a thick set.*

Proof. If A is syndetic or thick it is trivial. A is not syndetic and not thick, since it is piecewise syndetic, $A - n_1 \cup ... \cup (A - n_k)$ is thick for some n_1,..,n_k.If $A - n_2 \cup .. \cup A - n_k$ is not thick, take $\mathbb{N} \setminus ((A - n_2) \cup .. \cup (A - n_k))$ it is syndetic and $A - n_1$=$(A - n_1 \cup .. \cup A - n_k)$ $\cap\, \mathbb{N} \setminus ((A - n_2) \cup .. \cup (A - n_k))$, it is intersection of syndetic and thick set.It is obvious to see that then A is also intersection of syndetic amd thick set. Assume $A - n_2 \cup .. \cup A - n_k$ is thick and $A - n_3 \cup .. \cup A - n_k$ is not thick, we apply above process .Then apply process many times until $A - n_k$ is thick. Then A is also thick and it is intersection of itself and N.

Assume A is intersection of thick C and syndetic set B. Since B is syndetic there exists finite set F such that $\mathbb{N} \subseteq \bigcup_{n \in F}(B - n)$.We prove this theorem for $\mid F \mid$=2 and remaining is similar. So $(B - n_1) \cup (B - n_2) \supseteq N$.We want to prove that $(B \cap C\ {-n_1}) \cup (B \cap C\ {-n_2})$ is thick. $(B \cap C\ {-n_1}) \cup (B \cap C\ {-n_2})$=$(B - n_1) \cap (C - n_1) \cup (B - n_2) \cap (C - n_2)$=$((B - n_1)$ $\cap\ (C - n_1) \cup (B - n_2)) \cap ((B - n_1) \cap (C - n_1) \cup (C - n_2))$=$((B - n_1) \cup (B - n_2)) \cap$ $((C - n_1) \cup (B - n_2)) \cap ((B - n_1) \cup (C - n_2)) \cap ((C - n_1) \cup (C - n_2)) \supseteq \mathrm{N} \cap (C - n_1) \cap$ $(C - n_2)$.Since $(C - n_1) \cap (C - n_2)$ is thick,$(B \cap C\ {-n_1}) \cup (B \cap C\ {-n_2})$ is thick. So A is piecewise syndetic. $\square$

Problem 4.

1. *Give an example of a two-coloring $\mathbb{N} = A_1 \cup A_2$ in which neither A_1 nor A_2 is syndetic.*
2. *Show that for all two-colorings $\mathbb{N} = A_1 \cup A_2$, either A_1 or A_2 is piecewise syndetic. (Hint: Either A_1 is thick, or not.)*
3. *Suppose A is piecewise syndetic. Show that for all two-colorings $A = A_1 \cup A_2$, either A_1 or A_2 is piecewise syndetic. (Hint: Use Problem 3 to write $A = S \cap T$. Define $A_1' = A_1 \cup (\mathbb{N} \setminus S)$. Consider two cases: either A_1' is thick, or it is not.)*
4. *Iterate the result of the previous problem to show that if A is piecewise syndetic and $A = \cup_{i=1}^{r} A_i$, the at least one A_i is piecewise syndetic. (We say that* piecewise syndeticity is partition regular.*)*

Proof. A_1=$\{1, 4, 5, 6, 11, 12, 13, 14, 15, ..\}$ A_2=$\{2, 3, 7, 8, 9, 10, 16, 17, 18, 19, 20, 21, ..\}$

Assume that A_1 is thick.Then A_1 is piecewise syndetic. If A_1 is not thick. A_2 is syndetic.Then A_2 is piecewise syndetic.

Take $S \backslash A_2$.If $S \backslash A_2$ is syndetic, then A_1=$(S \backslash A_2)$ $\cap$ T is piecewise syndetic. If $S \backslash A_2$ is not syndetic then $N \backslash (S \backslash A_2)$ is thick.A_2= $(N \backslash (S \backslash A_2))$ $\cap S$ is piecewise syndetic.

A=$A_1 \cup A_2 \cup ... \cup A_k$.$A_1$ or $A_2 \cup ... \cup A_k$ is piecewise syndetic. If A_1 is piecewise syndetic we are done , if $A_2 \cup ... \cup A_k$ is piecewise syndetic , then A_2 is piecewise syndetic or $A_3 \cup ... \cup A_k$ is piecewise syndetic.We apply same process until A_k and we are done. $\square$

Problem 5. *Show that if A is piecewise syndetic, then there exists $n \in \mathbb{N}$ such that $A \cap (A - n)$ is piecewise syndetic. (Hint: Since A is piecewise syndetic, there exists a finite set $F \subseteq \mathbb{N}$ such that the set $T := \cup_{f \in F}(A - f)$ is thick. Show first that for all $k \in \mathbb{N}$, the set $A \cap (T - k)$ is piecewise syndetic. Then, use the definition of T and the partition regularity of piecewise syndeticity from Problem 4.)*

Proof. Assume that $A = B \cap C$, B is syndetic and C is thick.We prove theorem for $\mid F_1 \mid$=2 and $\mid F_2 \mid$=2 and remaining is similar.F_1 is finite set for which $T := \cup_{f \in F}(A - f)$ is thick and F_2 is finite set for which $\mathbb{N} \subseteq \bigcup_{n \in F}(B - n)$.Consider $B \cap C \cap (((B - n_1) \cap (C - n_1)) \cup$

$((B-n_2)\cap(C-n_2)) \cup ((B-n_3)\cap(C-n_3)) \cup ((B-n_4)\cap(C-n_4))$). $(B-n_1)\cap(C-n_1)$ $\cup$ $(B-n_2) \cap (C-n_2)$ is thick and $(B-n_3)$ $\cup$($B-n_4) \supseteq$ N.

$B \cap C \cap (((B-n_1)\cap(C-n_1)) \cup ((B-n_2)\cap(C-n_2)) \cup ((B-n_3)\cap(C-n_3)) \cup ((B-n_4)\cap(C-n_4)))$=$B\cap C \cap (((B-n_1) \cup (B-n_2)) \cap ((B-n_2) \cup (C-n_1)) \cap ((B-n_1)$ $\cup$ $(C-n_2)) \cap ((C-n_1) \cup (C-n_2)) \cup (((B-n_3) \cup (B-n_4)) \cap ((B-n_3) \cup (C-n_4)) \cap ((B-n_4) \cup (C-n_3)) \cap ((C-n_3) \cup (C-n_4))) \supseteq (B\cap C) \cap$ N $\cap (C-n_3) \cap (C-n_4)$ and due to thickness of $C \cap (C-n_3) \cap (C-n_4)$, $(B\cap C) \cap$ N $\cap (C-n_3) \cap (C-n_4)$ is intersection of syndetic and thick set, so $A \cap ((A-n_1) \cup (A-n_2) \cup (A-n_3) \cup (A-n_4))$ is piecewise syndetic. $(A\cap(A-n_1)) \cup (A\cap(A-n_2)) \cup (A\cap(A-n_3)) \cup (A\cap(A-n_4))$=$A \cap ((A-n_1) \cup (A-n_2) \cup (A-n_3) \cup (A-n_4))$ is piecewise syndetic. Therefore, one of four terms in union - $(A\cap(A-n_1)) \cup (A\cap(A-n_2)) \cup (A\cap(A-n_3)) \cup (A\cap(A-n_4))$ is piecewise syndetic. □

Problem 6. *Prove that the following statements are equivalent:*

1. *(van der Waerden's theorem) At least one piece of any finite partition of $\mathbb{N}$ contains arbitrarily long arithmetic progressions.*
2. *Every piecewise syndetic set contains arbitrarily long arithmetic progressions.*

2.1. Some open problems

Definition 2.2. The *piecewise syndeticity constant* of a set A is the cardinality of the smallest set F for which (2.2) holds. (If A is thick, then the piecewise syndeticity constant of A is zero. If A is not piecewise syndetic, then the piecewise syndeticity constant of A is ∞.) Denote the piecewise syndeticity constant of A by $\mathfrak{P}(A)$.

I don't know the answer either of the following questions. (It seems to me that it is necessary to understand solutions to both Problem 4 and Problem 5 before being able to answer either of these!)

Question 2.3. *Is*

$$\sup_{A \text{ s.t. } \mathfrak{P}(A)\leq 2} \ \min_{n\in\mathbb{N}} \ \mathfrak{P}\big(A\cap(A-n)\big) < \infty,$$

where the first supremum is over all sets $A \subseteq \mathbb{N}$ with $\mathfrak{P}(A) \leq 2$? Equivalently: does there exist $M \in \mathbb{N}$ such that for all $A \subseteq \mathbb{N}$ with $\mathfrak{P}(A) \leq 2$, there exists $n \in \mathbb{N}$ such that $\mathfrak{P}(A\cap(A-n)) \leq M$?

Proof. Assume $\mathfrak{P}(A)$=2.Then there exists n_1 and n_2 ($n_1 < n_2$) such that $(A-n_1) \cup (A-n_2)$ is thick.Denote n_1 by m and n_2 by m+k.$(A-m) \cup (A-m-k)$ is thick. Consider (($A \cap (A-n)$)-m) $\cup$ (($A \cap (A-n)$)-m-k) $\cup$ (($A \cap (A-n)$)-m-2k). We take n as 2k. So this expression turns (($A \cap (A-2k)$)-m) $\cup$ (($A \cap (A-2k)$)-m-k) $\cup$ (($A \cap (A-2k)$)-m-2k).Since ($A \cap B$)-c=$(A-c) \cap (B-c)$,(($A \cap (A-2k)$)-m) $\cup$ (($A \cap (A-2k)$)-m-k) $\cup$ (($A \cap (A-2k)$)-m-2k)=$((A-m) \cap (A-2k-m)) \cup ((A-m-k) \cap (A-m-3k)) \cup ((A-m-2k) \cap (A-m-4k))$.By applying De Morgan law,$((A-m) \cap (A-2k-m)) \cup ((A-m-k) \cap (A-m-3k)) \cup ((A-m-2k) \cap (A-m-4k))$=$(A-m-k \cup A-m \cup A-m-2k) \cap (A-m-k \cup A-m \cup A-m-4k) \cap (A-m-2k \cup A-m \cup A-m-3k) \cap (A-m-3k \cup A-m \cup A-m-4k) \cap (A-m-k \cup A-m-2k) \cap (A-m-2k \cup A-m-3k) \cap (A-m-2k \cup A-m-k \cup A-m-4k) \cap (A-m-2k \cup A-m-3k \cup A-m-4k) \supseteq (A-m \cup A-m-k) \cap (A-m-k \cup A-m-2k) \cap (A-m-2k \cup A-m-3k) \cap (A-m-3k \cup A-m-4k)$.Since $A-m \cup A-m-k$ is thick $(A-m \cup A-m-k) \cap (A-m-k \cup$

$A-m-2k) \cap (A-m-2k \cup A-m-3k) \cap (A-m-3k \cup A-m-4k)$=$(A-m \cup A-m-k)$ $\cap$ $((A-m \cup A-m-k)$-k) $\cap$ $((A-m \cup A-m-k)$-2k) $\cap$ $((A-m \cup A-m-k)$-3k) is thick.Therefore,$((A-m) \cap (A-2k-m)) \cup ((A-m-k) \cap (A-m-3k)) \cup ((A-m-2k)$ $\cap$ $(A-m-4k))$ is thick and there exists n=2k such that $\mathfrak{P}(A \cap (A-n)) \leq 3$. $\square$

Nurlan, your answer to Question 2.3 is very interesting. Here are two followup questions:

1. In the argument, the quantity n is a function of k. Is there an absolute constant N (say, $N = 10^{10}$) such that n can always be taken to be less than N?
2. Does a similar argument work when 2 is replaced by 3 in the statement of Question 2.3?

[[DGG]]

Yes,if $(A) \cup (A-m)$ is thick and m > 1, then we can take n as 1.Actually, we take consider of these:Since $(A-m) \cup (A-m-k)$ is thick, we have arbitrarily long intervals.There is a question which parts can make up these intervals? Assume $A \cup (A-2)$ is thick. We only take care of parts of this set for which we have arbitrarily long intervals.For example, assume that A=1,2,5,6,101,102,105,106,109,110,... For first part $A \cup A-2$ makes 1,2,3,4,5,6 , then 101,102,103,104,105,106,107,108,109,110.In first part, there are 6 consecutive elements, but in second 10.So, we observe that there should be these type of parts which make up arbitrarily long intervals.We write 0's for no element parts and 1's for element parts.1,2,5,6 turns 110011. If we have $A \cup A-2$ is thick , then we have only 0011 type of parts. If we have 00011 it is impossible to get consecutive elements by union of A-2.On the other hand, if we have 011 or 00111 it is trivial to get consecutive elements after union of A-2.So,A should contain 0011 parts infinitely many to get arbitrarily long intervals. After intersection of A with A-1, we get this part, 0001. To get consecutive elements we should make union of $A \cap A-1$ with $(A \cap A-1$ -1),$(A \cap A-1)$-2 and $(A \cap A-1)$-3.0001 turns 1111 and we get this in infinitely many parts ,so thick set.

Assume $A \cup A-m$ is thick.Then by previous observation, we should have 0^m1^m infinitely many.By intersection of A with A-1 we have 0..0 m+1 times and 1..1 m-1 times.It is easy to see that after making union of $A \cap A-1$ with 3 sets we have consecutive 2m elements.We have not only 1..1 2m times , we have 1..1 arbitrarily long because we have 0^m1^m in first part and $0^m1^m0^m1^m...0^m1^m$ arbitrarily long.

If $A \cup A-m \cup A-m-k$ is thick we should have 1's at least 1/2 of 0's in infinitely many parts. For example,if we have 0, 14 times and 1, 6 times, then it is impossible to make this all consecutive 1's by two union.So we should at least have 0 m+k times 1 (m+k)/2.After making intersection of A with A-1 we have 0 m+k+1 times 1 (m+k)/2-1 times.It is easy to see that after 4 unions we have thick set.For example, worst case occur when we have $A \cup A-2 \cup A-4$ is thick.Then we should have 000011.After intersection with A-1 we have 000001 and we should have at least 5 unions to make thick.Observe that we should have one finite set to make set thick.There can be parts 0 m+k times and 1 (m+k)/2 times, yes , but we can have also 0 m times 1 m times and 0 m+k times and 1 m+k times.We should consider all 3 sets.It is obvious that to make first part all consecutive 1's is harder than last 2.Also we may use different sets for all 3.So we should calculate worst case and cardinality of finite set to make all consecutive 1's and multiply by 3(in this case). Generally, Assume that $A \cup ... \cup A-n_k$ is thick k sets.Then the worst case is 0, n_k times and 1, 2 times .Then we should have n_k+1 union with $A \cap A-1$.There are 3 cases in $A \cup A-m \cup A-n$, by induction there are 2^{k-1} -1 cases.So if $A \cup A-n_1 \cup ... \cup A-n_k$ is thick and n_1 is greater than 1 then $\mathfrak{P}(A \cap (A-1)) \leq ((2^{k-1})$-1)($n_k$+1)

Here is a related question that I don't know how to answer.

Question 2.4. *Is there an $M \in \mathbb{N}$ such that for all 2-colorings $\mathbb{N} = A \cup B$, either $\mathfrak{P}(A) \leq M$ or $\mathfrak{P}(B) \leq M$. More generally, is it true that for all $k, r \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for all piecewise syndetic sets $A \subseteq \mathbb{N}$ with $\mathfrak{P}(A) \leq k$ and all r-colorings $A = \cup_{i=1}^{r} A_i$, there exists $i \in \{1, \ldots, r\}$ such that $\mathfrak{P}(A_i) \leq M$?*

I can show the following: a positive answer to Question 2.4 implies a positive answer to Question 2.3. If Question 2.3 has a positive answer, then there are some interesting open directions in which to take it (for example, "topological Khintchine"-type results and a topological analogue of an example of Ruzsa).

3. Density Ramsey theory

Definition 3.1. The *lower and upper asymptotic densities* of $A \subseteq \mathbb{N}$ are

$$\underline{d}(A) = \liminf_{N \to \infty} \frac{|A \cap \{1, \ldots, N\}|}{N} \quad \text{and} \quad \overline{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{1, \ldots, N\}|}{N}.$$

If $\underline{d}(A) = \overline{d}(A)$, then the *density* of A exists and is denoted by $d(A)$.

Problem 7. *Give an example of a set $A \subseteq \mathbb{N}$ with $\underline{d}(A) = 0$ and $\overline{d}(A) = 1$.*

Problem 8.
1. *If $\overline{d}(A) > 1/2$ and $\overline{d}(B) > 1/2$, does it follow that $A \cap B \neq \emptyset$?*
2. *If $\overline{d}(A) > 1/2$, show that for all $n \in \mathbb{N}$, $A \cap (A - n) \neq \emptyset$.*
3. *If $\underline{d}(A) > 1/2$ and $\underline{d}(B) > 1/2$, does it follow that $A \cap B \neq \emptyset$?*

Problem 9. *Show that $A \cap (A - n) \neq \emptyset$ if and only if $n \in A - A$. Combine this observation with one of the results in Problem 8 to show that if $\overline{d}(A) > 1/2$, then $\mathbb{N} \subseteq A - A$. Does the same conclusion hold if we only assume that $\overline{d}(A) = 1/2$?*

Problem 10.
1. *Show that if $\overline{d}(A) > 0$, then there exists $n \in \mathbb{N}$ such that $\overline{d}(A \cap (A - n)) > 0$.*
2. *Iterate the previous problem to show that if $\overline{d}(A) > 0$, then there exist $n_1, n_2 \in \mathbb{N}$ such that*

$$\overline{d}\big(A \cap (A - n_1) \cap (A - n_2) \cap (A - n_1 - n_2)\big) > 0.$$

3. *Show that for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\overline{d}(A \cap (A - n)) > \overline{d}(A)^2 - \epsilon$.*

Definition 3.2. The *upper Banach density* of $A \subseteq \mathbb{N}$ is

$$d^*(A) = \limsup_{N \to \infty} \max_{n \in \mathbb{N}} \frac{|A \cap \{n, \ldots, n + N - 1\}|}{N}.$$

Problem 11. *Show that the limit supremum in the definition of the upper Banach density is actually a limit. Then, show that*

$$d^*(A) = \lim_{N \to \infty} \limsup_{n \in \mathbb{N}} \frac{|A \cap \{n, \ldots, n + N - 1\}|}{N}.$$

Problem 12. *Prove that the following statements are equivalent:*
1. *(Szemerédi's theorem) Every set of positive upper asymptotic density contains arbi-*

trarily long arithmetic progressions.

2. *Every set of positive upper Banach density contains arbitrarily long arithmetic progressions.*