A Refined Mean Field Approximation for Synchronous Population Processes

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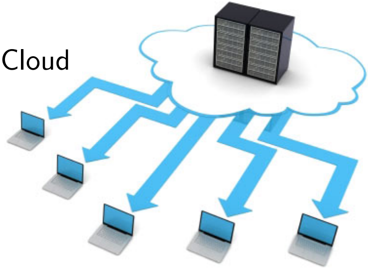
How to characterize emerging behavior starting from a stochastic model of interacting objects?

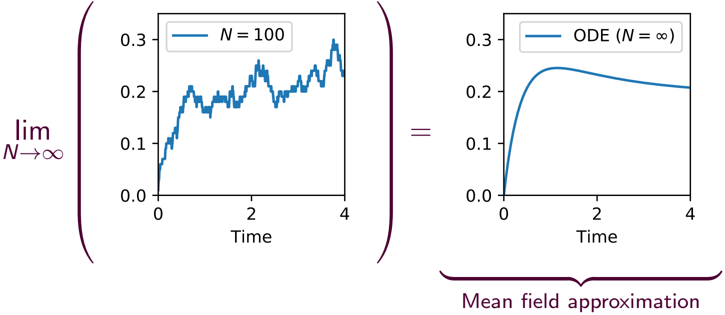
Wireless



Cloud

Evacuation

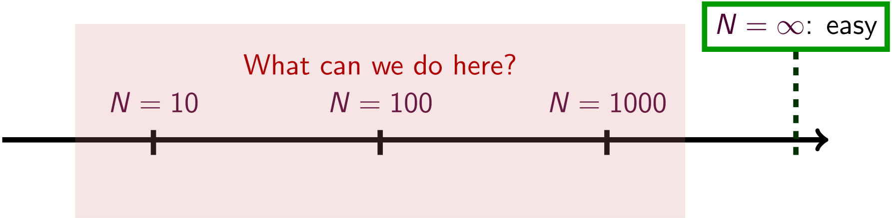
Thanks to the law of large numbers : Some systems simplify as they grow



Applications :

- Communication protocols (MPTCP, Simgrid)
- Mean field games (Adversarial classification)
- Performance of load balancing / caching algorithms
- Stochastic approximation / learning
- Theoretical biology

We can study large systems. What about moderate sizes?



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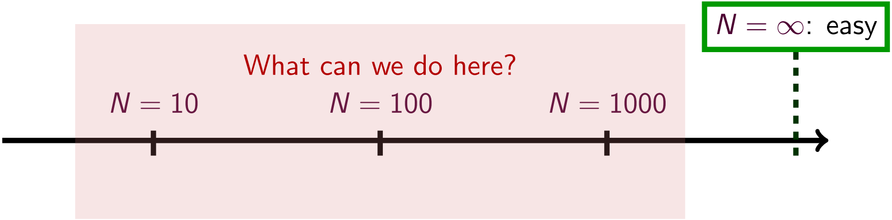

For many systems, asymptotically:

$$Perf(N) \approx Perf(\infty) + \frac{1}{N} V$$

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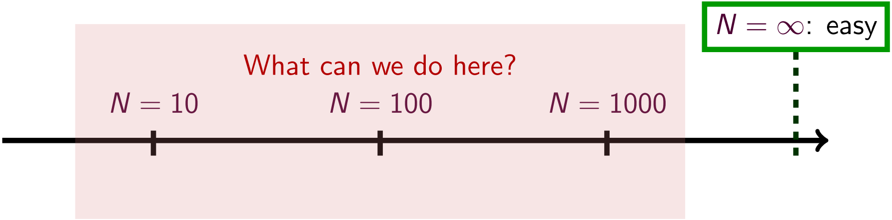

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Mean field approximation

Refined mean field approximation

Outline

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1. Mean field approximations of synchronous populations
2. The refined mean field
3. Does it always work?
4. Conclusion and recap

Discrete-time mean field models

Population of N objects (discrete time, discrete space).

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There are two cases:

- Asynchronous case (BLB08,K81) – One (or a few) objects change state at each time step. (see talk at SIGMETRICS)
  - Continuous-time mean field approximation
- Synchronous case (LB+07) – All object simultaneously change state (this talk).
  - Discrete-time mean field approximation

BLB08 Michel Benaïm, Jean-Yves Le Boudec: A class of mean field interaction models for computer and communication systems. Perform. Eval. 2008

K81 Thomas G. Kurtz : Approximation of Population Processes. 1981

LB+07 Jean-Yves Le Boudec, David D. McDonald, Jochen Mundinger: A Generic Mean Field Convergence Result for Systems

Our synchronous model is similar to the one of [LB+07]

At each time step : all objects perform an independent transition :

$$K_{ij}(x) = \mathbf{P}\left[\text{An object in state } i \text{ changes to state } j \mid X(t) = x\right].$$

The mean field approximation is given by :

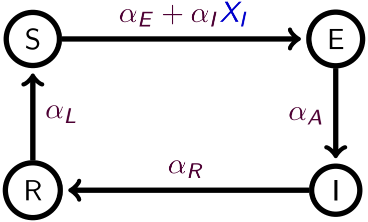
$$\mu(t+1) = \mu(t)K(\mu(t)) =: \Phi_1(\mu(t))$$

The mean field approximation is asymptotically exact (LB+07) :

$$X(t) = \mu(t) + o_{N\to\infty}(1).$$

Example : a simple infection model (SEIR model)

Evolution of one individual:



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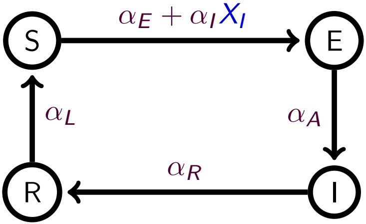
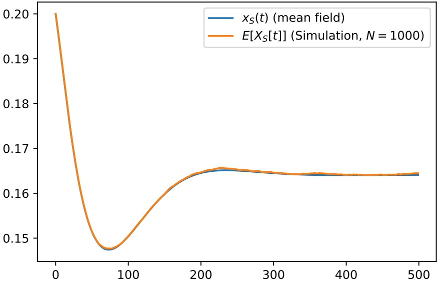


We plot $\mathbb{E}[X_S(t)]$ as a function of time :



$N = 1000$

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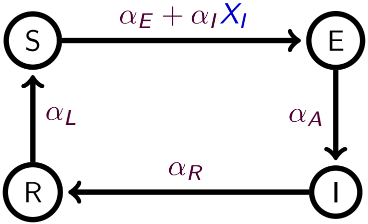

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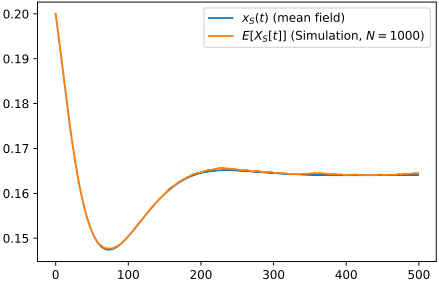

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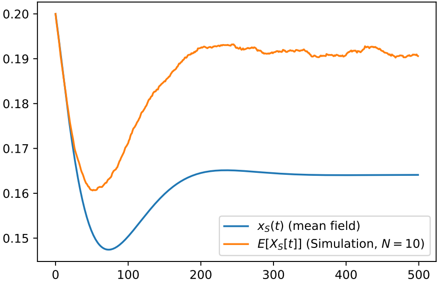

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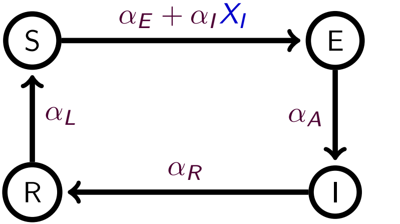

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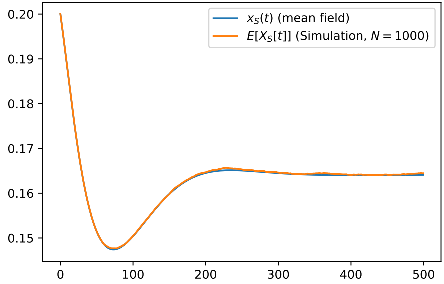

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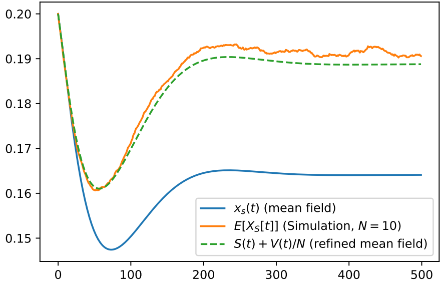

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Outline

First result : transient regime

We assume that $\Phi_1 : x \mapsto xK(x)$ is twice differentiable and we define

$$A(t) = (D\Phi_1)(\mu(t)) \text{ and } B(t) = (D^2\Phi_1)(\mu(t))$$

$$\Gamma_{jk}(x) = \sum_{i=1}^{n} x_i K_{ij}(x) \left(\mathbf{1}_{j=k} - K_{ik}(x)\right)$$

Let $V(0) = 0$, $W(0) = 0$ and

$$\begin{aligned} V(t+1) &= A(t)V(t) + \frac{1}{2}B(t) \cdot W(t) \\ W(t+1) &= \Gamma(\mu(t)) + A(t)W(t)A(t)^T. \end{aligned} \tag{1}$$

Theorem (Transient Regime)

For any time t : $\mathbb{E}[X(t)] = \mu(t) + \frac{1}{N}V(t) + o(1/N)$.

Second result : steady-state

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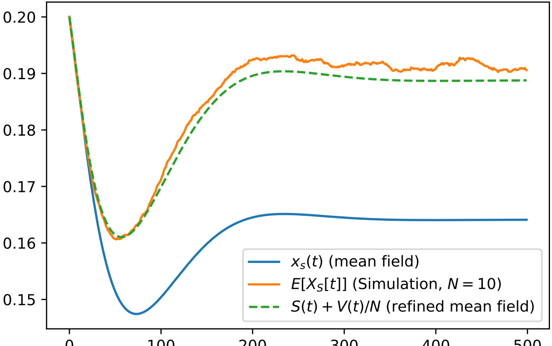
Theorem

If the mean field approximation has a unique exponentially stable fixed point, then the previous theorem holds uniformly in time.

In particular in steady state: $\mathbb{E}[X] = \pi + \frac{1}{N}\left(\lim_{t\to\infty} V(t)\right) + o(1/N)$.



Refined m.f. $\mu(t) + \frac{V_t}{10}$

Mean field $\mu(t)$

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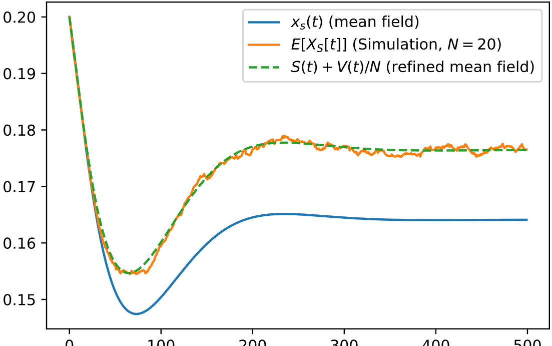

Refined m.f. $\mu(t) + \frac{V_t}{20}$

Mean field $\mu(t)$

Where does the $O(1/N)$-term come from?

The mean field approximation is $\mu(t+1)=\mu(t)K(\mu(t))$ with $\mu(0)=M(0)$.

$$\mathbb{E}\left[X(t+1) \mid X(t)=\mu(t)\right] = \mu(t)K(\mu(t)) = \mu(t+1)$$

Hence :

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$\mathbb{E}\left[X(t+1)\right] = \mathbb{E}\left[X(t)K(X(t))\right] \approx \mathbb{E}\left[X(t)\right] K(\mathbb{E}\left[X(t)\right])$ Mean field approx.

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Now, in addition :

$$\mathrm{cov}(X(t), X(t) \mid X(t)=\mu(t)) = \frac{1}{N}\Gamma(\mu(t)).$$

The refined mean field is derived by taking the $1/N$-term into account.

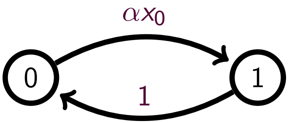
Outline

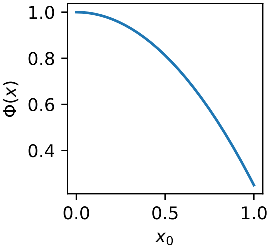
Positive side

Many models satisfy the assumptions (see paper).

- Refined model is easy to compute (linear algebra)
- It improves the accuracy for not so large values of N.

Limit of the approach on an example



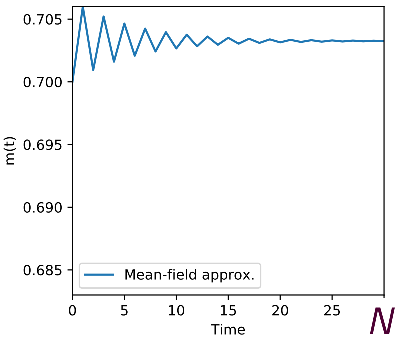


$$\Phi_1(x_0) = 1 - \alpha(x_0)^2$$

PROBLEM?

- There is a unique fixed point π.
- It is an attractor *iff* $\alpha \leq 0.75$.
- It is exponentially stable *iff* $\alpha < 0.75$.

Example (continued) : numerical illustration



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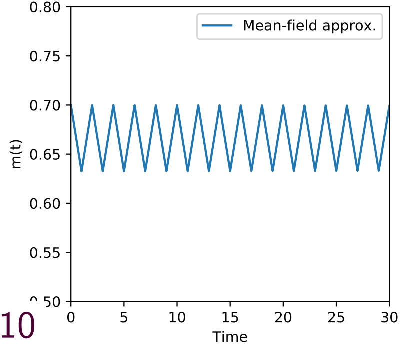

$a = 0.6$ (exponentially stable)

$a = 0.75$ (not exp. stable).

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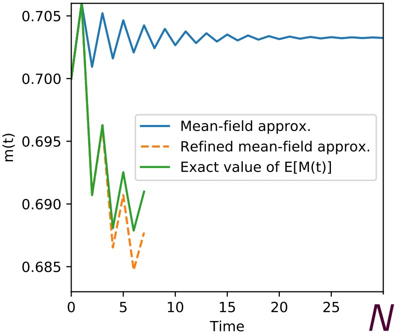

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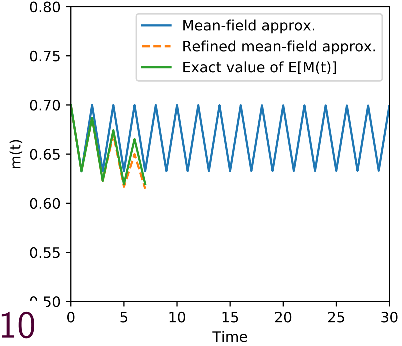

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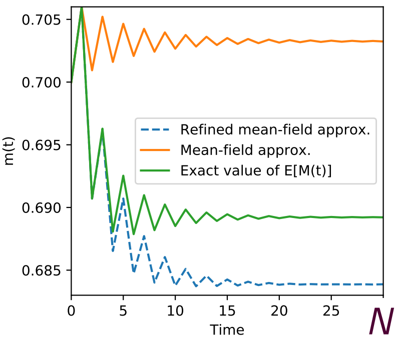

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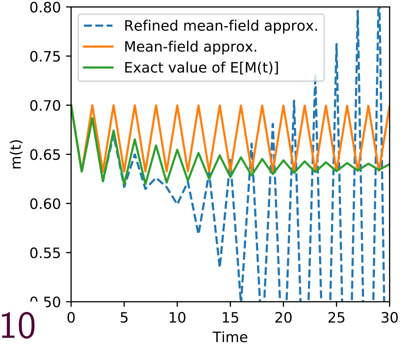

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Does not work for $t \gg N$

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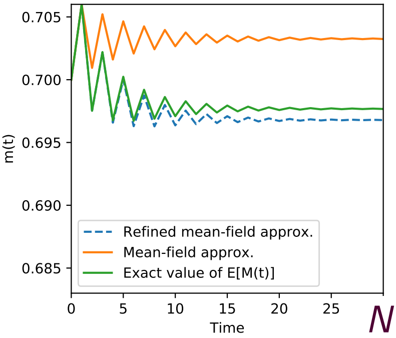
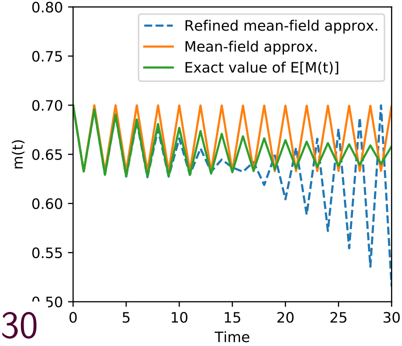


$N = 30$



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Outline

Recap

1. The traditional mean field approximation considers

$$X(t) \approx \mu(t),$$

where $\mu(t) = \lim_{N\to\infty} X(t)$.

2. Our approach : we focus on $\mathbb{E}[X(t)]$ and we show that there exists $V(t)$ such that :

$$\mathbb{E}[X(t)] = \underbrace{\mu(t) + \frac{V(t)}{N}}_{\text{Refined approximation}} + o(1/N)$$

It also works for $\mathbb{E}[h(X(t))]$.

3. $V(t)$ is easy to evaluate and for small N the refined approximation greatly improves the accuracy compared to the classical mean field approximation.

To go further :

It also works for continuous time models (*e.g.*: two-choice), see SIGMETRICS paper.

Future work:

- Find a relevant model (more general?) of synchronous population.
- Find guidelines where the method is applicable or not (example : Non-exponentially stable systems)

Main references :

- A Refined Mean Field Approximation of Synchronous Discrete-Time Population Models by Gast, Latella and Massink. To appear in *Performance Evaluation*.
  Paper is reproducible : `https://github.com/ngast/RefinedMeanField_SynchronousPopulation`
- A Refined Mean Field Approximation by Gast and Van Houdt. To appear in SIGMETRICS 2018 `https://hal.inria.fr/hal-01622054/` `https://github.com/ngast/rmf_tool/`