A Refined Mean Field Approximation for Synchronous Population Processes

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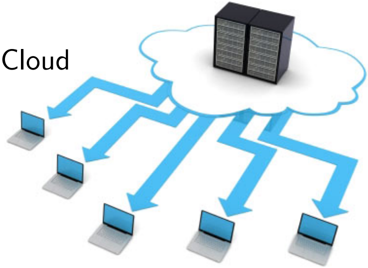
MAMA Workshop, 2018

How to characterize emerging behavior starting from a stochastic model of interacting objects?

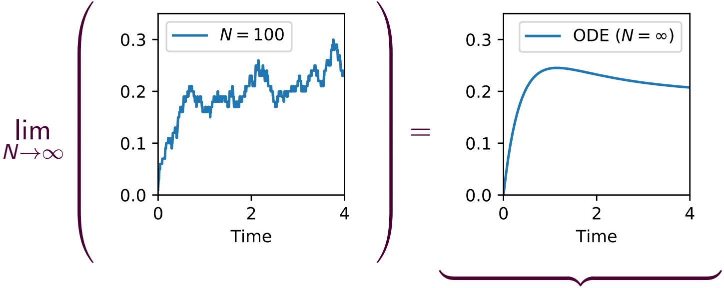
Wireless



Cloud

Evacuation

Thanks to the law of large numbers : Some systems simplify as they grow

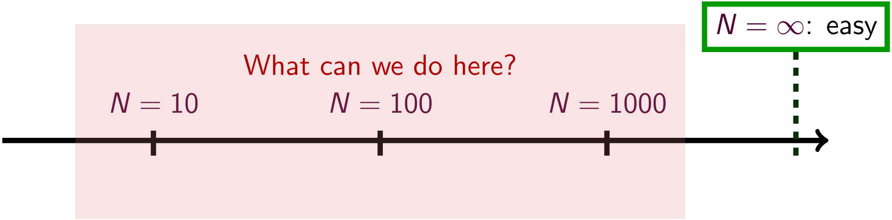


Mean field approximation

Applications :

- Communication protocols (MPTCP, Simgrid)
- Mean field games (Adversarial classification)
- Performance of load balancing / caching algorithms
- Stochastic approximation / learning
- Theoretical biology

We can study large systems. What about moderate sizes?



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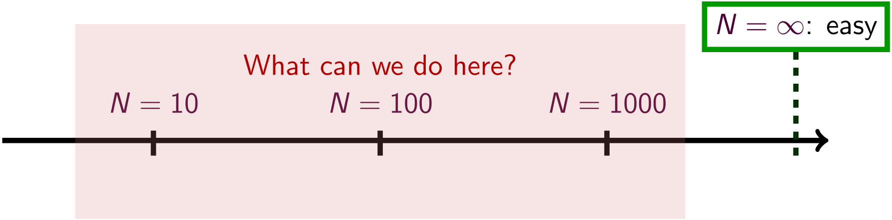

For many systems, asymptotically:

$$Perf(N) \approx Perf(\infty) + \frac{1}{N}V$$

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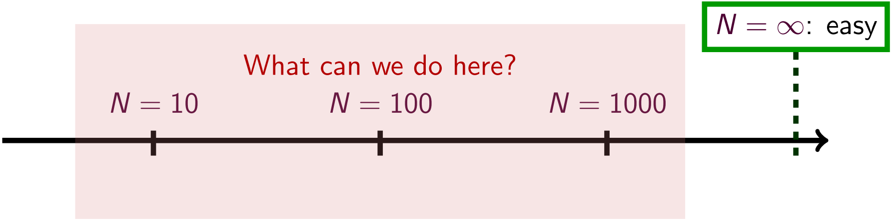

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Mean field approximation

Refined mean field approximation

Outline

1. Mean field approximations of synchronous populations

2. The refined mean field

3. Does it always work?

4. Conclusion and recap

Outline

Discrete-time mean field models

Population of N objects (discrete time, discrete space).

$$X_i(t) = \text{fraction of object in state } i \text{ at time } t$$

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There are two cases:

- Asynchronous case (BLB08,K81) – One (or a few) objects change state at each time step. (see talk at SIGMETRICS)
  - Continuous-time mean field approximation
- Synchronous case (LB+07) – All object simultaneously change state (this talk).
  - Discrete-time mean field approximation

BLB08 Michel Benaïm, Jean-Yves Le Boudec: A class of mean field interaction models for computer and communication systems. Perform. Eval. 2008

K81 Thomas G. Kurtz : Approximation of Population Processes. 1981

LB+07 Jean-Yves Le Boudec, David D. McDonald, Jochen Mundinger: A Generic Mean Field Convergence Result for Systems

Our synchronous model is similar to the one of [LB+07]

At each time step : all objects perform an independent transition :

$$K_{ij}(x) = \mathbf{P}\left[\text{An object in state } i \text{ changes to state } j \mid X(t) = x\right].$$

The mean field approximation is given by :

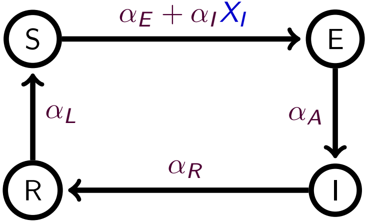
$$\mu(t+1) = \mu(t)K(\mu(t)) =: \Phi_1(\mu(t))$$

The mean field approximation is asymptotically exact (LB+07) :

$$X(t) = \mu(t) + o_{N\to\infty}(1).$$

Example : a simple infection model (SEIR model)

Evolution of one individual:



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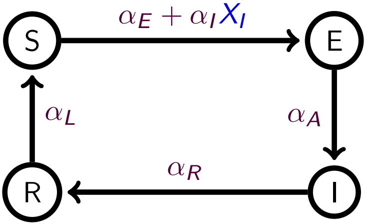
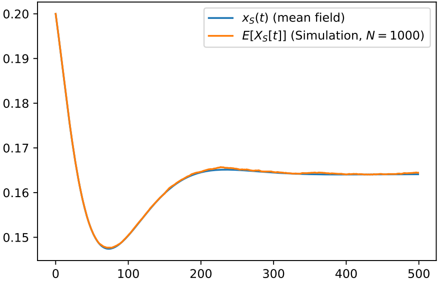


We plot $\mathbb{E}[X_S(t)]$ as a function of time :



$N = 1000$

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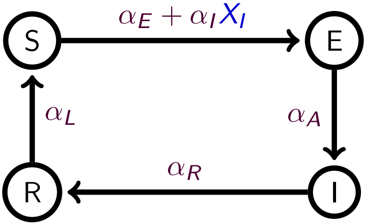

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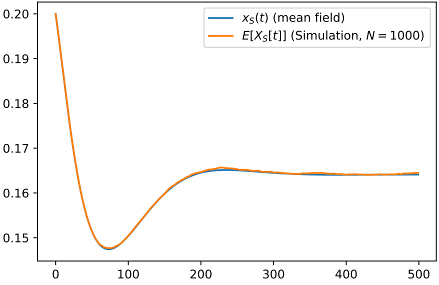

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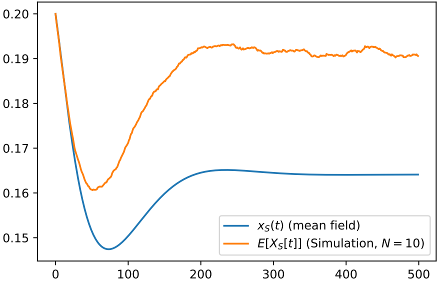

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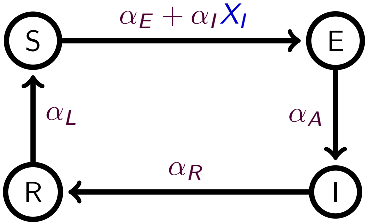

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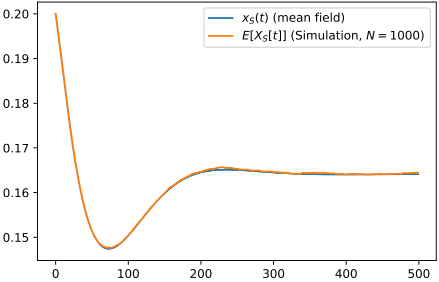

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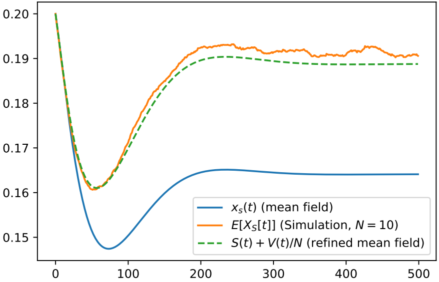

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Outline

First result : transient regime

We assume that $\Phi_1 : x \mapsto xK(x)$ is twice differentiable and we define

$$A(t) = (D\Phi_1)(\mu(t)) \text{ and } B(t) = (D^2\Phi_1)(\mu(t))$$

$$\Gamma_{jk}(x) = \sum_{i=1}^{n} x_i K_{ij}(x) \left(\mathbf{1}_{j=k} - K_{ik}(x)\right)$$

Let $V(0) = 0$, $W(0) = 0$ and

$$\begin{aligned} V(t+1) &= A(t)V(t) + \frac{1}{2}B(t) \cdot W(t) \\ W(t+1) &= \Gamma(\mu(t)) + A(t)W(t)A(t)^T. \end{aligned} \tag{1}$$

Theorem (Transient Regime)

For any time t : $\mathbb{E}\left[X(t)\right] = \mu(t) + \frac{1}{N}V(t) + o(1/N)$.

Second result : steady-state

We assume that in addition, the mean field approximation has a unique fixed point that is exponentially stable : $|\Phi_t(m) - \pi| \leq ae^{-bt}$.

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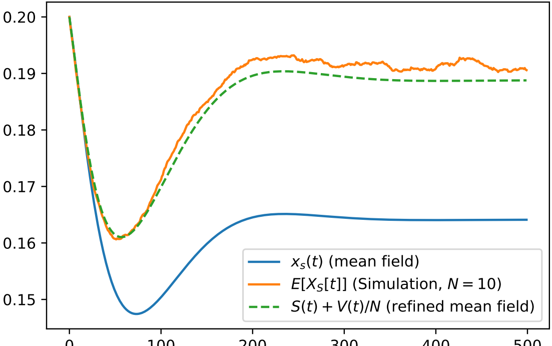
Theorem

If the mean field approximation has a unique exponentially stable fixed point, then the previous theorem holds uniformly in time.

In particular in steady state: $\mathbb{E}[X] = \pi + \frac{1}{N}\left(\lim_{t\to\infty} V(t)\right) + o(1/N)$.



Refined m.f. $\mu(t) + \frac{V_t}{10}$

Mean field $\mu(t)$

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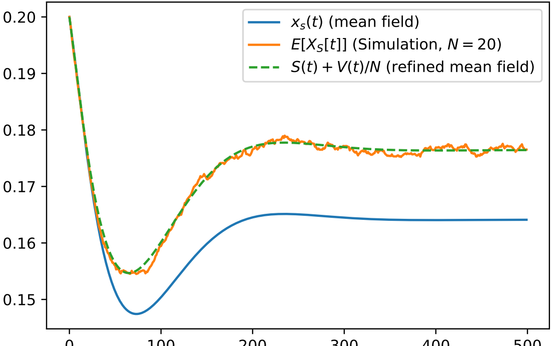

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Result for steady-state:

	N	10	20	30	50	∞ (mf
$\mathbb{E}[X_S]$ (Simulation)	0.191	0.177	0.175	0.169	0.166	–
$\mu_S(\infty) + \frac{V}{N}$ (Refined mf)	0.189	0.176	0.172	0.169	0.167	0.164

Where does the $O(1/N)$-term come from?

The mean field approximation is $\mu(t+1)=\mu(t)K(\mu(t))$ with $\mu(0)=M(0)$.

$$\mathbb{E}\left[X(t+1) \mid X(t)=\mu(t)\right] = \mu(t)K(\mu(t)) = \mu(t+1)$$

Hence :

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Hence :

$$\mathbb{E}\left[X(t+1)\right] = \mathbb{E}\left[X(t)K(X(t))\right] \approx \mathbb{E}\left[X(t)\right] K(\mathbb{E}\left[X(t)\right]) \quad \text{Mean field approx.}$$

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Now, in addition :

$$\mathrm{cov}(X(t),X(t)\mid X(t)=\mu(t))=\frac{1}{N}\Gamma(\mu(t)).$$

The refined mean field is derived by taking the $1/N$-term into account.

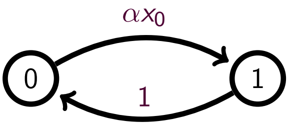
Outline

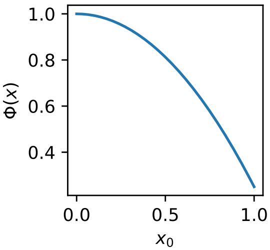
Positive side

Many models satisfy the assumptions (see paper).

- Refined model is easy to compute (linear algebra)
- It improves the accuracy for not so large values of N.

Limit of the approach on an example

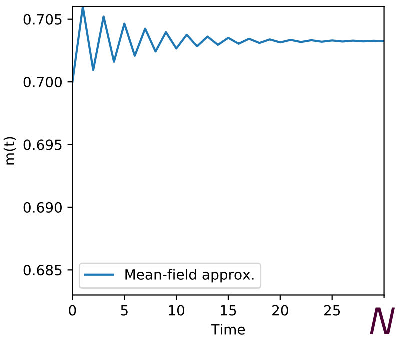




$$\Phi_1(x_0) = 1 - \alpha(x_0)^2$$

- There is a unique fixed point π.
- It is an attractor *iff* $\alpha \leq 0.75$.
- It is exponentially stable *iff* $\alpha < 0.75$.

Example (continued) : numerical illustration



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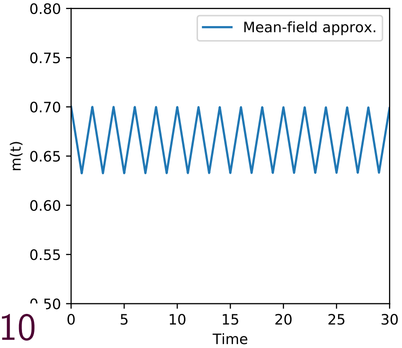
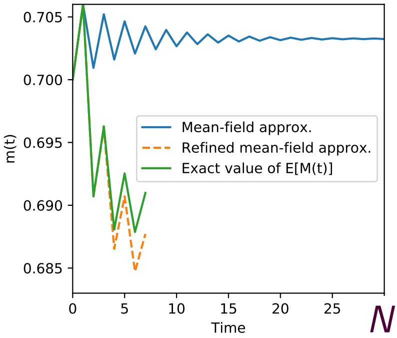
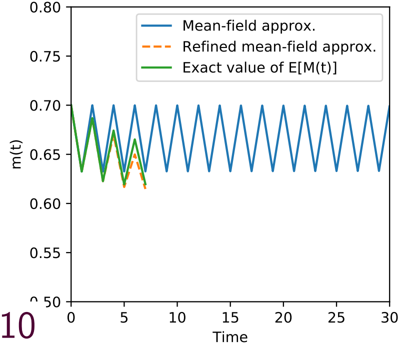


$a = 0.6$ (exponentially stable)

$a = 0.75$ (not exp. stable).

Example (continued) : numerical illustration





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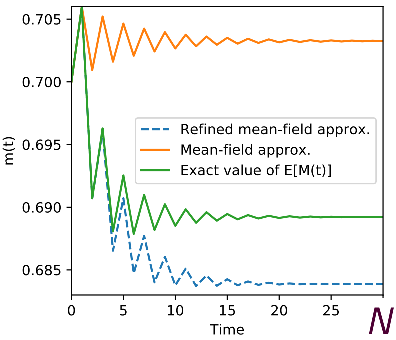

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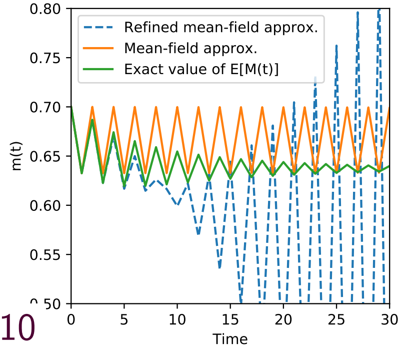

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WORKS uniformly in t

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Does not work for $t \gg N$

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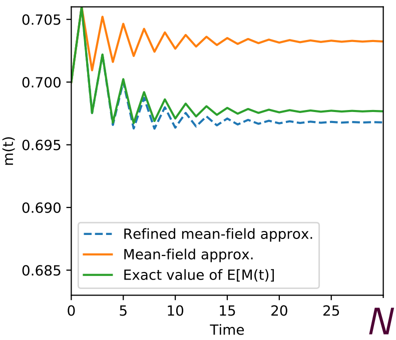
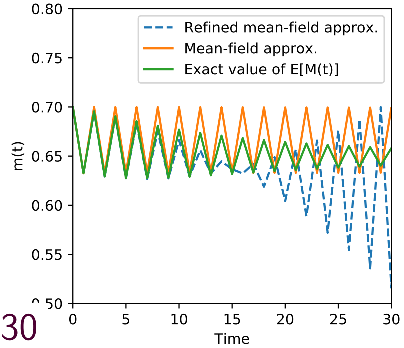


$N = 30$



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Outline

Recap

1. The traditional mean field approximation considers

$$X(t) \approx \mu(t),$$

   where $\mu(t) = \lim_{N\to\infty} X(t)$.

2. Our approach : we focus on $\mathbb{E}[X(t)]$ and we show that there exists $V(t)$ such that :

$$\mathbb{E}[X(t)] = \underbrace{\mu(t) + \frac{V(t)}{N}}_{\text{Refined approximation}} + o(1/N)$$

   It also works for $\mathbb{E}[h(X(t))]$.

3. $V(t)$ is easy to evaluate and for small N the refined approximation greatly improves the accuracy compared to the classical mean field approximation.

To go further :

It also works for continuous time models (*e.g.*: two-choice), see SIGMETRICS paper.

Future work:

- Find a relevant model (more general?) of synchronous population.
- Find guidelines where the method is applicable or not (example : Non-exponentially stable systems)

Main references :

- A Refined Mean Field Approximation of Synchronous Discrete-Time Population Models by Gast, Latella and Massink. To appear in *Performance Evaluation*.
  Paper is reproducible : `https://github.com/ngast/RefinedMeanField_SynchronousPopulation`
- A Refined Mean Field Approximation by Gast and Van Houdt. To appear in SIGMETRICS 2018 `https://hal.inria.fr/hal-01622054/` `https://github.com/ngast/rmf_tool/`