A Refined Mean Field Approximation for Synchronous Population Processes

Nicolas Gast (Inria) Diego Latella (CNR-ISTI) Mieke Massink (CNR-ISTI)

ABSTRACT

Mean field approximation is a popular method to study the behaviour of stochastic models composed of a large number of interacting objects. When the objects are asynchronous, the mean field approximation of a population model can be expressed as an ordinary differential equation. When the objects are synchronous the mean field approximation is a discrete time dynamical system. In this paper, we focus on the latter. We show that, similarly to the asynchronous case, the mean field approximation of a synchronous population can be refined by a term in $1/N$. Our result holds for finite time-horizon and steady-state. We provide two examples that illustrate the approach and its limit.

1. INTRODUCTION

The idea behind mean field approximation is to replace the study of the original stochastic system by the one of a, much simpler, deterministic dynamical system. Mean field approximation can be applied to a variety of systems [11]; it can be shown to be asymptotically optimal as the number of objects in the system goes to infinity [9, 3, 1] and is often very accurate also for systems of moderate size, composed of $N \approx 100$ objects.

The mean field approximation of a given model is constructed by considering the limit of the original stochastic model as the number of objects N goes to infinity. There can be two types of limits. The first type arises when the dynamics of the objects are asynchronous. In this case the mean field approximation is given by a continuous time dynamical system (often a system of ordinary differential equations) – this is the most studied case *e.g.* [9, 1, 2]. The second type arises when the objects are synchronous. In this case the mean field approximation is a discrete time dynamical system [3, 5]. We focus on the latter.

In this paper, we consider a variant of the (synchronous) population models considered in [3, 5, 10]. Each object evolves in a finite state-space and $M_i^{(N)}(t)$ denotes the proportion of objects in a state i at time t. The classical result of [3] states if $M^{(N)}(0) = m$, then for any time t the vector $M^{(N)}(t)$ converges almost surely as N grows to a deterministic quantity $\mu(t)$ that satisfies the recurrence equation

$$\mu(t+1) = \mu(t)\mathbf{K}(\mu(t)) \tag{1}$$

with $\mu(0) = m$. Our contribution consists in computing the rate of convergence : We show that there exists a time-dependent vector $V(t)$ such that

$$\mathbb{E}\left[M^{(N)}(t)\right] = \mu(t) + \frac{1}{N}V(t) + o\left(\frac{1}{N}\right). \tag{2}$$

We show that $V(t)$ satisfies a linear dynamical system that involves the first and second derivative of the function $\Phi : m \mapsto m\mathbf{K}(m)$. Moreover, when Φ has a unique fixed point $\mu(\infty)$ that is globally exponentially stable, then the same result holds for the steady-state.

We call the quantity $\mu(t) + V(t)/N$ the *refined* mean field approximation. As opposed to the classical mean field approximation, it depends on the system size N. We use two examples to show that :

- If the mean field approximation has a unique attractor that is exponentially stable, then our refined model is more accurate than the classical approximation uniformly in $t \in \mathbb{R} \cup \{\infty\}$.
- When the mean field approximation does *not* have an exponentially stable attractor, the improved accuracy only holds for a *finite* time horizon.

Our results extend the recent results of [7]. The authors of [7] study the steady-state of asynchronous stochastic models (that therefore have a continuous-time mean field approximation). There are two differences in our work : First we focus on *synchronous* objects; Second we obtain results also for the *transient* regime. Note that the results of [7] and the one of the current paper follow from a series of recent results concerning the rate of convergence of stochastic models to their mean field approximation [4, 8, 12].

An extended version of the current paper has been accepted for publication in [6]. This paper and the simulations it contained are fully reproducible : `https://github.com/ngast/RefinedMeanField_SynchronousPopulation`.

2. MODEL AND RESULTS

We consider a system of N identical interacting objects; (N is called the *size* of the system). Each object evolves in a finite state space and the time is slotted. The vector $M^{(N)}(t)$ denotes the occupancy measure at time t : $M_j^{(N)}(t)$ is the *fraction* of objects in state j at t.

At each time step $t \in \mathbb{N}$, each object performs a local transition. The transition probabilities of an object state depend on the current local state of the object and may depend also on $M^{(N)}(t)$. We denote by $K_{ij}(m)$ the probability for the object to jump from state i to state j in the system given that $M^{(N)}(t) = m$. We assume that, given the occupancy measure, the transitions made by the objects are independent. Our model is identical to the one of [3] up to the fact that the authors of [3] add a continuous resource to the model and allow object transition matrix $\mathbf{K}$ to depend also on the size N of the system. The results presented in this paper could be easily extended to the more general case.

The classical result (Theorem 4.1 of [3]) shows that if $M^{(N)}(0)$ converges almost surely to the deterministic limit $\mu(0)$ as N goes to infinity, then for any time $t > 0$, $M^{(N)}(t)$ converges almost surely to $\mu(t)$ defined in Equation (1).

2.1 First Main Result : Transient Behaviour

Let Φ_t be the function defined recursively by $\Phi_1(m) = mK(m)$ and $\Phi_{t+1}(m) = \Phi_1(\Phi_t(m))$. We denote by $(D\Phi_1)(m)$ and $(D^2\Phi_1)(m)$ the first and second derivative of the function Φ_1 evaluated in m.

Theorem 1. *Assume that the function Φ_1 is twice differentiable and that $M^{(N)}(0)$ converges weakly to $\mu(0)$. Let A_t and B_t be respectively the $n\times n$ matrix $A_t = (D\Phi_1)(\mu(t))$ and the $n \times n \times n$ tensor $B_t = (D^2\Phi_1)(\mu(t))$. Then*

$$\lim_{N\to\infty} N\mathbb{E}\left[M^{(N)}(t) - \Phi_t(M^{(N)}(0))\right] = V_t,$$

where V_t is a vector and W_t is an $n \times n$ matrix defined by

$$\begin{aligned} V_{t+1} &= A_t V_t + \tfrac{1}{2} B_t \cdot W_t \\ W_{t+1} &= \Gamma(\mu(t)) + A_t W_t A_t^T, \end{aligned} \tag{3}$$

with $V_0 = 0$, $W_0 = 0$ and $\Gamma(m)$ is the following $n\times n$ matrix:

$$\begin{aligned} \Gamma_{jj}(m) &= \textstyle\sum_{i=1}^n m_i \mathbf{K}_{ij}(m)(1 - \mathbf{K}_{ij}(m)) \\ \Gamma_{jk}(m) &= -\textstyle\sum_{i=1}^n m_i \mathbf{K}_{ij}(m)\mathbf{K}_{ik}(m) \end{aligned}$$

The main idea is to consider a Taylor expansion of $\mathbb{E}[h(\Phi_1(m))]$ around $\Phi_1(m)$ for any function h (see [6]).

2.2 Second Main Result : Steady-State

Mean field approximation can also be used to characterise the steady-state behaviour of a population model when the mean field approximation has a unique attractor. Here, we show how to refine this model when the mean field has an *exponentially stable* attractor, *i.e.* a point $\mu(\infty)$ such that

- $\mu(\infty)$ is an attractor: For any m : $\lim_{t\to\infty} \Phi_t(m) = \mu(\infty)$.
- $\mu(\infty)$ is exponentially stable : there exists $a, b > 0$ s.t. for all m in a neighbourhood of $\mu(\infty)$: $\|\Phi_t(m) - \mu(\infty)\| \leq ae^{-bt}\|m - \mu(\infty)\|$.

Theorem 2. *Assume that $M^{(N)}$ has a unique stationary distribution (for each N), that the function Φ_1 is twice differentiable and that the flow has a unique exponentially stable attractor $\mu(\infty)$. Then there exists a $n \times 1$ vector V_∞ and a $n \times n$ matrix W_∞ such that the constants V_t and W_t defined in Theorem 1 satisfy:*

$$\lim_{t\to\infty} V_t = V_\infty \qquad \text{and} \qquad \lim_{t\to\infty} W_t = W_\infty$$

Moreover

(i) W_∞ is the unique solution of the discrete-time Lyapunov equation:

$$A_\infty W A_\infty^T - W + \Gamma(\mu(\infty)) = 0$$

and $V_\infty = \frac{1}{2}(I - A_\infty)^{-1} B_\infty W_\infty$ with $A_\infty = D\Phi_1(\mu(\infty))$, $B_\infty = D^2\Phi_1(\mu(\infty))$ and I is the identity matrix.

(ii) We can exchange the limits :

$$\begin{aligned} &\lim_{N\to\infty}\lim_{t\to\infty} N\big(\mathbb{E}[M^{(N)}(t)] - \Phi_t(M^{(N)}(0))\big) \\ &= \lim_{t\to\infty}\lim_{N\to\infty} N\big(\mathbb{E}[M^{(N)}(t)] - \Phi_t(M^{(N)}(0))\big) = V_\infty. \end{aligned}$$

3. FIRST EXAMPLE : SEIR

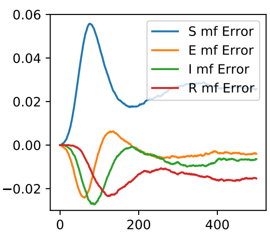
In this section we provide a simple example that illustrates the results for the refined mean field model of the simple computer epidemic SEIR example presented in [2]. Each object in the model consists of four local states: Susceptible (S), Exposed (E), Infected (I) (and active) and Recovered (R).

Its discrete time evolution is given by the following probability transition matrix $\mathbf{K}$ in which m_S, m_E, m_I and m_R denote the fraction of objects in the system that are in local state S, E, I and R:

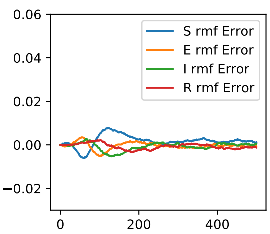
$$\mathbf{K}(m) = \begin{pmatrix} 1-(\alpha_e+\alpha_i m_I) & \alpha_e+\alpha_i m_I & 0 & 0 \\ 0 & 1-\alpha_a & \alpha_a & 0 \\ 0 & 0 & 1-\alpha_r & \alpha_r \\ \alpha_l & 0 & 0 & 1-\alpha_l \end{pmatrix}$$

In other words, a susceptible becomes exposed with probability $(\alpha_e + \alpha_i m_I)$ – *i.e.*, α_e denotes the external and α_i the internal infection probability –; An exposed node activates his infection with probability α_a; An infected recovers with probability α_r; and α_l is the probability to loose the protection against infection.

To give an idea of how the refined mean field approximation improves the accuracy compared to the classical mean field approximation, we plot in Figure 1 the difference between the two approximations with respect to the simulation : On the left panel, we plot $\mathbb{E}[M^{(N)}(t)] - \mu(t)$; On the right panel we plot $\mathbb{E}[M^{(N)}(t)] - (\mu(t) + V(t)/N)$ (in both cases for $N = 10$). The expectation was computed by using an average of 50,000 runs of a stochastic simulation of the model. The values of the parameters are : $\alpha_e = 0.01, \alpha_i = 0.08, \alpha_r = 0.02, \alpha_l = 0.01$ and $\alpha_a = 0.04$. and the initial state was $M_S(0) = 1$ and $M_E(0) = M_I(0) = M_R(0) = 0$.



Error of mean field approx.



Error of the refined approx.

Figure 1: SEIR model: Quantification of the error of the mean field or refined mean field approximation ($N = 10$).

We observe that the refined mean field approximation (right panel) is an order of magnitude closer to the value obtained by simulation. This figure illustrates that Theorem 1 is not just valid asymptotically, but actually it refines the classical mean field approximation for relatively small values of N. In this case, the system has a unique attractor and the refined approximation can be used for estimating the steady-state expected values (see Table 1).

State	S	E	I	R
Simulation ($N = 10$)	0.191	0.115	0.231	0.462
Refined mean field ($N = 10$)	0.189	0.116	0.232	0.464
Mean field ($N = 10$)	0.164	0.119	0.239	0.478

Table 1: SEIR model: Comparison of the accuracy of the mean field and refined mean field approximation for the steady-state proportion of objects in states S, E, I or R.

4. ACCURACY VERSUS TIME

In the previous example, the mean field limit has an exponentially stable attractor, which implies that the accuracy of the mean field approximation is uniform in time (Theorem 2(ii)). Here, we show that this is no longer the case if the model does not have an exponentially stable attractor.

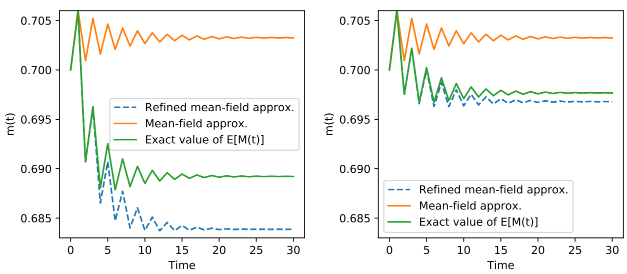
We consider a system with N objects in which each object is in state 0 or 1. An object in state 1 goes to state 0 with probability 1 and an object in state 0 goes to 1 with probability αm_0, where $\alpha \in (0, 1)$ is a parameter. The transition matrix K is therefore

$$K(m) = \begin{bmatrix} 1 - \alpha m_0 & \alpha m_0 \\ 1 & 0 \end{bmatrix}$$

The function $m \mapsto mK(m)$ has a unique fixed point whose first component is $\mu_0(\infty) = (\sqrt{1 + 4\alpha} - 1)/(2\alpha)$. This fixed point is exponentially stable if and only if $\alpha < 0.75$.

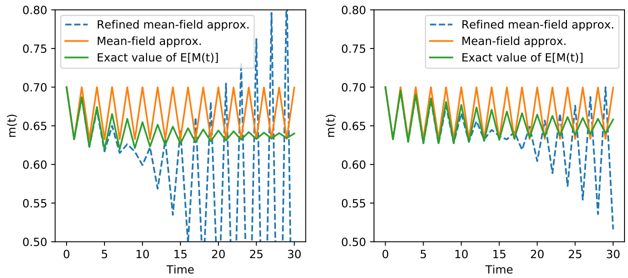
In Figure 2 and Figure 3, we plot the mean field $\mu(t)$ and refined mean field approximation $\mu(t) + V(t)/N$ as well an exact value of $\mathbb{E}[M(t)]$ for $N = 10$ and $N = 30$. The initial value is $m(0) = 0.7$. The exact value of $\mathbb{E}[M(t)]$ was computed by a numerical method that uses the fact that the system with N objects can be described by a Markov chain with $N + 1$ states.

These figures show that the refined approximation always improves the accuracy compared to the classical mean field approximation for small values of t, both for $\alpha = 0.6$ and $\alpha = 0.75$. The situation for large values of t is quite different. On the one hand, when the fixed point is exponentially stable ($\alpha = 0.6$, Figure 2), the refined approximation is very accurate for all values of t. On the other hand, when the fixed point is not exponentially stable ($\alpha = 0.75$, Figure 3), the refined approximation seems to be unstable and is not a good approximation of $\mathbb{E}[M(t)]$ for values of t that are too large compared to N ($t > 7$ for $N = 10$ or $t > 12$ for $N = 30$).



(a) $N = 10$ (b) $N = 30$

Figure 2: Exponentially stable case ($\alpha = 0.6$).



(a) $N = 10$ (b) $N = 30$

Figure 3: Non-exponentially stable case ($\alpha = 0.75$).

5. CONCLUSION AND EXTENSIONS

In this paper, we have shown that it is possible to adapt the refined mean field approximation proposed in [7] to the case of synchronous population processes, both for transient and steady-state state. This approximation is more accurate than the classical mean field approximation in many cases. Yet, when the mean field approximation does not have an exponentially stable attractor, this new approximation must be handle with care. We are currently working on extending this methodology to study the transient regime of asynchronous population processes, in which case Equation (3) can be replaced by an ordinary differential equation.

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