1) $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$\frac{\partial f}{\partial x_1} = 4x_1 - 4x_2 \qquad \frac{\partial^2 f}{\partial x_1^2} = 4$

$\frac{\partial f}{\partial x_2} = -4x_1 + 3x_2 + 1 \qquad \frac{\partial^2 f}{\partial x_2^2} = 3$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -4$

Location of zero gradient:

$4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$

$-4x_1 + 3(x_1) + 1 = 0 \Rightarrow x_1 = 1,\ x_2 = 1$

$H(x_1, x_2) = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} = H(1,1)$

$\begin{vmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 16 = 0$

$\lambda^2 - 7\lambda - 4 = 0 \Rightarrow \lambda_1 = 0.531$

$\lambda_2 = -7.531$

$\boxed{\lambda_1 > 0 > \lambda_2 \Rightarrow \text{indefinite hessian}}$

Directions of downslopes:

$f(x_1, x_2) \approx f(1,1) + \cancel{\nabla f(1,1)^T \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}} + \frac{1}{2}[\partial x_1\ \partial x_2] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$

$= f(1,1) + \frac{1}{2}[\partial x_1\ \partial x_2] \begin{bmatrix} 4\partial x_1 - 4\partial x_2 \\ -4\partial x_1\ 3\partial x_2 \end{bmatrix}$

$= f(1,1) + \frac{1}{2}(4\partial x_1^2 - 4\partial x_1\partial x_2 - 4\partial x_1\partial x_2 + 3\partial x_2^2)$

$= f(1,1) + \frac{1}{2}(4\partial x_1^2 - 8\partial x_1\partial x_2 + 3\partial x_2^2)$

$= f(1,1) + \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2)$

The directions of downslopes are such $(\partial x_1, \partial x_2)$ that

$$f(x_1, x_2) - f(1,1) = \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2) < 0$$

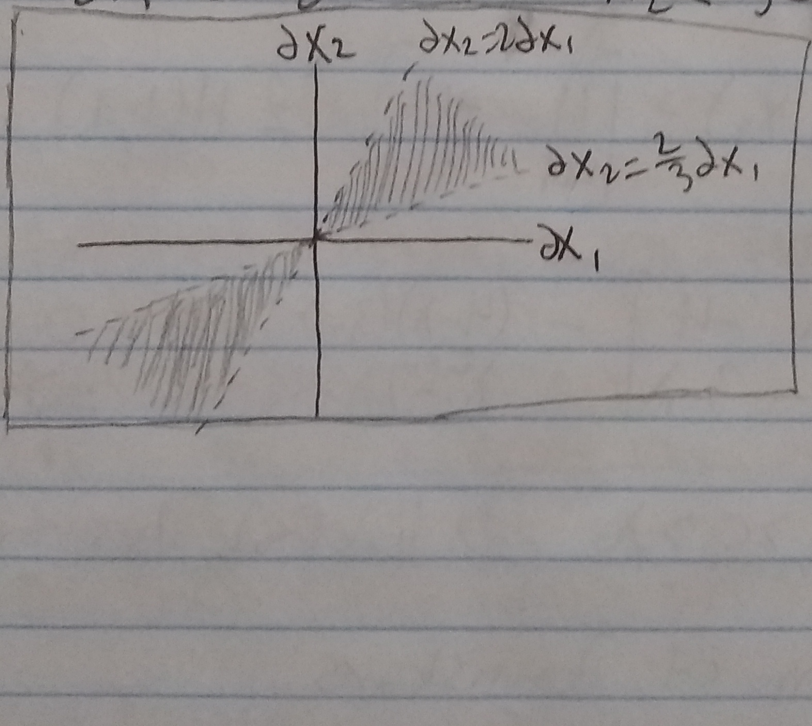
or $$\boxed{(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2) < 0}$$

case 1: $2\partial x_1 - \partial x_2 > 0 \Rightarrow \partial x_2 < 2\partial x_1$

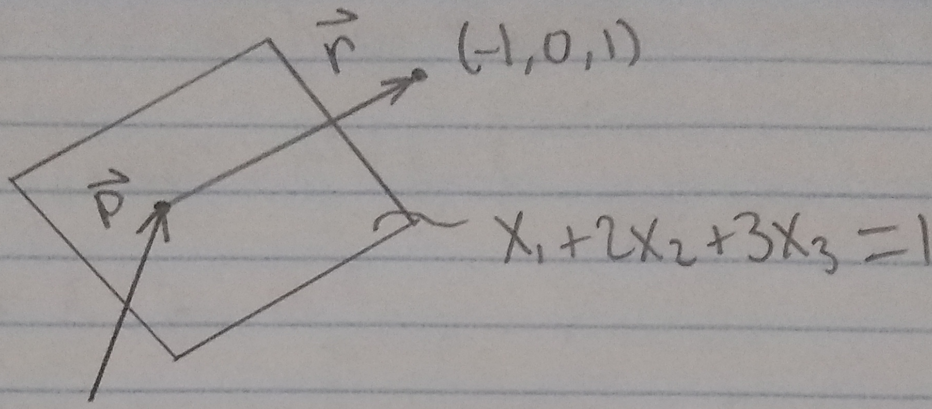
$2\partial x_1 - 3\partial x_2 < 0 \qquad \partial x_2 > \frac{2}{3}\partial x_2$

case 2: $2\partial x_1 - \partial x_2 < 0 \Rightarrow \partial x_2 > 2\partial x_1$

$2\partial x_1 - 3\partial x_2 > 0 \qquad \partial x_2 < \frac{2}{3}\partial x_2$



2) a)




$$\vec{p}(x_2,x_3)=\begin{bmatrix}1-2x_2-3x_3\\ x_2\\ x_3\end{bmatrix} \qquad \vec{r}=\begin{bmatrix}-2+2x_2+3x_3\\ -x_2\\ 1-x_3\end{bmatrix}$$

$$f(x_2,x_3)=\|\vec{r}\|=(-2+2x_2+3x_3)^2+x_2^2+(1-x_3)^2$$

Problem: $\min\limits_{x_2,x_3} f(x_2,x_3)$

$$\frac{\partial f}{\partial x_2}=2(-2+2x_2+3x_3)(2)+2x_2=10x_2+12x_3-8$$

$$\frac{\partial f}{\partial x_3}=2(-2+2x_2+3x_3)(3)+2(1-x_3)(-1)$$
$$=-12+12x_2+18x_3-2+2x_3=12x_2+20x_3-14$$

$$\frac{\partial f}{\partial x_2^2}=10 \quad \frac{\partial f}{\partial x_3^2}=20 \quad \frac{\partial f}{\partial x_2 \partial x_3}=12$$

$$H=\begin{bmatrix}10 & 12\\ 12 & 20\end{bmatrix}$$

$(10-\lambda)(20-\lambda)-144=0$

$\lambda^2-30\lambda+56=0$

$\lambda_1=28,\ \lambda_2=2 \Rightarrow \boxed{\text{P.D./convex}}$

3) Prove that a hyperplane is a convex set:

$$S = \{x \in \mathbb{R}^n : a^T x = c\}$$

Let $x_1, x_2 \in S$

$$a^T[\lambda x_1 + (1-\lambda)x_2] = \lambda a^T x_1 - \lambda a^T x_2 + a^T x_2$$

$$= \cancel{\lambda c} - \cancel{\lambda c} + c = c$$

$\therefore \lambda x_1 + (1-\lambda)x_2 \in S \quad \forall x_1, x_2 \in S$
S is convex.

4) Illumination Problem: $\min_{p} \max_{k} \{h(a_k^T p, I_t)\}$

S.T. $0 \leq p_i \leq p_{max}$

$p := [p_1, \dots, p_n]^T$ are the power output of n lamps.
$a_k$ $(k=1,\dots,m)$ are fixed parameters for m mirrors.
$I_t$ is the target intensity level.

$$h(I, I_t) = \begin{cases} I_t/I & \text{if } I \leq I_t \\ I/I_t & \text{if } I_t \leq I \end{cases}$$

a) Show that the problem is convex:

$$\frac{\partial h}{\partial p} = \frac{\partial h}{\partial I} \cdot \frac{\partial I}{\partial p} = \frac{\partial h}{\partial I} \cdot a_k$$

$$\frac{\partial h}{\partial p} = \begin{cases} -I_t/I^2 \cdot a_k \\ 1/I_t \cdot a_k \end{cases} = \begin{cases} \frac{-I_t}{(a_k^T p)^2} \cdot a_k \\ 1/I_t \cdot a_k \end{cases}$$

$$\frac{\partial^2 h}{\partial p^2} = \begin{cases} \frac{2I_t}{(a_k^T p)^3} a_k a_k^T \\ 0 \end{cases} \qquad \frac{2I_t}{(a_k^T p)^3} \geq 0$$

$$w^T a_k a_k^T w = (a_k^T w)^2 \geq 0 \Rightarrow a_k a_k^T \text{ is P.S.D.}$$

$\frac{\partial^2 h}{\partial p^2}$ is P.S.D $\Rightarrow h$ is convex, $\max_k \{h_k\}$ is convex

$$0 \leq p_i \leq p_{max} \Rightarrow \begin{cases} -p \leq 0 \\ p - p_{max} \leq 0 \end{cases} \quad \text{Linear/convex}$$

Objective and constraints are all convex.

b) With the additional requirement $\sum_{i=1}^{n} p_i < p^*$, will there be a unique solution?

Yes, because $[1,1,\ldots 1] \cdot p - p^* < 0$ is convex.

c) If we require no more than 10 lamps to be switched on, will there be a unique solution?

No, the selection of lamps could change to produce the same optimized solution

5) $c^*(y) = \max_x \{xy - c(x)\}$.
$f(x,y) = xy - c(x)$ is linear (convex) w.r.t. $y$: $\left(\frac{d^2 f}{dy^2} = 0\right)$.
The maximum of a set of convex functions (one for every possible $x$) is also convex. Therefore, $c^*(y)$ is a convex function with respect to $y$.