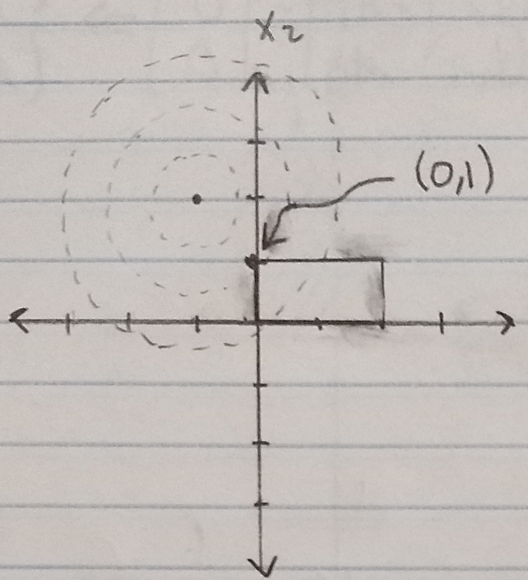
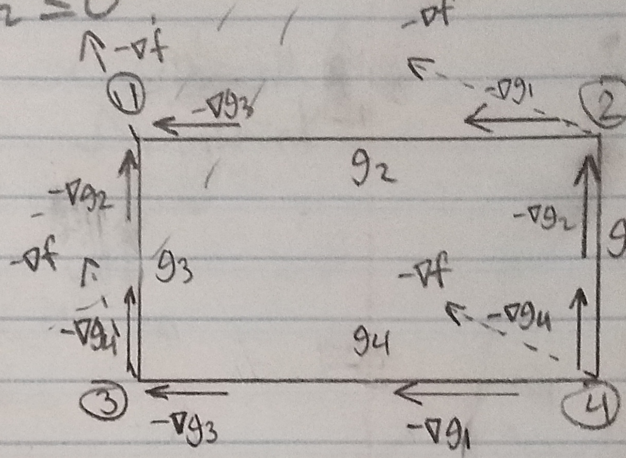
1) $\min f(x) = (x_1+1)^2 + (x_2-2)^2$

s.t. $g_1 = x_1 - 2 \leq 0$, $g_3 = -x_1 \leq 0$

$g_2 = x_2 - 1 \leq 0$, $g_4 = -x_2 \leq 0$





$$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\nabla_x L = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $x_1 - 2 = 0$ then $\mu_1 > 0$; if $x_1 - 2 < 0$ then $\mu_1 = 0$

if $x_2 - 1 = 0$ then $\mu_2 > 0$; if $x_2 - 1 < 0$ then $\mu_2 = 0$

if $-x_1 = 0$ then $\mu_3 > 0$; if $-x_1 < 0$ then $\mu_3 = 0$

if $-x_2 = 0$ then $\mu_4 > 0$; if $-x_2 < 0$ then $\mu_4 = 0$

1) $\mu_1 = \mu_4 = 0$, $x_2 = 1, x_1 = 0$ $\quad \begin{bmatrix} 2(x_1+1) - \mu_3 \\ 2(x_2-2) + \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \mu_3 = 2 \\ \mu_2 = 2 \end{cases}$ ✓

2) $\mu_3 = \mu_4 = 0$, $x_2 = 1, x_1 = 2$ $\quad \begin{bmatrix} 2(x_1+1) + \mu_1 \\ 2(x_2-2) + \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \mu_1 = -6 \\ \mu_2 = 2 \end{cases}$ ✗ ✓

3) $\mu_1 = \mu_2 = 0$ $\quad \begin{bmatrix} 2(x_1+1) - \mu_3 \\ 2(x_2-2) - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \mu_3 = 2 & \checkmark \\ \mu_4 = -4 & \times \end{cases}$

$x_2 = x_1 = 0$

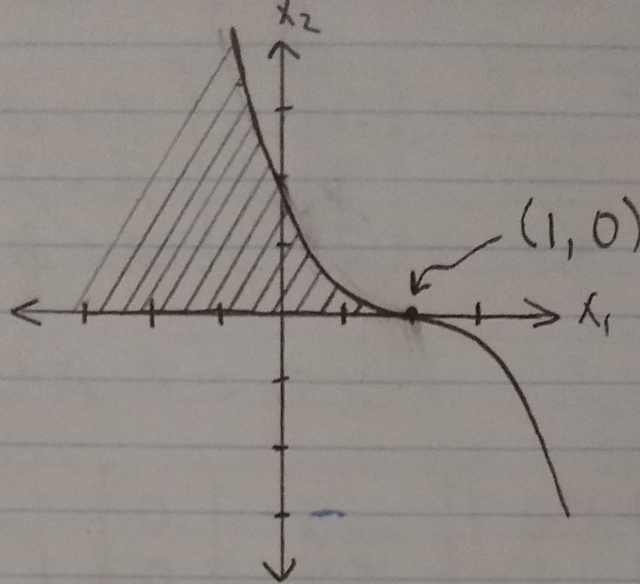
4) $\mu_3 = \mu_2 = 0$ $\quad \begin{bmatrix} 2(x_1+1) + \mu_1 \\ 2(x_2-2) - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \mu_1 = -6 & \times \\ \mu_4 = -4 & \times \end{cases}$

$x_2 = 0, x_1 = 2$

2) $\min f = -x_1$

s.t. $g_1 = x_2 - (1-x_1)^3 \leq 0$

$g_2 = -x_2 \leq 0$



$$L = -x_1 + \mu_1(x_2 - (1-x_1)^3) + \mu_2(-x_2)$$

$$\nabla_x L = \begin{bmatrix} -1 + 3\mu_1(1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $x_2 - (1-x_1)^3 = 0$ then $\mu_1 > 0$; if $x_2 - (1-x_1)^3 < 0$ then $\mu_1 = 0$

if $-x_2 = 0$ then $\mu_2 > 0$; if $-x_2 < 0$ then $\mu_2 = 0$

1) $x_2 - (1-x_1)^3 = 0$, $\mu_1 > 0$ $\Rightarrow x_1 = 1$ $\begin{bmatrix} -1 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ X

$-x_2 = 0$, $\mu_1 > 0$

2) $x_2 - (1-x_3)^3 = 0$, $\mu_1 > 0$ $\Rightarrow \mu_1 - 0 > 0$ X

$-x_2 < 0$, $\mu_2 = 0$

3) $x_2 - (1-x_3)^3 < 0$, $\mu_1 = 0$ $\Rightarrow -1 = 0$ X

$-x_2 < 0$ $\mu_2 = 0$

4) $x_2-(1-x_1)^3 < 0$, $\mu_1 = 0$ $\Rightarrow \mu_1 - \mu_2 < 0$ X
$-x_2 = 0$, $\mu_2 > 0$

A solution cannot be found based on optimality conditions because the gradients of g_1 and g_2 do not span the gradient of the objective f.

3) $\max f = x_1x_2 + x_2x_3 + x_1x_3$
s.t. $h = x_1 + x_2 + x_3 - 3 = 0$

$$L = -x_1x_2 - x_2x_3 - x_1x_3 + \lambda(x_1+x_2+x_3-3)$$

$$\nabla_x L = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla_\lambda L = x_1 + x_2 + x_3 - 3 = 0$$

$x_1 = x_2 = x_3$, by symmetry

$3x - 3 = 0 \Rightarrow \boxed{x_1 = x_2 = x_3 = 1}$

$-1 - 1 + \lambda = 0 \Rightarrow \boxed{\lambda = 2}$

$$L_{xx} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$ Eigenvalues: $-2, 1, 1$ Not P.S.D.

$$dx^T L_{xx} dx = [dx_1 \; dx_2 \; dx_3] \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$= -2dx_1dx_2 - 2dx_1dx_3 - 2dx_2dx_3$$

$$\frac{dh}{dx}dx = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = 0$$

$$\Rightarrow dx_1 + dx_2 + dx_3 = 0$$
$$dx_1 = -dx_2 - dx_3$$

$$-2\left[(-dx_2 - dx_3)dx_2 + (-dx_2 - dx_3)dx_3 + dx_2 dx_3\right]$$
$$= -2\left[-dx_2^2 - dx_2 dx_3 - \cancel{dx_2 dx_3} - dx_3^2 + \cancel{dx_2 dx_3}\right]$$
$$= 2(dx_2^2 + dx_2 dx_3 + dx_3^2)$$
$$= 2\left[\left(dx_2 + \tfrac{1}{2}dx_3\right)^2 + \tfrac{3}{4}dx_3^2\right] > 0$$ dx_1, dx_2, dx_3 cannot all be zero.

L_{xx} is convex for all feasible perturbations, so $x_1 = x_2 = x_3 = 1$, $f = 3$ is the solution.

5) Garbage truck routing problem: N nodes to visit, where cost of moving from node i to j is C_{ij}, or ∞ if there is no direct path.
Site 0 is the beginning and end. Minimize the total cost.

$X(t) = [X_1, X_2, \dots X_N]^T$ $X_i = 0$ or 1 State tensor

$L(X(t), X(t-1)) = C_{ij}$ or ∞ Loss function

$F = \sum_{t=1}^{N} L(X(t), X(t-1))$ Objective

$\min_{X} F(X)$

s.t. $\sum_{t=1}^{T} X(t) \geq 1_{n \times 1}$ All nodes visited at least once

$X_1(0) + X_1(T) \geq 2$ First and last node are $i = 1$.