Lecture Notes EMSE 4765:
DATA ANALYSIS - Probability Review

Chapter 3: Conditional Probability and Independence

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THE GEORGE WASHINGTON UNIVERSITY

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Text Book: A Modern Introduction to Probability and Statistics, Understanding Why and How

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3 Conditional Probability and Independence

3.1 Conditional Probability…

- Knowing that an event has occurred **sometimes forces us** to **reassess the probability of another event**; the new probability is called **the conditional probability given the event.**

- **Example:** Typically, the stock prices of companies move up and down in a similar pattern depending on **"overall market behavior".** This market behavior is captured **by market indices**, such as the **Dow Jones Index, SP500 Index or the Nasdaq Index.** You are considering investing in the stock of a particular company. All three market indices have shown a recent decline. **Are you more or less inclined to invest in this company now given this information?**

Regardless of your answer, you base your answer on assessing :

$$Pr(Stock\,goes\,up|Market\,is\,down) - Pr(Stock\,goes\,up)$$

3 Conditional Probability and Independence

3.1 Conditional Probability. . .

Die Experiment: **Those outcomes that are even :** $E = \{2, 4, 6\}$.

Die Experiment: **Those outcomes that are odd :** $O = \{1, 3, 5\}$.

Die Experiment: **Those outcomes that are prime:** $P = \{1, 2, 3, 5\}$

Suppose you know that the outcome of the die is prime, what is now the probability that the outcome of the die is odd?

Answer: $\frac{3}{4} = \boldsymbol{Pr(O|P)}$**. How does one calculate** $\boldsymbol{Pr(O|P)}$**?**

$$O \cap P = \{1, 3, 5\} \Rightarrow Pr(O \cap P) = \mathbf{3/6}$$

$$Pr(P) = \frac{\mathbf{4}}{\mathbf{6}}. \text{ Hence, apparently: } Pr(O|P) = \mathbf{(3/6)}/\mathbf{(4/6)} = \frac{\mathbf{3}}{\mathbf{4}}$$

3 Conditional Probability and Independence

3.1 Conditional Probability...

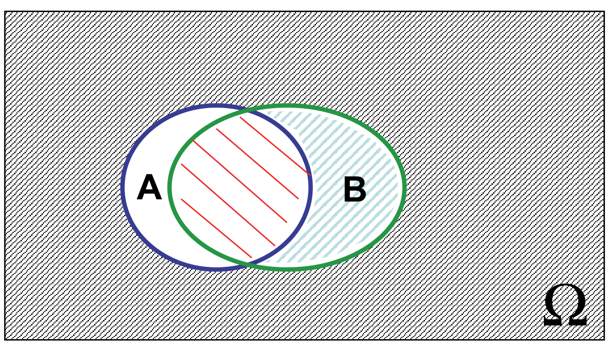
Definition. The conditional probability of A given B is given by:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}, \text{ provided } Pr(B) > 0.$$

Exercise: Show that $Pr(A|B) + Pr(A^c|B) = 1$

Solution: Using the definition of conditional probability we have

$$Pr(A|B) + Pr(A^c|B) = \frac{Pr(A \cap B)}{Pr(B)} + \frac{Pr(A^c \cap B)}{Pr(B)} = \frac{Pr(A \cap B) + Pr(A^c \cap B)}{Pr(B)}$$



However: $\{A \cap B\}$ and $\{A^c \cap B\}$ are disjoint and :

$$\{A \cap B\} \cup \{A^c \cap B\} = B \Rightarrow$$
$$Pr(A \cap B) + Pr(A^c \cap B) = Pr(B) \Rightarrow$$
$$Pr(A|B) + Pr(A^c|B) = Pr(B)/Pr(B) = 1$$

The Multiplication Rule. For any events A and B:

$$Pr(A \cap B) = Pr(A|B)Pr(B).$$

Example: Recall the die experiment

$$P = \{1, 2, 3, 5\}, E = \{2, 4, 6\}, \; P \cap E = \{2\}$$

Note that: $Pr(P \cap E) = \frac{1}{6} \neq Pr(P) \times Pr(E) = \frac{4}{6} \times \frac{1}{2} = \frac{1}{3}$

But: $Pr(P \cap E) = \frac{1}{6} = Pr(E|P) \times Pr(P) = \frac{1}{4} \times \frac{4}{6} = \frac{1}{6}$

3 Conditional Probability and Independence

3.4 Independence. . .

Definition. An event A is called independent of event B if
$$Pr(A|B) = Pr(A)$$

Coin Toss Experiment: $H =$ "Heads", $T =$ "$Tails$"

Die Experiment: $E = \{2, 4, 6\}$, $P = \{1, 2, 3, 5\}$

Are H and E independent events? Answer: Yes

Are P and E independent events? Answer: No

Are H and T independent events? Answer: No

Dependent events are more common than independent events!

3 Conditional Probability and Independence

3.4 Independence...

Independence: To show that **A and B are independent** it suffices to prove *just one* of the following:

$$Pr(A|B) = Pr(A), Pr(B|A) = Pr(B), Pr(A \cap B) = Pr(A)Pr(B)$$

where A may be replaced by A^c and B replaced by B^c, or both. **If one of these statements holds, all of them are true.** If two events are **not independent**, they are called **dependent**.

Example: Recall the die experiment $P = \{1, 2, 3, 5\}, E = \{2, 4, 6\}$, $P \cup E = \Omega,\ P \cap E = \{2\}$

$$Pr(P \cap E) = \tfrac{1}{6} \neq Pr(P) \times Pr(E) = \tfrac{4}{6} \times \tfrac{1}{2} = \tfrac{1}{3} \Rightarrow P \text{ and } E \text{ are dependent}$$

3 Conditional Probability and Independence

3.4 Independence...

- $Pr(A|B) = Pr(A) \Rightarrow Pr(A \cap B) = Pr(A|B)Pr(B) = Pr(A)rP(B)$

Hence: $\boldsymbol{A}$ **independent of** $\boldsymbol{B} \Leftrightarrow \boldsymbol{Pr(A \cap B) = Pr(A)Pr(B)}$.

- $Pr(A|B) = Pr(A) \Rightarrow Pr(B|A) = \frac{Pr(A|B)Pr(B)}{Pr(A)} = \frac{Pr(A)Pr(B)}{Pr(A)} = Pr(B)$

Hence: $\boldsymbol{A}$ **independent of** $\boldsymbol{B} \Leftrightarrow \boldsymbol{B}$ **independent of** $\boldsymbol{A}$.

- $Pr(A|B) = Pr(A) \Leftrightarrow 1 - Pr(A|B) = 1 - Pr(A) \Leftrightarrow Pr(A^c|B) = Pr(A^c)$.

Hence: $\boldsymbol{A}$ **independent of** $\boldsymbol{B} \Leftrightarrow \boldsymbol{A^c}$ **independent of** $\boldsymbol{B}$

Two VERY IMPORTANT RULES that help probability computations
Both use conditional probabilities.

LOTP $\equiv$ Computing a probability through conditioning on several disjoint events that make up the whole sample space Ω.

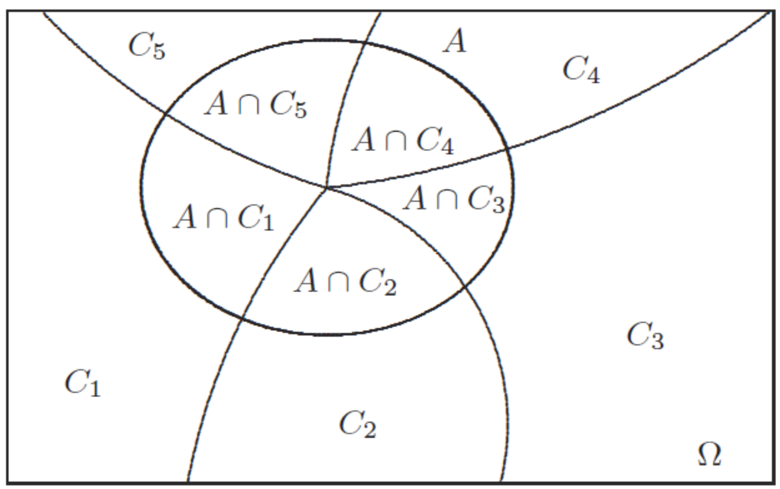


Fig. 3.2. The law of total probability (illustration for $m = 5$).

The Law of Total Probability: Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$. The probability of an arbitrary event A can then be expressed as:

$$Pr(A) = Pr(A \cap C_1) + Pr(A \cap C_2) + \ldots + Pr(A \cap C_m) \Leftrightarrow$$
$$Pr(A) = Pr(A|C_1)Pr(C_1) + Pr(A|C_2)Pr(C_2) + \ldots + Pr(A|C_m)Pr(C_m).$$

Die Experiment: **Those outcomes that are even :** $E = \{2, 4, 6\}$.

Die Experiment: **Those outcomes that are odd :** $O = \{1, 3, 5\}$.

Die Experiment: **Those outcomes that are prime:** $P = \{1, 2, 3, 5\}$

$$Pr(P) = Pr(P|O)Pr(O) + Pr(P|E)Pr(E) = 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6}$$

Bayes' Rule: Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$. The conditional probability of C_i given an arbitrary event A can then be expressed as:

$$Pr(C_i|A) = \frac{Pr(A \cap C_i)}{Pr(A)} = \frac{Pr(A|C_i)P(C_i)}{Pr(A|C_1)Pr(C_1) + \ldots + Pr(A|C_m)Pr(C_m)}$$

Die Experiment: $E = \{2, 4, 6\}$, $O = \{1, 3, 5\}$, $P = \{1, 2, 3, 5\}$

$$Pr(E|P) = \frac{Pr(P|E)Pr(E)}{Pr(P)} = \frac{Pr(P|E)Pr(E)}{Pr(P|O)Pr(O) + Pr(P|E)Pr(E)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{4}{6}} = \frac{1}{6} \cdot \frac{6}{4} = \frac{1}{4}$$