

1 Question 1: Player skill graphical models

1.a

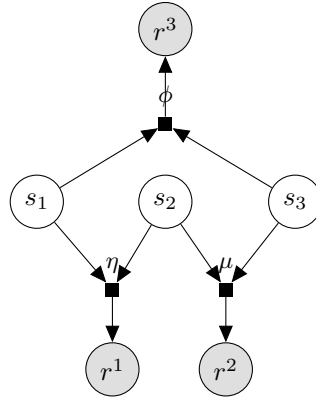


Figure 1: Factor graph of simple player skill model when observing three games between three players. The games are denoted as the factors ϕ , μ and η . Game outcomes are denoted as $r^{(k)}$, where $k \in \{1, 2, 3\}$ being the different games played. The result of the games, denoted $r^{(k)} \in \{0, 1\}$ with 0 indicating a loss and 1 indicating a win. The players' skills are identified as s_1 , s_2 and s_3 , where the index denotes the ID of the player $\in \{1, 2, 3\}$ respectively.

1.b

1.b.1 $I(s_1, s_2 | r^{(2)})$

There are 2 paths leading from s_1 to s_2 , namely $s_1 - \eta - s_2$ and $s_1 - \phi - s_3 - \mu - s_2$. The former path is blocked, as η has two incoming edges and neither η , nor $r^{(1)}$ (being all descendants) are in the conditioning set (being $r^{(2)}$). The latter is also blocked, but the case is slightly more complicated. Firstly, μ has two incoming edges, however, its descendant is in the conditioning set ($r^{(2)}$), therefore, the path is not blocked there. Secondly, ϕ also has two incoming edges and neither ϕ , nor $r^{(3)}$ are in the conditioning set, therefore, the path is blocked there. Thus, as the two paths are blocked $\Rightarrow I(s_1, s_2 | r^{(2)})$.

1.b.2 $I(s_1, s_2 | r^{(2)}, r^{(3)})$

As identified in Section 1.b.1, there are 2 paths. The path $s_1 - \eta - s_2$ is still blocked, as η , nor any of its descendants are in the conditioning set. The path $s_1 - \phi - s_3 - \mu - s_2$, however, is now unblocked, as for all factors and variables with 2 incoming edges, there is a child that belongs

to the conditioning set, namely $r^{(3)}$ is a child of ϕ and $r^{(2)}$ is a child of μ , both of them being in the conditioning set.

1.c

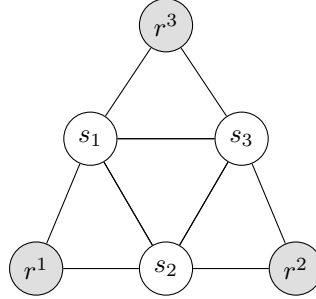


Figure 2: Markov network of simple player skill model when observing three games between three players. The observed and latent variables are denoted in a similar way, as Figure 1. Game outcomes are denoted as $r^{(k)}$, where $k \in \{1, 2, 3\}$ being the different games played. The result of the games, denoted $r^{(k)} \in \{0, 1\}$ with 0 indicating a loss and 1 indicating a win. The players' skills are identified as s_1 , s_2 and s_3 , where the index denotes the ID of the player $\in \{1, 2, 3\}$ respectively.

1.d

2 Question 2: Gaussian player skill model

2.a

From the figure, one can express the joint probability distribution as follows:

$$p(p_b, p_w, s_b, s_w) = \frac{1}{Z} \phi_1(p_w) \phi_2(s_w, p_w) \phi_3(s_b) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r) \quad (1)$$

Thereafter, using chain rule, $p(p_b, p_w | s_b, s_w)$ can be expressed as follows:

$$p(p_b, p_w | s_b, s_w) = \frac{p(p_b, p_w, s_b, s_w)}{p(s_b, s_w)} = \frac{p(p_b, p_w, s_b, s_w)}{\int_{p_b, p_w} p(p_b, p_w, s_b, s_w)} = \quad (2)$$

$$\frac{\frac{1}{Z} \phi_1(s_w) \phi_2(s_w, p_w) \phi_3(s_b) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)}{\int_{p_b, p_w} \frac{1}{Z} \phi_1(s_w) \phi_2(s_w, p_w) \phi_3(s_b) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)} \quad (3)$$

$$\frac{\frac{1}{Z} \phi_1(p_w) \phi_2(s_w, p_w) \phi_3(p_b) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)}{\frac{1}{Z} \phi_1(p_w) \phi_3(p_b) \int_{p_b, p_w} \phi_2(s_w, p_w) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)} \quad (4)$$

$$\frac{\phi_2(s_w, p_w) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)}{\int_{p_b, p_w} \phi_2(s_w, p_w) \phi_4(s_b, p_b) \phi_5(p_w, p_b, r)} \quad (5)$$

$$\frac{\phi_2(s_w, p_w) \phi_4(s_b, p_b)}{\int_{p_b, p_w} \phi_2(s_w, p_w) \phi_4(s_b, p_b)} \quad (6)$$

$$\phi_2(s_w, p_w) \phi_4(s_b, p_b) \quad (7)$$

2.b

2.b.1 $\mathbb{E}[\theta \mid s_b, s_w]$

$$\mathbb{E}[\theta \mid s_w, s_b] = \frac{1}{\sqrt{2}\beta} \mathbb{E}[(p_w - s_w) + (p_b - s_b) \mid s_w, s_b] \quad (8)$$

$$\frac{1}{\sqrt{2}\beta} \left(\mathbb{E}[p_w + p_b \mid s_w, s_b] - s_w - s_b \right) \quad (9)$$

$$\frac{1}{\sqrt{2}\beta} \left(\mathbb{E}[p_w \mid s_w, s_b] + \mathbb{E}[p_b \mid s_w, s_b] - s_w - s_b \right) \quad (10)$$

$$\frac{1}{\sqrt{2}\beta} \left(s_w + s_b - s_w - s_b \right) = 0 \quad (11)$$

2.b.2 $\mathbb{E}[\psi \mid s_b, s_w]$

$$\mathbb{E}[\psi \mid s_w, s_b] = \frac{1}{\sqrt{2}\beta} \mathbb{E}[(p_w - s_w) - (p_b - s_b) \mid s_w, s_b] \quad (12)$$

$$\frac{1}{\sqrt{2}\beta} \left(\mathbb{E}[p_w - p_b \mid s_w, s_b] - s_w + s_b \right) \quad (13)$$

$$\frac{1}{\sqrt{2}\beta} \left(\mathbb{E}[p_w \mid s_w, s_b] - \mathbb{E}[p_b \mid s_w, s_b] - s_w + s_b \right) \quad (14)$$

$$\frac{1}{\sqrt{2}\beta} \left(s_w - s_b - s_w + s_b \right) = 0 \quad (15)$$

2.b.3 $\mathbb{E}[\theta^2 \mid s_b, s_w]$

$$\mathbb{E}[\theta^2 \mid s_w, s_b] = \frac{1}{2\beta^2} \mathbb{E}[(p_w - s_w) + (p_b - s_b)]^2 \mid s_w, s_b] \quad (16)$$

$$\frac{1}{2\beta^2} \left(\underbrace{\mathbb{E}[(p_w - s_w)^2 \mid s_w, s_b]}_A + \underbrace{2\mathbb{E}[(p_w - s_w)(p_b - s_b) \mid s_w, s_b]}_B + \underbrace{\mathbb{E}[(p_b - s_b)^2 \mid s_w, s_b]}_C \right) \quad (17)$$

$$A = \mathbb{E}[p_w^2 \mid s_w, s_b] - 2s_w \mathbb{E}[p_w \mid s_w, s_b] + s_w^2 \quad (18)$$

$$A = \text{Var}(p_w) + \mathbb{E}[p_w \mid s_w, s_b]^2 - 2s_w^2 + s_w^2 = \text{Var}(p_w) = \beta^2 \quad (19)$$

$$B = 2\mathbb{E}[p_b p_w \mid s_w, s_b] - 2s_b \mathbb{E}[p_w \mid s_w, s_b] - 2s_w \mathbb{E}[p_b \mid s_w, s_b] + 2s_b s_w \quad (20)$$

$$B = 2 \left(\mathbb{E}[p_b \mid s_w, s_b] \mathbb{E}[p_w \mid s_w, s_b] + \text{Cov}(p_b, p_w) \right) - 2s_b s_w - 2s_w s_b + 2s_b s_w \quad (21)$$

$$B = 2 \left(\mathbb{E}[p_b | s_w, s_b] \mathbb{E}[p_w | s_w, s_b] + \text{Cov}(p_b, p_w) \right) - 2s_b s_w \quad (22)$$

$$B = 2 \left(s_b s_w + \underbrace{\text{Cov}(p_b, p_w)}_{p_w \text{ and } p_b \text{ are CI} \Rightarrow \text{Cov}(p_w, p_b) = 0} \right) - 2s_b s_w = 0 \quad (23)$$

$$C = \mathbb{E}[p_b^2 | s_w, s_b] - 2s_b \mathbb{E}[p_b | s_w, s_b] + s_b^2 \quad (24)$$

$$C = \text{Var}(p_b) + \mathbb{E}[p_b | s_w, s_b]^2 - 2s_b^2 + s_b^2 = \text{Var}(p_b) = \beta^2 \quad (25)$$

From Equation 17:

$$\frac{1}{2\beta^2} (A + B + C) = \frac{\beta^2 + 0 + \beta^2}{2\beta^2} = 1 \quad (26)$$

2.b.4 $\mathbb{E}[\psi^2 | s_b, s_w]$

$$\mathbb{E}[\psi^2 | s_w, s_b] = \frac{1}{2\beta^2} \mathbb{E}[(p_w - s_w) - (p_b - s_b)]^2 | s_w, s_b] \quad (27)$$

$$\frac{1}{2\beta^2} \left(\underbrace{\mathbb{E}[(p_w - s_w)^2 | s_w, s_b]}_D - 2 \underbrace{\mathbb{E}[(p_w - s_w)(p_b - s_b) | s_w, s_b]}_E + \underbrace{\mathbb{E}[(p_b - s_b)^2 | s_w, s_b]}_F \right) \quad (28)$$

$$D = \mathbb{E}[p_w^2 | s_w, s_b] - 2s_w \mathbb{E}[p_w | s_w, s_b] + s_w^2 \quad (29)$$

$$D = \text{Var}(p_w) + \mathbb{E}[p_w | s_w, s_b]^2 - 2s_w^2 + s_w^2 = \text{Var}(p_w) = \beta^2 \quad (30)$$

$$E = 2\mathbb{E}[p_b p_w | s_w, s_b] - 2s_b \mathbb{E}[p_w | s_w, s_b] - 2s_w \mathbb{E}[p_b | s_w, s_b] + 2s_b s_w \quad (31)$$

$$E = 2 \left(\mathbb{E}[p_b | s_w, s_b] \mathbb{E}[p_w | s_w, s_b] + \text{Cov}(p_b, p_w) \right) - 2s_b s_w - 2s_w s_b + 2s_b s_w \quad (32)$$

$$E = 2 \left(\mathbb{E}[p_b | s_w, s_b] \mathbb{E}[p_w | s_w, s_b] + \text{Cov}(p_b, p_w) \right) - 2s_b s_w \quad (33)$$

$$E = 2 \left(s_b s_w + \underbrace{\text{Cov}(p_b, p_w)}_{p_w \text{ and } p_b \text{ are CI} \Rightarrow \text{Cov}(p_w, p_b) = 0} \right) - 2s_b s_w = 0 \quad (34)$$

$$F = \mathbb{E}[p_b^2 | s_w, s_b] - 2s_b \mathbb{E}[p_b | s_w, s_b] + s_b^2 \quad (35)$$

$$F = \text{Var}(p_b) + \mathbb{E}[p_b | s_w, s_b]^2 - 2s_b^2 + s_b^2 = \text{Var}(p_b) = \beta^2 \quad (36)$$

From Equation 28:

$$\frac{1}{2\beta^2} (D - E + F) = \frac{\beta^2 - 0 + \beta^2}{2\beta^2} = 1 \quad (37)$$

2.b.5 $\mathbb{E}[\psi\theta \mid s_b, s_w]$

$$\mathbb{E}[\psi\theta \mid s_w, s_b] = \frac{1}{2\beta^2} \mathbb{E}[(p_w - s_w)^2 - (p_w - s_w)(p_b - s_b) + (p_w - s_w)(p_b - s_b) - (p_b - s_b)^2 \mid s_w, s_b] \quad (38)$$

$$\mathbb{E}[\psi\theta \mid s_w, s_b] = \frac{1}{2\beta^2} \mathbb{E}[(p_w - s_w)^2 - (p_w - s_w)(p_b - s_b) + (p_w - s_w)(p_b - s_b) - (p_b - s_b)^2 \mid s_w, s_b] \quad (39)$$

$$\mathbb{E}[\psi\theta \mid s_w, s_b] = \frac{1}{2\beta^2} \left(\underbrace{\mathbb{E}[(p_w - s_w)^2 \mid s_w, s_b]}_G - \underbrace{\mathbb{E}[(p_b - s_b)^2 \mid s_w, s_b]}_H \right) \quad (40)$$

$$G = \mathbb{E}[p_w^2 \mid s_w, s_b] - 2s_w \mathbb{E}[p_w \mid s_w, s_b] + s_w^2 \quad (41)$$

$$G = \text{Var}(p_w) + \mathbb{E}[p_w \mid s_w, s_b]^2 - 2s_w^2 + s_w^2 = \text{Var}(p_w) = \beta^2 \quad (42)$$

$$H = \mathbb{E}[p_b^2 \mid s_w, s_b] - 2s_b \mathbb{E}[p_b \mid s_w, s_b] + s_b^2 \quad (43)$$

$$H = \text{Var}(p_b) + \mathbb{E}[p_b \mid s_w, s_b]^2 - 2s_b^2 + s_b^2 = \text{Var}(p_b) = \beta^2 \quad (44)$$

From Equation 40:

$$G - H = \frac{\beta^2 - \beta^2}{2\beta^2} = 0 \quad (45)$$

2.b.6 $\mathbb{P}[\theta, \psi \mid s_b, s_w]$

$\mathbb{P}(\theta \mid s_b, s_w)$ is a distribution, however, as it is composed of p_w, p_b, s_w, s_b , that are all Gaussian distributions, $\mathbb{P}(\theta \mid s_b, s_w)$ is also a Gaussian distribution. Its mean($\mathbb{E}[\theta \mid s_b, s_w]$) is 0 and its variance is 1 ($\mathbb{E}[\theta^2 \mid s_b, s_w]$), therefore, $\mathbb{P}(\theta \mid s_b, s_w)$ is a standard normal Gaussian distribution.

$\mathbb{P}(\psi \mid s_b, s_w)$ is a distribution, however, as it is composed of p_w, p_b, s_w, s_b , that are all Gaussian distributions, $\mathbb{P}(\psi \mid s_b, s_w)$ is also a Gaussian distribution. Its mean($\mathbb{E}[\psi \mid s_b, s_w]$) is 0 and its variance is 1 ($\mathbb{E}[\psi^2 \mid s_b, s_w]$), therefore, $\mathbb{P}(\psi \mid s_b, s_w)$ is a standard normal Gaussian distribution.

Based on the 2 aforementioned statements, the joint distribution $\mathbb{P}(\theta, \psi \mid s_b, s_w)$ is a joint standard multivariate Gaussian distribution, expressed as follows:

$$\mathbb{P}(\theta, \psi \mid s_b, s_w) = \frac{1}{\sqrt{(2\pi)^2 \mid \Sigma \mid}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right) \quad (46)$$

With the following x, μ and Σ :

$$x = \begin{bmatrix} \theta \\ \psi \end{bmatrix} \quad (47)$$

$$\mu = \begin{bmatrix} \mathbb{E}[\theta \mid s_b, s_w] \\ \mathbb{E}[\psi \mid s_b, s_w] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (48)$$

$$\Sigma = \begin{bmatrix} \text{Var}(\theta) & \text{Cov}(\theta, \psi) \\ \text{Cov}(\theta, \psi) & \text{Var}(\psi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (49)$$

The $\text{Cov}[\theta, \psi \mid s_b, s_w] = 0$ as $\mathbb{E}[\psi\theta \mid s_b, s_w] = 0$, $\mathbb{E}[\psi \mid s_b, s_w] = 0$ and $\mathbb{E}[\theta \mid s_b, s_w] = 0$, meaning that θ and ψ are independent.

Leading to the following form:

$$\mathbb{P}(\theta, \psi \mid s_b, s_w) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(\theta^2 + \psi^2)\right) \quad (50)$$

2.c

For $r = \mathbb{1}[p_b > p_w]$

We can express p_b and p_w in terms of θ and ψ .

$$p_b = -\sqrt{2}\beta\psi + p_w - s_w + s_b \quad (51)$$

$$p_w = \sqrt{2}\beta\theta + s_w - p_b + s_b \quad (52)$$

Substituting Equation 51 in Equation 52 results in:

$$p_w = \sqrt{2}\beta\theta + s_w + \sqrt{2}\beta\psi - p_w + s_w - s_b + s_b \quad (53)$$

$$p_w = \frac{\sqrt{2}\beta(\theta + \psi) + 2s_w}{2} \quad (54)$$

Substituting Equation 52 in Equation 51 results in:

$$p_b = -\sqrt{2}\beta\psi + \sqrt{2}\beta\theta + s_w - p_b + s_b - s_w + s_b \quad (55)$$

$$p_b = \frac{\sqrt{2}\beta(\theta - \psi) + 2s_b}{2} \quad (56)$$

Substituting Equation 54 and 56 in $r = \mathbb{1}[p_b > p_w]$:

$$\frac{\sqrt{2}\beta(\theta - \psi) + 2s_b}{2} > \frac{\sqrt{2}\beta(\theta + \psi) + 2s_w}{2} \quad (57)$$

$$\sqrt{2}\beta(\theta - \psi) + 2s_b > \sqrt{2}\beta(\theta + \psi) + 2s_w \quad (58)$$

$$-\sqrt{2}\beta\psi + 2s_b > \sqrt{2}\beta\theta + 2s_w \quad (59)$$

$$2s_b + 2s_w > 2\sqrt{2}\beta\psi \quad (60)$$

$$\mathbb{P}(r = 1 \mid \theta, \psi, s_b, s_w) = \mathbb{1}\left[\frac{s_b + s_w}{\sqrt{2}\beta} > \psi\right] \quad (61)$$

2.d

$$\mathbb{P}(r = 1|s_b, s_w) = \frac{\mathbb{P}(r = 1, s_b, s_w)}{\mathbb{P}(s_b, s_w)} \quad (62)$$

$$\mathbb{P}(r = 1|s_b, s_w) = \frac{\int_{\theta, \psi} \mathbb{P}(r = 1, \theta, \psi, s_b, s_w)}{\mathbb{P}(s_b, s_w)} \quad (63)$$

$$\mathbb{P}(r = 1|s_b, s_w) = \frac{\int_{\theta, \psi} \mathbb{P}(r = 1|\theta, \psi, s_b, s_w) \mathbb{P}(\theta, \psi, s_b, s_w)}{\mathbb{P}(s_b, s_w)} \quad (64)$$

$$\mathbb{P}(r = 1|s_b, s_w) = \frac{\int_{\theta, \psi} \mathbb{P}(r = 1|\theta, \psi, s_b, s_w) \mathbb{P}(\theta, \psi|s_b, s_w) \mathbb{P}(s_b, s_w)}{\mathbb{P}(s_b, s_w)} \quad (65)$$

Substituting Equation 61 and 50 and using x , μ and Σ from Equation 47, 48 and 49:

$$\mathbb{P}(r = 1|s_b, s_w) = \int_{\theta, \psi} \mathbb{1} \left[\frac{s_b + s_w}{\sqrt{2}\beta} > \psi \right] \mathcal{N}(x|\mu, \Sigma) = \Phi \left[\frac{s_b - s_w}{\sqrt{2}\beta} \right] \quad (66)$$

2.e

3