Imperial College London
BSc/MSci EXAMINATION June 2016

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL ANALYSIS

For 1st-Year Physics Students

Friday 10^{th} June 2016: 14:00 to 16:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Consider a map $f : A \to B$ for sets A, B. Give a precise definition for this map to be *injective*, to be *surjective* and to be *bijective*.

[6 marks]

(ii) Give the definition of the *cardinality* $|A|$ of a finite set A.

Define a *countable set*.

[3 marks]

(iii) Given two sets A and B, both possibly infinite, define the cardinality relations;

- $|A| = |B|$
- $|A| \leq |B|$

In the case the two sets are finite prove the above relations are consistent with your definition of cardinality from part ii).

[6 marks]

(iv) Let A be the set of real numbers in the interval $(0, 1)$ whose infinite decimal form consists of only ones and twos (except the leading zero before the decimal point). So using infinite decimal notation;

$$A = \left\{a \in \mathbb{R} | a = 0.n_1 n_2 n_3 \ldots \ , \quad n_i \in \{1, 2\} \quad \forall \quad i \in \mathbb{N}^+\right\} \ .$$

Consider $2^{\mathbb{N}^+}$, the power set of $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. Show that $|A| \leq \left|2^{\mathbb{N}^+}\right|$.

Hint: carefully construct an appropriate map using infinite decimals.

[5 marks]

[Total 20 marks]

2. (i) Given a non-empty set $A \subseteq \mathbb{R}$ state the definition of an upper bound of A, a maximum of A, and the supremum of A.

[6 marks]

(ii) Suppose $A \subset \mathbb{R}$ is non-empty and has an upper bound.

What does the *completeness theorem* imply about A?

Prove this using infinite decimals.

[8 marks]

(iii) Use completeness of $\mathbb{R}$ to prove the following proposition;

Proposition: For any two reals $a, b \in \mathbb{R}$ so that $0 < a < b$, there exists a positive number $n \in \mathbb{N}^+$ satisfying $b \leq na$.

Hint: try a contradiction argument.

[6 marks]

[Total 20 marks]

3. (i) Give an $\epsilon - N$ definition of the statement that a sequence (x_n) of real numbers converges to a limit x.

[5 marks]

(ii) Suppose (u_n) and (v_n) are convergent sequences with $u_n \to u$ and $v_n \to v$ as $n \to \infty$. Consider the sequence (z_n), where $z_n = u_n - v_n$ for all $n \in \mathbb{N}^+$. Prove using $\epsilon - N$ that this is convergent with $z_n \to u - v$ as $n \to \infty$.

[5 marks]

(iii) Use completeness of $\mathbb{R}$ to give an $\epsilon - N$ proof of the following proposition;

Proposition: A decreasing sequence that is bounded from below is convergent.

[5 marks]

(iv) Consider the sequence (x_n) with $x_1 = a$, $x_2 = b$ for some $a, b \in \mathbb{R}$ and,

$$\frac{1}{x_n} = \frac{1}{x_{n-1}} + \frac{1}{x_{n-2}} , \quad n > 2 .$$

In the case $0 < b < a$, prove the sequence converges.

Hint: use the proposition in part iii).

[5 marks]

[Total 20 marks]

4. (i) For a function $f : I \to \mathbb{R}$, where I is an open interval, give $\epsilon - \delta$ definitions of;

- The limit of f at a point in I.
- Continuity of f at a point in I.
- Continuity of f over the whole interval I.

[5 marks]

(ii) Consider two continuous functions $u : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ on an open interval I. Prove using $\epsilon - \delta$ that the function $y : I \to \mathbb{R}$, with $y(x) = u(x) + v(x)$, is also continuous.

[5 marks]

(iii) Prove using $\epsilon - \delta$ that the constant function $g : \mathbb{R} \to \mathbb{R}$, with $g(x) = a$ for some $a \in \mathbb{R}$, is continuous.

Likewise prove that $h : \mathbb{R} \to \mathbb{R}$, with $h(x) = x$, is continuous.

[5 marks]

(iv) Give an $\epsilon - \delta$ proof that the product of two continuous functions over $\mathbb{R}$ is continuous.

Prove by induction that $f : \mathbb{R} \to \mathbb{R}$ with,

$$f(x) = \sum_{n=0}^{N} a_n x^n ,$$

for $N \in \mathbb{N}$ and coefficients $a_n \in \mathbb{R}$, is continuous.

[5 marks]

[Total 20 marks]

5. (i) Consider a series $\sum_{k=1}^{\infty} a_k$. State the definition of convergence for the series. If it does converge, what is its value?

[5 marks]

(ii) Consider $x \in \mathbb{R}$ with $x \neq 1$. Prove using induction that,

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \, , \quad n \in \mathbb{N} \, .$$

[5 marks]

(iii) Prove for which $y \in \mathbb{R}$ the *geometric series* $\sum_{k=0}^{\infty} y^k$ converges, and determine its value.

[You may state without proof how y^n behaves as $n \to \infty$.]

[5 marks]

(iv) Prove that the function $f(x) = \frac{1}{1-x}$ is *analytic* at $x = 0$.

[5 marks]

[Total 20 marks]