Imperial College London
BSc/MSci EXAMINATION June 2012

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL ANALYSIS

For 1st Year Physics Students

Friday, 15th June 2012: 14:00 to 16:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the 3 answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Consider the sets $A = \{1,2,3\}$, $B = \{1,2\}$, $C = \{1,3\}$, $D = \{2,3\}$, $E = \{1\}$, $F = \{2\}$, $G = \{3\}$, $H = \emptyset$. Simplify the following expressions. In each case the answer should be one of the sets $A, B \cdots H$.

(a) $A \cap B$

(b) $A \cup B$

(c) $A \cap (B \cap C)$

(d) $(C \cup A) \cap B$

(e) $A \backslash B$

(f) $C \backslash A$

(g) $(D \backslash F) \cup (F \backslash D)$

(h) $G \backslash A$

(i) $A \cup ((B \backslash C) \backslash F)$

(j) $((B \cup C) \cap C) \cup H$

[10 marks]

(ii) Let $A_n = \{a_1, a_2, \cdots a_n\}$ be a finite set with $|A_n| = n$. Write down the complete list of subsets of A_n for the cases $n = 1,2,3$.

[3 marks]

(iii) Show that the total number of possible subsets of A_n, defined in part (ii), is 2^n. (Use any method of proof). [4 marks]

(iv) Prove that $(A \cup B)' = (A' \cap B')$ algebraically. Draw the corresponding Venn diagram.

[3 marks]

[Total 20 marks]

2. (i) Let p be a positive integer. Show that if p^2 is divisible by 3, then p is divisible by 3. [3 marks]

(ii) Use the result of part (i) to show that $\sqrt{3}$ is irrational. [3 marks]

(iii) Use the fundamental theorem of arithmetic to show that $\sqrt{5}$ is irrational. [3 marks]

(iv) Show that every real number that may be expressed as a periodic decimal is rational. [4 marks]

(v) Find the rational expression, in lowest terms, for the periodic decimal

$$x = 0.36363636....$$

[2 marks]

(vi) For each of the following claims, state whether it is true or false. If it is true, prove it and if it is false find a counterexample.

(a) If a and $b \neq 0$ are both rational then a/b is rational.

(b) If a and b are both irrational then a/b is irrational.

(c) If a and b are both rational then $a+b$ is rational.

(d) If a is irrational and b is rational then $a+b$ is irrational.

(e) If a and b are both irrational then $a+b$ is irrational.

[5 marks]

[Total 20 marks]

3. (i) Consider the map $f : D \to T$ for sets D, T. Explain what it means for the map f to be surjective, injective or bijective. Draw diagrams illustrating each type of map. [3 marks]

(ii) Explain what it means for an infinite set to be countable. What does it mean to be uncountable? [3 marks]

(iii) Show that if sets A and B are countable, then so is $A \times B$. [3 marks]

(iv) Show that the union of two countable sets is countable. Hence show that if C is uncountable and D is countable then $C \setminus D$ is uncountable. [3 marks]

(v) State without proof whether each of the following sets is countable or uncountable:

(a) the integers, $\mathbb{Z}$;

(b) the rational numbers, $\mathbb{Q}$;

(c) the real numbers, $\mathbb{R}$;

(d) the irrationals, $\mathbb{R} \setminus \mathbb{Q}$;

(e) $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers;

(f) the power set of $\mathbb{N}$, $\mathbb{P}(\mathbb{N})$;

(g) the set of all real numbers with only the digits 3 and 8 in their decimal expansion;

(h) the set of points $\{x | x \in \mathbb{R}, x^2 \in \mathbb{N}\}$.

[8 marks]

[Total 20 marks]

4. (i) Give the precise definition of the statement that the sequence of real numbers $\{a_n\}$ converges to a number a. Define the terms *monotonically decreasing* sequence and *bounded* sequence. [3 marks]

(ii) Show that if a sequence of real numbers $\{a_n\}$ is monotonically decreasing and bounded from below, then it converges. How might this result be changed if the sequence consisted of rational numbers? [7 marks]

(iii) Consider the sequence defined by

$$a_k = \max\{\sin 1, \sin 2 \cdots \sin k\}$$

where max denotes the largest element of the set. Use a calculator to compute the first eight terms in the sequence. Show that this sequence converges. What does it converge to? [4 marks]

(iv) State the definition of a Cauchy sequence. Under what conditions is a Cauchy sequence convergent? [2 marks]

(v) Consider the sequence a_n defined to be the approximation to n decimal places of $\sqrt{2}$, hence the sequence has the form

$$\begin{aligned} a_1 &= 1.4 \\ a_2 &= 1.41 \\ a_3 &= 1.414 \\ &\cdots \\ a_n &= 1.c_1 c_2 \cdots c_n 000 \cdots \end{aligned}$$

where the c_k take integer values from 0 to 9. Show that this sequence is a Cauchy sequence. [4 marks]

[Total 20 marks]

5. (i) Show that the infinite series $\sum_{n=1}^{\infty} 1/n$ is divergent. [2 marks]

(ii) Show that the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent. [2 marks]

(iii) Consider the infinite series $\sum_{n=1}^{\infty} a_n$ and define the quantity

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Prove that the series converges if $\rho < 1$. What can you say about the series in the cases $\rho > 1$ and $\rho = 1$? [8 marks]

(iv) Consider the Taylor expansion

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Use the ratio test to show that the series converges absolutely for $|x| < 1$. What happens when $x = -1$? [3 marks]

(v) Consider the series for $\ln 2$ obtained by setting $x = +1$ in part (iv):

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Give a simple argument, using the result of part (ii), to show that this series converges. [2 marks]

(vi) Show that the terms in the series for $\ln 2$ in part (v) may be reordered to give the result $\frac{1}{2}\ln 2$. Which general result about infinite series is this an example of?

[3 marks]

[Total 20 marks]

6. (i) State the definition of a continuous function $f : \mathbb{R} \to \mathbb{R}$.

[2 marks]

(ii) State the relationship between continuity of a function f at a point and the convergence of sequences a_n to that point.

[2 marks]

(iii) Use the results of part (ii) to show that the function

$$\begin{aligned} \theta(x) &= 1 \text{ for } x \geq 0 \\ \theta(x) &= 0 \text{ for } x < 0 \end{aligned}$$

is not continuous at $x = 0$.

[4 marks]

(iv) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the relation

$$f(x+y) = f(x) + f(y) \tag{1}$$

Assuming that f is differentiable, show that its most general form is

$$f(x) = Kx$$

where K is a constant.

[4 marks]

(v) Now deduce the most general form of the function f defined in Eq.(1) assuming only that it is continuous (but not necessarily differentiable). [Hint: Use induction to prove that $f(nx) = nf(x)$, explore different choices of x, and use the results of part (ii).]

[8 marks]

[Total 20 marks]