Imperial College London

BSc/MSci EXAMINATION June 2013

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL ANALYSIS

For 1st-Year Physics Students

Friday, 14th June 2013: 14:00 to 16:00

Answer THREE questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Let A and B be sets. State the definitions of their union $A \cup B$, intersection $A \cap B$, and set difference $A \setminus B$. [3 marks]

(ii) Consider the sets $A = \emptyset$ (the empty set), $B = \{\emptyset\}$, $C = \{\,\emptyset, \{\emptyset\}\,\}$, $D = \{\,\emptyset, \{\emptyset\}, \{5, 6\}\,\}$, $E = \{4, 5, 6\}$, $F = \{\,\{4\}, 5, 6\,\}$, $G = \{5, 6\}$, $H = \{5, 6, 7\}$, $I = \{4\}$, $J = \{\,\{4\}\,\}$.

What is the cardinality of J? [1 mark]

Are the following statements true or false?

(a) $I \subset F$,

(b) $D \setminus G = C$,

(c) $A \cup B = C$.

[3 marks]

(iii) Simplify the following expressions involving the sets $A, B \cdots J$ from part (ii). In each case the answer should be one of the sets $A, B \cdots J$.

(a) $E \cap I$,

(b) $E \setminus H$,

(c) $A \cup G$,

(d) $(E \setminus G) \setminus I$,

(e) $(D \cap C) \setminus A$,

(f) $(E \cap G) \cup (I \setminus H)$,

(g) $F \setminus J$,

(h) $I \cap J$,

(i) $(G \setminus H) \cup (E \setminus I)$,

(j) $B \cup C$.

[5 marks]

(iv) Use Venn diagrams to prove

$$(X \cap Y) \cup (X \setminus Y) \cup (Y \setminus X) = X \cup Y$$

for any sets X and Y. [4 marks]

(v) No android that wears a hat is paranoid.

No android without a jetpack will say hello.

No nonparanoid android has rusty wheels.

No android without a hat has a jetpack.

Therefore, No android with rusty wheels will say hello.

Is this deduction logically correct? Explain your answer using Venn diagrams, or any other method. [4 marks]

[Total 20 marks]

2. (i) Consider the map $f : A \to B$ for sets A, B. What are the domain and codomain of f? Define the image of f. [2 marks]

Define what it means for f to be injective, surjective and bijective. [3 marks]

Give an example of a map that is neither injective nor surjective [1 mark]

(ii) Consider map $f : A \to B$ for finite sets A, B. Are the following statements true or false? If the statement is false, give a counterexample. If the statement is true, prove it.

(a) If $|A| \leq |B|$ then f is injective

(b) If f is injective then $|A| \leq |B|$.

(c) If f is surjective then $|A| > |B|$.

(d) If $|A| > |B|$ then f is not injective.

[Note: $|X|$ denotes the cardinality of a set X.]

[9 marks]

(iii) There are N people at a party where $N > 2$. Is it possible that no two people have exactly the same number of friends at the party? Explain your answer. Hint: consider two sets, the set of the people at the party and the set of numbers of friends at the party a person can have; then consider an appropriate map between them. [5 marks]

[Total 20 marks]

3. (i) Let A and B be sets. What does it mean to say that the cardinality of A is equal to the cardinality of B?

State the definition of a countable set.

[2 marks]

(ii) State whether each of the following sets is countable or uncountable and prove your answer

(a) the set of all even natural numbers; [2 marks]

(b) $\mathbb{Z}$, the integers; [2 marks]

(c) $\mathbb{Q}$, the rational numbers; [3 marks]

(d) $\mathbb{R}$, the real numbers; [5 marks]

(e) the set of irrational numbers; [2 marks]

(iii) Let X be the set of all possible sentences containing finitely many words (in any language) that can be typed on a standard keyboard. Is X countable or uncountable? Prove your answer [4 marks]

[Total 20 marks]

4. (i) Use mathematical induction to prove the following statements:

(a)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\ , \quad \forall n \in \mathbb{N}$$

[5 marks]

(b) $n^5 - n$ is a multiple of 5, $\forall n \in \mathbb{N}$. [5 marks]

(c)

$$x^n - 1 = (x-1)\left(\sum_{k=0}^{n-1} x^k\right)\ , \quad \forall n \in \mathbb{N}\,.$$

[5 marks]

(ii) Prove that if $2^n - 1$ is prime then n is prime. Hint: use part (c) above.

[5 marks]

[Total 20 marks]

5. (i) Give the precise definition of the statement that the sequence of real numbers $\{x_n\}$ is convergent to a limit x. [2 marks]

(ii) (a) Let $x_n = n^{-2}$. Prove that $\{x_n\}$ converges [4 marks]

(b) Let $y_n = b^n$ where b is a real number. What is the condition on b that guarantees that $\{y_n\}$ is convergent? Prove that when b satisfies the condition, $\{y_n\}$ converges. [4 marks]

(c) Let $\{u_n\}$ and $\{v_n\}$ be convergent sequences. Prove that $z_n = u_n v_n$ is a convergent sequence [5 marks]

(iii) Explain what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point $a \in \mathbb{R}$. Give the rigorous definition and also explain in words what it means. [3 marks]

(iv) Give an example of a continuous function and an example of a discontinuous function. For each of them, give a physical system that it might be appropriate to use the function to describe. [2 marks]

[Total 20 marks]

6. (i) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. [4 marks]

(ii) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent. You may use the fact that a geometric series $\sum_{k=1}^{\infty} \beta^k$ converges for $|\beta| < 1$ and the comparison test. [4 marks]

(iii) Consider the infinite series $\sum_{k=1}^{\infty} a_k$. We define the quantity

$$\rho = \lim_{k\to\infty} |a_k|^{\frac{1}{n}} \ .$$

Prove that the series converges absolutely if $\rho < 1$. You may use the fact that a geometric series $\sum_{k=1}^{\infty} \beta^k$ converges for $|\beta| < 1$ and the comparison test.

What can you say about the series in the cases $\rho > 1$ and $\rho = 1$? [8 marks]

(iv) Achilles and the tortoise have a race over 100 metres. The tortoise crawls at a constant speed and Achilles runs five times as fast as the tortoise crawls. Achilles gives the tortoise a 50 metre head start. At the time Achilles reaches the 50 metre mark, the tortoise has reached a new position, x_1, ahead of Achilles. At the time Achilles reaches x_1, the tortoise is ahead again at position x_2, *etc.* So by the time Achilles reaches position x_k the tortoise has moved to x_{k+1}. Find a formula for x_k and hence find the position on the track at which Achilles overtakes the tortoise [4 marks]

[Total 20 marks]