# Imperial College London

# BSc/MSci EXAMINATION June 2014

*This paper is also taken for the relevant Examination for the Associateship*

# ELECTRICITY & MAGNETISM AND RELATIVITY

**For First-Year Physics Students**

Wednesday 11 June 2014, 14:00 – 16:00

*Answer all questions.*
*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

**General Instructions**

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

**You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.**

## SECTION Electricity and magnetism

**1.** (i) An electric dipole (dipole 1) is formed by two charges $+Q_1$ and $-Q_1$ situated on the $x$-axis at $x = -a/2$ and $x = +a/2$ respectively. A second dipole (dipole 2) is formed by two further charges of magnitude $Q_2$, also on the $x$-axis at $x = X \pm b/2$. The dipoles are widely separated (so that $X \gg a, b$).

(a) If the second dipole is in its *stable* (lower energy) alignment with respect to the first, draw a diagram showing the directions of the two dipole moment vectors. [2 marks]

(b) Consider the first pair of charges above ($+Q_1$ and $-Q_1$). Show that for $x \gg a$ and $y = z = 0$ the electric field is given by

$$\mathbf{E} \cong \frac{\mathbf{p}_1}{2\pi\varepsilon_0 x^3}$$

where $\mathbf{p}_1 = -Q_1 a\hat{\mathbf{i}}$. [4 marks]

(c) Determine the approximate potential energy difference between the opposite orientations of dipole 2 in the field of dipole 1. [4 marks]

(d) Find the electric fields at $x = X \pm b/2$ (i.e. the fields experienced by the charges in the second dipole) [3 marks]

(ii) An isolated thick, perfectly conducting, initially uncharged spherical shell has inner and outer radii of $a$ and $b$ ($a < b$). A positive point charge of magnitude $Q$ is placed at the centre of the shell.

(a) Determine the electric field at a distance $r$ from the centre of the spherical shell for $r < a$, $a \leq r \leq b$ and $r > b$. [4 marks]

(b) The outside of the conducting shell is now connected to ground. Determine the total charge on the spherical shell and justify your answer. [2 marks]

(iii) A long cylindrical capacitor is formed from two concentric conducting cylindrical shells of radii $a$ and $b > a$. A total charge of $+Q$ is placed on the inner cylinder, with a total charge $-Q$ on the outer cylinder.

(a) Ignoring the edge effects determine the electric field for $r < a$, $a \leq r \leq b$, and $r > b$ where $r$ is the radial distance from the centre. [3 marks]

(b) Calculate the potential for $a \leq r \leq b$ and hence show that this configuration has a capacitance *per unit length* given by

$$C = \frac{2\pi\epsilon_o}{\ln(b/a)}$$

[3 marks]

[Total 25 marks]

**2.** (i) The Law of Biot and Savart gives the differential contribution $\boldsymbol{dB}$ to the magnetic field at a point $P$ from a current element $I\boldsymbol{dl}$ as

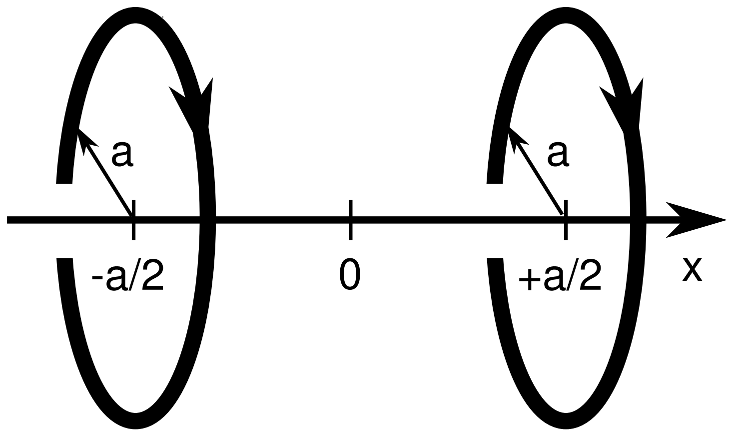
$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\boldsymbol{dl}\times\hat{\boldsymbol{r}}}{r^2}$$

where $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ is the position of $P$ relative to the current element. Use this law to show that the magnitude of the magnetic field on the axis of a circular loop of radius $a$ at a distance $h$ from the centre is

$$B = \frac{\mu_o I a^2}{2\left(a^2 + h^2\right)^{3/2}}$$

where $I$ is the current flowing in the loop. [7 marks]

(ii) The centres of a pair of Helmholtz coils, each of radius $a$ and carrying a current $I$ (in the same direction), lie on the $x$-axis at $x = \pm a/2$ as shown in the following diagram:




Determine the magnitude of the magnetic field at a general point on the axis. [4 marks]

(iii) An infinite straight wire of circular cross-section (radius $a = 0.02$ m) carries a total current $I = 2$ A. The current density within the wire is uniform.

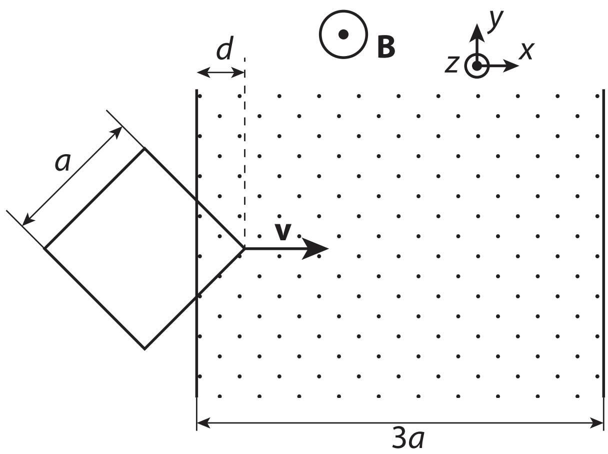
(a) Show that the magnitude of the magnetic field inside the wire varies with radial distance $r$ as $B = 1\times10^{-3}\,r$ Tesla and indicate on a sketch the direction of the magnetic field vector. [5 marks]

(b) Find the magnitude of the magnetic field outside the wire. [4 marks]

[Total 20 marks]

**3.** (i) State Faraday's induction law and discuss how Lenz's law is also expressed by it. [3 marks]

(ii) Consider the arrangement shown in the figure below.




A perfectly conducting wire loop forming a square, whose side is oriented at 45° with respect to the $\boldsymbol{i}$ direction, is being dragged into a homogeneous and static magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{k}}$ with a velocity $\boldsymbol{v} = v\hat{\boldsymbol{i}}$. The spatial extent of the magnetic field in the $x$ direction is $3a$ while it is infinite in the y direction. Calculate the electromotive force (emf) induced in the wire loop as a function of the wire position $d$ for $0 \leq d \leq a/\sqrt{2}$. From this result deduce or calculate the emf for $a/\sqrt{2} \leq d \leq 5a$. Plot the emf for $0 \leq d \leq 5a$ [8 marks]

(iii) For an electrostatic field

$$emf = \oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0.$$

Why is the value of this integral non-zero in the present case? [2 marks]

(iv) In moving the wire frame, work is being done to overcome the Lorentz force due to the current induced in the wire. Given that the resistance of the wire loop is $R$, find and plot the Lorentz force as a function of the wire position $d$ for $0 \leq d \leq 5a$. [12 marks]

[Total 25 marks]

## SECTION Relativity

**4.** A spaceship travels with $\beta = 0.7$ from Earth to exoplanet X at a distance 8 lightyears away. The velocity of X with respect to Earth is negligible.

(i) (a) What is the distance (in lightyears) between Earth and X as measured by the spaceship?

(b) Owing to time dilation, an observer on Earth would see a clock on the spaceship run more slowly than his own. To explain this, draw a light clock onboard the spaceship in both Earth's and the spaceship's inertial frames, and show that a time interval $\Delta t = \gamma \Delta t'$, where the prime denotes the spaceship's frame. What is the journey time of the spaceship in Earth's frame of reference, and in the spaceship's frame of reference?

[10 marks]

(ii) (a) Draw a space-time diagram in the Earth's frame of reference of the spaceship travelling from Earth to X, and annotate the world lines of Earth, exoplanet X, and the spaceship. Clearly label your axes, including any relevant numbers.

(b) Unbeknownst to the inhabitants of planet Earth, an intelligent population on exoplanet X have been observing Earth. As soon as they see the spaceship leaving Earth, the people on X send a light signal to the space ship to establish communication. Subsequently, there is a continuous exchange of light-encoded messages between the spaceship and X. Assume no delay between the reception of one message, and transmission of the next. Sketch the paths of these light signals in your spacetime diagram and calculate when the spaceship's first response arrives at planet X.

[10 marks]

(iii) The spaceship sends its messages using single-wavelength light of $\lambda = 632.8$ nm. Owing to the velocity of the spaceship relative to exoplanet X, the light received on X is Doppler-shifted.

(a) Explain which two physical effects contribute to the relativistic Doppler effect.

(b) Using a sketch of a moving light emitter, or otherwise, show that the relativistic Doppler effect is given by:

$$\lambda_R = \lambda \sqrt{\frac{1-\beta}{1+\beta}} \quad ,$$

where $\lambda_R$ is the wavelength of the light as observed on exoplanet X. What is the wavelength of the light as observed by the inhabitants of exoplanet X?

[10 marks]

[Total 30 marks]