Imperial College London

BSc/MSci EXAMINATION June 2015

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM AND RELATIVITY

For 1st-Year Physics Students

Wednesday, 10th June 2015: 14:00 to 16:00

Answer all questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) A point charge of $-2.9\,\mu$C rests at the origin. Find the electric field strength and direction and electrostatic potential due to this point charge at 3.1 m north of the origin. Another point charge of $+1.5\,\mu$C is placed at this northern position. Find the mutual force between the two charges and the potential energy shared between them. [5 marks]

(ii) Explain, with the use of diagrams, what is meant by charging by induction [5 marks]

(iii) An electron moving in outer space has a component of velocity of $0.01c$ in the direction of a magnetic field of magnitude $B = 10^{-9}$ T, and a component perpendicular to the field. Describe with the aid of a diagram the trajectory of the particle. Calculate the number of revolutions the particle makes on traveling between two points one light year apart along the direction of the field. [5 marks]

(iv) A long cable of circular cross section and diameter 2 cm has a long cylindrical hole of diameter 1 mm drilled in it parallel to the cable axis. The distance between the axis of the hole and the cable axis is 5 mm. If the cable has a uniform steady current density of $10^5\ \mathrm{A \cdot m^{-2}}$ flowing through it, calculate the magnetic field at the centre of the cable. (Hint: use the principle of superposition.) [5 marks]

(v) A circular wire loop of radius a carries a current I. Show that the magnitude of the magnetic field at distance z from the centre of the loop and along its axis is given by $B = \mu_0 m / \left[2\pi(a^2 + z^2)^{3/2}\right]$, where m is the magnitude of the loop's magnetic moment. Draw a sketch indicating the direction of current in the loop and the direction of the magnetic field. [5 marks]

[Total 25 marks]

2. Relativity:

(i) (a) Explain what it means in relation to different inertial frames when we say (i) a quantity is conserved, and (ii) a quantity is invariant.

(b) Consider the Lorentz transformation:

$$\begin{aligned} t' &= \gamma(t - vx/c^2), \\ x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where all the symbols have their usual meaning.
Show that $S^2 = (ct)^2 - (x^2 + y^2 + z^2)$ is an invariant quantity.

[8 marks]

(ii) (a) Consider two events with 4-positions (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) respectively. What can you say about
$\Delta S^2 = (ct_1 - ct_2)^2 - \left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right)$ when the two events are space-like separated? What if they are time-like separated? Sketch a space-time diagram and annotate the regions that are (i) space-like and (ii) time-like separated from event O at $t_o = 0$ and $x_o = 0$.

(b) Explain how the order of two events in different inertial frames is affected if the events are (i) space-like and (ii) time-like separated.

[10 marks]

(iii) A particle with rest-mass $140\,\mathrm{MeV}/c^2$ is moving in a lab with $\beta = 0.4$ in the x-direction.

(a) The lab is 200 m long. How long does it take for the particle to cross the lab in the lab frame? How long does it take in the particle's rest frame?

(b) What is the total energy and the kinetic energy of the particle in the lab frame? Calculate the same two quantities for the particle's rest frame as well.

(c) Two of these particles collide head-on and stick together to create a third particle with no resulting velocity in the lab frame. What is the mass of the resulting particle? What is its momentum in the rest frame of the incident particle that was moving in the x direction?

[12 marks]

[Total 30 marks]

3. (i) Write down an expression for the magnitude of the electric field strength of a static point charge Q at a distance r from the charge.

The point charge is enclosed by a closed, non-spherical, surface S. By considering the flux $\mathbf{E}.d\mathbf{S}$ of electric field due to the charge across a small element of the surface, show that Gauss' law holds, i.e., the total outward flux across the surface is given by Q/ϵ_0. [5 marks]

(ii) Consider a positive point charge $+Q_1$ at the centre of a uniformly charged sphere of radius R and of total negative charge $-Q_2$. Use Gauss' law to show that the electric field strength as a function of distance r from the centre of the sphere is given by

$$E(r) = \frac{Q_1}{4\pi\epsilon_0}\frac{1}{r^2} - \frac{Q_2}{4\pi\epsilon_0}\frac{r}{R^3} \qquad \text{for } r \leqslant R\,,$$

$$E(r) = \frac{Q_1 - Q_2}{4\pi\epsilon_0}\frac{1}{r^2} \qquad \text{for } r > R\,,$$

and sketch $E(r)$ vs. r for the case when $|Q_1| = |Q_2|$ [7 marks]

(iii) Assuming that the electric potential has a value of 0 at infinity, find a general expression for the electric potential as a function of r for the case when $|Q_1| = |Q_2|$ and verify that it takes the value $\frac{5}{8}\frac{Q_1}{4\pi\epsilon_0}\frac{1}{R}$ when $r = \frac{1}{2}R$.

A hydrogen atom can be modelled as a positive point charge of value $+e$ surrounded by a uniformly charged sphere of total charge $-e$ and radius 53 pm. Compute the speed a proton would have to be fired at a hydrogen atom to penetrate the electron cloud to half its depth (i.e., 26.5 pm) according to this model. Give two reasons why this model is an approximation and suggest whether the proton would have to be fired faster or slower to reach the depth stated in a real experiment. [8 marks]

[Total 20 marks]

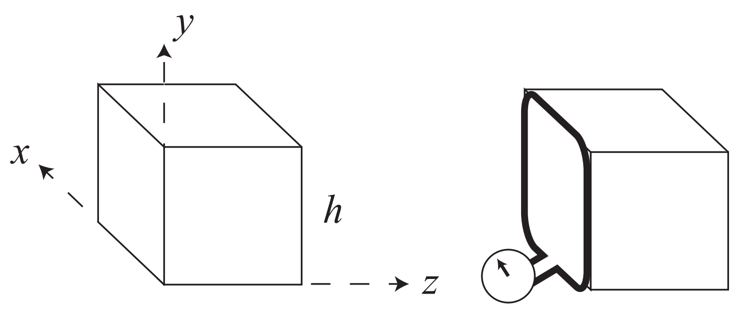
4. (i) Define magnetic flux ϕ_B, in terms of the magnetic field **B** and a surface made up of the vector surface elements $d\mathbf{S}$.

In Lenz's law, written as $\mathcal{E} = -\partial\phi_B/\partial t$, what does the symbol $\mathcal{E}$ represent? Explain the significance of the minus sign. [5 marks]

A cube of side h is cut from a meteorite of unknown composition. Bursts of magnetic field are noted to pass through the sides of the cube. With the block aligned to the axes as shown in Fig. (a), a field burst that peaks at $t = 0$ is given by

$$\mathbf{B}(x, y, z, t) = \left(a + by^2\right) e^{-t^2/\tau^2} (\mathbf{i} + \mathbf{j} + \mathbf{k}) \ ,$$

where a, b and τ are positive constants and the formula is valid for all time, t. The field is detected by placing a square wire frame aligned to a face of the cube (e.g. the x-y face at $z = 0$ in Fig. (b)), and measuring the current flowing through the wire as a function of time.



(a) (b)

(ii) Show that the current through the wire placed on the face parallel to $z = 0$ is given by

$$I_{xy}(z = 0, t) = \frac{2th^2}{R\tau^2} e^{-t^2/\tau^2} \left(a + \frac{bh^2}{3}\right) \ ,$$

where R is the resistance of the wire. [6 marks]

(iii) Sketch $I_{xy}(z = 0)$ as a function of time and indicate the direction of current flow at $t = \tau$ on a sketch of Fig. (b). [5 marks]

(iv) Give expressions for the flux ϕ_B linking the wire loop when it is placed on each of the six cube faces.

Measurements of current are made on a wire loop placed in turn on each of the six faces of the cube. The flux through each face is inferred and the results agree, within experimental error, to those predicted by the expressions calculated above. State, giving reasons, whether the results obtained from such measurements would be sufficiently ground breaking to warrant publication in *Nature*, the world's premier science journal. [9 marks]

[Total 25 marks]