Imperial College London

BSc/MSci EXAMINATION June 2014

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL ANALYSIS

For 1st-Year Physics Students

Friday, 13th June 2013: 14:00 to 16:00

Answer THREE questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) Use proof by contradiction to prove that there are an infinite number of prime numbers. [5 marks]

(ii) Let n be a prime number. Use proof by contradiction to prove that $\sqrt{n}$ is irrational. [5 marks]

(iii) Use mathematical induction to prove that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \qquad \text{for all} \quad n \in \mathbb{N}^{+}\,.$$

[5 marks]

(iv) Use proof by contradiction to prove that the equation $x^5 + x^4 + x^3 + x^2 + 1 = 0$ has no rational solution. Hint: if $\frac{p}{q}$ is a rational number such that p and q have no common factor, then either p and q are both odd or one is odd and the other even.

[5 marks]

[Total 20 marks]

2. (i) Let A and B be sets. What does it mean to say that the cardinality of A is equal to the cardinality of B?

State the definition of a countable set.

[2 marks]

(ii) State whether each of the following sets is countable or uncountable and prove your answer.

(a) $\mathbb{Z}$, the integers [2 marks]

(b) $A \times B$, where A and B are countable sets [4 marks]

(c) $\mathbb{R}$, the real numbers (hint: Cantor's diagonal argument) [5 marks]

(iii) State without proof whether each of the following sets is countable or uncountable.

(a) $\mathbb{Q}$, the set of all rational numbers

(b) the set of all irrational numbers

(c) the set of all subsets of $\mathbb{N}^+$

(d) the set of all solutions of polynomial equations with integer coefficients (the so-called algebraic numbers)

[2 marks]

(iv) An alien civilisation has a language in which the set of all letters, $\mathcal{A}$ is a countable (infinite) set. In this language, any finite string of letters is a possible word. For example if a, b, c are 3 elements of $\mathcal{A}$, then abc, $abab$, $bbbcccc$ are examples of possible words. A word can be any finite length.

Is the set of all words with one letter countable or uncountable? Is the set of all words with two letters countable or uncountable? Is the set of all words with n letters – where $n \in \mathbb{N}^+$ – countable or uncountable? Is the set of all words countable or uncountable?

Prove your answers. [5 marks]

[Total 20 marks]

3. (i) Let A and B be sets. State the definitions of their union $A \cup B$, intersection $A \cap B$, and set difference $A \setminus B$. [3 marks]

(ii) Consider the sets $\emptyset$ (the empty set), $S_1 = \{\emptyset\}$, $S_2 = \{\emptyset, S_1\}$ and $S_3 = \{\emptyset, S_1, S_2\}$.

What is the cardinality of S_1?

State without proof whether each of the following statements is true or false.

(a) $\emptyset \subseteq \emptyset$

(b) $\emptyset \in S_1$

(c) $\emptyset \subseteq S_1$

(d) $S_1 \in S_1 \cap S_2$

(e) $S_1 \subseteq S_1 \cap S_2$

(f) $S_1 \cap S_2 \cap S_3 = \emptyset$

[4 marks]

(iii) Consider the sets $A = \{0, 1, 2\}$, $B = \{0, 1\}$, $C = \{0, 2\}$, $D = \{1, 2\}$, $E = \{0\}$, $F = \{1\}$, $G = \{2\}$, $H = \emptyset$ (the empty set). Simplify the following expressions. In each case the answer should be one of the sets $A, B, \ldots H$.

(a) $A \cap C$

(b) $A \cup D$

(c) $A \cap (C \cap D)$

(d) $(F \cup A) \cap E$

(e) $A \backslash D$

(f) $G \backslash A$

(g) $(B \backslash F) \cup (F \backslash B)$

(h) $(A \backslash B) \cap (A \backslash H)$

(i) $A \cap ((B \backslash C) \backslash F)$

(j) $((D \cup C) \cap C) \cup H$

[5 marks]

(iv) Let X be a set and $P(X)$ is the set of all subsets of X (the power set of X). The binary operations, $\triangle$ and $\star$, are defined on elements of $P(X)$ as follows

- $A \triangle B = (A \setminus B) \cup (B \setminus A)$
- $A \star B = A \cap B$

for all $A, B \in P(X)$.

Prove each of the following statements.

(a) $A \star X = A \quad$ for all $A \in P(X)$

(b) $(A \triangle B) \triangle C = A \triangle (B \triangle C) \quad$ for all $A, B, C \in P(X)$

(c) $A \star (B \triangle C) = (A \star B) \triangle (A \star C) \quad$ for all $A, B, C \in P(X)$

(d) $B \star \Big[X \triangle \big(A \triangle (A \star B) \big) \Big] = B \quad$ for all $A, B \in P(X)$

[8 marks]

[Total 20 marks]

4. (i) Consider the map $f : A \to B$ for sets A, B. What are the domain and target (or codomain) of f? Define the image (or range) of f.

Define what it means for f to be injective, surjective and bijective.

[5 marks]

(ii) $f : A \to B$ is a map from set A to set B. Are the following statements true or false? If the statement is false, give a counterexample. If the statement is true, prove it.

(a) If f is injective and not surjective then $|A| < |B|$.

(b) If $|A| > |B|$ then f is surjective.

(c) If A and B are finite and $|A| > |B|$ then f is not injective.

(d) If f is surjective, there exists an inverse map $g : B \to A$ such that $g(f(x)) = x$ for all $x \in A$.

[Note: $|X|$ denotes the cardinality of a set X.]

[10 marks]

(iii) An octagonal dinner table has one place setting at each of its eight sides. Each place setting is labelled with the name of one of the eight guests. The diners sit down without looking at the name labels and find that no-one is sitting in the correct place. The table can be rotated while the guests remain in the chairs where they first sat down. Prove that the table can be rotated so that at least two guests have their correct name labels in front of them.

You may want to consider the set of all guests, the set of all positions of the table and an appropriate map.

[5 marks]

[Total 20 marks]

5. (i) Give the precise definition of the statement that the sequence of real numbers $\{x_n\}$ is convergent to a limit x. [2 marks]

(ii) (a) Let $x_n = n^{-a}$ where $a > 0$. Prove that $\{x_n\}$ converges. [4 marks]

(b) Let $\{u_n\}$ and $\{v_n\}$ be convergent sequences. Prove that $z_n = u_n + v_n$ is a convergent sequence. [4 marks]

(c) Let $\{y_n\}$ be a monotonic decreasing sequence i.e. $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}^+$. You are given that $\{y_n\}$ is bounded below *i.e.* there exists $z \in \mathbb{R}$ such that $z < y_n$ for all $n \in \mathbb{N}^+$. Using the completeness property of the real numbers, prove that $\{y_n\}$ is a convergent sequence. [5 marks]

(iii) Explain what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point $a \in \mathbb{R}$. Give the rigorous definition and also explain in words what it means.

[3 marks]

(iv) Give an example of each of the following

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at $x = 0$.

(b) A function $g : \mathbb{R} \to \mathbb{R}$ that is continuous at $x = 0$ but not differentiable at $x = 0$.

[2 marks]

[Total 20 marks]