Imperial College London

BSc/MSci EXAMINATION June 2015

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL ANALYSIS

For 1st-Year Physics Students

Friday, 12th June 2015: 14:00 to 16:00

Answer THREE questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) Let A and B be sets. What does it mean to say that the cardinality of A is equal to the cardinality of B?

State the definition of a countable set.

[2 marks]

(ii) State whether each of the following sets is countable or uncountable and prove your answer.

(a) $A \cup B$, where A and B are countable sets [3 marks]

(b) $\mathbb{Q}$, the rationals [5 marks]

(c) $F = \{f \mid f \text{ is a map, } f : \mathbb{N}^+ \to \{0,1\}\}$, the set of all maps from the natural numbers $\mathbb{N}^+$ to the set $\{0,1\}$ [5 marks]

(iii) Prove that the cardinality of the open interval between 3 and 4 is equal to the cardinality of the set of real numbers, i.e. $|\{x \in \mathbb{R} \mid 3 < x < 4\}| = |\mathbb{R}|$.

[5 marks]

[Total 20 marks]

2. (i) Let A and B be sets. State the definitions of their union $A \cup B$, intersection $A \cap B$, and set difference $A \setminus B$. [3 marks]

(ii) Consider the sets $X_1 = \emptyset$ (the empty set), $X_2 = \mathbb{Z}$ (the integers), $X_3 = \{\emptyset\}$, $X_4 = \{\mathbb{Z}\}$, $X_5 = \{\emptyset, \mathbb{Z}\}$.

State whether each of the following statements is true or false.

(a) $X_5 = X_1 \cup X_2$

(b) $X_1 \in X_3$

(c) $X_1 \subseteq X_4$

(d) $2 \in X_4$

(e) $X_1 \cap X_5 = X_3$

(f) $X_5 \setminus X_2 = X_3$

(g) $X_2 \subseteq X_5$

(h) $X_2 \cap X_5 = X_1$

[8 marks]

(iii) Let V be a finite nonempty set and let 2^V be its power set, *i.e.* the set of all subsets of V and let $\mathbb{Z}_2 = \{0, 1\}$. f is a map, $f : 2^V \to \mathbb{Z}_2$ which satisfies the following conditions:

$$\begin{aligned} &f(X \cap Y) = f(X)f(Y) \ \text{ for all } \ X, Y \in 2^V \\ &f(\emptyset) = 0 \\ &f(V) = 1 \end{aligned}$$

Prove the following statements.

(a) If $\{C_1, C_2, \ldots C_n\}$ is a set of subsets of V such that $f(C_k) = 1$ for all $k = 1, 2 \ldots n$, then
$$f(C_1 \cap C_2 \cap \ldots C_n) = 1\,.$$
Use proof by induction for this part.

(b) If $f(A) = 1$ and $f(B) = 1$ then the intersection of A and B is not the empty set.

(c) If $f(A) = 1$ then $f(V \setminus A) = 0$.

(d) If $f(A) = 1$ and $A \subseteq B$ then $f(B) = 1$.

(e) If $f(A) = 1$ and $f((V \setminus A) \cup B) = 1$ then $f(B) = 1$.

[9 marks]

[Total 20 marks]

3. (i) Consider the map $f : A \to B$ for sets A, B. Define what it means for f to be injective, surjective and bijective.

[2 marks]

(ii) There is something wrong with each of the following definitions of functions. In each case state why it is not a function (there may be more than one problem) and change the domain and/or target so that the new definition gives a well-defined function (there may be many ways to do this, you only need to give one). In each case, then state whether the function you have defined is injective and whether it is surjective.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

(b) $g : \mathbb{R} \to \{y \in \mathbb{R} \,|\, 0 \leq y \leq 2\pi\}$, $g(x) = \sin^{-1} x (= \arcsin x)$.

[6 marks]

(iii) $f : A \to B$ is a map from set A to set B. Are the following statements true or false? If the statement is true, prove it.

(a) If f is not a surjection then $|A| \leq |B|$.

(b) If $|A| = |B|$ then f is a bijection.

(c) If A and B are finite and $|A| > |B|$ then f is not injective.

[Note: $|X|$ denotes the cardinality of a set X.]

[5 marks]

(iv) (a) Let Y be a finite set of natural numbers. Y has $N + 1$ elements. Prove that there is at least one pair of elements of Y, $n, m \in Y$ with $n \neq m$, such that their difference, $n - m$ is divisible by N. [5 marks]

(b) Prove that for any natural number N, there is a natural number divisible by N whose only digits (in base 10) are 0's and 3's. Hint: use the previous result.

[2 marks]

[Total 20 marks]

4. (i) Give the precise mathematical definition of the statement that the sequence of real numbers $\{x_n\}$, $n \in \mathbb{N}^+$, converges to the limit x. [2 marks]

(ii) Use challenge-response proofs to prove each of the following.

(a) Let $x_n = \frac{\cos n}{\log(n+1)}$ for $n \in \mathbb{N}^+$. Prove that $\{x_n\}$ converges to 0. [4 marks]

(b) Let $\{u_n\}$ and $\{v_n\}$ be convergent sequences. Prove that $z_n = u_n - v_n$ is a convergent sequence. [4 marks]

(c) Let $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ be sequences such that $p_n < r_n < q_n, \forall n \in \mathbb{N}^+$. $\{p_n\}$ and $\{q_n\}$ are convergent and both converge to the *same* limit. Prove that $\{r_n\}$ is also convergent to that same limit. [5 marks]

(iii) Give the definition of an *open* subset of $\mathbb{R}$.

Let $\{x_n\}$ be a sequence that converges to the limit x as $n \to \infty$. Prove that for each open subset $S \subset \mathbb{R}$ such that $x \in S$, there exists $N \in \mathbb{N}^+$ such that $x_n \in S$ for $n > N$.

[5 marks]

[Total 20 marks]

5. (i) (a) A series is, formally, a sum of infinitely many real numbers: $\sum_{n=1}^{\infty} a_n$. What condition does the series have to satisfy to be *convergent*? What condition does the series have to satisfy to be *absolutely convergent*?

(b) State the comparison test for a convergent series.

[5 marks]

(ii) Consider the infinite series $\sum_{n=1}^{\infty} a_n$. We define the quantity

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| .$$

Prove that the series, $\sum_{n=1}^{\infty} a_n$, converges if $\rho < 1$. You may use the comparison test and assume the convergence properties of the geometric series.

What can you say about the series in the cases $\rho > 1$ and $\rho = 1$? [8 marks]

(iii) Consider the Taylor series expansion

$$\sum_{n=0}^{\infty} y_n x^n$$

where $\{y_n\}$, for $n \in \mathbb{N}$, is a convergent sequence, *i.e.* there exists a real number y such that $y_n \to y$ as $n \to \infty$.

(a) Use the ratio test to show that if $|y| > 0$ then the series converges absolutely for $|x| < 1$.

(b) Prove that the series also converges absolutely for $|x| < 1$ if $y = 0$.

[7 marks]

[Total 20 marks]