Mathematics for Computer Science - Problem set 01

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September 28^{th} 2020

Problem 1: We have **u** and **v** are **orthogonal unit vectors**. Show that **u + v** is orthogonal to **u - v**.

Let $u, v \in \mathbb{R}^n$. Since u and v are unit vector, we assume:

$$u = \begin{bmatrix} 1, & 0, & 0, & ... & 0 \end{bmatrix}$$
$$v = \begin{bmatrix} 0, & 1, & 0, & ... & 0 \end{bmatrix}$$

As to prove u + v is orthogonal to u - v, we have to show that the dot product between u + v and u - v is 0

We have:

$$u + v = \begin{bmatrix} 1, & 1, & 0, & 0, & ... & 0 \end{bmatrix}$$
$$u - v = \begin{bmatrix} 1, & -1, & 0, & ... & 0 \end{bmatrix}$$

And

$$\sum_{i=1}^{n} (u + v)_i \cdot (u - v)_i$$
$$= (1 \cdot 1) + [1 \cdot (-1)] + (0 \cdot 0) + \cdots + (0 \cdot 0) = 0$$

Since the dot product between u + v and u - v is 0, then u + v is orthogonal to u - v

Problem 2: Prove that nullspaces $\mathbf{N}(A^\top A) = \mathbf{N}(A)$

Let A denote a $\mathbb{R}^{m \times n}$ matrix

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\Rightarrow A^\top = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Since $\mathbf{N}(A)$ is the null space of A. Take $x \in \mathbf{N}(A)$, we have

$$A \times x = 0$$

Multiply two sides of the equation by $A^\top$, we have

$$A^\top \times A \times x = A^\top \times 0$$

$$\Leftrightarrow A^\top \times A \times x = 0$$

$$\Rightarrow x \in \mathbf{N}(A^\top A)$$

Then

$$\mathbf{N}(A) \subseteq \mathbf{N}(A^\top A) \tag{1}$$

Moreover $\mathbf{N}(A^\top A)$ is the null space of $A^\top A$. Take $x \in \mathbf{N}(A^\top A)$

$$A^\top A \times x = 0$$

Multiply two sides of the equation by $x^\top$, we have

$$x^\top \times A^\top A \times x = 0$$

$$\Leftrightarrow (A \times x)^\top \times (A \times x) = 0$$

$$\|Ax\| = 0$$

$$\Rightarrow Ax = 0$$

$$\Rightarrow A^\top A \subseteq A \tag{2}$$

From (1) and (2), we have $\begin{cases} A \subseteq A^\top A \\ A^\top A \subseteq A \end{cases}$

$\Rightarrow \mathbf{N}(A) = \mathbf{N}(A^\top A)$

$\rightarrow$ Q.E.D

Problem 3

We know for sure that a symmetric matrix must be a square one

Let A denote a $\mathbb{R}^{n\times n}$ matrix

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

We have the column space and the row space of A: $\mathbf{C}(A)$ and $\mathbf{C}(A^\top)$, respectively

$\mathbf{C}(A)$ is the combination of all the columns in A and $\mathbf{C}(A^\top)$ is the combination of all the rows in A, also.

Hence

$$\mathbf{C}(A) = \mathbf{C}(A^\top)$$

However, the elements in A can be linearly independent of each other, which leads to the asymmetry of A. Or in other words, $A \neq A^\top$

Indeed, take a closer look to the example:

Let A be given a 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The column space of A is

$$\mathbf{C}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and the row space of A is

$$\mathbf{C}(A^\top) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Thus

$$\mathbf{C}(A) = \mathbf{C}(A^\top)$$

However, since $A \neq A^\top$, then A does not have to be symmetric if $\mathbf{C}(A) = \mathbf{C}(A^\top)$

Problem 4

Let A denote a 2×2 matrix

$$A = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

Since

$$A \times \mathbf{N}(A) = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \times \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$$

Then the null space of A is

$$\mathbf{N}(A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

However

$$A^2 = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \times \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which leads to the null space of A^2 is

$$\mathbf{N}(A^2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ (in which } x_1, x_2 \in \mathbb{R})$$

Hence

$$\mathbf{N}(A) \subseteq \mathbf{N}(A^2)$$

which means it is not *always* true that $\mathbf{N}(A) = \mathbf{N}(A^2)$