QCVN Group Meeting

Haar measure & Unitary t-design
– Towards URB for qutrit

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Complete protocol of URB

A standard unitary randomized benchmarking consists of

1. Generate RB sequences

$$\mathcal{S}_{i_m} = \bigodot_{j=1}^{m+1} (\Lambda_{i_j,j} \odot C_{i_j}) \tag{1}$$

2. For each sequence, calculate survival probability

$$\mathrm{Tr}[E_\psi \mathcal{S}_{i_m}(\rho_\psi)] \tag{2}$$

3. Average over random realizations to find the *average* fidelity

$$F_{seq}(m, \psi) = \mathrm{Tr}\left(E_\psi \mathcal{S}_m(\rho_\psi)\right) \tag{3}$$

4. Fit the results to the model of exponential decay.

Generate RB sequences

Definition

A sequence of $m+1$ quantum operations with the first m operations chosen *uniformly* at random from some group $\mathcal{G} \in U(d)$ and the final operation $(m+1)$ chosen so that the net sequence is the identity operation.

- We primarily focus on $C_3 \in U(3^n)$, because they can be realized efficiently on both quantum & classical hardware.
- For each length m, we choose K_m RB sequences. Each sequence contains *m random* element C_{i_j} *sampled uniformly* from C_3.
- The $C_{i_{m+1}}$ element is defined as $(C_{i_1} \cdots C_{i_m})^{-1}$.

Measure survival probability

The survival probability is defined by

$$\mathrm{Tr}[E_\psi \mathcal{S}_{i_m}(\rho_\psi)] \tag{4}$$

where ρ_ψ is the initial state (SPAM absorbed) and E_ψ is the POVM element (having off-diagonal non-zero entries). If noise-free,

$$\rho_\psi = E_\psi = |\psi\rangle\langle\psi|$$

In practice, survival probabilities are probabilities that the qutrit go back to the ground state $|0\rangle$. For example, in the noise-free situation,

$$\mathrm{Tr}[E_\psi \mathcal{S}_{i_m}(\rho_\psi)] = p(0) \tag{5}$$

Measure survival probability

Theorem

The survival probability of the qutrit is the probability we obtain the initial state (or ground state if we prepare $|0\rangle$).

Proof. Suppose we prepare a statistical ensemble ρ, taking into account the SPAM error. Then the evolution of such system is governed by

$$\rho_\psi \to \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger \tag{6}$$

The probability of getting result m is

$$p(m) = \sum_i p(m|i) p_i \tag{7}$$

$$= \sum_i \langle\psi_i| U^\dagger E_{|\psi\rangle} U |\psi_i\rangle p_i \tag{8}$$

$$= \mathrm{Tr}\left[E_{|\psi\rangle} \sum_i p_i U |\psi_i\rangle\langle\psi_i| U^\dagger \right] \tag{9}$$

$$= \mathrm{Tr}\left[E_{|\psi\rangle} \mathcal{S}_{i_m}(\rho_\psi) \right] \tag{10}$$

If we intended to prepare an initial ground state, then

$$\mathrm{Tr}\left[E_{|0\rangle} \mathcal{S}_{i_m}(\rho_{|0\rangle}) \right] = p(0) \qquad \square$$

Average sequence fidelity and fit results

Average over K_m random realizations of the sequence to find the average sequence fidelity

$$F_{seq}(m, |\psi\rangle) = \mathrm{Tr}[E_\psi S_{K_m}(\rho_\psi)] \tag{11}$$

$$= \mathrm{Tr}\left[E_\psi \frac{1}{K_m} \sum_{i_m} \mathcal{S}_{i_m}(\rho_\psi)\right] \tag{12}$$

Repeat for different values of m and fit the results for the averaged sequence fidelity to the model

$$F^{(0)}(m, |\psi\rangle) = A_0 \alpha^m + B_0 \tag{13}$$

where A_0 and B_0 absorb SPAM error. α is called the *Error per Clifford (EPC)*, relating to the average error-rate.

Results

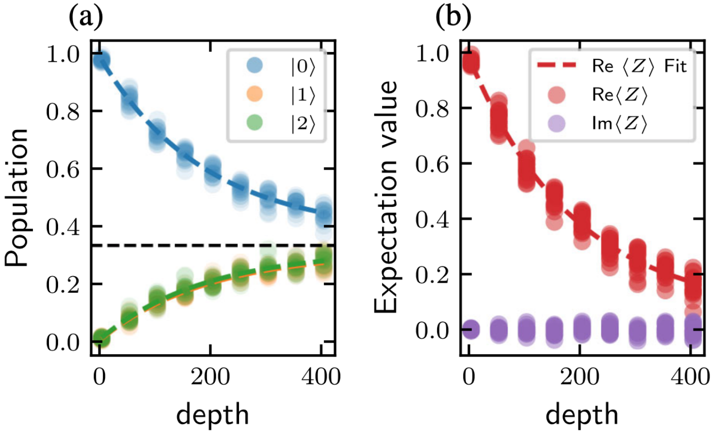

Figure: Qutrit randomized benchmarking [Morvan *et al.* Phys. Rev. Lett.126.210504 (2021)].

Intuition behind URB

- By sampling uniformly noisy gates, we create a *depolarizing channel*, characterized by the probability α of the qutrit not being turned into a maximally mixed state $I/3$.
- After a sequence of m gates, where the error per gate rate is α, then the resulting density matrix is

$$\rho_f^m = \alpha^m \rho_i + (1 - \alpha^m)/3 \cdot I \tag{14}$$

- Suppose we start with $|0\rangle$, and the entire sequence is equivalent to the identity operator. The probability of successfully measuring $|0\rangle$ is

$$p(0) = \begin{cases} \alpha^m [\rho_i]_{00} + 1 - \alpha/3 \\ \alpha^m + 1 - \alpha^m/3 = 2/3\alpha^m + 1/3 \end{cases} \tag{15}$$

The mathematical theory behind

- The exponential decay model is not a result of repeating gate in a sequential manner, but *uniformly randomized gates from the Clifford group*. These Clifford gates are noisy, thus $C_{i_j} \odot \Lambda$, where Λ constitutes a depolarizing channel.
- Formally speaking, taking an average over a finite group C_3 of a quantum channel Λ constitutes a twirl.
- Twirling over $U(3)$ yeilds exactly the same result as the Clifford group, because the Clifford group $C_3 \in U(3)$ is a *unitary 2-design* of the unitary group.
- Buzzing words: *uniformly-randomized*, *unitary t-design*, *twirling over a finite group*.

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The motivation of measure theory

The theory of random matrices involves extensively the generalised concept of *measure*.

- In $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$, think about length, area, and volume.
- What about higher dimensions?
- What about other spaces?
- What about vector spaces, e.g $\mathcal{M}_{3x3}$.

Definition (Loosely speaking)

Measure μ tells us how mathematical objects (points, matrices, functions, etc.) are distributed in a mathematical set or space–some place is immensily dense, some is not.

Example 1: Triple integral of volume V

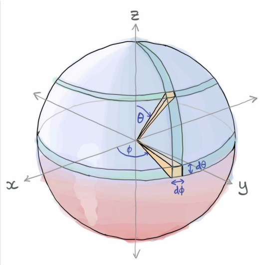


Figure: The measure μ takes into account that these infinitesimal volume is not uniformly distributed over $\mathbb{R}^3$.

The volume of the sphere is *properly measured* when including the measure factor μ

$$V = \int_0^r \int_0^{2\pi} \int_0^{\pi} \mu d\rho d\phi d\theta, \qquad (16)$$

where $\mu = \rho^2 \sin\theta$ weights portions of the sphere differently depending on where they are in the space.

The Haar measure

- Similar to points in spherical coordinates uniquely defined by (ρ, ϕ, θ), unitary matrices are uniquely defined by three parameters, e.g (θ, ϕ, λ) for every element in $U(2)$.
- For every dimension N, the unitary matrices of size $N \times N$ constitute the unitary group $U(N)$. Operations on $U(N)$ requires proper measure μ, or the *Haar measure*.
- For an N-dimensional system, the Haar measure tells us how to weight the elements of $U(N)$. As an example, suppose f is a function acts on $V \in U(N)$, the integral over the group is

$$\int_{V \in U(N)} f(V) d\mu_N(V) \tag{17}$$

where the analytical expression of μ_N is desired.

Haar measure in $U(2)$

Problem (Sampling)

Given the structure of the group $U(2)$, sample elements of the unitary group $U(2)$ in a properly uniform manner.

- The mathematical space we're dealing with is $U(2)$.
- The operation is sampling.
- An alternative angle of looking: sampling quantum *states* uniformly at random.

$$\rho_{|0\rangle} \to U_{\mu_N} |\psi(\theta,\varphi)\rangle\langle\psi(\theta,\varphi)| U_{\mu_N}^{\dagger} \tag{18}$$

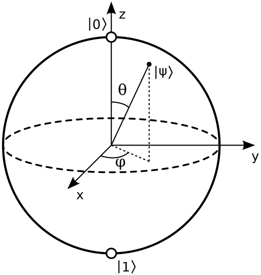


Figure: $SU(2) \subseteq U(2) \cong SO(3)$.

Visualization of state sampling

- Note that the density matrix of a single-qubit state can be expanded in the form

$$\rho = \frac{1}{2}(\hat{I} + \mathbf{r} \cdot \hat{\sigma}), \tag{19}$$

where $\mathbf{r}(r_x, r_y, r_z)$ is the Bloch vector, uniquely defines a mixed state. The Bloch vector component is $r_\alpha = \mathrm{Tr}(\sigma \cdot \rho) \in \mathbb{R}^3$.

- The explicit matrix representation of any arbitrary single-qubit operator $U(2)$ is

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & e^{i(\lambda+\phi)} \cos(\theta/2) \end{pmatrix}, \tag{20}$$

hence each element of $U(2)$ is effectively defined by θ, ϕ, λ.

Visulization of state sampling

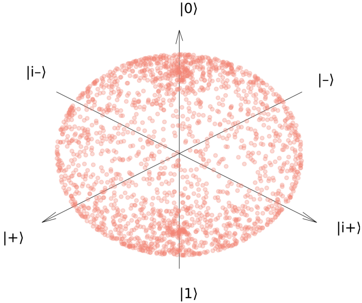


Figure: $d\mu_2 = d\theta d\phi d\lambda$. Not Haar-random.

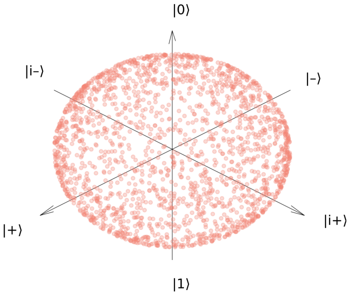
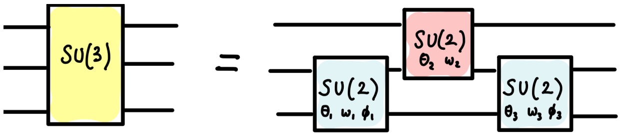


Figure: $d\mu_2 = \sin\theta d\theta d\phi d\lambda$. Haar-random.

Haar measure in $U(N)$

- It's hard to visualize when $N = d^n$. We thus remain focus on 3^1.
- From the study of photonics, we know that we can decompose any $SU(N)$ operation recursively by sandwiching an $SU(2)$ between two $SU(N-1)$ [H. de Guise *et al.*, Phys. Rev. A 97 022328 (2018)].
- For $N = 3^1$, the exact Haar measure $d\mu_3$ follows accordingly



$$d\mu_3 = \sin\theta_1 \sin\theta_2 \sin^2(\theta_2/2) \sin\theta_3 \Pi_{i=1}^{3} d\theta_i d\phi_i d\lambda_i \qquad (21)$$

Final note on Haar measure

- Unlike $SU(2)$, sampling uniformly over $SU(3)$ requires us to sample θ_i from *different distribution*.
- This suggests the higher dimension of the unitary group, the more parameters we need to care about. Generally speaking an N-dimensional unitary requires at least $N^2 - 1$ parameters.
- Other methods of sampling uniformly are thus desired, e.g. *Haar-random matrices from QR decomposition* [F. Mezzaddri, arXiv:math-ph/0609050v2 (2007)].
- Haar measure is invariant under unitary transformation, i.e.

$$\begin{aligned}\int_{V\in U(N)} f(WV)d\mu_N(V) &= \int_{V\in U(N)} f(VW)d\mu_N(V) \\ &= \int_{V\in U(N)} f(V)d\mu_N(V)\end{aligned}$$

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The problem of efficient randomization

Sampling from N-dimensional unitary group requires us to keep track of $N^2 - 1$ parameters, or $d^{2n} - 1$.

	1	2	3	...	n
Qubit ($d = 2$)	2	15	63	...	$2^{2n} - 1$
Qutrit ($d = 3$)	8	80	728	...	$3^{2n} - 1$

Table: The number of parameters we need to specify a $SU(d^n)$ operator. The exponential scaling hinders us from scalability.

Problem (Efficient sampling)

How to scalably and efficiently sample random unitaries in a uniform manner?

Motivation: spherical t-design

Problem (Averaging P over a sphere)

What is the average A of a d variables polynomial over the surface of a real d-dimensional unit sphere $\mathcal{S}(R^d)$?

$$A = \int_{\mathcal{S}(R^d)} P_t(u) d\mu(u) \tag{22}$$

- This integration–in principle–can be numerically calculated, provided we have the proper measure $d\mu(u)$.
- What if one could alternatively approximate A by uniformly sampling a sufficiently large number of points on the sphere?
- What if one could alternatively **calculate exactly** A by uniformly sampling a sufficiently large number of points on the sphere?

Motivation: spherical t-design

Definition (Spherical t-design)

Let $P_t : \mathcal{S}(R^d) \to \mathbb{R}$ be a polynomial in d variables with all terms homogeneous in degree at most t. A set $X = \{x | x \in \mathcal{S}(R^d)\}$ is a spherical t-design if

$$\frac{1}{|X|} \sum_{x \in X} P_t(x) = \int_{\mathcal{S}(R^d)} P_t(u) d\mu(u) \tag{23}$$

holds for all possible P_t, where $d\mu$ is the uniform, normalized spherical measure. A spherical t-design is also a k-design for all $k < t$.

1. One natural question is how do we find X, provided f?
2. The other question concerns with the structure of the set (group) $X/\mathcal{G}$ itself.

Unitary t-design

Instead of averaging polynomials P_t over spheres, we consider polynomials that are functions of the entries of $U \in U(N)$.

Definition (Unitary t-design)

Let $P_{t,t}(U)$ be a polynomial with homogeneous degree at most t in d variables in the entries of a unitary matrix U, and degree t in the complex conjugates of those entries. A unitary t-design is a set of K unitaries $\{U_k\}$ such that

$$\frac{1}{K}\sum_{k=1}^{K} P_{t,t}(U_k) = \int_{U(d)} P_{t,t}(U) d\mu(U) \tag{24}$$

holds for all possible $P_{t,t}$ and where $d\mu$ is the uniform Haar measure.

Unitary t-design in action: Haar measure

Let us revisit our efficient randomization problem. The ultimate goal of randomization is to calculate the fidelity. More precisely, the *average fidelity* of a quantum channel can be calculated by twirling over the Haar measure of the resprective Lie group.

Definition (Twirling $\mathcal{E}$ over a Haar-measure)

Suppose a quantum channel $\mathcal{E}$. The average fidelity with respect to the full set of Haar-random states $U_{\mu_N} \rho U_{\mu N}^\dagger$ is *twirling channel* $\mathcal{E}$,

$$\bar{F}_{\mathcal{E}} = \int_{U(d)} d\mu(U) U^\dagger \mathcal{E}(U \rho U^\dagger) U \tag{25}$$

Unitary t-design in action: Fidelity

Using the result of [J. Emerson *et al.*, arXiv:quant-ph/0606161v2 (2012)], we conclude that twirling any quantum channel over a unitary t-design is equivalent to twirling over a Haar-measure unitary group.

$$\frac{1}{K}\sum_{j=1}^{K} U_j^\dagger \mathcal{E}(U_j\rho)U_j = \int_{U(d)} d\mu(U)U^\dagger \mathcal{E}(U\rho U^\dagger)U \qquad (26)$$

- Note that the inner product involves two U and two $U^\dagger$, thus implies this unitary t-design is a *unitary 2-design*.
- What is the unitary 2-design then?

Theorem ($\mathcal{C}_3^n$)

The multi-qutrit Clifford group $\mathcal{C}_3^n$ forms a unitary 2-design.

A result from [M. Nielsen, arXiv:quant-ph/0205035v2 (2002)] prove that twirling a quantum channel $\mathcal{E}$ over a Haar measure yields a depolarizing channel.

Corollary

Twirling over the multi-qutrit Clifford group yields a depolarizing channel.

From this, we informally confirm the aformentioned results of URB,

$$\mathcal{E}_{dep}(\rho_i) = p\rho_i + (1-p)\frac{I}{3^n} \tag{27}$$

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Conclusion

1. The metrics we concern is the average fidelity of a quantum channel $\mathcal{E}$. This average fidelity can be calculated by twirling that channel over Haar-random unitary group [J. Emerson *et al.*, arXiv.quan-ph/0503243].
2. This is better than QPT, but still not scalable and efficient. We can however get around by using the unitary t-design.
3. Twirling over a unitary t-design is equivalent to twirling over Haar-random unitary group, and yields a depolarizing channel.
4. The depolarizing channel has an exponential decay with respect to gate length m.

Next week: The Clifford group $\mathcal{C}_3$ on single-qutrit.