# APM 503 Instructor Set Homework

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## Problem 5.4.1

Let $I$ be bounded interval and $Z$ a Banach space.
Let $(f_n)$ be a sequence of diferentiable functions $f_n : I \to Z$ such that $\sum_{n=1}^{\infty} f_n(x^o)$ converges in $Z$ for some $x^o \in I$ and there exists a sequence of positive numbers $(M_n)$ such that $f_n'(x)\| \leq M_n$ for all $n \in \mathbb{N}$ and $x \in I$ and $\sum_{n=1}^{\infty} M_k$ converges in $\mathbb{R}$.

Show that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in I$ and privdes a differentiable function $f : I \to Z$ such that $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$, with the convergence of the latter series being also uniform for $x \in I$.

**Proof** Since there exists a sequence of positive numbers $(M_n)_{n=1}^{\infty}$ such that $\|f_n'(x)\| \leq M_n$ for all $n \in \mathbb{N}$ and for all $x \in I$, and since $\sum_{n=1}^{\infty} M_n < \infty$, it follows from the Weierstraß test that $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly for $x \in I$ and provides a bounded function $f' : I \to Z$. By definition of uniform convergence for the series, the partial sums $h_n = \sum_{k=1}^{n} f_k'(x)$ converge uniformly for $x \in I$. It then follows by Theorem 2.33 that $(f_k'(t))$ converges as $k \to \infty$ uniformly in $x \in I$. Thus, theorem 5.16 can be applied. By theorem 5.16, $(f_k(x))$ converges as $k \to \infty$ uniformly in $x \in I$ to a differentiable function $f : I \to Z$ and furthermore, $f'(t) = \lim_{k\to\infty} f_k'(x)$ for all $x \in I$.