MAT 442 Homework 10

Nickolas Gregory Dodd

Serial number: 12

October 27, 2015

5.2.8

Suppose that $A \epsilon M_{nxn}(F)$. has two distinct eigenvalues, $\lambda 1$ and $\lambda 2$ and that $\dim(E_{\lambda 1})$ = n - 1. Prove that A is diagonalizable.

Proof: By Theorem 5.7, it follows that $\dim(E_{\lambda 2}) \geq 1$. Therefore, there exists a non-zero eigenvector in $E_{\lambda 2}$. Now, since $\dim(E_{\lambda 1})$ = n - 1, it follows that we can choose a basis of eignevectors β for $E_{\lambda 1}$. Hence, by Theorem 5.8, v $\cup$ β is a linearly independent subset of V of length n = dimV, so v $\cup$ β is a basis of eigenvectors. Thus, by Theorem 5.1, A is diagonalizable.

5.2.11b

Let A be an n x n matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following:
(b) $\det(A) = (\lambda_1)^{m_1}\ (\lambda_2)^{m_2}\ \ldots\ (\lambda_k)^{m_k}$.

Proof: Since the characteristic polynomial (and thus it's roots) is independent of the choice of basis, we know that the characterstic polynomial of A is the same as the characteristic polynomial of the similar upper triangular matrix, B. Let Q be the similar sized matrix such that A = QBQ^{-1}. Now, $\det(A) = \det(QBQ^{-1}) = \det(Q)\det(B)\det(Q^{-1}) = \det(B) = (\lambda_1)^{m_1}\ (\lambda_2)^{m_2}\ \ldots\ (\lambda_k)^{m_k}$ where λ_i appears m_i times on B's diagonal, which follows since the determinant of a triangular matrix is the product of its diagonal entries.

5.2.12

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a)

Recall that for an eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 8 of ection 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

Proof: Let E_λ be the eigenspace of T corresponding to λ and $E_{\lambda^{-1}}$ be the eigenspace of T^{-1} corresponding to λ^{-1}. Let v ϵ E_λ. Then T(v) = λv and so v = λT^{-1}(v). This means that T^{-1}(v) = λ^{-1}v and v ϵ $E_{\lambda^{-1}}$. Conversely, if v ϵ $E_{\lambda^{-1}}$, we have T^{-1}(v) = λ^{-1}v and so v = λ^{-1}T(v). Hence, T(v) = λv and v ϵ E_λ. Therefore, the two spaces are the same.

(b)

Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Proof: By part (a) above, if T is diagonalizable and invertible, then the eigenbasis of T obtained by diagonalizing T will also be the eigenbasis of T^{-1} corresponding to the inverse eigenvalues. To see this, let A be the representation of T. Then since, T is diagonalizable, A is diagonalizable, and hence is similar to a diagonal matrix D. Let Q be the similar sized matrix whose columns are composed of the eigenbasis vectors of A. Then, $A = Q^{-1}DQ$. The inverse matrix, representing T^{-1} is given by $A^{-1} = QD^{-1}Q^{-1}$. Since D is diagonal, $[D^{-1}]_{ii} = 1 \ / \ [D]_{ii}$, which agrees with part (a) above. Thus, T^{-1} is diagonalizable.

5.2.13

Let $A \ \epsilon \ M_{nxn}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvlaue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspace for A and A^t respectively.

(a)

Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

Proof: Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} ..Then, for \lambda = 0,$ E_λ = span(0, 1) while for A^t, E'_λ = span (1, 1).

(b)

Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.

Proof: We have that, dim (E_λ) = dim N(A - λI) = N(A - λI)^t) = N(A^t - λI) = $\dim(E'_\lambda)$.

(c)

Prove that if A is diagonalizable, then A^t is also diagonalizable.

Proof: By Theorem 5.9, A is diagonalizable iff the characteristic polynomial of A splits and for each eigenvalue λ_i, mult(λ_i) = dim E_{λ_i}. Now, since the characteristic polynomial of A and A^t are the same, if A is diagonalizable, then the characteristic polynomial of A^t also splits. By part (b) above, we also know that the dimensions of the corresponding eigenspaces E'_{λ_i} are the same, thus the multiplicities for the corresponding eigenvalues are the same. Then, it follows from Theorem 5.9 that A^t is diagonalizable.

5.2.18a

Prove that if T and U are simultaneously diagonalizable linear operators, then T and U commute (i.e., $TU = UT$).

Proof: Let β be as in the preamble to this exercise in the text. That is, β is a basis for V which makes both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Now, consider the matrix multiplication $[T]_\beta[U]_\beta$. Since both matrices are diagonal, we simply multiply their diagonal elements together. Therefore, letting $A = [T]_\beta$ and $B = [U]_\beta$, we have

C$_{ii}$ = $A_{ii}B_{ii}$ for 1 $\leq$ i $\leq$ dim(V). (1)

By associativy of scalar multiplication,

$C_{ii} = A_{ii}B_{ii} = B_{ii}A_{ii}$ (2)

, so $C = AB = BA$. Substituting back in, this means that $[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$. Hence, it follows that T and U commute.

5.20

Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} W_i = V$$

.

Prove that V is the direct sum of $W_1, W_2, \ldots, W_k$ iff dim(V) =

$$\sum_{i=1}^{k} dim(W_i)$$

.

Proof: Attached to back of assignment on handwritten paper.

5.4.3

Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspace are T-invariant.

(a)

0 and V.

Proof: Since for any $v \epsilon V$, T(0) = 0 and T(v) ϵ V, the claim holds for any linear operator T.

(b)

N(T) and R(T).

Proof: For any $v \epsilon$ N(T), by definition we have that T(v) = 0 ϵ N(T). Also, for any $v \epsilon$ R(T), we have T(v) ϵ R(T) by definition. Thus the claim holds for any linear operator T.

(c)

E_λ, for any eigenvalue λ of T.

Proof: If $v \epsilon E_\lambda$, where v is a non-zero eigenvector corresponding to eigenvalue λ, we have T(v) = λv. Since T(λv) = λT(v) = λ^2v, it follows that T(v) ϵ E_λ.

5.3.5

Let T be alinear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

Proof: Let $U_{k k \epsilon K}$ be a collection fo T-invariant subspaces of V. Let $U^* = \bigcap_{k \epsilon K} U_k$. For every u ϵ U^*, we have T(u) ϵ U_k for every $k \epsilon K$. So it follows that T(v) ϵ U^* as well.