Graph Theory Notes

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This is an on going project. Use at your own risk. There are bound to be typos. Corrections are appreciated, and rewarded with small amounts of extra credit, especially when they indicate mathematical understanding.

These notes are meant to enhance, not replace, the lectures and class discussions. They are intended to be concise records of proofs, to free students from the need to take careful notes during class. However the motivation for these proofs is left to the lectures and discussions in class.

CHAPTER 1

Introduction

1.1. Graphs

Formally a *graph* is an ordered pair $G = (V, E)$ where E is an irreflexive, symmetric, binary relation on V. Since E is symmetric there is no need to keep track of the order of pairs $(x, y) \in E$; since it is irreflexive there are no ordered singletons (x, x) in E. This leads to a more intuitive formulation. We take E to be a set of unordered pairs of elements from V. Elements of V are called *vertices*; elements of E are called *edges*. If $x, y \in V$ are vertices and $\{x, y\} \in E$ is an edge we usually (but not always) denote $\{x, y\}$ by the shorthand notation xy. So $xy = \{x, y\} = \{y, x\} = yx$. The vertices x and y are called *ends*, or *endpoints*, of the edge xy. The ends x and y of an edge xy are said to be *adjacent* and the end x is said to be *incident* to the edge xy. We also say that x and y are *joined* or *linked* (not connected) by the edge xy. Two edges are said to be *adjacent* if they have a common end. In this course, all graphs have a finite number of vertices, unless it is explicitly stated that they have infinitely many. Graphs are illustrated by drawing dots for vertices and joining adjacent edges by lines or curves.

Our definition of graph is what the text calls a *simple graph*. Most of the time we will only be interested in simple graphs, and so we begin with the simplest definition. When necessary, we will introduce the more complicated notions of *directed graphs* and *multigraphs,* but here is a quick hint. A directed graph $G = (V, E)$ is any binary relation (not necessarily irreflexive or symmetric) on V. In other words E is any set of *ordered* pairs of vertices. If E is a *multiset* then G is called a directed multigraph. If E is a set of subsets of V then G is called a hypergraph.

The study of graph theory involves a huge number of of parameters—see the front and back inside covers of the text. This can be quite daunting. My strategy is to introduce these parameters as they are needed. Please feel free to interrupt lectures to be reminded of their meanings. Most of the time my notation will agree with the text, and I will try to emphasize differences. Next we introduce some very basic notation.

Given a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. Set $|G| := |V(G)|$ and $\|G\| := |E(G)|$. This is not standard, and instead the book uses $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Suppose $v \in V(G)$ is a vertex of G. Define

$$N_G(v) := \{w \in V(G) : vw \in E(G)\}; \qquad E_G(v) := \{e \in E(G) : v \text{ is an end of } e\}.$$

The set $N_G(v)$ is called the (open) *neighborhood* of v, and its elements are called *neighbors* of v. So a vertex w is a neighbor of v iff it is adjacent to v. When there is no confusion with other graphs the subscript G is often dropped. The closed neighborhood of v is $N[v] := N(v) \cup \{v\}$—we dropped the subscript. The set $E_G(v)$ is the set of edges incident to v; again, we may drop the subscript G. The text does not provide notation for this set. For simple graphs $|N(v)| = |E(v)|$. However for multigraphs this may not hold, since two vertices might be joined by several edges. With this in mind, define the degree of a vertex v to be $d_G(v) := |E_G(v)|$, but note that for simple graphs $d_G(v) = |N_G(v)|$. A graph is k-regular if every vertex has degree k.

We use the following set theoretic notation. The sets of natural numbers, integers and positive integers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}^+$. For $n \in \mathbb{N}$ set $[n] := \{1, 2, \ldots, n\}$; in particular $[0] = \emptyset$. For a set X and an element y, set $X+y := X \cup \{y\}$ and $X-y := X \smallsetminus \{y\}$. Finally, $\binom{X}{n}$ is the set of all n-element subsets of X.

1.2. Proofs by Mathematical Induction

Most proofs in graph theory involve mathematical induction, or at least the Least Element Axiom. Here we quickly review this technique. Also see the discussion in the text on pages 19–20, and especially the *induction trap* on page 42.

Axiom (Least Element Axiom). *Every nonempty set of natural numbers has a least element.*

Let S be a set of natural numbers and $B = \mathbb{N} \smallsetminus S$. We would like to prove that $S = \mathbb{N}$ and $B = \emptyset$. Here is a way to organize the proof.

Theorem 1 (Principle of Induction). *Suppose $S \subseteq \mathbb{N}$, and every $n \in \mathbb{N}$ satisfies,*

$$\text{if } k \in S \text{ for every natural number } k < n, \text{ then } n \in S. \tag{1.2.1}$$

Then $S = \mathbb{N}$.

Proof. Consider any set $S \subseteq \mathbb{N}$ such that (1.2.1) holds for all $n \in \mathbb{N}$, and let $B = \mathbb{N} \smallsetminus S$. Arguing by contradiction, assume that B is nonempty. Then it has a least element l. Since l is the least element of B, every natural number less than l is in S. Applying (1.2.1) to l yields $l \in S$, a contradiction. □

Using the Principle of Induction to prove that $S = \mathbb{N}$, it suffices to prove (1.2.1) holds for all natural numbers n. Notice that the hypothesis of (1.2.1) always holds for $n = 0$ (why?), and so, if (1.2.1) holds for $n = 0$ then $0 \in S$. Thus in applications, checking (1.2.1) for $n = 0$ is usually a special case. Here is an example.

Theorem 2. *Every natural number greater than 1 has a prime factor.*

Proof. Let $S = \{n \in \mathbb{N} : n \leq 1 \text{ or } n \text{ has a prime factor}\}$. It suffices to show (1.2.1). Consider any $n \in \mathbb{N}$ such that $k \in S$ for every natural number less than n. We must show $n \in S$. If $n \leq 1$ then $n \in S$ by definition. So suppose $n \geq 2$. If n is prime then it is a prime factor of itself, and so it is in S. Otherwise, there exist integers a, b such that $1 < a, b < n$ and $ab = n$. Since $a < n$, we have $a \in S$. Since $1 < a$ this means that a has a prime factor p. Since p is a factor of a and a is a factor of n, p is a (prime) factor of n. □

HW 1. Let $S = \{n \in \mathbb{N} : \forall k \in \mathbb{N}(k < n \to k \text{ is even})\}$. List the elements of S. For what $n \in \mathbb{N}$ is (1.2.1) false?

1.3. Ramsey's Theorem for Graphs

Ramsey's Theorem is an important generalization of the Pigeonhole Principle. Here we only consider its simplest version applied to graphs. In the past it was presented as part of MAT 415, but it has been moved to MAT 416 because its presentation benefits from the language of graph theory.

Let $G = (V, E)$ be a graph, and suppose $X \subseteq V$. The set X is a *clique in G* if $xy \in E$ for all distinct vertices $x, y \in X$. It is an *independent set*, or *coclique*, in G if $xy \notin E$ for all vertices $x, y \in X$. A clique (coclique) X is a b-clique (b-coclique) if $|X| = b$. Let $\omega(G) := \max\{|X| : X \textit{ is a clique in } G\}$, *and* $\alpha(G) := \max\{|X| : X \textit{ is a coclique in } G\}$.

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is an *induced subgraph* of G if $H \subseteq G$ and $E(H) = \{xy \in E(G) : x \in V(H) \textit{ and } y \in V(H)\}$. For $X \subseteq V$, $G[X]$ is the induced subgraph of G that has vertex set X. The *complement* of G is the graph, $\overline{G} := (V(G), \overline{E}(G))$, where $\overline{E} := \{xy : xy \notin E \text{ and } x, y \in V(G)\}$.

THEOREM 3 (Ramsey's Theorem 8.3.7,11). *For all graphs G and $a, b \in \mathbb{N}$, if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.*

PROOF. Argue by induction on $n = a + b$. (That is, let S be the set of natural numbers n such that for all positive integers a, b if $n = a + b$, and G is a graph with $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$. Show that for all $n \in \mathbb{N}$, if $k \in S$ for all $k \in \mathbb{N}$ with $k < n$ then $n \in S$.) Consider any $n = a + b$ with $a, b \in \mathbb{Z}^+$, and any graph G with $|G| \geq 2^{a+b-2}$.
Base step: $\min\{a, b\} = 1$. Since $|G| \geq 2^{a+b-2} \geq 1$, G has a vertex v. Since $\{v\}$ is both a clique and an independent set, both $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So we are done regardless of whether $a = 1$ or $b = 1$.
Induction Step: $\min\{a, b\} \geq 2$ (so $a - 1, b - 1 \in \mathbb{Z}^+$). (We assume the induction hypothesis: the theorem holds for all $a', b' \in \mathbb{Z}^+$ with $a' + b' < a + b$.) Let $v \in V(G)$. Then

$$1 + d_G(v) + d_{\overline{G}}(v) = |G| \geq 2^{a+b-2} = 2^{a+b-3} + 2^{a+b-3}.$$

By the pigeonhole principle, either $d_G(v) \geq 2^{a+b-3}$ or $d_{\overline{G}}(v) \geq 2^{a+b-3}$.
Case 1: $d_G(v) \geq 2^{a+b-3}$. Set $H := G[N_G(v)]$. Then $|H| = d_G(v) \geq 2^{a-1+b-2}$. By the induction hypothesis H contains an $(a-1)$-clique X or a b-coclique Y. In the latter case Y is a b-coclique in G. In the former case $X + v$ is an a-clique in G.
Case 2: $d_{\overline{G}}(v) \geq 2^{a+b-3}$. Set $H := G[N_{\overline{G}}(v)]$. Then $|H| \geq 2^{a+b-1-2}$. By the induction hypothesis H contains an a-clique X or a $(b-1)$-coclique Y. In the former case, X is an a-clique in G. In the latter case $Y + v$ is a b-coclique in G. □

For $a, b \in \mathbb{Z}^+$, define $\mathrm{Ram}(a, b)$ to be the least integer n such that every graph G with $|G| \geq n$ satisfies $\omega(G) \geq a$ or $\alpha(G) \geq b$. By Theorem 3, $\mathrm{Ram}(a, b)$ exists and satisfies $\mathrm{Ram}(a, b) \leq 2^{a+b-2}$.

HW 2. Prove that for $a, b \in \mathbb{Z}^+$ with $a, b \geq 2$:

(1) $\mathrm{Ram}(a, b) = \mathrm{Ram}(b, a)$.
(2) $\mathrm{Ram}(a, 1) = 1$.
(3) $\mathrm{Ram}(a, 2) = a$.
(4) $\mathrm{Ram}(3, 3) = 6$.
(5) $\mathrm{Ram}(a, b) \leq \mathrm{Ram}(a - 1, b) + \mathrm{Ram}(a, b - 1)$.

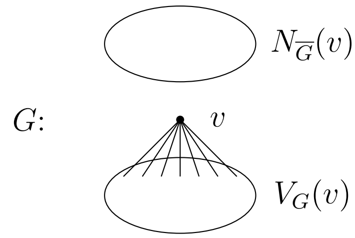


FIGURE 1.3.1. Ramsey's Theorem

(6) (+) $\mathrm{Ram}(3,4) = 9$, (tricky, see Proposition 8).
(7) $\mathrm{Ram}(4,4) \leq 18$.
(8) (+) $\mathrm{Ram}(4,4) = 18$.

It is known that $\mathrm{Ram}(4,5) = 25$ and $43 \leq \mathrm{Ram}(5,5) \leq 49$.

THEOREM 4 (General Ramsey Theorem). *For all $r, c, a_1, \dots, a_c \in \mathbb{Z}^+$ there exists $n \in \mathbb{Z}^+$ such that for all sets V with $|V| \geq n$ and functions $f : \binom{V}{r} \to [c]$ there exist $i \in [c]$ and $H \subseteq V$ with $|H| = a_i$ such that $f(S) = i$ for all $S \in \binom{V}{r}$.*

1.4. Graph Isomorphism and the Reconstruction Conjecture

In order to study graph theory we need to know when two graphs are, for all practical purposes, the same.

DEFINITION 5. Two graphs G and H are *isomorphic* if there exists a bijection

$$f : V(G) \to V(H) \text{ such that for all } x, y \in V(G), xy \in E(G) \text{ iff } f(x)f(y) \in E(H).$$

In this case we say that f is an isomorphism from G to H and write $G \cong H$. The isomorphism relation is an equivalence relation on the class of graphs. The equivalence classes of this relation are called *isomorphism types.* In graph theory we generally do not differentiate between two isomorphic graphs. We say that H is a *copy* of G to mean that $G \cong H$.

HW 3. Let $f : V(G) \to V(H)$ be an isomorphism between two graphs G and H. Prove carefully that $d_G(v) = d_H(f(v))$ for all $v \in V(G)$.

HW 4. Here we *do* distinguish between isomorphic graphs. Let $V = \{v, w, x, y, z\}$ be a set of five vertices, and $\mathcal{G} = \{G : G \text{ is a graph with } V(G) = V \text{ and } \|G\| = 4\}$. Determine (with proof) $|\mathcal{G}|$ and the number of isomorphism types of $\mathcal{G}$.

If x is a vertex of a graph G then $G - x$ is the induced subgraph $G[V(G) - x]$. The graph $G - x$ is called a vertex deleted subgraph of G.

DEFINITION 6. A *complete set of vertex deleted subgraphs* of a graph $G = (V, E)$ is a set $\mathcal{G}$ such that there exists a bijection $\psi : V \to \mathcal{G}$ with $\psi(x) \cong G - x$ for all $x \in V$. See Figure 1.4.1. *A* complete set of vertex deleted subgraphs $\mathcal{G}$ of G is also called a *deck* of G, and the elements of $\mathcal{G}$ are called cards.

Notice that G has infinitely many decks, but each deck $\mathcal{G}$ of G satisfies $|\mathcal{G}| = |G|$. We cannot determine $V(G)$ from $\mathcal{G}$, but the following famous conjecture asks whether we can determine the isomorphism type of G from $\mathcal{G}$.

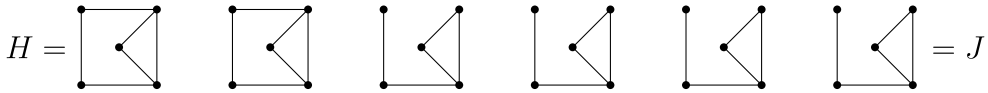


FIGURE 1.4.1. The vertex deleted subgraphs of a graph G. What is $|G|$? What is $\|G\|$? What is the isomorphism type of G?

CONJECTURE 7 (Reconstruction Conjecture 1.3.12). *If two graphs have the same complete set of vertex deleted subgraphs then they are isomorphic.*

For a graph $G = (V, E)$ and pair $(v, e) \in V \times E$, set

$$\iota(v,e) := \begin{cases} 1 & \text{if } e \in E(v) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 8 (Handshaking 1.3.3.). *Every graph $G := (V, E)$ satisfies $\sum_{v \in V} d(v) = 2\|G\|$. In particular, G has an even number of vertices with odd degree.*

PROOF.

$$\sum_{v\in V} d(v) = \sum_{v\in V}\sum_{e\in E}\iota(v,e) = \sum_{e\in E}\sum_{v\in V}\iota(v,e) = \sum_{e\in E} 2 = 2\|G\|.$$

□

PROPOSITION 9 (1.3.11). *For every graph $G = (V, E)$ with $|G| \geq 3$ and every vertex $v \in V$,*

$$\|G\| = \frac{\sum_{v\in V}\|G-v\|}{|G|-2} \qquad \text{and} \qquad d_G(v) = \|G\| - \|G-v\|.$$

PROOF. Every edge $e \in E$ satisfies $e \in E(G - v)$ if and only if $\iota(v, e) = 0$. Thus

$$\sum_{v\in V}\|G-v\| = \sum_{v\in V}\sum_{e\in E}(1-\iota(v,e)) = \sum_{e\in E}\sum_{v\in V}(1-\iota(v,e)) = \sum_{e\in E}(|G|-2) = \|G\|(|G|-2).$$

So the first equality holds. The second equality follows from $E = E(G - v) \cup E(v)$. □

EXAMPLE 10. Suppose $\mathcal{G}$ is the deck consisting of 2 copies of H and 4 copies of J as shown in Figure 1.4.1. Find (with proof) a graph G such that if the deck of G' is $\mathcal{G}$ then $G' \cong G$.

SOLUTION. Let $G = (V, E)$ be an arbitrary graph for which $\mathcal{G}$ is a deck. Then $|G| = |\mathcal{G}| = 6$. Using Proposition 9, we have:

$$\|G\| = (2\|H\| + 4\|J\|)/(|G|-2) = (2\cdot 6 + 4\cdot 5)/4 = 8.$$

Consider $x \in V$ with $G - x \cong H$. Then $d_G(x) = 8 - \|H\| = 2$. By inspection, $G - x$ has a vertex z with two adjacent neighbors w_1, w_2 on a 4-cycle $w_1w_2w_3w_4w_1$. As J does not contain a 4-cycle, $G - z \ncong J$, and so $G - z \cong H$. Thus $d(z) = 2$. As $\Delta(G) = 3$, we already know $N(z) = \{w_1, w_2\}$, $N(w_1) = \{w_2, w_4, z\}$, and $N(w_2) = \{w_2, w_3, z\}$. So there are only two possibilities for the two neighbors of x, and $N(x) = \{w_3, w_4\}$.

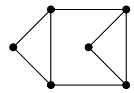


FIGURE 1.4.2. Discovered graph G

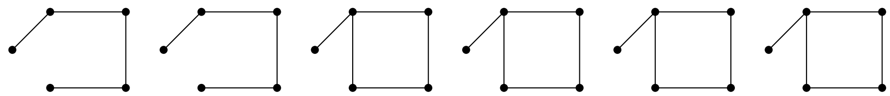
$\mathcal{G} =$ 

FIGURE 1.4.3. HW 6

Now suppose G' is another graph with deck $\mathcal{G}$. The argument above shows that G' has a vertices $x', z', w_1', \dots w_4'$ such that $w_1'w_2'w_3'w_4'w_1'$ is a 4-cycle, $N(z') = \{w_1', w_2'\}$, $N(w_1') = \{w_2', w_4', z'\}$, $N(w_2') = \{w_2', w_3', z'\}$, and $N(x') = \{w_3', w_4'\}$. It follows that

$$x \mapsto x', z \mapsto z', w_1 \mapsto w_1', \dots, w_4 \mapsto w_4'$$

is an isomorphism from G to G'. See Figure 1.4.2. $\square$

HW 5. (*) The *degree sequence* of a graph $G = (V, E)$ is a nondecreasing sequence of integers $d_1, \dots, d_{|G|}$ such that $d_i = d(v_i)$ for all $i \in [|G|]$ and $V = \{v_1, \dots v_{|G|}\}$. For example the degree sequence for the graph in Figure 1.4.2 is $2, 2, 3, 3, 3, 3$. Give a (small) example (with proof) of two graphs that have the same degree sequence, but are not isomorphic.

HW 6. (*) Find a graph G such that the graphs shown in Figure 1.4.3 form a set $\mathcal{G}$ of vertex deleted subgraphs (deck) of G. Prove that if $\mathcal{G}$ is also a set of vertex deleted subgraphs of H then $G \cong H$.

HW 7. (*) A graph is *regular* if all its vertices have the same degree. Prove that if two regular graphs have the same deck then they are isomorphic.

1.5. Some Important Graphs and Graph Constructions

A *path* is a graph $P = (V, E)$ such that V can be ordered as $v_1, \dots, v_{|P|}$ so that $E = \{v_iv_{i+1} : i \in [|G| - 1]\}$. The *length* of the path P is $\|P\|$. A path with only one vertex is possible; such paths are said to be *trivial.* Clearly, any two paths with the same length are isomorphic (HW 11). We use the notation P_n to denote a fixed path of length $n - 1$. Then if P is a path of length $n - 1$, we say that P is a copy of P_n, or more carelessly $P = P_n$. We write $v_1v_2 \dots v_n$ (without commas) to denote a copy of P_n whose edge set is $\{v_iv_{i+1} : i \in [n - 1]\}$. For a path $P = v_1 \dots v_n$ set $v_iP = v_i \dots v_n$, $v_iPv_j = v_i \dots v_j$, and $Pv_j = v_1 \dots v_j$. The vertices v_1 and v_n are called the *ends* of P_n. The other vertices are *internal* vertices. Suppose G is a graph and $X \subseteq V(G)$. A path $P \subseteq G$ is an X-path if its ends, but not its internal vertices, are contained in X. An $\{x, y\}$-path is usually called an x, y-path. The *distance* $d(x, y)$ between x and y is the length of the shortest x, y-path.

A *cycle* is a graph C formed by adding the additional edge v_1v_n to a path $v_1v_2 \dots v_n$ with $n \geq 3$. Again, the length of C is $\|C\|$. Clearly any two cycles with the same length are isomorphic. We use the notation C_n for a fixed cycle of length n. We write $v_1v_2 \dots v_nv_1$ to denote a copy of C_n whose edge set is $\{v_iv_{i\oplus 1} : i \in [n]\}$, where $\oplus$ denotes addition modulo n. The *girth* of a graph G is the length of its shortest cycle C with $C \subseteq G$, if there is one; otherwise the *girth* is infinity. The *circumference* of G is the length of the longest cycle C with $C \subseteq G$, if there is one. Otherwise the circumference is zero.

A *complete graph* is a graph $K = (V, E)$ such that $xy \in E$ for all $x, y \in V$. We use the notation K_n for a fixed complete graph with n vertices. Notice that the vertices of a complete graph are a clique. Then $\overline{K}_n$ is a graph with n vertices and no edges, and the vertices of $\overline{K}_n$ are a coclique. We call $\overline{K}_n$ the *empty graph on n vertices*. Now we introduce some notation not in the text: $K(A, B)$ denotes the graph (V, E) such that $V = A \cup B$ and $E = \{ab : a \neq b \wedge (a, b) \in A \times B\}$. Then $K(A, A)$ denotes a complete graph whose vertex set is A; we abbreviate this by $K(A)$. Finally, for $a, b \in \mathbb{Z}^+$, let $K_{a,b}$ denote a graph of the form $K(A, B)$, where $|A| = a$, $|B| = b$, and $A \cap B = \emptyset$. Such a graph is called a *complete bipartite graph.* (We will have more to say about bipartite graphs shortly.)

Let $G = (V, E)$ and $H = (W, F)$ be graphs. Define the *sum* of G and H by

$$G + H := (V \cup W), E \cup F)$$

and the *join* of G and H by

$$G \vee H := (G + H) + K(V, W).$$

The k-th power of G is the graph $G^k = (V, E^k)$, where $E^k = \{uv : d(u, v) \leq k\}$.

The *Petersen* graph has the form $(\binom{[5]}{2}, \{AB : A \cap B = \emptyset, A, B \in \binom{[5]}{2}\})$.

HW 8. (*) Prove that every graph G with $|G| < \|G\|$ contains a P_4.

DEFINITION 11. A decomposition of a graph G is a set of subgraphs such that each edge of G appears in exactly one subgraph of the set.

EXAMPLE 12. K_4 can be decomposed into two P_4's; it can also be decomposed into three P_3's, and into $K_3, K_{1,3}$.

HW 9. Prove that K_n decomposes into three isomorphic subgraphs if and only if $n + 1$ is not divisible by 3.

1.6. Connection in graphs

DEFINITION 13 (1.2.2). A *walk* (in a graph G) is a sequence (list) $W = v_1v_2 \dots v_n$ of not necessarily distinct vertices such that v_iv_{i+1} is an edge for each $i \in [n-1]$. If the vertices of W are distinct then W is just a path. If $v_1 = v_n$ then W is *closed*; otherwise it is *open.* If W is open then v_1 and v_n are its *ends*, and all other vertices are *internal.* If all the edges v_iv_{i+1} are distinct then W is a *trail.* The *length* of W is $n - 1$. The walk W is a u, v-walk (trail, path) if $u = v_1$ and $v = v_n$. The trivial walk v_1 of length 0 is closed. We say that W contains a walk W' if there exists a subsequence $W' = v_{i_1}v_{i_2} \dots v_{i_s}$ such that W' is a walk, and each edge $v_{i_h}v_{i_{h+1}}$ of W has the form v_jv_{j+1} for some $j \in [n-1]$. We use the notation Wv_j, v_iWv_j, v_iW to indicate the *subwalks* $v_1 \dots v_j, v_i \dots v_j, v_i \dots v_n$. *Also* $W^* := v_nv_{v-1} \dots v_1$.

LEMMA 14 (1.2.5). *Every u, v-walk $W = v_1 \dots v_n$ contains a u, v-path.*

PROOF. We argue by induction on the length l of W. If W is a path then we are done. Otherwise, there exist $i, j \in [n]$ with $i < j$ such that $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1}$ and $W' = v_1 \dots v_i v_{j+1} \dots v_n$ is a shorter u, v-walk contained in W. By induction W' contains a u, v-path P, and P is also contained in W. $\square$

DEFINITION 15 (1.2.6). Let $G = (V, E)$ be a graph. Vertices $x, y \in V$ are *connected* (even if $xy \notin E$) if there is a u, v-walk. The graph G is *connected* if all vertices x and y are connected. The *connection relation* is the set of ordered pairs (u, v) of G with u and v connected.

PROPOSITION 16. *The connection relation is an equivalence relation.*

HW 10. (-) Let G be a graph with $x, y, z \in V(G)$. Prove that if G contains an x, y-path and a y, z-path then it contains an x, z-path. Be careful; it is not completely trivial.

DEFINITION 17 (1.2.8). A *component* of G is a subgraph $H = G[X]$ induced by an equivalence class X of the connection relation.

HW 11. $(+)$ Prove that if two connected graphs G and H on n vertices both have degree sequence $1, 1, 2, \dots, 2$ (two ones followed by $n - 2$ twos) then they are isomorphic.

HW 12. (*) Let $P \subseteq G$ be an x, y-path. Prove that $G[P]$ contains an x, y-path Q with $Q = G[Q]$.

HW 13. (*) Prove that any two paths P and Q with maximum length in a connected graph have a common vertex.

1.7. Bipartite graphs

DEFINITION 18. Let $E(A, B)$ denote the set of edges with one end in A and one end in B, and abbreviate $E(\{x\}, B)$ as $E(x, B)$. Let $\|e, B\| = |E(x, B)|$ and $\|A, B\| = \sum_{a \in A} |E(a, B)|$, that is the number of edges in $E(A, B)$ counting those edges with both ends in A twice. A graph $G = (V, E)$ is *bipartite* if it has a *bipartition*, that is a partition of V into one or two independent sets. This means that $E = \emptyset$ or there exists a partition $\{A, B\}$ of V ($V = A \cup B$, $A \cap B = \emptyset$) such that both A and B are independent, or equivalently $E = E(A, B)$. Notice that a graph is bipartite if and only if it is a subgraph of a complete bipartite graph.

Many theorems in graph theory assert the existence of some special structure in a graph—say a bipartition. To show that a particular graph has such a structure it is enough to make a lucky random guess, and check that your guess provides the structure. In general, it is much harder to show that a graph does not have the desired structure. Typically this would require an exhaustive search of exponentially many possibilities—say all $2^{|G|}$ partitions of the vertices into at most two parts. However for some structures we can prove the existence of *obstructions* with the property that every graph either has the structure or it has an obstruction, but not both. In this case, a lucky guess of an obstruction provides a proof that the structure does not exist. Theorem 23 is an example of this phenomenon.

DEFINITION 19. A path, cycle, trail, walk W is *even* (*odd*) if its length is even (odd).

LEMMA 20. *Every odd closed walk $W = v_1 \dots v_n v_1$ contains an odd cycle.*

Proof. Argue by induction on the length of W. If W is a cycle we are done. Otherwise, as $\|W\| \geq 3$, there exist integers $1 \leq i < j \leq n$ with $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1} \in E$ and $W' := v_1 \dots v_i v_{j+1} \dots v_n v_1$ and $W'' := v_i v_{i+1} \dots v_j$ are shorter closed walks, whose lengths sum to the length of W. So one of them must be odd. By the induction hypothesis, the odd one contains an odd cycle, which is also contained in W. □

Observe that if $xy \in E$ then xyx is an even closed walk that does not contain any cycle.

Theorem 21 (1.2.18). *A graph $G = (V, E)$ is bipartite iff it contains no odd cycle.*

Proof. *Necessity.* Suppose G is bipartite with bipartition $\{A, B\}$. It suffices to show that if $C \subseteq G$ is a cycle then it is even. Since G is bipartite, $E(C) \subseteq E \subseteq E(A, B)$. So each edge $e \in E(C)$ has exactly one end in A. Thus the length of C is the even number

$$\|C\| = \sum_{e \in E(C)} \sum_{v \in A \cap V(C)} \iota(v, e) = \sum_{v \in A \cap V(C)} \sum_{e \in E(C)} \iota(v, e) = \sum_{v \in A \cap V(C)} d_C(v) = 2|A \cap V(C)|.$$

Sufficiency. Suppose G contains no odd cycle. It suffices to show that each component of G is bipartite (why?). So consider a component H of G, and let $x \in V(H)$. Set

$$A := \{v \in V(H) : \text{ there exists an odd } x, v\text{-walk in } H\} \text{ and}$$
$$B := \{v \in V(H) : \text{ there exists an even } x, v\text{-walk in } H\}.$$

Since H is a component of G, it is connected, and so $A \cup B = V(H)$. If there exists $v \in A \cap B$ then there exists an odd x, v-walk P and an even x, v-walk Q. Then $W = xPvQ^*x$ is an odd closed walk. By Lemma 20 there exists an odd cycle $C \subseteq W$. Since this is impossible, $A \cap B = \emptyset$. Similarly, if $uv \in E \smallsetminus E(A, B)$ then there exist walks xPu and xQv with the same parity. Thus $W = xPuvQ^*x$ is an odd closed walk; so there exists an odd cycle $C \subseteq W$, another contradiction. We conclude that $\{A, B\}$ is a bipartition of G. □

HW 14. (*) Prove that a 3-regular graph G decomposes into $K_{1,3}$'s if and only if G is bipartite.

HW 15. (*) Prove that a graph G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|H|$ for all $H \subseteq G$.

1.8. Dirac's Theorem

Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. An A, B-*walk* is a walk whose first vertex is in A, whose last vertex is in B and whose interior vertices are in neither A nor B. If $A = \{a\}$ or $B = \{b\}$, we may shorten this notation to an a, B-walk or an A, b-walk. Similarly, if $A = B$, we may shorten it to a B-walk. Also, if $H \subseteq G$ an H-walk is a $V(H)$-walk, etc.

The *minimal degree* of G is $\delta(G) = \min\{d(v) : v \in V\}$. Similarly, the *maximum degree* of G is $\Delta(G) = \max\{d(v) : v \in V\}$. If $\delta(G) = r = \Delta(G)$ then G is *r-regular.*

An *embedding* of H into G is an isomorphism from H to a subgraph of G. If there exists an embedding of H in G then we say that H can be *embedded* in G, or that H is embeddable in G. A subgraph $H \subseteq G$ is said to be a *spanning* subgraph of G if $V(H) = V(G)$. A spanning cycle of G is called a *hamiltonian* cycle. If G contains a hamiltonian cycle, G is said to be *hamiltonian.*

Many questions in graph theory have the following form: Given two graphs G and H with $|H| = |G|$ what "local" conditions on G ensure that H is embeddable in G? If G is

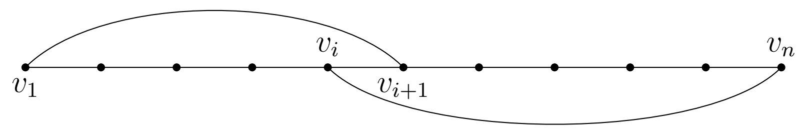


FIGURE 1.8.1. Hamiltonian cycle

complete then trivially H is embeddable in G. This is guaranteed by the local condition $\delta(G) = |G| - 1$, but in many cases we can do much better. Corollaries 23 and 24 below are examples.

We have seen that not only can the question of whether a graph is bipartite be answered positively with proof by a luck guess—the bipartition, it can also be answered negatively with proof by a lucky guess—an odd cycle. The question of whether a graph is hamiltonian can also be answered positively with proof by a lucky guess—the hamiltonian cycle. However there is no guessing method known for deciding with proof that a graph has no hamiltonian cycle, and it is strongly believed that there is no such method.

Intuitively, if a graph has enough edges—for instance if it is complete—then it is hamiltonian. Here are some ways of quantifying what "enough" means.

THEOREM 22. *Every connected graph $G = (V, E)$ with $|G| \geq 3$ contains a path or cycle of length at least $l = \min\{|G|, d(x) + d(y) : xy \notin E\}$.*

PROOF. Let $P = v_1 \dots v_t$ be a path in G with maximum length. We first prove:

$$\|P\| \geq l \text{ or } G \text{ contains a cycle with } V(C) = V(P). \tag{1.8.1}$$

Suppose $\|P\| \leq l - 1$. If G is complete then, $t = |G| \geq 3$. Else G has two nonadjacent, connected vertices x and y. Let Q be an x, y-path. Then $t \geq \|Q\| + 1 \geq 3$. Anyway $t \geq 3$. If $v_1 v_t \in E$ then, as $t \geq 3$, $C := v_1 P v_t v_1$ is the desired cycle. Otherwise, $v_1 v_t \notin E$. So

$$t = \|P\| + 1 \leq l \leq d(v_1) + d(v_t). \tag{1.8.2}$$

Since P is maximum, $N(v_1), N(v_t) \subseteq P - v_t$. Let

$$X = \{i \in [t-1] : v_1 v_{i+1} \in E\} \text{ and } Y = \{i \in [t-1] : v_t v_i \in E\}.$$

Then $|X| = d(v_1)$ and $|Y| = d(v_t)$. By (1.8.2) and inclusion-exclusion:

$$\begin{gathered} t - 1 \geq |X \cup Y| = |X| + |Y| - |X \cap Y| \geq t - |X \cap Y| \\ |X \cap Y| \geq 1. \end{gathered}$$

Let $i \in X \cap Y$ (Figure 1.8.1). Then the cycle $C = v_1 v_{i+1} P v_t v_i P v_1$ spans P, proving (1.8.1).

Since $l \leq |G|$ and $|C| = \|C\|$, it suffices to show $|G| = |C|$. Otherwise, there exists $x \in V \smallsetminus V(C)$. Since G is connected, there is an x, C-path $Q = x \dots u_1$. Choose notation so that $C = u_1 \dots u_t u_1$. Then $P' = x Q u_1 u_2 \dots u_t$ is a longer path than P, a contradiction. $\square$

COROLLARY 23 (Dirac's Theorem (1952) 7.2.8). *If $\delta(G) \geq \frac{1}{2}|G| > 1$ then G is hamiltonian.*

Proof. By Theorem 22, it suffices to prove that G is connected. Consider any distinct vertices x and y. Then

$$|G| \geq |N[x] \cup N[y]| = |N[x]| + |N[y]| - |N[x] \cap N[y]| \geq |G| + 2 - |N[x] \cap N[y]|$$
$$|N[x] \cap N[y]| \geq 2.$$

So x is connected to y by a path of length at most 2. □

Here is a weaker, but slightly less local condition that ensures a graph is hamiltonian.

Corollary 24 (Ore's Theorem 7.2.9 (1960)). *If G is a graph with $|G| \geq 3$ and $d(x) + d(y) \geq |G|$ for all distinct nonadjacent vertices x and y then G is hamiltonian.*

Proof. As in the proof of Corollary 23, G is connected, since for distinct vertices x and y either $xy \in E$ or $|N[x]| + |N[y]| \geq |G| + 2$. So we are done by Theorem 22. □

Pósa (when he was in high school) posed a conjecture extending Dirac's Theorem:

Conjecture 25 (Pósa 1963). *If $\delta(G) \geq \frac{2}{3}|G|$ then G contains the square (2-power) of a hamiltonian cycle.*

The next Theorem answers a related question.

Theorem 26 (Genghua Fan & Kierstead 1996). *If $\delta(G) \geq \frac{2|G|-1}{3}$ then G contains the square of a hamiltonian path. The degree condition is best possible.*

The next conjecture generalizes Corollary 23 and Conjecture 25.

Conjecture 27 (Seymour 1974). *If $\delta(G) \geq \frac{k}{k+1}|G|$ then G contains the k-th power of a hamiltonian cycle.*

Seymour's Conjecture was proved for sufficiently large graphs (more vertices than the number of electrons in the known universe when $k \geq 3$).

Theorem 28 (Komlós, Sárközy & Szemerédi 1998). *For every integer k there exists an integer n such that Conjecture 27 is true for graphs G with $|G| \geq n$.*

The next theorem improves the bound on n when $k = 3$.

Theorem 29 (Chau, DeBiasio & Kierstead 2011). *Conjecture 25 is true for graphs G with $|G| \geq 2 \times 10^8$.*

HW 16. Prove that $K_{a,a-1}$ is not hamiltonian for any integer $a > 1$. More generally, prove that if $\alpha(G) > \frac{1}{2}|G|$ then G is not hamiltonian. Determine $\delta(K_{a,a-1})$.

HW 17. For disjoint sets $A, B, \{v\}$, let $G = \overline{K}(A, B) + K(v, A \cup B)$. Prove that G is not hamiltonian. Determine $\delta(G)$ when $|A| = |B|$.

HW 18. Let $P = v_1 \dots v_t \subseteq G$ be a maximum path and $x \in V(G - P)$. Prove that $\|\{v_t, x\}, P\| \leq t - 1$.

HW 19. Prove that if $\delta(G) \geq \frac{|G|-1}{2}$ then G has a hamiltonian path.

HW 20. (+) Let G be an X, Y-bigraph with $|X| = |Y| = k$ and $\delta(G) \geq \frac{k+1}{2} \geq 2$. Prove that G contains a hamiltonian cycle. [Hint: First prove that if G contains a maximal path $P = P_t$ then $G[P]$ contains a cycle of length at least $2\lfloor \frac{t}{2} \rfloor$. Then show that G contains a hamiltonian path.] For all k give an example to show that the bound on the minimum degree cannot be lowered.

1.9. Even graphs and Euler's Theorem

Sometimes when proving a statement by induction it is easier to prove a stronger statement. This phenomenon is called the *inventors paradox.* The reason this is possible is that while more must be proved, the induction hypothesis provides more to base an argument on. In this section we see an elementary example of this. It is easier to prove our result for multigraphs. For this we need to revise our definition of the degree of a vertex.

DEFINITION 30. An edge of a multigraph is called a link if it has two distinct ends, and a loop if both ends are the same vertex. The *degree* $d(v)$ of a vertex of a multigraph is the number of links incident to v plus twice the number of loops incident to v. In a multigraph a cycle is there may be cycles of length 1—one loop—and cycles of length 2—two links between the same two vertices.

This definition is designed so that each edge is counted twice when you sum the degrees of a graph, and so Lemma 8 is also true for multigraphs.

DEFINITION 31. A multigraph is *eulerian* if it has a closed trail containing all edges. (Note that $T = v$ is closed, since its only vertex is its first and its last.) Such a trail is said to be an eulerian trail. A multigraph is *even if every* vertex has even degree.

For $H \subseteq G$ and $v \in V(G) \setminus V(H)$, set $d_H(v) := 0$.

FACT 32. *If H and G are even graphs with $H \subseteq G$ then $H' := G - E(H)$ is even.*

PROOF. Since G and H are even, every $v \in V(G)$ satisfies

$$d_{H'}(v) = d_G(v) - d_H(v) \equiv 0 \mod 2.$$

$\square$

PROPOSITION 33. *Let $T = v_1...v_t$ be a trail in a multigraph $G = (V, E)$. Then $d_T(v)$ is even for every vertex v, except that if T is open then $d_T(v_1)$ and $d(v_t)$ are odd.*

PROOF. If T is closed, let $T' = T$ and $G' = G$; if T is open and $v_nv_1 \notin E(T)$, let $T' := Tv_nv_1$ and $G' = G$; if T is open and $v_nv_1 \in E(T)$ then let v_nv_1 be a new edge, $T' = Tv_nv_1$ and $G' = G + v_nv_1$. Regardless, T' is closed. It suffices to show that $d_{T'}(v_i)$ is even for all $v \in V$, since $d_T(v_i) \not\equiv d_{T'}(v_i) \mod 2$ if and only if $v_i \in \{v_1, v_n\}$ and $v_1 \neq v_n$. Argue by induction on t that this is true for all multigraphs and all closed trails of length at most t.

If T' is a cycle, or $t = 1$, then $d_{T'}(v_i) \in \{0, 2\}$ for every $v \in V(G)$. Otherwise, there exist $1 < i < j \leq t$ with $v_i = v_j$. Let $T_1 = v_1T'v_iv_{j+1}T'v_1$ and $T_2 = v_iT'v_j(= v_i)$. Then T_1 and T_2 are both closed trails, shorter than T', and every edge of T' is in exactly one of T_1 and T_2. By the induction applied to T_1 and T_2,

$$d_{T'}(v) = d_{T_1}(v) + d_{T_2}(v) \equiv 0 + 0 \equiv 0 \mod 2.$$

$\square$

THEOREM 34 (Euler (1736) 1.2.26). *A multigraph G is eulerian iff it has at most one nontrivial component and it is even.*

PROOF. *Necessity.* Suppose G has a Eulerian trail T. Since T is connected it only contains edges from one component. Since T contains all edges, G has only one nontrivial component. Since T is closed and contains all edges of G, Proposition 33 implies every vertex of G has even degree (possibly 0).

Sufficiency. Suppose that G has at most one nontrivial component H and every vertex has even degree. Let $T = v_1 \dots v_t$ be a maximum length trail in G. Then T is closed: Otherwise v_t is incident to an odd number of edges of T by Proposition 33. Since $d(v_t)$ is even it is incident to some edge $v_t v$ that is not in T. So we can extend T to $T^+ = v_1 T v_t v$, contradicting the maximality of T.

It remains to show that $E(H) \subseteq E(T)$. Otherwise there is an edge $ab \in E(H) \smallsetminus E(T)$. Since H is connected there is an $\{a, b\}, T$-path P with no edge in T. Choose notation so that $P = b \dots v_i$. By definition, $a \notin V(P)$ (but maybe $b = v_i$). Since T is closed, $T^+ = abPv_iTv_i$ is a longer trail than T, a contradiction. □

LEMMA 35. *Every graph G with $\delta(G) \geq 2$ contains a cycle.*

PROOF. Let $P = v_1 \dots v_t$ be a maximum path in G. Then $N(v_t) \subseteq V(P)$. So there exist $i < t - 1$ such that $v_t v_i \in E(G)$. Thus $v_i P v_t v_i$ is a cycle contained in G. □

COROLLARY 36 (1.2.25). *If G is an even graph with $\|G\| > 0$ then G contains a cycle.*

PROOF. Some component H of G contains an edge. Since H is connected, $\delta(H) \geq 1$. Since G is even this can be strengthened to $\delta(H) \geq 2$. So by Lemma 35, $H \subseteq G$ contains a cycle. □

SECOND PROOF OF THEOREM 34 (SUFFICIENCY). Suppose G is even and has at most one nontrivial component G'. We argue by induction on $\|G'\|$. If G' is a cycle or $\|G'\| = 0$, then the cycle or any vertex is the Eulerian trail.

Otherwise, by Corollary 36, G' contains a cycle C. Let H be a nontrivial component of $G' - E(C)$ (maybe $H = G' - E(C)$), and set $H' = G' - E(H)$. Both H and H' are even. Also H' is connected, since all components of $G' - E(C)$ that are contained in H' are connected to each other in H' by edges of C. Moreover, $\|H\| \leq \|G'\| - \|C\| < \|G'\|$ and $\|H'\| = \|G'\| - \|H\| < \|G'\|$. So H and H' are nonempty even connected graphs with $\|G'\| = \|H\| + \|H'\|$. By the induction hypothesis H and H' contain Eulerian trails T and T'. Moreover, T contains a vertex $v_1 \in C$ and T' contains all vertices of C. Choose notation so that $T = v_1 \dots v_n v_1$ and $T' = v_1 u_2 \dots u_m v_1$. Then $v_1 T v_n v_1 T' u_m v_1$ is an Eulerian trail in G', and G. □

THEOREM 37 (1.2.33). *A connected graph G with exactly q vertices of odd degree decomposes into* $\max\{1, \frac{q}{2}\}$ *trails.*

PROOF. By Lemma 8, q is even. Let G^+ be the result of adding a new vertex v^+ to G so that $N(v^+)$ is the set of vertices with odd degree in G. Since q is even, and every $v \in V(G)$ satisfies $d_{G^+}(v) \equiv d_G(v) + 1 \mod 2$ if and only if $d_G(v)$ is odd, G^+ is even. By Theorem 34, G^+ has an Eulerian Trail T. Removing v^+ partitions T into $\frac{q}{2}$ trails that decompose G. □

Alternatively, we could have proved Theorem 37 by adding q edges connecting disjoint pairs of odd degree vertices.

CHAPTER 2

Cut-vertices, -edges and trees

Definition 38 (1.2.12.). A *cut-vertex* is a vertex in a graph G is a vertex such that $G - v$ has more components than G. Similarly, a *cut-edge* is an edge of G such that $G - e$ has more components than G.

Notice that a vertex v is a cut vertex of G if and only if $H - v$ is not connected, where H is the component of G (maybe $H = G$) containing v. Similarly an edge e is a cut-edge of G if and only if $H - e$ is not connected, where H is the component of G containing e. So G is not a cut-edge if its ends are connected in $G - e$.

Theorem 39 (1.2.14.). *An edge $e = xy$ in G is not a cut-edge iff it belongs to a cycle.*

Proof. First suppose e is not a cut edge. Then there exists an x, y-path P in $G - e$. So $xPyx$ is a cycle in G. Now suppose e is on a cycle C in G. Then $x(C - e)y$ is a path connecting the ends of e, and so e is not a cut-edge. □

Theorem 40. *The ends of a maximal path $P \subseteq G$ with $\|P\| \geq 1$ are not cut-vertices of G.*

Proof. Let x be an end of P and $y \in V(P) - x$. By maximality, $N[x] \subseteq P$. Suppose u is a vertex in the component H of G containing x. It suffices to show that there exists a u, y-walk W in $H - v_1$. Since H is connected, there exists a u, y walk W' in H. If $x \notin W$ then set $W = W'$. Otherwise the predecessor of x on W is a vertex $v \in N(v_1) \subseteq P$; set $W = uW'vPy$. □

Definition 41. A graph is *acyclic* if it contains no cycle. Acyclic graphs are also called *forests*. A connected acyclic graph is called a *tree*. A *leaf* is a vertex v with $d(v) = 1$. We say that a graph G satisfies (A) if it is acyclic, (C) if it is connected, and (E), if $|G| = \|G\| + 1$.

Lemma 42. *A graph G with $\|G\| \geq 1$ has at least two leaves if it satisfies (A) or both (C) and (E).*

Proof. First suppose that G is acyclic. Let $P = v_1 \dots v_t$ be a maximum path in G. Since G has an edge, $v_1 \neq v_t$. Since P is maximum and acyclic $N(v_1) = \{v_2\}$ and $N(v_t) = \{v_{t-1}\}$. So v_1 and v_t are distinct leaves.

Now suppose that G satisfies (C) and (E). Let L be the set of leaves in G. Since G is connected and has an edge, $\delta(G) \geq 1$. Since G satisfies (E),

$$2|G| - |L| \leq \sum_{v \in V(G)} d(v) = 2\|G\| = 2|G| - 2.$$

So $|L| \geq 2$. □

Lemma 43. *Suppose G is a graph with a leaf l and $G' = G - l$. Then each condition (A), (C), (E) is satisfied by G iff it is satisfied by G'.*

Proof. Suppose G is acyclic. Since removing a vertex cannot create a cycle G' is acyclic. Now suppose G' is acyclic. Since every vertex in a cycle has degree 2, adding a leaf l cannot create a cycle, and so G is acyclic.

Suppose G' is connected. Since l has a neighbor in $V(G')$, G is connected. Now suppose G is connected. Since $d(l) = 1$, there is a maximal path P with an end l. Thus l is not a cut-vertex, and so G' is connected.

Since $|G| = |G'| + 1$ and $\|G\| = \|G'\| + 1$, G satisfies (E) iff G' does. □

Theorem 44 (2.1.4). *If a graph G satisfies at least two of the conditions (A), (C), and (E) then it satisfies all three.*

Proof. Argue by induction on $|G|$.
Base Step: $|G| = 1$. By inspection, G satisfies all of (A), (C) and (E).
Induction Step: $|G| \geq 2$. Since G satisfies (C) or (E), it has an edge. Since G satisfies (A) or both (C) and (E), it has a leaf l by Lemma 42. Let $G' = G - l$. By Lemma 43, G' satisfies the same two conditions that G does. By the induction hypothesis G' satisfies all three conditions. By Lemma 43, G does also. □

For emphasis, we state that Theorem 44 implies every tree T satisfies $|T| = \|T\| + 1$.

Theorem 45 (2.1.4). *G is a tree iff there is exactly one path between any two vertices.*

Proof. By Lemma 14, G is connected if and only if there is a path between any two vertices. So aruing by contraposition, it suffices to prove: if G has a cycle then (D), where (D) is the staement that G has distinct vertices x and y with distinct x, y-paths; if (D) then G has a non-cut edge; if G has a non-cut-edge then G has a cycle.

Suppose G has a cycle C with edge xy. Then xy and $x(C - xy)y$ are distinct x, y-paths. Suppose P and Q are distinct x, y-paths. Then there exists $xy \in E(P) \triangle E(Q)$. Then xy is a non-cu-edge. By Theorem 39, if xy is a non-cut-edge then G contains a cycle. □

A spanning subgraph of G that is a tree is called a *spanning tree* of G.

Corollary 46 (2.1.5). *Let $T = (V, E)$ be a tree. Then*

(1) Removing an edge disconnects T.
(2) Adding an edge $xy \notin E$ with $x, y \in V$ to T creates a unique cycle.
(3) Every connected graph G contains a spanning tree.

Proof. (1) Since T is acyclic, every edge is a cut edge by Theorem 39.

(2) By Theorem 45 there is a *unique* x, y-path $P \subseteq T$. So $P + xy$ is a cycle in $T' := T + xy$. Consider any cycle $D \subseteq T'$. Since T is acyclic, $xy \in E(D)$. By the uniqueness of P, $D - xy = P$; so $D = P + xy$.

(3) Let F be a connected spanning subgraph of G with as few edges as possible. It exists because G is a candidate. By the minimality of F, every edge of F is a cut-edge. By Theorem 39, no edge of F is on a cycle; so F is acyclic. Thus F is a spanning tree of G.

(3, another proof) Let F be an acyclic spanning subgraph of G with as many edges as possible. It exists because the empty spanning subgraph is a candidate. By definition F is spanning and acyclic. It remains to show that it is connected. Since G is connected, it suffices to show that for any edge $uv \in E(G - F)$ there exists a u, v-path in F. By the maximality of F, $F + uv$ contains a cycle C. Since F is acyclic, $uv \in C$. Thus $u(C - uv)v$ is a uv-path in F.

(3, a third proof) Let F be a maximum subtree of G, i.e., a connected, acyclic subgraph with as many vertices as possible. It suffices to show that F is spanning. Otherwise, there exists $v_1 \in V(G-F)$. Since G is connected there exists a v_1, F-path $v_1 \dots v_t$. Then $v_t \in V(F)$ and $v_{t-1} \notin V(F)$. So $F^+ = F + v_{t-1} + v_t v_{t-1}$ is a graph with a leaf v_{t-1}. By Lemma 43, F^+ is a tree. Since $F^+ \subseteq G$ and $|F^+| > |F|$, it contradicts the maximality of F.

(3, a fourth proof) Argue by induction on $|G|$. If $|G| = 1$ then G itself is a tree. So suppose $|G| > 1$. Let v be a non-cut-vertex. It exists by Lemma 40. Then $G' := G - v$ is connected. By the induction hypothesis it contains a spanning tree F'. Let $e \in E(v)$. It exists because G is connected and $|G| \geq 2$. Then $T := T' + v + e$ is a connected, spanning subgraph of G that satisfies $|T| = \|T\| + 1$, and so T is a spanning tree of G by Theorem 44. □

PROPOSITION 47 (2.1.5). *Suppose T and T' are spanning trees of a graph G. Then for every $e = ab \in E(T) \smallsetminus E(T')$*

(1) there exists $e' \in E(T') \smallsetminus E(T)$ such that $T - e + e'$ is a spanning tree of G; and
(2) there exists $e' \in E(T') \smallsetminus E(T)$ such that $T' + e - e'$ is a spanning tree of G.

PROOF. (1) Since T' is connected, it has an a, b-path $P := (a =)v_1 \dots v_t(= b)$. Let i be the least index such that there is no a, v_i path in $T - ab$. Since ab is the unique ab path in T, t is a candidate for i, and so i exists. Also $i \neq 1$. Set $e' = v_{i-1}v_i$. Then $e' \in P' \subseteq T'$ and $e' \notin T - e$. Also $e' \neq e$ as $e \notin T'$. Since e' is a cut-edge in $T^* := T - e + e'$, it is not contained in a cycle of T^*. Also T is acyclic. So T^* is acyclic. Thus, since $|T^*| = |T| = \|T\| + 1 = \|T^*\| + 1$, Theorem 44 implies T^* is a tree.

(2) By Corollary 46(2), $T' + e$ contains a unique cycle C. Since T is acyclic, $C \nsubseteq T$; let $e' \in E(C - T) \subseteq E(T')$. Then $T^* := T' + e - e'$ is acyclic, since e' is an edge of the unique cycle in $T' + e$. By Theorem 44, T^* is a tree, since $|T^*| = |T| = \|T\| + 1 = \|T^*\| + 1$. □

PROPOSITION 48. *If T is a tree with k edges and G is a nontrivial graph with $\delta(G) \geq k$ then G contains a copy of T, i.e., a subgraph isomorphic to T.*

PROOF. Argue by induction on k.
Base Step: $k = 0$. Then $T \cong K_1$ so $T \cong G[\{v\}]$ for any vertex v.
Induction Step: $k > 1$. Let l be a leaf of T. Then $T' := T - l$ is a tree with $\|T'\| = k - 1$. By the induction hypothesis there exists $H' \subseteq G$ with $H' \cong T'$. Let p be the unique neighbor of l in T, and let x be the image of p in H'. Since $|H'| = \|H'\| + 1 = k$ and x is not adjacent to itself, x has at most $k - 1$ neighbors in H'. Since $\delta(G) \geq k$, there exists $y \in N_G(x) \smallsetminus V(H')$. Set $H = H' + y + xy$. Then $H \subseteq G$ and we can extend the isomorphism between T' and H' to an isomorphism between T and H by mapping l to y. □

HW 21. (+) Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ iff $\sum d_i = 2n - 2$.

HW 22. (*) Every tree is bipartite. Prove that every tree has a leaf in its larger partite set, and in both partite sets if they have equal size.

HW 23. (*) Let T be a tree such that $2k$ of its vertices have odd degree. Prove that T decomposes into k paths. Is it easier to prove this for forests?

HW 24. (+) Let T be a tree with even order. Prove that T has exactly one spanning subgraph such that every vertex has odd degree.

HW 25. (+) A *root* of a graph G is a special vertex r. A spanning tree T of a graph with root r is normal if every edge $xy \in E(G)$ satisfies either $x \in rTy$ or $y \in rTx$. Prove that every connected graph with root r has a normal spanning tree [Hint: Prove the stronger statement that every path P with end r is contained in a normal spanning tree.]

HW 26. (+) Let $\mathcal{T}$ be a set of subtrees of a tree G such that $T \cap T' \neq \emptyset$ for all $T, T' \in \mathcal{T}$. Prove that $\bigcap \mathcal{T} \neq \emptyset$.

HW 27. (*) A *directed graph* is a binary relation $G = (V, E)$. So the edges are *ordered* pairs. We still write xy for the directed edge (x, y). Let $E_G^+(x) := \{xy \in E\}$, $d_G^+(x) := |E_G^+(x)|$, and $\delta^+(G) := \min_{v \in V} d_G^+(v)\}$. A *directed cycle* $C = (V, E)$ is a digraph with the form $V = \{v_1, \ldots, v_s\}$ and $E = \{v_i v_{i \oplus 1} : i \in [s]\}$. This directed cycle is also denoted by $v_1 \ldots v_s v_1$ when it is clear form the context that the edges are directed.

Prove that a directed graph G has a directed cycle if $\delta^+(G) \geq 1$. [Hint: You might want to define the notion of a directed path.]

CHAPTER 3

Matchings

Definition 49. A *matching* is a set of edges with no common ends. A *maximal* matching is a matching that cannot be enlarged by adding an edge. A *maximum* matching is matching with maximum size among all matchings in the graph. A vertex is said to be M-*saturated* if and only if it is the end of an edge in M; otherwise it is M-*unsaturated*, and a set of vertices X is said to be M-*saturated* if every $x \in X$ is M-saturated. The matching M is *perfect* if every vertex is M-saturated

Definition 50. Given a matching M in a graph $G = (V, E)$, an M-*alternating* path is a path P such that each vertex $v \in V(P)$ is incident to at most one edge in $E(P) \smallsetminus M$. Such a path is M-*augmenting* if its ends are not M-saturated.

Theorem 51 (3.1.10 Berge). *A matching M in a graph $G = (V, E)$ is not maximum in G iff G has an M-augmenting path.*

Proof. Suppose P is an M-augmenting path. Then

$$M' = M \triangle E(P) =_{def} (M \smallsetminus E(P)) \cup (E(P) \smallsetminus M)$$

is a larger matching.

Now suppose M is not maximum. Choose a maximum matching M'. Let H be the spanning subgraph with edge set $M \triangle M'$. Then $M \smallsetminus E(H) = M' \smallsetminus E(H)$. Since each vertex is incident to at most one edge of each matching, $\Delta(H) \leq 2$, and the components of H are alternating paths and even cycles. Since $|M| < |M'|$,

$$|M \cap E(H)| = |M| - |M \smallsetminus E(H)| < |M'| - |M' \smallsetminus E(H)| = |M' \cap E(H)|.$$

So one component of H has more edges from M' than M. Such a component must be an M-augmenting path. □

HW 28. (*) Two players Alice and Bob play a game on a graph G. Alice begins the game by choosing any vertex. All other plays consist of the player, whose turn it is, choosing an unchosen vertex that is joined to the last chosen vertex. The winner is the last player to play legally. Prove that Alice has a winning strategy if G has no perfect matching, and Bob has a winning strategy if it does.

3.1. Bipartite matching

A bipartite G with bipartition $\{X, Y\}$ is called an X, Y-bigraph. For $S \subseteq X$ set $N(S) := \bigcup_{v \in S} N(v)$. For a function $f : A \to B$ and $S \subseteq A$, let $f(S) := \{y \in B : \exists x \in S(f(x) = y)\}$ be the range of f restricted to S.

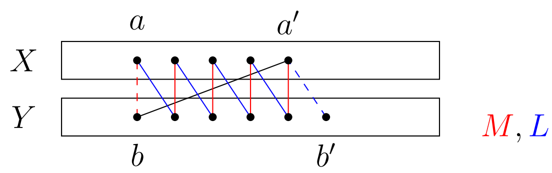


FIGURE 3.1.1. X, Y-bigraph $H \subseteq G$

THEOREM 52 (3.1.11 Hall's Theorem [1935]). *An X, Y-bigraph G has a matching that saturates X iff*

$$|S| \leq |N(S)| \text{ for all } S \subseteq X. \tag{3.1.1}$$

PROOF. If M is a matching saturating X and $S \subseteq X$ then $|S| = |E(S, Y) \cap M| \leq |N(S)|$; so (3.1.1) holds.

Suppose (3.1.1) holds for some X, Y-bigraph with no matching saturating X; among such graphs choose G with $|G|$ minimal, and subject to this $\|G\|$ maximal. By minimality, (1) $N(X) = Y$. Also (2) all $a \in X$ satisfy $N(a) \neq Y$: else, since $G - a$ has a matching saturating $X - a$ and there is an unsaturated vertex $b \in Y$ by (3.1.1), G has a matching saturating X.

Let $a \in X$; by (2) there is $b \in Y$ with $ab \notin E$. Since $G + ab$ satisfies (3.1.1), maximality implies $G + ab$ has a matching M^+ saturating X. By (1) there is $a' \in X$ with $a'b \in E$; by (2) there is $b' \in Y$ with $a'b' \notin E$. Again by maximality, $G + a'b'$ has a matching L^+ saturating X. Set $M := M^+ - ab$ and $L := L^+ - a'b'$; let H be the spanning subgraph of G with edge set $M \triangle L + a'b$ (Figure 3.1.1). Then $\Delta(H) \leq 2$, a is the unique vertex in X with $d_H(a) = 1$, and $|E(v) \smallsetminus M| \leq 1$ if $v \neq b$. Thus the component of H containing a is an a, v-path P with $v \in Y$. Either P or aPb, if $b \in P$, is an M-augmenting path; as M already saturates $X - a$, G has a matching saturating X. □

A graph (or multigraph) is k-regular if every vertex has degree k. Recall that the degree of a vertex in a multigraph is the number (counting multiplicities) of edges incident to it. A *perfect matching* in a graph G is a matching such that there are no M-unsaturated vertices in G. A 1-factor in G is a 1-regular spanning subgraph.

COROLLARY 53 (3.1.13). *Every k-regular bipartite multigraph G has a perfect matching.*

PROOF. Suppose G is an k-regular X, Y-bimultigraph. Then

$$k|X| = |E(X, Y)| = k|Y|.$$

It follows that $|X| = |Y|$. Thus it suffices to show that G has a matching that saturates X. By Hall's Theorem, it suffices to check 3.1.1. Consider any subset $S \subseteq X$. Then

$$k|S| = |E(S, Y)| = |E(S, N(S))| \leq |E(X, N(S)| = k|N(S)|.$$

So $|S| \leq |N(S)|$. □

HW 29. (*) Let $\mathcal{S} = \{S_i : i \in I\}$ be a family of sets. A *system of distinct representatives* (sdr) for $\mathcal{S}$ is a sequence $(a_i : i \in I)$ of distinct elements such that $a_i \in A_i$ for all $i \in I$. Prove that if $\mathcal{S}$ is finite then $\mathcal{S}$ has an sdr if and only if $|J| \leq |\bigcup_{j \in J} S_j|$ for all $J \subseteq I$.

HW 30. (*) Prove that there is an injection $f : \binom{[2k+1]}{k+1} \to \binom{[2k+1]}{k}$ such that $f(S) \subseteq S$ for all $S \in \binom{[2k+1]}{k+1}$.

HW 31. (*) Let $\mathcal{P} = \{P_1, \dots, P_t\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_t\}$ be two partitions of a set S into t subsets of size k. Prove that $\mathcal{P}$ and $\mathcal{Q}$ have a common sdr, that is there exists an sdr $(a_i : i \in [t])$ of $\mathcal{P}$ a permutation $\sigma : [t] \to [t]$ such that $(a_{\sigma(i)} : i \in [t])$ is an sdr of $\mathcal{Q}$.

HW 32. (+) Suppose G is an X, Y-bigraph with $\delta(G) \geq 1$ such that every edge xy with $x \in X$, satisfies $d(x) \geq d(y)$. Prove that G has a matching that saturates every vertex in X. [Hint: Consider assigning each edge xy with $x \in X$ the weight $w(xy) = \frac{1}{d(x)}$.]

HW 33. (+) For $k, n \in \mathbb{N}$, let G be an A, B-bigraph with $|A| = n = |B|$ such that $\delta(G) \geq k$, and for all $X \subseteq A, Y \subseteq B$, if $|X|, |Y| \geq k$ then $|E(X, Y)| \neq \emptyset$. Prove: G has a perfect matching.

HW 34. (+) Prove that for all partitions $\mathcal{P} = \{P_i : i \in [n]\}$ and $\mathcal{Q} = \{Q_i : i \in [n]\}$ of a region S of area n into parts of area 1 there exists a permutation $\sigma : [n] \to [n]$ such that area $(P_i \cap Q_{\sigma(i)}) \geq f(n)$ for all $i \in [n]$, where

$$f(n) := \begin{cases} \frac{4}{(n+1)^2} & \text{if } n \text{ is odd} \\ \frac{4}{n(n+2)} & \text{if } n \text{ is even} \end{cases}.$$

Also show that the function f is optimal.

HW 35. (*) Prove that every bipartite graph has a matching of size $\frac{\|G\|}{\Delta(G)}$.

Definition 54. A *cover* of a graph G is a subset $Q \subseteq V(G)$ that contains at least one end of every edge.

Let C be an odd cycle with $\|C\| = 2k + 1$. Since C is 2-regular, every m-set $Q \subseteq V(G)$ covers at most $2|Q|$ edges. Thus every vertex cover of C has at least $k + 1$ vertices. On the other hand, every m-matching in C has $2m$ ends; so $m \leq k$.

Theorem 55 (König, Egerváry [1931] 3.1.16). *If $G = (V, E)$ is a graph with a maximum matching M and a minimum cover W then $|M| \leq |W|$; if G is bipartite then $|M| = |W|$.*

Proof. Choose an arbitrary orientation $\overrightarrow{G}$ of G. Since W is a cover, every edge of M is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e) \in e \cap W$, and if $e \subseteq W$ then $g(e)$ is the head of e in $\overrightarrow{G}$. Since M is a matching, no vertex of W can be incident two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose G is an X, Y-bigraph. Let U be the set of unsaturated vertices in X. Set $m = \{(x, y) \in X \times Y : xy \in M\}$. Since the ends of M in X are distinct, m is a function with domain $X \smallsetminus U$. Since the ends of m in Y are distinct, m is an injection.

If $U = \emptyset$ then X is a cover with $|W| \leq |X| = |M| \leq |W|$; so suppose $U \neq \emptyset$. Letting $A \subseteq V(G)$ be the set of ends of alternating paths starting in U, set $S := A \cap X$, $\overline{S} := X \smallsetminus S$, $T := A \cap Y$, and $\overline{T} := Y \smallsetminus T$. Then $U \subseteq S$ (witnessed by trivial paths). Consider any alternating path $P = v_0 \dots v_n$ with $v_0 \in U$. If i is even then $v_i \in S$, and if also $i \neq 0$ then $v_{i-1}v_i \in M$; if i is odd then $v_i \in T$. We first show:

(3.1.2) $\qquad$ (i) $N(S) \subseteq T$ and (ii) $T \subseteq m(S \smallsetminus U)$.

(i) Let $z \in N(S)$; say $wz \in E(S,z)$. Then there is an alternating path $Q = y_0 \dots y_{2k} (= w)$ with $y_0 \in U$ and $y_{2k-1}w \in M$. Either $z \in V(Q)$ or Qwz is an alternating path starting in U. Regardless, $z \in T$, proving (3.1.2.i).

(ii) Let $z \in T$. Then there is an alternating path $P = y_0 \dots y_{2k+1} (= z)$ with $y_0 \in U$ and $y_{2k}z \notin M$. Since M is maximum, G has no augmenting path. So z is saturated; say $zx \in M$. Then Pzx is an alternating path; so $x \in S \smallsetminus U$, and $z \in m(S \smallsetminus U)$, proving (3.1.2.ii).[1]

The set $W = \overline{S} \cup T$ is a vertex cover of G: Suppose $uv \in E = E(X,Y)$. If $v \in V \smallsetminus W$ then $v \in S \cup \overline{T}$. By (3.1.2.1), if $v \in S$ then $u \in T$, and if $v \in \overline{T}$ then $u \in \overline{S}$.

It remains to show that $|W| \leq |M|$. Since $U \subseteq S$, every vertex in $\overline{S}$ is M-saturated. By (3.1.2.ii), every vertex in T is M-saturated. So (a) every vertex of W is M-saturated. Since m is an injection, (3.1.2.ii) implies $M \cap E(\overline{S}, T) = \emptyset$. So (b) no edge of M has both ends in W. Using (a) define a function $f : W \to M$ by $f(w) := e \in E(w) \cap M$ (e is unique, by the definition of matching). By (b) f is an injection. Thus $|W| \leq |M|$. □

3.2. General matching

Notice that if H is a component of a graph G and $|H|$ is odd then G does not have a perfect matching.

DEFINITION 56. Let $\mathcal{C}_G$ be the set of components of the graph G. A component with an odd number of vertices is said to be an *odd component*. Let $\mathcal{O}_G$ be the set of odd components of G and $o(G) = |\mathcal{O}_G|$. The graph G is factor critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$. A set S is matchable into $\mathcal{O}_{G-S}$ if there exists a matching M such that each edge $e \in M$ has one end in S and one end in an odd component, and at most one vertex of each odd component is saturated.

THEOREM 57 (Tutte [1947] 3.3.3). *A graph $G = (V,E)$ has a perfect matching iff*

$$o(G - S) \leq |S| \text{ for all } S \subseteq V. \tag{3.2.1}$$

PROOF. Suppose G has a perfect matching. For all $S \subseteq V$, all odd components of $G - S$ must have vertices matched to distinct vertices in S. So $o(G-S) \leq |S|$.

Now suppose (3.2.1) holds for some graph G_0 with no perfect matching. Among all such graphs, choose one $G = (V,E)$ with $\|G\|$ maximum subject to $|G| = |G_0|$. Adding any edge cannot cause G to violate (3.2.1), so it must create a perfect matching. Applying (3.2.1) to $S = \emptyset$ shows that G has no odd component; so $|G|$ is even. Let $U := \{v \in V : N[v] = V\}$. For a contradiction we show that G has a perfect matching.

First suppose $G - U$ consists of disjoint complete subgraphs. By (3.2.1) $|U| \geq o(G-U)$; so one vertex in each odd component can be matched to a vertex in U. The remaining vertices of $G - U$ are in disjoint even cliques, and the remaining vertices of U form an even clique, since $|G|$ is even. Each of the even cliques has a perfect matching. Combining these matchings yields a perfect matching of G.

Otherwise some component of $G - U$ contains two nonadjacent vertices, and so a path aba' with $aa' \notin E$. Since $b \notin U$, there exists $b' \in V$ with $bb' \notin E$. By maximality, $G + aa'$ and $G + bb'$ have perfect matchings M^+ and L^+, respectively. Set $M := M^+ - aa'$ and $L := L^+ - bb'$. The component of $M \triangle L$ containing a is a path $P = a \dots x$. Either P is M-augmenting, or P is M-alternating and x is L-unsaturated. The latter case implies baP

[1](i) and (ii) imply the sufficiency of Hall's Criteria (3.1.1): $|N(S)| \leq |S \smallsetminus U|$, so (3.1.1) fails if $U \neq \emptyset$.

is L-augmenting if $x \neq b$, and Pxa' is M-augmenting if $x = b$. Anyway, G has an M- or L-augmenting path, and so a matching M^* with $|M^*| = |M^+| = |L^+|$. Thus M^* is perfect.

Here is an alternative line through the last paragraph proposed by Theo Molla:

Otherwise some component of $G - U$ contains two nonadjacent vertices, and so a path aba' with $aa' \notin E$. Since $b \notin U$, there exists $b' \in V$ with $bb' \notin E$. By maximality, $G + aa'$ and $G + bb'$ both have perfect matchings. So G has matchings M and L with a, a' the only M-unsaturated vertices and b, b' the only L-unsaturated vertices. Let $H \subseteq G$ be a subgraph with $E(H) = M \bigtriangleup L + aba'$. Since b and b' are the only vertices of H with odd degree, they are in the same component of H; say $P = b' \dots xyb \subseteq H$. Then $E(b'Py) \subseteq M \bigtriangleup L$ and either $yb \in M$ or $y \in \{a, a'\}$ and $xy \in L$. Anyway, P is an L-augmenting path in G. So G has a perfect matching $L^* := L \bigtriangleup E(P)$. □

The next theorem is a generalization of Tutte's Theoerm. We will give two proofs. The first builds on the previous proof. The second starts from scratch.

THEOREM 58 (Tutte [1947] 3.3.3+3.3.7). *Let $G = (V, E)$ be a graph with a maximum matching M. Then the number of M-unsaturated vertices of G is equal to*

$$d := \max_{S \subseteq V} o(G - S) - |S|. \tag{3.2.2}$$

FIRST PROOF. If $d = 0$ then we are done by Theorem 57. Otherwise, using $S = \emptyset$ shows $d > 0$. Let $G^+ := G \vee K_d$ with $Q = V(K_d)$ and choose $S \subseteq V$ so that $d = o(G - S) - |S|$. Then G^+ is connected and

$$|G^+| = |G| + d \equiv o(G - S) + |S| + d = 2d + 2|S| \equiv 0 \mod 2.$$

Consider $T \subseteq V(G^+)$. If $o(G^+ - T) - |T| > 0$ then $G^+ - T$ has an odd component. Thus $T \neq \emptyset$, and so $o(G^+ - T) \geq 2$. Hence $Q \subseteq T$. So every odd component of $G^+ - T$ is an odd component of $G - (T - Q)$. It follows that

$$o(G^+ - T) - |T| = o(G - (T - Q)) - |T \smallsetminus Q| - |Q| \leq d - d \leq 0.$$

So by Theorem 57, G^+ has a perfect matching M^+. Then $M := M^+ \smallsetminus E(Q)$ is a matching in G with at most $|Q| = d$ unsaturated vertices. □

SECOND PROOF. For any set $S \subseteq V$ and matching M, there are at least $o(G - S) - |S|$ unsaturated vertices, since each odd component $H \subseteq G - S$ has an M-unsaturated vertex, unless $M \cap E(S, V(H)) \neq \emptyset$, and there are at most $|S|$ such edges in M. So it suffices to show that there exists a set $S \subseteq V$ and a matching M with exactly $o(G - S) - |S|$ unsaturated vertices.

Argue by induction on $|G|$. For the base step $|G| = 1$, let $S = \emptyset$. Then $o(G - S) - |S| = 1$ and the only vertex of G is unsaturated by any matching. Now consider the induction step.

Choose a set $S \subseteq V$ so that $o(G - S) - |S|$ is maximum, and subject to this, $|S|$ is also maximum. We will prove the following three claims:

CLAIM (1). Every component of $G - S$ is odd.

PROOF. Suppose $H \in \mathcal{C}_{G-S}$ with $|H|$ even. Choose a non-cut vertex x (end of a maximal path) of H, and set $S' = S + x$. Then

$$\mathcal{O}_{G-S'} = \mathcal{O}_{G-S} + (H - x) \text{ and } |S'| = |S| + 1.$$

Thus $o(G - S) - |S| = o(G - S') - |S'|$, contradicting the choice of S, since $|S| < |S'|$. □

Claim (2). Every odd component of $G-S$ is factor critical.

Proof. Consider any $H \in \mathcal{O}_{G-S}$ and any vertex $x \in V(H)$. We must show that $H' = H - x$ has a perfect matching. By the induction hypothesis, it suffices to show that $o(H'-T) - |T| \le 0$ for all $T \subseteq V(H')$. So consider any such T, and set $S' = S \cup T + x$. Then $|S'| = |S| + |T| + 1 > |S|$, and so by the choice of S

$$o(G-S) - |S| > o(G-S') - |S'|.$$

Since $T + x \subseteq V(H)$,

$$\mathcal{O}_{G-S'} = (\mathcal{O}_{G-S} - H) \cup \mathcal{O}_{H'-T}.$$

So

$$o(G-S) - |S| > o(G-S') - |S'| = o(G-S) - 1 + o(H'-T) - |S| - |T| - 1$$
$$2 > o(H'-T) - |T|.$$

Moreover, by Claim (1), H is an odd component, and so $|H'|$ is even. Thus

$$0 \equiv |H'| \equiv o(H'-T) + |T| \mod 2.$$

Hence $1 \ne o(H'-T) - |T|$, and so $0 \ge o(H'-T) - |T|$. □

Claim (3). S is matchable into $\mathcal{O}_{G-S}$.

Proof. Let H be the $S, \mathcal{O}_{G-S}$-bigraph with edge set

$$F := \{xD : x \in S, D \in \mathcal{O}_{G-S} \text{ and } N(x) \cap V(D) \ne \emptyset\}.$$

It suffices to show that H has a matching that saturates S. For this we apply Hall's Theorem.

Consider any set $T \subseteq S$. Since $|\emptyset| = 0 = |N(\emptyset)|$, we may assume that $T \ne \emptyset$. Thus $|S| > |S'|$, where $S' := S - T$. By the choice of S

$$o(G-S) - |S| \ge o(G-S') - |S'| \ge o(G-S) - N_H(T) - |S| + |T|$$
$$|N_H(T)| \ge |T|.$$

□

Finally, we obtain a matching M as follows. By Claim (3) there is a matching M_0 that saturates S and one vertex of $|S|$ odd components. For each $H \in \mathcal{O}_{G-S}$ choose a vertex v_H, and if possible, choose v_H so that it is M_0-saturated. Next use Claim (2) to obtain matchings M_H of $H - v_H$ for every odd component $H \in \mathcal{O}_{G-S}$. Then

$$M := M_0 \cup \bigcup_{H \in \mathcal{O}_{G-S}} M_H$$

is matching of G. Using Claim (1), it saturates every vertex of G except those $o(G-S) - |S|$ vertices v_H that are not saturated by M_0. □

A graph $G = (V, E)$ is transitive if for all $x, y \in V$ there is an automorphism φ of G with $\varphi(x) = y$. For example C_n, $K_{n,n}$ and the Petersen graph, but not P_n, are transitive.

HW 36. (*) Prove that a transitive graph does not have a cut vertex.

HW 37. (+) Prove that if G is a connected, transitive graph with $|G|$ even then G has a perfect matching. (Lovasz has conjectured that every connected transitive graph has a hamiltonian path; there are only four known examples of such graphs that are not hamiltonian.) [Hint: Use Theorem 58 to show that if G does not have a perfect matching then some, but not all, of its vertices have the property that they are saturated in every maximum matching.]

3.3. Applications of Matching Theorems

A graph is k-regular if every vertex has degree k. A *cubic* graph is a 3-regular graph. A k-regular spanning subgraph of a graph G is called a *k-factor* of G. Thus the edge set of a 1-factor of G is a perfect matching. A cut-edge is also called a *bridge*. A *bridgeless* graph is a graph without cut-edges. It need not be connected.

Theorem 59 (Petersen [1891] 3.3.8). *Every bridgeless cubic graph $G = (V, E)$ contains a 1-factor.*

Proof. By Tutte's Theorem, it suffices to show that $o(G - S) \leq |S|$ for every subset $S \subseteq V$. Fix any such S and consider any $H \in \mathcal{O}_{G-S}$. Since G is cubic and $|H|$ is odd,

$$3|H| = \sum_{v \in V(H)} d(v) = 2\|H\| + |E(V(H), S)| \equiv 1 \mod 2.$$

It follows that $|E(V(H), S)|$ is odd, and since G is bridgeless, $|E(V(H), S)| \geq 3$. Thus

$$3o(G - S) \leq |E(S, V \smallsetminus S)| \leq 3|S|,$$

and so $o(G \smallsetminus S) \leq |S|$. $\square$

Theorem 60 (Petersen [1891] 3.3.9). *Every regular graph with positive even degree has a 2-factor.*

Proof. Suppose $G = (V, E)$ is $2k$-regular with $k \in \mathbb{Z}^+$. It suffices to show that each component of G has a 2-factor, so we may assume G is connected. By Euler's Theorem 34, G has an Eulerian trail $T = v_1 \dots v_n (= v_1)$. Let D be an orientation of G obtained by directing each edge $e = xy$ as $\overrightarrow{e} = \overrightarrow{xy} := (x, y)$ iff there exists $i \in [n]$ such that $x = v_i$ and $y = v_{i+1}$.

Let $V' = \{v' : v \in V\}$ and $V'' = \{v'' : v \in V\}$ be sets of new vertices, disjoint from V and each other. Let H be the V', V''-bigraph defined by $x'y'' \in E(H)$ iff $\overrightarrow{xy} \in E(D)$. Then H is k-regular. So by the Corollary 53, H has a perfect matching M. Let $F = \{xy \in E : x'y'' \in M\}$. Then (V, F) is a 2-factor of G: for each $y \in V$ there exists a unique x such that $x'y'' \in M$ and a unique z such that $y'z'' \in M$. Since T is a trail, and $xy, yz \in T$, we have $xy \neq yz$. (If G were a multigraph then it could be that $x = z$, where xy and yz were parallel edges.) $\square$

HW 38. (*) Prove that a 3-regular graph has a 1-factor iff it decomposes into copies of P_4.

HW 39. (+) Let M be a matching in a graph G with an M-unsaturated vertex u. Prove that if G has no M-augmenting path starting at u then G has a maximum matching L such that u is L-unsaturated.

HW 40. (+) A graph is *claw-free* if it does not contain an induced $K_{1,3}$. Prove that a connected claw-free graph of even order has a 1-factor. Find (easy) a small counter example if the graph is not connected.

HW 41. (+) Let G be a k-regular graph with $|G|$ even that remains connected when any $k-2$ edges are removed. Prove that G has a 1-factor.

CHAPTER 4

Connectivity

DEFINITION 61. A *separating set* or *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* $\kappa(G)$ is the minimum size of a vertex set S such that $G - S$ has more than one component or only one vertex. A graph G is k-connected if $k \leq \kappa(G)$.

Note that it is not possible to disconnect a complete graph by removing vertices. The definition sets the connectivity of a complete graph K equal to $|K| - 1$.

DEFINITION 62. A *disconnecting set* of edges in a graph G is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The *edge-connectivity* $\kappa'(G)$ of G is the minimum size of a disconnecting set of edges. It is k-edge-connected if $k \leq \kappa'(G)$.

Following the text, we write $[S, T]$ for $E(S, T)$. An *edge cut* in G is a set of edges of the form $[S, \overline{S}]$, where $\emptyset \neq S \neq V(G)$ and $\overline{S}$ denotes $V(G) \smallsetminus S$.

4.1. Basics

THEOREM 63 (Whitney [1932] 4.1.9). *Every graph $G = (V, E)$ satisfies*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

PROOF. Choose a vertex v with $d(v) = \delta(G)$. Then $E(v)$ is a disconnecting set of edges of size $\delta(G)$, and so $\kappa'(G) \leq \delta(G)$.

For the first inequality, consider a minimum edge cut $[S, \overline{S}]$; so $|[S, \overline{S}]| = \kappa'(G)$. Note that $\kappa(G) \leq |G| - 1$. First suppose that every vertex in S is adjacent to every vertex in $\overline{S}$. So

$$\kappa'(G) = |[S, \overline{S}]| = |S||\overline{S}| \geq |G| - 1 \geq \kappa(G).$$

Else there are $x \in S$ and $y \in \overline{S}$ with $xy \notin E$. Define $f : [S, \overline{S}| \to V$ by $f(e) = z$ if $e = xz$; else $f(e) \in e \cap (S - x)$. So $f(e) \in e \smallsetminus \{x, y\}$. Every x, y-path P contains an edge $e \in [S, \overline{S}]$, and $f(e)$ is an interior vertex of P. So range(f) separates x from y. Thus

$$\kappa'(G) = |[S, \overline{S}]| \geq |\text{range}(f)| \geq \kappa(G).$$

□

THEOREM 64. *Every cubic graph $G = (V, E)$ satisfies $\kappa(G) = \kappa'(G)$.*

PROOF. Put $\kappa := \kappa(G)$ and $\kappa' := \kappa'(G)$. If $\kappa = 0$ then G is disconnected, and so $\kappa' = 0$ also. If $\kappa = 3$ then $3 \leq \kappa \leq \kappa' \leq \delta(G) = 3$, and again $\kappa = \kappa'$. So assume $\kappa \in [2]$. Let S be a separating set with $|S| = \kappa$, and let H_1, H_2 be two components of $G - S$. Since S is minimum, every vertex $v \in S$ has a neighbor in each H_i. Since G is cubic, there exists i such that v has a unique neighbor w_v in H_i. Choose such a w_v, preferring $w_v \in H_1$, and set $F := \{vw_v : v \in S\}$. Then $|F| = |S|$. Moreover F is a disconnecting set of edges: F disconnects H_1 from S unless $w_v \notin H_1$ for some $v \in S$. In this case v has two neighbors in H_1, by the preference for H_1, and so no neighbors in S. Thus F separates $H_1 + v$ from H_2. So $\kappa' \leq |F| = |S| = \kappa \leq \kappa'$. □

LEMMA 65 (Expansion Lemma 4.2.3). *If G is k-connected and G' is obtained from G by adding a new vertex x with at least k neighbors in G then G' is k-connected.*

PROOF. Since $|G'| = |G|+1$, it suffices to show that G' does not have a $(k-1)$-separating set. Consider any $(k-1)$-set S. Then $G - S$ is connected and x has a neighbor in $G - S$, so $G' - S$ is connected. □

HW 42. Prove that an r-connected graph on an even number of vertices with no induced subgraph isomorphic to $K_{1,r+1}$ has a 1-factor.

4.2. Low Connectivity

DEFINITION 66. Suppose H is a subgraph of G. Recall that a path $P \subseteq G$ is an H-path if its ends, but not it internal vertices are in H. Let $\mathcal{P} := P_0, P_1, \dots, P_t$ be a sequence of subgraphs of G, and set $G_i := P_0 \cup \cdots \cup P_{i-1}$. Then $\mathcal{P}$ is a 2-*witness for G* if P_0 is a cycle, P_i is a G_i-path in G for all $i \in [t]$, and every $e \in E(G)$ is an edge of exactly one P_i.

THEOREM 67 (Whitney [1932] 4.2.8). *A graph $G = (V, E)$ is 2-connected iff it has a 2-witness set.*

PROOF. First suppose G is 2-connected. Then $\delta(G) \geq 2$, and so G contains a cycle C. Let $H \subseteq G$ be a maximal subgraph such that H has a 2-witness $P_0, \dots, P_t$. It exists because C is a candidate. It suffices to show that $H = G$.

Suppose $v_0 \in V(G - H)$. Since G is connected, there exists a v_0, H-path $Q = v_0 \dots v_s$. Since G is 2-connected, $G - v_s$ is connected. So there exists a v_{s-1}, H path P in $G - v_s$. Then $P_{t+1} := v_s v_{s-1} P$ is an H-path in $H + P_{t+1}$, contradicting the maximality of H. We conclude that H is a spanning subgraph of G.

Now suppose $xy \in E(G - H)$. Then xy is an H-path of $H + xy$, contradicting the maximality of H. So H is an induced, spanning subgraph of G. Thus $G = H$.

Now suppose G has a 2-witness $P_0, \dots, P_t$. Argue by induction on t that G is 2-connected. The base step $t = 0$ is easy since the cycle P_0 is 2-connected. So consider the induction step $t \geq 1$. By the induction hypothesis, $H := \bigcup_{i=0}^{t-1} P_i$ is 2-connected. Put $P_t := v_1 \dots v_s$. Consider any $x \in V(G)$. We must show that $G - x$ is connected. Since H is 2-connected, $H - x$ is connected. Also every vertex of $P_t - x$ is connected to a vertex $v \in \{v_1, v_s\}$ in $G - x$, even if $x \in V(P_t)$. It follows that G is 2-connected. □

DEFINITION 68. Let $e = xy$ be an edge in a graph G, and fix a new vertex v_e. The graph $G \cdot e$ obtained by *contracting* e is defined by

$$G \cdot e := (G \cup K(v_e, N_G(\{x, y\}))) - x - y.$$

Note that if P' is a path in $G \cdot e$ then either P' is a path in G or $v_e \in V(P')$. In the latter case we can obtain a path in G by replacing v_e by one of x, y, xy, yx. If P is a path in G then either P is a path in $G \cdot e$ or one or both of x, y are in $V(P)$. In the latter case we can obtain a path P' in $G \cdot e$ by replacing one of x, y, xPy, yPx by v_e.

LEMMA 69 (Thomassen [1980] 6.2.9). *Every* 3*-connected graph* G *with* $|G| \geq 5$ *has an edge* e *such that* $G \cdot e$ *is* 3*-connected.*

PROOF. Suppose not. Consider any edge xy. Since $G' := G \cdot xy$ is not 3-connected and $|G'| \geq 4$, G' has a separating 2-set S. Observe that $v_{xy} \in S$: Otherwise v_{xy} is in a component H of $G' - S$ and $S \subseteq V(G)$. Choose w in another component of $G' - S$; then $w \in G$. For a contradiction it suffices to show that every $w, \{x, y\}$-path $P = w \ldots u'u$ in G contains a vertex of S, implying $\kappa(G) \leq |S| = 2$. Fix P; then $P' := wPu'v_{xy}$ is a w, v_{xy} path in G'. Since S separates w from v_{xy} in G', there exists $v \in V(P') \cap S \subseteq V(P)$.

So S has the form $S = \{v_{xy}, z\}$. Observe that $\{x, y, z\}$ is a separating 3-set in G: Choose vertices u, v in distinct components of $G' - S$. Then $u, v \in V(G)$. Consider any u, v-path $P \subseteq G$. If P contains neither x nor y then $P \subseteq G' - v_{xy}$, and so $z \in V(P)$.

So far the edge $xy \in E$ and the 2-separating set S are arbitrary. Now choose xy and $S = \{v_{xy}, z\}$ as above so that $G - \{x, y, z\}$ has a component H that is as large as possible among all possible choices of xy and S. Let H' be another component of $G - \{x, y, z\}$. Since $\{x, y, z\}$ is a minimal separating set, each of x, y, z has a neighbor in each of H, H'. Let u be a neighbor of z in H'. Then $G \cdot uz$ has a separating set $\{v_{uz}, v\}$, and $\{u, v, z\}$ is a separating set for G.

Put $H^+ := G[V(H) + x + y]$. Then H^+ is connected, and $u, z \notin V(H^+)$. Thus $H^* := H^+ - v$ is disconnected, since otherwise H^* is a component of $G - \{u, v, z\}$ with $|H^*| = |H + \{x, y\} - v| \geq |H| + 1$, contradicting the choice of xy, S, H. Let U be a component of H^* containing neither x nor y (there are at least two components, and x and y are in the same component, since they are adjacent), and consider $a \in V(U)$. Since $a \in V(H)$, it has no neighbors in $G - \{x, y, z\} - H = G - H^+ - z$. Since $a \in U$, it has no neighbors in $H^+ - v - U$. Thus $N(a) \smallsetminus U \subseteq \{v, z\}$. So $\{v, z\}$ separates U from H', contradicting $\kappa(G) \geq 3$. $\square$

DEFINITION 70. A sequence of graphs $G_0, \ldots, G_s$ is a 3*-witness for* G iff

(1) $G_0 = K_4$ and $G_s = G$; and
(2) for each $i \in [s]$ there is an edge $xy \in E(G_i)$ such that $G_{i-1} = G \cdot xy$ and $d_{G_i}(x), d_{G_i}(y) \geq 3$.

THEOREM 71. *A graph* G *is* 3*-connected iff it has a* 3*-witness.*

PROOF. First suppose that G is 3-connected. Then $|G| \geq 4$. We show by induction on $|G|$ that G has a 3-witness. Suppose $|G| = 4$. If $xy \notin E(G)$ then $V(G) \smallsetminus \{x, y\}$ is a 2-set that separates x from y, a contradiction. So $G = K_4$, and $G_0 = K_4 = G$ is a 3-witness for G. Otherwise, $|G| \geq 5$. By Lemma 69, there exists an edge $xy \in E(G)$ such that $G \cdot xy$ is 3-connected. Since G is 3-connected, $d(x), d(y) \geq 3$. By induction, $G \cdot xy$ has a 3-witness $G_0, \ldots, G_s$. So $G_0, \ldots, G_s, G$ is a 3-witness for G.

Now suppose $G_0, \ldots, G_s$ is a 3-witness for G. We show by induction on s that G is 3-connected. If $s = 0$ then $K_4 = G_0 = G$ is 3-connected. Otherwise, for some edge $xy \in E(G)$, both $G_{s-1} = G \cdot xy$ and $d_G(x), d_G(y) \geq 3$. By induction $G \cdot xy$ is 3-connected. Suppose for a contradiction that S is a 2-separator in G. If $S = \{x, y\}$ then v_{xy} is a cut vertex of $G \cdot xy$, a contradiction. So there is a component H of $G - S$ that contains one, say x, of x and y, and another component H' that contains neither x nor y. Since $d_G(x) \geq 3$ and $N(x) \subseteq V(H) \cup S$, x has a neighbor v in H. It is not y, since then S separates v_{xy} from

H' in $G \cdot xy$. If $y \notin S$ then S separates v from H' in $G \cdot xy$; otherwise $S' := S - y + v_{xy}$ separates v from H' in $G \cdot xy$. Regardless, we have a contradiction. $\square$

The last paragraph of the above proof is subtle. If $d_G(x) < 3$ then we could have $S = N(x)$, and $V(H) = \{x\}$. Then $H - x$ is not a component of $G \cdot xy - S'$ because $H - x$ has no vertices.

CONJECTURE 72 (Lovasz). *There exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$ and $f(k)$-connected graphs G and all vertices $x, y \in V(G)$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ is an x, y-path and $G[V_2]$ is k-connected.*

HW 43. Show that Conjecture 72 is true in the case $k = 1$ with $f(1) := 3$.

4.3. Menger's Theorem

DEFINITION 73. Let A and B be subsets of vertices in a graph $G = (V, E)$. An *A, B-path* is a path whose first vertex is in A, whose last vertex is in B and whose internal vertices are in neither A nor B. Let $l(A, B)$ be the maximum size of a set of disjoint A, B-paths. An *A, B-separating set (or A, B-separator)* is a set of vertices S such that $G - S$ has no A, B-paths. Let $k(A, B)$ be the minimum cardinality of an A, B-separating set.

Here is another fundamental theorem. We give two proofs.

THEOREM 74 (Menger 1927 4.2.17). *Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. Then the size $l := l(A, B)$ of a maximum set of disjoint A, B-paths is equal to the size $k := k(A, B)$ of a minimal A, B-separating set.*

PROOF 1. ($l \leq k$) If $\mathcal{P}$ is a set of disjoint A, B-paths and S is an A, B-separator then S must contain at least one vertex of each path, and each vertex of S is on at most one path of $\mathcal{P}$. Thus the function $f : \mathcal{P} \to S$ defined by setting $f(P)$ equal to the first $x \in S \cap V(P)$ is an injection; so $|\mathcal{P}| \leq |S|$. Choosing $\mathcal{P}$ maximum and S minimum yields the inequality.

($k \leq l$) For a set of A, B-paths $\mathcal{P}$ let $\text{end}(\mathcal{P})$ denote the set of ends in B of paths in $\mathcal{P}$. It suffices to show (*) if $\mathcal{P}'$ is a set of disjoint A, B-paths with $|\mathcal{P}'| < k$ then there exists a set $\mathcal{P}$ of disjoint A, B-paths such that $|\mathcal{P}| = |\mathcal{P}'| + 1$ and $\textit{end}(P') \subseteq \textit{end}(\mathcal{P})$. Argue by induction on $|G'|$, where $G' := G - B$. If $|G'| = 0$ then $A \subseteq B$. So the A, B-paths are exactly the paths consisting of a single vertex of A, and $k = |A|$. Thus (*) holds.

Suppose $|G'| > 0$, and fix a set $\mathcal{P}'$ of disjoint A, B-paths with $|\mathcal{P}'| < k$. Since $|\text{end}(\mathcal{P}')| = |\mathcal{P}'| < k$, there is an A, B-path Ry' in $G - \text{end}(\mathcal{P}')$. If $R \cap \bigcup \mathcal{P}' = \emptyset$ then put $\mathcal{P} := \mathcal{P}' + R$. If not, let x be the last vertex of R that is in $\bigcup \mathcal{P}'$; say $x \in P \in \mathcal{P}'$ and y is the end of P in B. Put $B' := B \cup V(xRy' \cup xPy)$ and $\mathcal{Q}' := \mathcal{P}' - P + Px$. Then $\text{end}(\mathcal{Q}') = \text{end}(\mathcal{P}') - y + x$. Since every A, B-path contains an A, B'-path, $k = k(A, B) \leq k(A, B')$. So by induction, there exists a set $\mathcal{Q}$ of $A, B^{'}$-paths and $y'' \in B'$ such that $|\mathcal{Q}| - 1 = |\mathcal{Q}'| = |\mathcal{P}'|$ and $\text{end}(\mathcal{Q}) = \text{end}(\mathcal{Q}') + y''$. So $x \neq y''$. Let x, y'' be the ends of $Q, Q'' \in \mathcal{Q}$. Set $\mathcal{P}_0 = \mathcal{Q} - Q - Q''$. If $y'' \in xPy$ then set $\mathcal{P} := \mathcal{P}_0 + QxRy' + Q''y''Py'$; if $y'' \in xRy'$ then set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''y''Ry'$; else set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''$. Evidently, $\mathcal{P}$ witnesses (*). $\square$

PROOF 2 ($k \leq l$). So it suffices to show $k \leq l$. Argue by induction on $\|G\|$.
Base Step: $\|G\| = 0$. Then every A, B-path is trivial. So $A \cap B$ is the maximum set of disjoint A, B-paths and the minimum A, B-separating set. Thus $l = |A \cap B| = k$.

Induction Step: $\|G\| \geq 1$. Let $e = xy \in E(G)$, and put $G' = G \cdot e$. For any $U \subseteq V$, define

$$U' = \begin{cases} U - \{x, y\} + v_e & \text{if } U \cap \{x, y\} \neq \emptyset \\ U & \text{otherwise} \end{cases},$$

and note that for every $T \subseteq V(G')$ there exists $S \subseteq V$ with $T = S'$. Every A', B' path P' in G' is an A, B-path in G, unless $v_e \in V(P')$, and in this case we can obtain an A, B path in G by replacing v_e by one of x, y, xy, yx. It follows that every set $\mathcal{P}'$ of disjoint A', B'-paths corresponds to a set $\mathcal{P}$ of disjoint A, B-paths with $|\mathcal{P}| = |\mathcal{P}'|$ (but not vice versa). So

$$l_{G'}(A', B') \leq l.$$

Also, if S is an A, B-separator in G if and only if S' is an A', B'-separator in G'. So

$$k_{G'}(A', B') \leq k \leq k_{G'}(A', B') + 1.$$

Choose a minimum A', B'-separator T in G'. If $k_{G'}(A', B') = k$ then by the induction hypothesis applied to G' we have:

$$k = k_{G'}(A', B') \leq l_{G'}(A', B') \leq l,$$

and we are done. Otherwise, $k = k_{G'}(A', B') + 1$. In this case $v_{xy} \in T$, and $T = S'$, where $S := T - v_{xy} + x + y$. In particular $xy \in G[S]$.

Set $G'' = G - e$. Since $e \in G[S]$,

$$k_G(A, S) = k_{G''}(A, S) \text{ and } k_G(B, S) = k_{G''}(B, S) \tag{4.3.1}$$

Since S separates A from B in G, every A, S-separator in G separates A from B, and so has size at least $|S|$, and a similar statement holds for B. So we have

$$k_G(A, S), k_G(S, B) \geq k. \tag{4.3.2}$$

Thus

$$\begin{aligned} &|S| \geq l_G(A, S) \geq l_{g''}(A, S) =_{i.h.} k_{G''}(A, S) =_{(4.3.1)} k_G(A, S) \geq_{(4.3.2)} k = |S| \text{ and} \\ &|S| \geq l_G(B, S) \geq l_{g''}(B, S) =_{i.h.} k_{G''}(B, S) =_{(4.3.1)} k_G(B, S) \geq_{(4.3.2)} k = |S|. \end{aligned}$$

Let $\mathcal{K}_A$ be a collection of $|S| = k$ disjoint A, S-paths and $\mathcal{K}_B$ be a collection of $|S|$ disjoint S, B-paths. Then for each $z \in S$ there is a unique A, z-path P_z and a unique z, B-path Q_z. If $v \in V(P_w) \cap V(Q_z)$ then $v \in S$, since otherwise PvQ is an A, B-walk in $G - S$, contradicting the fact that S is an A, B-separator. Thus $w = v = z$, and so $\{P_z z Q_z : z \in S\}$ is a collection of $|S| = k$ disjoint A, B-paths. $\square$

Definition 75. Let a and b be distinct vertices in a graph G. An a, b-separator is a set $S \subseteq V(G) - \{a, b\}$ such that there are no a, b-paths in $G - S$. Two a, b-paths are internally disjoint if they have no internal vertices in common. Let $\lambda(a, b)$ be the maximum number of internally disjoint a, b-paths and $\kappa(a, b)$ be the minimum size of an a, b-separating set (if it exists).

Corollary 76 (4.2.17). *If x and y are nonadjacent vertices of a graph G then $\lambda(a, b) = \kappa(a, b)$.*

Proof. Let $A = N(a)$ and $B = N(b)$. Then any A, B-path $P' \subseteq G' = G - \{a, b\}$ can be extended to an a, b-path $P = aP'b \subseteq G$ and any a, b-path $Q \subseteq G$ contains an A, B-path $Q' = Q - a - b \subseteq G'$. Thus S is an a, b-separator of G iff it is an A, B-separator of G' and the maximum number of internally disjoint a, b-paths in G is equal to the maximum number of disjoint A, B-paths in G'. Applying Menger's Theorem to G' we see that the minimum size of an A, B-separating set is equal to the size of a maximum set of disjoint A, B-paths. The corollary follows. □

Theorem 77 (4.2.21). *Every graph $G = (V, E)$ satisfies $\kappa(G) = t := \min_{a,b\in V} \lambda(a, b)$.*

Proof. Choose a, b with $t = \lambda(a, b)$. First we show that $\kappa(G) \geq t$. If G is complete then

$$t = \lambda(a, b) = 1 + (|G| - 2) = |G| - 1 = \kappa(G),$$

since ab is an a, b-path, and acb is also an a, b-path for all $c \in V - a - b$. Otherwise G has a separating $\kappa(G)$-set S. Let x, y be vertices in distinct components of $G - S$. Then

$$t = \lambda(a, b) \leq \lambda(x, y) \leq |S| = \kappa(G).$$

Now we show that $t \geq \kappa(G)$. If $ab \notin E$ then by Corollary 76, there exists an a, b-separating set S with $|S| = t$; so $t \geq \kappa(G)$. Otherwise, $ab \in E$; set $G' = G - ab$. Then $\lambda_G(a, b) = \lambda_{G'}(a, b) + 1$, since ab is an a, b-path.

We first show that $\kappa(G') + 1 \geq \kappa(G)$. Suppose $k := \kappa(G') < \kappa(G)$. Then G' has a separating k-set S, and S is not a separating set of G. It follows that $G' - S$ has exactly two components X and Y with (say) $a \in V(X)$ and $b \in V(Y)$. If $|X| > 1$ then $S + a$ is a separator of G with size $k + 1 \geq \kappa(G)$. A similar argument holds, if $|Y| > 1$. Otherwise $|G| = k + 2$, and so $\kappa(G) \leq |G| - 1 = k + 1$.

By Corollary 76, we have

$$\kappa(G) \leq \kappa(G') + 1 \leq \kappa_{G'}(a, b) + 1 = \lambda_{G'}(a, b) + 1 = \lambda_G(a, b) = t.$$

□

Definition 78 (4.2.18). The line graph $H = L(G)$ of a graph $G = (V, E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

Definition 79. Let x and y be distinct vertices in a graph G. An x, y-edge-cut is a set of edges F such that there are no x, y-paths in $G - F$; let $\kappa'(x, y)$ be the size of a minimum x, y-edge-cut. Two x, y-paths are edge-disjoint if they have no common edges; let $\lambda'(x, y)$ be the maximum size of a set of edge-disjoint x, y-paths.

Theorem 80 (4.2.19). *Let $G = (V, E)$ be a graph with distinct vertices $x, y \in V$. Then $\kappa'(x, y) = \lambda'(x, y)$.*

Proof. Set $G' = G + x' + xx' + y' + yy'$, where x' and y' are new vertices. Then $\kappa'_G(x, y) = \kappa'_{G'}(x, y)$ and $\lambda'_G(x, y) = \lambda'_{G'}(x, y)$. A set of edges disconnects x from y in G' iff the corresponding set of vertices separates xx' from yy' in $L(G')$. Moreover, edge disjoint x, y-paths in G' correspond to internally disjoint xx', yy'-paths in $L(G')$. Thus

$$\kappa'_G(x, y) = \kappa'_{G'}(x, y) = \kappa_{L(G')}(xx', yy') = \lambda_{L(G')}(xx', yy') = \lambda'_{G'}(x, y) = \lambda'_G(x, y).$$

□

DEFINITION 81 (4.2.22). Let $G = (V, E)$ be a graph with $x \in V$ and $U \subseteq V$. An x, U-fan is a set $\mathcal{F}$ of x, U-paths such that $|\mathcal{F}| = |U|$ and $F \cap F' = \{x\}$ for all $F, F' \in \mathcal{F}$.

THEOREM 82 (4.2.23). *A graph $G = (V, E)$ is k-connected if and only if $|G| \geq k + 1$ and G has an x, U-fan for all $x \in V$ and all k-sets $U \subseteq V - x$.*

PROOF. (Sketch) If G is k-connected then there exist k disjoint $N(x), U$-paths in $G - x$. Adding x yields the desired fan.

Conversely, the hypothesis implies $\delta(G) \geq k$, and for all x and y, there exist k disjoint $N(x), N(y)$-paths. It follows that

$$\kappa(G) = \min_{x \neq y \in V} \lambda(x, y) \geq k.$$

$\square$

THEOREM 83 (HW 4.2.24). *Let $G = (V, E)$ be a k-connected graph with $k \geq 2$. Then for any k-set $S \subseteq V$ there is a cycle $C \subseteq G$ with $S \subseteq V(C)$.*

PROOF. Let $C \subseteq G$ be a cycle containing as many vertices of S as possible. It exists because $\delta(G) \geq \kappa(G) \geq 2$. We claim that $S \subseteq V(C)$. Otherwise, let $v \in V \smallsetminus V(C)$. Then $|S \cap V(C)| < k$. Arguing by contradiction, it suffices to find a cycle containing S$\cap V(C) + v$.

Orient C cyclically as $\overrightarrow{C}$. Let $t = \min\{k, |C|\}$, and let $\mathcal{F}$ be a $v, V(C)$-fan with $|\mathcal{F}| = t$. Set $F = \bigcup \mathcal{F}$, and let $x_1, \ldots, x_t$ be a sequence of the leaves of F in cyclic order around $\overrightarrow{C}$, and set $P_i = x_i \overrightarrow{C} x_{i+1}$. Then there exists $i \in [k]$ such that P_i contains no internal vertices from S: If $t = |C|$ this is true for all $i \in [k]$; otherwise it follows by the pigeonhole principle, since $t = k > |S \cap V(C)|$. So $D = x_{i+1} \overrightarrow{C} x_i F v F x_{i+1}$ is a cycle containing $S \cap V(C) + v$. $\square$

HW 44. (*) Every 2-connected graph G has a cycle of length at least $\min\{|G|, 2\kappa(G)\}$.

HW 45. (*) If G is a 2-connected graph with $\alpha(G) \leq \kappa(G)$ then G is hamiltonian.

HW 46. (+) Let G be a 2-connected graph that does not induce $K_{1,3}$. Then G has a cycle of length at least $\min\{|G|, 4\kappa(G)\}$.

HW 47. (+) Show that Conjecture 72 is true in the case $k = 2$ with $f(2) := 5$. [Hint: Choose P so that $G - P$ contains the biggest 2-connected subgraph possible.]

HW 48. (+) Let $G = (V, E)$ is a graph with $x \in V$ and $Y, Z \subseteq V$ and $4 = |Y| = |Z| - 1$. Suppose $\mathcal{Q} = \{Q_y : y \in Y\}$ is an x, Y-fan in G, where each Q_y is an x, y-path. Similarly, suppose $\mathcal{R} = \{R_z : z \in Z\}$ is an x, Z-fan in G, where each R_z is an x, z-path. Prove: There exists a $x, (Y + z)$-fan in G for some $z \in Z$. [Hint: Add a new vertex w whose neighborhood is Z and apply a theorem.]

CHAPTER 5

Graph coloring

DEFINITION 84. Let $G = (V, E)$ be a graph and C be a set (of colors). A *proper C-coloring* of G is a function $f : V \to C$ such that for all vertices $x, y \in V$ if $xy \in E(G)$ then $f(x) \neq f(y)$. If k is a positive integer, we say that f is a proper k-coloring if it is a proper $[k]$-coloring. The chromatic number $\chi(G)$ is the least k such that G has a proper k-coloring. In this case G is said to be k-chromatic. If G has a k-coloring then it is said to be k-colorable. In this chapter we will assume that all colorings are proper unless otherwise stated. For $i \in C$, $f^{-1}(i)$ is called a color class.

PROPOSITION 85 (HW). *Every graph G satisfies $\omega(G), \frac{|G|}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$.*

HW 49. Prove Proposition 85.

5.1. Examples

EXAMPLE 86 (5.2.3 Mycielski [1955]). For every positive integer k there exists a graph G_k with $\omega(G_k) \leq 2$ and $\chi(G_k) = k$.

PROOF. We argue by induction on k. For $k \leq 2$ let $G_k = K_k$. Now suppose $k \geq 3$ and we have constructed $G_{k-1} = (V_{k-1}, E_{k-1})$ as required. We first construct $G_k = (V_v, E_k)$ as follows: Let $V'_{k-1} = \{v' : v \in V_{k-1}\}$ be a set of new vertices, x_k be a new vertex, and put

$$V_k = V_{k-1} \cup V'_{k-1} + x_k \text{ and,}$$

$$E_k = E_{k-1} \cup \{uv' : uv \in E_{k-1}\} \cup \{x_k v' : v' \in V'_{k-1}\}.$$

So $N(v') \cap V_{k-1} = N(v) \cap V_{k-1}$ for all $v \in V_{k-1}$.

Suppose $\omega(G_k) \geq 3$, and choose $Q = K_3 \subseteq G_k$. Then $k \geq 3$. Since $N(x_k) = V'_{k-1}$ is independent, and $\delta(Q) = 2$, both $|V'_{k-1} \cap Q| \leq 1$ and $x_k \notin Q$. Since $\omega(G_{k-1}) =_{i.h.} 2$, there is exactly one $v' \in V'_{k-1} \cap Q$. Hence $N(v') \cap Q = N(v) \cap Q$, and so $Q - v' + v$ is a K_3 in G_{k-1}, a contradiction.

Notice that $\chi(G_k) \leq k$: If $k \leq 2$ this is obvious; otherwise G_{k-1} has a $(k-1)$-coloring f' by the induction hypothesis. We can extend f' to a k-coloring f of G_k by setting $f(v') = k$ (k is the new color), and $f(x_{k+1}) = 1$ (1 is an old color).

Finally we show that $\chi(G_k) \geq k$. If $k \leq 2$ this is obvious. For $k > 2$ it suffices to show that every $(k-1)$-coloring g of $G_k - x_k$ satisfies $g(V'_{k-1}) = [k-1]$, since then x_k will require a new color. Suppose not; say $\alpha \in [k-1] \smallsetminus g(V'_{k-1})$, and assume (wolog) that $\alpha = k-1$. Define a $(k-2)$-coloring h of G_{k-1} by:

$$h(v) = \begin{cases} g(v) & \text{if } g(v) \neq k-1 \\ g(v') & \text{if } g(v) = k-1 \end{cases}.$$

We claim that h is proper: Suppose $uv \in E_{k-1}$. Then $g(u) \neq g(v)$ since g is proper. If $g(u) \neq k-1 \neq g(v)$ then

$$h(u) = g(u) \neq g(v) = h(v).$$

Otherwise, exactly one of u, v is colored with $k-1$ by g; say $g(v) = k-1$. Since $uv' \in E_{k+1}$,

$$h(u) = g(u) \neq g(v') = h(v).$$

This contradicts the induction hypothesis that $\chi(G_{k-1}) \geq k-1$. □

HW 50. (*) Let G_k be the graph in Example 86. Prove that G_k is critical, i.e., $\chi(G_k - e) < \chi(G_k)$ $(= k)$ for all $e \in E_k$. [Hint: There are three types of edges to consider.]

HW 51. (*) Let G be graph, and suppose any two odd cycles $C, C' \subseteq G$ have a common vertex. Prove that $\chi(G) \leq 5$.

HW 52. (*) Let $P = \{v_1, \dots, v_n\}$ be a path, and suppose $G = (V, E)$ is a graph such that V is a subset of the set of subpaths of P, and $E = \{RQ \in V : R \cap Q \neq \emptyset\}$. Prove that $\chi(G) = \omega(G)$. [Hint: Order V so that if $Q, R \in V$ (subpaths of P), the first vertex of Q is v_i, the first vertex of R is v_j, and $i < j$ then Q precedes R, and color V in this order.]

HW 53. (*) Let $G = (V, E)$ be a k-colorable graph, and let P be a set of vertices such that the distance $d_G(x, y)$ between any two points in P is at least 4. Prove that any $[k+1]$-coloring of $G[P]$ can be extended to a $[k+1]$-coloring of G.

5.2. Brooks' Theorem

PROPOSITION 87. *Let $G = (V, E)$ be a graph, and $A, B \subseteq V$. If $A \cap B$ is a separator and $G[A \cap B]$ is complete, then $\chi(G) \leq k := \max \chi(G[A]), \chi(G[B])$.*

PROOF. Let $A \cap B := \{v_1, \dots v_l\}$. Then $l \leq \omega(G) \leq k$, since $G[A \cap B]$ is complete $\{A_1, \dots, A_k\}$ be a partition (allowing empty sets) of A into independent sets, and $\{B_1, \dots, B_k\}$ be a partition (allowing empty sets) of B into k independent sets. This is possible since $\chi(G[A]), \chi(G[B]) \leq k$. Each independent set contains at most one vertex of $A \cap B$. So we may assume $A_i \cap B_i = \{v_i\}$. Put $C_i := A_i \cup B_i$, and note that C_i is independent, since $C_i \cap (A \cap B) = \{v_i\}$ and $A \cap B$ is a separator. Then $\{C_1, \dots, C_k\}$ is a partition (allowing empty sets) of V into k independents sets. □

LEMMA 88. *Let $G = (V, E)$ be a connected graph, and $v \in V$. There exists an ordering $v_1, \dots, v_{|G|}$ of V such that (*) $v = v_{|G|}$ and for all $i \in [n-1]$ there exists $j \in [n] \smallsetminus [i]$ with $v_i v_j \in E$.*

PROOF. Argue by induction on $n := |G|$. If $n = 1$ the only possible order works; so assume $n > 1$. Let P be a maximum path, and choose an end u of P with $u \neq v$. Since u is not a cut-vertex, $G - u$ is connected. By induction, there exists an ordering $v_2, \dots, v_n$ of $V - u$ satisfying (*) for $G - u$. So $v_1 := u, v_2, \dots, v_n$ satisfies (*) for G. □

Define a *b-obstruction* to be K_b, or, if $b = 3$, an odd cycle, and let (non-standard) $\omega^*(G)$ be the largest integer b such that G contains a b-obstruction. Then $\omega^*(G) \leq \chi(G) \leq \Delta(G) + 1$.

THEOREM 89 (Brooks (1941)). *Every graph satisfies $\chi(G) \leq \max\{\omega^*(G), \Delta(G)\}$.*

Proof. Set $\Delta := \Delta(G)$, $\chi := \chi(G)$, $\omega^* := \omega^*(G)$, and argue by induction on $|G|$. Since $\chi \leq \Delta + 1$, it suffices to show that $\omega^* = \Delta + 1$ or $\chi \leq \Delta$. If $\Delta \leq 1$ then $\omega^* = \Delta + 1$. If $\Delta = 2$ then $2 \leq \omega^* \leq \chi \leq 3$; if $\omega^* = 2$ then $\chi \leq 2$ as G has no odd cycle. So assume $3 \leq \omega^* \leq \Delta$.

By Proposition 87, assume G is 2-connected. Let S be a maximal independent set, and put $G' := G - S$. So $\Delta(G') < \Delta$, since every vertex of G' has a neighbor in S. If $\omega^*(G') < \Delta$ then $(\Delta - 1)$-color G' by induction, and use a new color for S. Else, consider a Δ-obstruction $Q \subseteq G'$. Choose $y \in S$ with $\|y, Q\| \geq 1$. Then $V(Q) \nsubseteq N(y)$, since $G[Q + y] \neq K_{\Delta+1}$, and if $Q \neq K_\Delta$ then $\Delta < |Q|$. As Q is connected, there is $wx \in E_Q(\overline{N}(y), N(y))$; let $w' \in N_Q(x) - w$.

Since G is 2-connected, there is a minimum y, Q-path $P := y \dots z$ in $G - x$. If $z \neq w$ then set $w^* = w$. Else $yw' \notin E$; set $w^* = w'$. Anyway $z \neq w^*$, $w^*x \in E$, and $w^*y \notin E$. Since $Q - w^*$ and $P - y$ are connected, $H := G[Q \cup P]$ has an ordering $L := w^*, y, v_1, \dots, v_t, x$, where each v_i has a neighbor to its right. Using induction, Δ-color $H' := G - H$ by f. Since

$$d_{H'}(w^*) + d_{H'}(y) \leq 2\Delta - d_H(w^*) - d_H(y) \leq 2\Delta - (\Delta - 1) - 2 < \Delta,$$

some color β is not used on $N(w^*) \cup N(y)$. Extend f to G by setting $f(w^*) = \beta = f(y)$ and coloring the remaining vertices in the order L. This is possible, since each v_i has at most $\Delta - 1$ colored neighbors when colored, and x has two neighbors w^*, y colored the same. □

Reed proved the conjecture for $\Delta > 10^{14}$.

Conjecture 90 (Borodin & Kostochka 1977). *If a graph G satisfies $8, \omega(G) < \Delta(G)$ then $\chi(G) < \Delta(G)$.*

HW 54. For a graph G let $\theta(G) = \max_{uv \in E(G)}(d(x) + d(y))$. Prove that if $2\omega(G), \theta(G) \leq 2r$ then $\chi(G) \leq r$.

5.3. Turán's Theorem

Definition 91. A graph is said to be *r-partite* if it is r-colorable. Saying r-partite instead of r-colorable tends to emphasize the partition into r independent sets provided by the r-coloring. These independent sets are called *parts.* The *complete r-partite $K_{n_1,\dots n_r}$* graph is the r-partite graph with r parts of sizes $n_1, \dots, n_r$ such that any two vertices in different parts are adjacent. The *Turán* graph $T_{n,r}$ is the complete r-partite graph on n vertices such that any two parts differ in size by at most one.

Lemma 92 (5.2.8). *Among all r-partite graphs on n vertices, $T_{n,r}$ has the most edges.*

Proof. Let G be an r-partite graph on n vertices with as many edges as possible; say $\mathcal{X}$ is an r-partition of G. Clearly, G is a complete r-partite graph. So, if $G \neq T_{n,r}$ then there exist parts $X, Y \in \mathcal{X}$ with $|X| - |Y| \geq 2$ and $x \in X$. Let G' be the complete r-partite graph with r-partition $\mathcal{X}' := \mathcal{X} - X - Y + (X - x) + (Y + x)$. Then

$$E(G') \supseteq E(G) - \{xy : y \in Y\} + \{xx' : x' \in X - x\}.$$

Thus

$$\|G'\| \geq \|G\| - |Y| + |X| - 1 \geq \|G\| + 1,$$

a contradiction. So $G \cong T_{n,r}$. □

Theorem 93 (5.2.9 Turán [1941]). *Among all graphs $G = (V, E)$ on n vertices with $\omega(G) \leq r$, the one with the most edges is $T_{n,r}$.*

PROOF. Evidently $T_{n,r}$ is a candidate. Argue by induction on r that if G satisfies $|G| = n$, $\omega(G) \leq r$, and $\|G\| \geq \|T_{n,r}\|$ then $G \cong T_{n,r}$. If $\omega(G) \leq 1$ then $G \cong T_{n,1}$; so suppose $r > 1$.

Choose $v \in V$ with $d(v) = \Delta := \Delta(G)$. Set $N := N(v)$, $G' := G[N]$, $S := V - N(v)$ and $G'' := G[S]$. Then $|G'| = \Delta$, and $\omega(G') \leq r - 1$, since $K + v$ is a clique in G for every clique K in G'. Set $H := T_{\Delta,r-1} \vee \overline{K}(S)$. Then H is an r-partite graph on n vertices, and $\omega(H) \leq r$, since any clique in H has at most $r-1$ vertices in $T_{\Delta,r-1}$ and one vertex in S. So

$$\begin{aligned}
\|G\| &= \|G'\| + \|G''\| + |E(N,S)| \\
&= \|G'\| + \sum_{v \in S} d_G(v) - \|G''\| \\
&\leq \|T_{\Delta,r-1}\| + \sum_{v \in S} d_G(v) - \|G''\| && \text{(induction)} \qquad (5.3.1) \\
&\leq \|T_{\Delta,r-1}\| + \Delta|S| && \text{(maximum degree)} \qquad (5.3.2) \\
&= \|H\| \\
&\leq \|T_{n,r}\| && \text{(Lemma 92)} \qquad (5.3.3)
\end{aligned}$$

Inequality (5.3.1) is strict unless $G' \cong T_{\Delta,r-1}$. Inequality (5.3.2) is strict unless $G'' = \overline{K}(S)$ and $G = G' \vee G''$. Inequality (5.3.3) is strict unless $H \cong T_{n,r}$. If $\|G\| \geq \|T_{n,r}\|$ then all three inequalities are tight, and so

$$G \cong T_{\Delta,r-1} \vee \overline{K}(S) \cong H \cong T_{n,r}.$$

□

HW 55. (*) Prove that if $\omega(G) \leq r$ then $\|G\| \leq (1 - 1/r)|G|^2/2$.

5.4. Edge Coloring

DEFINITION 94. Let $G = (V,E)$ be a graph. A *proper k-edge-coloring* of G is a function $f : E \to [k]$ such that $f(e) = f(e')$ implies that e and e' are not adjacent ($e \cap e' = 0$). The *chromatic index* $\chi'(G)$ of G is the least k such that G has a proper k-edge-coloring. In this section we will assume that all edge colorings are proper. Note that this is not the case when we consider Ramsey Theory.

HW 56. (*) Let P be the Petersen graph and $v \in V(P)$. Determine $\chi'(P - v)$.

THEOREM 95 (7.1.17 König [1916]). *Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.*

PROOF. Argue by induction on $\Delta = \Delta(G)$. The base step $\Delta = 1$ is trivial since G has no adjacent edges, and so all edges can receive the same color. So consider the induction step $\Delta > 1$. First observe that it suffices to find a Δ-regular bipartite multigraph H with $G \subseteq H$: By Hall's theorem H has a perfect matching M. Color all edges in $M \cap E(G)$ with color Δ, and set $G^* = G - M$. Then $\Delta(G^*) = \Delta - 1$, and so by the induction hypothesis, we can $(\Delta - 1)$-edge-color G^*. This yields a Δ-edge-coloring of G.

It remains to construct H. Suppose G has bipartition $\{A, B\}$ with $|A| \leq |B|$. Form G' by adding new vertices to A, but no new edges, to form a new set A' with $|A'| = |B|$. We will construct H so that it has bipartition $\{A', B\}$. Then we will have $\|H\| = \Delta|B|$. Argue by induction on $l(G') = \Delta|B| - \|G'\|$. If $l = 0$ we are done. Otherwise

$$\sum_{v \in A'} d_{G'}(v) = \|G'\| = \sum_{v \in B} d_{G'}(v) = \Delta|B| - l(G') > 0.$$

Thus there exist vertices $a \in A'$ and $b \in B$ such that $d_{G'}(a), d_{G'}(b) < \Delta$. Set $G^+ = G' + e$, where e is a new, possibly parallel edge, joining a and b. Then $l(G^+) < l(G')$, and so we are done by the induction hypothesis. □

Now we consider edge coloring of general graphs. The fundamental result is Theorem 97 due to Vizing. The following lemma does most of the work in its proof.

LEMMA 96. *Suppose $G = (V, E)$ is a simple graph with $\Delta(G) \le k \in \mathbb{N}$, and $v \in V$. If $\chi'(G - v) \le k$ and $d(x) = k$ for at most one $x \in N(v)$ then $\chi'(G) \le k$.*

PROOF. Argue by induction on k. If $k = 1$ then E is a matching, and so $\chi'(G) \le 1$. Now suppose $k > 1$. For a function $f : E \to [k]$ and $\alpha \in [k]$, set

$$f(x) := [k] \smallsetminus \{f(e) : e \in E(x)\} \text{ and } f_\alpha := \{x \in N(v) : \alpha \in f(x)\}.$$

By adding edges and vertices to G, we may assume $k-1 \le d(x) \le k = d(v)$ for all $x \in N(v)$, and $d(y) = k$ for exactly one $y \in N(v)$. So $|f(x)| = 2$ for all $x \in N(v) - y$ and $|f(y)| = 1$. Choose a k-edge-coloring f of $G' := G - v$ maximizing $T(f) := \{\beta \in [k] : 1 \le |f_\beta| \le 2\}$.

Suppose $|f_\alpha| \ne 1$ for all $\alpha \in [k]$. Since $\sum_{\alpha \in [k]} |f_\alpha| = \sum_{x \in N(v)} |f(x)| = 2k - 1$, there exist $\beta, \gamma \in [k]$ with $|f_\beta| = 0$ and $|f_\gamma| \ge 3$; say $w \in f_\gamma$. Set $G_{\beta,\gamma} = (V - v, E_{\beta,\gamma})$, where $E_{\beta,\gamma} = \{e \in E : f(e) \in \{\beta, \gamma\}\}$. Then the component of $G_{\beta,\gamma}$ containing w is a path P with ends w and (say) z, where $f(z) \cap \{\beta, \gamma\} \ne \emptyset$. Obtain a new k-edge coloring f' of G' by exchanging colors γ and β on the edges of P. Then $f'(u) = f(u)$ for $u \in V(G') \smallsetminus \{w, z\}$, and $f'(w) = f(w) - \gamma + \beta$. Thus $w \in f'_\beta \subseteq \{w, z\}$. Hence $T(f) \subsetneq T(f') + \beta$, a contradiction.

So $f_\alpha = \{z\}$ for some $z \in N(v)$ and $\alpha \in [k]$; say $\alpha = k$. Set $M = f^{-1}(k) + vz$. Since neither z nor v are incident to any edges colored k, M is a matching. Put $H := G - M$. Since $f_k = \{z\}$ and $vz \in M$, every vertex of $N[v]$ is M-saturated. So $d_H(x) \le k-1$ for every $x \in N_H(v)$, and equality holds at most once. Since $f^{-1}(k) \subseteq M$, f is a $(k-1)$-coloring of $H - v$, and $\Delta(H - v) \le k-1$. So $\Delta(H) \le k-1$. By induction, $\chi'(G) \le \chi'(H) + 1 \le_{i.h.} k$. □

THEOREM 97 (7.1.10 Vizing (1964)). *Every graph $G = (V, E)$ satisfies $\chi'(G) \le \Delta(G) + 1$.*

PROOF. Set $k := \Delta(G) + 1$ and argue by induction on $|G|$. If $|G| = 1$ then $\chi'(G) \le 1 = k$. Otherwise choose $v \in V$. By induction, $\chi'(G - v) \le k$, and so by Lemma 96, $\chi'(G) \le k$. □

THEOREM 98 (Full Vizing (1964)). *Every multigraph M satisfies $\chi'(M) \le \Delta(M) + \mu(M)$.*

HW 57 (*). Let G be a graph with $\Delta(G) = k$. Put $X = \{v \in V(G) : d(v) = k\}$. Prove that if $G[X]$ is acyclic then $\chi'(G) \le k$. [Hint: Use Lemma 96.]

HW 58 (+). Prove Theorem 98.

CONJECTURE 99 (Goldberg (1973), Seymour (1979)). *Every multigraph M with $\chi'(M) \ge \Delta(M) + 2$ satisfies $\chi'(M) = \max_{H \subseteq M} \lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor} \rceil$.*

HW 59 (*). Prove that $\chi'(M) \ge \max_{H \subseteq M} \lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor} \rceil$.

If H is the line graph of a simple graph G then H contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ (a K_5 missing one edge). Also, $\chi(H) = \chi'(G)$ and $\omega(H) = \Delta(G)$, unless $\Delta(G) = 2$ and $\omega(G) = 3$. So the following theorem (with an extra observation for the case $\Delta(G) = 2 < \omega(G)$) extends Vizing's Theorem for simple graphs.

THEOREM 100 (Kierstead & Schmerl 1983). *Every graph H that contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ satisfies $\chi(H) \leq \omega(H) + 1$.*

5.5. List Coloring

DEFINITION 101. Let $G = (V, E)$ be a graph and C a set of colors. We write 2^C for the power set of C. A *list assignment* for G is a function $f : V \to 2^C$. One should think of $f(v) \subseteq C$ as the set of colors that are available for coloring the vertex v. A *k-list assignment* is a list assignment f such that $|f(v)| = k$ for all $v \in V$. Given a list assignment f, an f-coloring is a proper coloring g such that $g(v) \in f(v)$ for all $v \in V$. In this case G is f-colorable. The graph G is *k-list-colorable* (also *k-choosable*) if for every k-list assignment f it is f-colorable. The *list-chromatic number* (also *choosability*) $\chi_l(G)$ *of* G is the least k such that it is k-list colorable.

EXAMPLE 102. Let $G = K_{t,t^t}$. Then $\chi(G) = 2$, but $\chi_l(G) \geq t + 1$.

PROOF. Let X, Y be a bipartition of G with $|X| = t$. Let f be a t-list assignment for G such that the vertices of X have disjoint lists of size t, and for each $\sigma \in \prod_{x \in X} f(x)$ there exists $y_\sigma \in Y$ with $f(y) = \text{range}(\sigma)$. Then for any f-coloring σ of $G[X]$, the vertex v_σ cannot be colored from the list $f(y_\sigma) = \sigma$. $\square$

DEFINITION 103. An *edge-list assignment* for G is a function $f : E \to 2^C$. One should think of $f(e) \subseteq C$ as the set of colors that are available for coloring the edge e. A *k-edge-list assignment* is a list assignment f such that $|f(e)| = k$ for all $e \in E$. Given an edge-list assignment f, an f-coloring is a proper edge-coloring g such that $g(e) \in f(e)$ for all $e \in E$. In this case, G is f-list-colorable. The graph G is *k-edge-list-colorable* (also *k-edge-choosable*) if for every k-edge-list assignment f, it is f-colorable. The *list-chromatic index* $\chi'_l(G)$ *of* G is the least k such that it is k-edge-list colorable.

CONJECTURE 104. *Every graph G satisfies $\chi'_l(G) = \chi'(G)$.*

DEFINITION 105. A *kernel* of a digraph $D = (V, A)$ is an independent set $S \subseteq V$ such that for every $x \in V \setminus S$ there exists $y \in S$ with $xy \in A$.

LEMMA 106 (8.4.29 Bondy & Boppana & Siegel). *Let $D = (V, A)$ be a digraph all of whose induced subgraphs have kernels. If f is a list assignment for D satisfying $d^+(v) < |f(v)|$ for all $v \in V$ then D has an f-coloring.*

PROOF. Argue by induction on $|D|$. If $V = \emptyset$ the conclusion is vacuously true. Otherwise, fix $v_0 \in V$. Since $|f(v_0)| > d^+(v_0) \geq 0$, there exists $\alpha \in f(v_0)$. Set $W = \{v \in V : \alpha \in f(v)\}$. Then $v_0 \in W$. By hypothesis $D[W]$ has a (nonempty) kernel S. Color every vertex in S with α. This is possible because S is independent and $\alpha \in f(v)$ for every vertex in S.

Now it suffices to f-color $D' = D - S$ so that no vertex in $D - S$ is colored α. For this purpose, let f' be the list assignment for D' defined by $f'(v) = f(v) - \alpha$. Since $|D'| < |D|$, using induction, it suffices to show that $|f'(v)| > d^+_{D'}(v)$ for all $v \in V \setminus S$.

If $v \notin W$ then $\alpha \notin f(v)$, and so

$$|f'(v)| = |f(v)| > d^+_D(v) \geq d^+_{D'}(v).$$

Else $v \in W$. Since S is a kernel of $D[W]$, there exists $w \in S = V \smallsetminus V(D')$ with $vw \in A$. So

$$|f'(v)| = |f(v) - \alpha| > d^+_D(v) - 1 \geq d^+_{D'}(v).$$

□

Theorem 107 (8.4.30 Galvin 1995). *Every X, Y-bigraph G satisfies $\chi'_l(G) = \Delta(G)$.*

Proof. Let $\Delta := \Delta(G)$ and set $H := L(G)$. Then $\chi'_l(G) = \chi_l(H)$ and $\chi'(G) = \Delta$ (Theorem 95). Fix a Δ-edge coloring $c : E(G) \to [\Delta]$.

Let L be a Δ-edge-list assignment for G; so L is a Δ-list assignment for H. Our plan is to apply Lemma 106 to H to show that it has an L-coloring f; then f is an L-edge-coloring of G. So it suffices to show H has an orientation $D := (E(G), A)$ such that (i) $\Delta^+(D) \leq \Delta - 1$ and (ii) every induced subgraph of D has a kernel.

Each $ee' \in E(H)$ satisfies $e \cap e' \subseteq X$ or $e \cap e' \subseteq Y$, but not both. Define D by putting

$$ee' \in A \text{ iff } (e \cap e' \subseteq X \wedge c(e) > c(e')) \vee (e \cap e' \in Y \wedge c(e) < c(e')).$$

Each $e \in E(G) = V(H)$ satisfies $d^+_H(e) \leq \Delta - 1$, since it has at most $c(e) - 1$ out-neighbors e' with $e \cap e' \subseteq X$ and $\Delta - c(e)$ out-neighbors e' with $e \cap e' \subseteq Y$. So (i) holds for D.

For (ii), consider any induced subgraph $D' := D[F]$, where $F \subseteq E(G) = V(D)$. Argue by induction on $|D'| \geq 1$. Let $X' := \{x \in X : E_G(x) \cap F \neq \emptyset\}$. For each $x \in X'$, choose $e_x \in E_G(x) \cap F$ with $c(e_x)$ minimum. Then $ee_x \in A$ for every $e \in E_G(x) \cap F - e_x$. If $Q = \{e_x : x \in X'\}$ is independent then it is a kernel of D'; else fix distinct $x, x' \in X'$ with $e_x \cap e_{x'} \neq \emptyset$. Then $e_x, e_{x'} \in E_G(y)$ for some $y \in Y$; say $c(e_x) < c(e_{x'})$. Let $D'' = D' - e_x$. By induction, D'' has a kernel S. If $e_{x'} \in S$ then S is a kernel for D', since $e_x e_{x'} \in A$. Otherwise, $e_{x'} e^* \in A$ for some $e^* \in S$. The choice of $e_{x'}$ implies $e_{x'} \cap e^* \nsubseteq X$. So $e_{x'} \cap e^* \subseteq Y$, and thus $e_x, e_{x'}, e^* \in E_G(y)$. So $c(e^*) > c(e_{x'}) > c(e_x)$; so $e_x e^* \in A$. Hence S is a kernel for D'. □

CHAPTER 6

Planar graphs

We have been informally drawing graphs in the Euclidean plane $\mathbb{R}^2$ since the start of the semester. Now we formalize the definition of a drawing of a graph in $\mathbb{R}^2$.

6.1. Very Basic Topology of the Euclidean Plane

Let $p, q \in \mathbb{R}^2$. The p, q-line segment $L_{p,q}$ is the subset of $\mathbb{R}^2$ defined by $L(p, q) := \{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$ and $\mathring{L}(p, q) := L(p, q) \setminus \{p, q\}$. For distinct points $p_0, \ldots, p_k \in \mathbb{R}^2$, the union $A(p_0, \ldots, p_k) := \bigcup_{i \in [k]} L(p_{i-1}, p_i)$ is a (polygonal) p_0, p_k-arc provided $L(p_{i-1}, p_i) \cap \mathring{L}(p_{j-1}, p_j) = \emptyset$ for all distinct $i, j \in [k]$. We say that p_0 and p_k are linked by $A(p_0, \ldots, p_k)$. If $A(p_0, \ldots, p_k)$ is an arc and $L(p_0, \ldots, p_k) \cap \mathring{L}(p_k, p_0) = \emptyset$ then $P(p_0, \ldots, p_k, p_0) := L(p_0, \ldots, p_k) \cup L(p_k, p_0)$ is a polygon. Note that arcs and polygons are closed in $\mathbb{R}^2$.

For $x \in \mathbb{R}^2$ the *open ball around x with radius r* is the set $B_r(x) := \{y \in \mathbb{R}^2 : \|x, y\| < r\}$. A set $U \subseteq \mathbb{R}^2$ is *open* if for all points $p \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$. In particular, $\mathbb{R}^2$ and $\emptyset$ are open. The complement of an open set is a *closed* set. The *frontier* of a set X is the set of all points $y \in \mathbb{R}^2$ such that $B_r(x) \cap X \neq \emptyset$ and $B_r(x) \setminus X \neq \emptyset$ for all $r > 0$. Note that if X is open, then its frontier lies in $\mathbb{R}^2 \setminus X$.

Let U be an open set. Two points $x, y \in U$ are linked in U if there exists an x, y-arc contained in U. The relation of being linked is an equivalence relation on U. Its equivalence classes are called *regions*. Regions are open: Suppose $R \subseteq U$ is a region and $x \in R$. Then there exists a $r > 0$ such that $B_r(x) \subseteq U$. Clearly every $y \in B_r(x)$ is linked to x in U, since $L(x, y) \subseteq B_r(x)$. So $B_r(x) \subseteq R$. A closed set X separates a region R if $R \setminus X$ has more than one region.

Suppose X is a finite union of points and arcs. Then $U := \mathbb{R}^2 \setminus X$ is open. Let R be a region of U, and F be the frontier of R.

THEOREM 108 (Jordan Curve Theorem for Polygons). *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has exactly two regions. Each of these regions has the entire polygon as its frontier.*

PROOF. *To be continued ...* □

6.2. Graph Drawings

Let $G = (V, E)$ be a (multi)graph. A *drawing* of G is a graph $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ such that $\widetilde{V} \subseteq \mathbb{R}^2$, each edge $e \in \widetilde{E}$ is an arc between its ends, and $\widetilde{G} \cong G$. So edges are no longer just pairs of vertices, but have their own identity and structure (we need this anyway to formally deal with different edges linking the same two vertices). It should be clear that every finite graph has a drawing. Moreover, by moving vertices slightly and readjusting edges, we can (and do) require the following additional properties for drawings, without restricting the set of (finite) graphs that can be drawn.

(1) No three edges have a common internal point.
(2) The only vertices contained in an edge are its endpoints.
(3) No two edges are tangent.
(4) No two edges have more than one common internal point.

A *plane* (multi)graph is a drawing of a (multi)graph that has no crossing, i.e., two edges that have a common internal vertex. A *planar* (multi)graph is a (multi)graph that has a plane drawing.

Let $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ be a plane (multi)graph. The faces of $\widetilde{G}$ are the regions of $\mathbb{R}^2 \setminus (\widetilde{V} \cup \bigcup \widetilde{E})$. The frontier of a face is called its *boundary*. A boundary is the union of edges and vertices of G. The edges of a face can be oriented to form its *boundary walk*. Note that a plane cycle is a polygon. By the Jordan Curve Theorem we have:

Proposition 109 (Top.). *A plane cycle is the boundary of exactly two faces.*

To be continued ...

6.3. Basic facts

Fact 110 (Top.). *Let G be a planar multigraph. Suppose $e \in E$ is not a loop. Then $G - e$ has one fewer faces than G iff e is not a cut-edge.*

Fact 111 (Top.). *Contracting a non-loop does not change the number of faces. Deleting a loop, reduces the number of faces by* 1.

Fact 112 (Top.). *Let G be a planar multigraph. Every non-cut-edge appears exactly once on the bounding walk of exactly two faces. Every cut-edge appears exactly twice on the bounding walk of exactly one face. In particular,*

$$\sum_{f \in F(G)} l(f) = 2\,\|G\|\,.$$

Theorem 113 (6.1.21 Euler's Formula (1758)). *Let G be a connected planar multigraph. Then*

$$|G| - \|G\| + |F(G)| = 2.$$

Proof. Argue by induction on $|G|$.
Base Step: $|G| = 1$. In this case all edges of G are loops. Argue by secondary induction on $\|G\|$. For the base step $\|G\| = 0$, note that G has one vertex and one face, and so

$$|G| - \|G\| + |F(G)| = 1 - 0 + 1 = 2.$$

For the induction step, set $G' := G - l$ for some loop l. Then using Fact 111:

$$|G| - \|G\| + |F(G)| = |G'| - (\|G'\| + 1) + (|F(G')| + 1) =_{i.h.} 2.$$

Induction Step: $|G| > 1$. Since G is connected, G has a non-loop edge e. Set $G' = G \cdot e$. By Fact 111,

$$|G| - \|G\| + |F(G)| = (|G'| + 1) - (\|G'\| + 1) + |F(G')| = 2.$$

□

Lemma 114. *Let $\widetilde{G}$ be a drawing of a* 2-connected *planar graph G. Then the boundary walk of every face is a cycle.*

Proof. By Theorem 67, G has an an ear decomposition $P_0, \ldots, P_h$. Argue by induction on h. For the base step $h = 0$, note that P_0 is a cycle that bounds the only two faces of $\widetilde{G}$ (Jordan Curve Theorem). Now consider the induction step. Let P_h have ends x, y, and note that $\mathring{P}_h$ is contained in some face f of $\widetilde{H} = \widetilde{G} - \mathring{P}_h$. By the induction hypothesis, every face of $\widetilde{H}$ is bounded by a cycle. Let $C = xv_1 \ldots v_a y v_{a+2} \ldots v_b x$ be the cycle that bounds f. Observe that $F(\widetilde{H}) - f \subseteq F(\widetilde{G})$. Using Euler's formula,

$$\begin{aligned} |F(\widetilde{G})| &= 2 - |\widetilde{G}| + \left\|\widetilde{G}\right\| = 2 - (|\widetilde{H}| + |P_h| - 2) + (\left\|\widetilde{H}\right\| + \|P_h\|) \\ &= (2 - |\widetilde{H}| + \left\|\widetilde{H}\right\|) - (|P_h| - \|P_h\|) + 2 \\ &= |F(\widetilde{H})| + 1. \end{aligned}$$

So $\mathring{P}_h$ divides $f \smallsetminus \mathring{P}_h$ into exactly two new faces. By the Jordan Curve Theorem, one must be bounded by the cycle $xyv_{a+2} \ldots v_b x$ and the other by the cycle $xv_1 \ldots v_a yx$. □

Theorem 115. *If G is a simple planar graph with at least three vertices then $\|G\| \leq 3|G| - 6$. Moreover, if G has girth greater than 3 then $\|G\| \leq 2|G| - 4$.*

Proof. We may assume that G is a maximal planar graph, i.e., it is not a spanning subgraph of any planar graph with more edges. Then G is connected, since otherwise we could add an edge between two components of G while maintaining planarity. Since G is connected and has at least three vertices, $\|G\| \geq 2$; since it is simple it has no parallel edges. Thus the length of every face is at least 3. By Fact 112,

$$2\|G\| = \sum_{f \in F(G)} l(f) \geq 3|F(G)|.$$

So $|F(G)| \leq \frac{2}{3}\|G\|$. By Theorem 113

$$2 = |G| - \|G\| + |F(G)| \leq |G| - \frac{1}{3}\|G\|,$$

and so $3|G| - 6 \geq \|G\|$.

Now suppose that the girth of G is greater than 3. Then every face boundary has length at least 4. So

$$2\|G\| = \sum_{f \in F(G)} l(f) \geq 4|F(G)|.$$

Thus $|F(G)| \leq \frac{1}{2}\|G\|$. By Theorem 113

$$2 = |G| - \|G\| + |F(G)| \leq |G| - \frac{1}{2}\|G\|,$$

and so $2|G| - 4 \geq \|G\|$. □

Corollary 116. *Neither K_5 nor $K_{3,3}$ is planar.*

Proof. If K_5 is planar then Theorem 115 yields the following contradiction:

$$10 = \|K_5\| \leq 3|K_5| - 6 = 9.$$

If $K_{3,3}$ is planar, then since it is bipartite, and so has girth greater than 3, Theorem 115 yields the contradiction:

$$9 = \|K_{3,3}\| \leq 2|K_{3,3}| - 4 = 8.$$

$\square$

Theorem 117 (6.2.2 Kuratowski (1930)). *A graph is planar iff it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.*

We will break the proof of Kuratowski's Theorem into smaller pieces. First we need some preparation. Call a subdivision of K_5 or $K_{3,3}$ a *K-graph* (for Kuratowski).

Lemma 118. *Let $e = xy$ be an edge of a graph G. If G contains no K-graph then $G \cdot e$ contains no K-graph.*

Proof. We prove the contrapositive. So suppose $G \cdot e$ contains a K-graph Q. Then $Q \subseteq G$ unless $v_e \in V(Q)$. So suppose $v_e \in V(Q)$. Note that $N_Q(v_e) \subseteq N_G(x) \cup N_G(y)$. First suppose that there exists $z \in \{x, y\}$ such that $|N_Q(v_e) \smallsetminus N_G(z)| \leq 1$. (This is the case if $d(v_e) \leq 3$.) Say $z = x$. Then replacing v_e by x or x, y, xy yields a subdivision of the K-graph Q. Otherwise, $d_Q(v_e) = 4$ and and $|N_Q(v_e) \smallsetminus N_G(z)| = 2$ for both $z \in \{x, y\}$. In this case Q is a subdivision of K_5. We show that G contains a subdivision of $K_{3,3}$! Let $Q' = Q - v_e + x + y + xy$, where x and y are new branch vertices, and (say) a, b, c, d are the remaining old branch vertices. Then Q' contains a subdivision $K_{3,3}$, where the branch vertices have the bipartition $\{\{x, a, b\}, \{y, c, d\}\}$. $\square$

Corollary 119. *If G is planar then G does not contain a K-graph.*

Proof. Suppose $Q \subseteq G$ is a K-graph. We show by induction on the number h of subdivision (degree 2) vertices in Q that G is not planar. The base step $h = 0$ is Corollary 116. For the induction step $h > 0$, consider a subdivision vertex x and one of its two neighbors y. Contracting xy yields a K-graph with one less subdivision vertex. Thus by the induction hypothesis $G \cdot xy$ is nonplanar, and so G is also nonplanar. $\square$

Lemma 120 (Top.). *Let $\widetilde{G}$ be a drawing of G with a face f. Then G can be redrawn as $\widetilde{H}$ so that the boundary of f is the boundary of the outer face in $\widetilde{G}$.*

Proof. Transfer $\widetilde{G}$ to a sphere, and then poke a hole in f. Stretching the resulting surface to the plane yields the desired drawing $\widetilde{H}$. $\square$

Let $G = (V, E)$ be a graph, and $S \subseteq V$. An S-lobe is a subgraph of the form $G[S \cup V(H)]$, where H is a component of $G - S$.

Lemma 121 (Top.). *If G is nonplanar, but every proper subgraph of G is planar, then G is 2-connected.*

Proof. Certainly G is connected since we can draw an edge between the outer faces of any two components of G. Otherwise G has a cut-vertex v. Let H be a $\{v\}$-lobe of G and $H' = G - (H - v)$. Then by the minimality of G, H and H' are planar. Clearly we can attach a drawing $\widetilde{H}$ of H with v on the outer face boundary to a face of a drawing $\widetilde{H'}$ of H' whose boundary contains v. This violates the hypothesis that G is nonplanar. $\square$

Lemma 122 (Top. Contrapositive, lobes). *Let G be a graph with subgraph $S = K_2 \subseteq G$. If G is nonplanar then there exists a nonplanar S-lobe of G.*

Proof. Argue by induction on the number h of components of $G-S$. The base step $h=1$ is trivial; so consider the the induction step. Let H be an S-lobe, and $G'=G-(H-S)$. If H is nonplanar then we are done, and if G' is nonplanar we are done by induction. Otherwise let $\widetilde{H}$ be a drawing of H with S on the boundary of the outer face and $\widetilde{G'}$ be a drawing of G'. We can draw G by deforming $\tilde{H}$ to fit in a face of $\widetilde{G'}$ containing S, a contradiction. □

Lemma 123. *If G is nonplanar and contains no K-graph (i.e., it is a counter example to Kuratowski's Theorem) and among such graphs $\|G\|$ is minimum then G is 3-connected.*

Proof. Suppose G is a minimum counterexample. Then G does not contain a K-graph and G is not planar. Thus no subgraph of G contains a K-subgraph. Since G is minimum, every proper subgraph of G is planar. So by Lemma 121, G is 2-connected.

Suppose for a contradiction G has a separating 2-set $S=\{x,y\}$; then it is a minimum separating set. Of course, $G^+ := G+xy$ is nonplanar. By Lemma 122, there exists a nonplanar S-lobe H^+ of G^+. Since S is a minimal separating 2-set, it has at least two neighbors in $G-H$. Thus $\|H^+\| \leq \|G^+\|-2 < \|G\|$. So H^+ is not a counterexample. It follows that H^+ must contain a K-graph Q. All of Q appears in $H := G[V(H^+)]$ except xy. Let $H' \neq H$ be a component of $G-S$. Then H' contains an x,y-path P. So $Q-xy+P$ is a K-graph contained in G, a contradiction. □

Theorem 124. *If G is a 3-connected graph that does not contain any K-graph then G is planar.*

Proof. We argue by induction on $|G|$. If $|G| \leq 4$ then G is planar, since $G \subseteq K_4$, and K_4 is planar. So suppose $|G| \geq 5$. By Theorem 69, G has an edge $e=xy$ such that $H=G\cdot e$ is 3-connected. By Corollary 118, H does not contain a K-graph. Thus by the induction hypothesis, H is planar. Let $\widetilde{H}$ be a drawing of H.

Let $H' = H - v_e$, and let f be the face of $\widetilde{H}-v_e$ that contains v_e. Since H' is 2-connected, Lemma 114 implies that f is bounded by a cycle C, say with orientation $\overrightarrow{C}$, and $(N_G(x)\cup N_G(y)) \smallsetminus \{x,y\} = N_H(v_e) \subseteq V(C)$. Choose the notation so that $d_G(x) \leq d_G(y)$. Obtain a drawing $\widetilde{G'}$ of $G'=G-y$ from $\widetilde{H}'$ by drawing x at the point corresponding to v_e in $\widetilde{H}$ and deleting the edges $v_e z$ with $z \in N_G(y) \smallsetminus N_G(x)$. Our goal is to extend $\widetilde{G}'$ to a drawing $\widetilde{G}$ of G by adding y and the edges in $E(y)$ to $\widetilde{G}'$.

Let $x_1,\dots,x_k$ be the neighbors of x in G' arranged in cyclic order around $\overrightarrow{C}$, and $U := N_G(y)-x$. If there exists an index i such that $U \subseteq C_i := V(x_i\overrightarrow{C}x_{i+1})$ then we can extend $\widetilde{G'}$ to $\widetilde{G}$ by drawing y in the face f' of $\widetilde{G'}$ bounded by $xx_iCx_{i+1}x$, and then drawing edges from y to each vertex of $U+x$. This is possible, because all vertices of $U+x$ appear on the boundary of f'. Otherwise for every index i there exists $u_i \in U$ such that $u_i \notin C_i$.

If there exists an index i and a vertex $v_i \in N(y)\cap V(C_i-x_i-x_{i+1})$ then the vertices

$$v_i, u_i, x, x_i, x_{i+1}, y$$

are the branch vertices of a subdivision of $K_{3,3}$ with bipartition $\{\{v_i,u_i,x\},\{x_i,x_{i+1},y\}\}$, where the edges of the cycle $x_iv_ix_{i+1}u_ix_i$ are represented by the paths

$$x_i\overrightarrow{C}v_i, v_i\overrightarrow{C}x_{i+1}, x_{i+1}\overrightarrow{C}u_i, u_i\overrightarrow{C}x_i.$$

(Draw the picture.) This is a contradiction.

Otherwise, $U \subseteq N_G(x)$. Since $d_G(x) \leq d_G(y)$, we have $U = N_G(x) - y$. Thus $S := N_G(x) - y$ separates xy from the rest of G. Since $|G| \geq 5$ and G is 3-connected, $|S| \geq 3$. Say $a, b, c \in S$. Then x, y, a, b, c are the branch vertices of a subdivision of a K_5, where the edges of the cycle a, b, c, a are represented by the paths $a\overrightarrow{C}b, b\overrightarrow{C}c, c\overrightarrow{C}a$. (Draw the picture.) This is a contradiction. □

Proof of Kuratowski's Theorem 124. First suppose G contains a K-graph. Then by Lemma 119, G is not planar. Now suppose, for a contradiction, that some nonplanar graph does not contain a K-subgraph. Let G be such a graph with the minimum number of edges. By Lemma 123, G is 3-connected. But then by Theorem 124, G is planar, a contradiction. □

Theorem 125 (8.4.32 Thomassen (1994)). *Every simple planar graph G is 5-list colorable.*

Proof. It suffices to prove the following more technical statement by induction on G.

Claim. Suppose $\widetilde{G}$ is a drawing of a simple planar graph G such that every interior face, has length three, the boundary of the outer face is a cycle C, and $xy \in E$ is an edge of C. If L is a list assignment for G such that

(1) $L(x) = \{\alpha\}$, $L(y) = \{\beta\}$, and $\alpha \neq \beta$,
(2) $|L(v)| \geq 3$ for all vertices v on the outer boundary, except x and y, and
(3) $|L(v)| \geq 5$ for all vertices not on the outer boundary,

then G has an L-coloring.

To see that the claim implies the theorem note that adding edges and vertices to G, and deleting colors from some lists of a list assignment L does not make it easier to L-color G. Moreover, we can add edges to a planar graph G so that every face is bounded by a C_3, and G remains planar. So it suffices to prove the claim.

Proof of Claim. Argue by induction on $|G|$. Note that $|G| \geq |C| \geq 3$. First consider the base step $|G| = 3$. Color x with α and y with β. The last vertex z has at least three colors in its list, and so we can color it with a color distinct from α and β.

Now consider the induction step $|G| > 3$. The outer face of $\widetilde{G}$ is bounded by a cycle $C = v_1v_2 \dots v_sv_1$ with $v_1 = x$ and $v_2 = y$.

Case 1. C has a chord v_iv_j with $i > j$. Let $C_1 = v_iv_{i+1} \dots v_jv_i$ and $C_2 = v_jv_{j+1} \dots v_iv_j$ be the two nonspanning cycles contained in $C + v_iv_j$. Let $\widetilde{G}_i$ be the plane graph formed by C_i and its interior. Then $\widetilde{G}_1 \cup \widetilde{G}_2 = G$, $\widetilde{G}_1 \cap \widetilde{G}_2 = \widetilde{G}[\{v_iv_j\}]$, and $x, y \in V(C_1)$. By the induction hypothesis, there exists an L-coloring f_1 of $\widetilde{G}_1$. Set $x' = v_i$, $\alpha' = f(x')$, $y' = v_j$, $\beta' = f(y')$, $L'(x') = \{\alpha'\}$, $L'(y') = \{\beta'\}$ and $L'(v) = L(v)$ for all vertices of $\widetilde{G}_2 - x' - y'$. Then by the induction hypothesis there exists an L'-coloring g_2 of $\widetilde{G}_2$. It follows that $f = g_1 \cup g_2$ is an L-coloring of G.

Case 2. C does not have a chord. Since every interior face is bounded by a C_3, $G[N(v_s)]$ contains a hamiltonian x, v_{s-1}-path P. Moreover, since C has no chords, the outer face of $\widetilde{G'} = \widetilde{G} - v_s$ is bounded by the cycle $C' = xPv_{s-1}C^*v_1$ $(= x)$. Of course, the interior faces of $\widetilde{G'}$ have length three. Let $\gamma, \delta \in L(v_s)$ be distinct colors not equal to α. Define a list assignment L' for L by

$$L'(v) = \begin{cases} L(v) - \gamma - \delta & \text{if } v \in V(P) - x - v_{s-1} \\ L(v) & \text{else} \end{cases}.$$

By the induction hypothesis $\widetilde{G'}$ has an L' coloring f'. Choose $\varepsilon \in \{\gamma, \delta\}$ such that $\varepsilon \neq f'(v_{s-1})$. Then f' can be extended to an L-coloring f of G by setting $f(v_s) = \varepsilon$.

$\square$

This completes the proof of the claim and the Theorem. $\square$

HW 60 (*). Let G be a simple planar graph with girth (length of the shortest cycle) k. Prove that $\|G\| \leq \frac{k}{k-2}(|G| - 2)$.

HW 61 (*). Prove that every simple planar graph G with $|G| \geq 4$ has at least four vertices with degree less than six.

HW 62 (+). Prove that every simple planar graph G with $\delta(G) = 5$ has a matching with at most $\frac{1}{5}|G|$ unsaturated vertices.

HW 63 (*). Prove that the vertices of a simple planar graph can be ordered so that every vertex is preceded by at most five of its neighbors. Similarly, prove that the vertices of every planar bipartite graph can be ordered so that each vertex is preceded by at most three of its neighbors.

HW 64 (*). Prove that every planar bipartite graph satisfies $\chi_l(G) \leq 4$. [Hint: Use the previous problem.

HW 65 (+). Prove that every orientation of every X, Y-bigraph has a kernel. [Hint: When is X a kernel?]

HW 66 (*). Prove that every bipartite planar graph G satisfies $\chi_l(G) \leq 3$. [Hint: Use the previous problem.]

CHAPTER 7

Extras

7.1. Lower Bounds on Ramsey's Theorem

THEOREM 126. *For every integer $k \geq 2$ there exists a graph G on k vertices such that $\omega(G) < k$ and $\alpha(G) < k$ and $|G| \geq \lfloor 2^{k/2-1/2} \rfloor$. In other words,* $\mathrm{Ram}(k,k) > 2^{k/2-1/2}$.

PROOF. Fix $k \geq 2$, and set $n = \lfloor 2^{k/2-1/2} \rfloor$. Let V be a set of n vertices, and $\mathcal{G}$ be the set of all graphs G with $V(G) = V$. So $G = (V, E) \in \mathcal{G}$ if and only iff $E \subseteq \binom{V}{2}$. Since there are $2^{\binom{n}{2}}$ choices for E,

$$|\mathcal{G}| = N := 2^{\binom{n}{2}}. \tag{7.1.1}$$

For $X \subseteq V$ with $|X| = k$, let $\mathcal{G}_X$ be the set of graphs in $\mathcal{G}$ such that X is a clique or co-clique. So if $G := (V, E) \in \mathcal{G}$ then $G \in \mathcal{G}_X$ iff $E \cap \binom{X}{2} \in \{\emptyset, \binom{X}{2}\}$ and $E \smallsetminus \binom{X}{2} \subseteq \binom{V}{2} \smallsetminus \binom{X}{2}$. There are two possibilities for the first conjunct and $2^{\binom{n}{2}-\binom{k}{2}}$ possibilities for the second. Thus

$$|\mathcal{G}_X| = 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} = 2N2^{-(k^2-k)/2}. \tag{7.1.2}$$

Any graph G in

$$\mathcal{G} \smallsetminus \bigcup_{X \in \binom{V}{k}} \mathcal{G}_X,$$

satisfies $\omega(G), \alpha(G) < k$ and $|G| = n$. So it suffices to prove $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| < |\mathcal{G}|$. Since (a) $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| \leq \binom{n}{k} |\mathcal{G}_X|$, (b) $\binom{n}{k} < \frac{n^k}{k!}$, and (c) $\frac{n}{2^{k/2-1/2}} \leq 1$, this follows from:

$$\begin{aligned}
|\mathcal{G}| - |\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| &\geq N - \binom{n}{k} \cdot 2N2^{-(k^2-k)/2} && ((a), (7.1.1), (7.1.2)) \\
&> N(1 - \frac{2}{k!} n^k 2^{-(k-1)k/2}) && (b) \\
&\geq N(1 - (\frac{n}{2^{k/2-1/2}})^k) \geq 0. && (c)
\end{aligned}$$

$\square$

7.2. Equitable Coloring

DEFINITION 127. An equitable k-coloring of a graph $G = (V, E)$ is a proper coloring $f : V \to [k]$ such that difference $||f^{-1}(i)| - |f^{-1}(j)||$ in the sizes of the the i-th and j-th color classes is at most 1 for all $i, j \in [k]$. In particular, every color is used if $|G| \leq k$.

THEOREM 128 (Hajnal & Szemerédi Theorem (1976)). *Every graph G with maximum degree at most r has an equitable $(r+1)$-coloring.*

The proof was long and sophisticated, and does not provide a polynomial time algorithm. Kierstead and Kostochka found a much simpler and shorter proof. This better understanding has led to many new results, several of which are stated below.

Let $\theta(G) = \max\{d(x) = d(y) : xy \in E(G)\}$.

THEOREM 129 (Kiestead & Kostochka (2008)). *For every $r \geq 3$, each graph G with $\theta(G) \leq 2r+1$ has an equitable $(r+1)$-coloring.*

THEOREM 130 (Kierstead, Kostochka, Mydlarz & Szemerédi). *There is an algorithm that constructs an equitable k-coloring of any graph G with $\Delta(G)+1 \leq k$, using time $O(r|G|^2)$.*

PROBLEM 131. Find a polynomial time algorithm for constructing the coloring in Theorem (129).

One might hope to prove an equitable version of Brooks' Theorem, but the following example shows that the statement would require special care: For r is odd, $K_{r,r}$ satisfies $\Delta(K_{r,r}) = r$ and $\omega(G) = 2$, but has no r-equitable coloring. Chen, Lih and Wu [**?**] proposed the following common strengthening of Theorem 128 and Brooks' Theorem.

CONJECTURE 132. *Let G be a connected graph with $\Delta(G) \leq r$. Then G has no equitable r-coloring if and only if either (a) $G = K_{r+1}$, or (b) $r = 2$ and G is an odd cycle, or (c) r is odd and $G = K_{r,r}$.*

Kierstead and Kostochka have proved the conjecture for $r \leq 4$, and also for $r \geq \frac{1}{4}|G|$.

Proof of Theorem 128. Let G be a graph with $\Delta(G) \leq r$. We may assume that $|G|$ is divisible by $r+1$: If $|G| = s(r+1) - p$, where $p \in [r]$ then set $G' := G + K_p$. Then $|G'|$ is divisible by $r+1$ and $\Delta(G') \leq r$. Moreover, the restriction of any equitable $(r+1)$-coloring of G' to G is an equitable $(r+1)$-coloring of G. So we may assume $|G| = (r+1)s$.

We argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial, so consider the induction step. Let u be a non-isolated vertex. By the induction hypothesis, there exists an equitable $(r+1)$-coloring of $G - E(u)$. We are done unless some color class V contains an edge uv. Since $\Delta(G) \leq r$, some color class W contains no neighbors of u. Moving u to W yields an $(r+1)$-coloring of G with all classes of size s, except for one *small* class $V^- := V - u$ of size $s-1$ and one *large* class $V^+ := W + u$ of size $s+1$. Such a coloring is called *nearly equitable.*

Given a nearly equitable $(r+1)$-coloring, define an auxiliary digraph $\mathcal{H}$, whose vertices are the color classes, so that UW is a directed edge if and only if some vertex $y \in U$ has no neighbors in W. In this case we say that y *witnesses* UW. Let $\mathcal{A}$ be the set of classes from which V^- can be reached in $\mathcal{H}$, $\mathcal{B}$ be the set of classes not in $\mathcal{A}$ and $\mathcal{B}'$ be the set of classes reachable from V^+ in $\mathcal{H}[\mathcal{B}]$. Set $a := |\mathcal{A}|$, $b := |\mathcal{B}|$, $b' := |\mathcal{B}'|$, $A := \bigcup \mathcal{A}$, $B := \bigcup \mathcal{B}$ and $B' := \bigcup \mathcal{B}'$. Then $r+1 = a+b$. Since every vertex $y \in B$ has a neighbor in every class of $\mathcal{A}$ and every vertex $z \in B'$ also has a neighbor in every class of $\mathcal{B} - \mathcal{B}'$,

$$(*) \qquad d_A(y) \geq a \text{ for all } y \in B \text{ and } d_{A \cup B \setminus B'}(z) \geq a+b-b' \text{ for all } z \in B'.$$

Case 0: $V^+ \in \mathcal{A}$. Then there exists a V^+, V^--path $\mathcal{P} = V_1, \ldots, V_k$ in $\mathcal{H}$. Moving each witness y_j of V_jV_{j+1} to V_{j+1} yields an equitable $(r+1)$-coloring of G. $\square$

We now argue by a secondary induction on b, whose base step $b = 0$ holds by Case 0. Also $|A| = as - 1$ and $|B| = bs + 1$. Now consider the secondary induction step.

A class $W \in \mathcal{A}$ is *terminal*, if every $U \in \mathcal{A} - W$ can reach V^- in $\mathcal{H} - W$. Let $\mathcal{A}'$ be the set of terminal classes, $a' := |\mathcal{A}'|$ and $A' := \bigcup \mathcal{A}'$. An edge wz is *solo* if $w \in W \in \mathcal{A}'$, $z \in B$ and $N_W(z) = \{w\}$. Ends of solo edges are *solo* vertices and *solo neighbors* of each other.

Order $\mathcal{A}$ as $V^-, X_1, \ldots, X_{a-1}$ so that each X_i has a previous out-neighbor.

Case 1: For some $a - b \leq i \leq a - 1$, class X_i is not terminal. Then some $X_j \in \mathcal{A}'$ cannot reach V^- in $\mathcal{H} - X_i$. So $j > i$ and X_j has no out-neighbors before X_i. In particular, $d^+_{\mathcal{A}}(X_j) < b$. Then for each $w \in X_j$, $d_A(w) \geq a - b$, and so $d_B(w) < 2b$. Let S be the set of solo vertices in X_j, and $D := X_j \setminus S$. If $v \in B - N_B(S)$ then v has no solo neighbor in X_j, and so has at least two neighbors in D. Thus $2b\,|D| > 2\,|B - N_B(S)|$. Using $|S| + |D| = s$ and $r\,|S| \geq |E(S, A)| + |N_B(S)|$,

$$bs + (a-1)\,|S| = b\,|D| + r\,|S| > |B - N_B(S)| + |E(S, A)| + |N_B(S)| > bs + |E(S, A)|\,.$$

Thus $(a-1)\,|S| > |E(S, A)|$, and so there exists $w \in S$ with $d_A(w) \leq a-2$. Thus w witnesses some edge $X_jX \in E(\mathcal{H}[A])$. Since $w \in S$, it has a solo neighbor $y \in B$.

Move w to X and y to X_j. This yields nearly equitable colorings of $G[A+y]$ and $G[B-y]$. Since X_j is terminal, $X + w$ can still reach V^-. Thus by Case 0, $G[A + y]$ has an equitable a-coloring. By (*), $\Delta(G[B - y]) \leq b - 1$. So by the primary induction hypothesis $G[B - y]$ has an equitable b-coloring. After combining these equitable colorings we are done.

Case 2: All the last b classes $X_{a-b}, \ldots, X_{a-1}$ are terminal. Then $a' \geq b$. For $y \in B'$, let $\sigma(y)$ be the number of solo neighbors of y. Similarly to (*),

$$r \geq d(y) \geq a + b - b' + d_{B'}(y) + a' - \sigma(y) \geq r + 1 + d_{B'}(y) + a' - b' - \sigma(y).$$

So $\sigma(y) \geq a' - b' + d_{B'}(y) + 1$. Let I be a maximal independent set with $V^+ \subseteq I \subseteq B'$. Then $\sum_{y \in I}(d_{B'}(y) + 1) \geq |B'| = b's + 1$. Since $a' \geq b$,

$$\sum_{y \in I} \sigma(y) \geq \sum_{y \in I}(a' - b' + d_{B'}(y) + 1) \geq s(a' - b') + b's + 1 > a's = |A'|\,.$$

So some vertex $w \in W \in \mathcal{A}'$ has two solo neighbors y_1 and y_2 in the independent set I.

Since the class Y of y_1 is reachable from V^+, we can equitably b-color $G[B - y_1]$. Let Y' be the new class of y_2. If w witnesses an edge WX of $G[A]$ then we are done by Case 1; otherwise we can move w to some class $U \subseteq B' - y_1$. Replacing w with y_1 in W to get W^* and moving w to U yields a new nearly equitable $(r + 1)$-coloring of G. If $U \in \mathcal{A}$ then we are done by Case 0; otherwise at least $a + 1$ classes, W^*, Z', and all $X \in \mathcal{A}' - W$, can reach V^-. In this case we are done by the secondary induction hypothesis.

APPENDIX A

Exceptional notation

$|G| = |V(G)| = n(G)$, number
$\|G\| = |E(G)| = e(G)$, number
$E(v) = \{vu : vu \in E\}$, edge set
$K(A, B) = (A \cup B, \{ab : a \neq b \wedge (a, b) \in A \times B\})$, graph
$K(A) = K(A, A)$, graph
$E_G(A, B) = \{ab \in E(G) : a \neq b \wedge (a, b) \in A \times B\}$, edge set

APPENDIX B

Standards

B.0.1. MAT 416—Level C.

(1) Theorem 3 (Ramsey's Theorem for graphs).
(2) Theorem 21 (Characterization of bipartite graphs).
(3) Theorem 34 (Euler's Theorem). You may use Lemma 33.
(4) Theorem 44 (Tree Theorem). Provide proofs for required parts of Lemma 42 and Lemma 43.
(5) Theorem 51 (Berge's theorem on maximum matchings).
(6) Corollary 53. You may use Hall's Theorem.
(7) Theorem 59 Petersens's Matching Theorem.
(8) Theorem 67 (Whitney's Theorem).
(9) Lemma 106 (Kernel Lemma).
(10) Euler's Formula Theorem 113 and Theorem 115.

B.0.2. MAT 416—Level A. All of MAT 416—Level C and all homework and the following list.

(1) Corollary 23 (Dirac's Theorem). Provide proofs for required parts of Theorem 22.
(2) Theorem 52 (Hall's Theorem).
(3) Theorem 55 (König's Theorem).
(4) Theorem 58 (Berge-Tutte Formula).
(5) Theorem 60 (Petersen's 2-Factor Theorem).
(6) Theorem 69 (Thomassen's Contraction Theorem).
(7) Theorem 74 (Menger's Theorem).
(8) Example 86 Mycielski's Construction.
(9) Theorem 89 (Brooks' Theorem).
(10) Theorem 93 (Turan's Theorem).
(11) Theorem 97 (Vizing's Theorem).
(12) Theorem Lemma 106 (Kernel Lemma).
(13) Theorem 107 (Galvin;s Theorem).
(14) Kuratowski's Theorem 117. You may assume: 120, 121, 122, 118, 119 and 114. You should be able to prove: 123 and 124.
(15) Thomassen's 5-Choosability Theorem 125.
(16) Lower bound on $\mathrm{Ram}(k, k)$ Theorem 126.

B.0.3. MAT—513. All of MAT 416—Level A and old qualifiers.

APPENDIX C

Matching card trick

Consider a deck of $2k+1$ cards numbered $1,\dots,2k+1$, and denoted by $[2k+1]$. The class chooses a *hand* H consisting of $k+1$ of these cards, and gives them to Professor A. Professor A looks at them, puts one of them in his pocket, and then has a student spread the remaining k cards face-up on a table. Professor B, who has observed none of this transaction, now enters the room, looks at the cards on the table and identifies the one in Professor A's pocket. How is this done?

Solution. Our arithmetic is done modulo $k+1$, and we use $k+1$ instead of 0 for the representative of its equivalence class. Arrange the cards of H in order as $c_1 < \cdots < c_{k+1}$. Let $x = \sum_{c\in H} c \mod k+1$. Professor A hides card c_x. When Professor B arrives, he sees that the cards $d_1 < \cdots < d_{k+1}$ in $[2k+1] \smallsetminus (H - c_x)$ are missing, and he calculates $y := \sum_{c\in H-x} = x - c_x$. The class is holding $c_x - 1 - (x-1) = -y$ cards less than c_x and Professor A is holding c_x. It follows that $c_x = d_{1-y}$, and Professor B can calculate the rhs.

Another way of saying this is that Professor B knows the missing cards $\overline{d}_1 > \cdots > \overline{d}_{k+1}$. Then $c_x = \overline{d}_{k+2-(1-y)} = \overline{d}_y$.