MA1512 TUTORIAL 1

Key Concepts – Chapter 1 Differential Equations

Theory

- **General solutions** to DE containing arbitrary constants.
- Substituting specific values into constants of general solutions produces a **particular solution**.
- The general solution of an n^{th} order DE will contain n arbitrary constants.

Solving 1st order DEs

<table>
<tr>
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Technique 1:

Separable Equations

$$y' = M(x)N(y)$$

$$\Rightarrow \int \frac{1}{N(y)}\,dy = \int M(x)\,dx$$

Question 1

</td>
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Technique 2 (1st Type): **Change of Variable**

Form: $y' = \mathrm{f}(ax + by + c)$

If $b \neq 0$, the equation will be reduced to a separable form.

Strategy: Substitution

Let $u = ax + by + \{c\}$, $u' = \frac{d}{dx}(ax + by + c) = a + by'$

Simplifying the equation into a separable form:

$$y' = \mathrm{f}(ax + by + c)$$
$$(u' - a)/b = \mathrm{f}(u)$$
$$u' = b\mathrm{f}(u) + a$$

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Technique 2 (2nd Type): **Change of Variable**

Form: $y' = \mathrm{f}(\frac{y}{x})$

Strategy: Substitution

Let $u = \frac{y}{x}$, $y' = u'x + ux' = u'x + u$

simplifying the equation into a separable form:

$$y' = \mathrm{f}\left(\frac{y}{x}\right)$$
$$u'x + u = \mathrm{f}(u)$$
$$\frac{1}{\mathrm{f}(u) - u}du = \frac{1}{x}dx$$

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Typical Applications

Context	Differential Equation	Solution
Hot object left in environment T is the temperature of object.	$\frac{dT}{dt} = -k(T - T_{env})$	$T = T_{env} + Ae^{-kt}$
Radioactive Decay x is the amount of substance.	$\frac{dx}{dt} = -kx$	$x = Ae^{-kt}$, $k = \frac{\ln 2}{t_{1/2}}$

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KEY CONCEPTS – CHAPTER 1 DIFFERENTIAL EQUATIONS

1st order DE (continued)

Technique 3: Integrating Factor

$$y' + P(x)y = Q(x)$$

Derivation:

Step 1 Define **Integrating Factor**:

$$R(x) = e^{\int^x P(s)\,ds}$$

$$R'(x) = \frac{d}{dx} e^{\int^x P(s)\,ds} = e^{\int^x P(s)\,ds} * \left(\int^x P(s)\,ds\right)'$$

$$= R(x)P(x)$$

Step 2 Multiply the DE by $R(x)$:

$$R(x)\frac{dy}{dx} + R(x)P(x)y = R(x)Q(x)$$

$$R(x)y' + R'(x)y = R(x)Q(x)$$

Step 3 Recognize the LHS as $(Ry)'$ according to product rule, then:

$$LHS = (Ry)', RHS = RQ$$

$$(Ry)' = RQ$$

Step 4 Integrate both sides simultaneously:

$$Ry = \int RQ$$

Technique 4: Bernoulli Equation

$$y' + p(x)y = q(x)y^n$$

Derivation:

Step 1 Divided by y^n:

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

Step 2 Substitution: Let $z = y^{1-n}$,

$z' = (1-n)y^{-n}y'$, i.e. $y^{-n}y' = (1-n)^{-1}z'$

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

$$(1-n)^{-1}z' + p(x)z = q(x)$$

Simplifying:

$$z' + (1-n)p(x)z = (1-n)q(x).$$

Step 3 Solve the first order linear differential equation using the integrating factor method.

Integrating Factor Method Solution Steps:

$$y' + P(x)y = Q(x)$$

Step 1 Find Integrating Factor:

$$R(x) = e^{\int^x P(s)\,ds}$$

Step 2 Transform the DE to

$$(Ry)' = RQ$$

Step 3 Integrate both sides simultaneously:

$$Ry = \int RQ + C \qquad (C \in \mathbb{R})$$

$$y = \frac{1}{R}\left(\int RQ\ dx + C\right)$$

Bernoulli's Equation Solution Steps:

$$y' + p(x)y = q(x)y^n$$

Step 1 Find out n

Step 2 Set $z = y^{1-n}$

Step 3 Directly transform to

$z' + (1-n)p(x)z = (1-n)q(x)$.

Step 4 Solve the DE using the integrating factor method

REMARK

1. This method applies to all **linear** 1st order DE

2. Remember to fit the DE into the standard form before finding the value of $P(x)$ and $Q(x)$.

REMARK

1. In this case, $n \neq 0, 1$. (Why?)

2. Remember to fit the DE into the standard form before finding the value of n, $p(x)$ and $q(x)$.

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KEY CONCEPTS – CHAPTER 1 DIFFERENTIAL EQUATIONS

Consider the following 2nd order linear ordinary differential equation **with Constant Real Coefficients**:

$$y'' + ay' + by = r(x)$$

- If $r(x) \equiv 0$, this is known as a **homogeneous DE**. Then, the general solution is given by:

$$y = C_1 y_1 + C_2 y_2$$

  where y_1 and y_2 are linearly independent solutions to the DE.

- If $r(x) \not\equiv 0$, the DE becomes **non-homogeneous**. Thus, the general solution is given by:

$$y = y_h + y_p = (C_1 y_1 + C_2 y_2) + y_p$$

  where y_h is the general solution to the homogeneous DE and y_p is the particular solution satisfying the non-homogeneous DE.

What is the underlying principle for the characteristic equation?

- **Homogenous 2nd Order Linear ODEs with Constant Real Coefficients**:

$$ay'' + by' + cy = 0, \qquad a, b, c \in \mathbb{R}$$

Step 1 Find out the characteristic equation $a\lambda^2 + b\lambda + c = 0$. Then solve for λ.

Step 2 Choose case based on λ.

Case A: λ_1, λ_2 real and distinct	Case B: λ real, repeated	Case C: $\lambda_{1,2} = \alpha \pm \beta i$
G.S.: $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$	G.S.: $y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$	G.S.: $y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

To solve non-homogeneous problems:

- **Method of Variation of Parameters**: $y'' + ay' + by = r(x), \;\; r(x) \not\equiv 0$

Step 1 Find the general solution to homogeneous DE $y'' + ay' + by = 0$ given by

$$y_h = C_1 y_1 + C_2 y_2.$$

Step 2 Find Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$.

Step 3 Use the formula to find u and v.

$$u = -\int \frac{y_2 r}{W(y_1, y_2)} dx \qquad v = \int \frac{y_1 r}{W(y_1, y_2)} dx \qquad y_p = u y_1 + v y_2$$

Step 4 The general solution is given by $y = (C_1 y_1 + C_2 y_2) + (u y_1 + v y_2) = y_h + y_p$.

- **Method of Undetermined Coefficients**: $y'' + ay' + by = r(x),\ \ r(x) \not\equiv 0$

Step 1 Find the general solution y_h to homogeneous DE $y'' + ay' + by = 0$.

Step 2 Choose case based on $r(x)$.

<table>
<tr><th><u>Case A</u>: $r(x)$ is a polynomial</th><th><u>Case B</u>: $r(x) = P(x)e^{kx}$</th><th><u>Case C</u>: $r(x) \equiv P(x)e^{\alpha x}\sin\beta x$ or $r(x) \equiv P(x)e^{\alpha x}\cos\beta x$</th></tr>
<tr><td rowspan="2">Guess y_p to be a polynomial with unknown constant coefficients with the same degree as $r(x)$.
• Otherwise, choose next higher powers.</td><td>Guess $y_p = u(x)e^{kx}$, where $u(x)$ is a polynomial, $k \in \mathbb{C}$.</td><td>Guess
$y_{p1} = ue^{(\alpha+i\beta)x} = ue^{kx}$.
If $r(x)$ has $\sin\beta x$, then $y_p = \operatorname{Im}\left[ue^{(\alpha+i\beta)x}\right]$.
If $r(x)$ has $\cos\beta x$, then $y_p = \operatorname{Re}\left[ue^{(\alpha+i\beta)x}\right]$.</td></tr>
<tr><td colspan="2">A. If the value of k is neither λ_1 or λ_2, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x))$.
B. If the value of k is the same as either simple root λ_1 or λ_2, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x)) + 1$.
C. If the value of k is the same as the repeated root λ, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x)) + 2$.</td></tr>
</table>

Step 3 The general solution is given by $y = y_h + y_p$.

Chapter 2 – Applications of ODEs

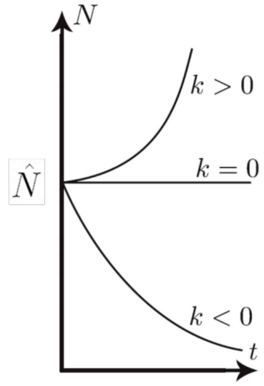
The Harmonic Oscillator

<table>
<tr><td>Type 1 Simple Harmonic Oscillator
$$m\ddot{x} + kx = 0,\text{ where } m, k > 0$$
m is the mass and k is the spring constant.
General Solution:
$$\boxed{\begin{aligned} x(t) &= C\cos\omega t + D\sin\omega t \\ &= A\cos(\omega t - \delta)\end{aligned}}$$
Angular frequency: ω; Amplitude: A;
Phase Angle: δ
Period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$; Frequency, $f = \frac{1}{T}$;</td>
<td>Type 2 Forced Harmonic Oscillator
$$m\ddot{x} + kx = F(t) = F_0\cos\alpha t$$
General Solution:
$$\boxed{x(t) = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos\alpha t}$$
Assume initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, we get particular solution:
$$\begin{aligned} x(t) &= \frac{\left(\frac{F_0}{m}\right)}{\omega^2 - \alpha^2}[\cos\alpha t - \cos\omega t] \\ &= \left[\frac{\left(\frac{2F_0}{m}\right)}{\alpha^2 - \omega^2}\sin\left(\frac{\alpha - \omega}{2}t\right)\right]\sin\left(\frac{\alpha + \omega}{2}t\right) \\ &= A(t)\sin\left(\frac{\alpha + \omega}{2}t\right)\end{aligned}$$
Resonance: $\alpha = \omega$
Particular Solution:
$$x(t) = \frac{F_0 t}{2m\omega}\sin(\omega t)$$</td></tr>
</table>

MA1512 TUTORIAL 4

KEY CONCEPTS – CHAPTER 3 MATHEMATICAL MODELLING

Malthus Model of Population:



$$\frac{dN}{dt} = (B - D)N = kN$$

- Solution: $N(t) = \widehat{N}e^{kt}, where\ \widehat{N}\ is\ the\ initial\ population$

Question How do different values of k affect the prediction of population growth using the Malthus Model?

Logistic Growth Model:

Assumption: $D \sim N, D = sN$

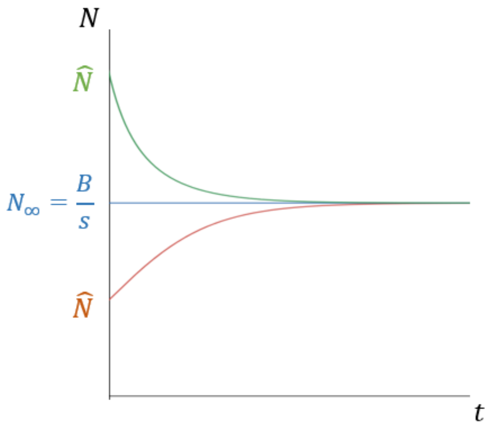
$$\frac{dN}{dt} = (B - D)N = (B - sN)N = -sN^2 + BN$$

- A logistic growth model has (1) increasing initial growth; (2) slower growth as saturation steps in; (3) growth stops upon maturation.
- Solution:

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\widehat{N}} - 1\right)e^{-Bt}}$$

where $N_\infty = \frac{B}{s}$ $(carrying\ capacity/logistic\ population)$ and $N(0) = \widehat{N}$.

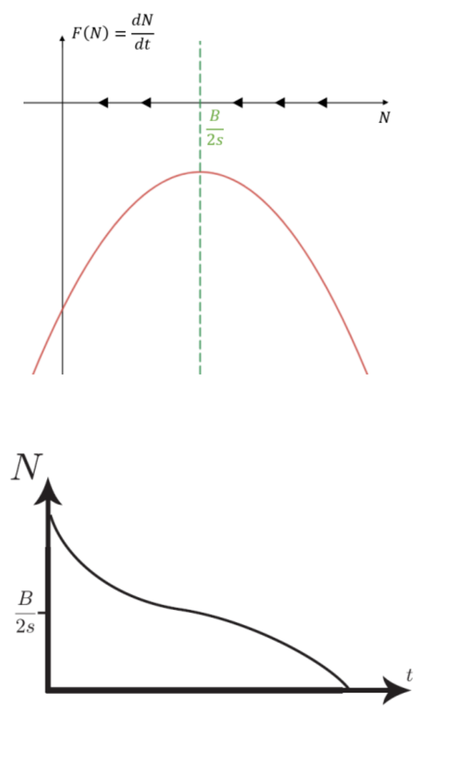
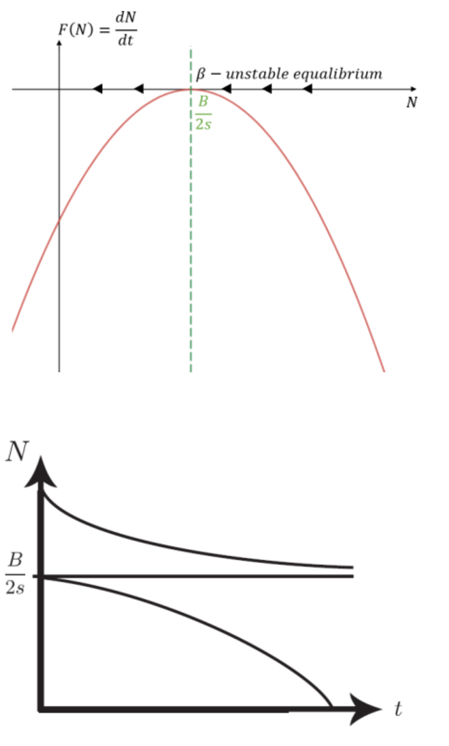
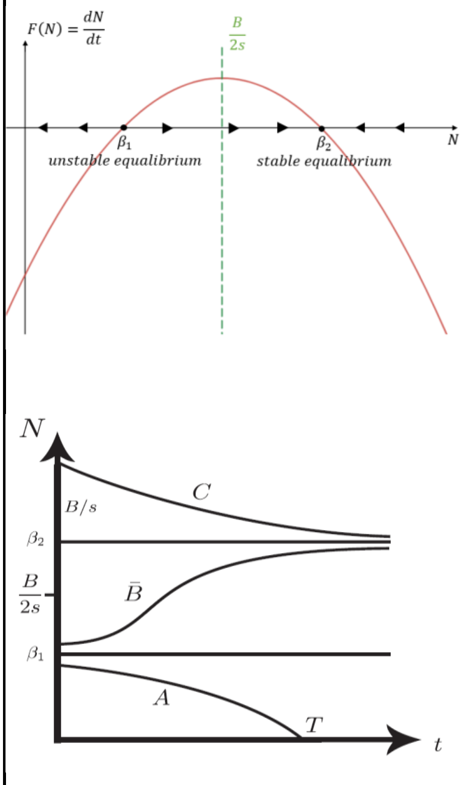
- The graph of N against t is shown below, with different initial starting values of N_0.



Harvesting Model:

$$\frac{dN}{dt} = BN - sN^2 - E$$

To find equilibrium solutions, we consider $BN - sN^2 - E = 0$. The discriminant $B^2 - 4sE$ is of concern, as it will tell us the number of equilibrium solutions we will get, as well as the behaviour of N with time t.

<table>
<tr>
<td><u>Case 1</u>: $B^2 - 4sE < 0 \Leftrightarrow E > \frac{B^2}{4s}$

</td>
<td><u>Case 2</u>: $B^2 - 4sE = 0 \Leftrightarrow E = \frac{B^2}{4s}$

</td>
<td><u>Case 3</u>: $B^2 - 4sE > 0 \Leftrightarrow E < \frac{B^2}{4s}$

</td>
</tr>
<tr>
<td><u>Extinction!</u></td>
<td><u>Extinction!</u>
$0 < x < \beta$: $\frac{dN}{dt} < 0, N\ decrease$
$x > \beta$: $\frac{dN}{dt} < 0, N\ decrease$</td>
<td>$0 < x < \beta_1$: $\frac{dN}{dt} < 0, N\ decrease$
$\beta_1 < x < \beta_2$: $\frac{dN}{dt} > 0, N\ increase$
$x > \beta_2$: $\frac{dN}{dt} < 0, N\ decrease$</td>
</tr>
</table>

KEY CONCEPTS – CHAPTER 4 LAPLACE TRANSFORM

- Laplace transform $L(f)$ is a mapping L which maps a function $f(t)$ to a function $F(s)$, where $F(s)$ is given by

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

- \The Laplace transform and inverse Laplace's transform has the **linearity** property (α and β are constants):

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g) \qquad L^{-1}(\alpha f + \beta g) = \alpha L^{-1}(f) + \beta L^{-1}(g)$$

- Solving **Initial** Value Problems

Step 1 Perform Laplace transform on the DE (in terms of t) and transform it into an **algebraic** subsidiary equation.

Step 2 Substitute in the appropriate initial values from the problem and make $L(f)$ the subject of the formula (in terms of s).

Step 3 Perform inverse Laplace transform on the resultant equation to obtain the solution (in terms of t) for the differential equation.

- The standard Laplace transforms and its inverses are presented in the table below.

$L(k) = \frac{k}{s}, k \in \mathbb{R}$	$L^{-1}\left(\frac{k}{s}\right) = k, k \in \mathbb{R}$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
$L(e^{at}) = \frac{1}{s-a}, s > a$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(\cos at) = \frac{s}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sin at) = \frac{a}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cosh at) = \frac{s}{s^2-a^2}, s > \|a\|$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
$L(\sinh at) = \frac{a}{s^2-a^2}, s > \|a\|$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
$L\big(f(t-a)\cdot u(t-a)\big) = e^{-as}\cdot F(s)$	$L^{-1}(e^{-as}F(s)) = f(t-a)\cdot u(t-a)$
$L\big(\delta(t-a)\big) = e^{-as}$	$L^{-1}(e^{-as}) = \delta(t-a)$
$L(y') = sL(y) - y(0)$	
$L(y'') = s^2L(y) - sy(0) - y'(0)$	
$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f), s > 0$	
Frequency-Shifting (s-shifting): $L\big(e^{ct}f(t)\big) = F(s-c)$	Time-shifting (t-shifting): $L^{-1}\big(e^{-as}F(s)\big) = f(t-a)\cdot u(t-a)$
$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$	$L^{-1}\left(\frac{1}{(s-c)^n}\right) = \frac{e^{ct}t^{n-1}}{(n-1)!}$
$L(e^{ct}\cos\omega t) = \frac{s-c}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t$
$L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin\omega t$

KEY CONCEPTS – CHAPTER 5 PARTIAL DIFFERENTIAL EQUATIONS

Separation of Variables

Step 1 Assume that the solution is in the form $u(x, y) = X(x) \cdot Y(y)$.

Step 2 Substitute the form above into the PDE.

Step 3 Perform separation of variables across equal signs, and equate it to a constant. (**Question** Why must we equate to a constant?)

Step 4 Separate the variables to obtain ODEs with their boundary conditions.

Step 5 Obtain solutions to both ODEs and combine them to give the solution u.

WAVE EQUATION	HEAT EQUATION:
$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \pi,\ \ t > 0$$ $$u(0, t) = 0,\ \ u(\pi, t) = 0,$$ $$u(x, 0) = f(x),\ \ u_t(x, 0) = 0.$$ Then, $\boxed{u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]}$ is a solution, where f is an odd extension with period 2π.	$$u_t = c^2 u_{xx},$$ $$u(0, t) = 0,\ \ u(L, t) = 0,$$ $$u(x, 0) = f(x).$$ Then $\boxed{u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}}$ is a solution.