Chapter 1: Differential Equations

Theory :

- General solutions to DE containing arbitrary constants.
- Substituting specific values into constants of general solutions produces a particular solution.
- The general solution of an nth order DE will contain n arbitrary constants.

Solving 1st order Des :

Technique 1:

Separable Equations

$$y' = M(x)N(y)$$

$$\Rightarrow \int \frac{1}{N(y)}\, dy = \int M(x)\, dx$$

Question 1

Technique 2 (1^{st} Type): **Change of Variable**

Form: $y' = \mathrm{f}(ax + by + c)$

If $b \neq 0$, the equation will be reduced to a separable form.

Strategy: Substitution

Let $u = ax + by + \{c\}$, $u' = \frac{d}{dx}(ax + by + c) = a + by'$

Simplifying the equation into a separable form:

$$y' = \mathrm{f}(ax + by + c)$$
$$(u' - a)/b = \mathrm{f}(u)$$
$$u' = b\mathrm{f}(u) + a$$

Technique 2 (2^{nd} Type): **Change of Variable**

Form: $y' = \mathrm{f}(\frac{y}{x})$

Strategy: Substitution

Let $u = \frac{y}{x}$, $y' = u'x + ux' = u'x + u$

simplifying the equation into a separable form:

$$y' = \mathrm{f}\left(\frac{y}{x}\right)$$
$$u'x + u = \mathrm{f}(u)$$
$$\frac{1}{\mathrm{f}(u) - u}\, du = \frac{1}{x}\, dx$$

Typical applications:

Context	Differential Equation	Solution
Hot object left in environment T is the temperature of object.	$\frac{dT}{dt} = -k(T - T_{env})$	$T = T_{env} + Ae^{-kt}$
Radioactive Decay x is the amount of substance.	$\frac{dx}{dt} = -kx$	$x = Ae^{-kt}$, $k = \frac{\ln 2}{t_{1/2}}$

1st order DE (continued):

Technique 3: Integrating Factor

$$y' + P(x)y = Q(x)$$

Derivation:

Step 1 Define **Integrating Factor**:

$$R(x) = e^{\int^x P(s)\, ds}$$

$$R'(x) = \frac{d}{dx} e^{\int^x P(s)\, ds} = e^{\int^x P(s)\, ds} * \left(\int^x P(s)\, ds\right)'$$
$$= R(x)P(x)$$

Step 2 Multiply the DE by $R(x)$:

$$R(x)\frac{dy}{dx} + R(x)P(x)y = R(x)Q(x)$$
$$R(x)y' + R'(x)y = R(x)Q(x)$$

Step 3 Recognize the LHS as $(Ry)'$ according to product rule, then:

$$LHS = (Ry)', RHS = RQ$$
$$(Ry)' = RQ$$

Step 4 Integrate both sides simultaneously:

$$Ry = \int RQ$$

Integrating Factor Method Solution Steps:

$$y' + P(x)y = Q(x)$$

Step 1 Find Integrating Factor:

$$R(x) = e^{\int^x P(s)\, ds}$$

Step 2 Transform the DE to

$$(Ry)' = RQ$$

Step 3 Integrate both sides simultaneously:

$$Ry = \int RQ + C \quad (C \in \mathbb{R})$$

$$y = \frac{1}{R}\left(\int RQ\, dx + C\right)$$

REMARK

1. This method applies to all **linear** 1^{st} order DE
2. Remember to fit the DE into the standard form

Technique 4: Bernoulli Equation

$$y' + p(x)y = q(x)y^n$$

Derivation:

Step 1 Divided by y^n:

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

Step 2 Substitution: Let $z = y^{1-n}$,

$z' = (1-n)y^{-n}y'$, i.e. $y^{-n}y' = (1-n)^{-1}z'$

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$
$$(1-n)^{-1}z' + p(x)z = q(x)$$

Simplifying:

$$z' + (1-n)p(x)z = (1-n)q(x).$$

Step 3 Solve the first order linear differential equation using the integrating factor method.

Bernoulli's Equation Solution Steps:

$$y' + p(x)y = q(x)y^n$$

Step 1 Find out n

Step 2 Set $z = y^{1-n}$

Step 3 Directly transform to

$z' + (1-n)p(x)z = (1-n)q(x)$.

Step 4 Solve the DE using the integrating factor method

REMARK

1. In this case, $n \neq 0, 1$. (Why?)
2. Remember to fit the DE into the standard form before finding the value of n, $p(x)$ and $q(x)$.

Homogenous 2nd Order Linear ODEs with Constant Real Coefficients:

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}$$

Step 1 Find out the characteristic equation $a\lambda^2 + b\lambda + c = 0$. Then solve for λ.

Step 2 Choose case based on λ.

Case A: λ_1, λ_2 real and distinct	Case B: λ real, repeated	Case C: $\lambda_{1,2} = \alpha \pm \beta i$
G.S.: $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$	G.S.: $y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$	G.S.: $y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

Non-homogenous 2nd Order Linear ODEs with constant real coefficients:

$$y'' + ay' + by = r(x), \quad r(x) \not\equiv 0$$

Method of Variation of Parameters:

Step 1 Find the general solution to homogeneous DE $y'' + ay' + by = 0$ given by

$$y_h = C_1 y_1 + C_2 y_2.$$

Step 2 Find Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$.

Step 3 Use the formula to find u and v.

$$u = -\int \frac{y_2 r}{W(y_1, y_2)}\, dx \qquad v = \int \frac{y_1 r}{W(y_1, y_2)}\, dx$$

$$y_p = uy_1 + vy_2$$

Step 4 General solution, $y = (C_1 y_1 + C_2 y_2) + (uy_1 + vy_2) = y_h + y_p$.

Method of Undetermined Coefficients:

Step 1 Find the general solution y_h to homogeneous DE $y'' + ay' + by = 0$.

Step 2 Choose case based on $r(x)$.

Case A: $r(x)$ is a polynomial	Case B: $r(x) = P(x)e^{kx}$	Case C: $r(x) \equiv P(x)e^{\alpha x}\sin\beta x$ or $r(x) \equiv P(x)e^{\alpha x}\cos\beta x$
Guess y_p to be a polynomial with unknown constant coefficients with the same degree as $r(x)$. • Otherwise, choose next higher powers.	Guess $y_p = u(x)e^{kx}$, where $u(x)$ is a polynomial, $k \in \mathbb{C}$.	Guess $y_{p1} = ue^{(\alpha+i\beta)x} = ue^{kx}$. If $r(x)$ has $\sin\beta x$, then $y_p = \mathrm{Im}\left[ue^{(\alpha+i\beta)x}\right]$. If $r(x)$ has $\cos\beta x$, then $y_p = \mathrm{Re}\left[ue^{(\alpha+i\beta)x}\right]$.
	A. If the value of k is **neither** λ_1 or λ_2, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x))$. **B.** If the value of k is the same as **either** simple root λ_1 or λ_2, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x)) + 1$. **C.** If the value of k is the same as the repeated root λ, then guess u to be a polynomial, and $\deg(u(x)) = \deg(P(x)) + 2$.	

Step 3 The general solution is given by $y = y_h + y_p$.

Chapter 2: Applications of ODEs

The Harmonic Oscillator

Type 1 Simple Harmonic Oscillator

$$m\ddot{x} + kx = 0, \text{ where } m, k > 0$$

m is the mass and k is the spring constant.

General Solution:

$$\boxed{\begin{aligned} x(t) &= C\cos\omega t + D\sin\omega t \\ &= A\cos(\omega t - \delta) \end{aligned}}$$

Angular frequency: ω; Amplitude: A;

Phase Angle: δ

Period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$; Frequency, $f = \frac{1}{T}$;

Type 2 Forced Harmonic Oscillator

$$m\ddot{x} + kx = F(t) = F_0\cos\alpha t$$

General Solution:

$$\boxed{x(t) = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos\alpha t}$$

Assume initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, we get particular solution:

$$x(t) = \frac{\left(\frac{F_0}{m}\right)}{\omega^2 - \alpha^2}[\cos\alpha t - \cos\omega t]$$
$$= \left[\frac{\left(\frac{2F_0}{m}\right)}{\alpha^2 - \omega^2}\sin\left(\frac{\alpha - \omega}{2}t\right)\right]\sin\left(\frac{\alpha + \omega}{2}t\right)$$
$$= A(t)\sin\left(\frac{\alpha + \omega}{2}t\right)$$

Resonance: $\alpha = \omega$

Particular Solution:

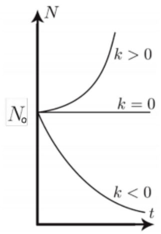
$$x(t) = \frac{F_0 t}{2m\omega}\sin(\omega t)$$

Chapter 3: Mathematical Modelling

Malthus Model of Population:

$$\frac{dN}{dt} = (B - D)N = kN$$

Solution: $N(t) = N_0 e^{kt}$, $N_0 =$ *initial population*



How do different values of k affect the prediction of population growth using the Malthus Model?

Logistic Growth Model:

Assumption: $\boldsymbol{D \sim N}$, $\boldsymbol{D = sN}$

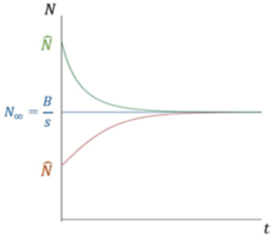
$$\frac{dN}{dt} = (B - D)N = (B - sN)N = -sN^2 + BN$$

Solution:

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right) e^{-Bt}} \quad (N_0 < N_\infty), \qquad N = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{N_0}\right) e^{-Bt}} \quad (N_0 > N_\infty)$$

where $N_\infty = B/s$ (*carrying capacity/logistic population*) and $N(0) = N_0$

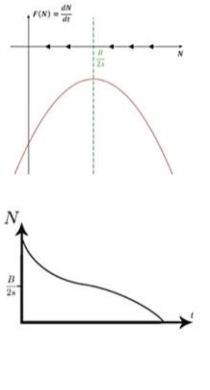
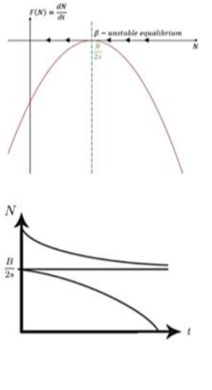
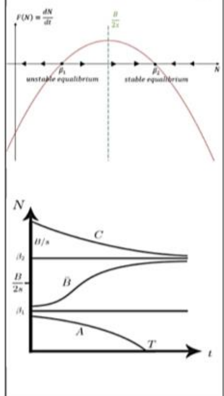
The graph of N against t is shown below, with different initial starting values of N_0.



Harvesting Model:

$$\frac{dN}{dt} = BN - sN^2 - E$$

Equilibrium solutions: $BN - sN^2 - E = 0$, discriminant $B^2 - 4sE$ is of concern, as it will tell us the number of equilibrium solutions and the behaviour of N with time t.

<u>Case 1</u>: $B^2 - 4sE < 0 \Leftrightarrow E > \frac{B^2}{4s}$	<u>Case 2</u>: $B^2 - 4sE = 0 \Leftrightarrow E = \frac{B^2}{4s}$	<u>Case 3</u>: $B^2 - 4sE > 0 \Leftrightarrow E < \frac{B^2}{4s}$
		
<u>*Extinction!*</u>	<u>*Extinction!*</u> $0 < x < \beta$: $\frac{dN}{dt} < 0, N$ *decrease* $x > \beta$: $\frac{dN}{dt} < 0, N$ *decrease*	$0 < x < \beta_1$: $\frac{dN}{dt} < 0, N$ *decrease* $\beta_1 < x < \beta_2$: $\frac{dN}{dt} > 0, N$ *increase* $x > \beta_2$: $\frac{dN}{dt} < 0, N$ *decrease*

<u>Chapter 4: The Laplace Transform</u>

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The Laplace transform and inverse Laplace's transform has the linearity property (α and β are constants):

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \quad L^{-1}(\alpha f + \beta g) = \alpha L^{-1}(f) + \beta L^{-1}(g)$$

Solving Initial Value Problems

Step 1 Perform Laplace transform on the DE (in terms of t) and transform it into an algebraic subsidiary equation.

Step 2 Substitute in the appropriate initial values from the problem and make $L(f)$ subject of formula (in terms of s).

Step 3 Perform inverse Laplace transform on the resultant equation to obtain the solution (in terms of t) for the differential equation.

Standard Laplace transforms and its inverses:

$L(k) = \frac{k}{s}, k \in \mathbb{R}$	$L^{-1}\left(\frac{k}{s}\right) = k, k \in \mathbb{R}$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
$L(e^{at}) = \frac{1}{s-a}, s > a$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(\cos at) = \frac{s}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sin at) = \frac{a}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cosh at) = \frac{s}{s^2-a^2}, s > \lvert a \rvert$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
$L(\sinh at) = \frac{a}{s^2-a^2}, s > \lvert a \rvert$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
$L(f(t-a) \cdot u(t-a)) = e^{-as} \cdot F(s)$	$L^{-1}(e^{-as}F(s)) = f(t-a) \cdot u(t-a)$
$L(\delta(t-a)) = e^{-as}$	$L^{-1}(e^{-as}) = \delta(t-a)$
$L(y') = sL(y) - y(0)$	
$L(y'') = s^2L(y) - sy(0) - y'(0)$	
$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}L(f), s > 0$	
Frequency-Shifting (s-shifting): $L(e^{ct}f(t)) = F(s-c)$	Time-shifting (t-shifting): $L^{-1}(e^{-as}F(s)) = f(t-a) \cdot u(t-a)$
$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$	$L^{-1}\left(\frac{1}{(s-c)^n}\right) = \frac{e^{ct}t^{n-1}}{(n-1)!}$
$L(e^{ct}\cos \omega t) = \frac{s-c}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos \omega t$
$L(e^{ct}\sin \omega t) = \frac{\omega}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin \omega t$

<u>Chapter 5: Partial Differential Equations</u>

Separation of variables:

Step 1 Assume that the solution in the form $u(x,y)=X(x)\cdot Y(y)$.

Step 2 Substitute the form above into PDE.

Step 3 Perform separation of variables across equal signs, and equate it to a constant, i.e. k.

Step 4 Separate the variables to obtain ODEs with their boundary conditions.

Step 5 Obtain solution to both ODEs and combine them to give the solution u.

<u>WAVE EQUATION</u>

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le \pi, \quad t > 0$$
$$u(0,t) = 0, \quad u(\pi,t) = 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Then, $\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$ is a solution, where f is an odd extension with period 2π.

<u>HEAT EQUATION:</u>

$$u_t = c^2 u_{xx},$$
$$u(0,t) = 0, \quad u(L,t) = 0,$$
$$u(x,0) = f(x).$$

Then $\boxed{u(x,t) = \sum_{n=1}^\infty b_n \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}}$ is a solution.