An overview of Hladký et al's (2021) work on inhomogeneous W-random graphs

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Overview

- Introduction to Graphons
- Construction of Inhomogeneous Random Graphs
- Graph Homomorphisms for Graphs and Graphons
- Homomorphism Density
- Conditional density for r-cliques
- K_r-regularity
- Dependency graphs and proof idea for Theorem 1.2b
- Hypergraphs, clique graphs and cycle densities

Graphons

Definition

A **graphon** is a bounded, symmetric and measurable function

$$W : [0,1]^2 \to [0,1] \quad \text{where } W(x,y) = W(y,x)\ \forall\, x,y \in [0,1].$$

Let $\mathcal{W}_0$ denote the space of all graphons.

- Intuitively, we may think of graphons as weighted symmetric graphs with uncountably many vertices, where the vertex set is $[0,1]$ and the weights are the values $W(x,y) = W(y,x)$.
- Graphons may also be thought of as the limit of graph sequences.
- If G is an unweighted graph, then fix $w_e = 1$ for each edge e.
- Given a graph G with n nodes, we may construct an empirical graphon.

Empirical Graphons

Definition

Let $G = (V, E)$ be a finite simple graph on n vertices, where for each edge $e \in E$, w_e denotes the weight of e.
For each $j \in \{1, \ldots, n\}$, define the interval I_j as $I_j := \left[\frac{j-1}{n}, \frac{j}{n}\right]$.
Then, the **empirical graphon of** G is defined as:

$$W^G(x, y) := \begin{cases} w_e & \text{if } e = (i, j) \in E, \quad (x, y) \in I_i \times I_j \\ 0 & \text{otherwise} \end{cases}$$

- For any graph $G = (V, E)$, the associated empirical graphon $W^G \in \mathcal{W}_0$ if $w_e \in [0, 1]$ for all $e \in E$

Empirical Graphons (cont.)

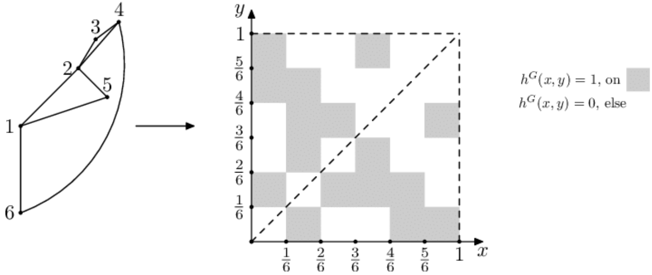


Figure: Example of an empirical graphon (Braunsteins et al. 2021)

Inhomogeneous Random Graphs

- Given a graphon W, we generate the random graph $\mathbb{G}(n, W)$ as follows:
  - Sample independently n numbers $U_1, \ldots, U_n \sim \text{Unif}(0, 1)$. Call these numbers **types** (continuous analog of node colorings).
  - Identify each uniform random variable U_j with a node $j \in [1..n]$, i.e. assign each node a type.
  - Any two nodes i, j in $\mathbb{G}(n, W)$ are connected by an edge (i, j) with probability $W(U_i, U_j)$
- If the graphon W is constant, i.e. $W(x, y) \equiv p \in [0, 1]$, then $\mathbb{G}(n, W)$ is identical to the ErdHos–Rényi random graph $\mathbb{G}(n, p)$.

Graph Homomorphisms

Definition

Let $F = (V', E')$ and $G = (V, E)$ be graphs.
A **graph homomorphism** from F to G is a map

$$\beta : V' \to V \quad \text{such that if } (i, j) \in E', \text{ then } (\beta(i), \beta(j)) \in E.$$

Write $F \to G$ if there exists a homomorphism from F to G.

- Intuition: the images of adjacent vertices remain adjacent
- Note that given any F and G, there may exist many possible homomorphisms $F \to G$
- A homomorphism $K_n \to G$ indicates that G contains an n-clique

Graph Homomorphisms (cont.)

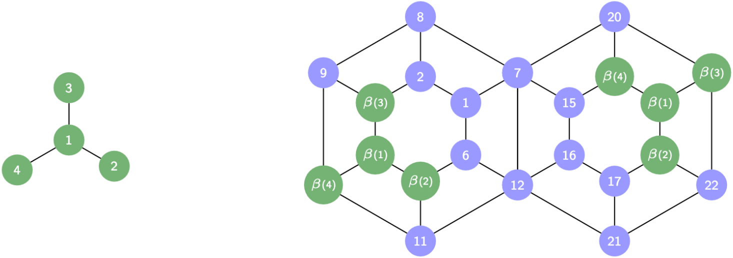


Figure: Example of multiple homomorphisms $F \to G$ (Ribeiro 2021)

- Let $\hom(F, G)$ denote the no. of homomorphisms $F \to G$

Homomorphism Density for Weighted Graphs

Definition

For a *weighted* graph $G = (V, E)$ on n nodes with adjacency matrix A, and a graph $F = (V', E')$ on k nodes, the **homomorphism density** of F in G is defined as:

$$t(F, G) = \frac{1}{n^k} \sum_{\substack{\beta : V' \to V \\ \text{graph hom.}}} \left(\prod_{(i,j) \in E'} [A]_{\beta(i),\beta(j)} \right)$$

where $[A]_{\beta(i),\beta(j)}$ denotes the $(\beta(i), \beta(j))$-th entry of A.

- Intuition: weight each homomorphism $\beta : V' \to V$ by the product of edge weights in the image of β
- For an unweighted graph, set all edge weights equal to 1

Homomorphism Densities for Graphons

Definition

For a graphon $W \in \mathcal{W}_0$ and a multigraph $H = (V, E)$ on n nodes, the **homomorphism density** of H in W is:

$$t(H, W) = \int_{[0,1]^n} \prod_{(i,j) \in E} W(x_i, x_j) \prod_{i \in V} dx_i$$

If H is a clique, then the homomorphism density can be defined as:

$$t(H, W) = \mathbb{E} \prod_{(i,j) \in V} W(U_i, U_j)$$

(Equation 6, Hladký et al. 2021)

- Similar to the definition for weighted graphs, where $W(x_i, x_j)$ is the weight of the edge (i, j)

Conditional Homomorphism Density

Definition (Equation 7, Hladký et al. 2021)

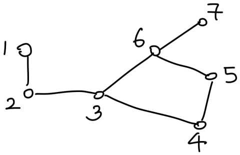
For an integer $l \leq k$, let J be an l-element subset of $[k] = \{1, 2, \ldots, k\}$.
Let H be a graph with vertex set $[k]$ where nodes in J are considered to be marked.
Then, given a vector of values $\mathbf{x} = (x_j)_{j \in J} \in [0, 1]^l$, define the conditional density as follows:

$$t_{\mathbf{x}}(H, W) = \mathbb{E}\left[\prod_{\{i,j\} \in E(H)} W(U_i, U_j) \,\middle|\, U_j = x_j : j \in J \right]$$

Relationship between $t_{\mathbf{x}}(H, W)$ and marked nodes J

- If $H = K_r$ is an r-clique, $t_{\mathbf{x}}(H, W)$ depends only on the cardinality of J and not on the elements of J (marked nodes)
- Consider the following example for an arbitrary non-clique graph H:

H



$$J = \{1, 2, 3\}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_j \in [0, 1]$$

$$t_{\vec{x}}(H, W) = \mathbb{E}\left[\prod_{\{i,j\} \in E(H)} W(U_i, U_j) \;\middle|\; U_j = x_j : j \in J\right]$$

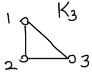
$$= \mathbb{E}\left[W(U_1, U_2)\, W(U_2, U_3)\, W(U_3, U_4)\, W(U_3, U_6) \;\middle|\; U_1 = x_1,\ U_2 = x_2,\ U_3 = x_3\right]$$

$$= \mathbb{E}\left[W(x_1, U_2)\, W(x_2, U_3)\, W(x_3, U_4)\, W(x_3, U_6)\right]$$

Relationship between $t_{\mathbf{x}}(H, W)$ and marked nodes J

- Now consider the following example for the 3-clique K_3:

K_3



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1, x_2 \in [0, 1]$$

$$J_1 = \{1, 2\}$$

$$\begin{aligned} t_{\vec{x}}(K_3, W) &= \mathbb{E}\left[W(U_1, U_2)W(U_1, U_3)W(U_2, U_3) \,\middle|\, U_1 = x_1,\ U_2 = x_2\right] \\ &= \mathbb{E}\left[W(x_1, x_2)W(x_1, U_3)W(x_2, U_3)\right] \\ &= \mathbb{E}\left[f_1(U_3)\right] \end{aligned}$$

Where $f_1(U_3) = W(x_1, x_2)W(x_1, U_3)W(x_2, U_3)$ ——— (1)

Now for $J_2 = \{2, 3\}$, we have:

$$\begin{aligned} t_{\vec{x}}(K_3, W) &= \mathbb{E}\left[W(U_1, U_2)W(U_1, U_3)W(U_2, U_3) \,\middle|\, U_2 = x_1,\ U_3 = x_2\right] \\ &= \mathbb{E}\left[W(U_1, x_1)W(U_1, x_2)W(x_1, x_2)\right] \\ &= \mathbb{E}\left[f_2(U_1)\right] \end{aligned}$$

$$\begin{aligned} \text{Where } f_2(U_1) &= W(U_1, x_1)W(U_1, x_2)W(x_1, x_2) \\ &= W(x_1, x_2)W(x_1, U_1)W(x_2, U_1) \quad \text{———} \quad (2) \end{aligned}$$

(by symmetry of W)

Since $U_1, U_3 \overset{i.i.d}{\sim} \mathrm{Unif}(0, 1)$, we have that equations (1) & (2) are equal.

K_r-free and K_r-regular graphons

- Let $K_r^\bullet$ and $K_r^{\bullet\bullet}$ denote K_r with one and two marked nodes respectively, with corresponding conditional homomorphism densities $t_x(K_r^\bullet, W)$ and $t_{x,y}(K_r^{\bullet\bullet}, W)$.

Definition

A graphon W is K_r-**free** if $t(K_r, W) = 0$ and **complete** if $t(K_r, W) = 1$ almost everywhere.

Definition (Equation 8, Hladký et al. 2021)

A graphon W is K_r-**regular** if for almost every $x \in [0, 1]$, we have:

$$t_x(K_r^\bullet, W) = t(K_r, W)$$

- We may view this notion as a generalization of regularity for graphs.

Degree Function of a Graphon

Definition

For a graphon W, the **degree function** $\deg_W : [0,1] \to [0,1]$ is defined as:

$$\deg_W(x) = \int_0^1 W(x,y)\, dy$$

- The degree function allows us to examine how the degree of a node varies as its type changes.
- In an Erdos–Rényi random graph $\mathbb{G}(n,p)$, a node has expected degree $(n-1) \cdot p$
- In $\mathbb{G}(n,W)$, if a node has type $x \in [0,1]$, then its expected degree is $(n-1) \cdot \deg_W(x)$

Definition

Say that a graphon W is **regular** if $\deg_W(x) \equiv d$ for some constant $d \in [0,1]$.

The Graphon $V_W^{(r)}$

- For any graphon W and $r \geq 2$, define the graphon $V_W^{(r)}$ as:

$$V_W^{(r)}(x, y) = t_{x,y}(K_r^{\bullet\bullet}, W)$$

- View $V_W^{(r)}(x, y)$ as the conditional density of r-cliques containing nodes with types x, y

Equivalence of K_r-regularity and regularity of $V_W^{(r)}$

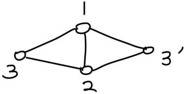
- W is K_r-regular $\iff$ $V_W^{(r)}$ is regular

$$\begin{aligned}
\deg_{V_W^{(r)}}(x) &= \int_0^1 V_W^{(r)}(x,y)\,dy \\
&= \int_0^1 t_{x,y}(K_r^{\bullet\bullet}, W)\,dy \\
&= t_x(K_r^{\bullet}, W) \\
&= t(K_r, W) \quad \text{(by } K_r\text{-regularity)} \\
&= t_r
\end{aligned}$$

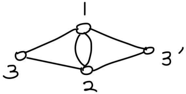
The Parameter $\sigma^2_{r,W}$

- Let $K_r \oplus_2 K_r$ denote the simple graph consisting of two r-cliques sharing 2 nodes (total of $2r - 2$ nodes)
- Let $K_r \ominus_2 K_r$ denote the multigraph obtained from $K_r \oplus_2 K_r$ where we duplicate the shared edge.

$K_3 \ominus_2 K_3$:



$K_3 \oplus_2 K_3$:



- (Equation 9, Hladký et al. 2021) We have that:

$$
\begin{aligned}
t_{x,y}(K_r \oplus_2 K_r, W) &= W(x,y)t_{x,y}(K_r \ominus_2 K_r, W) \\
&= (t_{x,y}(K_r^{\bullet\bullet}, W))^2 \\
&= (V_W^{(r)}(x,y))^2
\end{aligned}
$$

Then define:

$$
\sigma^2_{r,W} := \frac{1}{2((r-2)!)^2}(t(K_r \ominus_2 K_r, W) - t(K_r \oplus_2 K_r, W))
$$

Statement of Theorem 1.2a-b

- Let W be a graphon. Fix $r \geq 2$ and let $t_r = t(K_r, W)$.
- Let $X_{n,r}$ denote the no. of r-cliques in $\mathbb{G}(n, W)$.

Theorem (Theorem 1.2 (abridged), Hladký et al. 2021)

(a) *If W is K_r-free or complete, then almost surely $X_{n,r} = 0$ or $X_{n,r} = \binom{n}{r}$ respectively.*

(b) *If W is not K_r-regular, then:*

$$\frac{X_{n,r} - \binom{n}{r} t_r}{n^{r-1/2}} \xrightarrow{d} \hat{\sigma}_{r,W} \cdot Z$$

where $Z \sim N(0,1)$ and
$\hat{\sigma}_{r,W} = \frac{1}{(r-1)!} \left(t(K_r \ominus K_r, W) - t_r^2 \right)^{1/2} > 0$

Dependency Graphs

- The proof of Theorem 1.2b uses a construction called dependency graphs.
- Given a collection of random variables $(Y_i : i \in I)$ for some index set I, create a **dependency graph** $\mathcal{G}$ with vertex set I
- For each vertex $i \in I$, let N_i denote the neighborhood of $i \in \mathcal{G}$
- Construct $\mathcal{G}$ such that:

$$\forall\ i \in I,\ \text{the random variable } Y_i \text{ is independent of } \{Y_j\}_{j \notin N_i}$$

- The dependency graph need not be unique for given $(Y_i)_{i \in I}$

Dependency Graphs (cont.)

- Consider the following example: let $I = \{1, \ldots, 5\}$, and for each $i \in I$, let Y_i be standard normal random variables.
- Suppose $\{Y_1, Y_2, Y_3\}$ are independent of Y_4 and Y_5 respectively, where $Y_4 \perp\!\!\!\perp Y_5$. Then $\mathcal{G}$ is given by:

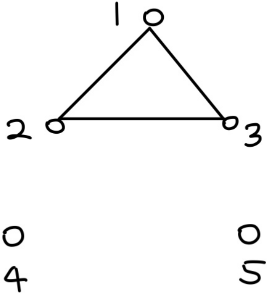


Figure: Example of a dependency graph for $I = \{1, \ldots, 5\}$

Dependency Graphs and the Wasserstein Distance

- Consider $d_{Wass}(X, Y)$, the Wasserstein distance between two random variables X, Y
- For $Z \sim N(0,1)$ and a sequence X_n of random variables:

$$d_{Wass}(X_n, Z) \to 0 \Longrightarrow X_n \xrightarrow{d} Z$$

Theorem (Theorem 2.2, Hladký et al. 2021)

Let $(Y_i : i \in I)$ be a finite collection of random variables where $\forall\ i \in I, \mathbb{E}[Y_i] = 0$ and $\mathbb{E}[Y_i^4] < \infty$. Let $\sigma^2 = \text{Var}\left[\sum_{i \in I} Y_i\right]$ and $Q = \sum_{i \in I} \frac{Y_i}{\sigma}$. Let $\mathcal{G}$ be a dependency graph for $(Y_i : i \in I)$, and let $D = \max_{i \in I} |N_i|$. Then, we have that:

$$d_{Wass}(Q, Z) \leq \frac{D^2}{\sigma^3} \sum_i \mathbb{E}[|Y_i|^3] + \frac{\sqrt{28} D^{3/2}}{\sqrt{\pi} \sigma^2} \sqrt{\sum_i \mathbb{E}[Y_i^4]}$$

Proof Idea for Theorem 1.2b

- Let $\binom{[n]}{r}$ denote the set of all size-r subsets of $[n]$.
- Consider the collection of random variables $(Y_R : R \in \binom{[n]}{r})$. Let $Y_r = I_r - \mathbb{E}[I_R] = I_r - t_r$, where I_r is the indicator random variable for the event where R induces a clique in $\mathbb{G}(n, W)$.
- Note that $E[Y_R] = 0$ for each R.
- Construct the dependency graph $\mathcal{G}$, where edges correspond to non-disjoint R_i, R_j, i.e. $R_i \cap R_j \neq \emptyset$.
- In $\mathcal{G}$, each neighbourhood N_R has the same size $D = \sum_{l=1}^{r} \binom{r}{l}\binom{n-r}{r-l} = O(n^{r-1})$
- Let $\sigma_n^2 = \mathrm{Var}\left[\sum_R Y_R\right] = \sum_{R_1,R_2} \mathbb{E}[Y_{R_1} Y_{R_2}]$
- For each $l \in \{1, \ldots, r\}$, the no. of ordered pairs R_1, R_2 s.t. $|R_1 \cap R_2| = l$ is $\binom{n}{l}\binom{n-l}{r-l}\binom{n-r}{r-l} = O(n^{2r-l})$
- One can show that $\sigma_n^2 \sim \hat{\sigma}_{r,W}^2 n^{2r-1}$.

Theorem 1.2b proof idea (cont.)

- Let $Q_n = \sum_{R \in \binom{[n]}{r}} \frac{Y_R}{\sigma_n}$.
- Applying Theorem 2.2 where we bound $\binom{n}{r} \leq n^r$ and examine powers of n, one can show that $d_{Wass}(Q_n, Z) = O(n^{-1/2}) \to 0$, i.e. $Q_n \xrightarrow{d} Z$.
- Applying Slutsky's Theorem, we have that:

$$\frac{\sum_{R \in \binom{[n]}{r}} Y_R}{n^{r-1/2}} = \frac{\sigma_n}{n^{r-1/2}} \cdot Q_n \xrightarrow{d} \hat{\sigma}_{r,W} Z$$

- Since $\sum_{R \in \binom{[n]}{r}} = X_{n,r} - \binom{n}{r} t_r$, this completes the proof.

Statement of Theorem 1.1(c)

- Suppose W is a K_r-regular graphon that is neither K_r-free nor complete. Then we have $t(x) = t_x(K_r, W) = t(K_r, W)$ for almost every $x \in [0, 1]$.
- Recall that X_n denotes the no. of r-cliques in $\mathbb{G}(n, W)$. Then the following holds:

Theorem (Theorem 1.1c (abridged), Hladký et al. 2021)

If $t(x)$ is constant and $t(x) \notin \{0, 1\}$, then there exist $c_0, c_1, \ldots \in \mathbb{R}$ such that $\sum_i c_i^2 \in (0, \infty)$ and:

$$\frac{X_n - \mathbb{E}[X_n]}{n^{r-1}} \xrightarrow{d} c_o Z_0 + \sum_{i \geq 1} c_i (Z_i^2 - 1)$$

where $Z_0, Z_1, \ldots$ are independent standard normal.

r-Uniform Hypergraphs, Clique Graphs

Definition

For $r \geq 2$, a r-**uniform hypergraph** $\mathcal{H}$ on a vertex set V is a collection of r-element subsets (**hyperedges**) of V.

Definition

Given a hypergraph $\mathcal{H}$, the **graph associated with** $\mathcal{H}$ (**clique graph** of $\mathcal{H}$) is a graph on the same vertex set, where each hyperedge S of $\mathcal{H}$ is replaced by a clique on S, with multiple edges replaced by single edges.

Loose Cycles (Hypergraph version of cycles)

- For $l \geq 2$, let $C_l^{(r)}$ be a r-uniform hypergraph with l hyperedges.
- To construct $C_l^{(r)}$, take the cycle graph C_l, and for each edge, insert an additional $r-2$ nodes, where all $l(r-2)$ new nodes are distinct.
- Then let $G_{l,r}$ be the graph associated with $C_l^{(r)}$.

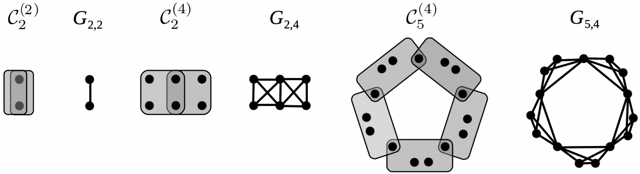


Figure: Examples of hypergraphs $C_l^{(r)}$ and their associated graphs $G_{l,r}$ (Hladký et al. 2021)

Cycle densities in $V_W^{(r)}$

- Assume that the nodes shared by consecutive r-cliques in $G_{l,r}$ have labels $1, \ldots, l$.
- For $l \geq 3$ and $r \geq 2$, one can show using Fubini's Theorem that:

$$t(C_l, V_W^{(r)}) = t(G_{l,r}, W)$$

- For $l = 2$, since C_2 is a multigraph consisting of two nodes linked by a double edge, we have:

$$t(C_2, V_W^{(r)}) = t(K_r \oplus_2 K_r, W)$$

Future directions

- Understand the proof of Theorem 1.2c (in particular the constructions involving hypergraphs and moment generating functions)

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