Programming in the Untyped λ-Calculus

Church & Scott Encodings, Y Combinator

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Motivation

The λ-calculus provides simple semantics for understanding functional abstraction.

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We can encode data purely within the untyped λ-calculus!

Remarks & notational conventions

- Function application is left-associative:

$$\text{Write } t_1\ t_2\ t_3 \text{ to denote } (t_1\ t_2)\ t_3$$

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- Bodies of lambda abstractions extend as far right as possible:

$$\text{Write } \lambda x.\ \lambda y.\ x\ y\ x \text{ to denote } \lambda x.\ (\lambda y.\ ((x\ y)\ x))$$

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- A term with no free variables is **closed**
- Closed terms are called **combinators**
  - Simplest combinator: the identity function `id`

$$\mathtt{id} = \lambda x.\ x$$

Agenda

1. Encoding simple datatypes

    Church Booleans

    Pairs

2. Church numerals

    Arithmetic operations

    Predecessor

    Testing equality

3. Y-combinator & recursion

    Factorial

4. Scott encodings

    Church vs Scott numerals

    Chruch vs Scott lists

Encoding simple datatypes

Church Booleans

Definition

Let $\mathtt{True}$ and $\mathtt{False}$ be represented by:

$$\mathtt{tru} = \lambda t.\, \lambda f.\, t$$
$$\mathtt{fls} = \lambda t.\, \lambda f.\, f$$

Note: $\mathtt{tru}$ & $\mathtt{fls}$ are normal forms!

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Note: $\mathtt{tru}$ & $\mathtt{fls}$ are normal forms!

Definition

The $\mathtt{test}$ combinator tests the truth value of a Boolean:

$$\mathtt{test} = \lambda l.\, \lambda m.\, \lambda n.\, l\ m\ n$$
$$\mathtt{test}\ \ \mathtt{tru}\, v\ w \to v$$
$$\mathtt{test}\ \ \mathtt{fls}\, v\ w \to w$$

The `test` combinator

Observe:

$$\mathit{test}\ b\ v\ w \longrightarrow\ b\ v\ w$$

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$$\mathtt{test}\ b\ v\ w \longrightarrow\ b\ v\ w$$

Example: (β-redexes underlined)

$$\begin{aligned}
\mathtt{test}\ \mathtt{tru}\ v\ w &\to \underline{(\lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n)\ \mathtt{tru}}\ v\ w \\
&\to \underline{(\lambda m.\ \lambda n.\ \mathtt{tru}\ m\ n)\ v}\ w \\
&\to \underline{(\lambda n.\ \mathtt{tru}\ v\ n)\ w} \\
&\to \mathtt{tru}\ v\ w
\end{aligned}$$

The `test` combinator (cont.)

Observe:

$$\begin{aligned} \mathtt{test}\ \mathtt{tru}\ v\ w &\longrightarrow v \\ \text{“if true then v else w”} &\longrightarrow v \end{aligned}$$

Example: (β-redexes are underlined)

$$\begin{aligned} \mathtt{test}\ \mathtt{tru}\ v\ w &\to \ldots \\ &\to \mathtt{tru}\ v\ w \\ &\to \underline{(\lambda t.\, \lambda f.\, t)\ v}\ w \\ &\to \underline{(\lambda f.\, v)\ w} \\ &\to v \end{aligned}$$

Similarly, $\mathtt{test}\ \mathtt{fls}\ v\ w \longrightarrow w$.
(“if false then v else w” $\longrightarrow$ w)

Conjunction

Intuition: $\mathit{and}\ b\ c \approx$ "if b then c else false"

Definition

$$\mathit{and} = \lambda b.\, \lambda c.\, b\ c\ \mathit{fls}$$

Conjunction

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Definition

$$and = \lambda b.\,\lambda c.\,b\ c\ fls$$

For Boolean values b, c, we have that:

$$and\ \ b\ \ c = \begin{cases} c & \text{if } b = tru \\ b & \text{if } b = fls \end{cases}$$

Conjunction

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Definition

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For Boolean values b, c, we have that:

$$and\ \ b\ \ c = \begin{cases} c & \text{if } b = tru \\ b & \text{if } b = fls \end{cases}$$

Examples:

$$\begin{aligned} and\ \ tru\ \ b &\to tru\ \ b\ \ fls \\ &\to b \end{aligned}$$

$$\begin{aligned} and\ \ fls\ \ b &\to fls\ \ b\ \ fls \\ &\to fls \end{aligned}$$

Disjunction

Intuition: $or\ b\ c \approx$ “if b then true else c”

Definition

$$or = \lambda b.\, \lambda c.\, b\ tru\ c$$

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Definition

$$or = \lambda b. \lambda c. b\ tru\ c$$

Examples:

$$\begin{aligned} or\ tru\ b &\to tru\ tru\ b \\ &\to tru \end{aligned}$$

$$\begin{aligned} or\ fls\ b &\to fls\ tru\ b \\ &\to b \end{aligned}$$

Negation

Intuition: $not\ b \approx$ "if b then false else true"

Definition

$$not = \lambda b.\, b\ fls\ tru$$

Negation

Intuition: $\mathit{not}\ b \approx$ "if b then false else true"

Definition

$$\mathit{not} = \lambda b.\, b\ \mathit{fls}\ \mathit{tru}$$

$$\begin{aligned}
\mathit{not}\ \mathit{tru} &\to (\lambda b.\, b\ \mathit{fls}\ \mathit{tru})\ \mathit{tru} \\
&\to \ \mathit{tru}\ \mathit{fls}\ \mathit{tru} \\
&\to \ \mathit{fls}
\end{aligned}$$

$$\begin{aligned}
\mathit{not}\ \mathit{fls} &\to (\lambda b.\, b\ \mathit{fls}\ \mathit{tru})\ \mathit{fls} \\
&\to \ \mathit{fls}\ \mathit{fls}\ \mathit{tru} \\
&\to \ \mathit{tru}
\end{aligned}$$

Pairs

Intuition: $(v, w) \approx$ "$\lambda b.$ if b then v else w"

$$\begin{aligned} \mathit{pair} &= \lambda v.\, \lambda w.\, \lambda b.\, b\ v\ w \\ \Longrightarrow \mathit{pair}\ \ v\ \ w &= \lambda b.\, b\ v\ w \end{aligned}$$

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$$\begin{aligned} \mathit{pair} &= \lambda v.\,\lambda w.\,\lambda b.\; b\; v\; w \\ \Longrightarrow \mathit{pair}\;\; v\;\; w &= \lambda b.\; b\; v\; w \end{aligned}$$

When applied to a Boolean b, $\mathit{pair}\;\; v\;\; w$ applies b to v and w:

$$\begin{aligned} \mathit{pair}\;\; v\;\; w\;\; \mathit{tru} &\to \mathit{tru}\;\; v\;\; w \\ &\to v \end{aligned}$$

$$\begin{aligned} \mathit{pair}\;\; v\;\; w\;\; \mathit{fls} &\to \mathit{fls}\;\; v\;\; w \\ &\to w \end{aligned}$$

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When applied to a Boolean b, $\mathtt{pair}\;\; v\;\; w$ applies b to v and w:

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$$\begin{aligned} \mathtt{pair}\;\; v\;\; w\;\; \mathtt{fls} &\to \mathtt{fls}\;\; v\;\; w \\ &\to w \end{aligned}$$

This motivates the projection functions $\mathtt{fst}$ & $\mathtt{snd}$:

$$\begin{aligned} \mathtt{fst} &= \lambda p.\, p\; \mathtt{tru} \\ \mathtt{snd} &= \lambda p.\, p\; \mathtt{fls} \end{aligned}$$

Pairs (cont.)

Example: (β-redexes underlined)

$$\begin{aligned}
\mathtt{fst\ (pair\ v\ w)} &\to \mathtt{fst}\ (\lambda b.\, b\, v\, w) \\
&\to \underline{(\lambda p.\, p\ \mathtt{tru})\, (\lambda b.\, b\, v\, w)} \quad \text{(by definition of } \mathtt{fst}\text{)} \\
&\to \underline{(\lambda b.\, b\, v\, w)\ \mathtt{tru}} \\
&\to \mathtt{tru\ v\ w} \\
&\to v
\end{aligned}$$

Church numerals

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Intuition: “A number n is a function that does something n times”

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Definition

Define the **Church numerals** $c_0, c_1, c_2, \ldots$ as follows:

$$
\begin{aligned}
c_0 &= \lambda s.\ \lambda z.\ z \\
c_1 &= \lambda s.\ \lambda z.\ s\ z \\
c_2 &= \lambda s.\ \lambda z.\ s\ (s\ z) \\
&\ldots
\end{aligned}
$$

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&\ldots
\end{aligned}
$$

Each $n \in \mathbb{N}$ is represented by a combinator c_n that takes arguments s and z ("successor" and "zero") and applies s to z for n times.

$$c_n = \lambda s.\, \lambda z.\ \langle \text{apply } s \text{ to } z \text{ for } n \text{ times} \rangle$$

Successor function

Definition

The **successor function** scc on Church numerals is defined as:

$$scc = \lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$$

Successor function

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The **successor function** scc on Church numerals is defined as:

$$scc = \lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$$

Intuition: $n + 1 \approx$ "apply s to z for n times, then apply s once more"

scc takes a Church numeral n and returns $\underbrace{\text{another Church numeral}}_{\substack{\text{function that takes } s, z \\ \text{\& applies } s \text{ repeatedly to } z}}$

Successor function (cont.)

Example: showing that “$scc\ 0 = 1$”:

$$
\begin{aligned}
scc\ \ c_0 &\to \underbrace{(\lambda n.\, \lambda s.\, \lambda z.\, s\ (n\ s\ z))}_{scc}\ \ \underbrace{(\lambda s.\, \lambda z.\, z)}_{c_0} \\
&\to\ \lambda s.\, \lambda z.\, s\ (\underbrace{(\lambda s.\, \lambda z.\, z)}_{c_0}\ s\ z) \\
&\to\ \lambda s.\, \lambda z.\, s\ (\underbrace{(\lambda z.\, z)}_{id}\ z) \\
&\to\ \lambda s.\, \lambda z.\, s\ z \\
&= c_1 && \text{(by definition of } c_1\text{)}
\end{aligned}
$$

Successor function (cont.)

Another way* to define the successor function:

$$scc_2 = \lambda n.\, \lambda s.\, \lambda z.\, n\ s\ (s\ z)$$

Intuition: "apply s to $(s\ \ z)$ for n times"

(as opposed to "applying s to z for $(n+1)$ times")

*TAPL Exercise 5.2.2

Addition of Church numerals

$$\begin{aligned} \mathit{plus} &= \lambda m.\, \lambda n.\, \lambda s.\, \lambda z.\, m\ s\ (n\ s\ z) \\ \Longrightarrow \underbrace{\mathit{plus}\ m\ n}_{m+n} &= \lambda s.\, \lambda z.\, m\ s\ (n\ s\ z) \end{aligned}$$

Addition of Church numerals

$$
\begin{aligned}
\mathtt{plus} &= \lambda m.\, \lambda n.\, \lambda s.\, \lambda z.\, m\ s\ (n\ s\ z) \\
\implies \underbrace{\mathtt{plus}\ m\ n}_{m+n} &= \lambda s.\, \lambda z.\, m\ s\ (n\ s\ z)
\end{aligned}
$$

<u>Intuition:</u> To compute $m + n$,

1. $\underbrace{\text{Apply } s \text{ iterated } n \text{ times to } z}_{n\ s\ z}$ …
2. … then $\underbrace{\text{apply } s \text{ to the result for } m \text{ more times}}_{m\ s\ (n\ s\ z)}$

Addition (cont.)

Recall: $c_1 = \lambda s.\, \lambda z.\, s\ z$

Example: Proving 1 + 1 = 2

$$
\begin{aligned}
\mathtt{plus}\ c_1\ c_1 &\to \lambda s.\, \lambda z.\, c_1\ s\ \underline{(c_1\ s\ z)} \\
&\to \lambda s.\, \lambda z.\, \underline{c_1\ s\ (s\ z)} \\
&\to \lambda s.\, \lambda z.\, s\ (s\ z) \\
&= c_2 && \text{(by definition of } c_2\text{)}
\end{aligned}
$$

Multiplication of Church numerals

Definition

$$\mathtt{times} = \lambda m.\, \lambda n.\, m\, (\mathtt{plus}\ \ n)\, c_0$$

$$\begin{aligned} m\,(\mathtt{plus}\ \ n)\, c_0 &\approx \text{“apply } \mathtt{plus}\ n \text{ iterated } m \text{ times to } c_0 \text{ (zero)”} \\ &\approx \text{“add together } m \text{ copies of } n\text{”} \end{aligned}$$

Multiplication (cont.)

Can we define multiplication without using $\mathtt{plus}$? Recall that:

$$\mathtt{times}\ m\ n \approx \text{“add together } m \text{ copies of } n\text{”}$$

*TAPL Exercise 5.2.3

*Here, $n\ s$ is akin to $\mathtt{plus}\ n$

Multiplication (cont.)

Can we define multiplication without using $\mathtt{plus}$? Recall that:

$$\mathtt{times}\ m\ n \approx \text{“add together } m \text{ copies of } n\text{”}$$

This motivates an alternate definition*:

$$\mathtt{times} = \lambda m.\ \lambda n.\ \lambda s.\ \lambda z.\ m\ (n\ s)\ z$$

Intuition: $m\ (n\ s)\ z \approx$ “apply $(n\ s)$ to z for m times”*

*TAPL Exercise 5.2.3
*Here, $n\ s$ is akin to $\mathtt{plus}\ n$

Multiplication example

$$\mathit{times} = \lambda x.\, \lambda y.\, \lambda a.\, x\ (y\ a)$$

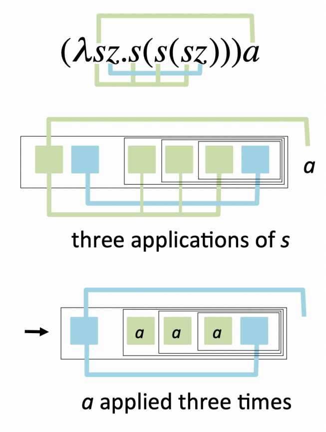
Compute 3×3:

$$\begin{aligned} \mathit{times}\ c_3\ c_3 &= (\lambda x.\, \lambda y.\, \lambda a.\, x\ (y\ a))\ c_3\ c_3 \\ &\to (\lambda a.\, c_3\ (c_3\ a)) \end{aligned}$$

Multiplication example (cont.)

Consider the term $(c_3\ a)$:

$$c_3 = \lambda s.\ \lambda z.\ s\ (s\ (s\ z))$$



Applying c_3 to a produces a function that applies a three times (Rojas)

Multiplication example (cont.)

Let **3a** denote (c_3 a). Now, consider c_3 (**3a**):

$$\lambda a.\ c_3\ (\mathbf{3a}) = \left(\lambda a.\ \underbrace{(\lambda s.\ \lambda b.\ s\ (s\ (s\ b)))}_{c_3}\ (\mathbf{3a})\right)$$

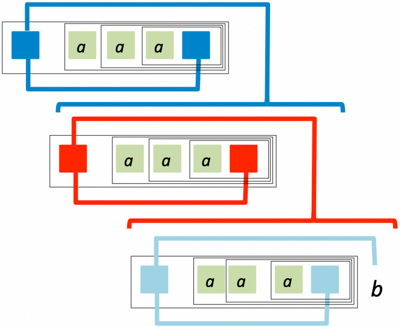
$$\rightarrow \lambda a.\ \lambda b.\ \mathbf{3a}\ (\mathbf{3a}\ (\mathbf{3a}\ b))$$

Applying c_3 to **3a** returns a function that applies **3a** three times
= applies a for (3 × 3) times

Example from Rojas (2015), *A Tutorial Introduction to the Lambda Calculus*

Multiplication example (cont.)

$$(\lambda ab.(3a)((3a)((3a)b)))$$



a applied 3 by 3 times to *b*

$$a(a(a(a(a(a(a(a(ab))))))))$$

c_3 applied to **3a**, visualized

Diagram from Rojas (2015), *A Tutorial Introduction to the Lambda Calculus*

How should we define `predecessor` for Church numerals?

Predecessor function

Strategy: Create a pair $(n - 1, n)$, then pick the 1st element of the pair

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We define two auxiliary functions:

$$zz = \mathtt{pair}\ c_0\ c_0$$
$$ss = \lambda p.\,\mathtt{pair}\ (\mathtt{snd}\ p)\,(\mathtt{plus}\ c_1\ (\mathtt{snd}\ p))$$

When applied to a pair (i, j), $\mathtt{ss}$ returns a pair $(j, j+1)$:

$$\mathtt{ss}\ \left(\mathtt{pair}\, c_i\ c_j\right) = \mathtt{pair}\ c_j\ c_{j+1}$$

Predecessor function

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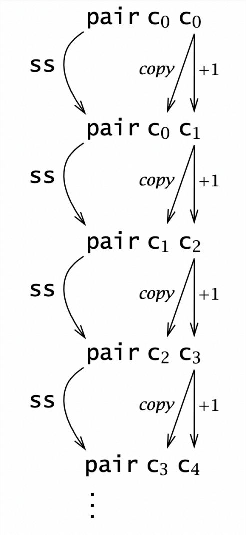
The predecessor function $\mathtt{prd}$ involves applying $\mathtt{ss}$ to $\mathtt{pair}\, c_0 c_0$ for m times, then projecting the 1st component:

$$\mathtt{prd} = \lambda m.\ \mathtt{fst}\,(m\ \mathtt{ss}\ \mathtt{zz})$$

Predecessor function



$prd \approx$ "apply ss to $pair\ c_0\ c_0$ for m times"

$$\approx \begin{cases} pair\ \ c_0\ c_0 & \text{when } m = 0 \\ pair\ \ c_{m-1}\ c_m & \text{otherwise} \end{cases}$$

Evaluating $prd\ c_n$ requires $O(n)$ steps!

(diagram from TAPL)

5-minute break

Roadmap for the next few slides

Aim: To represent `factorial` in the untyped λ-calculus

To do this, we need to discuss the following:

1. Testing if a Church numeral $\stackrel{?}{=} 0$
2. Equality of Church numerals
3. Y-comabintor & recursion

Testing if a Church numeral $\stackrel{?}{=} 0$

Definition

$$\mathtt{isZero} = \lambda m.\, m\, (\lambda x.\, \mathtt{fls})\, \mathtt{tru}$$

Example (β-redexes underlined):

$$\begin{aligned}
\mathtt{isZero}\, c_0 &= (\lambda m.\, m\, (\lambda x.\, \mathtt{fls})\, \mathtt{tru})\, c_0 \\
&= \underline{(\lambda m.\, m\, (\lambda x.\, \mathtt{fls})\, \mathtt{tru})\, (\lambda s.\, \lambda z.\, z)} \quad \text{(by definition of } c_0\text{)} \\
&\rightarrow \underline{(\lambda s.\, \lambda z.\, z)\, (\lambda x.\, \mathtt{fls})}\, \mathtt{tru} \\
&\rightarrow \underline{(\lambda z.\, z)\, \mathtt{tru}} \\
&\rightarrow \mathtt{tru}
\end{aligned}$$

Equality of Church numerals

Intuition: $m == n \iff (m - n) == 0 \wedge (n - m) == 0$

Definition

The *equal* function tests two Church numerals for equality, returning a Church Boolean:

```
equal = λm. λn.
          and (isZero (m prd n))
            (isZero (n prd m))
```

m prd n ≈ “applying the predecessor function for *m* times on *n*”

≈ “m *minus* n”

Y-combinator & recursion

How do we represent recursion?

Ω-combinator

Definition

The **divergent combinator** Ω is:

$$\Omega = (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$$

Ω-combinator

Definition

The **divergent combinator** Ω is:

$$\Omega = (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$$

Let's try to β-reduce Ω:

$$\begin{aligned}(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) &\to (x\ x)\Big[\ x\ \coloneqq\ (\lambda x.\ x\ x)\ \Big] \\ &\to (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\end{aligned}$$

We get what we started with!

A λ-term is **divergent** if it has no β-normal form.

Y-combinator

Definition

The **fixpoint combinator** is the term

$$\mathbf{Y} = \lambda f.\, (\lambda x.\, f\ (x\ x))\, (\lambda x.\, f\ (x\ x))$$

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$$\mathbf{Y} = \lambda f.\,(\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))$$

$$\begin{aligned}
\mathbf{Y}\,F &= \Big(\lambda f.\,(\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))\Big)\,F \\
&\to (\lambda x.\, F\,(x\,x))\,(\lambda x.\, F\,(x\,x)) \\
&\to F\,\Big(\underbrace{(\lambda x.\, F\,(x\,x))\,(\lambda x.\, F\,(x\,x))}_{\mathbf{Y}\,F}\Big) \\
&\to F\,(\mathbf{Y}\,F)
\end{aligned}$$

Y-combinator

Definition

The **fixpoint combinator** is the term

$$\mathbf{Y} = \lambda f.\, (\lambda x.\, f\ (x\ x))\ (\lambda x.\, f\ (x\ x))$$

$$\begin{aligned}
\mathbf{Y}\ F &= \Big(\lambda f.\, (\lambda x.\, f\ (x\ x))\ (\lambda x.\, f\ (x\ x))\Big)\ F \\
&\to (\lambda x.\, F\ (x\ x))\ (\lambda x.\, F\ (x\ x)) \\
&\to F\ \Big(\underbrace{(\lambda x.\, F\ (x\ x))\ (\lambda x.\, F\ (x\ x))}_{\mathbf{Y}\ F}\Big) \\
&\to F\ (\mathbf{Y}\ F)
\end{aligned}$$

Say that $\mathbf{Y}\ F$ is a **fixed point** of the function F:

$$\mathbf{Y}\ F = F\ (\mathbf{Y}\ F)$$

Y-combinator

We can use **Y** to achieve recursive calls to F:

$$\begin{aligned} \mathbf{Y}\ F &= F\ (\mathbf{Y}\ F) \\ &= F\ (F\ (\mathbf{Y}\ F)) \\ &= \ldots \end{aligned}$$

Factorial

Definition

Using Church numerals, we define the factorial function as:

$$\begin{aligned} fact = \lambda f.\lambda n.\ & \mathit{if}\ \mathit{isZero}\ n\ \mathit{then}\ c_1 \\ & \mathit{else}\ \mathit{times}\ n\ \left(f\ (\mathit{prd}\ n)\right) \end{aligned}$$

where $n \in \mathbb{N}$ & f is the function to call in the body

Factorial (cont.)

Use $\mathbf{Y}$ to achieve recursive calls to fact:

$$
\begin{aligned}
(\mathbf{Y}\ \mathit{fact})\ c_1 = (\mathit{fact}\ (\mathbf{Y}\ \mathit{fact}))\ c_1 \\
&\to \mathit{if\ equal}\ c_1\ c_0\ \mathit{then}\ c_1\ \mathit{else\ times}\ c_1\ \left((\mathbf{Y}\ \mathit{fact})\ c_0\right) \\
&\to \mathit{times}\ c_1\ \left((\mathbf{Y}\ \mathit{fact})\ c_0\right) \\
&\to \mathit{times}\ c_1\ \left(\mathit{fact}\ (\mathbf{Y}\ \mathit{fact})\ c_0\right) \\
&\to \mathit{times}\ c_1\ \left(\mathit{if\ equal}\ c_0\ c_0\ \mathit{then}\ c_1\right. \\
&\qquad\qquad \left.\mathit{else\ times}\ c_0\ ((\mathbf{Y}\ \mathit{fact})\ (\mathit{prd}\ c_0))\right) \\
&\to \mathit{times}\ c_1\ c_1 \\
&\to c_1
\end{aligned}
$$

Factorial (cont.)

Instead of using the Y-combinator, we can also define factorial using the U-combinator. ⏩ (see appendix)

Scott encodings

Scott encodings

Consider the following algebraic data types in Haskell:

```
data Nat = Zero | Succ Nat
data List a = Nil | Cons a (List a)
```

Scott encodings

Consider the following algebraic data types in Haskell:

```
data Nat = Zero | Succ Nat
data List a = Nil | Cons a (List a)
```

Scott encodings allow us to encode ADTs as λ-terms.

Scott numerals

Definition

$$
\begin{aligned}
\mathtt{zero} &= \lambda z.\, \lambda s.\, z \\
\mathtt{scc} &= \lambda n.\, \lambda z.\, \lambda s.\, s\ n
\end{aligned}
$$

Intuition: Arguments distinguish between different cases

Church vs Scott numerals

How do the Church & Scott encodings differ?

Church vs Scott numerals

How do the Church & Scott encodings differ?

Church	Scott
$zero = \lambda s.\, \lambda z.\, z$	$zero = \lambda z.\, \lambda s.\, z$
$scc = \lambda n.\, \lambda s.\, \lambda z.\, s\ (n\ s\ z)$	$scc = \lambda n.\, \lambda z.\, \lambda s.\, s\ n$

Church vs Scott numerals

Church	Scott
$scc = \lambda n.\, \lambda s.\, \lambda z.\, s\ (n\ s\ z)$	$scc = \lambda n.\, \lambda z.\, \lambda s.\, s\ n$
folds continuation threaded throughout structure	*case analysis* continuation unwraps one layer only

Church vs Scott numerals

Church	Scott
$\lambda s.\, \lambda z.\, z$ $\lambda s.\, \lambda z.\, s\ z$ $\lambda s.\, \lambda z.\, s\ (s\ z)$ $\lambda s.\, \lambda z.\, s\ (s\ (s\ z))$	$\lambda z.\, \lambda s.\, z$ $\lambda z.\, \lambda s.\, s\ (\lambda s.\, \lambda z.\, z)$ $\lambda z.\, \lambda s.\, s\ (\lambda s.\, \lambda z.\, s\ (\lambda s.\, \lambda z.\, z))$ $\lambda z.\, \lambda s.\, s\ (\lambda s.\, \lambda z.\, s\ (\lambda s.\, \lambda z.\, s\ (\lambda s.\, \lambda z.z)))$
“apply s, iterated n times”	“apply s on the preceding Scott numeral”

Church vs Scott encodings: Predecessor

Church: $O(n)$	**Scott**: $O(1)$
$\mathit{prd} = \lambda m.\, \mathtt{fst}\,(m\ ss\ zz)$ where $zz = \mathtt{pair}\ c_0\ c_0$, $ss = \lambda p.\, \mathtt{pair}\ (\mathtt{snd}\ p)\ (\mathtt{plus}\ c_1\ (\mathtt{snd}\ p))$	$\mathit{prd} = \lambda n.\, n\ \mathtt{zero}\ (\lambda p.\, p)$

Predecessor can be expressed more succintly using Scott encodings!

Church encoding for lists

Definition

$$\mathtt{nil} = \lambda n.\, \lambda c.\, n$$

$$\mathtt{cons} = \lambda x.\, \lambda l.\, \lambda n.\, \lambda\, c.\, c\ x\ (l\ n\ c)$$

(akin to $\mathtt{foldr}$)

Church encoding for lists

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$$\mathtt{nil} = \lambda n.\, \lambda c.\, n$$

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(akin to $\mathtt{foldr}$)

$x \approx$ "head"

$l \approx$ "tail"

$n \approx$ case for $\mathtt{nil}$

$c \approx$ case for $\mathtt{cons}$

Church encoding for lists (cont.)

Definition

$$\mathtt{nil} = \lambda n.\, \lambda c.\, n$$

$$\mathtt{cons} = \lambda x.\, \lambda l.\, \lambda n.\, \lambda\, c.\, c\ x\ (l\ n\ c)$$

(akin to $\mathtt{foldr}$)

Example:

$$x : y : z : [] \quad \approx \quad \lambda c.\, \lambda n.\, (c\ x\ (c\ y\ (c\ z\ n)))$$

Scott encoding for lists

Definition

$$\mathtt{nil} = \lambda n.\, \lambda c.\, n$$
$$\mathtt{cons} = \lambda x.\, \lambda l.\, \lambda n.\, \lambda c.\, c\ x\ l$$

$x \approx$ "head"

$\mathtt{l} \approx$ "tail"

$n \approx$ case for $\mathtt{nil}$

$c \approx$ case for $\mathtt{cons}$

Church vs Scott lists

Church	Scott
$cons = \lambda x.\, \lambda l.\, \lambda n.\, \lambda\, c.\; c\; x\; (l\; n\; c)$	$cons = \lambda x.\, \lambda l.\, \lambda n.\, \lambda c.\; c\; x\; l$ (much simpler!)

$x \approx$ "head"

$l \approx$ "tail"

$n \approx$ case for nil

$c \approx$ case for $cons$

Church vs Scott encodings

- Encodings only differ for recursive datatypes

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Church vs Scott encodings

- Encodings only differ for recursive datatypes
- **Church**: defines how functions should be folded over an element of the type
- **Scott**: uses “case analysis”, recursion not immediately visible
  - Simpler representation (for certain functions)
  - **Y**-combinator needed for other operations

Further reading:
Jansen (2013), *Programming in the λ-Calculus: From Church to Scott and Back*

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Appendix

Appendix: Defining `factorial` using the U-combinator

Instead of using the **Y**-combinator, we can also define *`factorial`* using the **U**-combinator.

Definition
The **U**-combinator applies its argument f to itself:

$$\mathbf{U} = \lambda f.\ f\ f$$

Appendix: Defining `factorial` using the U-combinator

Recall the definition of factorial:

$$
\begin{aligned}
fact = \lambda f.\lambda n.\ & if\ equal\ n\ c_0\ then\ c_1 \\
& else\ times\ n\ \left(f\ (prd\ n)\right)
\end{aligned}
$$

See this link for worked examples

Appendix: Defining `factorial` using the U-combinator

Recall the definition of factorial:

$$\begin{aligned} fact = \lambda f.\lambda n.\ & if\ equal\ n\ c_0\ then\ c_1 \\ & else\ times\ n\ \left(f\,(prd\ n)\right) \end{aligned}$$

We can define factorial using **U** as follows:

$$\begin{aligned} fact = \mathbf{U}\Big(\lambda f.\lambda n.\ & if\ isZero\ n\ then\ c_1 \\ & else\ times\ n\ \left(\mathbf{U}\, f\,(prd\ n)\right)\Big) \end{aligned}$$

See this link for worked examples

Appendix: More on the U-combinator

It turns out that we can define **Y** using **U**:

$$
\begin{aligned}
\mathbf{U} &= \lambda f.\; f\; f \\
\mathbf{Y} &= \lambda g.\; \mathbf{U}\left(\lambda f.\; g\; (\underline{\mathbf{U}\; f})\right) \\
&\to \lambda g.\; \underline{\mathbf{U}\left(\lambda f.\; g\; (f\; f)\right)} \\
&\to \underbrace{\lambda g.\; \left(\lambda f.\; g\; (f\; f)\right)\left(\lambda f.\; g\; (f\; f)\right)}_{\substack{\text{definition of } \mathbf{Y} \text{ we saw on slide 32} \\ \text{(up to } \alpha\text{-equivalence)}}}
\end{aligned}
$$

◂ Back to main presentation

Appendix: CBV vs CBN

- **Call-by-value** (CBV): given an application $(\lambda x.\ e_1)\ e_2$, make sure e_2 is a *value* before applying the abstraction
  - Reduce a redex only when its RHS has already been reduced to a value
- **Call-by-name** (CBN): Apply the function as soon as possible
  - No reductions are allowed inside abstractions
- TAPL & this presentation both use CBV.

Back to main presentation