Calculus Workshop

Applications of Derivatives Problem Set Solutions

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Answer the following questions to the best of your ability. Feel free to work with anyone in the cohort, though I would encourage attempting on your own first to make sure you fully understand the concepts.

Derivative Rules

1) Explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (constant, multiple, sum, product, etc.) you would use. [For extra practice, have a try at solving them!]

   a) $f(x) = \sqrt{(x+3)}$

Chain rule, but the $\frac{du}{dx} = 1$ so it would look like we only need the power rule. We need to keep $(x+3)$

$$f(x) = \sqrt{(x+3)} \tag{1}$$

$$f'(x) = \frac{1}{2\sqrt{(x+3)}} \tag{2}$$

b) $h(a) = \frac{\exp^{-a}}{a^2-5}$

First we would use Quotient rule then to solve for the individual derivatives, exponent with chain rule and then power rule

$$h(a) = \frac{\exp^{-a}}{a^2-5} \tag{3}$$

$$h'(a) = \frac{-(a^2-5)e^{-a} - (2a)e^{-a}}{(a^2-5)^2} \tag{4}$$

c) $g(t) = \left(\frac{5t^3+2}{2(t^2-2)}\right)^6$

First chain rule, but in the u-substitution we have to use the quotient and power rules

$$g(t) = \left(\frac{5t^3+2}{2(t^2-2)}\right)^6 \tag{5}$$

$$g(t) = u(t)^6 \qquad u = \frac{5t^3+2}{2(t^2-2)} \tag{6}$$

$$g'(t) = 6(u(t))^5 \qquad \frac{du}{dx} = \frac{2(t^2-2)15t^2 - 4t(5t^3+2)}{(2(t^2-2))^2} \tag{7}$$

$$g'(t) = 6(u(t))^5 * \frac{2(t^2-2)15t^2 - 4t(5t^3+2)}{(2(t^2-2))^2} \tag{8}$$

$$g'(t) = 6\left(\frac{5t^3+2}{2(t^2-2)}\right)^5 \frac{2(t^2-2)15t^2 - 4t(5t^3+2)}{(2(t^2-2))^2} \tag{9}$$

2) On the graph, label all regions where the following statements are true:

a) $f'(x) = 0$

At the max and min $x = -2$ and $x = 2$

b) $f'(x) > 0$

$-\infty < x < -2$ and $0 < x < \infty$

c) $f'(x) < 0$

$-2 < x < 0$

d) [Bonus] $f''(x) > 0$

We need to look for inflection points in the graph where it starts accelerating more or less. This occurs at $x = -1$. To the left the rate of deceleration is negative so $f''(x) < 0$ and the to the right $f''(x) > 0$

e) [Bonus] $f''(x) < 0$

See above

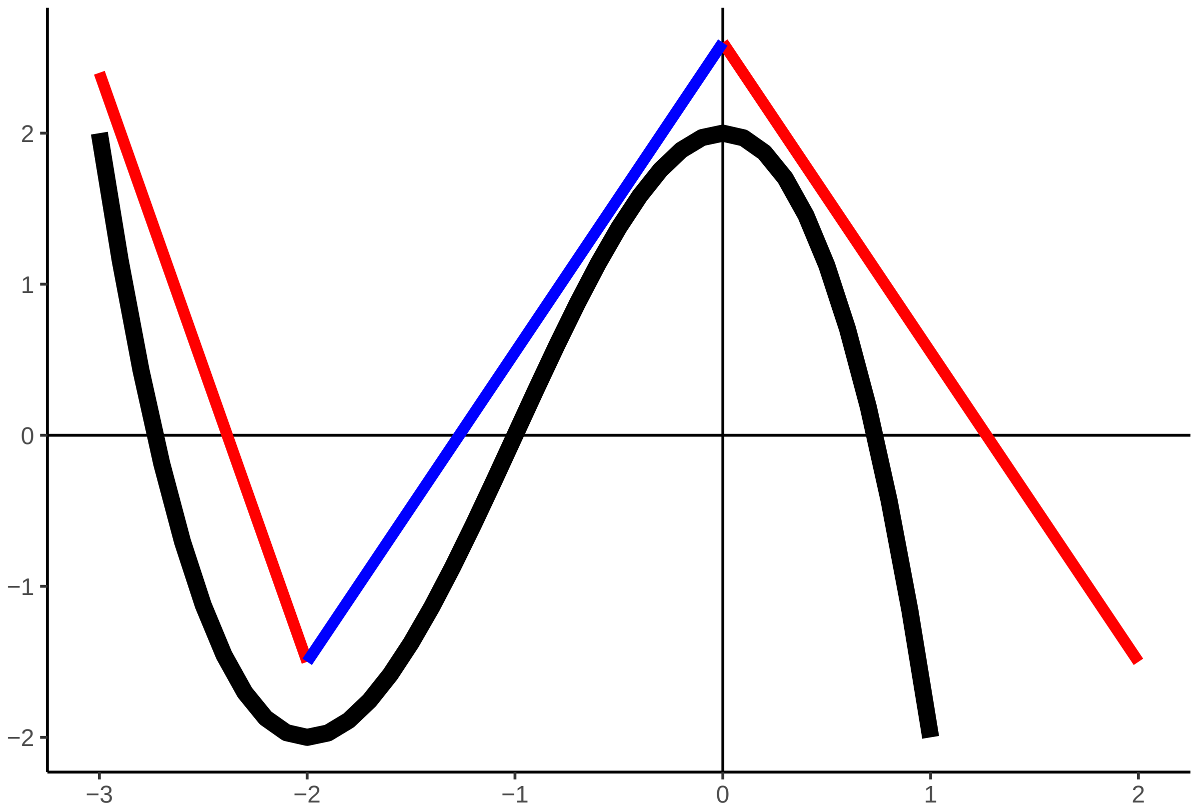

Figure 1: The color bars correspond to the verbal statements and their colors in answer key above

3) Find the derivative of these equations

a) $f(x) = \exp^{x^2+3}$

$$\begin{align} f(x) &= e^{x_3^2} \tag{10} \\ f'(x) &= 2xe^{x^2+3} \tag{11} \end{align}$$

b) $f(x) = \sqrt{(}2x)((x^3 - 4x + 15))$

$$\begin{align} f(x) &= \sqrt{(2x)}((x^3 - 4x + 15)) \tag{12} \\ f'(x) &= \frac{1}{\sqrt{2x}}(x^3 - 4x + 15) + \sqrt{2x}(3x^2 - 4) \tag{13} \end{align}$$

c) $f(x) = \frac{x^3+2x}{x-1}$

$$\begin{align} f(x) &= \frac{x^3 + 2x}{x - 1} \tag{14} \\ f'(x) &= \frac{(x-1)(3x^2+2) - (x^3+2x)(1)}{(x-1)^2} \tag{15} \end{align}$$

4) Find the critical points of the function and determine whether they are maximum or minimum.

$$2x^3 - 5x^2 + 3x - 12$$

$$\begin{align} f'(x) &= 6x^2 - 10x + 3 = 0 & &\text{Use the Quadratic formula to solve} \tag{16} \\ x &\approx 1.27 & x &\approx 0.39 \tag{17} \\ f''(x) &= 12x - 10 \tag{18} \\ f''(1.27) &> 0 & f''(0.39) &< 0 \tag{19} \end{align}$$

Based on the first and second order conditions, there are two optimas for this function at $x \approx 1.27$ and $x \approx 0.39$. Plugging into the second order conditions we can see that $x \approx 0.39$ is a maximum and $x \approx 1.27$ is a minimum.

5) Let's solve the canonical Gordon-Schaefer fisheries production model. This model formed the basis of fisheries management for half a century and is still a widely used model. It starts by defining the change in the abundance or biomass of fish by the logistic growth equation:

$$\frac{dB}{dt} = rB(t)(1 - \frac{B(t)}{K}) \tag{20}$$

where B is biomass, t is time, r is the intrinsic growth rate, and K is the carrying capacity. Next, fishers catch fish based on their effort E, technological ability q (also called the catchability coefficient), and the amount of fish B. For simplicity, we use an easy multiplicative form:

$$Y = qEB$$

This catch or harvest rate takes away from the growth of the fish stock, thus we can subtract it from the logistic growth equation 1.

$$\frac{dB}{dt} = rB(t)(1 - \frac{B(t)}{K}) - Y \tag{21}$$

a) Fishery managers are interested in finding the equilibrium level of biomass and harvest. We can set $\frac{dB}{dt} = 0$. Solve for the equilibrium biomass in terms of K,q,r,E.

Solution

$$0 = rB(1 - \frac{B}{K}) - qEB \tag{22}$$

$$qEB = rB(1 - \frac{B}{K}) \tag{23}$$

$$qE = r(1 - \frac{B}{K}) \tag{24}$$

$$\frac{rB}{K} = r - qE \tag{25}$$

$$B = K(1 - \frac{qE}{r}) \tag{26}$$

b) Sub the equilibrium biomass you found in A into the harvest equation.

Solution

$$Y = qEB \tag{27}$$

$$Y = qEK(1 - \frac{qE}{r}) \tag{28}$$

c) What effort leads to a maximum sustainable yield? Maximize the equation from b) with respect to effort E.

We can use either the product rule or distribute the E. It's easier algebra in this case to distribute the terms then take the derivative

$$Y = qEK - \frac{q^2E^2K}{r} \tag{29}$$

$$\frac{dY}{dE} = qK - \frac{2q^2EK}{r} = 0 \tag{30}$$

$$qK = \frac{2q^2EK}{r} \tag{31}$$

$$\frac{r}{2q} = E^* \tag{32}$$

d) Interpret what you found in c). What are the two variables that drive fisher harvest levels?

The amount of effort that leads to the most amount of catch is driven entirely by the rate of growth of fish stocks and the ability of fishers to catch them. If q catchability goes up, fishers become more efficient and need less effort to reach max harvest. Faster growing species allow for more growth and thus more effort to catch them.

e) Calculate what the maximum sustainable yield by substituting the optimal harvest level into the harvest equation.

$$Y_{MSY} = \frac{qrK}{2q} - \frac{q^2(\frac{r}{2q})^2K}{r} \tag{33}$$

$$Y_{MSY} = \frac{rK}{2} - \frac{rK}{4} \tag{34}$$

$$Y_{MSY} = \frac{Kr}{4} \tag{35}$$