UC SANTA BARBARA

Bren Calculus Workshop

Limits and Intro to Derivative

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Team Review

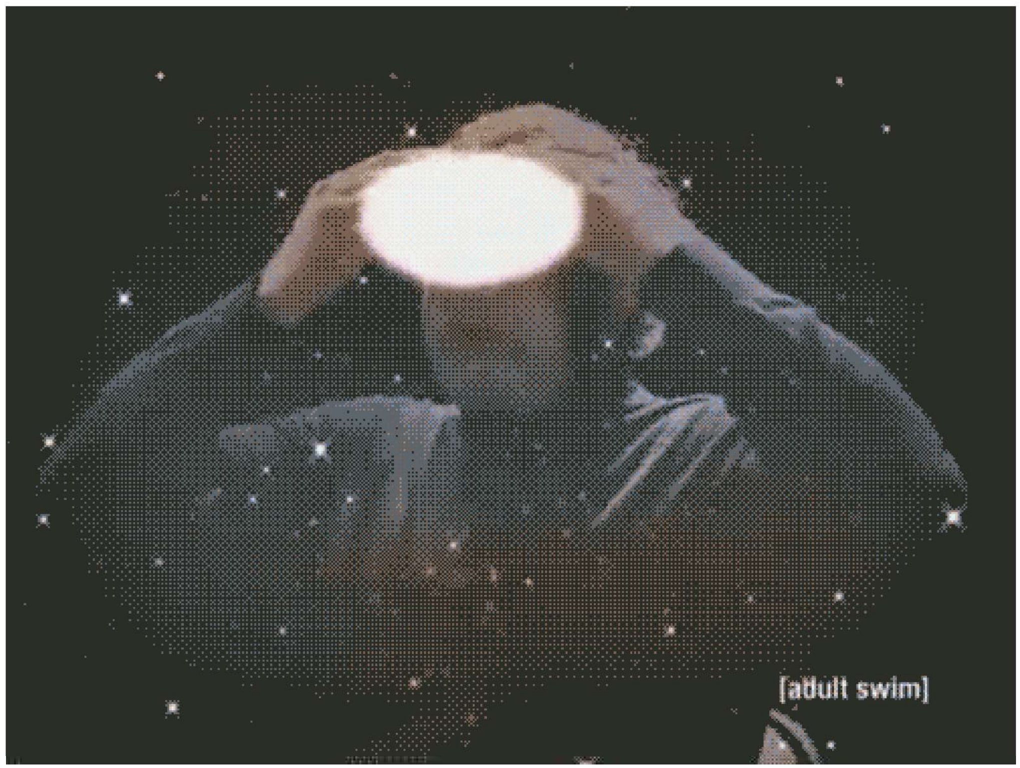
How did everyone feel about the problem set?

Anything still confusing?

Discuss with Team

Disclaimer

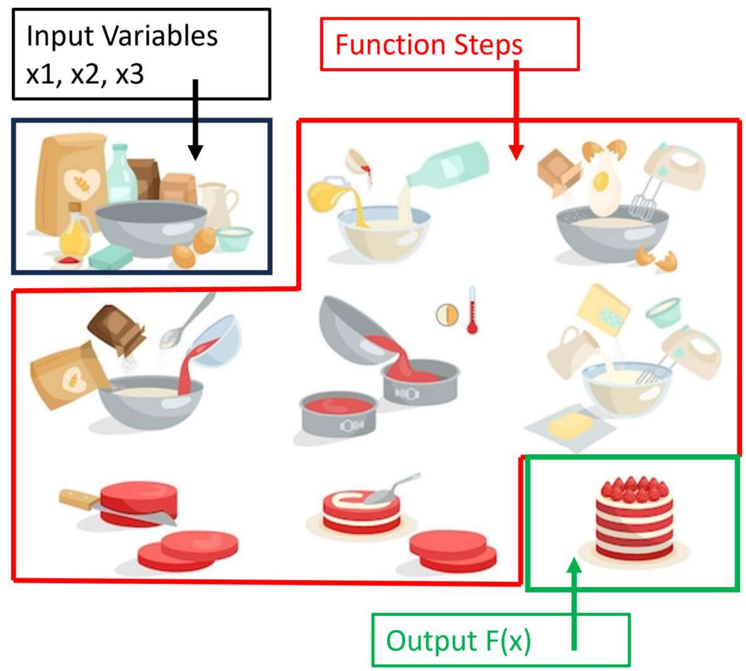
Today is pretty concept heavy you might end up feeling like this:



Functions are like baking receipes

- Assemble all your ingredients (independent variables like x_1, x_2)
- Follow the instructions to mix, bake, and decorate (Instructions of function)
- End up with final product (f(x))

Typically we use the notation $f(x) = x$, but we can always use different representations like $g(x) = x$ or $y = x$



Properties of functions

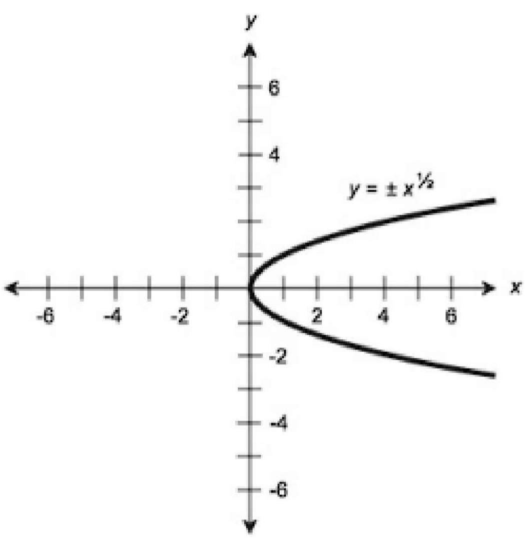
For each combination of independent variables, there is exactly one value of the dependent variable.

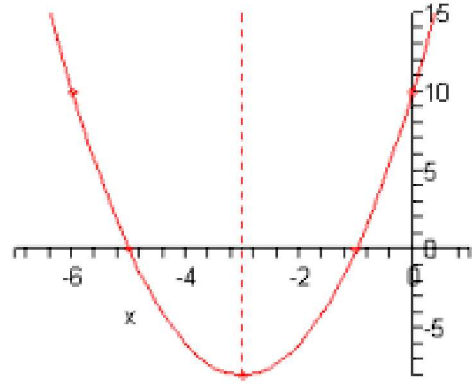
- $f(1) \neq \{2, 3\}$ i.e. if I put 1 into the function I can't have both 2 and 3 come out
- Vertical line test

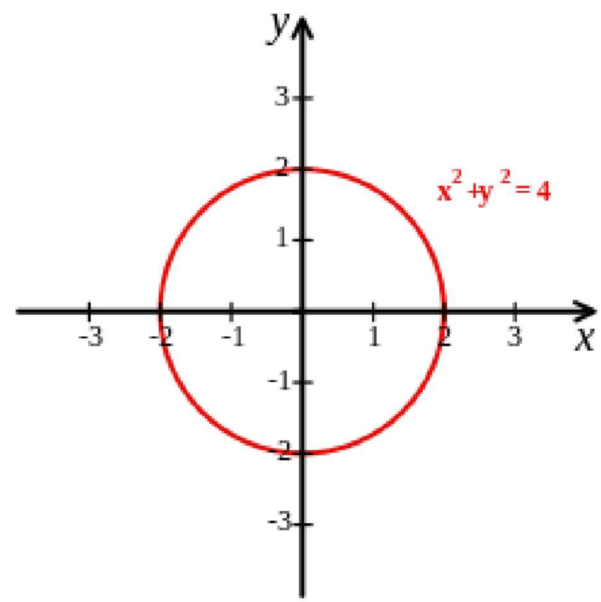
Functions can be continuous or discontinuous

- Continuous: For every value of x, f(x) returns a number
- Discontinuous: Parts are undefined

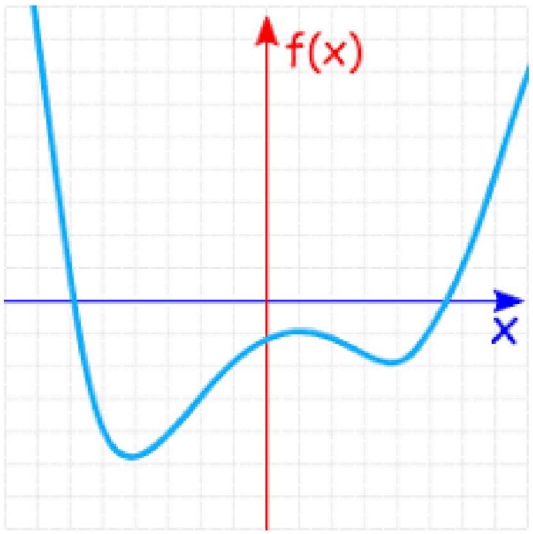
Are these functions?





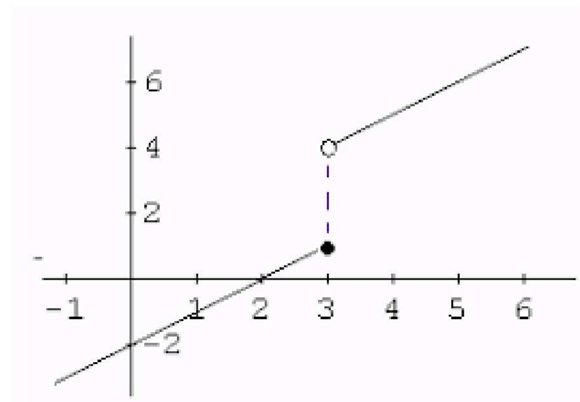


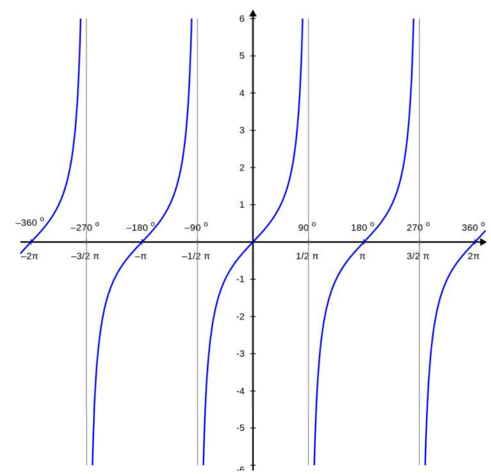
Continuous



Most function we deal with will be continuous

Discontinuous





Limits

Definition:

The value a function approaches as the input approaches a specific value

$$\lim_{x \to c} f(x) = L$$

Verbally:

"The limit of the function f(x) as x approaches the value c is L"

Start with an example

What is limit of $f(x) = 2x^2 - 4$ as x approaches 2?

Strategy to solve: Evaluate f(x) at x that get closer and closer to 2

Quick sanity check, what should $f(2) = ?$

$f(2) = 4$

x approaches 2 from negative direction

x approaches 2 from positive direction

x	1.5	1.75	1.95	1.99	**2.0**	2.01	2.05	2.25	2.5
$f(x)$	0.5	2.12	3.60	3.92	**4.0**	4.08	4.41	6.13	8.5

f(x) approaches 4 from negative direction

f(x) approaches 4 from positive direction

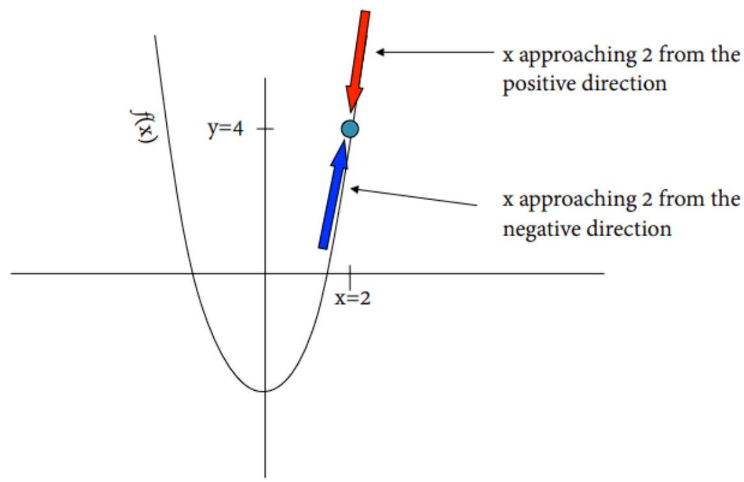
From both directions it looks like f(x) converges to 4

$$\lim_{x \to c}(2x^2 - 4) = 4$$

Graphical Solution

Finding limits of functions when f(x) is continuous at c is easy.

$$\lim_{x \to c} f(x) = f(c)$$



What happens with discontinuous functions?

Try with your group the same approach for:

$$\lim_{x \to 2} f(x) = \frac{|x-2|}{x-2}$$

Hint: Start very close to 2 like 1.9 and 2.1

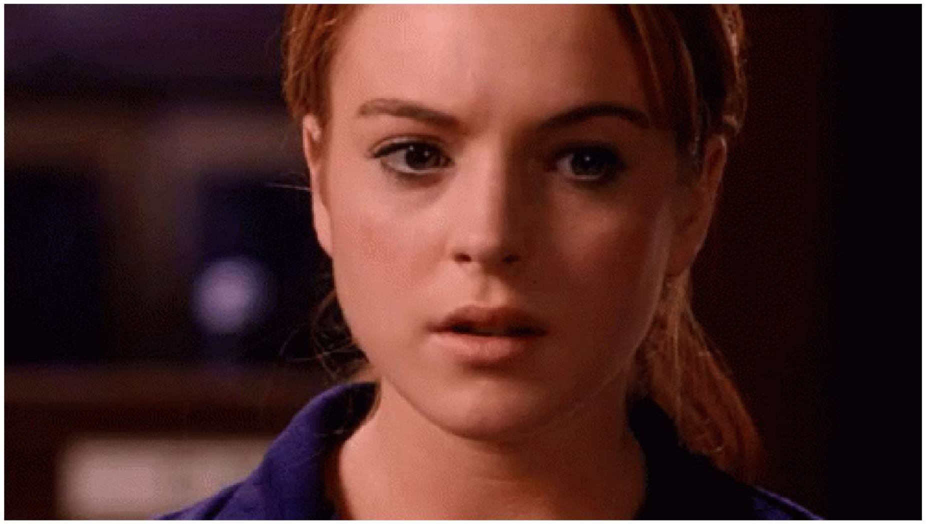
x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	-1	-1	-1	?	1	1	1

We broke the universe

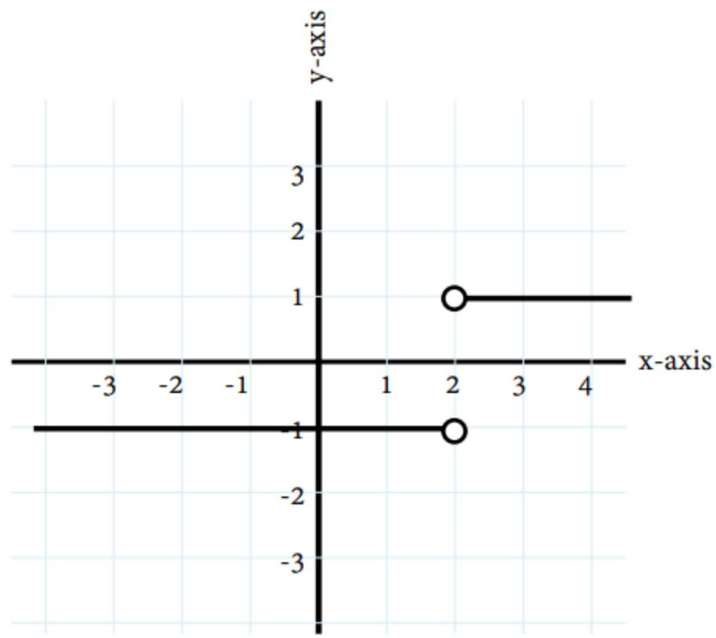
Dividing by zero is impossible

The function approaches different values from either side

Therefore...



Clearer on the graph



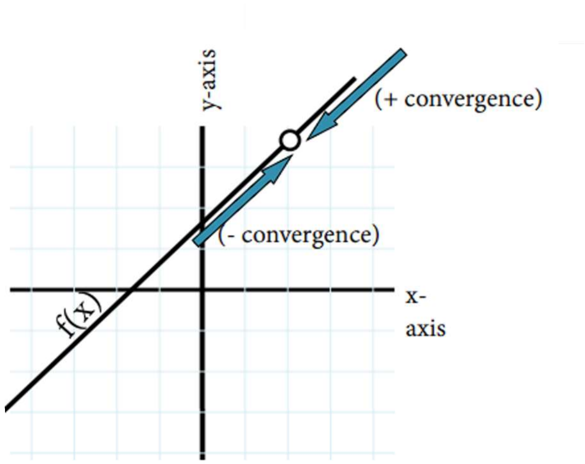
Technically the limit is "Undefined", because it does approach a finite value. Does not exist happens when the limits spiral off to infinity in both directions (tan graph)

Defined Limit at an undefined point

Discontinuous functions can still have limits

Find the limit of $f(x) = x + 1, x \neq 2$ as x approaches 2

x	1.9	1.99	1.999	--	2.001	2.01	2.1
f(x)	2.9	2.99	2.999	--	3.001	3.01	3.1



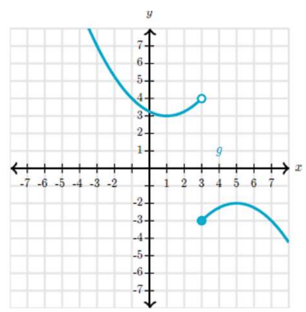
x=2 may not exist, but we can still find a limit because it consistently approaches 3 from both directions

Team Assessment

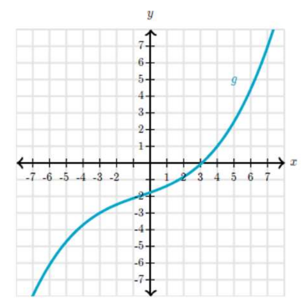
Work with you team to complete these tasks Part 1:

1) Match the limit expression to the graph

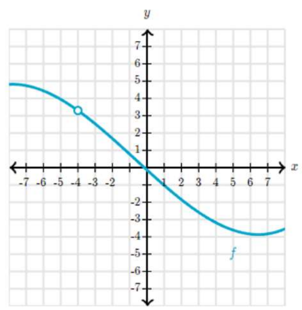
A)



B)



C)



1)

$$\lim_{x\to 3} g(x) = 0$$

2)

$$\lim_{x\to -4} g(x) = 3$$

3)

$$\lim_{x\to 3} g(x) = \text{DNE}$$

Work with your team to complete these tasks Part 2:

2) Can you think of examples where discontinuous function might exist in environmental science?

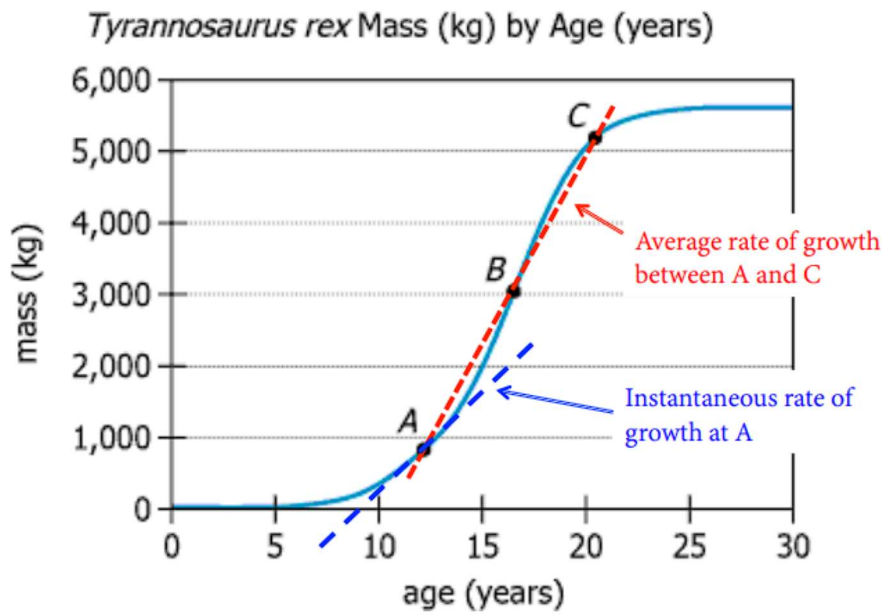
3) Choose as a team to draw an example of one of these statements

- $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ are both infinite
- $\lim_{x \to 3} f(x) = 2$, but $f(3) = 0$
- $\lim_{x \to 5^-} f(x) = 4$ and $\lim_{x \to 5^+} f(x) = 2$
- $\lim_{x \to -3} f(x) = -5$ but $f(-3) = -5$

Introduction to Derivatives

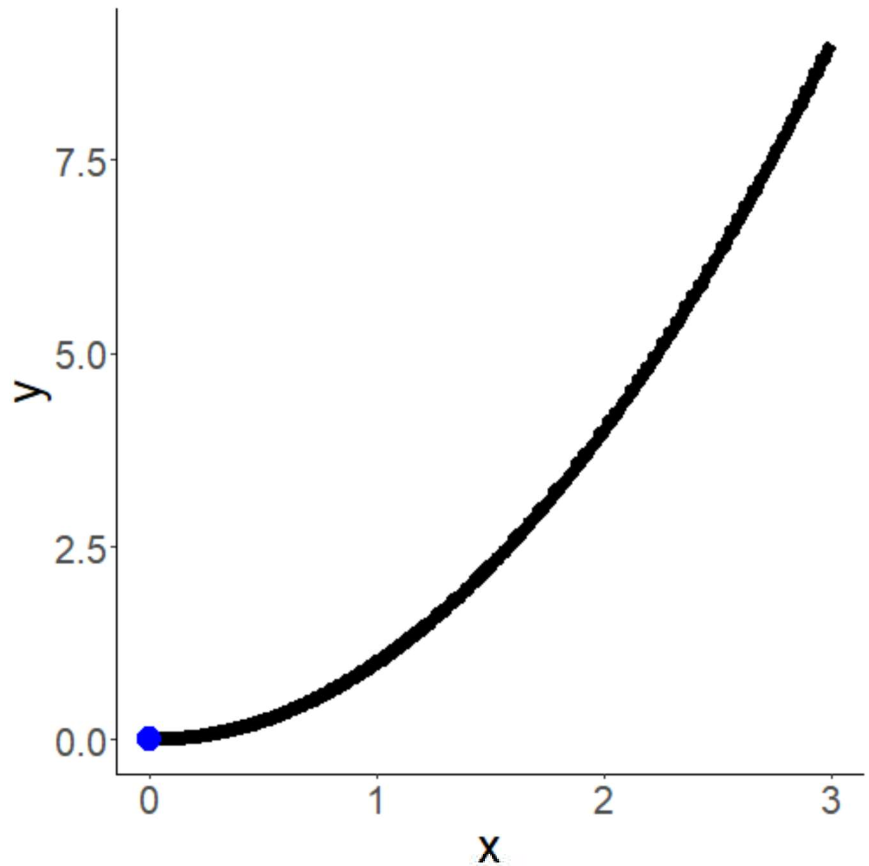
Putting it all together

Recall average rate of change and instantaneous



Taking the average rate of change to a set limit will eventually converge to the instantaneous.

Walk Through Example



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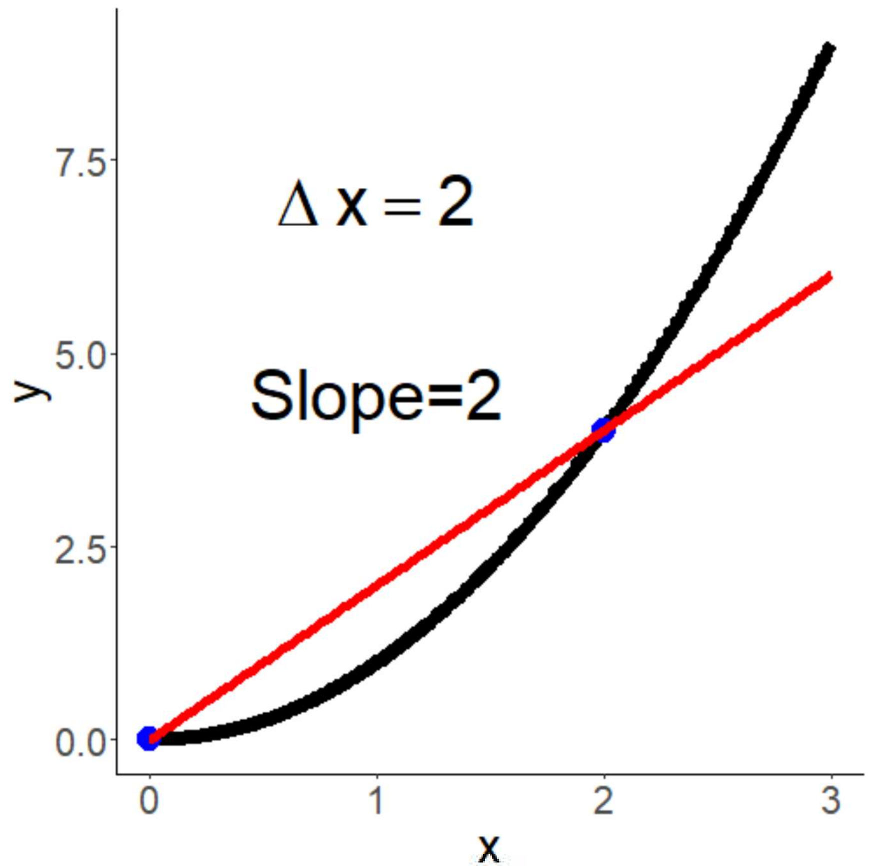

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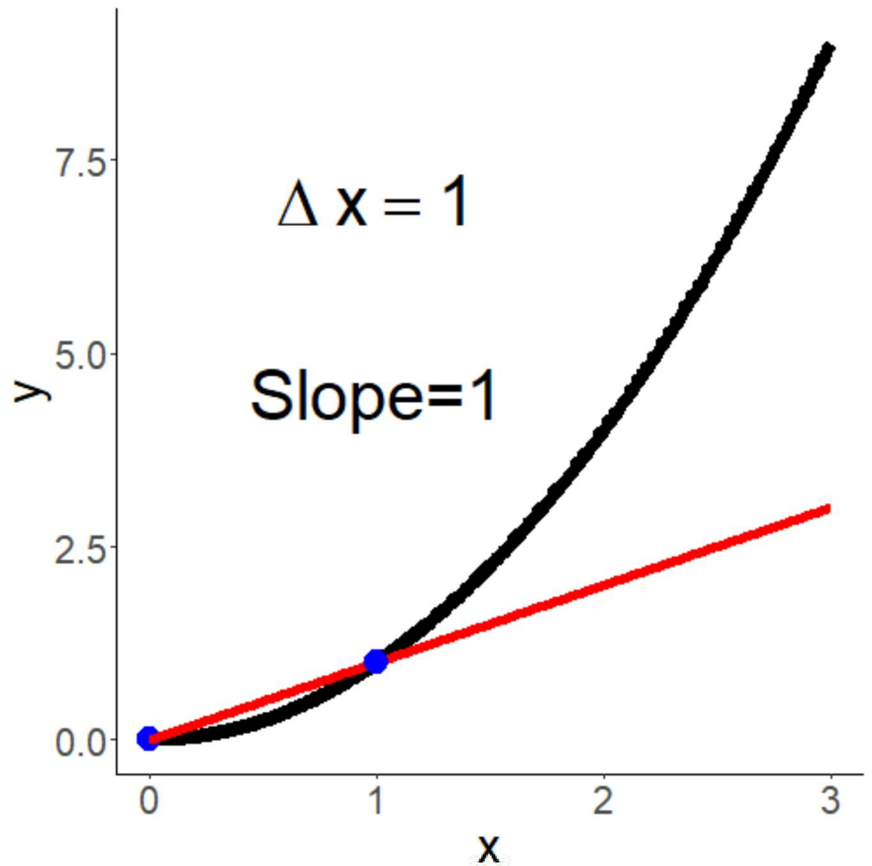
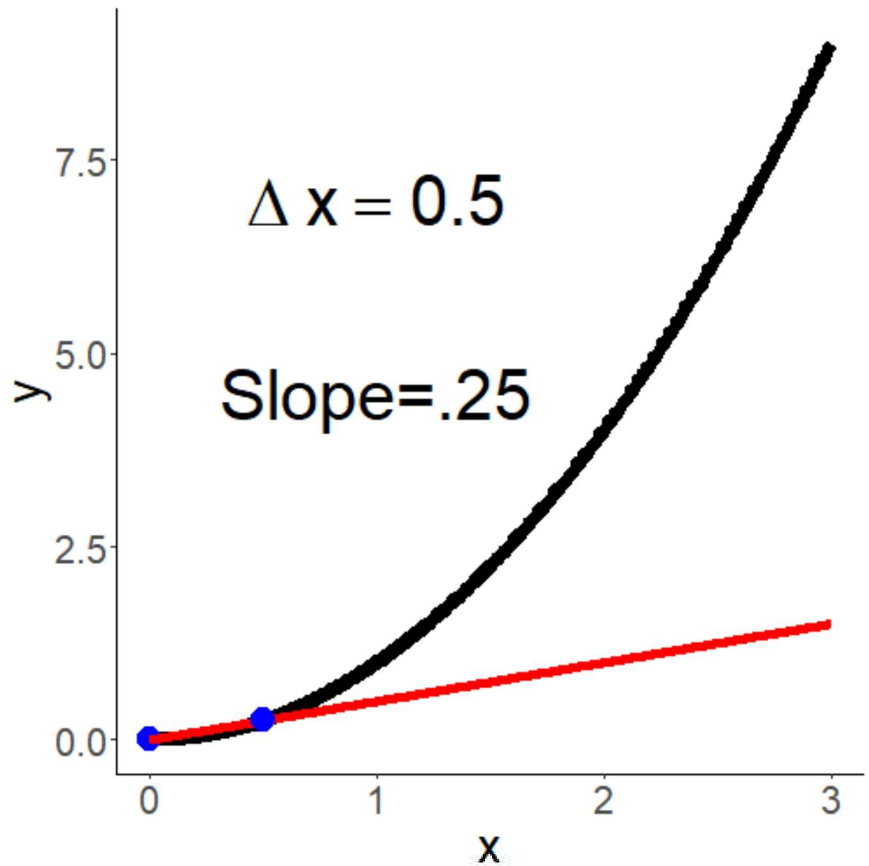
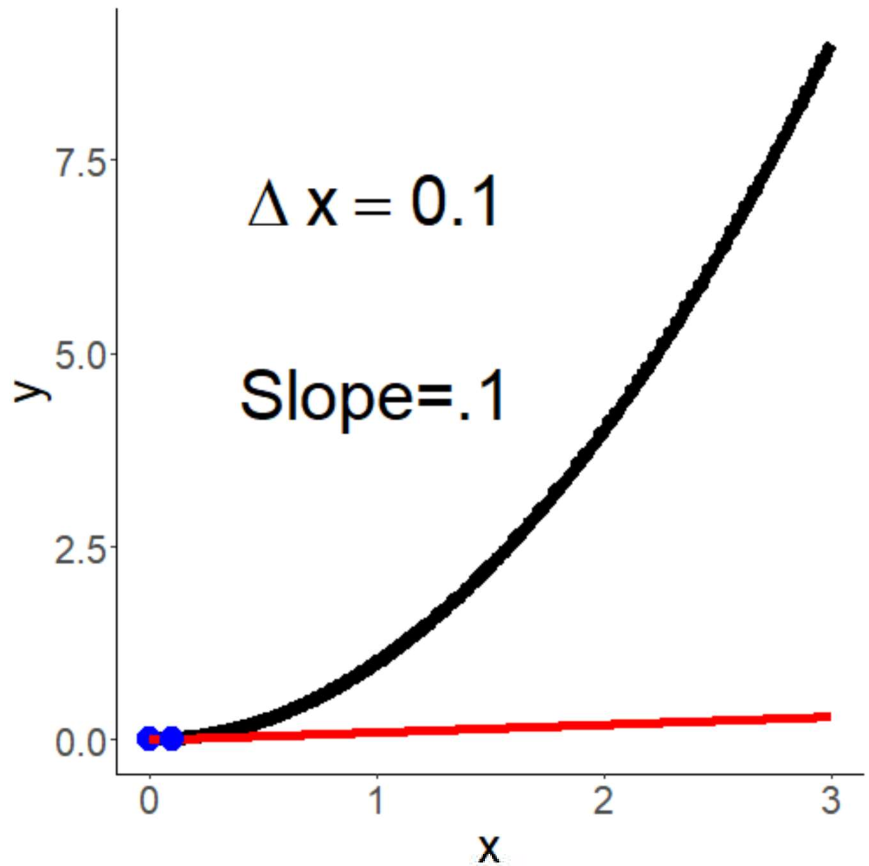


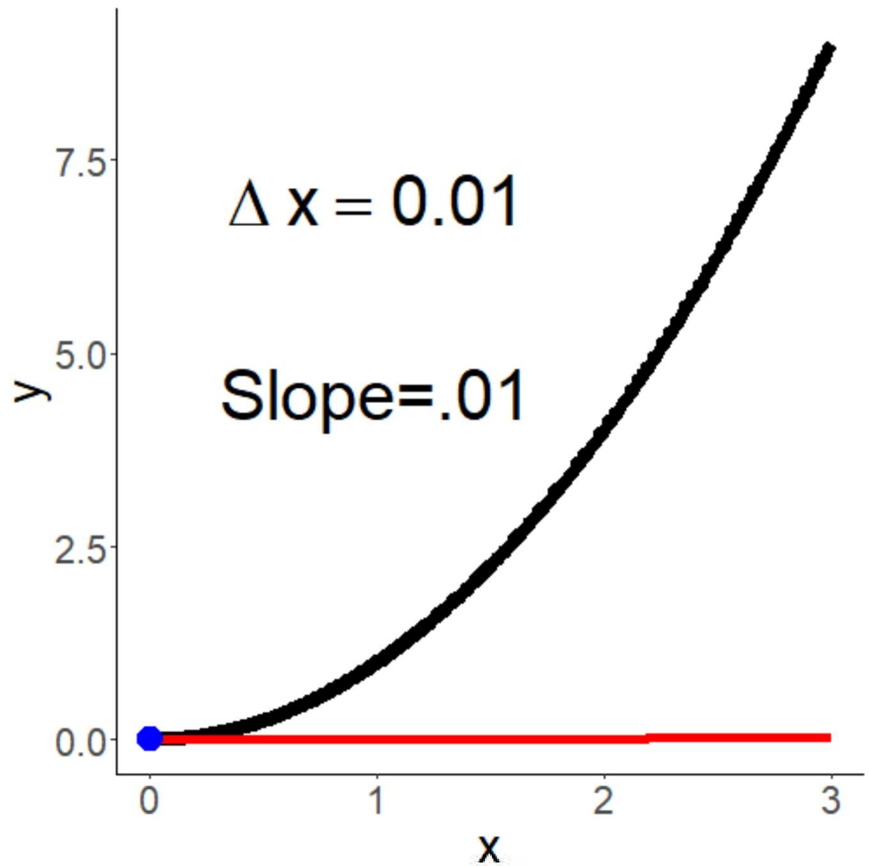
Walk Through Example



Walk Through Example

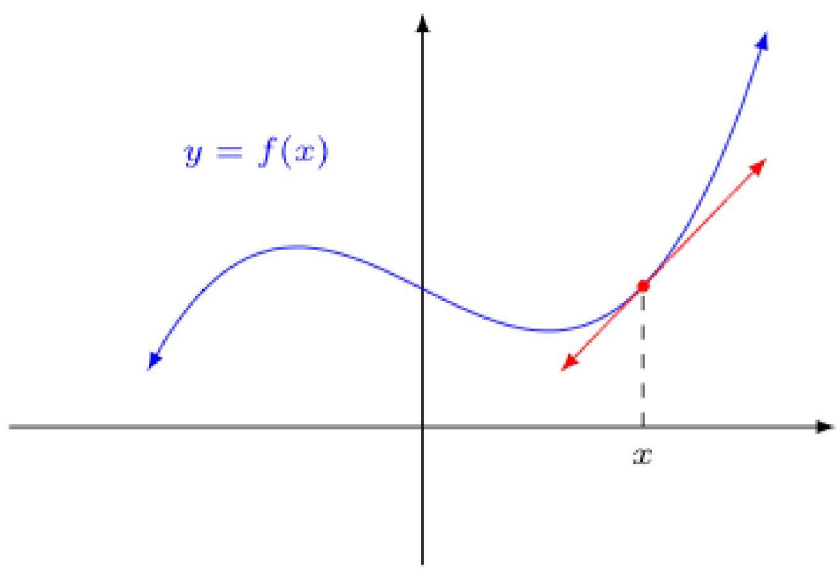


Walk Through Example



Tangent Lines

What if we set $\Delta x = 0$? Then we would have a slope line that only touches at x=0.



These are tangent lines

Put our example in math notation

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Choose a point along the function $(x, f(x))$

Choose a different point on the function Δx away $(x + \Delta x, f(x + \Delta x))$

Add these into the slope equation

$$\text{slope} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivative Definition

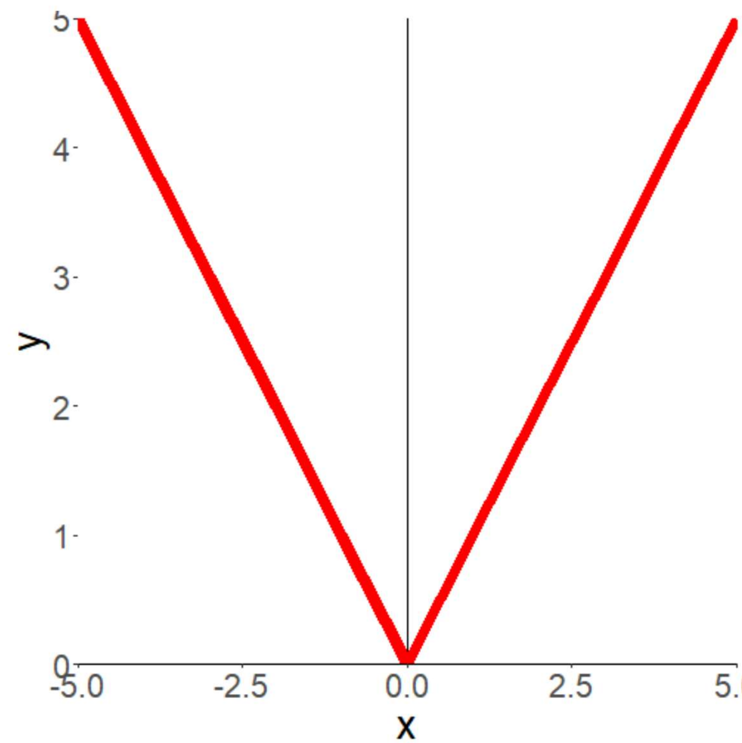
$$f'(x) = \lim_{\Delta x \to 0} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Most common notation

$$f'(x), \text{ or } \frac{dy}{dx}$$

Not all functions are differentiable

- All differentiable functions are continuous, but not all continuous functions are differentiable
- The absolute value function is one example $y = |x|$



Calculus and Derivatives are the study of change

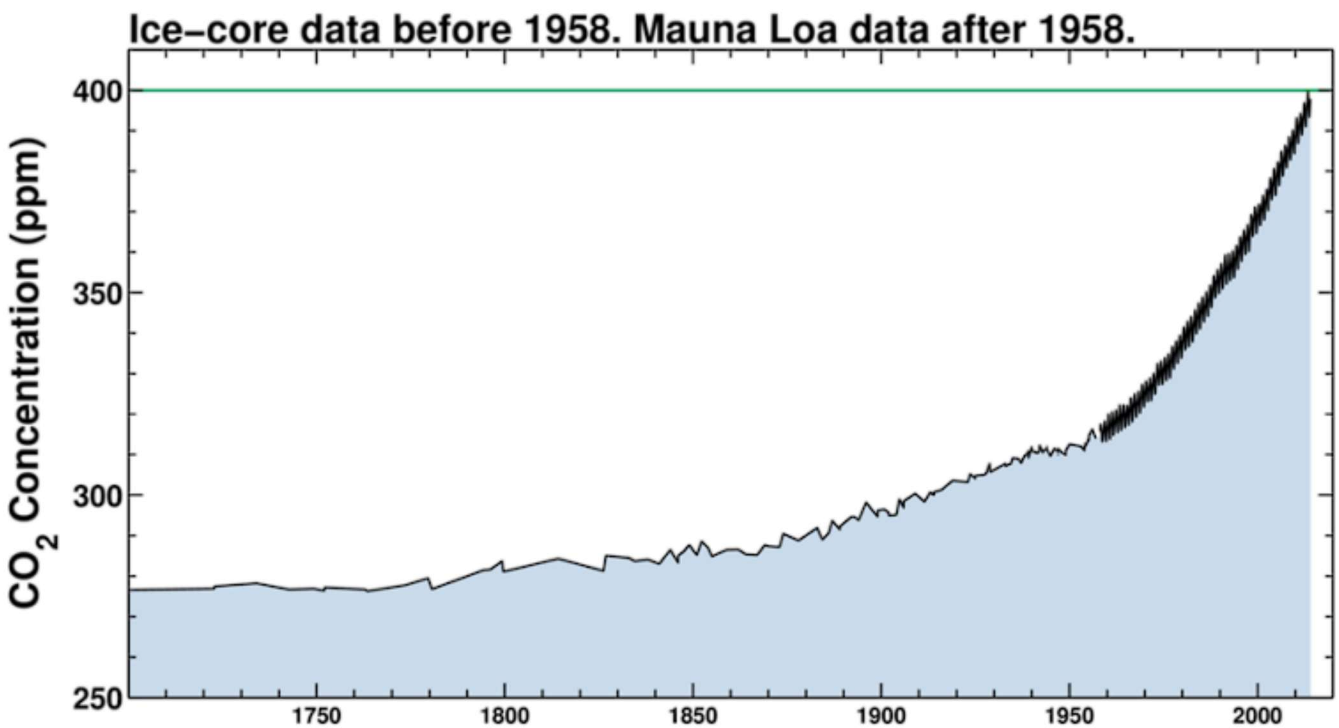


Image source: The Keeling Curve

Rules for Differentiation

Constant Rule

- If f is constant, then for all x, $f'(x) = 0$

$$y = a \qquad \frac{dy}{dx} = 0$$

Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Examples

$$y = 100 \qquad y = 5x^5 \qquad y = \frac{1}{x^2}$$

Rules for Differentiation

Sum and Difference Rules

$$\frac{d}{dx} = [f(x) + g(x)] = f'(x) + g'(x)$$
$$\frac{d}{dx} = [f(x) - g(x)] = f'(x) - g'(x)$$

Looks really scary. All it says, if the function has pieces that are added or subtracted you can take the derivative of each individual piece.

Example

$$y = x^2 + 8x + 4 \qquad y = x^3 - x^2 + x - 15$$

Team Assessment

1) As a team, list 5 fields of environmental science where studying the rate of change and derivatives would be important

2) Which rules should you use to take these derivatives?

$$\text{A) } f(x) = 3x^4 \qquad \text{B) } y = 4x^2 + 3x - 16$$

3) Find the derivatives of these functions

$$\text{A) } y = 3x^2 \qquad \text{B) } h(x) = y = 7x + 4 \qquad \text{C) } g(y) = \sqrt{y}$$

4) Discuss why derivatives need continuity. It might help to think about why some continuous functions don't have derivatives.