UC SANTA BARBARA

Bren Calculus Workshop

Intro to Integration

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Team Review

How did everyone feel about the problem set?

Anything remaining confusing?

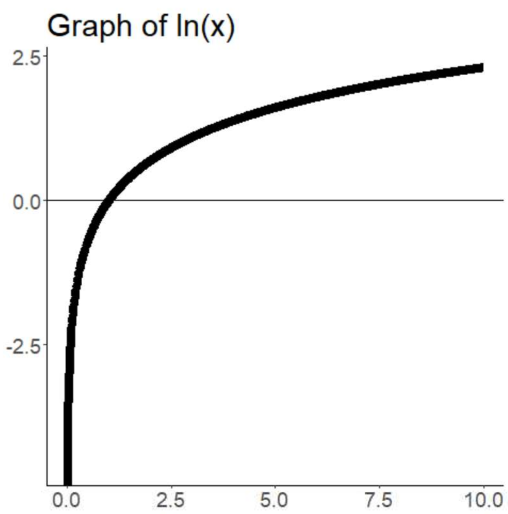
Discuss with Team

Natural Logartihms

Inverse of exponential

By Definition:

$$e^{\ln(x)} = x$$



Properties:

$$\ln(e^x) = x$$

$$\ln(xy) = \ln x + \ln y$$

$$\ln(\tfrac{x}{y}) = \ln x - \ln y$$

$$\ln(x^y) = y \ln x$$

Derivative of Natural Log

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Math Uses:

Allows us to more easily handle exponents (e^x)

Surprisingly simple derivative

Really important in Integration

Real World Uses:

Time and Growth Problems

Differential Equations

Examples of Natural Logs

Solve for t in this equation:

$$A = Pe^{rt}$$

$$\frac{A}{P} = e^{rt}$$

$$\ln(\frac{A}{P}) = rt$$

$$\frac{\ln(A) - \ln(P)}{r} = t$$

Find the Derivative

$$y = \ln(x^2)$$

$$\begin{aligned} y &= \ln(u) & u &= x^2 \\ \frac{dy}{du} &= \frac{1}{u} & \frac{du}{dx} &= 2x \\ \frac{dy}{du}\frac{du}{dx} &= \frac{2x}{u} & &\text{By Chain Rule} \\ \frac{dy}{dx} &= \frac{2x}{x^2} \\ \frac{dy}{dx} &= \frac{2}{x} \end{aligned}$$

Team Assessment

1) Solve these equation

$$\text{Differentiate A) } f(x) = x \ln x \qquad \text{B) } 5 + \ln(3x) = 7 \qquad \text{C) } e^{5x-0.2} = 10$$

2) You contribute monthly to your retirement a company. After leaving the company, you no longer contribute, but the fund continues to accrue interest until you can remove it without penalty at age 55.

The increase in your funds is given by:

$$S = 122000e^{0.032t}$$

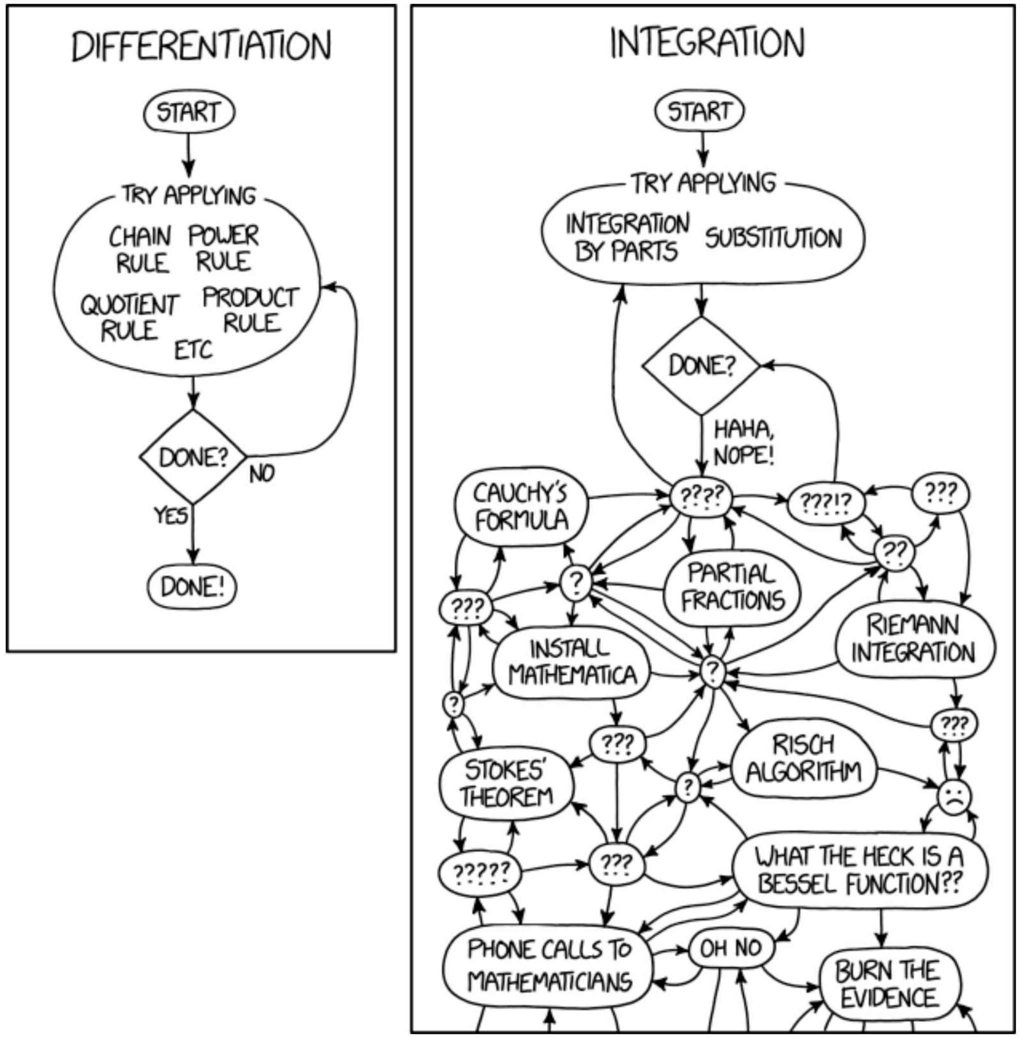
,

where S is the amount in your retirement account t years after leaving the company.

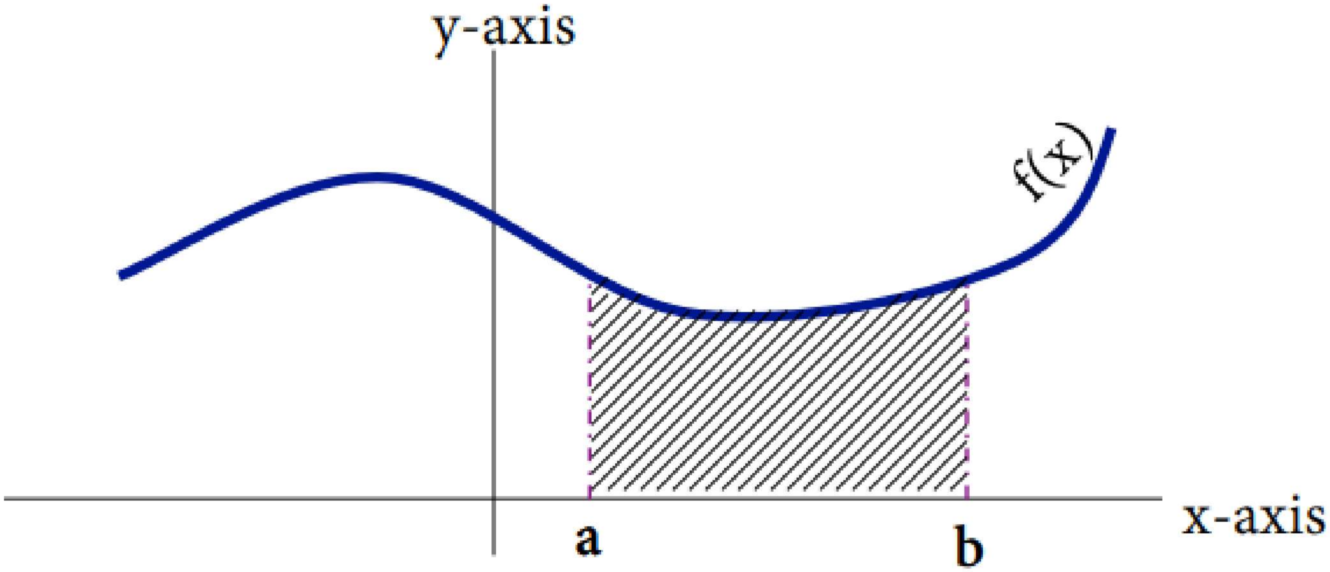
How many years after you quit will your retirement account be earning $8000/year in interest?

3) Why can't logarithms have a negative x value?

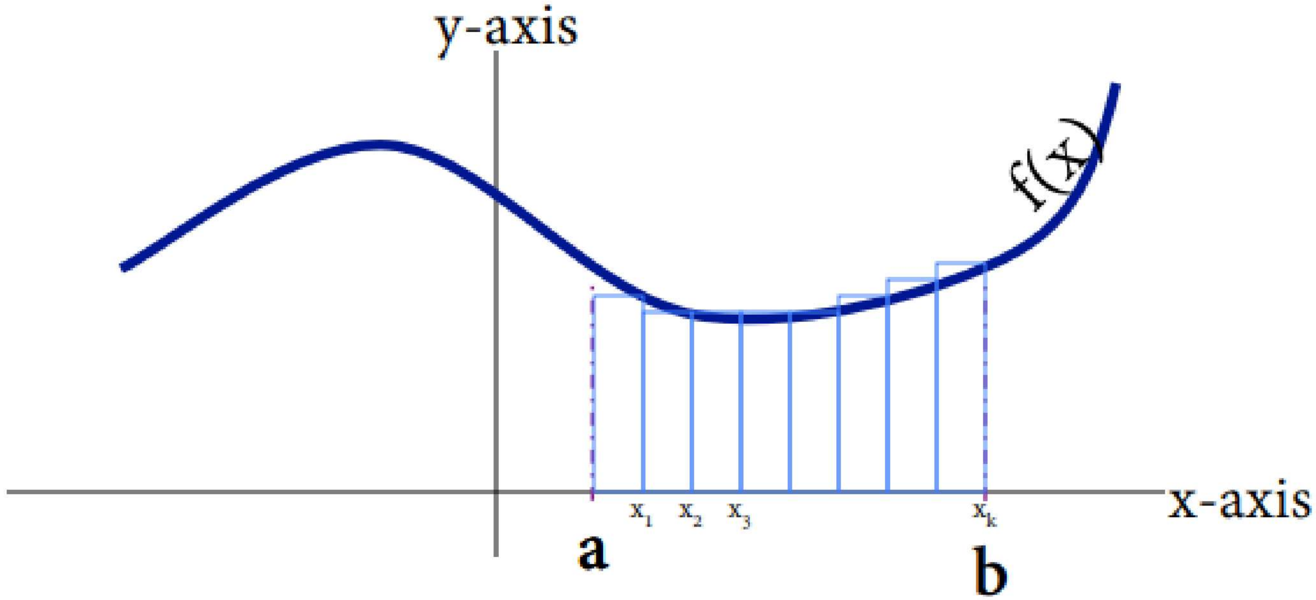
Integration Conceptually


DIFFERENTIATION
START
TRY APPLYING
CHAIN RULE
POWER RULE
QUOTIENT RULE
PRODUCT RULE
ETC
DONE?
NO
YES
DONE!
INTEGRATION
START
TRY APPLYING
INTEGRATION BY PARTS
SUBSTITUTION
DONE?
HAHA, NOPE!
CAUCHY'S FORMULA
????
???!?
???
??
?
PARTIAL FRACTIONS
RIEMANN INTEGRATION
INSTALL MATHEMATICA
RISCH ALGORITHM
STOKES' THEOREM
?????
WHAT THE HECK IS A BESSEL FUNCTION??
PHONE CALLS TO MATHEMATICIANS
OH NO
BURN THE EVIDENCE

How do we find the area under this curve?



We can approximate the area by summing up easy to calculate rectangles



How do we make the rectangles?

1. Start at one end of the interval $[a, b]$
2. Evaluate the function at $f(a)$
3. Step away from a by some small amount called Δx
4. Multiply $f(a)\Delta x$
5. Repeat the same steps above, but keep moving the function evaluation to new intervals
6. Sum up all (k) rectangles

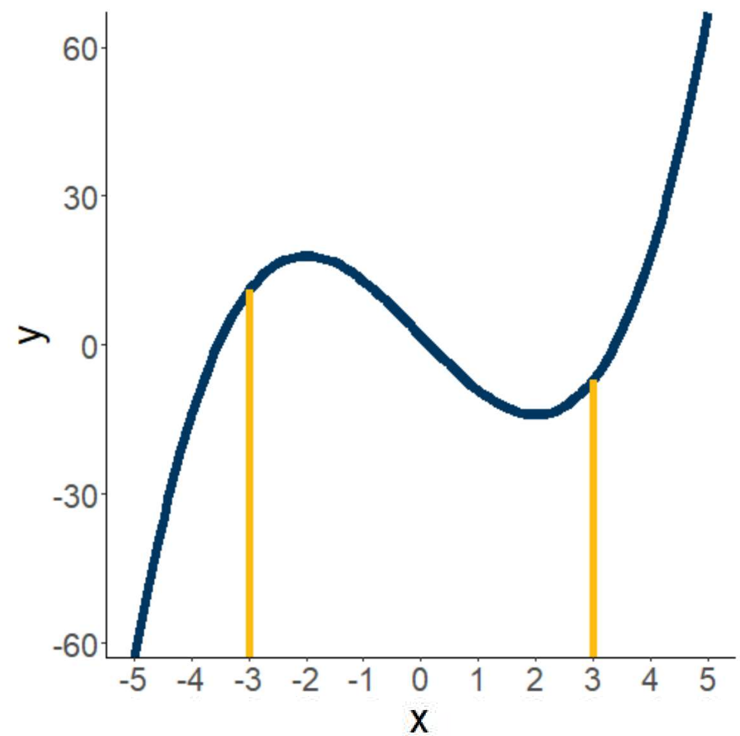
$$\text{Area} = f(x_1)\Delta x + f(x_2)\Delta x + \ldots f(x_k)\Delta x$$

This is called a **Reimann Sum**

Example of Reimann Sum

Approximate the area under the curve from $[-3, 3]$ with $\Delta x = 1$ and $k = 6$

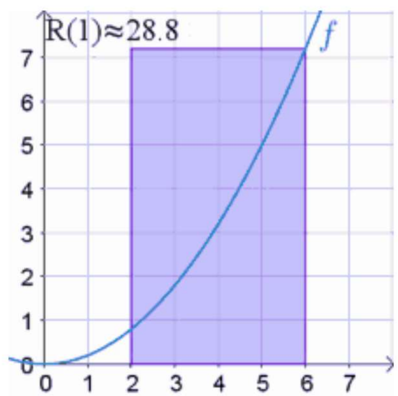
x	y
-3	11
-2	18
-1	13
0	2
1	-9
2	-14
3	-7



Expanding Reimann Sums

What if we try to make Δx really small and the number of rectangles really big $(k \to \infty)$?

Our estimation of the Area will get better and better with more rectangles



Take the limits to infinty

Take the area calculation we did before, but with infinitely many ks

$$\lim_{k \to \infty} R(f(x), \Delta x) = \text{True Area}$$

We formally call this an integral with the following notation:

$$\int_a^b f(x)dx$$

Verbally:

The Integral of f(x) from x=a to x=b with respect to x

How do we calculate?

Taking infinite sums is tedious to impossible by hand

Luckily the dx portion hints at the way to solve integrals

Fundamental Theorem of Calculus

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by:

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)$ over $[a, b]$

FToC means Integral=Anti-Derivative

If we integrate a derivative, we should get the same function back as the original vice-versa

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \qquad \text{Differentiation is the inverse of integration}$$

$$\int f'(x) = f(x) + C \qquad \text{Integration is the inverse of differentiation}$$

Where the heck did that C come from?

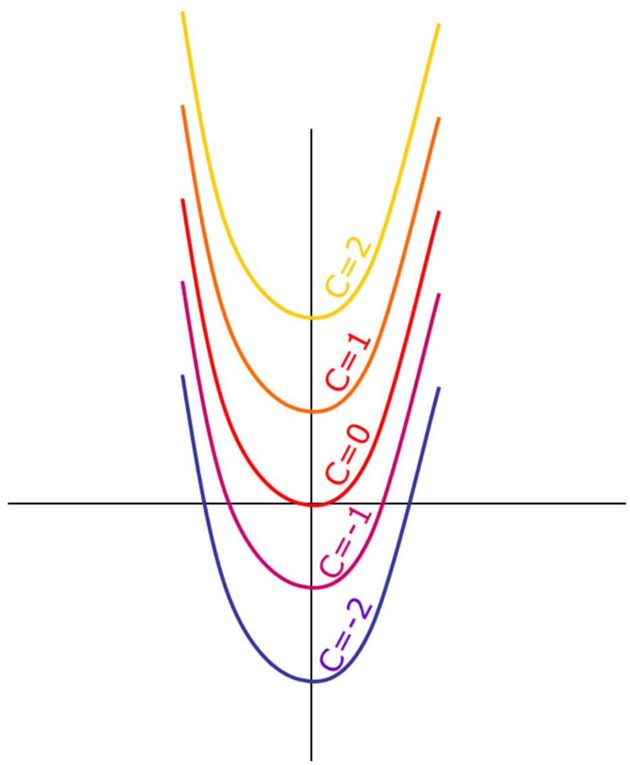
Start with $x^2 - 4x + 1$

$\frac{d}{dx} = 2x - 4$

If we integrate, we'll have $x^2 - 4x + C$

There is no way of knowing what the x-intercept should be without additional information

We have to solve with an initial value problem



Why is Integration Useful?

- Convert from rate of changes to total values
- Solve Differential Equations
    - i.e $\frac{dy}{dx}=2x-4$
    - Logistic Growth is one example
- Total area of curves
    - Useful in policy evaluations

Integration is tough

Before integrating ask yourself:

1. What am I trying to solve?
2. Does it make sense to take an integral?

Rules of Integration

Follow many of the same rules for derivatives

Sum and Difference rules

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Rules of Integration

Constant rules

$$\int k dx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

Second part says we can move multiples outside the integration to simplify.

Rules of Integration

Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \mathrm{n} \neq 1$$

Workhorse rule of integration

Examples

$$f(x) = 4 \qquad g(x) = 2x^2 \qquad f(x) = x^3 - 4x$$

Initial Value Problems

To find the C terms, we need extra information

Often times that comes from being given an *initial value*

The start of the period we know that $y(0) = a$ or $y(12) = b$

We use this information to solve for what C should be

IVP Example

The marginal cost of producing x units of a product is:

$$\frac{dC}{dx} = 25 - 0.02x$$

Where C is the cost (\$), and x is the number of units produced. Given that producing 2 units of product costs $10, what is the complete cost equation?

$$\begin{aligned} C(x) =& 25x - \frac{.02x^2}{2} + D && \text{Solve with Power Rule} \\ 10 =& 25(2) - \frac{0.02(2)^2}{2} + D && \text{Sub in x=2} \\ -39.96 =& D \\ C(x) =& 25x - \frac{0.2x^2}{2} - 39.96 \end{aligned}$$

Definite vs indefinte Integrals

Definite integrals have start and end values

We must evaluate between the intervals by

$$\int_a^b x dx$$

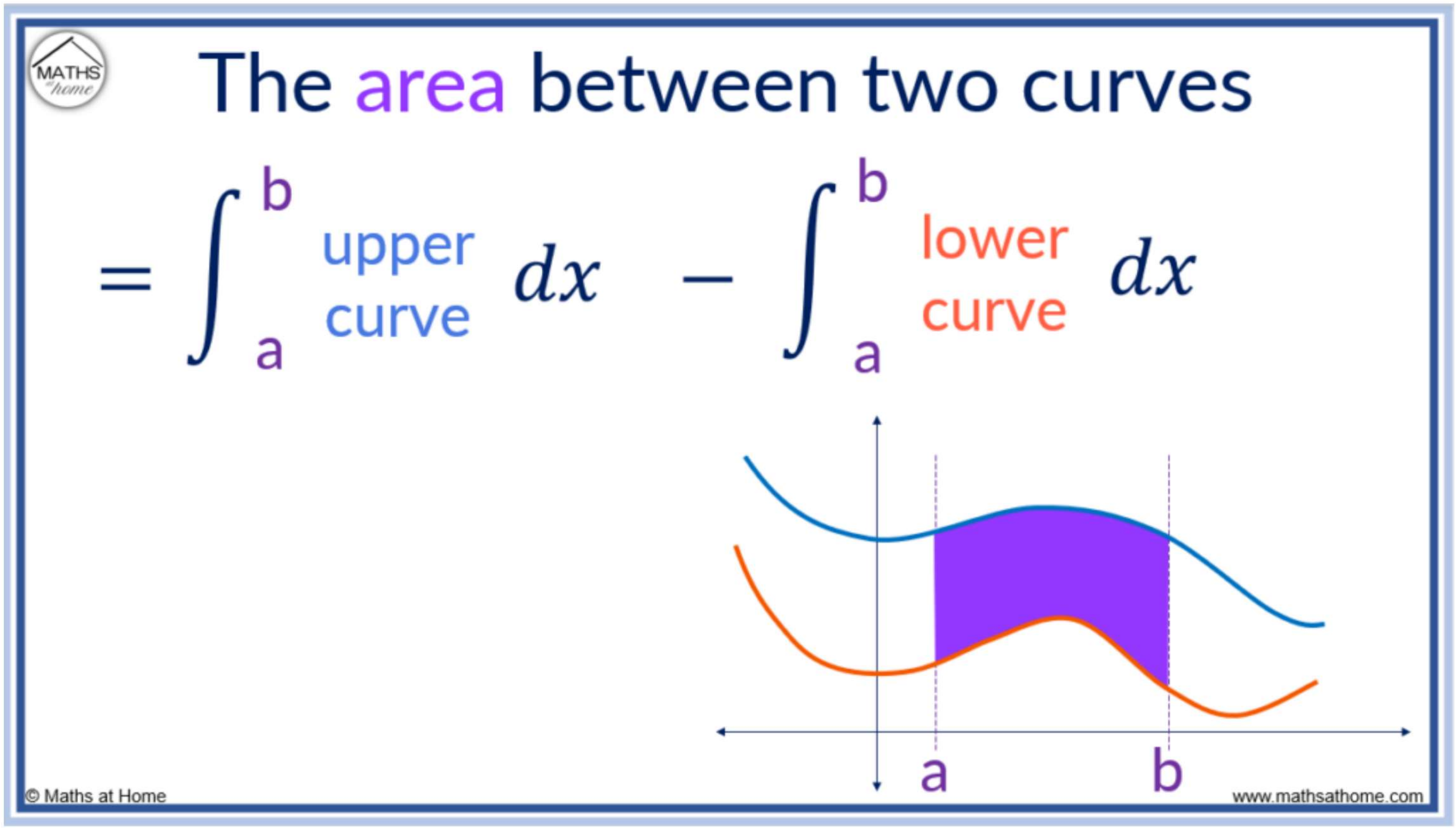
$$\frac{1}{2}x^2|_a^b$$

$$\frac{1}{2}(b)^2 - \frac{1}{2}(a)^2$$

Definite Integral example

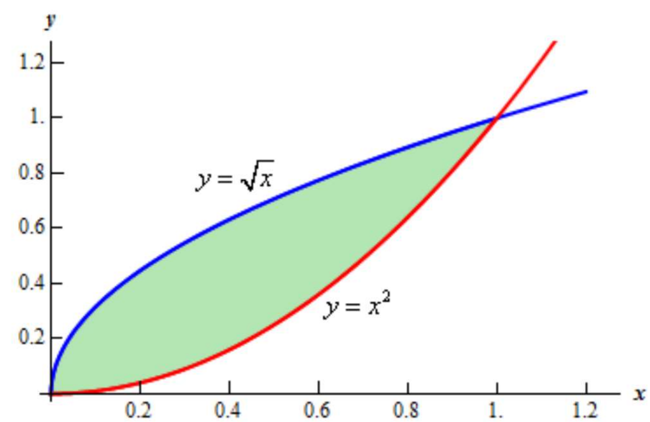
$$\int_{-1}^{2} 3x^2 dx$$
$$x^3|_{-1}^{2}$$
$$(2)^3 - (-1)^3 = 9$$

Area between curves



Area between curves example

Find the area of the shaded region in the graph below



Area between curves example

$$\int_0^1 \sqrt{x} = \frac{2}{3}x^{\frac{3}{2}}$$
$$\int_0^1 x^2 = \frac{1}{3}x^3$$

Then get the area of each by solving the definite

$$\frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}} = \frac{2}{3}$$
$$\frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$$

Then subtract the upper from the lower

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Team Assessment

1) Explain and demonstrate how to approximate the area under the curve $f(x) = -9x^3 + 3x - 9$ on the interval [4,10] using a Riemann sum with $k = 3$

2) Take the Integrals

$$\text{A) } y = \frac{3}{x^2}, y(0) = 5 \qquad \text{B) } g(t) = 3t^5 - 2t^3 + 16t - 7 \qquad \text{C) } \int_2^4 \frac{1}{2}x$$

3) A model for the rate of change in ozone concentrations over time between 1962-1984 is given by $\frac{dC}{dt} = 2t + 20$. Where C is the ozone concentration (ppm) and t is the elapsed time in years. Given that in 1964 the ozone concentration was 30 ppm, what was the ozone concentration in 1982?