Supplementary Materials for "Absence of spin-boson quantum phase transition for multi-level charge qubits"

MICROSCOPIC PARAMETERS FOR THE CHARGE-BOSON MODEL

In this section, we review one possible method for establishing the microscopic Hamiltonian from the circuit model, Eq. (4) of the main text. The starting point is the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\vec{\Phi}}^{\intercal}\boldsymbol{C}\dot{\vec{\Phi}} - \frac{1}{2}\vec{\Phi}^{\intercal}\mathbb{1}_{\boldsymbol{L}}\vec{\Phi} + E_J\cos(\Phi_0) - n_g\dot{\Phi}_0, \tag{S1}$$

written in units of $\hbar = 2e = 1$, and with the capacitance and inductance matrices :

$$\boldsymbol{C} = \begin{bmatrix} C_J + C_c & -C_c & & & \\ -C_c & C + C_c + C_g & -C & & \\ & -C & 2C + C_g & -C & \\ & & -C & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}, \quad \mathbb{1}_{\boldsymbol{L}} = \frac{1}{L}\begin{bmatrix} 0 & 0 & & & \\ 0 & 1 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}. \tag{S2}$$

We denoted here $C_J = C_s + C_g$ the combination of the shunting capacitance C_s and ground capacitance C_g of the qubit (see Fig. 1 of the main text).

Diagonalisation. A half-infinite chain is here assumed. We want to find a basis where both capacitance and inductance quadratic forms are diagonal. Since $\boldsymbol{C}$ is positive definite, this is equivalent to solving $\mathbb{1}_{\boldsymbol{L}}\boldsymbol{P} = \boldsymbol{C}\boldsymbol{P}\boldsymbol{\omega}^2$ (a generalized eigenvalue problem). Under the change of basis $\vec{\Phi} = \boldsymbol{P}\vec{\phi}$, the Lagrangian is :

$$\mathcal{L} = \frac{1}{2}\dot{\vec{\phi}}^{\intercal}\boldsymbol{P}^{\intercal}\boldsymbol{C}\boldsymbol{P}\dot{\vec{\phi}} - \frac{1}{2}\vec{\phi}^{\intercal}\boldsymbol{P}^{\intercal}\boldsymbol{C}\boldsymbol{P}\boldsymbol{\omega}^2\vec{\phi} + E_J\cos\left(\sum_l P_{0l}\phi_l\right) - n_g\sum_l P_{0l}\dot{\phi}_l. \tag{S3}$$

$\boldsymbol{P}^{\intercal}\boldsymbol{C}\boldsymbol{P}$ is diagonal, because it commutes with $\boldsymbol{\omega}^2$. We can then scale the eigenvectors to have $\boldsymbol{P}^{\intercal}\boldsymbol{C}\boldsymbol{P} = \mathbb{1}$, and we reach Eq. (2) of the main text. Written explicitly, the generalized eigenvalue problem gives :

$$\text{site } i = 0: \quad P_{1l} = \frac{C_J + C_c}{C_c}P_{0l}, \tag{S4}$$

$$\text{site } i = 1: \quad P_{2l} = P_{1l}\left(1 - \frac{C_f}{C}\frac{\omega_{k_l}^2}{\omega_0^2 - \omega_{k_l}^2}\right), \quad \text{with} \quad C_f = \frac{C_cC_J}{C_c + C_J} + C_g \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}, \tag{S5}$$

$$\text{site } i > 1: \quad P_{i+1\,l} + P_{i-1\,l} = P_{il}\left(2 - \frac{C_g}{C}\frac{\omega_{k_l}^2}{\omega_0^2 - \omega_{k_l}^2}\right), \tag{S6}$$

where we parametrize the wavevector $k_l = \pi l/N_{\text{modes}}$ of mode number l, with N_{modes} the number of modes (or sites in the chain). The eigenproblem is almost invariant by translation, except for the boundary condition. We assume that the solution obeys the form : $P_{jl} = N_l\cos\left((j-1)k_l + \theta_l\right)$, with N_l the normalisation factor, and θ_l a phase shift due to the boundary. Note that P_{0l} follows instead condition (S4), and therefore is not part of the parametrization, hence the '$(j-1)$' labeling of the sites. In the half-infinite chain limit $N_{\text{modes}} \to \infty$, the wavenumbers continuously fill the Brillouin zone : $k_l \in [0, \pi[$. The dispersion relation is obtained by injecting the solution in the bulk equation (S6), resulting in the standard expression:

$$\omega_{k_l}^2 = \frac{4}{LC_g}\frac{\sin^2(k_l/2)}{1 + 4\,(C/C_g)\sin^2(k_l/2)}. \tag{S7}$$

Besides the dispersion relation, we also need P_{0l}, which appears in the coupling term between the charge qubit and the modes of the chain. The phase shift θ_l is imposed by the boundary equation (S5), used together with dispersion relation, and reads:

$$\tan\theta_l = \tan\left(\frac{k_l}{2}\right)\left(2\frac{C_f}{C_g} - 1\right). \tag{S8}$$

Finally, we have to compute the normalization factors. First, one should note that the $k = 0$ eigenfrequency is 0. Equations (S4), (S5), (S6) do not hold in this case, such that matrix elements P_{i0} must be computed by normalization of $\boldsymbol{P}^\intercal \boldsymbol{C}\boldsymbol{P}$, which leads to $P_{00} = 1/\sqrt{C_J + C_c}$, $\quad P_{i0} = 0\ \forall i > 0$. This zero mode is then localized on the zeroth site: we recognize the qubit degree of freedom, which will be singled out as in the main text by a change of variable. The other normalisation factors are computed from the following matrix elements :

$$\forall l, l' \neq 0, \quad \sum_{i,j=1}^{\infty} P_{il} C_{ij} P_{jl'} = \left(C_g \sum_{i=1}^{\infty} P_{il} P_{il'} + \frac{C_c C_J}{C_c + C_J} P_{1l} P_{1l'} \right) \left(1 + 4 \frac{C}{C_g} \sin^2(k_l/2) \right). \tag{S9}$$

This matrix must be diagonal, so we collect only the terms proportional to $\delta_{l,l'}$. Expanding the first part as

$$\begin{aligned} \sum_{i=1}^{\infty} P_{il} P_{il'} = \frac{1}{2} N_l N_{l'} \sum_{j=0}^{\infty} \quad & \cos(j(k_l + k_{l'})) \cos(\theta_l + \theta_{l'}) + \cos(j(k_l - k_{l'})) \cos(\theta_l - \theta_{l'}) \\ & - \sin(j(k_l + k_{l'})) \sin(\theta_l + \theta_{l'}) + \sin(j(k_l - k_{l'})) \sin(\theta_l - \theta_{l'}). \end{aligned} \tag{S10}$$

Using the identity $\lim_{n\to\infty} \sum_{j=0}^{n} \cos(jk_l) = 1/2 + n\delta_{l,0}$ we can deduce the normalisation factor N_l. With equation (S4), we get an analytic expression of the couplings for a large but finite number of modes N_{modes}:

$$P_{00} = \frac{1}{\sqrt{C_J + C_c}}, \quad P_{0l} = \frac{C_c}{C_J + C_c} \sqrt{\frac{2}{N_{\text{modes}}}} \left(C_g + 4C \sin^2(k_l/2) \right)^{-\frac{1}{2}} \left(1 + \left(2\frac{C_f}{C_g} - 1 \right)^2 \tan^2(k_l/2) \right)^{-\frac{1}{2}}. \tag{S11}$$

These expressions match the results from the numerical diagonalization of the Lagrangian (S1) performed with a finite number N_{modes} of sites, as soon as $N_{\text{modes}} \gtrsim 10$.

Hamiltonian form. Once in diagonal form, the Lagrangian (S3) reads

$$\mathcal{L} = \frac{1}{2} \sum_k \left(\dot{\phi}_k^2 - \omega_k^2 \phi_k^2 \right) + E_{\text{J}} \cos \left(\sum_k P_{0k} \phi_k \right) - n_g \dot{\phi}_0.$$

The Hamiltonian expression is obtained by Legendre transformation, using conjugate momenta $N_k = \partial \mathcal{L} / \partial \dot{\phi}_k$:

$$\mathcal{H} = \sum_k N_k \dot{\phi}_k - \mathcal{L} = \frac{1}{2} \sum_k \left((N_k + n_g P_{0k})^2 + \omega_k^2 \phi_k^2 \right) - E_{\text{J}} \cos \left(P_{0k} \phi_k \right). \tag{S12}$$

Canonical quantization is now straightforward : all dynamical variables are promoted to operators, obeying the commutation rule $[\phi_k, N_l] = i\delta_{kl}$. In this Hamiltonian expression, the qubit degree of freedom does not appear explicitly. It can be reinstated following a change of variables that conserves the commutation rules:

$$\begin{cases} \varphi = \sum_k P_{0k} \phi_k \\ \varphi_m = \phi_m \end{cases}, \begin{cases} n = N_0 / P_{00} \\ n_m = N_m - (P_{0m}/P_{00}) N_0 \end{cases}, \tag{S13}$$

where $\vec{n}$ (resp. $\vec{N}$) is the vector of charges conjugate to $\vec{\varphi}$ (resp. $\vec{\phi}$). As a result, we obtain the Hamiltonian:

$$\hat{H} = (P_{00}^2 + \sum_m P_{0m}^2)(\hat{n} - n_g)^2 + 2(\hat{n} - n_g) \sum_m P_{0m} \hat{n}_m + \frac{1}{2} \sum_m \left(\hat{n}_m^2 + \omega_m^2 \hat{\varphi}_m^2 \right) - E_{\text{J}} \cos(\hat{\varphi}). \tag{S14}$$

It is noteworthy that the change of variables does not complicate this expression, thanks to the fact that $\omega_0 = 0$ (this is a general feature of our model, because the qubit mode does not participate in the inductance matrix $\mathbf{L}$). Otherwise, $\omega_0^2 \hat{\phi}_0^2$ would have transformed into:

$$\omega_0^2 \hat{\phi}_0^2 = \left(\frac{\omega_0}{P_{00}} \right)^2 \hat{\varphi}^2 + \left(\frac{\omega_0}{P_{00}} \right)^2 \left(\sum_m P_{0m} \hat{\varphi}_m \right)^2 - 2 \left(\frac{\omega_0}{P_{00}} \right)^2 \hat{\varphi} \sum_m \hat{\varphi_m}, \tag{S15}$$

the right-hand side terms being respectively interpreted as qubit inductive energy, a diamagnetic 'A^2' term, and a supplementary coupling between the qubit and the array. If present, these terms would violate phase compactness. Finally, the array normal modes are expressed in terms of creation/annihilation operators, defined by:

$$\hat{\varphi}_m = \frac{1}{\sqrt{2\omega_m}} (\hat{a}_m^\dagger + \hat{a}_m), \quad \hat{n}_m = i \sqrt{\frac{\omega_m}{2}} (\hat{a}_m^\dagger - \hat{a}_m) \quad \text{and} \quad [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}. \tag{S16}$$

Spectral density. The bath spectral density gives a more convenient tool to describe this system with a small number of relevant parameters. It is defined as a continuous function of frequency, $J(\omega) = \pi \sum_l g_{k_l}^2 \delta(\omega - \omega_{k_l})$. The g_k couplings are defined in the main text as $g_{k_l} = \sqrt{\omega_{k_l}/2} P_{0l}$. All the dependencies in the wave number k_l can be expressed in terms of ω_k using the dispersion relation (S7). We also change sums over modes to integrals in the limit $N_{\text{modes}} \to \infty$:

$$\frac{1}{N_{\text{modes}}} \sum_l F[k_l] \to \frac{1}{\pi} \int_0^\pi \mathrm{d}k \; F[k], \tag{S17}$$

$$J(\omega) = N_{\text{modes}} \frac{\mathrm{d}k}{\mathrm{d}\omega} [g(\omega)]^2 = \left(\frac{C_c}{C_c + C_J} \right)^2 \sqrt{\frac{L}{C_g}} \; \omega \; \frac{\sqrt{1 - \omega^2/\omega_P^2}}{1 + \omega^2 \left(1/\omega_Q^2 - 1/\omega_P^2 \right)} \theta(\omega_P - \omega), \tag{S18}$$

with $\omega_P = \omega_0 / \sqrt{1 + 4C/C_g}$ the plasma frequency of the chain, and $\omega_Q = \omega_P \sqrt{1 + {}^{4C}/_{C_g}} / ({}^{2C_f}/_{C_g} - 1)$ a characteristic frequency related to the qubit. From Eq. (S18), it is clear that the denominator is never vanishing within the band $\omega \in [0, \omega_P]$ (otherwise $J(\omega)$ would be singular). Using the expression for ω_P and ω_Q, and $1/\omega_J^2 \equiv 1/\omega_Q^2 - 1/\omega_P^2$, we recover the low frequency cutoff of the qubit ω_J defined in the main text.

For small frequencies, $J(\omega)$ obeys the so-called Ohmic behavior, $J(\omega) \simeq 2\pi\alpha\omega$, which defines the coupling strength α. A higher frequencies, $J(\omega)$ quickly vanishes as $1/\omega^2$, provided $\omega_J \ll \omega_P$. Most experimental devices verify this condition. On the other hand, if $\omega_J \gg \omega_P$, $J(\omega)$ displays a square-root hard cut-off at $\omega = \omega_P$. In many cases, the exact form of the cut-off is not relevant, and we replace it by an exponential cut-off at $\omega_c = \min\{\omega_J, \omega_P\}$, $J(\omega) = 2\pi\alpha\omega \exp(-\omega/\omega_c)$, which is the spectral function used in the main text to perform the numerical computations.

Microscopic derivation of the electrostatic bound on dissipation. The charging energy of our microscopic circuit explicitly reads

$$E_c = \frac{1}{8} \left(C_J + C_c - \frac{C_c^2}{C_c + \frac{C_g}{2} + \sqrt{C_g \left(\frac{C_g}{4} + C \right)}} \right)^{-1}. \tag{S19}$$

This result can be obtained by noting that $E_c = \boldsymbol{C}_{00}^{-1}/8$ and analytically inverting the capacitance matrix. From equation (5) in the main text one can reach the same result, recast into

$$8E_c = \frac{1}{C_c + C_J} + 2\pi\alpha \frac{\omega_J^2}{\omega_P} \left(\sqrt{1 + \frac{\omega_P^2}{\omega_J^2}} - 1 \right) \geqslant 2\pi\alpha \frac{\omega_J^2}{\omega_P} \left(\sqrt{1 + \frac{\omega_P^2}{\omega_J^2}} - 1 \right), \tag{S20}$$

by integrating the spectral function (S18) over ω. Thus the dissipation strength α obeys the inequality

$$\alpha \leqslant \frac{4E_c}{\pi\omega_P} \left(\sqrt{1 + \frac{\omega_P^2}{\omega_J^2}} + 1 \right). \tag{S21}$$

When $\omega_J \ll \omega_P$ (which is the typical situation for realistic devices), the electrostatic bound $\alpha \leqslant 4E_c/(\pi\omega_J)$ assumes the same form as in the main text (albeit with a different numerical prefactor), while for $\omega_J \gg \omega_P$ the bound reads $\alpha \leqslant 8E_c/(\pi\omega_P)$.

COMPACT ANSATZ WAVEFUNCTION FOR CHARGE SENSITIVE CIRCUITS

We build in this section the compact ansatz step by step, following the outline of the main text. While our approach is completely generic to charge sensitive superconducting circuits, we focus here on the charge-boson Hamiltonian:

$$\hat{H} = 4E_c(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi} + (\hat{n} - n_g) \sum_k i g_k (\hat{a}_k^\dagger - \hat{a}_k) + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k, \tag{S22}$$

where the discrete sums over the wave vector k run on the Brillouin zone $[0, \pi]$. The renormalized linear approximation, that holds only in the deep transmon regime where the phase fluctuations are much smaller that 2π, is used as a linearized parent Hamiltonian for our variational trial state. It reads:

$$\hat{H}_{\text{SCHA}} = 4E_c\hat{n}^2 + \frac{E_J^\star}{2}\hat{\varphi}^2 + i\hat{n} \sum_k g_k (\hat{a}_k^\dagger - \hat{a}_k) + \sum_k \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + {}^1/_2 \right). \tag{S23}$$

Here the charge offset n_g was gauged out since the phase is uncompact in the linear approximation, and $E_J^\star$ is a free parameter used in the variational method, after compactification is applied. This Hamiltonian can be brought to diagonal form by a Bogoliubov rotation mixing the qubit degree of freedom and bosons from the environment. We use greek symbols to denote jointly the qubit and bosons variables, such that $\hat{n}_\mu = (\hat{n}, \hat{n}_1, \hat{n}_2, \ldots)$. Then, up to rescaling of $\hat{\varphi}$ and $\hat{n}$,

$$\begin{aligned} \hat{H}_{\text{SCHA}} &= \tfrac{1}{2}\textstyle\sum_\mu \hat{\varphi}_\mu \hat{\varphi}_\mu + \tfrac{1}{2}\sum_{\sigma\rho} \hat{n}_\sigma \boldsymbol{M}_{\sigma\rho} \hat{n}_\rho \\ &= \textstyle\sum_\mu \Omega_\mu (\hat{b}_\mu^\dagger \hat{b}_\mu + \tfrac{1}{2}). \end{aligned}, \qquad \text{where } \boldsymbol{M} = \begin{bmatrix} 8E_J E_c & g_1\sqrt{2E_J\omega_1} & g_2\sqrt{2E_J\omega_2} & \cdots \\ g_1\sqrt{2E_J\omega_1} & \omega_1^2 & & \\ g_2\sqrt{2E_J\omega_2} & & \omega_2^2 & \\ \vdots & & & \ddots \end{bmatrix} \tag{S24}$$

with Ω_μ the eigen-frequencies of the linear system and $b_\mu, b_\mu^\dagger$ its raising/lowering operators. The last equation is obtained by diagonalization of matrix $\boldsymbol{M}$, which assumes an arrowhead form; efficient numerical algorithms exists for such eigenvalue problems, as well as perturbative series expansion. We can decompose the qubit variables on this new basis. To do so, we define the coefficients u_μ and v_μ such that

$$\hat{n} = i\sum_\nu v_\nu(\hat{b}_\nu^\dagger - \hat{b}_\nu) \quad \text{and} \quad \hat{\varphi} = \sum_\nu u_\nu(\hat{b}_\nu^\dagger + \hat{b}_\nu). \tag{S25}$$

We then enforce the periodic boundary conditions (compactification) by repeatedly displacing the ground state of H_{SCHA}, noted $|0\rangle$, by an integer times 2π:

$$|0_{\circlearrowleft}\rangle = \sum_{w\in\mathbb{Z}} e^{i2\pi w\hat{n}}|0\rangle, \quad \text{where } \hat{b}_\mu|0\rangle = 0 \quad \forall\mu. \tag{S26}$$

Regularisation. With such a definition, $|0_{\circlearrowleft}\rangle$ has infinite norm, because the associated wave function is both periodic and defined over $\mathbb{R}$. Indeed, by re-indexing sums over winding numbers,

$$\langle 0_{\circlearrowleft}|0_{\circlearrowleft}\rangle = \sum_{v,w\in\mathbb{Z}}\int_{\mathbb{R}} \mathrm{d}\varphi\, \langle 0|\varphi - 2\pi v\rangle\langle\varphi + 2\pi w|0\rangle = \langle 0|0_{\circlearrowleft}\rangle \left(\sum_{v\in\mathbb{Z}} 1\right), \tag{S27}$$

which is clearly infinite. A way out is to restrict the wave-function over the interval $[0, 2\pi[$, which is in fact equivalent to simply drop the infinite factor in the last expression:

$$\begin{aligned} \langle 0|0_{\circlearrowleft}\rangle \left(\sum_{v\in\mathbb{Z}} 1\right) &\xrightarrow[\text{to } [0,2\pi[]{\text{restricted}} \sum_{v,w\in\mathbb{Z}}\int_0^{2\pi} \mathrm{d}\varphi\, \langle 0|\varphi - 2\pi v\rangle\langle\varphi + 2\pi w|0\rangle \\ &= \sum_{v,w\in\mathbb{Z}}\int_{2\pi v}^{2\pi(v+1)} \mathrm{d}\varphi\, \langle 0|\varphi\rangle\langle\varphi + 2\pi(v+w)|0\rangle = \langle 0|0_{\circlearrowleft}\rangle. \end{aligned} \tag{S28}$$

The last line is obtained with a relabeling of the sums $w' = w + v$, and patching the integrals together to get back an integral on $\mathbb{R}$. The same trick can be used for expectation values of any operator $\hat{\mathcal{O}}$, provided that it is itself 2π-periodic, which means that $[\hat{\mathcal{O}}, \sum_w \exp(i2\pi w\hat{n})] = 0$.

Aharonov-Casher phases. We already mentioned that n_g acts as a gauge potential on the system. It can usually be removed from the Hamiltonian by a gauge transformation $\hat{U} = \exp(in_g\hat{\varphi})$, which however affects the boundary condition on the phase (unless the model is not compact). Under such a transformation, $\hat{H}_{\text{SCHA}}(n_g) = \hat{U}^\dagger \hat{H}_{\text{SCHA}}(0)\hat{U}$. It can be checked that $\hat{U}^\dagger|0\rangle$ is an eigenstate of $\hat{H}_{\text{SCHA}}(n_g)$. For a non-compact model, the gauge has no observable effect, but the compactification process will change this state of affairs, since

$$\sum_{w\in\mathbb{Z}} e^{i2\pi w\hat{n}}\hat{U}^\dagger|0\rangle = \hat{U}^\dagger \sum_{w\in\mathbb{Z}} e^{i2\pi w(\hat{n}-n_g)}|0\rangle. \tag{S29}$$

As an example, the effect on the ansatz norm is :

$$\langle 0_{\circlearrowleft}|0_{\circlearrowleft}\rangle = \sum_{v,w\in\mathbb{Z}} \langle 0|e^{-i2\pi v(\hat{n}-n_g)}\hat{U}\hat{U}^\dagger e^{i2\pi w(\hat{n}-n_g)}|0\rangle = \sum_{w\in\mathbb{Z}} e^{-i2\pi w n_g}\langle 0|e^{i2\pi w\hat{n}}|0\rangle, \tag{S30}$$

using $\hat{U}\hat{U}^\dagger = \mathbb{1}$ and the same infinite factor canceling argument as before. The offset charge effect is seen as an Aharonov-Casher phase that depends on the winding number. The interference between different winding numbers will create an observable effect due to the gauge. The same argument can be used for the expectation value of any gauge invariant operator $\hat{\mathcal{O}}$, *i.e.* $[\hat{\mathcal{O}}, \hat{U}] = 0$.

$\mathbf{E}_J^\star$ optimisation. The anharmonicity of the cos-shaped potential tends to soften the phase confinement compared to quadratic potential, thus enhancing the zero point phase fluctuations. Having built an ansatz adapted to the specifics of the problem, we can use it as starting point for a variational method. The free parameter is the effective stiffness of the potential $E_J^\star$ in the linearized Hamiltonian (S23). We need to compute the energy expectation value of the full Hamiltonian (S22) within the compactified ground state of the linearized Hamiltonian (S23). It is noteworthy that $\hat{H}$ is 2π-periodic, but not gauge invariant. Instead, we make use of $\hat{U}\hat{H}(n_g)\hat{U}^\dagger = \hat{H}(0)$. Using once again the decomposition (S25),

$$\frac{\langle 0_\circlearrowleft|\hat{H}|0_\circlearrowleft\rangle}{\langle 0_\circlearrowleft|0_\circlearrowleft\rangle} = \sum_\mu \frac{\Omega_\mu}{2} - E_J\frac{u^2}{2} - E_J\left(\sum_{w\in\mathbb{Z}} e^{-2(\pi w)^2v^2 - i2\pi w n_g}\right)^{-1} \sum_{w\in\mathbb{Z}}\left(\frac{(\pi w)^2}{2} + (-1)^w e^{-\frac{u^2}{2}}\right) e^{-2(\pi w)^2v^2 - i2\pi w n_g}. \tag{S31}$$

The $E_J^\star$ dependence is contained in u^2 and v^2. The full numerical procedure consists, at every step, in a minimization of the expression (S31) over $E_J^\star$, using a fast diagonalization the arrowhead $\boldsymbol{M}$ matrix (S24) for the new value of $E_J^\star$, and the computation of the two scalars u^2 and v^2. We then compute the energy expectation value (S31) with a number of terms in the sums over w controlled by v^2. Since the $w \neq 0$ terms are exponentially suppressed, the sums are rapidly convergent. In practice, we need at most $w \sim 10$ terms when the Josephson energy is close to the breaking point of the method, $E_J/E_c \sim 1$. Crucially, u^2 is independent of the number of modes. Overall, the complexity of the whole procedure is $\mathcal{O}(N_{\text{modes}}^2)$, with N_{modes} the total number of modes in the chain. The quality of the Ansatz is found to be excellent, see Fig. S1 for a comparison of the ground state energy obtained in the full NRG simulation.

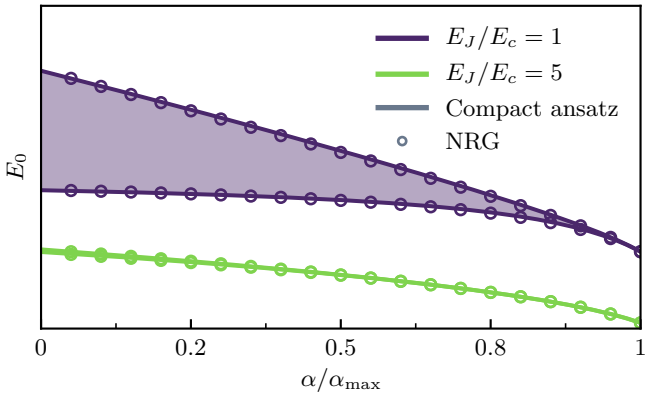

FIG. S1. Ground state energy bands associated to the offset charge n_g, both for a multi-level charge qubit (narrower band at the bottom, at $E_J/E_c = 5$) and the crossover regime (broaded band at the top, at $E_J/E_c = 1$). The analytical expression (S31) from the compact Ansatz (lines) compares quantitatively to the full NRG simulation (dots).

Zero point phase fluctuations. Since $\cos\hat{\varphi}$ is both 2π-periodic and gauge invariant, its expectation value is evaluated using the two previous tricks. Then, expressing every operator in terms of $\hat{b}_\mu^\dagger, \hat{b}_\mu$ with (S25) :

$$\begin{aligned}\langle 0_\circlearrowleft|\cos\hat{\varphi}|0_\circlearrowleft\rangle &= \frac{1}{2}\sum_{\pm,w\in\mathbb{Z}} e^{-i2\pi w n_g}\langle 0|\exp\Big(\pm i\sum_\sigma u_\sigma(\hat{b}_\sigma^\dagger + \hat{b}_\sigma)\Big)\exp\Big(-2\pi w\sum_\rho v_\rho(\hat{b}_\rho^\dagger - \hat{b}_\rho)\Big)|0\rangle \\ &= e^{-\frac{1}{2}\sum_\nu(u_\nu)^2}\sum_w(-1)^w e^{-2(\pi w)^2\sum_\mu(v_\mu)^2 - i2\pi w n_g}.\end{aligned} \tag{S32}$$

Note that since the state norm isn't unity, normalization is necessary, by a factor $\langle 0_\circlearrowleft|0_\circlearrowleft\rangle = \sum_w \exp(-2(\pi w)^2 v^\mu v_\mu)$. Clearly, $v^2 \equiv \sum_\mu (v^\mu)^2$ weights the corrections from non-zero winding numbers. It vanishes when $E_J \to \infty$, providing a pure harmonic oscillator behavior in this limit. At finite E_J, it sets the number of windings taken into account to reach required numerical accuracy. In the same fashion, $u^2 \equiv \sum_\mu (u^\nu)^2$ renormalizes the bare Josephson energy, $E_J' = E_J\exp(-u^2/2)$ (note that the previously defined term $E_J^\star$ appears only in the linearized Hamiltonian used to derive the ansatz, and acts only as a variational parameter).

PERTURBATION THEORY AT SMALL COUPLING STRENGTH.

One of the charge-boson model's striking features is the different responses of the junction's phase fluctuations $\langle\cos\hat{\varphi}\rangle$ to coupling strength α, depending of the E_J/E_c regime, as shown by Fig. 3 of the main text. Broadly speaking, the environment damps the phase fluctuations of the dissipative multi-level charge qubits, but enhances those of the dissipative two-level system. As already emphasized, this behavior cast doubt on the two-level description of dissipative multi-level qubits. Arguably, while this feature is correctly described by both NRG and our compact ansatz, a simple perturbative analysis in α should already be able to discriminate between these two regimes, and pin-point the break down of the two-level approximation.

We employ time-independent perturbation theory at second order, with $(\hat{n}-1/2)\sum_k ig_k(\hat{a}_k^\dagger-\hat{a}_k)$ as the perturbation. We denote $|\psi_n\rangle$ the eigenstates of the bare qubit, E_n their energies. Then,

$$\langle\cos\hat{\varphi}\rangle \simeq \langle 0|\cos\hat{\varphi}|0\rangle + \sum_{\substack{k\\ n\neq 0\\ m\neq 0}} g_k^2 \frac{\langle\psi_n|\hat{n}|\psi_0\rangle\langle\psi_0|\hat{n}|\psi_m\rangle}{(E_0-E_n-\omega_k)(E_0-E_m-\omega_k)}\langle\psi_n|\cos\hat{\varphi}|\psi_m\rangle - \sum_{\substack{k\\ n\neq 0}} g_k^2 \frac{|\langle\psi_n|\hat{n}|\psi_0\rangle|^2}{(E_0-E_n-\omega_k)^2}\langle\psi_0|\cos\hat{\varphi}|\psi_0\rangle$$
$$+ 2\sum_{\substack{k,j,\\ i\neq 0}} g_k^2 \frac{\langle\psi_0|\cos\hat{\varphi}|\psi_i\rangle\langle\psi_j|\hat{n}|\psi_0\rangle\langle\psi_j|\hat{n}|\psi_i\rangle}{(E_0-E_i)(E_0-E_J-\omega_k)}. \tag{S33}$$

This expression takes into account the multi-level nature of the qubit. It can be reduced by restricting the sum on bare qubit levels to the most significant element:

$$\langle\cos\hat{\varphi}\rangle \simeq \langle 0|\cos\hat{\varphi}|0\rangle + \sum_k g_k^2 \frac{|\langle\psi_1|\hat{n}|\psi_0\rangle|^2}{(E_0-E_1-\omega_k)^2}\langle\psi_1|\cos\hat{\varphi}|\psi_1\rangle - \sum_k g_k^2 \frac{|\langle\psi_1|\hat{n}|\psi_0\rangle|^2}{(E_0-E_1-\omega_k)^2}\langle\psi_0|\cos\hat{\varphi}|\psi_0\rangle$$
$$+ 2\sum_k g_k^2 \frac{\langle\psi_1|\hat{n}|\psi_0\rangle\langle\psi_1|\hat{n}|\psi_2\rangle}{(E_0-E_2)(E_0-E_1-\omega_k)}\langle\psi_2|\cos\hat{\varphi}|\psi_0\rangle. \tag{S34}$$

The first and second term correspond to the two-level approximation (in the limit $\alpha\to 0$). However, the third term adds the contribution from the third qubit level into the mix. Indeed, this term is mostly responsible for the qualitative change between dissipative two- and multi-level qubits when E_J/E_c is increased, as shown by the Fig. S2.

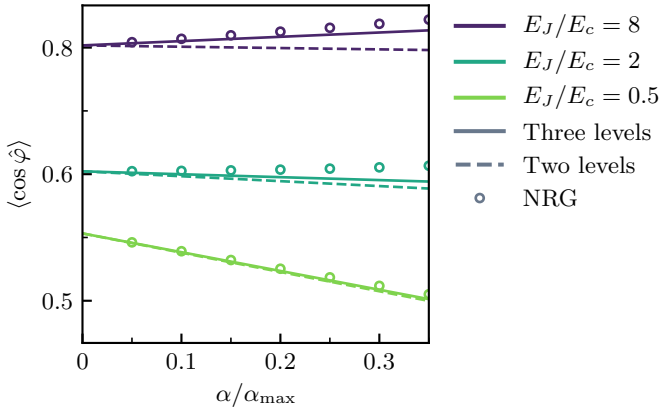

FIG. S2. NRG estimates for phase fluctuations as a function of α (circles), compared to the two and three levels approximation within first order perturbation theory (dashed and solid lines respectively). The two levels approximation (dashed lines) clearly fails for tmulti-level qubits, where the slope changes of sign, while bringing the third level improves the agreement to NRG.