Dynamical equations of a qubit coupled to a cavity decaying into a bosonic bath – via SPIN-BOSON

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(Dated: 6th March 2024)

I. GENERAL ALGORITHM

We start with the following wavefunction

$$|\Psi\rangle = \sum_{i}^{N_q-1} \sum_{n}^{\text{ncs}} p_{i,n} |i\rangle |z_{i,n}\rangle \tag{1}$$

Here $p_{i,n}$ and $z_{i,n}^p$ are all complex and time dependent variational parameters.

The Lagrangian is given by:

$$\mathscr{L} = \langle\Psi|\frac{i}{2}\overleftrightarrow{\partial_t} - \hat{H}|\Psi\rangle \tag{2}$$

Explicitely:

$$\begin{aligned}\langle\Psi|\overrightarrow{\partial}_t|\Psi\rangle &= \left(\sum_m p_m^\star \langle z_m|\right)\overrightarrow{\partial}_t\left(\sum_n p_n |z_n\rangle\right) \\ &= \sum_{mn} p_m^\star \langle z_m|z_n\rangle \left(\dot{p}_n - \frac{1}{2}p_n\left(\sum_p \dot{z}_n^p z_n^{p\star} + z_n^p \dot{z}_n^{p\star} - 2z_m^{p\star}\dot{z}_n^p\right)\right)\end{aligned} \tag{3}$$

where we have used:

$$\langle z_n|\overrightarrow{\partial}_t|z_m\rangle = -\frac{1}{2}\left(\sum_p \dot{z}_m^p z_m^{p\star} + z_m^p \dot{z}_m^{p\star} - 2z_n^{p\star}\dot{z}_m^p\right)\langle z_n|z_m\rangle$$

Since we have that:

$$\langle\Psi|\overleftarrow{\partial_t}|\Psi\rangle = \langle\Psi|\overrightarrow{\partial_t}|\Psi\rangle^\star, \tag{4}$$

we obtain:

$$\mathscr{L} = \frac{i}{2}\sum_{mn}\langle z_m|z_n\rangle\left[p_m^\star\dot{p}_n - p_n\dot{p}_m^\star - \frac{1}{2}p_m^\star p_n\left(\sum_p \dot{z}_n^p z_n^{p\star} + z_n^p\dot{z}_n^{p\star} - 2z_m^{p\star}\dot{z}_n^p - \dot{z}_m^{p\star}z_m^p - z_m^{p\star}\dot{z}_m^p + 2z_n^p\dot{z}_m^{p\star}\right)\right] - \langle\Psi|\hat{H}|\Psi\rangle \tag{5}$$

The Euler-Lagrange equations are:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{p}_j^\star} - \frac{\partial\mathscr{L}}{\partial p_j^\star} = 0 \quad \text{and} \qquad \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{z}_j^{p\star}} - \frac{\partial\mathscr{L}}{\partial z_j^{p\star}} = 0. \tag{6}$$

After $p_j^\star$ variation we get

$$\boxed{\sum_m\left(\dot{p}_m - \frac{1}{2}p_m\kappa_{mj}\right)M_{jm} = -i\frac{\partial E}{\partial p_j^\star} \equiv P_j} \tag{7}$$

After $z_j^{p\star}$ variation we get

$$\sum_m p_m p_j^\star \dot{z}_m^p M_{jm} - \frac{1}{4}\sum_m \left(2\dot{p}_m - p_m \kappa_{mj}\right) p_j^\star (z_j^p - 2z_m^p) M_{jm} + \frac{1}{4}\sum_m \left(2\dot{p}_m^\star - p_m^\star \kappa_{mj}^\star\right) p_j z_j^p M_{mj} = -i\frac{\partial E}{\partial z_j^{p\star}} \tag{8}$$

where we have defined:

$$M_{jm} = \langle z_j | z_m \rangle \tag{9}$$

$$\kappa_{mj} = \sum_p \dot{z}_m^p z_m^{p\star} + \dot{z}_m^{p\star} z_m^p - 2z_j^{p\star} \dot{z}_m^p \tag{10}$$

Using (7) to simplify (8), we get:

$$\boxed{\sum_m p_m \dot{z}_m^p M_{jm} + \sum_m (\dot{p}_m - \frac{1}{2} p_m \kappa_{mj}) z_m^p M_{jm} = Z_j^p,} \tag{11}$$

where we have defined:

$$Z_j^p = -i\left[\frac{\partial E}{\partial z_j^{p\star}}\frac{1}{p_j^\star} + \frac{1}{2}\left(\frac{\partial E}{\partial p_j^\star} + \frac{\partial E}{\partial p_j}\frac{p_j}{p_j^\star}\right) z_j^p\right] \tag{12}$$