$\boxed{\text{All coherent states approach}}$

Hamiltonian : transmon + cavity + bath

$$H = \underbrace{4E_c\,\hat{n}^2 + E_J - E_J \cos\varphi}_{\text{transmon}} + \underbrace{\omega_c\, a^\dagger a}_{\text{cavity}} + \underbrace{g\,\hat{n}(a+a^\dagger)}_{\text{coupling}}$$

$$+ \underbrace{\sum_k \omega_k\, a_k^\dagger a_k}_{\text{bath}} + \underbrace{\sum_k \gamma_k (a_k^\dagger a + a^\dagger a_k)}_{\text{leaking}}$$

Linearisatom: we replace $E_J - E_J \cos\varphi$ by $\frac{E_J}{2}\varphi^2$

$\Rightarrow H = H_0 + H_1$

with $\left|\begin{array}{l} H_0 \text{ the linearized part} \\ H_1 = E_J - E_J\cos\varphi - \frac{E_J}{2}\varphi^2 \end{array}\right.$

We put H_0 in normal modes:

$$\boxed{H_0 = \sum_k \epsilon_k\, b_k^\dagger b_k}$$

with $\varphi = \sum_k \varphi_k (b_k^\dagger + b_k)$

$$\Rightarrow \boxed{H_1 = E_J - E_J \cos\left[\sum_k \varphi_k (b_k^\dagger + b_k)\right] - \frac{E_J}{2}\left[\sum_k \varphi_k (b_k^\dagger + b_k)\right]^2}$$

Coherent state: they are expressed on the normal mode basis $\Rightarrow$ exact solution of the linear problem

$$|f\rangle = e^{\sum_k (f_k b_k^\dagger - f_k^* b_k)}|0\rangle$$

and the general state is:

$$|\Psi\rangle = \sum_{n=1}^{N_{cs}} p_n\, |f^{(n)}\rangle$$

To compute the overlap, we need

$$\langle f^{(n)} | \cos\varphi | f^{(m)} \rangle = \frac{1}{2} \langle f^{(n)} | e^{i\varphi} | f^{(m)} \rangle + (\varphi \to -\varphi)$$

$$= \frac{1}{2} \langle f^{(n)} | e^{i \sum_k \varphi_k (b_k^+ + b_k)} | f^{(m)} \rangle + (\varphi \to -\varphi)$$

$$= \frac{1}{2} \langle f^{(n)} | e^{i \sum_k \varphi_k b_k^+} e^{i \sum_k \varphi_k b_k} | f^{(m)} \rangle e^{-\frac{1}{2} \sum_k \varphi_k^2} + (\varphi \to -\varphi)$$

$$= \frac{1}{2} \langle f^{(n)} | e^{i \sum_k \varphi_k f_k^{(n)*}} e^{i \sum_k \varphi_k f_k^{(m)}} | f^{(m)} \rangle e^{-\frac{1}{2} \sum_k \varphi_k^2} + (\varphi \to -\varphi)$$

$$= \frac{1}{2} \langle f^{(n)} | f^{(m)} \rangle e^{i \sum_k \varphi_k (f_k^{(n)*} + f_k^{(m)})} e^{-\frac{1}{2} \sum_k \varphi_k^2} + (\varphi \to -\varphi)$$

$$= \langle f^{(n)} | f^{(m)} \rangle \cos\left[\sum_k \varphi_k [f_k^{(n)*} + f_k^{(m)}] \right] e^{-\frac{1}{2} \varphi_k^2}$$

Similarly we need:

$$\langle f^{(n)} | \left(\sum_k \varphi_k (b_k^+ + b_k) \right)^2 | f^{(m)} \rangle$$

$$= \langle f^{(n)} | \sum_{kk'} \varphi_k \varphi_{k'} (b_k^+ + b_k)(b_{k'}^+ + b_{k'}) | f^{(m)} \rangle$$

$$= \langle f^{(n)} | \sum_{kk'} \varphi_k \varphi_{k'} (b_k^+ b_{k'}^+ + \delta_{kk'} + b_{k'}^+ b_k + b_k^+ b_{k'} + b_k b_{k'}) | f^{(m)} \rangle$$

$$= \langle f^{(n)} | \cdot \sum_{kk'} \varphi_k \varphi_{k'} (f_k^{(n)*} f_{k'}^{(n)*} + \delta_{kk'} + f_{k'}^{(n)*} f_k^{(m)} + \dots$$

$$\dots + f_k^{(n)*} f_{k'}^{(m)} + f_k^{(m)} f_{k'}^{(m)} | f^{(m)} \rangle$$

$$= \langle f^{(n)} | f^{(m)} \rangle \left(\sum_k \varphi_k \left[f_k^{(n)*} + f_k^{(m)} \right] \right)^2$$

$$+ \langle f^{(n)} | f^{(m)} \rangle \sum_k \varphi_k^2$$