Multimode Schrödinger Cats

* Wigner function of a mode in the bath

We use the spin boson Hamiltonian:

$$H = \frac{\Delta}{2}\sigma_x + \sum_k \omega_k a_k^\dagger a_k - \frac{\sigma_z}{2}\sum_k g_k (a_k^\dagger + a_k)$$

The multimode Schrödinger cat reads:

$$|\Psi\rangle = |\uparrow\rangle \sum_{n=1}^{N_{cs}} p_n |f_n\rangle + |\downarrow\rangle \sum_{n=1}^{N_{cs}} q_n |h_n\rangle$$

with $|f_n\rangle = \prod_{k=1}^{N} e^{f_{k,n} a_k^\dagger - f_{k,n}^* a_k} |0\rangle$

Note that we work in the displacement basis, not the qbit basis.

Assuming $\langle\Psi|\Psi\rangle = 1$, the Wigner function of mode a_k reads:

$$\boxed{W_k^{\sigma\sigma}(\alpha) = \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \langle\Psi| e^{\lambda a_k^\dagger - \lambda^* a_k} |\sigma\rangle\langle\sigma|\Psi\rangle}$$

where we projected diagonally on $|\sigma\rangle\langle\sigma|$.

Note that we could also define an off-diagonal Wigner with a $|\sigma\rangle\langle-\sigma|$ projection:

Let's compute $W_k^{\psi\psi}(\alpha)$ explicitely.

$$W_k^{\psi\psi}(\alpha) = \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \sum_{n,m} p_n^* p_m \langle f_n | e^{\lambda a_k^\dagger - \lambda a_k} | f_m \rangle$$

We need the overlap:

$$\langle f_n | e^{\lambda a_k^\dagger - \lambda^* a_k} | f_m \rangle \overset{\text{from Baker-Campbell}}{=} \prod_{q \neq k} \langle f_{qn} | f_{qm} \rangle \times \cdots$$

$$\cdots \times \langle f_{kn} | f_{km} + \lambda \rangle e^{\frac{\lambda f_{km}^* - \lambda^* f_{km}}{2}}$$

$$\cdots = \prod_{q \neq k} \langle f_{qn} | f_{qm} \rangle e^{-\frac{|f_{kn}|^2}{2} - \frac{|f_{km}+\lambda|^2}{2} + f_{kn}^*(f_{km}+\lambda) + \frac{\lambda f_{km}^* - \lambda^* f_{km}}{2}}$$

$$\cdots = \prod_{q} \langle f_{qn} | f_{qm} \rangle e^{-\frac{\cancel{f_{km}^* \lambda} + f_{km}\lambda^*}{2} + f_{kn}^*\lambda + \frac{\cancel{\lambda f_{km}^*} - \lambda^* f_{km}}{2} - \frac{|\lambda|^2}{2}}$$

$$\cdots = \prod_{q} \langle f_{qn} | f_{qm} \rangle e^{f_{kn}^*\lambda - f_{km}\lambda^* - \frac{|\lambda|^2}{2}}$$

We now perform the λ integration:

$$W_k^{\psi\psi}(\alpha) = \int \frac{d^2\lambda}{\pi^2} \sum_{n,m} p_n^* p_m \langle f_n | f_m \rangle e^{-\frac{|\lambda|^2}{2}} e^{\lambda(f_{kn}^* - \alpha^*) - \lambda^*(f_{km} - \alpha)}$$

$$\boxed{W_k^{\psi\psi}(\alpha) = \frac{2}{\pi} \sum_{n,m} p_n^* p_m e^{-2(f_{kn}^* - \alpha^*)(f_{km} - \alpha)} \langle f_n | f_m \rangle}$$

with $\langle f_n | f_m \rangle = e^{\sum_q \left[-\frac{|f_{qn}|^2}{2} - \frac{|f_{qm}|^2}{2} + f_{qn}^* f_{qm} \right]}$

the full multimode overlap.

* Wigner function of the cavity mode

The cavity mode is $a^\dagger = \sum_k \gamma_k^* a_k^\dagger$

$$\Rightarrow W_{cav}^{\phi\phi}(\alpha) = \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \langle\psi| e^{\lambda \sum_k \gamma_k^* a_k^\dagger - \lambda^* \sum_k \gamma_k a_k} |\phi\rangle\langle\phi|\psi\rangle$$

$$\ldots = \sum_{nm} p_n^* p_m \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \langle f_n | e^{\lambda \sum_k \gamma_k^* a_k^\dagger - \lambda^* \sum_k \gamma_k a_k} | f_m \rangle$$

$$\ldots = \sum_{nm} p_n^* p_m \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \prod_q \langle f_{qn} | f_{qm} + \lambda \gamma_q \rangle \times \ldots$$

$$\ldots \times e^{\frac{\lambda \gamma_q^* f_{qn}^* - \lambda^* \gamma_q f_{qm}}{2}}$$

$$\ldots = \sum_{nm} p_n^* p_m \int \frac{d^2\lambda}{\pi^2} e^{\lambda^*\alpha - \lambda\alpha^*} \langle f_n | f_m \rangle e^{\sum_k f_{kn}^* \lambda \gamma_k^* - f_{km} \lambda^* \gamma_k} \times \ldots$$

$$\ldots \times e^{-\frac{1}{2} \sum_k |\lambda|^2 |\gamma_k|^2}$$

But due to $[a, a^\dagger] = 1$, we have $\sum_k |\gamma_k|^2 = 1$

$$\Rightarrow \boxed{W_{cav}^{\phi\phi}(\alpha) = \frac{2}{\pi} \sum_{nm} p_n^* p_m \, e^{-2\left(\sum_k f_{kn}^* \gamma_k^* - \alpha^*\right)\left(\sum_k f_{km} \gamma_k - \alpha\right)} \langle f_n | f_m \rangle}$$

* Renyi entropy of qbit ⊕ bath mode

We compute first $S_{q\oplus k} = 1 - \mathrm{Tr}\left[\hat{\rho}^2_{q\oplus k}\right]$

where we trace out all modes $q \neq k$, keeping

the qbit ⊕ mode k as subsystem.

For this purpose, we compute the density

matrix $\hat{\rho}_{q\oplus k}$ in the joint qbit space $|\sigma\rangle$

and Fock space $|\ell\rangle_k = \dfrac{(a^\dagger_k)^\ell}{\sqrt{\ell!}}|0\rangle$

so that we get $\mathrm{Tr}\,\hat{\rho}^2_{q+k}$ from a regular matrix trace.

$$\left[\hat{\rho}_{q\oplus k}\right]^{\sigma\sigma'}_{\ell\ell'} = \langle\Psi|\Big[|\ell\rangle_k |\sigma\rangle\langle\sigma'|\,{}_k\langle\ell'|\Big]|\Psi\rangle$$

$$\ldots = \sum_n \Big[p_n^* \langle f_n|\langle\uparrow| + q_n^*\langle h_n|\langle\downarrow|\Big]\,|\ell\rangle_k|\sigma\rangle \times \ldots$$

$$\ldots \times \langle\sigma'|\,{}_k\langle\ell'| \sum_m \Big[p_m|f_m\rangle|\uparrow\rangle + q_m|h_m\rangle|\downarrow\rangle\Big]$$

$$\ldots = \sum_{nm} p_n^* p_m\,\delta_{\sigma\uparrow}\delta_{\sigma'\uparrow}\langle f_n|\ell\rangle_k\,{}_k\langle\ell'|f_m\rangle$$

$$+ p_n^* q_m\,\delta_{\sigma\uparrow}\delta_{\sigma'\downarrow}\langle f_n|\ell\rangle_k\,{}_k\langle\ell'|h_m\rangle$$

$$+ q_n^* p_m\,\delta_{\sigma\downarrow}\delta_{\sigma'\uparrow}\langle h_n|\ell\rangle_k\,{}_k\langle\ell'|f_m\rangle$$

$$+ q_n^* q_m\,\delta_{\sigma\downarrow}\delta_{\sigma'\downarrow}\langle h_n|\ell\rangle_k\,{}_k\langle\ell'|h_m\rangle$$

We need:

$$\langle f_n | l\rangle_k \, {}_k\langle l' | f_m\rangle = \prod_{q\neq k} \langle f_{nq} | f_{mq}\rangle \langle f_{nk} | \frac{(a_k^\dagger)^l}{\sqrt{l!}} |0\rangle \cdots$$

$$\cdots \times \langle 0 | \frac{(a_k)^{l'}}{\sqrt{l'!}} | f_{mk}\rangle$$

$$\cdots = \prod_{q\neq k} \langle f_{nq} | f_{mq}\rangle \frac{(f_{nk}^*)^l}{\sqrt{l!}} \frac{(f_{mk})^{l'}}{\sqrt{l'!}} \underbrace{\langle f_{nk} | 0\rangle \langle 0 | f_{mk}\rangle}_{e^{-\frac{1}{2}|f_{nk}|^2 - \frac{1}{2}|f_{mk}|^2}}$$

$$\cdots = \langle f_n | f_m\rangle \frac{(f_{nk}^*)^l}{\sqrt{l!}} \frac{(f_{mk})^{l'}}{\sqrt{l'!}} e^{-f_{nk}^* f_{mk}}$$

Checking the Trace is unity:

$$\mathrm{Tr}\, \hat{\rho}_{q\otimes k} = \sum_\sigma \sum_l \left[\hat{\rho}_{q\otimes k}\right]^{\sigma\sigma}_{ll} = \sum_{nm} p_n^* p_m \langle f_n | f_m\rangle \overbrace{\sum_{l=0}^{\infty} \frac{(f_{nk}^* f_{mk})^l}{l!}}^{e^{f_{nk}^* f_{mk}}} \times \cdots$$

$$\cdots \times e^{-f_{nk}^* f_{mk}} + q_n^* q_m \langle h_n | h_m\rangle \underbrace{\sum_{l=0}^{\infty} \frac{(h_{nk}^* h_{mk})^l}{l!} e^{-h_{nk}^* h_{mk}}}_{1}$$

$$\Rightarrow \mathrm{Tr}\, \hat{\rho}_{q\otimes k} = \sum_{nm} p_n^* p_m \langle f_n | f_m\rangle + q_n^* q_m \langle h_n | h_m\rangle$$

$$= \langle \Psi | \Psi\rangle = 1 \quad \text{by normalization.}$$

OK

Now we get the entropy:

$$S_{q\oplus k} = 1 - \sum_{\sigma\sigma'} \sum_{\ell\ell'} \left[\hat{\rho}_{q\oplus k}\right]^{\sigma\sigma'}_{\ell\ell'} \left[\hat{\rho}_{q\oplus k}\right]^{\sigma\sigma'}_{\ell'\ell}$$

$$S_{q\oplus k} = 1 - \sum_{\ell\ell'} \sum_{\sigma\sigma'} \left| \left[\hat{\rho}_{q\oplus k}\right]^{\sigma\sigma'}_{\ell\ell'} \right|^2$$

$$\boxed{\begin{aligned} S_{q\oplus k} = 1 &- \sum_{\ell\ell'} \left| \sum_{nm} p_n^* p_m \langle f_n | f_m \rangle e^{-f_{nk}^* f_{mk}} \frac{(f_{nk}^*)^\ell (f_{mk})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\ &- \sum_{\ell\ell'} \left| \sum_{nm} p_n^* q_m \langle f_n | h_m \rangle e^{-f_{nk}^* h_{mk}} \frac{(f_{nk}^*)^\ell (h_{mk})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\ &- \sum_{\ell\ell'} \left| \sum_{nm} q_n^* p_m \langle h_n | f_m \rangle e^{-h_{nk}^* f_{mk}} \frac{(h_{nk}^*)^\ell (f_{mk})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\ &- \sum_{\ell\ell'} \left| \sum_{nm} q_n^* q_m \langle h_n | h_m \rangle e^{-h_{nk}^* h_{mk}} \frac{(h_{nk}^*)^\ell (h_{mk})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \end{aligned}}$$

* Renyi entropy of qbit ⊕ cavity mode

We generalize the above formula using the cavity mode $a^\dagger_{cav} = \sum_k \gamma_k^* a_k^\dagger$

and noting
$$\begin{cases} \bar{f}_n \equiv \sum_k f_{nk} \gamma_k \\ \bar{h}_n \equiv \sum_k h_{nk} \gamma_k \end{cases}$$

$$
\begin{aligned}
S_{q\oplus cav} = 1 &- \sum_{\ell\ell'} \left| \sum_{nm} p_n^* p_m \langle f_n | f_m \rangle \, e^{-\overline{f_n}^* \overline{f_m}} \frac{(\overline{f_n}^*)^\ell (\overline{f_m})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\
&- \sum_{\ell\ell'} \left| \sum_{nm} p_n^* q_m \langle f_n | h_m \rangle \, e^{-\overline{f_n}^* \overline{h_m}} \frac{(\overline{f_n}^*)^\ell (\overline{h_m})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\
&- \sum_{\ell\ell'} \left| \sum_{nm} q_n^* p_m \langle h_n | f_m \rangle \, e^{-\overline{h_n}^* \overline{f_m}} \frac{(\overline{h_n}^*)^\ell (\overline{f_m})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2 \\
&- \sum_{\ell\ell'} \left| \sum_{nm} q_n^* q_m \langle h_n | h_m \rangle \, e^{-\overline{h_n}^* \overline{h_m}} \frac{(\overline{h_n}^*)^\ell (\overline{h_m})^{\ell'}}{\sqrt{\ell!\,\ell'!}} \right|^2
\end{aligned}
$$

with

$$
\begin{cases}
\langle f_n | f_m \rangle = e^{-\frac{1}{2}\sum_k \left[|f_{nk}|^2 + |f_{mk}|^2 - 2 f_{nk}^* f_{mk} \right]} \\
\overline{f_n} = \sum_k f_{nk} \gamma_k
\end{cases}
$$