Dynamical equations of a qubit coupled to a cavity decaying into a bosonic bath – via SPIN-BOSON

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The quantromon Lagrangian is given by:

$$L = \frac{1}{2}\left(\frac{C_q}{2}\right)\dot{\Phi}_1^2 + \frac{1}{2}\left(\frac{C_q}{2}\right)\dot{\Phi}_3^2 + \frac{1}{2}C_r(\dot{\Phi}_3 - \dot{\Phi}_1)^2 + E_J\cos\left(\frac{\Phi_1}{\phi_0}\right) + E_J\cos\left(\frac{\Phi_3}{\phi_0}\right) - \frac{1}{2L}(\Phi_1 - \Phi_3 - \Phi)^2 \tag{1}$$

Defining $\Phi_q = \frac{1}{2}(\Phi_1 + \Phi_3)$, and $\Phi_q = \frac{1}{2}(\Phi_1 - \Phi_3)$, we get:

$$L = \frac{1}{2}C_q\dot{\Phi}_q^2 + \frac{1}{2}(C_q + 4C_r)\dot{\Phi}_r^2 + 2E_J\cos\left(\frac{\Phi_q}{\phi_0}\right)\cos\left(\frac{\Phi_r}{\phi_0}\right) - \frac{2}{L}(\Phi_r - \Phi/2)^2 \tag{2}$$

The quantromon Hamiltonian is given by:

$$H = \frac{Q_q^2}{2C_q} + \frac{Q_r^2}{2(C_q + 4C_r)} - 2E_J\cos\left(\frac{\Phi_q}{\phi_0}\right)\cos\left(\frac{\Phi_r}{\phi_0}\right) + \frac{2}{L}(\Phi_r - \Phi/2)^2 \tag{3}$$

Defining $E_{C_q} \equiv e^2/(2C_q)$, $E_{C_r} \equiv e^2/(2(4C_r + C_q))$, $E_{L_r} \equiv 4\phi_0^2/L$, and setting $\Phi = 0$, we get:

$$H = 4E_{C_q}n_q^2 + 4E_{C_r}n_r^2 - 2E_J\cos(\varphi_q)\cos(\varphi_r) + \frac{E_{L_r}}{2}\varphi_r^2 \tag{4}$$

Linearising the cavity cosine:

$$H = 4E_{C_q}n_q^2 - 2E_J\cos(\varphi_q) + 4E_{C_r}n_r^2 + E_J\cos(\varphi_q)\varphi_r^2 + \frac{E_{L_r}}{2}\varphi_r^2 \tag{5}$$

Define the resonator ladder operators such that:

$$\begin{aligned} \varphi_r &= \left(\frac{2E_{C_r}}{E_{L_r}}\right)^{1/4}(a^\dagger + a) = \sqrt{\eta_r}(a^\dagger + a) \qquad \text{where} \qquad \eta_r = \sqrt{\frac{2E_{C_r}}{E_{L_r}}} \\ n_r &= i\left(\frac{E_{L_r}}{32E_{C_r}}\right)^{1/4}(a^\dagger - a) \end{aligned} \tag{6}$$

Defining $\omega_r = \sqrt{8E_{C_r}E_{L_r}}$, we get:

$$H = 4E_{C_q}n_q^2 - 2E_J\cos(\varphi_q) + \omega_r a^\dagger a + E_J\eta_r\cos(\varphi_q)(a^2 + a^{\dagger 2} + 2a^\dagger a + 1), \tag{7}$$

Defining the transmon effective Josephson energy $E_{J_q} = E_J(2 - \eta_r)$:

$$H = 4E_{C_q}n_q^2 - E_{J_q}\cos(\varphi_q) + \omega_r a^\dagger a + E_J\eta_r\cos(\varphi_q)(a^2 + a^{\dagger 2} + 2a^\dagger a), \tag{8}$$

Defining the transmon ladder operators such that:

$$\begin{aligned} \varphi_q &= \left(\frac{2E_{C_q}}{E_{J_q}}\right)^{1/4}(q^\dagger + q) = \sqrt{\eta_q}(q^\dagger + q) \qquad \text{where} \qquad \eta_q = \sqrt{\frac{2E_{C_q}}{E_{J_q}}} \\ n_q &= i\left(\frac{E_{J_q}}{32E_{C_q}}\right)^{1/4}(q^\dagger - q) \end{aligned} \tag{9}$$

Defining , and $g_{q-r} \equiv E_J\eta_r$:

$$H = 4E_{C_q}n_q^2 - E_{J_q}\cos(\varphi_q) + \omega_r a^\dagger a + g_{q-r}\cos(\varphi_q)(a^2 + a^{\dagger 2} + 2a^\dagger a), \tag{10}$$

Rewriting the transmon operators in its eigenbasis, adding drive and decay, the full Hamiltonian becomes:

$$\begin{aligned}\hat{H} = \sum_{i=0}^{N_q-1} \omega_{\rm qb}^i |i\rangle\langle i| + \omega_r a^\dagger a + \sum_{i,j} g_{ij} |i\rangle\langle j| (a^2 + {a^\dagger}^2 + 2a^\dagger a) \\ + \sum_{k=1}^{N} \omega_k \hat{d}_k^\dagger \hat{d}_k + a^\dagger \sum_{k=1} \gamma_k d_k + a \sum_{k=1} \gamma_k^\star d_k^\dagger \\ + A_{\rm d} \cos(\omega^{\rm drive} t)(a + a^\dagger)\end{aligned} \tag{11}$$

where $g_{ij} = g_{q-r} \langle i| \cos(\varphi_q) |j\rangle$.

I. DIAGONALISATION OF THE FREE BOSONIC MODES

By combining the field modes together:

$$a_0^\dagger \equiv a^\dagger \quad \text{if} \quad p = k = 0, \tag{12}$$

$$a_p^\dagger \equiv d_k^\dagger \quad \text{if} \quad p = k \neq 0, \tag{13}$$

we can obtain:

$$H = \sum_{i=0}^{N_q-1} \omega_{\rm qb}^i |i\rangle\langle i| + \left(a_0^2 + (a_0^\dagger)^2 + a_0^\dagger a_0\right) \sum_{i,j}^{N_q-1} g_{i,j}^{\rm qb-cav} |i\rangle\langle j| + \sum_{p,p'} h_{pp'} a_p^\dagger a_{p'} + A(t)(a_0 + a_0^\dagger), \tag{14}$$

with

$$\begin{aligned} h_{00} &= \omega_{\rm cav} \\ h_{kk} &= \omega_{\rm bath}^k \\ h_{0k} &= h_{k0}^\star = \gamma_k \\ h_{pp'} &= 0 \quad \text{else.} \end{aligned} \tag{15}$$

Diagonalising the matrix $h_{pp'}$ provides normal modes b_p of the problem:

$$\sum_{pp'} h_{pp'} a_p^\dagger a_{p'} = \sum_p \omega_p b_p^\dagger b_p, \tag{16}$$

In doing so, we have defined the ladder operators in in the new basis:

$$b_\sigma = \sum_\mu O_{\sigma\mu}^T a_\mu \tag{17}$$

Conversely,

$$a_0 = \sum_\mu O_{0\mu} b_\mu \quad \text{and} \quad a_k = \sum_\mu O_{k\mu} b_\mu. \tag{18}$$

The matrix O denotes the transfer matrix used to go from the original basis to the new basis. It verifies:

$$D = O^T h O. \tag{19}$$

Hence we obtain the following Hamiltonian:

$$H = \sum_{i=0}^{N_q-1} \omega_{\rm qb}^i |i\rangle\langle i| + \left(a_0^2 + (a_0^\dagger)^2 + a_0^\dagger a_0\right) \sum_{i,j}^{N_q-1} g_{i,j}^{\rm qb-cav} |i\rangle\langle j| + \sum_p \omega_p b_p^\dagger b_p + A(t) \sum_p O_{0p}(b_p + b_p^\dagger), \tag{20}$$

where we have defined the coupling $g_{i,i+1}^p$ between the transmon and the normal-mode p as

$$g_{i,j}^p = g_{i,j}^{\rm qb-cav} O_{0p}. \tag{21}$$

II. GENERAL ALGORITHM

We start with the following wavefunction

$$|\Psi\rangle = \sum_{i}^{N_q-1} \sum_{n}^{\text{ncs}} p_{i,n} |i\rangle |z_{i,n}\rangle \tag{22}$$

Here $p_{i,n}$ and $z_{i,n}^p$ are all complex and time dependent variational parameters.

The Lagrangian is given by:

$$\mathscr{L} = \langle\Psi|\frac{i}{2}\overleftrightarrow{\partial_t} - \hat{H}|\Psi\rangle \tag{23}$$

Explicitely:

$$\begin{aligned}\langle\Psi|\overrightarrow{\partial}_t|\Psi\rangle &= \left(\sum_m p_m^\star \langle z_m|\right)\overrightarrow{\partial}_t\left(\sum_n p_n |z_n\rangle\right)\\ &= \sum_{mn} p_m^\star \langle z_m|z_n\rangle\left(\dot{p}_n - \frac{1}{2}p_n\left(\sum_p \dot{z}_n^p z_n^{p\star} + z_n^p \dot{z}_n^{p\star} - 2z_m^{p\star}\dot{z}_n^p\right)\right)\end{aligned} \tag{24}$$

where we have used:

$$\langle z_n|\overrightarrow{\partial}_t|z_m\rangle = -\frac{1}{2}\left(\sum_p \dot{z}_m^p z_m^{p\star} + z_m^p \dot{z}_m^{p\star} - 2z_n^{p\star}\dot{z}_m^p\right)\langle z_n|z_m\rangle$$

Since we have that:

$$\langle\Psi|\overleftarrow{\partial_t}|\Psi\rangle = \langle\Psi|\overrightarrow{\partial_t}|\Psi\rangle^\star, \tag{25}$$

we obtain:

$$\mathscr{L} = \frac{i}{2}\sum_{mn}\langle z_m|z_n\rangle\left[p_m^\star\dot{p}_n - p_n\dot{p}_m^\star - \frac{1}{2}p_m^\star p_n\left(\sum_p \dot{z}_n^p z_n^{p\star} + z_n^p\dot{z}_n^{p\star} - 2z_m^{p\star}\dot{z}_n^p - \dot{z}_m^{p\star}z_m^p - z_m^{p\star}\dot{z}_m^p + 2z_n^p\dot{z}_m^{p\star}\right)\right] - \langle\Psi|\hat{H}|\Psi\rangle \tag{26}$$

The Euler-Lagrange equations are:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{p}_j^\star} - \frac{\partial\mathscr{L}}{\partial p_j^\star} = 0 \quad \text{and} \qquad \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{z}_j^{p\star}} - \frac{\partial\mathscr{L}}{\partial z_j^{p\star}} = 0. \tag{27}$$

After $p_j^\star$ variation we get

$$\boxed{\sum_m\left(\dot{p}_m - \frac{1}{2}p_m\boldsymbol{\kappa}_{mj}\right)M_{jm} = -i\frac{\partial E}{\partial p_j^\star} \equiv P_j} \tag{28}$$

After $z_j^{p\star}$ variation we get

$$\sum_m p_m p_j^\star \dot{z}_m^p M_{jm} - \frac{1}{4}\sum_m\left(2\dot{p}_m - p_m\boldsymbol{\kappa}_{mj}\right)p_j^\star(z_j^p - 2z_m^p)M_{jm} + \frac{1}{4}\sum_m\left(2\dot{p}_m^\star - p_m^\star\boldsymbol{\kappa}_{mj}^\star\right)p_j z_j^p M_{mj} = -i\frac{\partial E}{\partial z_j^{p\star}} \tag{29}$$

where we have defined:

$$M_{jm} = \langle z_j | z_m \rangle \tag{30}$$

$$\kappa_{mj} = \sum_p \dot{z}_m^p z_m^{p\star} + \dot{z}_m^{p\star} z_m^p - 2 z_j^{p\star} \dot{z}_m^p \tag{31}$$

Using (28) to simplify (29), we get:

$$\boxed{\sum_m p_m \dot{z}_m^p M_{jm} + \sum_m (\dot{p}_m - \frac{1}{2} p_m \kappa_{mj}) z_m^p M_{jm} = Z_j^p,} \tag{32}$$

where we have defined:

$$Z_j^p = -i \left[\frac{\partial E}{\partial z_j^{p\star}} \frac{1}{p_j^\star} + \frac{1}{2} \left(\frac{\partial E}{\partial p_j^\star} + \frac{\partial E}{\partial p_j} \frac{p_j}{p_j^\star} \right) z_j^p \right] \tag{33}$$

From here on we only derive the equations for $\dot{y}_n$, as those $\dot{z}_n^p$ can be guessed from the former.

From Eqs. (28) and (32), we get:

$$\begin{aligned}
\sum_j M_{nj}^{-1} P_j &= \dot{p}_n - \frac{1}{2} \sum_{mj} p_m \kappa_{mj} M_{nj}^{-1} M_{jm} \\
&= \dot{p}_n - \frac{1}{2} p_n \Big(\sum_q \dot{z}_n^q z_n^{q\star} + \dot{z}_n^{q\star} z_n^q \Big) + \sum_{mj} M_{nj}^{-1} M_{jm} p_m \Big(\sum_q z_j^{q\star} \dot{z}_m^q \Big)
\end{aligned} \tag{34}$$

$$\sum_j M_{nj}^{-1} Z_j^p = p_n \dot{z}_n^p + \dot{p}_n z_n^p - \frac{1}{2} p_n z_n^p \Big(\sum_q \dot{z}_n^q z_n^{q\star} + \dot{z}_n^{q\star} z_n^q \Big) + \sum_{mj} M_{nj}^{-1} M_{jm} p_m z_m^p \Big(\sum_q z_j^{q\star} \dot{z}_m^q \Big) \tag{35}$$

From here we can obtain:

$$\sum_j M_{nj}^{-1} Z_j^p - z_n^p \sum_j M_{nj}^{-1} P_j = p_n \dot{z}_n^p + \sum_{mj} M_{nj}^{-1} M_{jm} p_m \Big(\sum_q z_j^{q\star} \dot{z}_m^q \Big) (z_m^p - z_n^p). \tag{36}$$

Hence:

$$z_i^{p\star} \sum_j M_{nj}^{-1} \Big(Z_j^p - z_n^p P_j \Big) = p_n z_i^{p\star} \dot{z}_n^p + \sum_{mj} M_{nj}^{-1} M_{jm} p_m \Big(\sum_q z_j^{q\star} \dot{z}_m^q \Big) (z_i^{p\star} z_m^p - z_i^{p\star} z_n^p). \tag{37}$$

Defining:

$$a_{in} = p_n \Big(\sum_p z_i^{p\star} \dot{z}_n^p \Big), \tag{38}$$

$$b_{in} = \sum_p z_i^{p\star} z_n^p, \tag{39}$$

$$A_{in} = \sum_j M_{nj}^{-1} \Big(\sum_p z_i^{p\star} (Z_j^p - z_n^p P_j) \Big), \tag{40}$$

we obtain an equation from Eq. (36) which do not depend on the mode index:

$$\boxed{a_{in} + \sum_{mj} M_{nj}^{-1} M_{jm} a_{jm} (b_{im} - b_{in}) = A_{in}.} \tag{41}$$

In order to solve (41), we define:

$$d_{in} \equiv \sum_l M_{il}^{-1} M_{ln} a_{ln}, \tag{42}$$

and use it to reexpress (41):

$$d_{in} + \sum_m \left(\sum_l M_{il}^{-1} M_{ln} (b_{lm} - b_{ln}) \right) d_{nm} = \sum_l M_{il}^{-1} M_{ln} A_{ln} \tag{43}$$

Hence we get:

$$\boxed{\sum_{mj}(\delta_{mn}\delta_{ij}+\alpha_{inm}\delta_{jn})d_{jm}=\sum_{l}M_{il}^{-1}M_{ln}A_{ln}} \tag{44}$$

where:

$$\alpha_{inm}=\sum_{l}M_{il}^{-1}M_{ln}(b_{lm}-b_{ln}) \tag{45}$$

Once we have solved for d_{in}, we get $\dot{z}_n^p$ and $\dot{p}_n$ from Eqs. (34) and (36):

$$\dot{p}_n=\sum_j M_{nj}^{-1}P_j+\frac{1}{2}p_n\Big(\sum_q \dot{z}_n^q z_n^{q\star}+\dot{z}_n^{q\star}z_n^q\Big)-\sum_m d_{nm} \tag{46}$$

$$\dot{z}_n^p=\frac{1}{p_n}\left(\sum_j M_{nj}^{-1}(Z_j^p-z_n^pP_j)-\sum_m d_{nm}(z_m^p-z_n^p)\right) \tag{47}$$

III. RELEVANT TERM EVALUATIONS

Let us now evaluate the terms on the RHS of the two dynamical equations.

First, the energy is given by:

$$\begin{aligned}
E=&\Big(\sum_{l,m}p_{l,m}^{\star}\langle l|\langle z_{l,m}|\Big)H\Big(\sum_{i,n}p_{i,n}|i\rangle|z_{i,n}\rangle\Big)\\
E=&\sum_{i,n,m}p_{i,m}^{\star}p_{i,n}\langle z_{i,m}|z_{i,n}\rangle\Big[\omega_i^{\mathrm{qb}}+\sum_{p=0}\omega_p z_{i,m}^{p\star}z_{i,n}^p+A(t)\sum_p O_{0,p}\Big(z_{i,m}^{p\star}+z_{i,n}^p\Big)\Big]\\
&+\sum_{i,l,n,m}g_{li}p_{l,m}^{\star}p_{i,n}\langle z_{l,m}|z_{i,n}\rangle\left(z_{l,m}^{0\star}+z_{i,n}^{0}\right)^2
\end{aligned} \tag{48}$$

$$\frac{\partial E}{\partial p_{s,j}^{\star}}=\sum_n p_{s,n}\langle z_{s,j}|z_{s,n}\rangle\Big[\omega_s^{\mathrm{qb}}+\sum_p\omega_p z_{s,j}^{p\star}z_{s,n}^p+A(t)\sum_p O_{0,p}\Big(z_{s,j}^{p\star}+z_{s,n}^p\Big)\Big] \tag{49}$$

$$+\sum_{i,n}g_{si}p_{i,n}\langle z_{s,j}|z_{i,n}\rangle\left(z_{s,j}^{0\star}+z_{i,n}^{0}\right)^2 \tag{50}$$

$$\begin{aligned}
\frac{\partial E}{\partial z_{s,j}^{q\star}}=&\sum_n p_{s,j}^{\star}p_{s,n}\langle z_{s,j}|z_{s,n}\rangle\Big[\omega_q z_{s,n}^q+A(t)O_{0,q}+(z_{s,n}^q-\tfrac{1}{2}z_{s,j}^q)\Big(\omega_s^{\mathrm{qb}}+\sum_{k=0}\omega_p z_{s,j}^{p\star}z_{s,n}^p+A(t)\sum_p O_{0,p}\big(z_{s,j}^{p\star}+z_{s,n}^p\big)\Big)\Big]\\
&-\frac{1}{2}\sum_n p_{s,n}^{\star}p_{s,j}\langle z_{s,n}|z_{s,j}\rangle z_{s,j}^q\Big[\omega_s^{\mathrm{qb}}+\sum_{p=0}\omega_p z_{s,n}^{p\star}z_{s,j}^p+A(t)\sum_p O_{0,p}\big(z_{s,n}^{p\star}+z_{s,j}^p\big)\Big]\\
&+\sum_{in}\Big[p_{s,j}^{\star}p_{i,n}\langle z_{s,j}|z_{i,n}\rangle\Big(2g_{s,i}O_{0,q}(z_{s,j}^{0\star}+z_{i,n}^0)+(z_{i,n}^q-\tfrac{1}{2}z_{s,j}^q)g_{s,i}\big(z_{s,j}^{0\star}+z_{i,n}^0\big)^2\Big) \qquad (51)\\
&\quad-\frac{1}{2}p_{i,n}^{\star}p_{s,j}\langle z_{i,n}|z_{s,j}\rangle z_{s,j}^q g_{i,s}\big(z_{i,n}^{0\star}+z_{s,j}^0\big)^2\Big] \qquad (52)
\end{aligned}$$

IV. EVALUATING THE ERROR BETWEEN THE POLARON ANSATZ AND THE EXACT SOLUTION

To check the accuracy of our wave-function, we monitor the norm of the following vector:

$$|\Phi\rangle = \left(i\frac{\overrightarrow{\partial_t}}{2} - i\frac{\overleftarrow{\partial_t}}{2} - H\right)|\Psi\rangle \tag{53}$$

$$\boxed{\langle\Phi|\Phi\rangle = -\frac{1}{2}\Re\big(\langle\Psi|\ \overrightarrow{\partial_t}\ \overrightarrow{\partial_t}\ |\Psi\rangle\big) + \frac{1}{2}\langle\Psi|\ \overleftarrow{\partial_t}\ \overrightarrow{\partial_t}\ |\Psi\rangle - 2\ \Im\big(\langle\Psi|\ \overleftarrow{\partial_t}\ H\ |\Psi\rangle\big) + \langle\Psi|H^2|\Psi\rangle} \tag{54}$$

Noting that:

$$\langle\alpha|\overleftarrow{\partial_t}|\beta\rangle = -\langle\alpha|\beta\rangle\,\frac{1}{2}\Big(\sum_p \dot{\alpha}_p\alpha_p^\star + \dot{\alpha}_p^\star\alpha_p - 2\beta_p\dot{\alpha}_p^\star\Big), \tag{55}$$

$$\langle\alpha|\overleftarrow{\partial_t}\ a_q^\dagger|\beta\rangle = \alpha_q^\star\,\langle\alpha|\overleftarrow{\partial_t}|\beta\rangle + \langle\alpha|\beta\rangle\,\dot{\alpha}_q^\star, \tag{56}$$

$$\langle\alpha|\overrightarrow{\partial_t}\ a_q^\dagger|\beta\rangle = \alpha_q^\star\,\langle\alpha|\overrightarrow{\partial_t}|\beta\rangle \tag{57}$$

$$\langle\alpha|a_q\overleftarrow{\partial_t}|\beta\rangle = \beta_q\,\langle\alpha|\overleftarrow{\partial_t}|\beta\rangle \tag{58}$$

we obtain

$$\begin{aligned}
\hat{B} &= (a^\dagger)^4 + a^4 + 8a^\dagger a + 6(a^\dagger)^2a^2 + 4(a^\dagger)^3a + 4a^\dagger a^3 + 4a^\dagger a^3 + 4(a^\dagger)^2 + 4a^2 + 2\\
\hat{D} &=\\
\hat{C} &= 2(a^\dagger)^3 + 2a^3 + 6a^\dagger a^2 + 6(a^\dagger)^2a + 4a^\dagger + 4a
\end{aligned}$$

$$\begin{aligned}
\langle\Psi|H^2|\Psi\rangle = \sum_{inm} p_{i,m}^\star p_{i,n}\,\langle z_{i,m}|z_{i,n}\rangle\Bigg[&\Big(\omega_{\rm qb}^i + \sum_p \omega_p z_{im}^{p\star}z_{in}^p + A(t)\sum_p O_{0p}(z_{im}^{p\star}+z_{in}^p)\Big)^2\\
&+\sum_p \omega_p^2 z_{im}^{p\star}z_{in}^p + \sum_p \omega_p A(t)O_{0p}(z_{im}^{p\star}+z_{in}^p) + A^2(t)\sum_p O_{0p}^2\Bigg]\\
+\sum_{i,j} p_i^\star p_j\,\langle z_i|z_j\rangle\Bigg[\sum_s g_{is}g_{sj}\,\langle z_i|\hat{B}|z_j\rangle &+ g_{ij}(\omega_{qb}^i+\omega_{qb}^j)(z_i^{\star 2}+z_j^2+z_i^\star z_j) + g_{ij}\,\langle z_i|\hat{D}|z_j\rangle + g_{ij}A_d\cos(\omega_d t)\,\langle z_i|\hat{C}|z_j\rangle\Bigg]
\end{aligned}$$

$$\langle\Psi|\ \overleftarrow{\partial_t}\ H\ |\Psi\rangle = \tag{59}$$

$$\begin{aligned}
\langle\Psi|\ \overleftarrow{\partial_t}\ \overrightarrow{\partial_t}\ |\Psi\rangle = \sum_i\sum_{m,n}\langle z_{i,m}|z_{i,n}\rangle\Bigg[&\dot{p}_{i,m}^*\dot{p}_{i,n} - \frac{1}{2}\dot{p}_{i,m}^*p_{i,n}\sum_p(\dot{z}_{i,n}^p z_{i,n}^{p*} + \dot{z}_{i,n}^{p*}z_{i,n}^p - 2\dot{z}_{i,n}^p z_{i,m}^{p*}) - \frac{1}{2}p_{i,m}^*\dot{p}_{i,n}\sum_p(\dot{z}_{i,m}^p z_{i,m}^{p*} + \dot{z}_{i,m}^{p*}z_{i,m}^p - 2\dot{z}_{i,m}^{p*}z_{i,n}^p)\\
&+p_{i,m}^\star p_{i,n}\Bigg[\sum_p \dot{z}_{i,m}^{p\star}\dot{z}_{i,n}^p + \frac{1}{4}\sum_{p,q}\Big(\dot{z}_{i,m}^p z_{i,m}^{p\star} + \dot{z}_{i,m}^{p*}z_{i,m}^p - 2\dot{z}_{i,m}^{p*}z_{i,n}^p\Big)\Big(\dot{z}_{i,n}^q z_{i,n}^{q\star} + \dot{z}_{i,n}^{q\star}z_{i,n}^q - 2\dot{z}_{i,n}^q z_{i,m}^{q\star}\Big)\Bigg]\Bigg]
\end{aligned}$$

$$\begin{aligned}
\langle\Psi|\ \overrightarrow{\partial_t}\ \overrightarrow{\partial_t}\ |\Psi\rangle &= \sum_i\sum_{m,n}p_{i,m}^\star\,\langle z_{i,m}|\overrightarrow{\partial_t}\Big[\dot{p}_{i,n}\,|z_{i,n}\rangle + p_{i,n}\overrightarrow{\partial_t}\,|z_{i,n}\rangle\Big] = \sum_i\sum_{m,n}p_{i,m}^\star\,\langle z_{i,m}|\Big[\ddot{p}_{i,n}\,|z_{i,n}\rangle + 2\dot{p}_{i,n}\overrightarrow{\partial_t}\,|z_{i,n}\rangle + p_{i,n}\overrightarrow{\partial_t}^2\,|z_{i,n}\rangle\Big]\\
&= \sum_i\sum_{m,n}p_{i,m}^\star\Big[\ddot{p}_{i,n}\,\langle z_{i,m}|z_{i,n}\rangle + 2\dot{p}_{i,n}\,\langle z_{i,m}|\overrightarrow{\partial_t}|z_{i,n}\rangle + p_{i,n}\,\langle z_{i,m}|\overrightarrow{\partial_t}^2\,|z_{i,n}\rangle\Big]\\
&= \sum_i\sum_{m,n}p_{i,m}^\star\,\langle z_{i,m}|z_{i,n}\rangle\Bigg[\ddot{p}_{i,n} - \dot{p}_{i,n}\Big(\sum_p \dot{z}_{i,n}^{p*}z_{i,n}^p + \dot{z}_{i,n}^p z_{i,n}^{p*} - 2\dot{z}_{i,n}^p z_{i,m}^{p*}\Big)\\
&\qquad + p_{i,n}\Bigg(\Big(\sum_p z_{i,m}^{p\star}\ddot{z}_{i,n}^p - \frac{1}{2}(\ddot{z}_{i,n}^{p\star}z_{i,n}^p + \ddot{z}_{i,n}^p z_{i,n}^{p\star} + 2\dot{z}_{i,n}^{p\star}\dot{z}_{i,n}^p)\Big) + \frac{1}{4}\Big(\sum_p \dot{z}_{i,n}^p z_{i,n}^{p\star} + \dot{z}_{i,n}^{p\star}z_{i,n}^p - 2\dot{z}_{i,n}^p z_{i,m}^{p\star}\Big)^2\Bigg)\Bigg]
\end{aligned}$$