Perfect squares are everywhere - Part 1

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In this article, we will discuss properties of perfect squares through a number of interesting examples.

Example (Example 1)

Cancelling the exponents yields

$$\frac{37^3+13^3}{37^3+24^3}=\frac{37+13}{37+24}=\frac{50}{61}.$$

which is correct.

Find the necessary and sufficient conditions for the positive integer triple (A, B, C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}.$$

Solution. For A, B, and C positive real numbers,

$$\begin{aligned}&\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}\Leftrightarrow\frac{A^3+B^3}{A+B}=\frac{A^3+C^3}{A+C}\Leftrightarrow A^2-AB+B^2=A^2-AC+C^2\\&\Leftrightarrow(B-C)(A-B-C)=0\Leftrightarrow\boxed{B=C}\text{ or }\boxed{A=B+C.}\end{aligned}$$

□

Example (Example 2)

For which natural number n is $2^8+2^{11}+2^n$ a perfect square?

Solution. Let m be an integer such that $m^2=2^8+2^{11}+2^n$. Note that:

$$2^8+2^{11}=2^8(1+2^3)=2^8\cdot 3^2=48^2.$$

Thus,

$$2^n=m^2-48^2=(m-48)(m+48).$$

Therefore there exists positive integers k, ℓ, such that

$$\left.\begin{aligned}m-48=2^k\quad&(1)\\m+48=2^\ell\quad&(2)\\k+\ell=n\quad&\end{aligned}\right\}(2)-(1)\Rightarrow 96=2^5\cdot 3=2^k(2^{\ell-k}-1)\Rightarrow k=5,\ell=7\Rightarrow\boxed{n=12.}$$

□

Example (Example 3)

Prove that for n positive integer the following number is a perfect square:

$$m = \underbrace{99\ldots9}_{n}\underbrace{00\ldots0}_{n}25.$$

Solution. Note that $\underbrace{99\ldots9}_{n} = 10^n - 1$, thus:

$$m = \underbrace{99\ldots9}_{n}10^{n+2} + 25 = 10^{2n+2} - 10^{n+2} + 25 = \boxed{(10^{n+1} - 5)^2.}$$

□

Example (Example 4)

Find all prime p such that $2p^4 - p^2 + 16$ is a perfect square.

Solution. Note that a perfect square has a remainder 0 or 1 when divided by 3. Thus

$$2p^4 \equiv 2p^2 \pmod 3 \Rightarrow 2p^4 - p^2 + 16 \equiv p^2 + 1 \pmod 3.$$

Since $2p^4 - p^2 + 16$ is a perfect square, so $p^2 + 1 \equiv 0$ or $1 \pmod 3 \Rightarrow \boxed{p = 3.}$ In this case $2p^4 - p^2 + 16 = 169 = 13^2$. □

Example (Example 5)

Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

Solution. Let $n \geq 0$ be an integer. Let $S(n,k) = (n+1)^2 + (n+2)^2 + \cdot + (n+k)^2$, where k is a positive integer, then:

$$S(n,k) = kn^2 + 2n(1 + 2 + \cdot + k) + (1^2 + 2^2 + \cdots + k^2) = kn^2 + nk(k+1) + \frac{k(k+1)(2k+1)}{6}$$

With $k = 1984$,

$$S(n, 1984) = 992(2n^2 + (2 \cdot 1985)n + 1985 \cdot 1323)$$

Since the second term of $S(n, 1984)$ is an odd number and $992 = 2^5 \cdot 31$, $S(n, 1984)$ is divisible by 2^5, but not by 2^6. Hence, it is not a perfect square. □