Perfect squares are everywhere - Part 2

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This article is the second part of the series on expedition to find *Perfect Squares*.

Example (Example 6)

a, b, c are positive integers such that ab and bc are perfect squares. Prove that ca is also a perfect square.

Solution. Let $ab = m^2, bc = n^2$, then $(ab)(bc) = (mn)^2$, so $(ac)b^2 = (mn)^2$. It is easy to see that if p is a prime factor of b, then it is also a prime factor of mn, thus by canceling these prime factors of b, then $\frac{(mn)^2}{b^2}$ is an integer and a perfect square. Hence ac must be a perfect square. □

Example (Example 7)

Prove that if x and y are sum of two perfect squares, then xy is also a sum of two perfect squares. In other words, if a, b, c, d integers such that $x = a^2 + b^2$, $y = c^2 + d^2$ then $xy = (a^2 + b^2)(c^2 + d^2)$ can be written as a sum of two perfect squares.

Prove that if x and y are sum of a perfect square and twice of another perfect square, then xy is also a sum of a perfect square and twice of another perfect square. In other words, if a, b, c, d integers such that $x = a^2 + 2b^2$, $y = c^2 + 2d^2$ then $xy(a^2 + 2b^2)(c^2 + 2d^2)$ can be written as a sum of a perfect square and twice of another perfect square.

Solution. See below,

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ac - bd)^2$$
$$(a^2 + 2b^2)(c^2 + 2d^2) = (ac + 2bd)^2 + 2(ac - bd)^2$$

□

Example (Example 8)

x, y, z are positive integers and their greatest common divisor is 1, such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Prove that $x + y$ is a perfect square.

Solution. It is easy to transform the given equation as follow,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \Rightarrow z(x+y) = xy \Rightarrow z^2 = (x-z)(x-y).$$

Let d be a common dividor of $x - z$ and $y - z$, then d^2 is a common divisor of z^2, thus $d \mid z$.

But $x = (x-z) + z,\ y = (y-z) + z$, therefore $d \mid x, d \mid y$, which means that d is a common divisor of x, y, z. Therefore $\gcd(x-z, y-z) = 1$.

Thus, there exist k, l positive integers such that $x - z = k^2$, $y - z = l^2$, or $(kl)^2 = z^2$, so $kl = z$, which means that:

$$x + y = (z + k^2) + (z + l^2) = k^2 + l^2 + 2kl = \boxed{(k+l)^2.}$$

□

Example (Example 9)

n is called *interesting* number if it can be written as $3x^2 + 32y^2$, where x, y are integers. Prove that if n is an *interesting* number, then $97n$ is *interesting* number too.

Solution. Note that

$$96n = 96 \cdot 3x^2 + 96 \cdot 32y^2 = 3(32)y^2 + 32(3x^2).$$

Thus, $96n$ is an *interesting* number. Now,

$$\begin{aligned} 97n = n + 96n &= \left[3x^2 + 32y^2\right] + \left[3(32)y^2 + 32(3x^2)\right] = 3[x^2 + (32y)^2] + 32[y^2 + (3x)^2] \\ &= 3[x^2 + 64xy + (32y)^2] + 32(y^2 - 6yx + (3x)^2] = 3(x + 32y)^2 + 32(y - 3x)^2. \end{aligned}$$

Thus $97n$ is an *interesting* number too. □

Fact. $n = ax^2 + by^2$, then $(ab+1)n = a(by - x)^2 + b(ax + y)^2$.

Example (Example 10)

Determine all perfect squares in the sequence $\{a_1, a_2, \ldots\}$, where

$$a_3 = 91,\ a_{n+1} = 10a_n + (-1)^n,\ \forall n \geq 0.$$

Solution. Note that $a_2 = 9,\ a_1 = 1,\ a_0 = 0$ are perfect square. By induction we can prove that, for all $n \geq 2$,

$$\begin{cases} a_2n \equiv 5 \pmod 8 \\ a_{2n+1} \equiv 3 \pmod 8 \end{cases}$$

Since every odd perfect square is congruent to 1 mod 8, thus none of them can be a perfect square. The answer is $\boxed{a_2 = 9,\ a_1 = 1,\ a_0 = 0}$. □