A picture is worth a thousand words - Part 4

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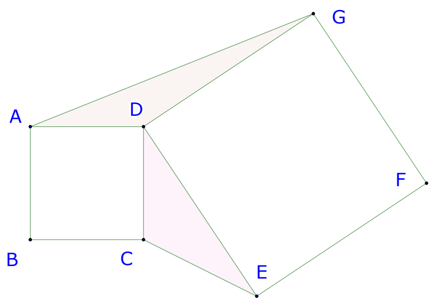
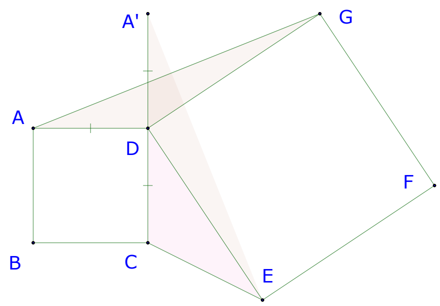
This article is the fourth part of the series on investigating a number of ways to *prove area equality without writing lengthy proof.*

Example (Example 16)

$ABCD$ and $DEFG$ are two squares as shown below, prove that

$$[ADG] = [CDE].$$

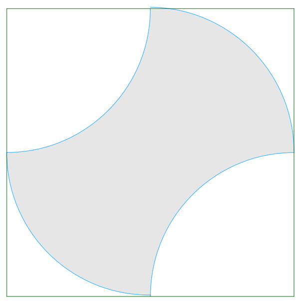


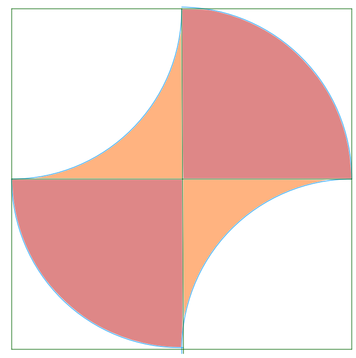


Proof. Rotate the triangle ADG clockwise around D. Because $\angle CDA = \angle EDG = 90°$, $DG = DE$, we receive the triangle $A'DE$. A', D, C are collinear (lie on the same line). Thus $[ADG] = [A'DE] = [DCE]$. □

Example (Example 17)

A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.

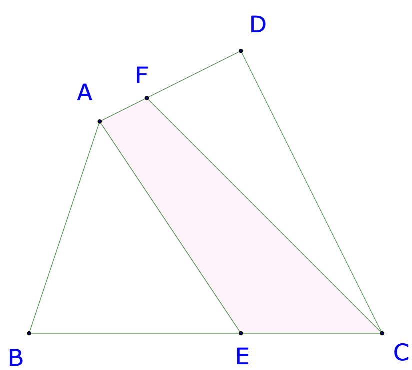


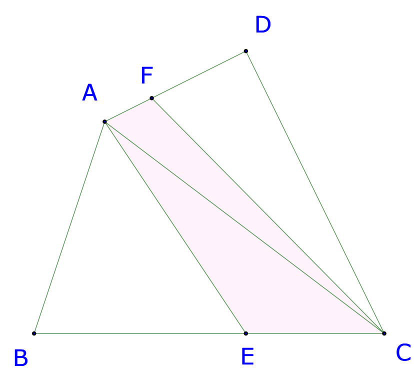


Proof. Similar to the Example 8 in the second part of this article series, but splitting into four regions, it is easy to see that they all make up two quarter square, thus the ratio is $\frac{1}{2}$. □

Example (Example 18)

$ABCD$ is a convex quadrilateral. E and F are points on BC and DA such that $BE = 2EC$, $DF = 2FA$. Find the ratio of the area of the shaded region to the area of the quadrilateral.





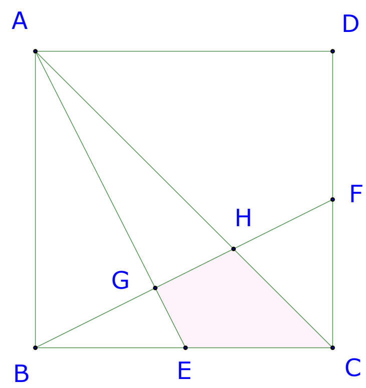
Proof. Connect AC. Since $BE = 2EC$ and $DF = 2AF$,

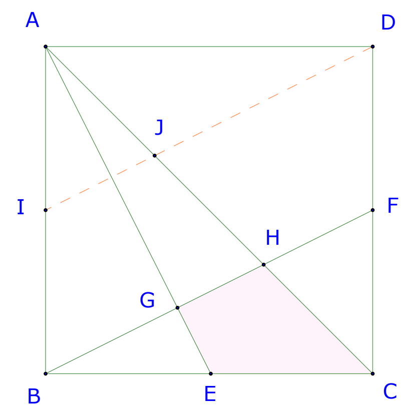
$$\frac{[ACE]}{[ABC]} = \frac{[CFD]}{[CAD]} = \frac{[ACE]+[CFD]}{[ABC]+[CDA]} = \frac{[ACE]+[CFD]}{[ABCD]}$$
$$\Rightarrow \frac{[AECF]}{ABCD} = 1 - \frac{[ACE]+[CFD]}{[ABCD]} = \frac{1}{3}.$$

□

Example (Example 19)

$ABCD$ is a square. E and F are midpoints of BC and CD. BF intersect AE and AC at G and H. Find the ratio of the area of the shaded region to the area of the square.





Proof. WLOG, let the side length of the square be 1. $BE = \frac{1}{2}$, $AE = \sqrt{AB^2 + BE^2} = \frac{\sqrt{5}}{2}$.

$$\triangle BGE \sim \triangle ABE \Rightarrow \frac{[BGE]}{[ABE]} = \left(\frac{BE}{AE}\right)^2 = \left(\frac{\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)^2 = \frac{1}{5} \Rightarrow [BGE] = \frac{1}{20}.$$

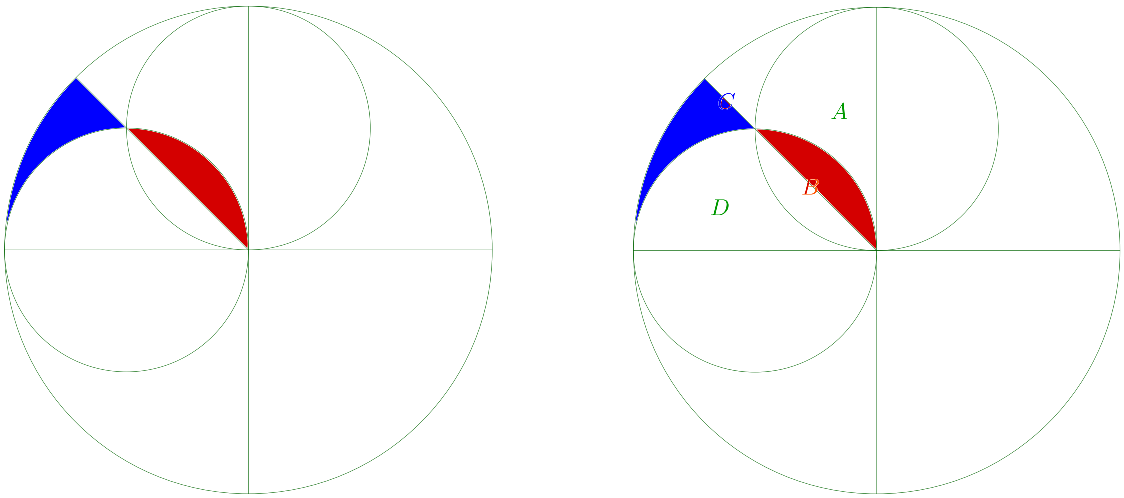
$$[CFH] = \frac{HF}{DJ}[CJD] = \frac{1}{4}[CJD] = \frac{1}{4}\frac{CJ}{AC}[ACD] = \frac{1}{6}[ACD] \Rightarrow [CFH] = \frac{1}{12}$$

$$\Rightarrow [GHCE] = [BCF] - [BGE] - [CFH] = \frac{1}{4} - \frac{1}{20} - \frac{1}{12} = \frac{7}{60}$$

Thus, the ratio is $\frac{7}{60}$. $\square$

Example (Example 20)

(*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Solution. Let A, B, C, and D denote the regions in the figure above. Let $[X]$ denotes the area of a region X.

$$[A] + [C] = \frac{1}{4}\pi(2r)^2 - ([B] + [D]) = \frac{1}{4}\pi(2r)^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi r^2 = [A] + [B]$$

Thus $[B] = [C]$, or $\boxed{\frac{1}{2}[B] = \frac{1}{2}[C].}$ $\square$