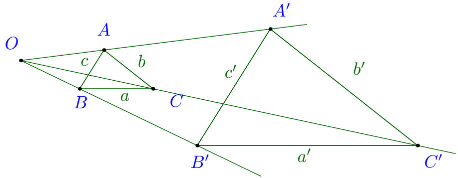
A **homothety** (or homothecy) is a transformation of space which dilates distances *with respect to a fixed point.*

A homothety can be an *enlargement* (resulting figure is larger), *identity* transformation (resulting figure is congruent), or a *contraction* (resulting figure is smaller).

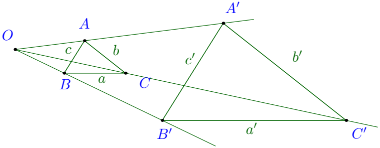
A homothety with center O and factor k sends point A to a point A', and

$$OA' = k \cdot OA.$$

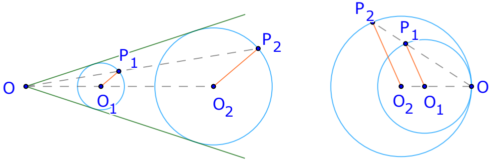


This is denoted by $\mathcal{H}_{(O,k)}$.

Let $\mathcal{H}_{(O,k)}$ be a homothety. For point A, $\mathcal{H}_{(O,k)}(A) = A' \Rightarrow O, A, A'$ colliner. Thus, the lines connecting each point of a polygon to its corresponding point of a homothetic polygon are all concurrent.

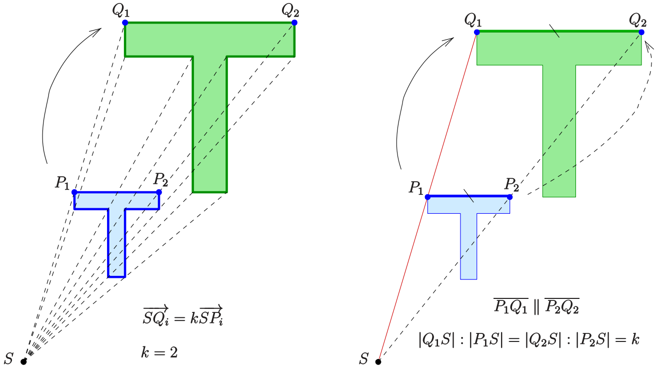


For line XY, $\mathcal{H}_{(P,k)}(AB) = A'B' \Rightarrow AB \parallel A'B'$.
For $\triangle ABC$, $\mathcal{H}_{(A,k)}(\triangle ABC) = \triangle AB'C' \Rightarrow \triangle ABC \sim \triangle AB'C'$.
The resulting image of a circle from a homothety is also a circle.



Let $\mathcal{H}_{(S,k)}$ be a homothety.

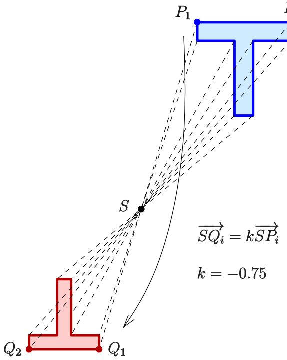
If $k > 0$, then the image and the original will be on the same side of the center, they are are scaled and translated similar to one another.



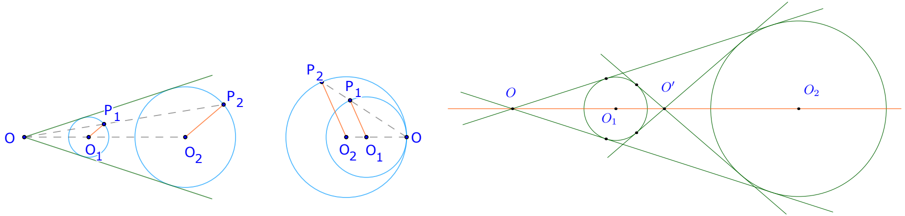
If $|k| > 1$, then the homothety is a magnification (enlargement); If $|k| < 1$, then it is a reduction (shrinking).

Let $\mathcal{H}_{(S,k)}$ be a homothety.

If $k < 0$, the image and the original will be on different sides of the center, i.e. the center will be between them. A homothety with factor $k = -1$ is a 180° rotation about the center.



Circles are geometrically similar to one another and *rotation invariant*. These two homothetic centers lie on the line joining the centers of the two given circles.

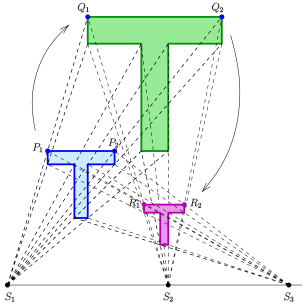


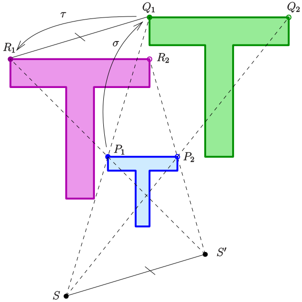
The common external tangents pass through the external homothetic center, while the common internal tangents pass through the internal homothetic center.

If the circles have the same radius (but different centers), they have no external homothetic center. If the circles have the same center, they have only one homothetic center and that is the common center of the circles.

Let $\mathcal{H}_{3\ (S_3,k_3)}$ be the compositions of $\mathcal{H}_{1\ (S_1,k_1)}$ and $\mathcal{H}_{2\ (S_2,k_2)}$ (below on the left):

$$\mathcal{H}_{3\ (S_3,k_3)} = \mathcal{H}_{2\ (S_2,k_2)} \circ \mathcal{H}_{1\ (S_1,k_1)} \Rightarrow S_3 \in S_1S_2,\ \ k_3 = k_1 \cdot k_2.$$





The composition of a homothety and a translation is a homothety (above on the right).

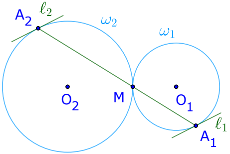
Example

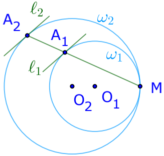
Circles ω_1 and ω_2 are tangent at M. A line through M intersects ω_1 and ω_2 at A_1 and A_2. Show that the tangent lines to ω_1 at A_1 and to ω_2 at A_2 are parallel.

Example

Circles ω_1 and ω_2 are tangent at M. A line through M intersects ω_1 and ω_2 at A_1 and A_2. Show that the tangent lines to ω_1 at A_1 and to ω_2 at A_2 are parallel.

Consider the homothety $\mathcal{H}(M, \pm\frac{r_2}{r_1})$, where r_1, r_2 are the radii of circles ω_1, ω_2 respectively, and *the minus sign is chosen in the case of exterior tangency* of the two circles (on the left), while *the plus sign is chosen in the case of interior tangency* of the two circles (on the right).



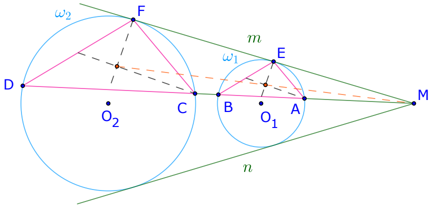


This transformation carries the circle ω_1 of radius r_1 into a circle of radius r_2, tangent to ω_1 at the point M; that is, it carries ω_1 into ω_2.
The point A_1 on the circle ω_1 is carried by this transformation into the point A_2 on the circle ω_2, and the tangent line ℓ_1 to ω_1 at A_1 is carried into the tangent line ℓ_2 to ω_2 at A_2.
Since **the line ℓ_2 is obtained from ℓ_1 by a homothety, the two lines are parallel**.

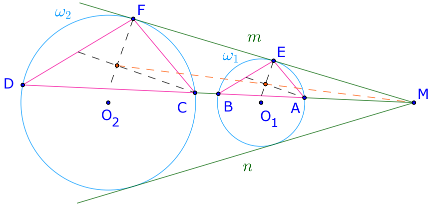
Example

Let ω_1 and ω_2 be two disjoint circles, neither inside the other. Let m be a common tangent to ω_1 and ω_2 and assume that ω_1 and ω_2 are both on the same side of m. Let n be another common tangent, with ω_1 and ω_2 both on the same side of n. Let M be the point of intersection of m and n. Let ℓ be a line through M meeting ω_1 in points A and B and meeting ω_2 in points C and D. Finally, let E be the point of tangency of m and ω_1 and let F be the point of tangency of m and ω_2. Prove that:

1. $\triangle ABE \sim \triangle CDF$.
2. $[ABE]/[CDF] = (r_1/r_2)^2$, where r_1 and r_2 are the radii of circles ω_1 and ω_2, respectively.
3. M is on the line through the centroids (intersection of the medians) of $\triangle ABE$ and $\triangle CDF$.



Consider the homothety with center M and coefficient $k = r_2/r_1$. This transformation carries the lines m and n onto themselves, and carries the circle ω_1 tangent to m and n and with radius r_1 onto a circle tangent to m and n and with radius r_2; that is, it carries ω_1 onto ω_2.

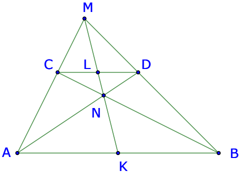


Also, the line ℓ is carried into itself, the segment AB is carried into CD, the point E into F and, consequently, **triangle ABE is carried onto triangle CDF**. From this it follows that these triangles are similar, and that the **similarity coefficient is** $k = r_2/r_1$; therefore $[ABE]/[CDF] = (r_1/r_2)^2$.

Finally, from the fact that triangle CDF is obtained from triangle ABE by a homothety with center M, it follows that **the line joining two corresponding points** of these triangles, for example, their centroids (the points of intersection of their medians), **passes through the point** M.

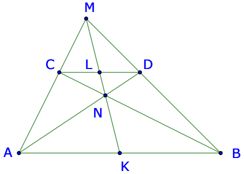
Example

Prove that the line joining the midpoints of the two parallel sides of a trapezoid passes through the point of intersection of the extensions of the other two sides, as well as through the point of intersection of the diagonals.



Example

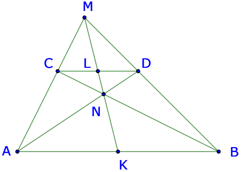
Prove that the line joining the midpoints of the two parallel sides of a trapezoid passes through the point of intersection of the extensions of the other two sides, as well as through the point of intersection of the diagonals.



The homothety with center at the point M of intersection of the extensions of sides AC and BD of trapezoid $ABDC$ and with coefficient CD/AB carries the segment AB onto the segment CD and carries the midpoint K of AB into the midpoint L of side CD Therefore, the line KL passes through the homothety center M.

Example

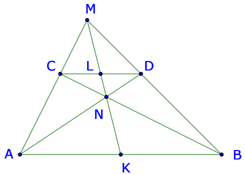
Prove that the line joining the midpoints of the two parallel sides of a trapezoid passes through the point of intersection of the extensions of the other two sides, as well as through the point of intersection of the diagonals.



The homothety with center at the point M of intersection of the extensions of sides AC and BD of trapezoid $ABDC$ and with coefficient CD/AB carries the segment AB onto the segment CD and carries the midpoint K of AB into the midpoint L of side CD Therefore, the line KL passes through the homothety center M.
The point K is also carried onto the point L by the homothety with center at the point N of intersection of the diagonals AD and BC of the trapezoid and with the (negative) coefficient $-CD/AB$. This transformation carries the segment AB onto CD. Therefore the line KL also passes through the point N.

Example

Prove that the line joining the midpoints of the two parallel sides of a trapezoid passes through the point of intersection of the extensions of the other two sides, as well as through the point of intersection of the diagonals.



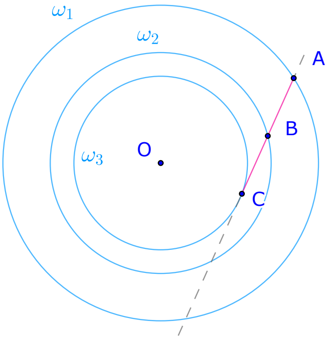
The homothety with center at the point M of intersection of the extensions of sides AC and BD of trapezoid $ABDC$ and with coefficient CD/AB carries the segment AB onto the segment CD and carries the midpoint K of AB into the midpoint L of side CD Therefore, the line KL passes through the homothety center M.

Is this true for non-parallelogram convex quadrilateral?

Example

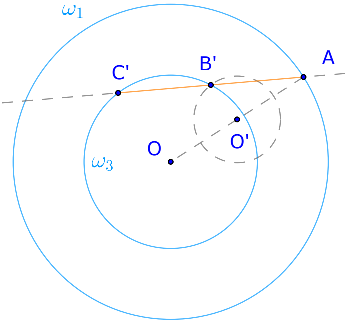
Three concentric circles ω_1, ω_2, and ω_3 are given.

Draw a line ℓ meeting ω_1, ω_2, and ω_3 in order in points A, B, and C such that $AB = BC$.



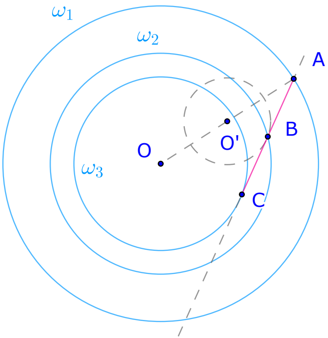
We first solve a problem with two concentric circles. Assume that $AB' = B'C$. Let O' be the midpoint of AO.

Then the homothety centred at A with similarity coefficient 2 brings B' to C' and O' to O. Thus it brings a circle centred at O' (with the radius equal to half of the radius of ω_3) to the circle ω_3.



Now, the homothety centred at A with similarity coefficient 2 brings O' to O, circle (O') to ω_3.

Thus it brings B - the intersection of (O') with ω_2 - to C - point on ω_3. Because of the similarity coefficient: $AB = BC$.

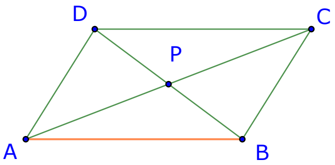


How about A, B, C, D on four concentric circles such that $AB = CD$?

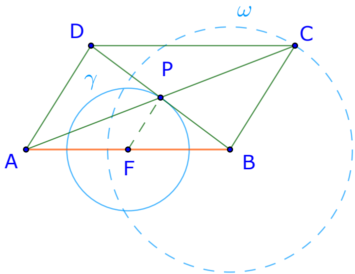
Example

Let a "hinged" parallelogram A BCD be given. More precisely, the lengths of the sides are fixed, and vertices A and B are fixed, but vertices C and D are movable.

As C and D move, what is the locus of point Q, the intersection of the diagonals?



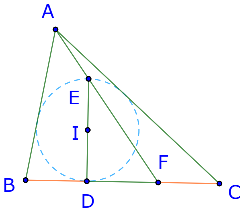
Since side BC does not change in length and B is fixed, C moves on a circle with center B as the "hinged" parallelogram moves. But Q is obtained from C by a central similarity transformation with center at the fixed point A and with coefficient $1/2$. Therefore Q moves on the circle γ obtained by this transformation (centred at F midpoint of AB with radius r half of BC) from the circle ω on which C moves.



Is this true for arbitrary convex quadrilateral?

Example

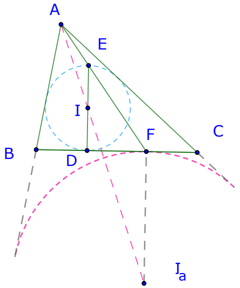
Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$



Example

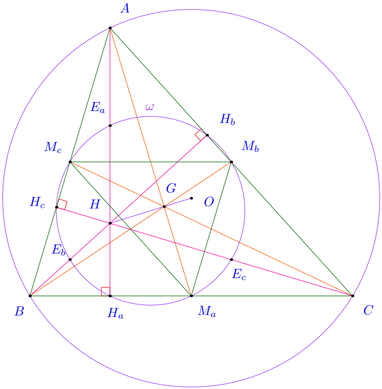
Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$

The homothety with center A that carries the incircle to an excircle (why?).
The diameter DE of the incircle is mapped to the diameter of the excircle perpendicular to BC.
It follows that E must get mapped to the point of tangency between the excircle and BC. Since the image of E must lie on the line AE, it must be F.
That is, the excircle is tangent to BC at F. Then, it follows easily that $BD = CF$ (how?).

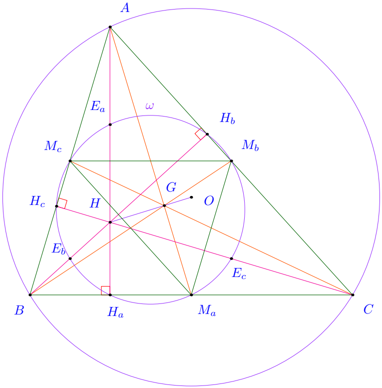


Example

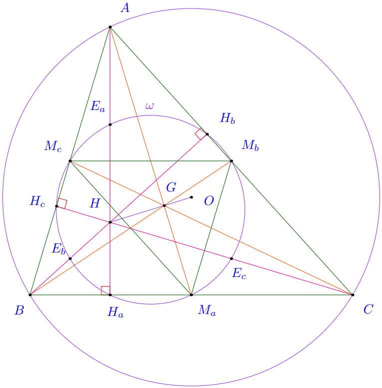
For a given triangle, prove that there exists a circle, which passes through: (i) the three feet of the altitudes, (ii) the three midpoints of the sides, and (iii) the three midpoints of the segments joining the vertices of the triangle to its orthocenter.



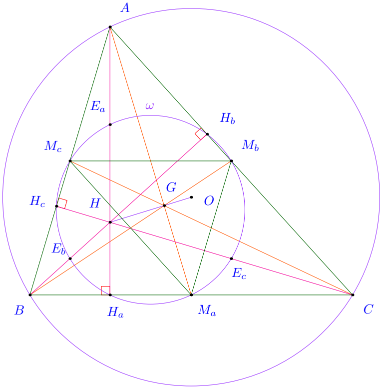
First, E_a, E_b, and E_c are midpoints of HA, HB, and HC.



Second, if X_a, X_b, and X_c be the intersection of AH_a, BH_b, and CH_c with the circle (ABC), then $HH_a = H_aX_a$, $HH_b = H_bX_b$, and $HH_c = H_cX_c$.

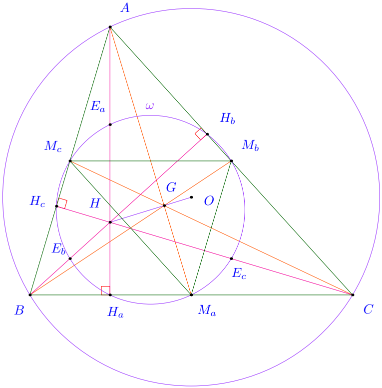


Third, if Y_a, Y_b, and Y_c be points such that $AHBY_c$, BHY_a, and $CHAY_b$ are parallelograms, then Y_a, Y_b, and Y_c are on the circle (ABC) (because for example $ACBY_a$ is cyclic)



Therefore the homothety centred at H with factor 2 maps E_a, E_b, and E_c to A, B, and C; H_a, H_b, and H_c to X_a, X_b, and X_c; and M_a, M_b, and M_c to Y_a, Y_b, and Y_c.

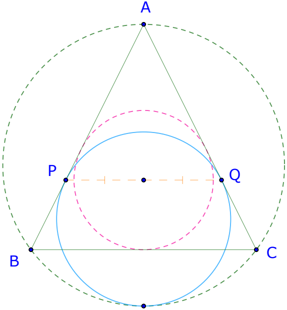
Hence, all nine points are on the circle ω.



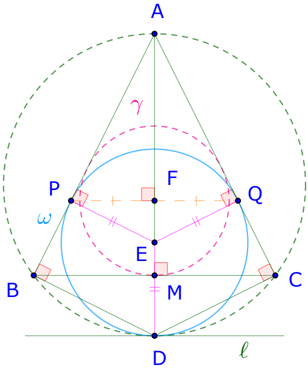
Example

In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q.

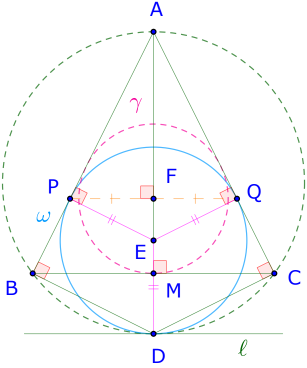
Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.



Let E be the center of the circle ω, which is tangent with AB, AC, and (ABC). Let D be the tangent point of the two circles, M be the midpoint of BC, and F be the midpoint of PQ. It is easy to see that A, F, E, M, D are collinear. Let $\mathcal{H}_{(A,k)}$ be a homothety centred at A and $\mathcal{H}_{(A,k)}(D) = M$. Since $AF/AM = AE/AD$, so $\mathcal{H}_{(A,k)}(E) = F$.

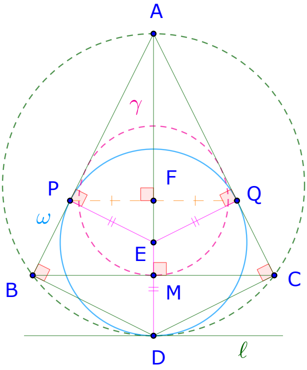


Let γ be the image of ω, $\gamma = \mathcal{H}_{(A,k)}(\omega)$. Since ω is tangent (ABC) at D, so both are tangent with line ℓ through D parallel with BC, thus γ tangent with the image of ℓ, which is line BC. Furthermore, the $\mathcal{H}_{(A,k)}(D) = M$, keeps B and C on rays AB and AC, and because ω is tangent to AB and AC, so γ is also tangent to AB and AC.



Therefore γ is tangent with all sides of $\triangle ABC$, so $\gamma = I_{\triangle ABC}$.
Therefore, the image of $\mathcal{H}_{(A,k)}(E)$ is I, the incentre of $\triangle ABC$, therefore $F \equiv I$.

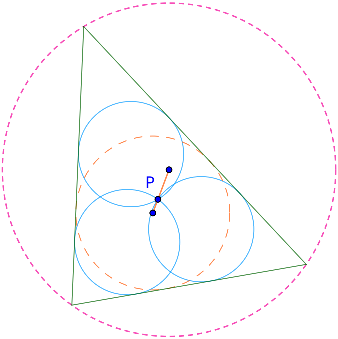
Hence, $\boxed{\text{the midpoint of } PQ \text{ is the center of the inscribed circle of the triangle } ABC.}$



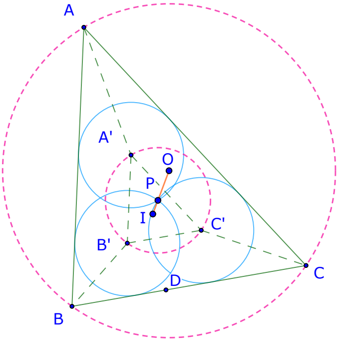
Example

Three congruent circles have a common point P and lie inside a given triangle. Each circle touches a pair of sides of the triangle.

Prove that the incenter and the circumcenter of the triangle and the point P are collinear.



Let A', B', C' be the centers of the circles. Since the radii are the same, so $A'B'$ is parallel to AB, $B'C'$ is parallel to BC, $C'A'$ is parallel to CA. Since AA', BB', CC' bisect $\angle A$, $\angle B$, $\angle C$, respectively, they concur at the incenter I of $\triangle ABC$.

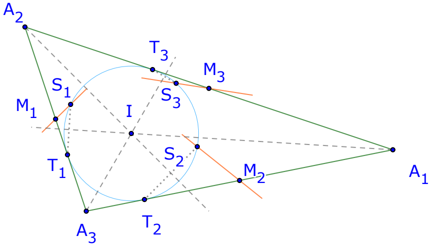


Note $P$is the circumcenter of $\triangle A'B'C'$ as it is equidistant from A', B', C'. Then the homothety with center I sending $\triangle A'B'C'$ to $\triangle ABC$ will send O to the circumcenter O of $\triangle ABC$. Therefore, I, P, O are collinear.

Example

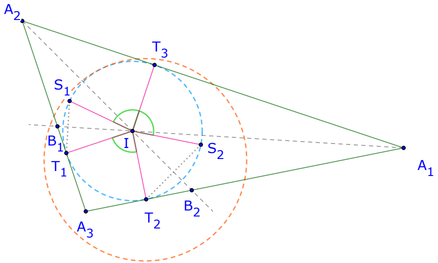
A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i.

Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.



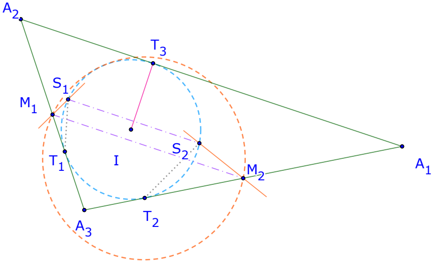
Let I be the incenter of $\triangle A_1A_2A_3$. Let B_1, B_2, B_3 be the points where the internal angle bisectors of $\angle A_1$, $\angle A_2$, and $\angle A_3$ meet sides a_1, a_2, a_3 respectively.

With respect to A_1B_1, the reflection of T_1 is S_1 and the reflection of T_2 is T_3. So $\angle T_3IS_1 = \angle T_2IT_1$. With respect to A_2B_2, the reflection of T_2 is S_2 and the reflection of T_1 is S_3. So $\angle T_3IS_2 = \angle T_1IT_2$. Then $\angle T_3IS_1 = \angle T_3IS_2$.



Since IT_3 is perpendicular to A_1A_2, we get S_2S_1 is parallel to A_1A_2.
Since A_1A_2 is parallel to M_2M_1, we get S_2S_1 is parallel to M_2M_1.
Similarly, S_3S_2 is parallel to M_3M_2 and S_1S_3 is parallel to M_1M_3.

Thus the sides of $\triangle S_1S_2S_3$ and $\triangle M_1M_2M_3$ are pairwise parallel.

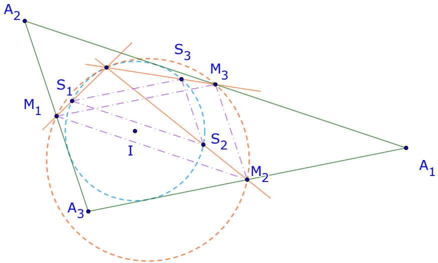


Now the circumcircle of $\triangle S_1S_2S_3$ is the incircle of $\triangle A_1A_2A_3$ and the circumcircle of $\triangle M_1M_2M_3$ is the nine point circle of $\triangle A_1A_2A_3$.

Since $\triangle A_1A_2A_3$ is not equilateral, these circles have different radii (why?).

Hence $\triangle S_1S_2S_3$ is not congruent to $\triangle M_1M_2M_3$, and because their sides are pairwise parallel, thus there is a homothety sending $\triangle S_1S_2S_3$ to $\triangle M_1M_2M_3$ (why?)

Then $M_1S_1, M_2S_2.$ and M_3S_3 concur at the center of the homothety.



Example

Two weather buoy float near the shore at points A and B. How can you draw a line through a point M on the shore parallel to AB?