Test Problems fore UMC K1 - Second Semester

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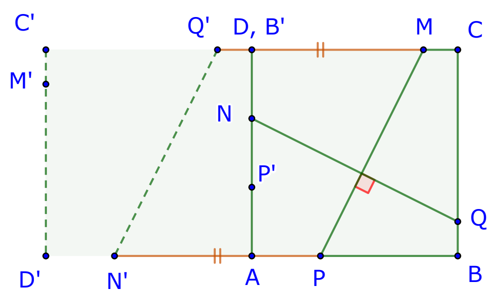
Problem 0.1 (Problem One)**.** $ABCD$ is a unit square. Points P, Q, M, and N are on sides AB, BC, CD, and DA, respectively, such that:

$$CM + AN + AP + CQ = 2.$$

Prove that $PM \perp QN$.

Proof. Consider the rotation $\mathcal{R}(A, 90°)$ around A anti-clockwise by $90°$, shown as below:

$$\mathcal{R}(A, 90°)(B) = D,\ \mathcal{R}(A, 90°)(C) = C',\ \mathcal{R}(A, 90°)(D) = D',\ \mathcal{R}(A, 90°)(Q) = Q',\ \mathcal{R}(A, 90°)(N) = N'.$$



By the property of the rotation:

$$AN = AN',\ CQ = C'Q' \implies PN' = PA + AN' = PA + AN = 2 - (CM + CQ) = CC' - CM - C'Q' = MQ'.$$

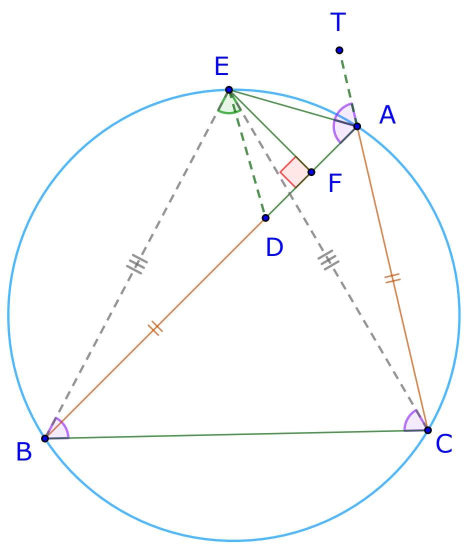
Thus, $PMQ'N'$ is a parallelogram, therefore $Q'N' \parallel MP$. By the property of the rotation, $QN \perp MP$. $\square$

Problem 0.2 (Problem Two)**.** In $\triangle ABC$, $AB > AC$. An external angle bisector of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at E. Let F be the foot of of the perpendicular from E to line AB. Prove that:

$$2AF = AB - AC.$$

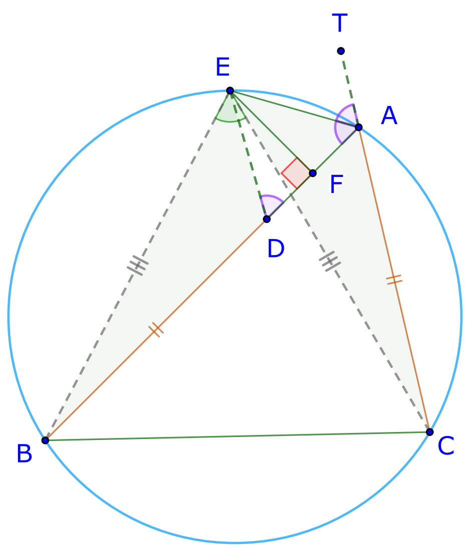
Proof. Consider the rotation $\mathcal{R}(E, \angle CEB)$ around E clockwise by $\angle CEB$.

$$\angle EBC = \angle EAT = \angle EAB = \angle ECB \implies EC = EB \implies \mathcal{R}(E, \angle CEB)(C) = B.$$



Let $\mathcal{R}(E, \angle CEB)(A) = D$. Since $\angle CAB = \angle CEB$ and $AB > AC$, thus $D \in AB$. Therefore:

$$\mathcal{R}(E, \angle CEB)(\triangle AEC) = \triangle DEB.$$



Furthermore

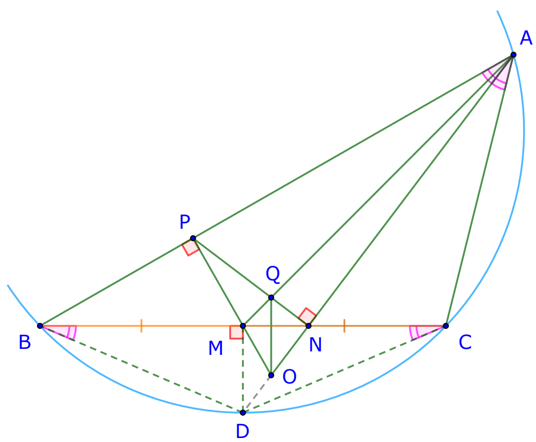
$$\angle DAE = \angle EAT = \angle EDA \implies \triangle AED \text{ is isosceles.}$$

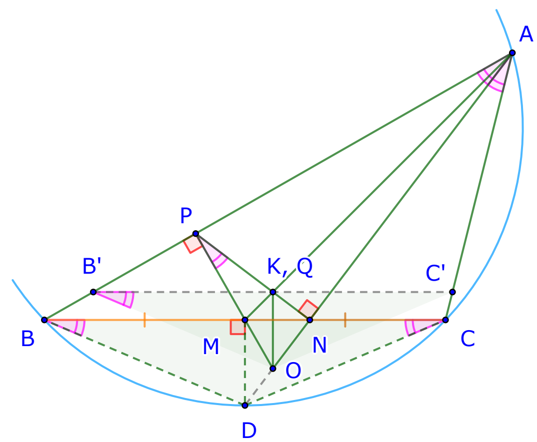
Now $EF \perp AD$, hence $2AF = AD = AB - BD = AB - AC$. $\square$

Problem 0.3 (Problem Three)**.** Let ABC be a triangle such that $AB > AC$. Let M and N be the intersections of the median and the angle bisector, respectively, from A to BC. Let Q and P be the points where the perpendicular at N to NA meets MA and BA, respectively. Let O be the point where the perpendicular at P to BA meets AN. Prove that $QO \perp BC$.

Proof. Let AN intersect the circumcircle of $\triangle ABC$ at D. Then

$$\angle DBC = \angle DAC = \frac{1}{2}\angle BAC = \angle DAB = \angle DCB \implies DB = DC \implies MD \perp BC.$$



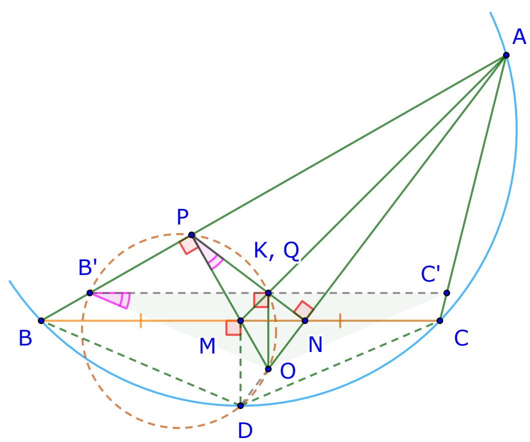


Consider the homothety with center A with factor $k = \frac{AO}{AD}$:

$$\mathcal{H}_{(\mathcal{A},\|)}(\triangle DBC) = \triangle OB'C' \implies OB' = OC',\ BC \parallel B'C'.$$

Let $B'C' \cap PN = K$, then:

$$\angle OB'K = \angle DBC = \angle DAB = 90^\circ - \angle AOP = \angle OPK \implies P, B', O, K \text{ are concyclic.}$$



Thus:

$$\angle B'KO = \angle B'PO = 90^\circ \implies B'K = C'K \implies K \in MA\ (BC \parallel B'C') \implies K \equiv Q.$$

Now $\angle B'KO = 90^\circ$ implies that $QO \equiv KO \perp B'C'$, hence $QO \perp BC$. $\square$