# Sequence

Ngo Van Minh

May 13, 2022

A sequence is an enumerated collection of objects (numbers, letters, functions, etc) in which repetitions are allowed and the order of their apperance matters

$$1,\ 2,\ 3,\ 4,\ 5, ...$$

The members of a sequence are called terms or elements. The sequence is called a finite sequence, if there are a *finite* amount of terms in a sequence. In other words, the sequence ends. Not all sequence ends, such sequence is called *infinite.* When using variables to represent a sequence, we often use the same letter with different subscripts to represent the terms. For example, we might represent a sequence with 5 terms as

$$a_1, a_2, a_3, a_4, a_5.$$

The expression $\{a_n\}$ denotes the sequence, where $a_n$ denotes a generic $n^{\text{th}}$ element of the sequence.

Some sequences have obvious patterns, such as

**The sequence of natural numbers**: $0,\ 1,\ 2,\ 3,\ 4, \ldots$

**The sequence of odd numbers**: $1,\ 3,\ 5,\ 7,\ 9, \ldots$

**The sequence of prime numbers**: $1,\ 3,\ 5,\ 7,\ 9, \ldots$

**The sequence of square numbers**: $1,\ 4,\ 9,\ 16,\ 25,\ 36, \ldots$ The $i^{\text{th}}$ is $a_i = i^2$.

**The sequence of cubic numbers**: $1,\ 8,\ 27,\ 64,\ 125, \ldots$ The $i^{\text{th}}$ is $a_i = i^3$.

**Fibonacci sequence**: $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$ The sequence starts with two terms 1 and 1, then each following term is equal to the sum of two previous terms.

**Arithmetic sequence**: $\{a_i\}$, where $a_i = a_1 + (i-1) \times d$. $d$ is the common difference between two consecutive terms. *For example,* $2,\ 5,\ 8,\ 11,\ 14, \ldots$ *is an arithmetic sequence with* $a_1 = 2$ *and* $d = 3$.

**Geometric sequence**: $\{a_i\}$, where $a_i = a_{i-1} \times r$. $r$ is the common ratio between two consecutive terms. It is easy to see that $a_i = a_1 \times r^{i-1}$.

**Sequence as a combination of twin numbers sequences**: a combination of two sequences $a_1,\ a_2,\ a_3, \ldots,$ and $b_1,\ b_2,\ b_3, \ldots$ results in $a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3, \ldots$.

**Mixed sequence**: a sequence without any particular rule or pattern.

**Sequence of words**: a sequence of words that has some logics based on their meanings.

---

**Problem. What is the missing number?**

$$\_,\ 1,\ 2,\ 6,\ 24,\ 120,\ 720.$$

*Solution.* By observation

$$1 = 1!,\ 2 = 1 \times 2 = 2!,\ 6 = 1 \times 2 \times 3 = 3!, 24 = 1 \times 2 \times 3 \times 4 = 4!,$$
$$120 = 1 \times 2 \times 3 \times 4 \times 5 = 5!, 720 = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 6!,$$

Thus, the missing term is 0!. The answer is $\boxed{1.}$ □

**Problem. What is the missing letter?**

$$\_,\ \text{O},\ \text{T},\ \text{T},\ \text{F},\ \text{F},\ \text{S},\ \text{S},\ \text{E},\ \text{N},\ \text{T}.$$

*Solution.* By observation, from the second term, each letter is the first letter of each word representing the first ten postitive integers

$$\underline{\text{O}}\text{ne},\ \underline{\text{T}}\text{wo},\ \underline{\text{T}}\text{hree},\ \underline{\text{F}}\text{our},\ \underline{\text{F}}\text{ive},\ \underline{\text{S}}\text{ix},\ \underline{\text{S}}\text{even},\ \underline{\text{E}}\text{ight},\ \underline{\text{N}}\text{ine},\ \underline{\text{T}}\text{en}.$$

Thus, the missing term is Zero. The answer is $\boxed{Z.}$ □

**Problem. Find the next term of the sequence**

$$13,\ 57,\ 91,\ 11,\ 31,\ 51,\ \_$$

*Solution.* By placing the terms one after another, we obtain

$$\underline{1}\ \underline{3}\ \underline{5}\ \underline{7}\ \underline{9}\ \underline{11}\ \underline{13}\ \underline{15}\ 1\ldots$$

By observation, it is a sequence of odd numbers. Thus, the missing term is 71. The answer is $\boxed{71.}$ □

**Problem. What is the missing letter?**

$$\text{B},\ \text{C},\ \text{E},\ \text{G},\ \text{K},\ \text{M},\ \_$$

*Solution.* The letters in the alphabet form a sequence with their positions in the sequence shown as below

| A | B | C | ... | Z |
|---|---|---|---|---|
| 1 | 2 | 3 | ... | 26 |

Now, by looking at the letters in the given sequence with their position in the alphabet

| B | C | E | G | K | M |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 |

It is easy to see that the sequence of their positions in the alphabet is a sequence of consecutive prime numbers, starting from 2. Thus, the missing letter is the one at the position 17 in the alphabet, which is Q. The answer is $\boxed{Q.}$ □

**Problem. Find the next term of the sequence**

$$311,\ 220,\ 233,\ 112,\ 202,\ 331,\ \_$$

*Solution.* By placing the terms one after another, we obtain

$$\underline{31122023}\ \underline{31122023}\ \underline{31}\ldots$$

The pattern 31122023 (December, 31, 2023) seems to repeat. Thus, by this observation, the missing term is 122. The answer is $\boxed{122.}$ □

**Problem. Find the next term of the sequence**

$$1,\ 1,\ 1,\ 3,\ 2,\ 5,\ 3,\ 7,\ 5,\ 9,\ 8,\ 11,\ 13,\ 13,\ \_$$

*Solution.* By observation, it can be seen that the sequence is a combination of twin numbers sequences

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ldots$$
$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ldots$$

The first sequence is the Fibonacci sequence and the second one is the sequence of consecutive odd numbers. Thus, by this observation, the missing term is the term after 13 in the Fibonacci sequence, which is $8 + 13 = 21$. The answer is $\boxed{21.}$ □

**Problem. Find the missing term of the sequence**

$$365824,\ \_,\ 85636,\ 5617,\ 658,\ 613,\ 64$$

*Solution.* By observation, the first term in the sequence can be split at the tens digits the into two parts,

$$3658 \mid 24$$

By reversing the digits of the first part and adding the digits of the second part, we obtain

$$8563 \mid 6 \Longrightarrow 85636$$

This is the second term. Using this observation for the terms we have

$$5617 \to 56 \mid 17 \to 65 \mid 8 \Longrightarrow 658$$
$$658 \to 6 \mid 58 \to 6 \mid 13 \Longrightarrow 613$$
$$613 \to 6 \mid 13 \to 6 \mid 4 \Longrightarrow 64$$

So the discovered rule seems to hold. Now, by applying this to the second term

$$85636 \to 856 \mid 36 \to 658 \mid 9 \Longrightarrow 6589$$

Thus, the missing term is 6589. The answer is $\boxed{6589.}$ □

**Problem. Find the next term of the sequence**

$$5,\ 6,\ 6,\ 7,\ 8,\ 10,\ 13,\ 18,\ \_$$

*Solution.* [First solution] By observation, from the second term, by subtracting each term from the previous one, we obtain

$$1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ \ldots$$

This is a Fibonacci sequence. The next term in this sequence is 8. Thus missing term in the original sequence is $18 + 8 = 26$. The answer is $\boxed{26.}$ □

---

*Solution.* [Second solution] By observation, by subtracting 5 from each term, we obtain

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$$

This is a Fibonacci sequence. The next term in this sequence is $8 + 13 = 21$. Thus missing term in the original sequence is $21 + 5 = 26$. The answer is $\boxed{26.}$ $\square$

Now, lets try to complete some sequences.

**Exercise.** Find the missing term in each of the sequence below

$$\begin{aligned}
&1,\ 2,\ 4,\ 7,\ 12,\ _\\
&0,\ 2,\ 6,\ 12,\ 22,\ 38,\ _\\
&0,\ 1,\ 10,\ 11,\ 100,\ 101,\ 110,\ 111,\ _\\
&6,\ 12,\ 15,\ 21,\ 24,\ 30,\ 33,\ 39,\ _\\
&1,\ 11,\ 21,\ 1211,\ 111221,\ 312211,\ _
\end{aligned}$$