Derivative

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Definition (Injective function)**.** An injective function (Figure 1), also known as **injection**, or **one-to-one function**, is a function that maps distinct elements of its domain to distinct elements of its codomain.

$$\text{If } f : X \mapsto Y \text{ is an injective if } \forall a, b \in X,\ f(a) = f(b) \Rightarrow a = b.$$

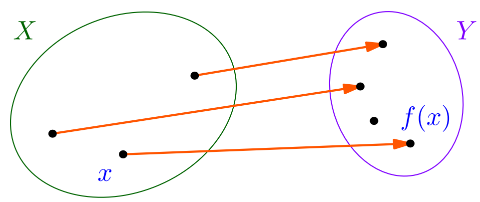

Figure 1: f is injective

Every element of the codomain is the image of at most one element of its domain. The term *one-to-one function* must not be confused with *one-to-one correspondence* that refers to *bijective* functions, which are functions such that each element in the codomain is an image of exactly one element in the domain.

Problem (Problem 1)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Assume that f is a one-to-one function. Show that

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

without using any differentiation rules.

> **Remark.** Assuming that g is a one-to-one function.

> **Remark.** There are a two notable facts:
>
> - Since we cannot use any differentiation rules, it is likely that *proof by definition* is the starting point.
> - The condition that g is a one-to-one function is something that the chain rule does not assume.
>
> Strategy: First, we look at the proof by definition how the chain rule is proved. Second, we try to figure out why the additional condition might be useful. Finally, we will *shorten* the official proof by using the condition.

Solution. For any x, note that

$$\frac{f(g(x+h)) - f(g(x))}{h} = \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right) \quad (*)$$

Since g is a one-to-one function, or $g(x+h) \neq g(x)$, none of the expressions on the left side has 0 as denominator, thus the limit of a product is a product of the limits,

$$\lim_{h\to 0}\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right) = \left(\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\cdot\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right).$$

Furthermore, g is differentiable, thus continuous, so

$$\lim_{h\to 0} g(x+h) = g(x) \Rightarrow \lim_{h\to 0} g(x+h) - g(x) = 0.$$

Therefore

$$\begin{aligned}\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} &= \lim_{h\to 0}\frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{g(x+h) - g(x)} \\ &= \lim_{H_h = g(x+h) - g(x)\to 0}\frac{f(g(x) + H_h) - f(g(x))}{H_h} = f'(g(x)).\end{aligned}$$

Now,

$$(f(g(x)))' = \left(\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\cdot\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right) = \boxed{f'(g(x))\cdot g'(x).}$$

$\square$

Problem (Problem 2a)**.** Let $a \in \mathbb{R}$. Let f be a function defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. If it is true, prove it directly Hint: often times, the easiest way to prove something is false is by providing a counter example and proving that counter example satisfies the required conditions.

(a) If the limit

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \text{ exists,}$$

then f is twice differentiable at $x = a$.

Remark. Note that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{1}{h}\left(\frac{f(a+h) - f(a)}{h} - \frac{f(a) - f(a-h)}{h}\right) \quad (*)$$

Thus the expression in (*) gives an *impression* that if f is (once) derivative then both left and right (first) derivatives would be the same, then the expression inside the parenthesis tends to 0, and the denominator outside of the parenthesis h also tends to 0. However *the rate of convergence might be different!* It means that if $f'(a+h)$ tends to $f'(a)$ in a different rate than h tends to 0, in other words *the second derivate of f from the left and right sides of a would have different values.*

Therefore, we just need to find a function that

- Let pick $a = 0$ for simplicity, f is differentiable at a,
- f is twice differentiable at $a-$ and $a+$ but the values should be different;
- f should be selected so that the given equation stands.

Solution. We show a counter example,

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Then for $a = 0$,

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h\to 0} \frac{h^2 - 2(0) + 0}{h^2} = 1.$$

However f is not twice differentiable, because

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \\ \text{does not exist}, & \text{if } x = 0 \end{cases}$$

□

Remark. You can create a differentiable function that is not twice differentiable with almost any two differentiable functions, $f(x)$ and $g(x)$. If $f'(x) \neq g'(x)$ where you want to join them (at a, say), multiply $f'(x)$ by $g'(a)$ and $g'(x)$ by $f'(a)$, then add a constant to one of the functions to make them equal at a. Now the function is continuous and differentiable at a, but not twice differentiable.

Problem (Problem 2b)**.** (b) If there exists a function $m(x)$ such that

$$f(x) - f(a) = m(x)(x - a),$$

then f is differentiable at $x = a$.

> **Remark.** Note that
>
> $$f(x) - f(a) = m(x)(x - a) \Rightarrow m(x) = \frac{f(x) - f(a)}{x - a} \quad (*)$$
>
> Since there is no requirement for $m(x)$, the limit $\lim_{x\to a} m(x)$ might not exist at all, thus the function f would not be differentiable.
>
> We just need to chose an f function that is not differentiable at a by having $\lim_{x\to a} f(x) \neq f(a)$.

Solution. We show a counter example,

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $m(x) = \frac{1}{x^2}$, for $a = 0$,

$$f(x) - f(a) = \frac{1}{x} - 0 = \frac{1}{x} = \frac{1}{x^2}(x - 0) = m(x)(x - a).$$

f is obviously not differentiable. ◻

> **Remark.** The assumption for the existence of
>
> $$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$
>
> is very strong. It requires the existence of the limits on both sides of a as well as the convergences result in the same value, which then should be the same as $f(a)$.

Problem (Problem 3)**.** Consider the function $f(x)$ given by the equation

$$f(x) = 2023 + \cfrac{2023}{2022 + \cfrac{2022}{2021 + \cfrac{2021}{\ddots + \cfrac{2}{1 + \cfrac{1}{x}}}}}$$

Find the equation of the line tangent to the graph of $f(x)$ at the point with $x-$coordinate 1.

Hint: Construct a sequence of functions $f_1,\ f_2,\ f_3,\ \ldots,\ f_{2023}$ such that $f_{2023} = f(x)$. Then use induction twice (to find $f'(1)$ and $f(1)$).

> **Remark.** If
> $$f_1(x) = 1 + \frac{1}{x},\ f_n(x) = n + \frac{n}{f_{n-1}(x)},\ \forall n \geq 2.$$
> then
> $$f_1(1) = 1 + \frac{1}{1} = 2,\ f_2(1) = 2 + \frac{2}{2} = 3,\ f_3(1) = 3 + \frac{3}{3} = 4.$$
> and
> $$f_n'(x) = -\frac{n}{(f_{n-1}(x))^2} \cdot f_{n-1}'(x) = (-1)^2 \frac{n(n-1)}{(f_{n-1}(x) f_{n-2}(x))^2} \cdot f_{n-2}'(x)$$

Solution. Let define the sequence of functions $f_1,\ f_2,\ f_3,\ \ldots,\ f_{2023}$, as follow:

$$f_1(x) = 1 + \frac{1}{x},\ f_n(x) = n + \frac{n}{f_{n-1}(x)},\ \forall n \geq 2.$$

It is easy to verify that $f_{2023}(x) = f(x)$.

First, we prove by induction that

> **Claim —**
> $$f_n(1) = n + 1,\ \forall n \geq 1 \quad (*)$$

Proof. For the base case $n = 1$,

$$f_1(1) = 1 + \frac{1}{1} = 2.$$

Let's assume that the hypothesis is true for n, or

$$f_n(1) = n + 1.$$

Then,

$$f_{n+1}'(1) = (n+1) + \frac{n+1}{f_n(1)} = (n+1) + \frac{n+1}{n+1} = n + 2.$$

Hence then hypothesis is true for all $n \geq 1$. ∎

Second, we prove by induction that

Claim —

$$f_n'(x) = \frac{(-1)^n n!}{(f_{n-1}(x)\cdot f_{n-2}(x)\cdots f_1(x)\cdot x)^2} = \frac{(-1)^n n!}{\left(x\prod_{k=1}^{n-1} f_k(x)\right)^2},\ \forall n\geq 2 \quad (**)$$

Proof. Note that

$$f_1'(x) = \left(1+\frac{1}{x}\right) = -\frac{1}{x^2} = \frac{(-1)^1\cdot 1!}{x^2}$$
$$f_n'(x) = -\frac{n}{(f_{n-1}(x))^2}\cdot f_{n-1}'(x),\ \forall n\geq 2$$

For the base case $n=2$,

$$f_2'(x) = -\frac{2}{(f_1(x))^2}\cdot f_1'(x) = -\frac{2}{(f_1(x))^2}\cdot\frac{(-1)^1\cdot 1!}{x^2} = \frac{(-1)^2 2!}{(xf_1(x))^2}.$$

Let's assume that the hypothesis is true for n, or

$$f_n'(x) = \frac{(-1)^n n!}{(x\prod_{k=1}^{n-1} f_k(x))^2},\ \forall n\geq 2.$$

Then,

$$f_{n+1}'(x) = -\frac{n+1}{(f_n(x))^2}\cdot f_n'(x) = -\frac{n+1}{(f_n(x))^2}\cdot f_n'(x)\cdot\frac{(-1)^n n!}{\left(x\prod_{k=1}^{n-1} f_k(x)\right)^2} = \frac{(-1)^{n+1}(n+1)!}{(x\prod_{k=1}^{n} f_k(x))^2}.$$

Hence then hypothesis is true for all $n\geq 2$. ■

Therefore, using both results (*) and (**),

$$f'(1) = f_{2023}'(1) = \frac{(-1)^{2023}2023!}{\left(1\cdot\prod_{k=1}^{2022} k+1\right)^2} = \frac{(-1)^{2023}2023!}{(2023!)^2} = -\frac{1}{2023!}$$

Thus the equation of the line tangent to the graph of $f(x)$ at the point with $x-$coordinate 1 is

$$y = f'(1)(x-1)+f(1) = -\frac{1}{2023!}(x-1)+2 = \boxed{-\frac{1}{2023!}x+\left(2+\frac{1}{2023!}\right)}.$$

□

Problem (Problem 4ab). Let $f: \mathbb{R} \to \mathbb{R}$ be a function. We say that a non-empty set $U \subseteq \mathbb{R}$ is **ajar** if

$$\forall u \in U, \exists r > 0 \text{ such that } (u-r, u+r) \subseteq U.$$

(a) Show that $(0, 2)$ is ajar.

(b) Find a set that is not ajar. You don't need to prove it.

> **Remark.** First, we prove it by using the definition, in other words, for an arbitrary $u \in U = (0, 2)$, let's determine an r such that
>
> $$(u-r, u+2) \subseteq U \Rightarrow 0 \leq u-r < u+r \leq 2 \Rightarrow r \leq u, r \leq 2-u \Rightarrow r = \min(u, 2-u)$$
>
> For the second question, let's investigate the definition,
>
> $$\forall u \in U, \exists r > 0 \text{ such that } (u-r, u+r) \subseteq U.$$
>
> $(u-r, u+r) \subseteq U$ means that some numbers smaller than u must be inside U. It means that if U contain a minimal or a maximal element, then it cannot be ajar, since no element smaller than the minimal (or larger than the maximal) would be in U.
>
> Let formally prove it. Assume the contrary, let $m \in U$ be the minimal element of U, then any number less than m is not element of U, which mean that for any r, the interval $(u-r, u+r)$ can not be contained entirely by U, precisely
>
> $$\forall r > 0, \forall v: \ m-r < v < m, v \notin U.$$
>
> Similarly, U should not have a maximal element.
>
> It seems that the open boundary is a (necessary) condition for ajar intervals. Would an **ajar** set be a set of non-overlapping open intervals?

Solution. Let an arbitrary $u \in U = (0, 2)$, then $0 < u < 2$. Let $r = \min(u, 2-u)$, then $r > 0$, and

$$0 \leq u-r \text{ and } u+r \leq 2 \Rightarrow 0 \leq u-r < u+r \leq 2 \Rightarrow (u-r, u+2) \subseteq (0, 2) = U.$$

Therefore $(0, 2)$ is ajar.

For the second question, let $U = [0, 2]$, then for $u = 0 \in U$,

$$\forall r > 0 \Rightarrow u-r = -r < 0 \Rightarrow (u-r, u+r) = (-r, r) \not\subseteq [0, 2] = U.$$

$\square$

Problem (Problem 4c)**.** In the following parts, we will use the two definitions:

For $A \subseteq \mathbb{R}$, we define $f(A) \ := \ \{f(a): \ a \in A\}$.

For $B \subseteq \mathbb{R}$, we define $f^{-1}(B) \ := \ \{x \in \mathbb{R}: \ f(x) \in B\}$.

Note that $f(A)$ and $f^{-1}(B)$ are both sets.

(c) Show that $\forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0$, such that

$$f((a-\delta, a+\delta)) \subseteq (f(a)-\epsilon, f(a)+\epsilon) \quad (*)$$

is equivalent to the definition of f is continuous everywhere.

Solution. Let rewrite the statement (*):

$$\forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall x \in (a-\delta, a+\delta): \ f(a)-\epsilon \leq f(x) \leq f(a)+\epsilon \quad (S1)$$

On the other hand, we say that f is continuous at a point a in the domain of f, if

$$\lim_{x \to a} f(x) = f(a) \Leftrightarrow \forall \epsilon > 0, \ \exists \delta > 0, \ \forall x: \ 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \quad (S2)$$

We prove the two statements are equivalent in two directions.

Claim — (S1) $\Rightarrow$ (S2).

Proof. Rewriting (S1) using the symbol γ, instead of ϵ, and ω instead of δ: for an arbitrary $a \in \mathbb{R}$,

$$\forall \gamma > 0, \exists \omega > 0, \forall x \in (a-\omega, a+\omega): \ f(a)-\gamma \leq f(x) \leq f(a)+\gamma \quad (1)$$

So, for an arbitrary $a \in \mathbb{R}$, we need to use (1) to find a value for δ base on an arbitrary ϵ so that S(2) stands.

Now, for a given arbitrary ϵ, by choosing $\gamma = \frac{1}{2}\epsilon$, we have, for $\gamma = \frac{1}{2}\epsilon$,

$$\exists \omega > 0, \forall x \in (a-\omega, a+\omega): \ f(a)-\epsilon < f(a) - \frac{1}{2}\epsilon \leq f(x) \leq f(a) + \frac{1}{2}\epsilon < f(a)+\epsilon$$
$$\Rightarrow \exists \omega > 0, \forall x: \ 0 < |x-a| < \omega \Rightarrow |f(x) - f(a)| < \epsilon < f(a) \quad (2)$$

Hence, by choosing $\gamma = \frac{1}{2}\epsilon$, $\delta = \omega$, and applied (1) as shown above, we received (2), which is

$$\forall \epsilon, \exists \delta > 0, \ \forall x: 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

This is exactly S2. ∎

Claim — (S2) $\Rightarrow$ (S1).

Proof. Rewriting (S2) using the symbol γ, instead of ϵ, and ω instead of δ: for an arbitrary $a \in \mathbb{R}$,

$$\forall \gamma > 0, \ \exists \omega > 0, \ \forall x: \ 0 < |x-a| < \omega \Rightarrow |f(x) - f(a)| < \gamma \quad (3)$$

So, for an arbitrary $a \in \mathbb{R}$, we need to use (1) to find a value for δ base on an arbitrary ϵ so that S(1) stands.

Hence, by choosing $\gamma = \epsilon$, $\delta = \omega$, and applied (3), we received

$$\forall \epsilon, \exists \delta > 0, \ \forall x \in (a-\delta, a+\delta): f(a)-\epsilon \leq f(x) \leq f(a)+\epsilon \quad (4)$$

Note that the inequalities in (4) are weakened from (3), which is perfectly correct to be done so. This is exactly S1. ∎

□

Problem (Problem 4d)**.** (d) Assume that f is continuous everywhere. Show that for any non-empty subset $U \subseteq \mathbb{R}$, if U is ajar, then $f^{-1}(U)$ is ajar.

Hint: To prove this, you need to prove and use these facts:

1. for all non-empty $A \subseteq \mathbb{R}$, $A \subseteq f^{-1}(f(A))$. This should be one-line proof.
2. $f^{-1}(B_1) \subseteq f^{-1}(B_2)$ if non-empty sets B_1, B_2 satisfies $B_1 \subseteq B_2 \subseteq \mathbb{R}$. This should be very short; and
3. the result from part (c)

Solution. We prove the two facts. First, for the inverse image

Claim — For all non-empty $A \subseteq \mathbb{R}$, $A \subseteq f^{-1}(f(A)) \quad (*)$

Proof. By definition,

$$x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}(f(A)) = \{x \in \mathbb{R} : f(x) \in f(A)\} \Rightarrow A \subseteq f^{-1}(f(A)).$$

■

Second, for the subsets

Claim — If non-empty sets B_1, B_2 satisfies $B_1 \subseteq B_2 \subseteq \mathbb{R}$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2) \quad (**)$

Proof. By definition, for non-empty sets B_1, B_2 satisfies $B_1 \subseteq B_2 \subseteq \mathbb{R}$,

$$\begin{aligned}&\forall e \in f^{-1}(B_1) \Rightarrow e\{x \in \mathbb{R} : f(x) \in B_1\} \Rightarrow f(e) \in B_1 \subseteq B_2 \Rightarrow e \in f^{-1}(B_2)\\ &\Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)\end{aligned}$$

■

Now, f is continuous, by part (c), for an arbitrary $a \in \mathbb{R}$,

$$\forall \epsilon > 0, \exists \delta > 0 : \ f((a-\delta, a+\delta)) \subseteq (f(a)-\epsilon, f(a)+\epsilon) \quad (1)$$

U is ajar, by definition,

$$\forall f(a) \in U, \exists \epsilon > 0 : \ (f(a)-\epsilon, f(a)+\epsilon) \subseteq U \quad (2)$$

Apply (**) on (1), then (2),

$$f^{-1}(f((a-\delta, a+\delta))) \subseteq f^{-1}((f(a)-\epsilon, f(a)+\epsilon)) \subseteq f^{-1}(U).$$

By (*)

$$(a-\delta, a+\delta) \subseteq f^{-1}(f((a-\delta, a+\delta))) \quad (3)$$

This means $f^{-1}(U)$ is ajar, by (3):

$$\forall a \in f^{-1}(U), \exists \delta > 0 : (a-\delta, a+\delta) \subseteq f^{-1}(U).$$

□

Problem (Problem 4e)**.** (e) Assume that for any non-empty subset $U \subseteq \mathbb{R}$, if U is ajar, then $f^{-1}(U)$ is ajar. Prove that f is continuous everywhere.

Hint: Let $a \in \mathbb{R}$ and $\epsilon > 0$, and consider the ajar set $(f(a) - \epsilon, f(a) + \epsilon)$ (you might assume this set is ajar without proof.) You may also prove and use these facts.

1. for all non-empty $B \subseteq \mathbb{R}$, $f(f^{-1}(B)) \subseteq B$.
2. $f(A_1) \subseteq f(A_2)$ if non-empty sets A_1, A_2 satisfies $A_1 \subseteq A_2 \subseteq \mathbb{R}$. Both proofs should be very short; and
3. the result from part (c)

Solution. We prove the two facts. First, for the inverse image

Claim — For all non-empty $B \subseteq \mathbb{R}$, $f(f^{1}(B)) \subseteq B \quad (*)$

Proof. By definition, $f(f^{-1}(B)) = \{f(a) : a \in f^{-1}(B)\}$, so

$$y \in f(f^{-1}(B)) \Rightarrow \exists x \in f^{-1}(B) : y = f(x) \Rightarrow \exists x \in \mathbb{R}, y = f(x) \in B \Rightarrow f(f^{1}(B)) \subseteq B.$$

∎

Second, for the subsets

Claim — If non-empty sets A_1, A_2 satisfies $A_1 \subseteq A_2 \subseteq \mathbb{R}: \ f(A_1) \subseteq f(A_2) \quad (**)$

Proof. By definition, for non-empty sets A_1, A_2 satisfies $A_1 \subseteq A_2 \subseteq \mathbb{R}$,

$$y \in f(A_1) = \{f(a) : a \in A_1\} \Rightarrow \exists a \in A_1, y = f(a) \Rightarrow a \in A_2, y = f(a) \Rightarrow y \in f(A_2) \Rightarrow f(A_1) \subseteq f(A_2).$$

∎

Let $a \in \mathbb{R}$ and $\epsilon > 0$, and consider the ajar set $U = (f(a) - \epsilon, f(a) + \epsilon)$.

$f^{-1}(U)$ is ajar,:

$$\forall a \in f^{-1}(U), \exists \delta > 0 : (a - \delta, a + \delta) \subseteq f^{-1}(U) \quad (1)$$

Apply (**) on (1),

$$f((a - \delta, a + \delta)) \subseteq f(f^{-1}(U)) \quad (2)$$

By (*) on (1),

$$f((a - \delta, a + \delta)) \subseteq f(f^{-1}(U)) \subseteq U = (f(a) - \epsilon, f(a) + \epsilon)$$

This means, for an arbitrary $a \in \mathbb{R}$,

$$\forall \epsilon > 0, \exists \delta > 0 : \ f((a - \delta, a + \delta)) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$$

By part (c), f is continuous. □