Test Problems fore UMC K1

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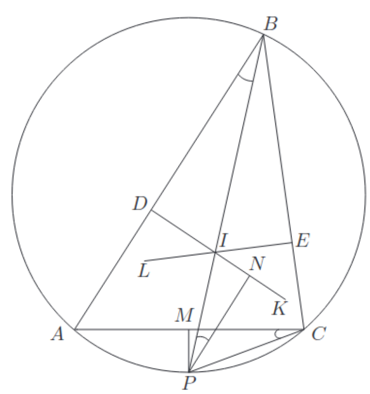
November 9, 2024

Problem 0.1 (Problem One)**.** Given a triangle ABC satisfying $AB + BC = 3 \cdot AC$. The incircle of triangle ABC has center I and touches the sides AB and BC at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Let P be the other intersection of BI with the circumcircle (ABC). Prove that the points A, C, K, L lie on a circle centred at P.

Proof. Let M be the midpoint of AC and N the projection of P to IK. Since $AB + BC = 3AC$, we get $BD = BE = AC$, so $BD = 2CM$.

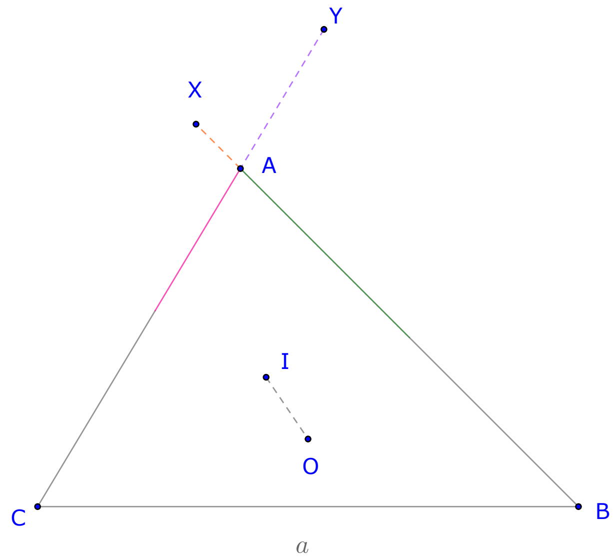
Furthermore, $\angle ABP = \angle ACP$, therefore $\triangle DBI$ and $\triangle MCP$ are similar in ratio 2.



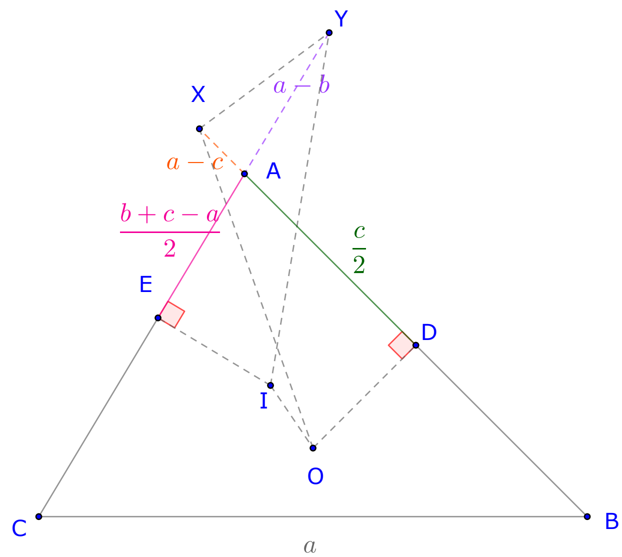
It is known that $PA = PI = PC$. Moreover, $\angle NPI = \angle DBI$, so that the triangles PNI and CMP are congruent. Hence $ID = 2PM = 2IN$; i. e. N is the midpoint of IK.

This shows that PN is the perpendicular bisector of IK, so $PC = PK = PI$. Analogously, $PA = PL = PI$. So P is the centre of the circle through A, K, I, L and C. $\square$

Problem 0.2 (Problem Two)**.** In triangle ABC, let BC be the longest side. Point X is chosen on side AB such that $BX = BC$. Similarly, point Y is chosen on AC such that $CY = BC$. Prove that OI is perpendicular to XY, where O and I are the circumcenter and incenter, respectively, of triangle ABC.



Proof. Let $AB = c, BC = a, CA = b$. Then



$$XA = a - c, AD = \frac{c}{2}, DX = DA + XA = \frac{c}{2} + (a - c) = a - \frac{c}{2}$$
$$OX^2 = OD^2 + DX^2 = OA^2 - AD^2 + DX^2 = OA^2 - \left(\frac{c}{2}\right)^2 + \left(a - \frac{c}{2}\right)^2 = OA^2 + a(a - c)$$
Similarly $OY^2 = OA^2 + a(a - b) \Rightarrow OY^2 - OX^2 = a(c - b)$.

Similarly for I,

$$YA = a - b, EA = \frac{c + b - a}{2}, EY = EA + YA = \frac{c + a - b}{2}$$
$$IY^2 = IE^2 + EY^2 = IA^2 - EA^2 + EY^2 = IA^2 - \left(\frac{c + b - a}{2}\right)^2 + \left(\frac{c + a - b}{2}\right)^2 = IA^2 + c(a - b)$$
Similarly $IX^2 = IA^2 + b(a - c) \Rightarrow IY^2 - IX^2 = a(c - b)$.

Hence, by the Perpendicularity Lemma OI is perpendicular to XY. $\square$

Problem 0.3 (Problem Three)**.** 25 flies are resting on the outdoor table in the garden, waiting for lunch to be served.

- It is known that for any 3 of them, 2 are at a distance less than 20 cm.
- There are at least a pair of flies that are further than 20 cm from each other.

Minh's mother gave him a fly swatter, shown below, with a hoop of radius 20 cm, With a single strike he can swat the flies where the hoop landed. In *at least* how many strikes can he swat all of them?

Assume that Minh is so fast that the flies do not have time for reaction during and between his lightning strikes.

Solution. If no 2 flies are further than 20 cm from each other, Minh can strike them all in 1 strike by aiming the center of the swatter at any fly. But this is not the case, so let's assume there are 2 flies, A and B, that are more than 20 cm apart.

Then, every other fly is either in a 20 cm radius of A or in a 20 cm radius of B. Out of the 23 remaining flies either at least 12 will be in the 20 cm radius of A or 12 will be in the 20 cm radius of B.

Swatting that the A or B fly with the center of the swatter kills at least 13. Thus, by $\boxed{2}$ strikes, he can swat them all. □