Solving Forty Two Problems by the Induction Principle - Part VI

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Problem 0.1 (Problem Twenty Eight)**.** Define a sequence (a_n) by

$$a_1 = 1, a_2 = 2, a_{n+2} = 2a_{n+1} - a_n + 2,\ \forall n \geq 1.$$

Prove that $\forall m \geq 1,\ a_m a_{m+1}$ is also a term of this sequence.

Solution. First we prove by induction that

Claim — $a_n = (n-1)^2 + 1$.

Proof. It is easy to verify the base case. For the inductive step:

$$2(n^2+1) - ((n-1)^2+1) + 2 = n^2 + 2n + 2 = (n+1)^2 + 1.$$

■

Now, $\forall m \geq 1$,

$$a_m a_{m+1} = ((m-1)^2+1)(m^2+1) = (m^2-2m+2)(m^2+1) = (m^2-m+1)^2+1 = a_{m^2-m+2}.$$

□

Problem 0.2 (Problem Twenty Nine)**.** Let $a_1, a_2, \ldots$ be a sequence with

$$a_1 = 1,\ a_{n+1} = \begin{cases} a_n - 2 \text{ if } a_n - 2 \notin \{a_1, a_2, \ldots, a_n\} \text{ and } a_n - 2 > 0, \\ a_n + 3 \text{ otherwise} \end{cases}$$

Prove that for every positive integer $k \geq 1$, there exist n such that

$$a_n = a_{n-1} + 3 = k^2.$$

Solution. It is easy to verify that by induction for a set of five numbers $5n+1, 5n+2, 5n+3, 5n+4, 5n+5$,

$$a_{5n+1} = 5n+1,\ a_{5n+2} = 5n+4,\ a_{5n+3} = 5n+2,\ a_{5n+4} = 5n+4,\ a_{5n+5} = 5n+3.$$

The required statement follows. □

Problem 0.3 (Problem Thirty)**.** Let x, y be real numbers such that the number $x+y$, x^2+y^2, x^3+y^3, x^4+y^4, are all integers. Prove that, for all $n \geq 5$, $x^n + y^n$ is an integer.

Solution. First note that $2xy = (x+y)^2 - (x^2+y^2)$ so this is an integer. Furthermore

$$2(x+y)^4 = 2(x^4+y^4) + 4(2xy)(x^2+y^2) + 3(2xy)^2 \Rightarrow 2 \mid 3(2xy)^2 \Rightarrow 2 \mid 2xy \Rightarrow xy \in \mathbb{Z} \quad (*)$$

Now, with (*) and the identity

$$x^{n+1} + y^{n+1} = (x+y)(x^n+y^n) - xy(x^{n-1}+y^{n-1})$$

it is easy to prove by induction that $n \geq 5$, $x^n + y^n$. ◻

Problem 0.4 (Problem Thirty One)**.** Let a and n be two positive integers such that $a^n - 1$ is divisible by n. Prove that the number $a+1, a^2+2, \ldots, a^n+n$ are all distinct modulo n.

Theorem (Euler's Theorem)

$a, n \in \mathbb{Z},\ \gcd(a,n) = 1 \Rightarrow a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.

Solution. We prove by induction. For $n = 1$ the case case is clear with single term. Assume now that the statement is true for all integers less than $n \geq 2$. Let k, the order of a modulo n, which is the least value such that $a^k \equiv 1 \pmod{n}$.

From Euler's Theorem, $k < n$, and from the condition given by the problem $n \mid a^n - 1$, we have $k \mid n$. By induction hypothesis,

$$a+1, a^2+2, \ldots, a^k+k$$

are distinct modulo k. We prove now the statement:

$$1 \leq x \neq y \leq n,\ a^x + x \not\equiv a^y + y \pmod{n}.$$

Let $x = kz + t, y = ku + v$, $(1 \leq t, uv \leq k, 0 \leq z, u < \frac{n}{k}$. Then

$$a^x \equiv a^t \pmod{n},\ a^y \equiv a^v \pmod{n}.$$

Case 1: $t \neq v$, then

$$a^x + x \equiv a^t + t \not\equiv a^v + v \equiv a^y + y \pmod{n}.$$

Case 2: $t = v$, then $z \neq u$,

$$a^x + x \equiv a^t + kz + t = a^t + ku + v + k(z-u) \equiv a^y + y + k(z-u) \not\equiv a^y + y \pmod{n}.$$

◻

Problem 0.5 (Problem Thirty Two)**.** Let $n \geq 1$, which is not divisible by 3. Show that

$$x^3 + y^3 = z^n$$

has at least one solution (x, y, z) where x, y, z are positive integers.

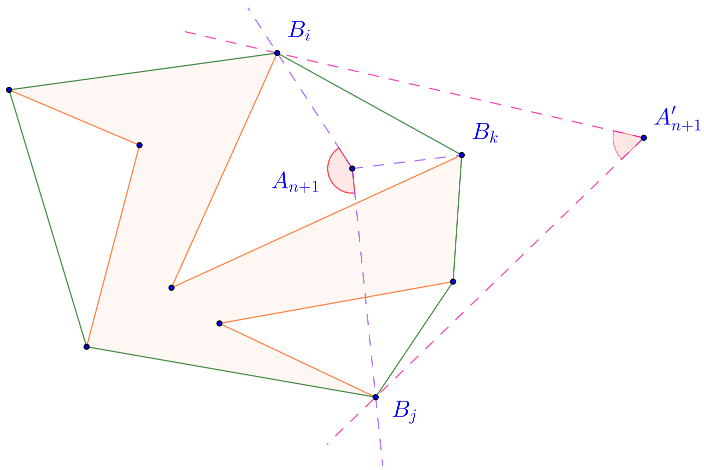
Solution. The base case $n = 1$ is obvious. For $n = 2$, we have $1^3 + 2^3 = 3^2$.

Now, let assume that (x_n, y_n, z_n) is a solution for the case n, then

$$(z_n)^{n+3} = (z_n^3)z_n^n = (z_n^3)(x_n^3 + y_n^3) = (z_n x_n)^3 + (z_n y_n)^3$$

Thus $(x_{n+3}, y_{n+3}, z_{n+3}) = (z_n x_n, z_n y_n, z_n)$ is a solution for $x^3 + y^3 = z^{n+3}$ case. ◻

Problem 0.6 (Problem Thirty Three)**.** If $A_1, A_2, \ldots, A_n$ are any points in the plane, with any three not collinear, then there is a convex polygon P such that some of A_i are vertices of P and the rest of the point are inside P. *Note that P is called the convex hull of $A_1, A_2, \ldots, A_n$.*



Solution. We prove by induction. The base case $n = 3$ is trivial.

Let's assume the hypothesis is true for n. Now, Let $B_1B_2 \ldots B_m$ ($m \leq n$) be the convex hull of the polygon $A_1A_2 \ldots A_n$, where $B_1, B_2, \ldots, B_m$ are some points of $A_1, A_2, \ldots, A_n$.

Consider point A_{n+1}. By the Extremal Principle there exists B_i, B_j such that the measure of angle $\angle B_iAB_j$ (less than $180°$) is maximal.

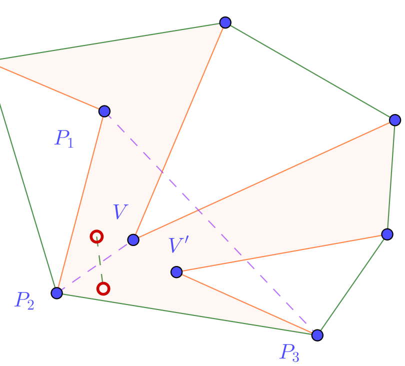
Case 1: There exists point B_k lies outside of this $\angle B_iA_{n+1}B_j$. B_i, B_j, B_k cannot lie the same half-plane, otherwise one of $\angle B_iA_{n+1}B_k$ or $\angle B_kAB_j$ is larger than $\angle B_iA_{n+1}B_j$, which contradicting the choice of $\angle B_iA_{n+1}B_j$. Thus B_k like outside of the same half-plane formed by the angle $\angle B_iA_{n+1}B_j$, which means that A_{n+1} is inside the triangle $\angle B_iB_kB_j$, or inside the convex hull $B_1, B_2, \ldots, B_m$. Thus, $B_1, B_2, \ldots, B_m$ is the convex hull of $A_1A_2 \ldots A_nA_{n+1}$.

Case 2: All B_k other than B_i, B_k lie inside of the $\angle B_iA'_{n+1}B_j$. Then, $B_1B_2B_iA_{n+1}B_j \ldots B_m$ is the convex hull of $A_1A_2 \ldots A_nA_{n+1}$. □

Problem 0.7 (Problem Thirty Four)**.** n is a positive integer, $n \geq 3$.

(a) Prove that any $n-$gon (not necessarily convex) can be cut into triangles by non-intersecting diagonals.

(b) Prove that the sum of the inner angles of any $n-$gon (not necessarily convex) is equal to $(n-2)180°$. Hence prove that the number of triangles into which an $n-$gon is cut by non-intersecting diagonals is $n-2$.



Solution. First, we use the Problem 33 for existence of a convex hull. Then

Claim — Any $n-$gon $\mathcal{P}$ ($n \geq 4$) has at least one diagonal that completely lies inside it.

Proof. Let us index the vertices $P_1P_2\dots P_n$. Consider the convex hull $\mathcal{H}$. $\mathcal{H}$ is a convex polygon, so it has a vertex, which will be one of the original vertices, WLOG, A_2. This ensures that the angle $P_1P_2P_3$ is less than $180°$, i.e. if we take triangle $\triangle P_1P_2P_3$ then part of the polygon $\mathcal{P}$ near P_2 is in the interior of $\mathcal{H}$.

Now if P_1P_3 is an interior diagonal, then we are done. Otherwise, we know that there exists exterior part of the polygon inside $\triangle P_1P_2P_3$. In particular, there should be some vertices of $\mathcal{P}$ in there (see V, V'). Let V be the vertex other than P_2 inside $\triangle P_1P_2P_3$, which is farthest away from the line P_1P_3,.

Then VP_2 lies in the interior of $\mathcal{P}$. Indeed, if there is a side of $\mathcal{P}$ that intersects VP_2, then one of the ends of is will be further away from P_1P_3 than V, but will still lie inside the triangle (see the points as hollow red circles) This would contradict the choice of V. Since the points inside the triangle near P_2 are in the interior of $\mathcal{P}$, then the whole segment is, which makes it an interior diagonal. ■

For the first question, we prove by induction. The base case of $n = 3$ is clear. Now, for the inductive step, an interior diagonal divide it into two non-overlapping polygons, each with the number of sides less than n. By the hypothesis, both polygons can be cut into triangles by non-intersecting diagonals, thus the union set of triangles are the ones that the $n-$gon is cut into by non-intersecting diagonals.

For the second question, the base case for the given hypothesis is clear. Now, for the inductive step, an interior diagonal divide it into two non-overlapping polygons, one with $k+1$ sides (k of the $n-$gon plus the diagonal), the other with $n-k-1$ sides ($n-k$ sides from the $n-$gon plus the diagonals). The sum of the inner angles of any $n-$gon now is equal to $(k-1)180° + (n-k-1)180° = (n-2)180°$.

By the total measure of the sum of all angles, it implies that the number of such triangles is $n-2$. □

Problem 0.8 (Problem Thirty Five)**.** Point O is inside (or on the boundary of) a convex $n-$gon $A_1A_2\dots A_n$. Prove that among the angles A_iOA_j ($1 \leq i \neq j \leq n$) there are not fewer than $n-1$ non-acute ones.

Solution. We prove by induction based on n. For $n = 3$ there is nothing to prove.

Let consider $P_1P_2\dots P_n$ polygon, $n \geq 4$. Let p, q, r be such indexes that O is inside $\triangle A_pA_qA_r$. Let $A_k \notin \{A_p, A_q, A_r\}$. By removing A_k we receive a $(n-1)-$gon that we can apply the induction hypothesis to receive $n-2$ non-acute angles. Note that none of these triangles has A_k as a vertex.

Point O now must be inside one of the triangles with vertices A_k and two of A_p, A_q, A_r. WLOG let $O \in \triangle A_kA_qA_r$. Then

$$\angle A_kOA_q + \angle A_kOA_p = 360° - \angle AqOA_r \geq 180°.$$

Hence, at least of of these two $\angle A_kOA_q, \angle A_kOA_p$ is a non-acute angles. Together with $n-2$ previously determined ones, they make $n-1$ non-acute angles. □