Nghia Doan

Math Club & Competitions
Victoria, BC, Canada

(1) **January 5**: Geometric Transformations II: Translations. Half Turns. Sum of Half Turns.

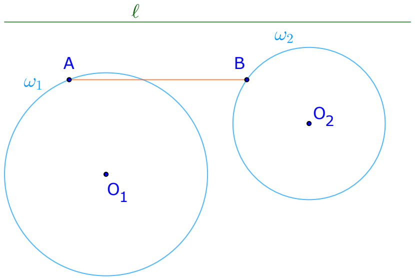
(2) **January 19**: Geometric Transformations III: Rotations by an Angle. Reflections over a Line.

(3) **February 9**: Geometric Transformations IV: Homothety.

Example

Given circles ω_1, ω_2, and line ℓ.

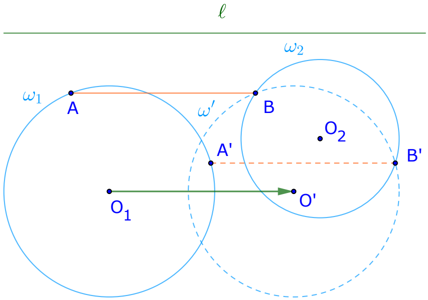
Construct a segment AB parallel with ℓ with a given length $AB = c$, such that $A \in \omega_1$ and $B \in \omega_2$.



Translate the circle ω_1 to ω' by a distance $O_1O' = c$ and $O_1O' \parallel \ell$.

The intersections (if any) ω_2 and ω' are B and B'.

They are the images of the translation of points A and A'. Thus $AB = A'B' = c$.

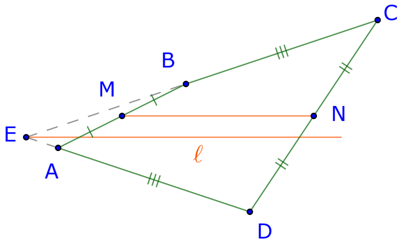


Example

$ABCD$ is a quadrilateral such that $AD = BC$. M and N are midpoints of AB and CD, respectively.

E is the intersection of (the extensions of) AB and CD.

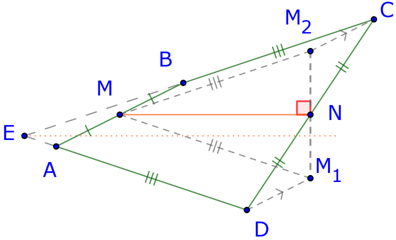
Prove that MN is parallel to the line ℓ, the angle bisector of $\angle AEB$.



Translate AD and BC to MM_1 and MM_2, respectively. Then DM_1 and CM_2 are the images of AM and BM by the translation, thus $DM_1 \parallel CM_2$ and $DM_1 = CM_2$. Thus, DM_1CM_2 is a parallelogram.

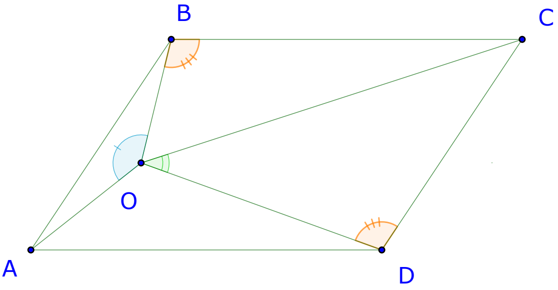
$\triangle NDM_1 \cong \triangle NCM_2$, thus M_1, N, M_2 are collinear. Therefore $NM_1 = NM_2$.

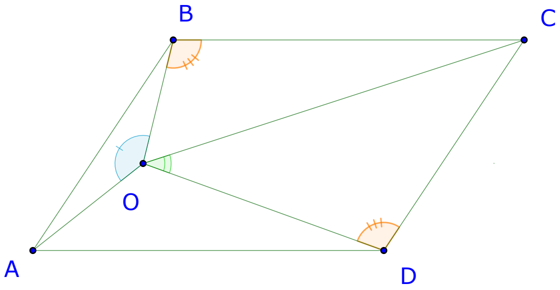
MM_1M_2 is an isosceles triangle, thus the median MN is the angle bisector, which is parallel to the angle bisector of $\angle AEB$.

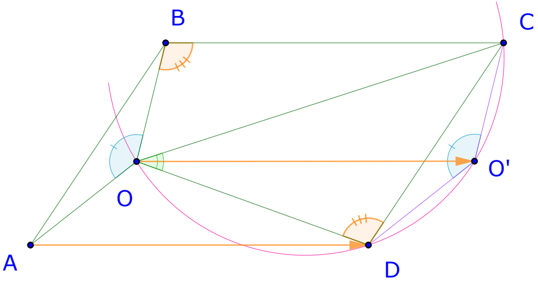


Example

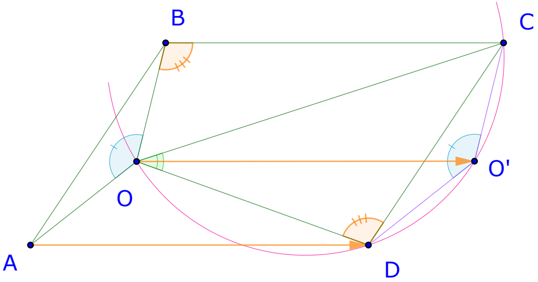
The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.




B
C
O
A
D

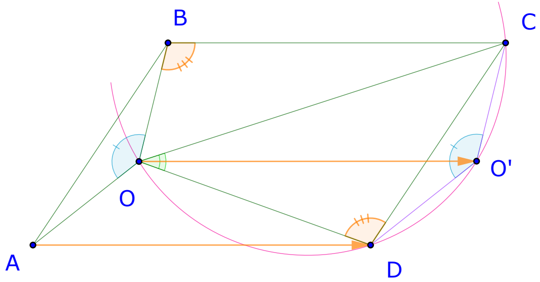


The translation by $\overrightarrow{AD}$ maps A to D, B to C, and O to O'.



The translation by $\overrightarrow{AD}$ maps A to D, B to C, and O to O'.

$ABCD$ is a parallelogram, $AD \parallel BC, AD = BC$. By the translation, $OO' \parallel AD, OO' = AD$, thus $OO' \parallel BC, OO' = BC$. Therefore $OBCO'$ is a parallelogram. It implies that $\angle OBC = \angle OO'C$.



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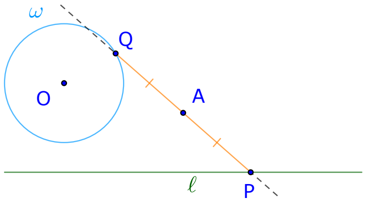
Since $\angle AOB + \angle COD = 180^\circ$, so $\angle DO'C + \angle COD = 180^\circ$, or $CODO'$ is cyclic. Therefore $\angle ODC = \angle OO'C$. Hence, $\boxed{\angle OBC = \angle ODC.}$

Example

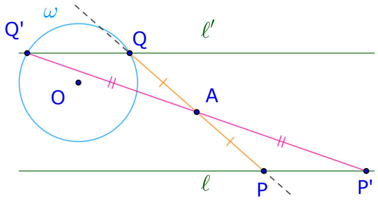
Construct a line through A intersecting line ℓ and circle ω at P and Q, respectively, such that $AP = AQ$.



Rotate the line ℓ half turn around A. Assume that ℓ', the image of ℓ, intersects ω at Q. Draw a line through A, Q intersects ℓ at P, then:

$$\frac{1}{2}\text{ turn} : P \to Q.$$

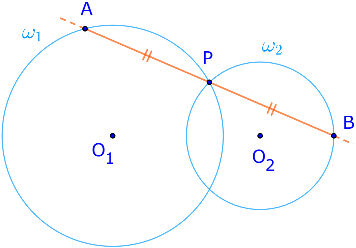
We have: (1) P is on ℓ, (2) Q is on ω $(\cap \ell')$, (3) A, P, Q are collinear, and (4) $AP = AQ$.



We have **at most two solutions** (why?)

Example

P is an intersection point of circles ω_1 and ω_2. Construct a line through P intersecting ω_1 and ω_2 at A and B, respectively, such that $AP = PB$.

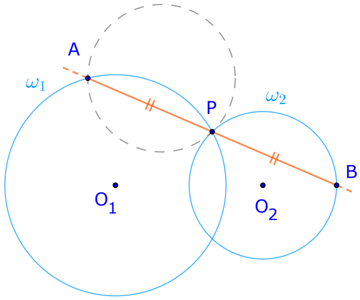


Let **rotate** ω_2 **half turn** (180°) around (or reflect ω_2 over point) P.

Let A be the other intersection of ω_1 and the image of ω_1 (the dotted circle) and B be the intersection of AP with ω_2, then:

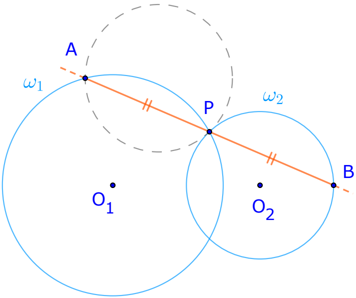
$$\frac{1}{2}\text{ turn} : B \to A.$$

We have: (1) A is on ω_1 $(\cap\, \omega_2')$, (2) B is on ω_2, (3) P, A, B are collinear, and (4) $AP = PB$.



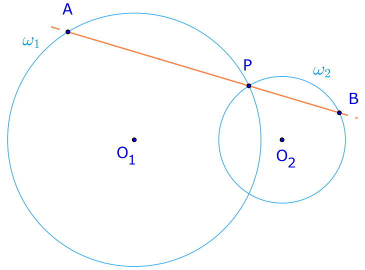
How many solutions?

1. If $|\omega_1 \cup \omega_2| = 2$, then we have 1 solution.
2. If $|\omega_1 \cup \omega_2| = 1$, then we have no solution (why?)
3. If $|\omega_1 \cup \omega_2| = 0$, and the two radii are the same then we have infinitely many solutions otherwise no solution (why?).

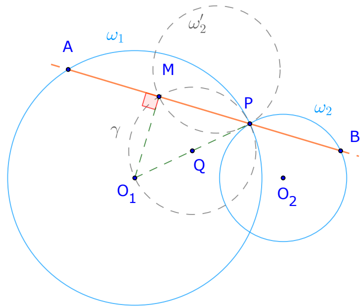


Example

P is an intersection point of circles ω_1 and ω_2. Construct a line through P intersecting ω_1 and ω_2 at A and B, respectively, such that $AP = 2PB$.



If M **is the midpoint of** AP**, then** $\angle OMP = 90^\circ$ **and** $MP = PB$. Thus M is the intersection of ω_2', the image of ω_2, and the circle γ diameter O_1P.

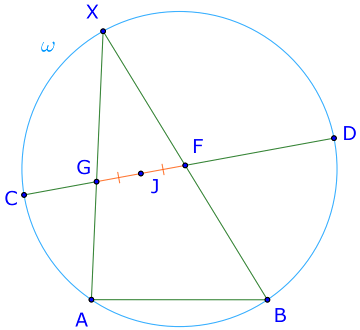


Thus we rotate ω_2 **half turn** about P. Then we draw the circle γ diameter O_1P. Their intersection is M. Line through MP intersects ω_1 and ω_2 at A and B respectively.

$$AM \overset{OM \perp MP}{=} MP \overset{B \to M}{=} PB \Rightarrow AP = 2PB.$$

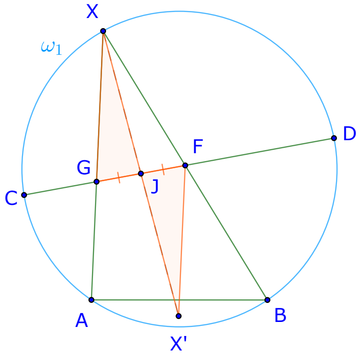
Example

AB and CD are chords of circle ω. J is a point on CD. Find point X on the circumference of ω such that $JG = JF$, where G and F are intersections of CD with XA and XB, respectively.

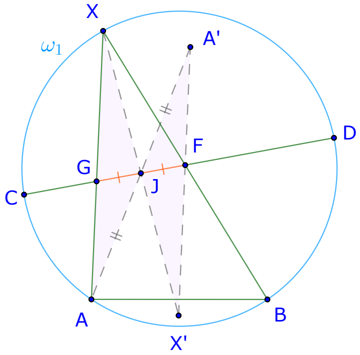


The condition $GJ = JF$ give us the idea to rotate X **half turn** about J to X'.

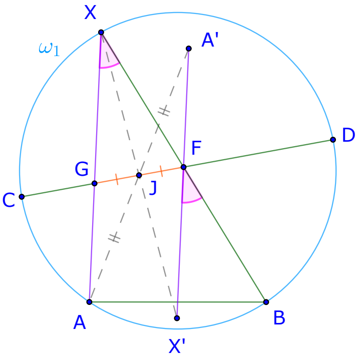
$\triangle XGJ \cong \triangle XFJ$ shows that $\angle XGJ = \angle JFX$, thus $FX' \parallel XA$. Or $\angle X'FB = \angle AXB = \widehat{AB}$.



We rotate A **half turn** about I to A'. Therefore, $AXA'X'$ is a parallelogram.



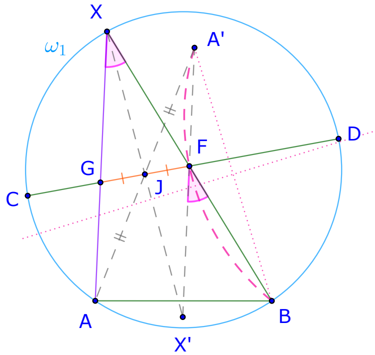
$A'X' \parallel XA$ thus X, F, A' are collinear.



$\angle A'FB = 180^\circ - \angle F'XB = 180^\circ - \angle AXB = 180^\circ - \frac{1}{2}\widehat{AB}.$

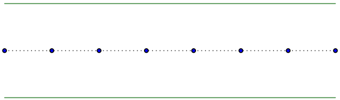
Hence, we first construct A', then F is the intersection the arc $\widehat{A'B}$ with measure $180^\circ - \frac{1}{2}\widehat{AB}$ (how to construct an arc knowing the measure of the angle subtending it?) with the chord CD.

Finally X is the intersection of BF with ω.



Example

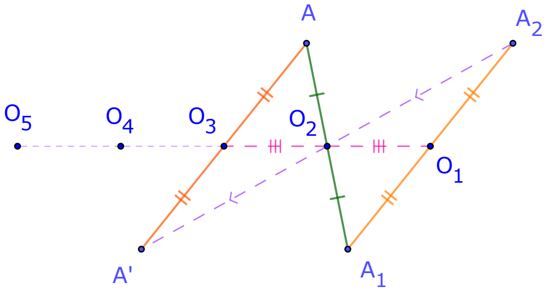
The strip formed by two parallel lines clearly has infinitely many centers of symmetry. Can a figure have more than one, but only a finite number of centers of symmetry (for example, can it have two and only two centers of symmetry)?



Assume that the figure $\mathcal{F}$ has two centers of symmetry, O_1 and O_2.

Then the point O_3, obtained from O_1 by a half turn about O_2 is also a center of symmetry of $\mathcal{F}$.

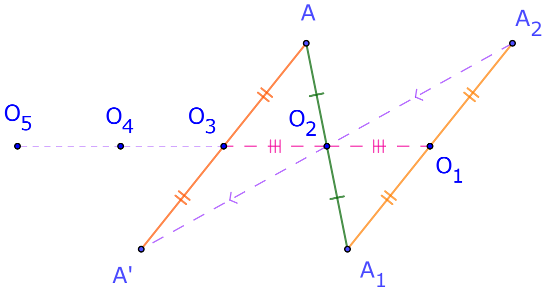
Indeed, if A is any point of $\mathcal{F}$, then the points A_1, A_2, and A', where A_1 is obtained from A by a half turn about O_2, A_2 from A_1 by a half turn about O_1, and A' from A_2 by a half turn about O_2, will also be points of $\mathcal{F}$ (since O_1 and O_2 are centers of symmetry).



But the point A' is also obtained from A by a half turn about O_3!

Indeed, the segments AO_3 and O_3A' are equal, parallel, and have opposite directions, since the pairs of segments (AO_3, A_1O_1), (A_1O_1, A_2O_1), $(A_2O_1, A'O_3)$ are equal, parallel, and have opposite directions.

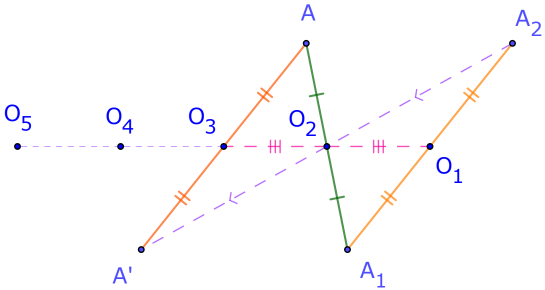
Thus if A is any point of $\mathcal{F}$, then the symmetric point A' obtained from A by a half turn about O_3 is also a point of $\mathcal{F}$, that is, O_3 is a center of symmetry of $\mathcal{F}$.



Similarly one shows that the point O_4 , obtained from O_2 by a half turn about O_3, and the point O_5, obtained from O_3 by a half turn about O_4, etc. are centers of symmetry.

Thus we see that if the figure $\mathcal{F}$ has two distinct centers of symmetry then it has infinitely many.

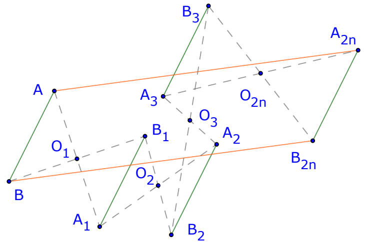
Now you can solve problem like this one **Prove that any circle has a single center!**



Example

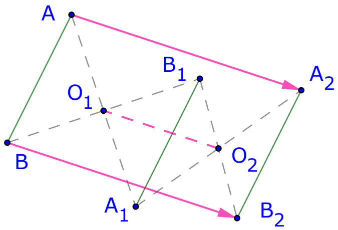
n is a positive integer. Let $O_1, O_2, \ldots, O_{2n}$ be points on the plane and AB is an arbitrary segment. Let segment A_1B_1 be obtained from AB by half turn about O_1, let A_2B_2 be obtained from A_1B_1 by half turn about O_2, ..., and finally let $A_{2n}B_{2n}$ be obtained from $A_{2n-1}B_{2n-1}$ by half turn about O_{2n} (see the figure for $n = 2$.)

Show that $AA_{2n} = BB_{2n}$.



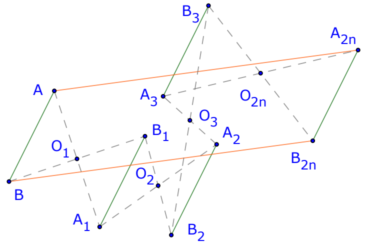
First, it is easy to see that **the sum of two half turns** around O_1 and O_2 is **a translation**:

$$AA_2 \parallel BB_2 \parallel O_1O_2 \quad \text{and} \quad AA_2 = BB_2 = 2O_1O_2.$$



Thus, for an **even** $2n$ **number of translations**, their sum is just **another translation**, hence

$$AA_{2n} = BB_{2n}.$$

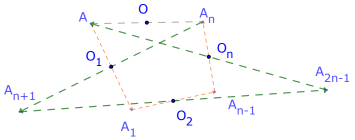


Is the conclusion still true if we have an **odd number of translations**? Why or why not?

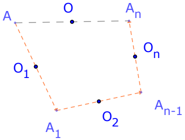
Example

n is a positive odd integer. Let $O_1, O_2, \ldots, O_n$ be points on the plane. Let an arbitrary point A be moved successively by half turns about $O_1, O_2, \ldots, O_n$ and then once again moved successively by half turns about the same points $O_1, O_2, \ldots, O_n$.

Show that the point A_{2n}, obtained as the result of these $2n$ half turns, coincides with the point A.



Since the **sum of an odd number of half turns** is **a half turn**, the point A_n, obtained from A by the n successive half turns about the points $O_1, O_2, \ldots, O_n$ can also be obtained from A by a single half turn about some point O.



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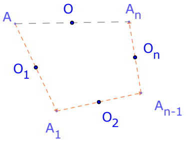
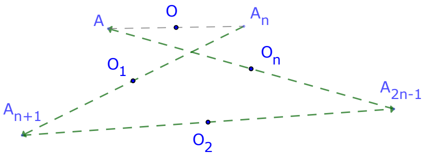
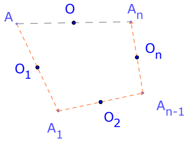
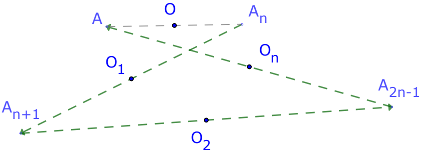




It is important to note that O **depends on** $O_1, O_2, \ldots, O_n$ **only** and not A.

Since the **sum of an odd number of half turns** is **a half turn**, the point A_n, obtained from A by the n successive half turns about the points $O_1, O_2, \ldots, O_n$ can also be obtained from A by a single half turn about some point O.





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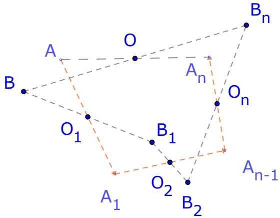
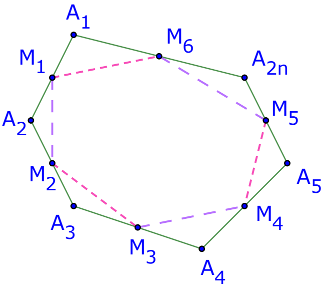


The point A_{2n} is obtained from A_n, by these same n half turns; therefore it can also be obtained

Example

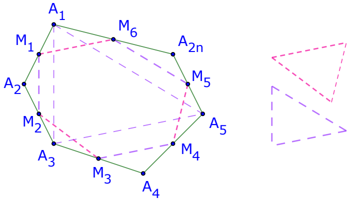
$A_1A_2\ldots A_{2n}$ is a $2n-$gon. $M_1, M_2, \ldots, M_{2n}$ are the midpoints of A_1A_2, A_2A_3, $\ldots$, $A_{2n}A_1$, respectively. Prove that there exists a $n-$gon whose sides are equal and parallel to the segments M_1M_2, M_3M_4, $\ldots$, $M_{2n-1}M_{2n}$ and there exists a $n-$gon whose sides are equal and parallel to the segments M_2M_3, $\ldots$, $M_{2n-2}M_{2n-1}$, $M_{2n}M_1$.



Note that by $2n$ half turns around $M_1, M_2, \ldots, M_{2n}$:

$$A_1 \to A_2 \to A_3 \to \cdots \to A_{2n} \to A_1.$$

The sum of two half turns around M_1 and M_2 is a translation $A_1 \to A_3$ with distance $A_1A_3 = 2M_1M_2$ similarly the sum of two half turns around M_3 and M_4 is a translation $A_3 \to A_5$ with distance $A_3A_4 = 2M_3M_4$ and so on.



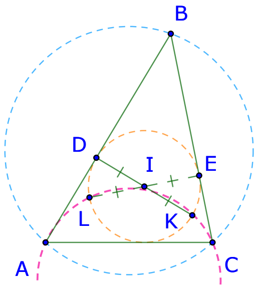
Furthermore after n translations: $A_1 \to A_1$, therefore the sum of them is an **identity transformation**, thus **the n translations** form a **close path** and therefore is an $n-$gon.

Hence, each of the sides is equal and parallel to the segments M_1M_2, M_3M_4, ..., $M_{2n-1}M_{2n}$.

Example

Given a triangle ABC satisfying $AB + BC = 3AC$. The incircle of triangle ABC has center I and touches the sides AB and BC at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I.

Prove that the points A, C, K, L lie on one circle.



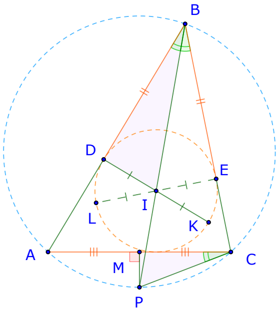
What do we have?

$$AB + BC = 3AC \Rightarrow \frac{1}{2}(AB + BC - CA) = CA.$$

This means that the tangent segment BD and BE is equal to CA!

Let P be the other intersection of B with (ABC). Let M be the midpoint of AC, then:

$$BD = AC = 2MC, \ \angle DBI = \angle MCP \Rightarrow \triangle DBI \sim \triangle MCP \text{ with similarity ratio 2.}$$

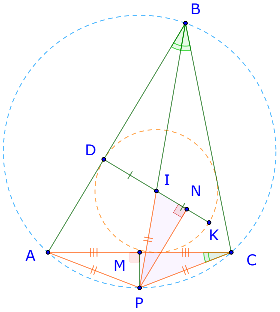


Let N be the foot of the altitude from P to IK. Now:

$$PI = PC\ (= PA)\ \text{(why?) and } \angle NIP = \angle DIB \Rightarrow \triangle PNI \cong \triangle CMP.$$

Thus,

$$\triangle DBI \sim \triangle NPI \text{ with similarity ratio 2.}$$



Now, by comparing corresponding segments: DI, NI:

$$DI = 2NI \Rightarrow KI = 2NI \Rightarrow \triangle IPK \text{ isosceles} \Rightarrow PK = PI, \text{ similarly } PL = PA.$$

Thus

$$PC = PK = PI = PL = PA.$$