Derivative II

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Problem (1a)**.** Let I be an open interval. Let f be a function defined on I. Let $a \in I$. Assume that f is continuous at a and that f is differentiable near a (except possibly at a).

We know that f has a vertical tangent line at a when $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \infty$ or $-\infty$.

Is each of the following claims true or false? If it is true, prove it directly.

If it is false, provide a counterexample and justify that it satisfies the required conditions.

(a) If $\lim_{x \to a} = \infty$ or $-\infty$, then f has a vertical tangent line at a.

Proof. By L'Hopital Rule,

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{f'(x)}{1} = \lim_{x \to a} f'(x) = \infty \text{ or } -\infty.$$

Therefore by definition, f has a vertical tangent line at a. The given statement is TRUE. $\square$

Problem (1b)**.** (b) If f has a vertical tangent line at a, then $\lim_{x\to a} f'(x) = \infty$ or $-\infty$.

Solution. We show a counterexample. Let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sqrt{|x|} + |x| \cos\left(\frac{1}{|x|}\right), \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

We prove that f satisfies the required conditions.

Claim — f is a continuous function.

Proof. If $x \neq 0$, since $|x|$, $\sqrt{|x|}$ and $\cos\left(\frac{1}{|x|}\right)$ are continuous functions so $f(x) = \sqrt{|x|} + |x|\cos\left(\frac{1}{|x|}\right)$ is continuous at every $x \neq 0$.

Now, we prove that f is continuous at 0.

For $\epsilon > 0$, since x is near 0 let's assume that $|x| < 1$. Let $\delta = \left(\frac{1}{2}\epsilon\right)^2$, then

$$|x - 0| < \delta \Rightarrow |x| < \left(\frac{1}{2}\epsilon\right)^2$$
$$|f(x) - f(0)| = \sqrt{|x|} + |x| \left|\cos\left(\frac{1}{|x|}\right)\right| \leq \sqrt{|x|} + |x| < 2\sqrt{|x|} < \epsilon.$$

Therefore f is a continuous function. ∎

Claim — f is differentiable at x where $x \neq 0$.

Proof. For $x > 0$, f is differentiable, since x, $\sqrt{x}$, and $\cos\left(\frac{1}{x}\right)$ are differentiable, so is f.

Since $f(-x) = f(x)$, so f is also differentiable for $x < 0$.

Thus f is differentiable at every $x \neq 0$. ∎

Now we prove that the limit $\lim_{x\to 0} f'(x)$ does not exist.

Note that for $x \neq 0$,

$$f(x) = \sqrt{x} + x\cos\left(\frac{1}{x}\right) \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{x}\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

Consider two sequences (a_n) and (bn), where $a_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$, $b_n = \dfrac{1}{\frac{3\pi}{2} + 2n\pi}$, $\forall n \in \mathbb{Z}^+$. Then

$$\sin\left(\frac{1}{a_n}\right) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1,\ \cos\left(\frac{1}{a_n}\right) = \cos\left(\frac{\pi}{2} + 2n\pi\right) = 0$$
$$\sin\left(\frac{1}{b_n}\right) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = -1,\ \cos\left(\frac{1}{b_n}\right) = \cos\left(\frac{3\pi}{2} + 2n\pi\right) = 0$$

Therefore

$$f'(a_n) = \frac{1}{2\sqrt{a_n}} + \frac{1}{a_n} > \frac{1}{a_n},\ \lim_{n\to\infty} \frac{1}{a_n}, = \infty \Rightarrow \lim_{n\to\infty} f'(a_n) = \infty$$
$$f'(b_n) = \frac{1}{2\sqrt{b_n}} - \frac{1}{b_n} < 1 - \frac{1}{\sqrt{b_n}},\ \lim_{n\to\infty} -\frac{1}{b_n}, = -\infty \Rightarrow \lim_{n\to\infty} f'(b_n) = -\infty$$

Hence, $\lim_{n\to\infty} f'(a_n) \neq \lim_{n\to\infty} f'(b_n) \Rightarrow \nexists \lim_{x\to 0} f'(x)$. The given statement is FALSE. □

Problem (2)**.** Let f be a function with domain (a, b). Assume f is (strictly) increasing and bounded on (a, b). Let S be the supremum of f, i.e. $S = \sup f(x) : x \in (a, b)$. Prove that

$$\lim_{x \to b^-} f(x) = S$$

Notes: You will need to use the definition of supremum of a function and the definition of $\lim_{x \to b^-} f(x)$. Do not make any unwarranted assumptions about the function f; for example, do not assume that f is continuous or that $\lim_{x \to b^-} f(x)$ exists.

Proof. Let $epsilon > 0$. We show that there exists $\delta > 0$, such that

$$\forall x : \; b - \delta < x < b \Rightarrow |f(x) - S| < \epsilon.$$

Note that,

$$|f(x) - S| < \epsilon \Leftrightarrow S - \epsilon < f(x) < S + \epsilon.$$

Since S is an upper bound for f on (a, b) therefore:

$$f(x) \leq S < S + \epsilon \quad (*)$$

S is the supremum of f, thus $S - \epsilon < S$ is not an upper bound for f on (a, b), so $\exists x_0 \in (a, b) : \; f(x_0) > S - \epsilon$.

$$f \text{ increases on } (a, b) \Rightarrow \forall x \in (x_0, b) : f(x) > f(x_0) > S - \epsilon \quad (**)$$

Now, let choose $\delta = b - x_0 > 0$, then from (*) and (**)

$$\forall x : \; x_0 = b - \delta < x < b \Rightarrow S - \epsilon < f(x_0) < f(x) < S + \epsilon \Rightarrow |f(x) - S| < \epsilon.$$

Thus,

$$\lim_{x \to b^-} f(x) = S$$

□

Problem (3a)**.** Let f be a function defined on $[a, b]$. Assume $\forall \epsilon > 0,\ \exists \delta > 0$ such that $\forall x, y \in [a, b]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Prove that f is integrable on $[a, b]$.

To prove this statement, you need to prepare several pieces and then put all pieces together.

(a) First, you must prove the function is continuous on $[a, b]$, i.e. for any $k \in [a, b]$, f is continuous at $x = k$. Note that when $k = a$ or $k = b$, it means f is continuous from the right at $x = a$ and continuous from the left at $x = b$.

Hint: Recall the $\epsilon - \delta$ definition of the function f is continuous at $x = k$.

Proof. First, for $a < k < b$, let $\epsilon > 0$, by the given condition $\exists \delta > 0$

$$\forall x:\ |x - k| < \delta \Rightarrow |f(x) - f(k)| < \epsilon.$$

This means that $\lim_{x \to k} f(x) = f(k)$, or f is continuous at $x = k$.

Second, for $k = a$, again $\forall \epsilon > 0,\ \exists \delta > 0$

$$\forall x:\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

Rewriting

$$\forall x:\ a < x < a + \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

This means that $\lim_{x \to a^+} f(x) = f(a)$, or f is continuous from the right at $x = a$.

Third, similarly with $k = b$, $\forall \epsilon > 0,\ \exists \delta > 0$

$$\forall x:\ |x - b| < \delta \Rightarrow |f(x) - f(b)| < \epsilon$$

Rewriting

$$\forall x:\ b - \delta < x < b \Rightarrow |f(x) - f(b)| < \epsilon$$

This means that $\lim_{x \to b^-} f(x) = f(b)$, or f is continuous from the left at $x = b$.

Therefore f is continuous on $[a, b]$. $\square$

Problem (3b)**.** (b) Second, you must verify the function is bounded on [a, b].

Proof. Let $\epsilon > 0$, then $\exists \delta > 0$

$$\forall x: \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Case 1: If $\delta \geq b - a$, then

$$\forall x \in \left[a, \frac{a+b}{2}\right] \Rightarrow |x-a| \leq \left|\frac{1}{2}(b-a)\right| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \Rightarrow f(a) - \epsilon < f(x) < f(a) + \epsilon$$
$$\forall x \in \left[\frac{a+b}{2}, b\right] \Rightarrow |x-b| \leq \left|\frac{1}{2}(b-a)\right| < \delta \Rightarrow |f(x) - f(b)| < \epsilon \Rightarrow f(b) - \epsilon < f(x) < f(b) + \epsilon$$

Thus

$$\forall x \in [a, b]: \ \max(f(a), f(b)) - \epsilon < f(x) < \min(f(a), f(b)) + \epsilon.$$

In other words, f is bounded.

Case 2: If $\delta < b - a$, then let $n = \left\lfloor \frac{2(b-a)}{\delta} \right\rfloor + 1$ so

$$\frac{2(b-a)}{\delta} < n \leq \frac{2(b-a)}{\delta} + 1 \Rightarrow \frac{n-1}{2}\delta \leq b - a < \frac{n}{2}\delta \Rightarrow a + \frac{n-1}{2}\delta \leq b < a + \frac{n}{2}\delta.$$

Thus, let's divide the interval $[a, b]$ into n sub-intervals as follow:

$$[a, b] = \left[a, a + \frac{1}{2}\delta\right] \cup \left[a + \frac{1}{2}\delta, a + (2)\frac{1}{2}\delta\right] \cup \cdots \cup \left[a + (n-1)\frac{1}{2}\delta, b\right]$$

Now, let $a_1 = a, a_2 = a + \frac{1}{2}\delta, \ldots, a_{n-1} = a + (n-1)\frac{1}{2}\delta, a_n = b$, then for $1 \leq i \leq n-1$,

$$\forall x \in [a_i, a_{i+1}] \Rightarrow |x - a_| \leq \frac{1}{2}\delta < \delta \Rightarrow |f(x) - f(a_i)| < \epsilon \Rightarrow f(a_i) - \epsilon < f(x) < f(a_i) + \epsilon$$

Therefore

$$\forall x \in [a, b]: \ \max_{i \in \{1,2,\ldots,n\}} f(a_i) - \epsilon < f(x) < \min_{i \in \{1,2,\ldots,n\}} f(a_i) + \epsilon.$$

In other words, f is bounded. $\square$

Problem (3c)**.** (c) Third, you establish a sufficient condition for integrability. Prove:

Let f be a bounded function on $[a, b]$.

IF $\forall \epsilon > 0$, there exists a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \epsilon$

THEN f is integrable on $[a, b]$.

Proof. Let $\epsilon > 0$, and choose a partition P that satisfies the condition. Then since

$$\begin{cases} L_P(f) \leq \underline{I_a}^b(f) = \sup\{L_P(f) \mid P \text{ is a partition of } [a, b]\} \\ \overline{I_a}^b(f) = \inf\{U_P(f) \mid P \text{ is a partition of } [a, b]\} \leq U_P(f) \end{cases}$$

$$\Rightarrow 0 \leq \overline{I_a}^b(f) - \underline{I_a}^b(f) \leq U_P(f) - L_P(f) < \epsilon \quad (*)$$

Since (*) hold for every $\epsilon > 0$, we must have $\overline{I_a}^b(f) - \underline{I_a}^b(f) = 0$, which means that f is integrable. □

Problem (3d)**.** (d) Finally, use the results of part(a), (b) and (c) to prove the statement.

Proof. Let f be a function defined on $[a, b]$. Assume $\forall \epsilon > 0,\ \exists \delta > 0$ such that $\forall x, y \in [a, b]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

By (a) f is continuous on $[a, b]$.

By (b) f is bounded on [a, b].

Now, let $\epsilon > 0$, then $\exists \delta$ such that

$$\forall x, y \in [a, b] : \ |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Let n be an integer such that $n > \frac{b-a}{\delta}$ Let chose a partition $P = \{I_1, I_2, \ldots, I_n\}$ of $[a, b]$ such that

$$I_k = [a + (k-1)\frac{b-a}{n}, a + (k)\frac{b-a}{n}],\ \forall k = 1, 2, \ldots, n \Rightarrow |I_k| = \frac{b-a}{n} < \delta.$$

Since f is bounded, then $\exists U_k = \sup\{f(x) | x \in I_k\},\ \exists L_k = \inf\{f(x) | x \in I_k\}$, and

$$\forall x, y \in I_k : \ |x - y| \leq |I_k| < \delta \Rightarrow U_k - L_k < \frac{\epsilon}{b - a}$$

Therefore, for $\epsilon > 0$, there exists a partition P of $[a, b]$ such that:

$$U_P(f) - L_P(f) = \sum_{k=1}^{n} (U_k - L_k) \cdot |I_k| < \frac{\epsilon}{b - a} \sum_{k=1}^{n} |I_k| = \epsilon.$$

By (c) f is integrable. □

Problem (4a)**.** In this question, we will prove a theorem from one of the videos:

Theorem 1. *If f, g are bounded, integrable functions on $[a, b]$, then so is $f + g$ and:*

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(a) Prove that $f + g$ is bounded on $[a, b]$.

Proof. f, g are bounded on $[a, b]$, then

$$\begin{cases} \sup_{x\in[a,b]}(f(x) + g(x)) \leq \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x) \\ \inf_{x\in[a,b]}(f(x) + g(x)) \geq \inf_{x\in[a,b]} f(x) + \inf_{x\in[a,b]} g(x) \end{cases}$$

Hence, $f + g$ is bounded on $[a, b]$. □

Problem (4b)**.** (b) Prove that for any partition P,

$$U_P(f + g) \leq U_P(f) + U_P(g) \quad \text{and} \quad L_P(f) + L_P(g) \leq L_P(f + g).$$

Proof. Let $P = \{I_1, I_2, \ldots, I_n\}$ be a partition of $[a, b]$, then

$$U_P(f + g) = \sum_{k=1}^{n} \sup_{x\in I_k}(f(x) + g(x)) \cdot |I_k| \leq \sum_{k=1}^{n} \sup_{x\in I_k} f(x) \cdot |I_k| + \sum_{k=1}^{n} \sup_{x\in I_k} g(x) \cdot |I_k| = U_P(f) + U_P(g)$$

Similarly

$$L_P(f + g) = \sum_{k=1}^{n} \inf_{x\in I_k}(f(x) + g(x)) \cdot |I_k| \geq \sum_{k=1}^{n} \inf_{x\in I_k} f(x) \cdot |I_k| + \sum_{k=1}^{n} \inf_{x\in I_k} g(x) \cdot |I_k| = L_P(f) + L_P(g)$$

□

Problem (4c)**.** (c) Prove that

$$\underline{I_a}^b(f) + \underline{I_a}^b(g) \le \underline{I_a}^b(f+g) \le \overline{I_a}^b(f+g) \le \overline{I_a}^b(f) + \overline{I_a}^b(g).$$

Hint: you may use the result of part (b) and the properties of the lower sums and upper sums.

Proof. Let $\epsilon > 0$, since $\overline{I_a}^b(f)$ is the infimum of the upper sums, there are partition Q, R such that

$$U_Q(f) \le \overline{I_a}^b(f) + \frac{1}{2}\epsilon \quad \text{and} \quad U_R(g) \le \overline{I_a}^b(g) + \frac{1}{2}\epsilon$$

Let $P = Q \cup R$, then by the properties of the upper sums,

$$U_P(f) < \overline{I_a}^b(f) + \frac{1}{2}\epsilon \quad \text{and} \quad U_P(g) < \overline{I_a}^b(g) + \frac{1}{2}\epsilon$$

Therefore

$$U_P(f) + U_P(g) < \overline{I_a}^b(f) + \overline{I_a}^b(g) + \epsilon.$$

Thus, by (b),

$$U_P(f+g) \le U_P(f) + U_P(g) < \overline{I_a}^b(f) + \overline{I_a}^b(g) + \epsilon.$$

And since

$$\overline{I_a}^b(f+g) \le U_P(f+g).$$

Hence,

$$\overline{I_a}^b(f+g) < \overline{I_a}^b(f) + \overline{I_a}^b(g) + \epsilon.$$

Since the inequality holds for arbitrary $\epsilon > 0$, we must have

$$\overline{I_a}^b(f+g) \le \overline{I_a}^b(f) + \overline{I_a}^b(g) \quad (1)$$

Similarly

$$\underline{I_a}^b(f) + \underline{I_a}^b(g) \le \underline{I_a}^b(f+g) \quad (2)$$

By the properties of the lower sums and upper sums,

$$\underline{I_a}^b(f+g) \le \overline{I_a}^b(f+g) \quad (3)$$

From (1), (2), and (3),

$$\underline{I_a}^b(f) + \underline{I_a}^b(g) \le \underline{I_a}^b(f+g) \le \overline{I_a}^b(f+g) \le \overline{I_a}^b(f) + \overline{I_a}^b(g).$$

$\square$

Problem (4d)**.** (d) Conclude that $f+g$ is integrable on $[a,b]$ and that

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Proof. For integrable f and g on $[a,b]$,

$$\overline{I_a}^b(f) + \overline{I_a}^b(g) = \underline{I_a}^b(f) + \underline{I_a}^b(g).$$

From (c)

$$\begin{aligned} &\overline{I_a}^b(f+g) \leq \overline{I_a}^b(f) + \overline{I_a}^b(g) = \underline{I_a}^b(f) + \underline{I_a}^b(g) \leq \underline{I_a}^b(f+g) \\ &\Rightarrow \overline{I_a}^b(f+g) = \underline{I_a}^b(f+g). \end{aligned}$$

Thus $f+g$ is integrable and since:

$$\begin{aligned} &\int_a^b (f(x)+g(x))dx = \overline{I_a}^b(f+g) = \underline{I_a}^b(f+g) \\ &\int_a^b f(x)dx + \int_a^b g(x)dx = \overline{I_a}^b(f) + \overline{I_a}^b(g) = \underline{I_a}^b(f) + \underline{I_a}^b(g) \end{aligned}$$

Hence,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

□