Perfect squares are everywhere - Part 4

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This article is the fourth part of the series on expedition to find *Perfect Squares*.

Example (Example 16)
n is an arbitrary positive integer. Find positive integer m such that

$$\binom{m}{2} = 3\binom{n}{4}.$$

Solution. First we prove the claim,

Claim — The product of 4 consecutive positive integers plus one is always a perfect square.

Proof. Let n be an arbitrary positive integer

$$n(n-1)(n-2)(n-3)+1 = n(n-3)\cdot(n-1)(n-2)+1 = (n^2-3n)(n^2-3n+2)+1 = \left((n^2-3n)+1\right)^2.$$

■

Now, by the given condition, then by the claim:

$$\binom{m}{2} = 3\binom{n}{4} \Leftrightarrow \frac{m(m-1)}{2} = \frac{3n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$
$$\Leftrightarrow 4m(m-1) = n(n-1)(n-2)(n-3) = \left((n^2-3n)+1\right)^2 - 1$$
$$\Leftrightarrow (2m-1)^2 = \left((n^2-3n)+1\right)^2 \Leftrightarrow m = \frac{1}{2}(n^2-3n+2) = \frac{(n-1)(n-2)}{2} = \boxed{\binom{n-1}{2}.}$$

□

Example (Example 17)

Prove that if the sum $x+y$ can be written as sum of a perfect square and thrice of another perfect square, then the sum x^3+y^3 can also be written as sum of a perfect square and thrice of another perfect square. In other words, if there exist a, b integers such that $x+y=a^2+3b^2$, then there exist c, d integers, such that

$$x^3+y^3=c^2+3d^2.$$

Solution. First we prove the claim,

Claim — Prove that if x and y are sum of a perfect square and thrice of another perfect square, then xy is also a sum of a perfect square and thrice of another perfect square. In other words, if a, b, c, d integers such that $x=a^2+3b^2$, $y=c^2+2=3d^2$ then $xy=(a^2+3b^2)(c^2+3d^2)$ can be written as a sum of a perfect square and thrice of another perfect square.

Proof. It is easy to verify that $(a^2+3b^2)(c^2+3d^2)=(ac+2=3bd)^2+3(ac-bd)^2$. ■

Now, let $x+y=a^2+3b^2$, we have $x^3+y^3=(x+y)(x^2-xy+y^2)$, where:

$$x^2-xy+y^2 \begin{cases} \left(\frac{x+y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2, & \text{if } x, y \text{ have the same parity} \\ \left(\frac{x}{2}-y\right)^2+3\left(\frac{y}{2}\right)^2, & \text{if } x, y \text{ have different parity} \end{cases}$$

Thus, x^2-xy+y^2, can be written as sum of a perfect square and thrice of another perfect square. Hence, x^3+y^3 can be written as sum of a perfect square and thrice of another perfect square. □

Example (Example 18)

Prove that for integers x, y such that

$$2x^2+x=3y^2+y,$$

then $2x+2y+1$ and $3x+3y+1$ are perfect squares.

Solution. First, note that:

$$(2x+2y+1)(x-y)=2x^2-2xy+2yx-2y^2+x-y=2x^2-2y^2+3y^2-2x^2=y^2$$
$$(3x+3y+1)(x-y)=3x^2-3xy+3yx-3y^2+x-y=3x^2-3y^2+3y^2-2x^2=x^2$$

Now, $(2x+2y+1)(3x+3y+1)(x-y)^2=(xy)^2$. If $xy=0$, then one of $x=0$ or $y=0$, which means from the given equality that then the other one of $x=0$ or $y=0$ is also 0 (why?) In any case both $2x+2y+1$ and $3x+3y+1$ are 1 and thus are perfect squares.

Now, suppose that $xy \neq 0$. Let $d \mid \gcd(2x+2y+1, 3x+3y+1)$, then $d \mid (3x+3y+1)-(2x+2y+1)=x+y$. Which means that $d \mid (2x+2y+1)-2(x+y)=1$. Therefore $\gcd(2x+2y+1, 3x+3y+1)=1$, and since $(2x+2y+1)(3x+3y+1)(x-y)^2=(xy)^2$, hence each of $2x+2y+1$ and $3x+3y+1$ is a perfect square. □

Example (Example 19)

Prove that the sum of squares of twelve positive consecutive integers is not divisible by the sum of them. In other words, if n non-negative integers, then

$$(n+1)+(n+2)+\cdots+(n+12) \nmid (n+1)^2+(n+2)^2+\cdots+(n+12)^2$$

Solution. [Solution 1] First, note that the sum of any three positive consecutive integers is divisible by 3, thus the sum of twelve positive consecutive integers is divisible by 3, too. On the other hand, the sum of squares of three consecutive integers when divided by 3 leaves a remainder of 2, the sum of twelve positive consecutive integers when divided by 3 has a remainder 2. The conclusion follows. □

Solution. [Solution 2] Note that

$$\begin{aligned}
&(n+1)+(n+2)+\cdots+(n+12)=12n+78 \\
&(n+1)^2+(n+2)^2+\cdots+(n+12)^2=12n^2+156n+650 \\
&\Rightarrow \frac{12n^2+156n+650}{12n+78}=\left(n+\frac{13}{2}\right)(12n+78)+143
\end{aligned}$$

The last expression shows that it cannot be an integer for any non-negative integer n. □

Example (Example 20)

Let N be a 16−digit number, where none of its digits can be $0, 1, 4$, or 9. Prove that there exist some of its consecutive digits such that their product is a perfect square.

Solution. Let $N=\overline{a_1a_2\ldots a_{16}}$. Consider the following sequence of 16 terms:

$$a_1,\ a_1a_2,\ ldots,\ a_1a_2\ldots a_{16} \quad (*)$$

Note that since each of $a_1, a_2, \ldots, a_{16}$ can only be $2, 3, 5, 6, 7, 8$, thus each of the term can only be factored as

$$2^{2\alpha_2}+\beta_2\cdot 3^{2\alpha_3}+\beta_3\cdot 5^{2\alpha_5}+\beta_5\cdot 7^{2\alpha_7}+\beta_7,$$

where α_i $(i=2,3,5,7)$ are non-negative integers, β_i $(i=2,3,5,7)$ are 0 or 1.

Thus, the (*) sequence has a one-on-one correspondence map to a sequence of four-digit binaries $\beta_2\beta_3\beta_5\beta_7$. If there is a 0000 binary, then it is the term in (*) which is the desired sequence of consecutive digits, whose product is a perfect square. If there is no such binary, then among 16 terms, two should have the same four-digit binaries $\beta_2\beta_3\beta_5\beta_7$. Let assume that they are $a_1a_2\ldots a_i$ and $a_1a_2\ldots a_j$, where $i<j$, then

$$\frac{a_1a_2\ldots a_j}{a_1a_2\ldots a_i}=a_{i+1}a_{i+2}\ldots a_j$$

has a 0000 binary representation of its prime factorization. This product is a perfect square.

Note that 16 is the best possible lower limit. For 15, the following number 232523272325232 is a counter example. □