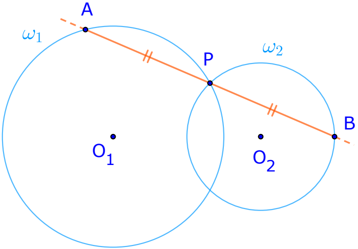
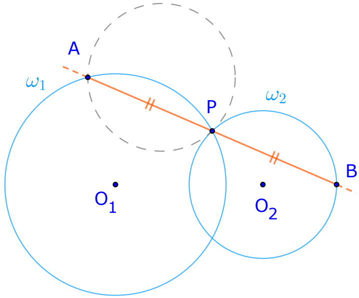
Example

P is an intersection point of circles ω_1 and ω_2. Construct a line through P intersecting ω_1 and ω_2 at A and B, respectively, such that $AP = PB$.

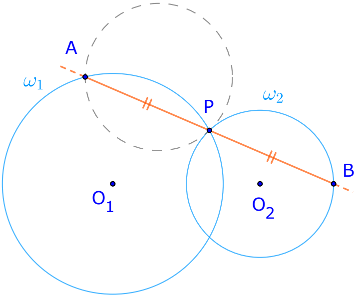


Let **rotate** ω_2 **half turn** (180°) or **reflect** ω_2 **over point** P. Let A be the other intersection of ω_1 and the image of ω_1 (the dotted circle); and B be the intersection of AP with ω_2.



Then A, P, B are collinear (why?) and A is on the circumference of the image of ω_2, thus A is the image of B: $B \to A$, thus $AP = BP$.

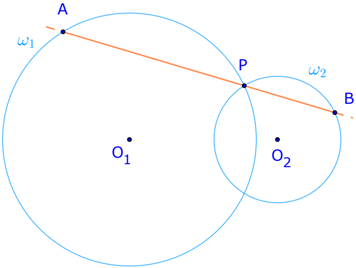
How many solutions?



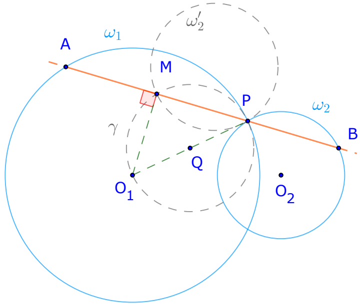
1. If $|\omega_1 \cup \omega_2| = 2$, then we have 1 solution.
2. If $|\omega_1 \cup \omega_2| = 1$, then we have no solution (why?)
3. If $|\omega_1 \cup \omega_2| = 0$, and the two radii are the same then we have infinitely many solutions otherwise no solution (why?).

Example

P is an intersection point of circles ω_1 and ω_2. Construct a line through P intersecting ω_1 and ω_2 at A and B, respectively, such that $AP = 2PB$.



The idea is **if** M **is the midpoint of** AP**, then** $\angle OMP = 90^\circ$ **and** $MP = PB$. Thus M is the intersection of ω_2', the image of ω_2 and the circle γ diameter O_1P.

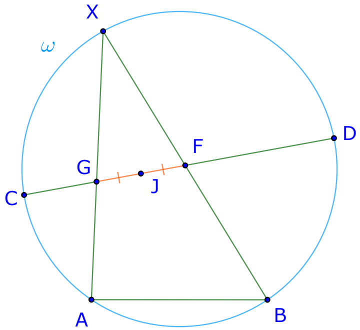


Thus we **rotate** ω_2 **half turn over point** P. Then we draw the circle γ diameter O_1P. Their intersection is M. Line through MP intersects ω_1 and ω_2 at A and B respectively.

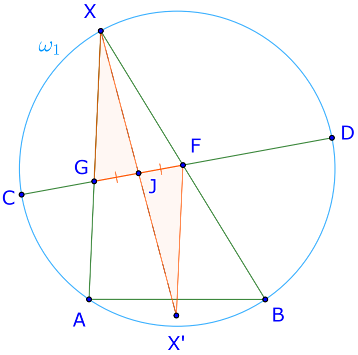
$$AM \overset{OM\perp MP}{=} MP \overset{B\to M}{=} PB \Rightarrow AP = 2PB.$$

Example

AB and CD are chords of circle ω. J is a point on CD. Find point X on the circumference of ω such that $JG = GF$, where G and F are intersections of CD with XA and XB, respectively.

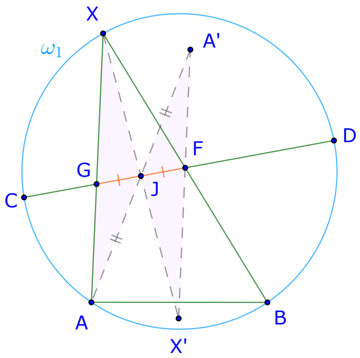


The condition $GJ = JF$ give us the idea to **rotate** X **half turn over point** I to X'.



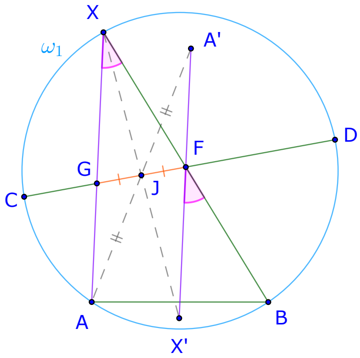
Congruent triangles $\triangle XGJ$ and $\triangle XFJ$ shows that $\angle XGJ = \angle JFX$, thus $FX' \parallel XA$. Furthermore $\angle X'FB = \angle AXB$.

We **rotate** A **half turn over point** I to A'.

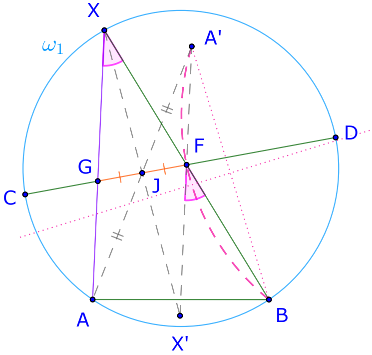


Therefore, $AXA'X'$ is a parallelogram.

$A'X' \parallel XA$ thus X, F, A' are collinear.



$\angle A'FB = 180^\circ - \angle F'XB = 180^\circ - \angle AXB = 180^\circ - \frac{1}{2}\widehat{AB}.$

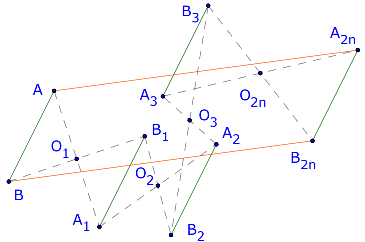


Hence, we first construct A', then F is the intersection the arc $\widehat{A'B}$ with measure $180^\circ - \frac{1}{2}\widehat{AB}$ (how to construct an arc knowing the measure of the angle subtending it?) with the chord CD. Finally X is the intersection of BF with ω.

Example

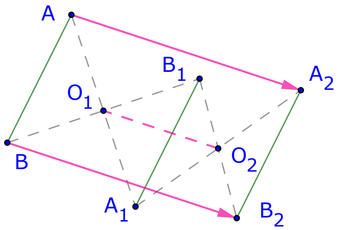
n is a positive integer. Let $O_1, O_2, \ldots, O_{2n}$ be points on the plane and AB is an arbitrary segment. Let segment A_1B_1 be obtained from AB by half turn about O_1, let A_2B_2 be obtained from A_1B_1 by half turn about O_2, ..., and finally let $A_{2n}B_{2n}$ be obtained from $A_{2n-1}B_{2n-1}$ by half turn about O_{2n} (see the figure for $n = 2$.)

Show that $AA_{2n} = BB_{2n}$.



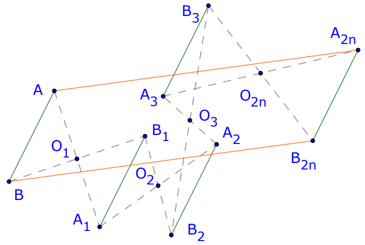
First, it is easy to see that **the sum of two half turns** around O_1 and O_2 is **a translation**:

$$AA_2 \parallel BB_2 \parallel O_1O_2 \quad \text{and} \quad AA_2 = BB_2 = 2O_1O_2.$$



Thus, for an **even $2n$ number of translations, their sum is just another translation**, hence

$$AA_{2n} = BB_{2n}.$$

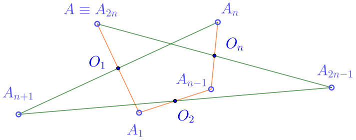


Is the conclusion still true if we have an **odd number of translations**? Why or why not?

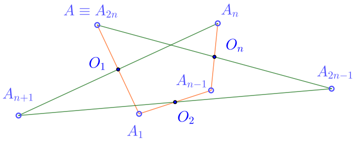
Example

n is a positive odd integer. Let $O_1, O_2, \ldots, O_n$ be points on the plane. Let an arbitrary point A be moved successively by half turns about $O_1, O_2, \ldots, O_n$ and then once again moved successively by half turns about the same points $O_1, O_2, \ldots, O_n$.

Show that the point A_{2n}, obtained as the result of these $2n$ half turns, coincides with the point A.



Since the **sum of an odd number of half turns** is **a half turn**, the point A_n, obtained from A by the n successive half turns about the points $O_1, O_2, \dots, O_n$ can also be obtained from A by a single half turn about some point O.



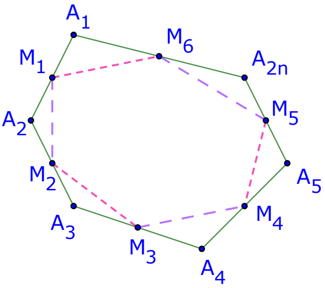
It is important to note that O **depends on** $O_1, O_2, \dots, O_n$ **only** and not A.

The point A_{2n} is obtained from A_n, by these same n half turns; therefore it can also be obtained from A_n, by the single half turn about the point O. But this means that A_{2n}, coincides with A, because of the two half turns around the same point O.

Is the conclusion still true if we have n as **even number**? Why or why not?

Example

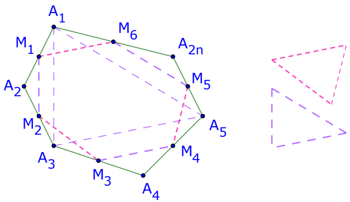
$A_1A_2\dots A_{2n}$ is a $2n-$gon. $M_1, M_2, \dots, M_{2n}$ are the midpoints of A_1A_2, A_2A_3, $\dots$, $A_{2n}A_1$, respectively. Prove that there exists a $n-$gon whose sides are equal and parallel to the segments M_1M_2, M_3M_4, $\dots$, $M_{2n-1}M_{2n}$ and there exists a $n-$gon whose sides are equal and parallel to the segments M_2M_3, $\dots$, $M_{2n-2}M_{2n-1}$, $M_{2n}M_1$.



Note that by $2n$ half turns around $M_1, M_2, \ldots, M_{2n}$:

$$A_1 \to A_2 \to A_3 \to \cdots \to A_{2n} \to A_1.$$

The sum of two half turns around M_1 and M_2 is a translation $A_1 \to A_3$ with distance $A_1A_3 = 2M_1M_2$ similarly the sum of two half turns around M_3 and M_4 is a translation $A_3 \to A_5$ with distance $A_3A_4 = 2M_3M_4$ and so on.

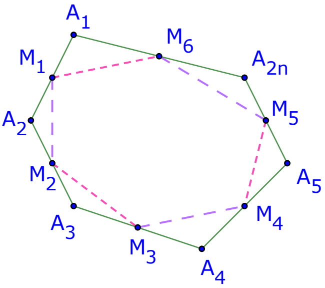


Furthermore after n translations: $A_1 \to A_1$, therefore the sum of them is an **identity transformation**, thus the n translations form a **close path** and therefore is an $n-$gon.

Hence, each of the sides is equal and parallel to the segments M_1M_2, M_3M_4, ..., $M_{2n-1}M_{2n}$.

Example

$A_1A_2\dots A_{2n}$ is a $2n-$gon. $M_1, M_2, \dots, M_{2n}$ are the midpoints of A_1A_2, A_2A_3, $\dots$, $A_{2n}A_1$, respectively. Prove that there exists a $n-$gon whose sides are equal and parallel to the segments M_1M_2, M_3M_4, $\dots$, $M_{2n-1}M_{2n}$ and there exists a $n-$gon whose sides are equal and parallel to the segments M_2M_3, $\dots$, $M_{2n-2}M_{2n-1}$, $M_{2n}M_1$.



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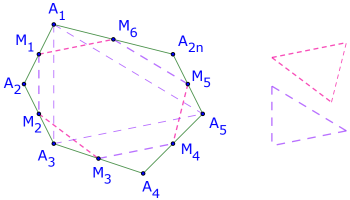


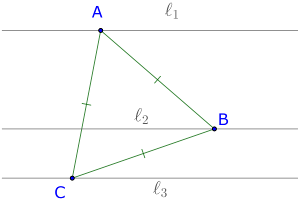
Furthermore after n translations: $A_1 \to A_1$, therefore the sum of them is an **identity transformation**, thus the n translations form a **close path** and therefore is an $n-$gon.

Hence, each of the sides is equal and parallel to the segments M_1M_2, M_3M_4, ..., $M_{2n-1}M_{2n}$.

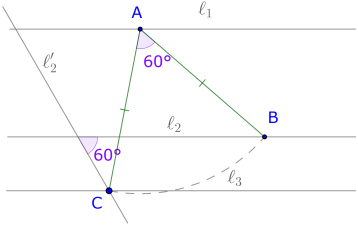
Example

Three parallel lines ℓ_1, ℓ_2, and ℓ_3 are given. A is a point on the line ℓ_1.

How can we determine points B and C on ℓ_2 and ℓ_3, respectively, such that ABC is an equilateral triangle.

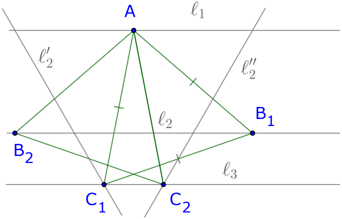


Assume that $\triangle ABC$ is equilateral, then **a rotation by** 60° **about** A **will carry** B **to** C.



That rotation also carries ℓ_2 (containing B) to ℓ_2'. The intersection of ℓ_2' and ℓ_3 is C.

Now we know how to do it. Rotate ℓ_2 by 60° about A to obtain ℓ_2'. The intersection of ℓ_2' with ℓ_3 is the position for C. B can be constructed easily as the intersection of circle centred at A radius AC.

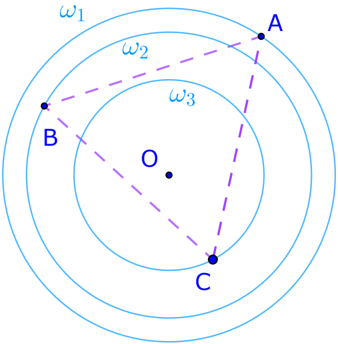


Note that there are two different solutions (why?)

Example

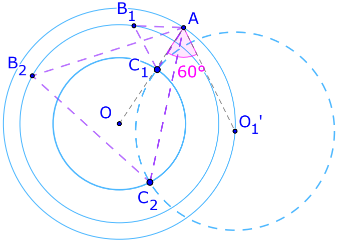
Three concentric circles ω_1, ω_2, and ω_3 are given. A is a point on the line ω_1.

How can we determine points B and C on ω_2 and ω_3, respectively, such that ABC is an equilateral triangle.



Pretty much the same as in the solution for the previous example.

Rotate ω_2 by 60° about A to obtain ω_2'. The intersection of ω_2' with ω_3 is the position for C. B can be constructed easily as the intersection of circle centred at A radius AC.

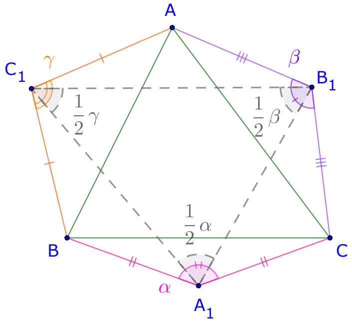


Note that there are **at most four** different solutions (why?).

Example

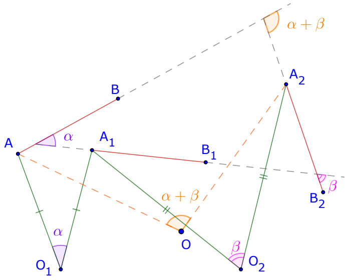
On the sides of an arbitrary triangle ABC, exterior to it, construct isosceles triangles BCA_1 ACB_1, CAB_1 with angles at the vertices A_1, B_1, and C_1, respectively equal to α, β and γ.

Prove that if $\alpha + \beta + \gamma = 360°$, then the angles of the triangle $A_1B_1C_1$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\gamma$, that is, they do not depend on the shape of the triangle ABC.



Let's take a look at a **sum of two rotations**:

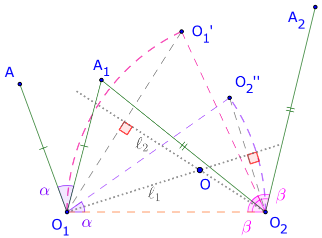
$$AB \overset{\text{rotate}(O_1,\alpha)}{\longrightarrow} A_1B_1 \overset{\text{rotate}(O_2,\beta)}{\longrightarrow} A_2B_2.$$



It is easy to see that the angle between A_2B_2 and AB is $\alpha + \beta$, thus it is a rotation by the angle $\alpha + \beta$, We need to determine the position of the center of rotation O.

Now, what happen with the centers O_1 and O_2:

$$O_1 \overset{\text{rotate}(O_1,\alpha)}{\rightarrow} O_1 \overset{\text{rotate}(O_2,\beta)}{\rightarrow} O_1' \quad \text{and} \quad O_2'' \overset{\text{rotate}(O_1,\alpha)}{\rightarrow} O_2 \overset{\text{rotate}(O_2,\beta)}{\rightarrow} O_2.$$

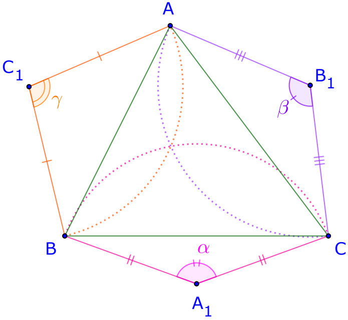


Therefore O is on both perpendicular bisectors of O_1O_1' and $O_2''O_2$.

Hence, $\angle OO_1O_2 = \frac{1}{2}\alpha$, $\angle OO_2O_1 = \frac{1}{2}\beta$.

First, point A is taken into itself by the sum of three rotations through the angles β, α, and γ $(\alpha + \beta + \gamma = 360^\circ)$ about the centers B_1, A_1, C_1:

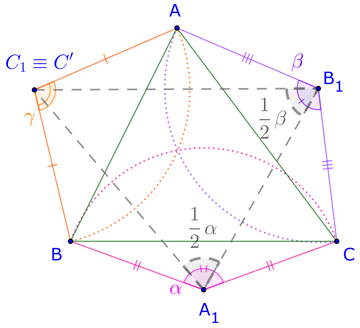
$$A \overset{\text{rotate}(B_1,\beta)}{\rightarrow} C \overset{\text{rotate}(A_1,\alpha)}{\rightarrow} B \overset{\text{rotate}(C_1,\gamma)}{\rightarrow} A.$$



Thus, the **sum of the these rotations** is the **identity transformation**.

Let C' be the center of the rotation equivalent to the sum of the rotations about B_1 and A_1. Then it is the rotation through $\alpha + \beta = 360^\circ - \gamma$ brings A to B.

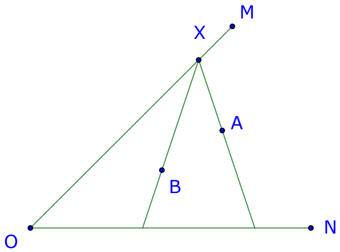
However, the rotation about C_1 through γ brings A to B in opposite direction. Since a rotation through an angle θ is the same as the rotation through an angle $360^\circ - \theta$ about the same center in the opposite direction, thus $C_1 \equiv C'$.



Therefore $\angle C_1A_1B_1 = \frac{1}{2}\alpha, \angle C_1B_1A_1 = \frac{1}{2}\beta$, and similarly $\angle B_1C_1A_1 = \frac{1}{2}\gamma$.

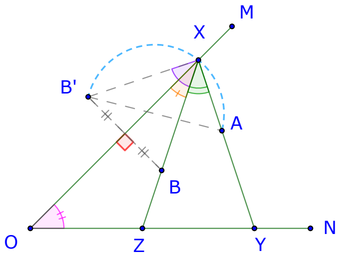
Example

$\angle MON$ is given, together with two points A and B. Find a point X on the side OM such that the triangle XYZ is isosceles: $XY = XZ$, where Y and Z are on the points of intersection of XA and XB with ON.



Let B' be the image of B over OM, then:

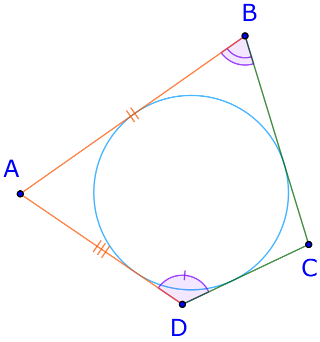
$\angle B'XA = \angle B'XB + \angle YXZ$, $\angle B'XB = 2\angle OXZ = 2(\angle XZY - \angle MON) \Rightarrow \angle B'XA = 180^\circ - \angle MON$.



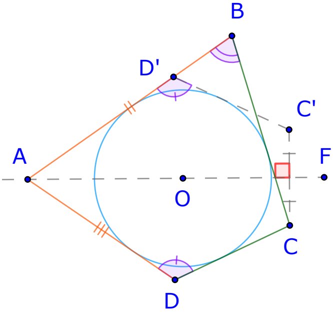
Thus, X is the intersection of OM with the arc constructed on the chord AB', that subtends an angle equal to $180^\circ - \angle MON$.

Example

Construct a quadrilateral $ABCD$ in which a circle can be inscribed, given the lengths of two adjacent sides AB and AD and the angles at the vertices B and D.



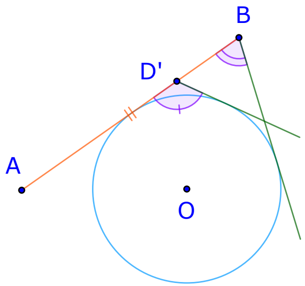
The key idea here is that the reflection of CD over the line through A and the center of the circle is a tangent to the circle!



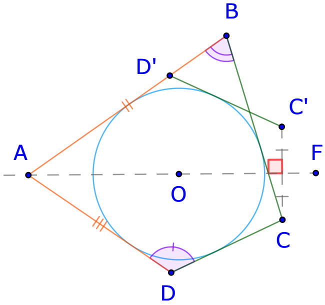
First, we start the construction by point A then segment AB, then segment $AD_1 = AD$ where D is on the line AB, same side as B in respect to A.

Second, because $\angle B$ and $\angle D_1 = \angle D$ are known, thus we can construct rays going from B and D_1.

Finally, we construct a circle tangents to all three lines.



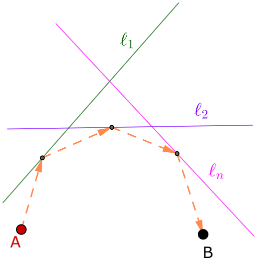
The rest is simple, we reflect D' and its ray over the line AO where O is the center of the circle. The reflected ray will intersect the ray from B at C. We are done.



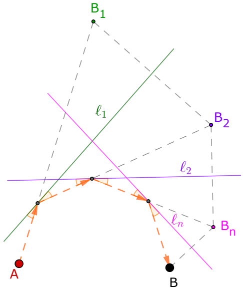
Example

A billiard ball bounces off a side of a billiard table in such a manner that the two lines along which it moves before and after hitting the sides are equally inclined to the side.

Suppose a billiard table were bordered by n lines $\ell_1, \ell_2, \ldots, \ell_n$. Let A and B be two given points on the billiard table. In what direction should one hit a ball placed at A so that it will bounce consecutively off the lines $\ell_1, \ell_2, \ldots, \ell_n$, and then pass through the point B (see the diagram below, where $n = 3$)?



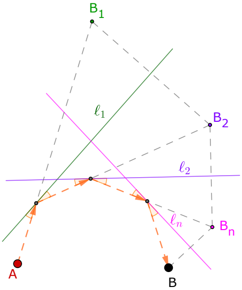
Assume that the problem has been solved, that is, that points $X_1, X_2, \ldots X_n$ have been found on the lines $\ell_1, \ell_2, \ldots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



It is easy to see that the point X_n, is the point of intersection of the line ℓ_n with the line $X_{n-1}B_n$, where B_n, is the image of B in ℓ_n, that is, the points B_n,X_n, X_{n-1} lie on a line.

But then the point X_{n-1} is the point of intersection of the line ℓ_{n-1} with the $X_{n-2}B_{n-1}$, where B_{n-1}, is the image of B_n in ℓ_{n-1} and so on.

Assume that the problem has been solved, that is, that points $X_1, X_2, \ldots X_n$ have been found on the lines $\ell_1, \ell_2, \ldots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



Here's the construction: Reflect the point B in l_n, obtaining the point B_n; next reflect B_n in l_{n-1} to obtain B_{n-1}, and so forth, until the image B_1 of the point B_2, in line ℓ_1 is obtained.

The point X_1, that determines the direction in which the billiard ball at A must be hit, is obtained as the point of intersection of the line ℓ_1 with the line AB_1. It is then easy to find the points $X_2, \ldots X_n$ with the aid of the points $B_2, \ldots B_n$ and X_1.