Eight ways to prove - Part 2

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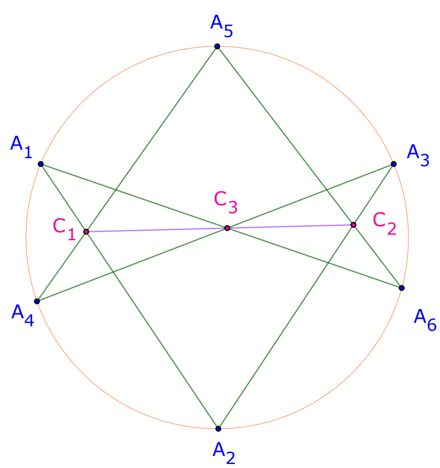
This article is the second part of the series on explore different topics of geometry to find a way for the same problem.

Example (IMO 2014, Problem 4)

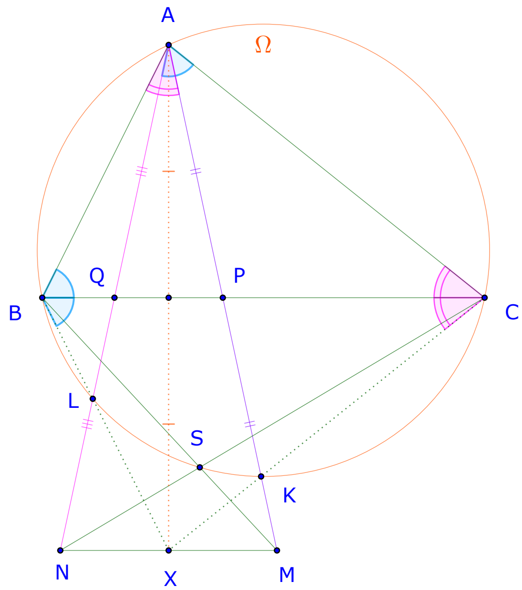
Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA, \angle CAQ = \angle ABC$. Let M and N be points on lines AP and AQ, respectively, such that P and Q are midpoints of AM and AN, respectively. Prove that the intersection S of BM and CN is on the circumference of $\triangle ABC$.

Theorem (Pascal's Theorem)

Let $\mathcal{P} = A_1A_2A_3A_4A_5A_6$ be a hexagon, $C_1 = A_1A_2 \cap A_4A_5$, $C_2 = A_2A_3 \cap A_5A_6$, $C_3 = A_3A_4 \cap A_6A_1$. Then $\mathcal{P}$ is a cyclic hexagon (which is circumscribed by a circle) if and only if C_1, C_2, C_3 are collinear.

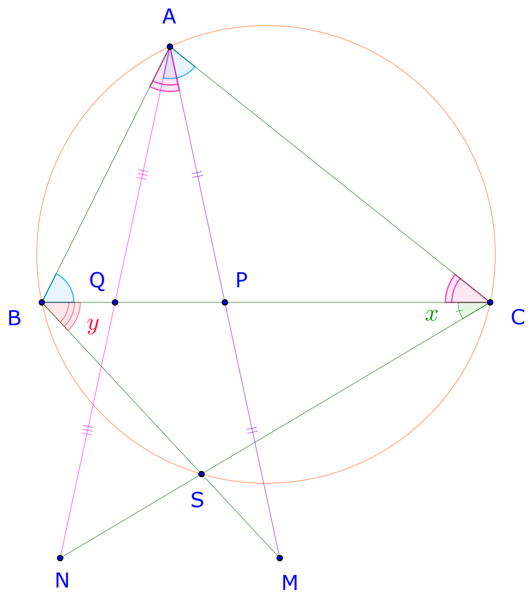


5^{th} ***proof based on Pascal's Theorem.*** Let Ω denote the circle. Let points $L = \Omega \cap AN$, $K = \Omega \cap AM$. Since $\angle LBC = \angle LAC = \angle CBA$ and $\angle KCB = \angle KAB = \angle BCA$, thus $X = BL \cap CK$ is an image of A by the reflection over the line BC.



Since Q and P are midpoints of AN and AM, so X must be on NM. Thus if $\mathcal{P} = A_1A_2A_3A_4A_5A_6$ denotes $ALBSCK$, then since $N = AL \cap SC$, $X = LB \cap CK$, and $M = BS \cap KA$ are collinear thus $ALBSCK$ is cyclic. Therefore S is on the circle (ABC). $\square$

6^{th} ***proof based on the Law of Sines***. First $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.



By the Law of Sines for $\triangle QCN$, and note that $\angle QNC = \angle AQC - x = \angle A - x$,

$$\frac{QN}{QC} = \frac{\sin \angle QCN}{\sin \angle QNC} = \frac{\sin x}{\sin(\angle A - x)},\ QN = QA,\ \frac{QA}{QC} = \frac{AB}{AC} \Rightarrow \frac{AB}{AC} = \frac{QA}{QC} = \frac{QN}{QC} = \frac{\sin x}{\sin(\angle A - x)} \quad (1)$$

Similarly for $\triangle PBM$, $\dfrac{AC}{AB} = \dfrac{\sin y}{\sin(\angle A - y)}$ (2).

From (1) and (2), and note that $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$,

$$1 = \frac{AB}{AC}\cdot\frac{AC}{AB} = \frac{\sin x}{\sin(\angle A - x)}\cdot\frac{\sin y}{\sin(\angle A - y)} \Rightarrow \sin(\angle A - x)\sin(\angle A - y) = \sin x \sin y$$

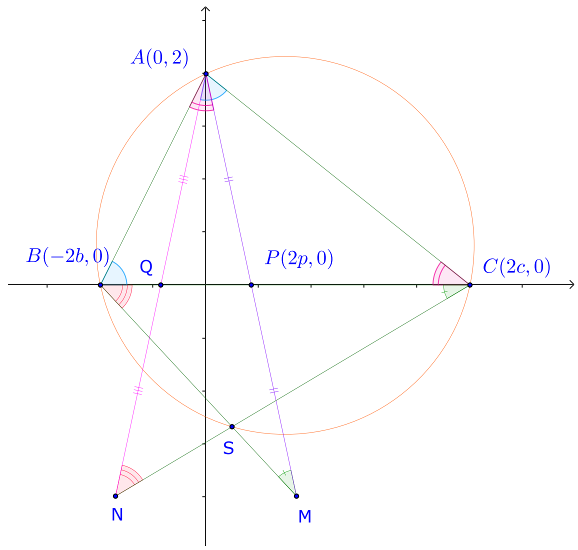
$$\Rightarrow \cos(y-x) - \cos(x+y-2\angle A) = \cos(x-y) - \cos(x+y) \Rightarrow \cos(2\angle A - (x+y)) = \cos(x+y) \quad (3)$$

Since $x < \angle A, y < \angle A$ and $\cos\alpha = \cos(-\alpha)$,

$$(3) \Rightarrow 2A - (x+y) = x+y \Rightarrow x+y = A \Rightarrow \angle BSC = 180^\circ - \angle A.$$

Therefore $ABSC$ is cyclic and S is on the circle (ABC). $\square$

7^th proof based on Analytical Geometry. Let $A(2,0)$, $B(-2b,0)$, $C(2c,0)$, $P(2p,0)$ $(0<b<p<c)$ be the coordinates. Let A, B, C be the measures of the $\angle A$, $\angle B$, $\angle C$, respectively;



$$\cot B = \frac{OB}{OA} = b, \cot C = c, \cot A = \cot\left(180^\circ - B - C\right) = -\cot\left(B + C\right) = \frac{1-bc}{b+c}$$
$$\frac{PO}{AO} = \cot A \Rightarrow p = \cot\angle APB = \frac{1-bc}{b+c}, \Rightarrow P\left(\frac{2(1-bc)}{b+c},0\right),\ Q\left(\frac{-2(1-bc)}{b+c},0\right),$$
$$P,\ Q \text{ are midpoints of } AM,\ AN \Rightarrow M\left(\frac{4(1-bc)}{b+c},2\right),\ N\left(\frac{-4(1-bc)}{b+c},-2\right).$$

The slope of line BM is $\dfrac{2-0}{\dfrac{4(1-bc)}{b+c}-(-2b)} = \dfrac{b+c}{b^2-bc+2}$. The equation of line BM is

$$y-0 = \left(\frac{b+c}{b^2-bc+2}\right)x - (-2b) \Rightarrow y = \left(\frac{b+c}{b^2-bc+2}\right)x + \frac{2b(b+c)}{b^2-bc+2}.$$

Let D be the circumcentre of (ABC). $D_x = \frac{1}{2}(B_x + C_x) = \dfrac{2c+(2b)}{2} = c-b$. Line AC is $y = -\dfrac{1}{c}x + 2$ the perpendicular bisector of AC has the slope c and is through $(c,1)$. Thus, its equation of is $y = cx + (1-c^2)$. Therefore $D_y = c(c-b) + (1-c^2) = 1-bc$. Furtheremore, the circumradius of (ABC) is:

$$R = \frac{AB\cdot BC\cdot CA}{4[ABC]} = \frac{2\sqrt{1+b^2}\cdot(2c-2b)\cdot 2\sqrt{1+4^2}}{4\cdot\frac{1}{2}(2)((2c-2b))} = \sqrt{(1+b^2)(1+c^2)}.$$

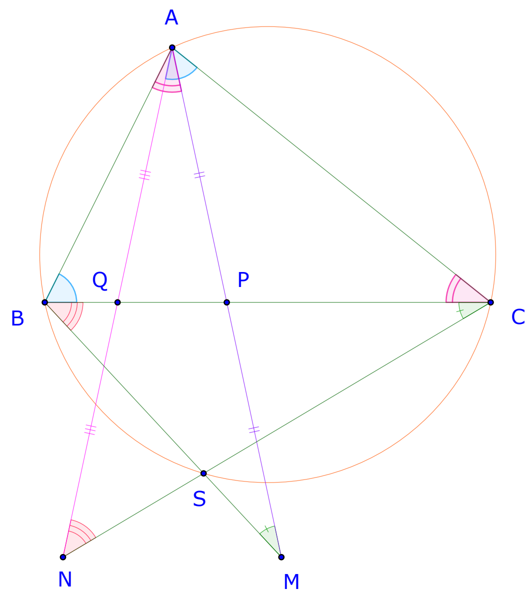
Thus, the equation of the circumcircle (ABC) is $[x-(c-b)]^2 + [y-(1-bc)]^2 = (1+b^2)(1+c^2)$. It is easy to verify that the intersection of BM with (ABC) is the same as the intersection of CN with (ABC), which is point S with coordinates symmetric in regards to b and c,

$$S\left(2\frac{(c-b)(2-bc)}{(c-b)^2+4}, -2\frac{(c+b)^2}{(c-b)^2+4}\right) \Rightarrow S = BM\cap CN \in (ABC)$$

□

8^{th} ***proof based on Complex Numbers.*** Let $X = a + ib$ complex number is represented by a point $X(a, b)$ on the complex plane. In short, we write X, and interpret it as both complex number and a number on the complex plane (of course depending on what context).

Note that for points X, Y on the complex plane then $X - Y$ is the *directed* distance of them (It is easy to derive by denote $X(x_1, x_2)$, $Y(y_1, y_2)$ then $(X - Y)(x_1 - x_2, y_1 - y_2)$.) Also note that $\overline{X}$ is the conjugate of X.



Now, $\triangle ABC \sim \triangle PBA$, ABC is anti-clockwise, while PBA is clockwise, thus they have different orientations, therefore

$$\frac{A-P}{B-P} = \overline{\left(\frac{C-A}{B-A}\right)} \quad (1)$$

Similarly

$$\frac{C-Q}{A-Q} = \overline{\left(\frac{C-A}{B-A}\right)} \quad (2)$$

But P and Q are midpoints of AM and AN, respectively, thus

$$(3) \quad \begin{cases} M - P = -(A - P) \\ N - Q = -(A - Q) \end{cases}$$

From (1), (2), and (3)

$$\frac{M-P}{B-P} = \frac{C-Q}{N-Q}, \text{ thus } \triangle MPB \sim \triangle CQN.$$

From here it is easy to see that $\angle BSC + \angle A = 180°$, thus $BM \cap CN \in (ABC)$. $\square$