The Extremal Principle

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In this article, we discuss the Extremal Principle and its applications. One of the simplest forms of the principle is as follow: "in a finite set of numbers, there is a number with minimal value, i.e. it is smaller than or equal to any other number in the set. Similarly there is a number with maximal value, i.e. it is larger than or equal to any other number in the set."

Proof by contradiction is an extremely useful tool when combining with the Extremal Principle, as you will see in below examples.

> **Example** (Dancing at a party)
>
> At a party no boy danced with all the girls, but each girl dances with at least one boy. Prove that there are two pairs of girl-boy (g_1, b_1) and (g_2, b_2) who danced with each other but g_1 did not dance with b_2 and g_2 did not dance with b_1.

Solution. Let b_1 be *the boy who danced with the maximum number of girls.* Then there is a girl g_2 who he did not danced with. For g_2 there is a boy b_2 that (g_2, b_2) danced together. Among the girls who danced with b_1 there is at least one g_1 who did not danced with b_2, otherwise b_2 danced with g_2 and all the girls that b_1 danced with, meaning b_2 danced with more girls than b_1, contradicting with the choice of b_1. $\square$

> **Example** (Infinity by contradiction)
>
> Ω is a set of points on the plane. Every point in Ω is a midpoint of two points in Ω. Show that Ω is infinite set.

Solution. Suppose that Ω is a finite set. According to the Extremal Principle, *there exists two points $A, B \in \Omega$, such that the distance AB is maximal.*

Now, since $B \in \Omega$, there exist two points $C, D \in \Omega$ so that B is the midpoint of CD.

[h]
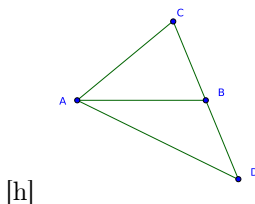
Since one of the angles $\angle ABC$, $\angle ABD$, says $\angle ABD$ is at least $90°$, thus in $\triangle ABD$, $AD > AB$. This contradicts the assumption that A, B are the two points in Ω, such that the distance AB is maximal.

Thus, there are no such two points A, B, so $\boxed{\Omega \text{ is infinite set.}}$ $\square$

> **Example** (How many olives did the knights eat?)
>
> At the dinner of King Anthony, several knights sits around a round table eating green olives. Minh, the Magician, made sure that each knight ate either twice as many olives or 10 olives less than his right neighbour. Is that possible that the knights could have eaten exactly 1001 olives?

Solution. Let assume that the knights have eaten exactly 1001 olives. Let choose the knight who *ate the smallest number of olives.* (If there are some of them, choose one.) His neighbour on the left, knight k, ate either 10 less or twice more. Since the knight we chose ate the smallest number of olives, then knight k ate twice as many. Therefore, knight k ate an even number of olives.

The neighbour on the left of knight k ate either twice as many olives or 10 olives less, hence he ate an even number of olives as well. Making the full circle, we'll end us with the first knight, who must have eaten an even number of olives as well.

Therefore, the total number of olives must be an even number. The number of olives eaten $\boxed{\text{cannot be 1001.}}$

$\square$

> **Example** (Chop the flies)
>
> 25 flies are resting on the outdoor table in the garden, waiting for lunch to be served. It is known that for any three of them, two are at a distance less than 20 cm; and there are at least a pair of flies that are further than 20 cm from each other.
>
> Minh's mother gave him a fly swatter, shown below, with a hoop of radius 20 cm, With a single strike he can swat the flies where the hoop landed. In *at least* how many strikes can he swat all of them? *Note that Minh is so fast that the flies do not have time for reaction during and between his lightning strikes.*

Solution. If no 2 flies are further than 20 cm from each other, Minh can strike them all in 1 strike by aiming the center of the swatter at any fly. But this is not the case, so let's assume there are 2 flies, A and B, that are more than 20 cm apart. Then, every other fly is either in a 20 cm radius of A or in a 20 cm radius of B. Out of the 23 remaining flies either at least 12 will be in the 20 cm radius of A or 12 will be in the 20 cm radius of B. Swatting that the A or B fly with the center of the swatter kills at least 13.

Thus, by $\boxed{2}$ strikes, he can swat them all.

$\square$