Eight ways to prove - Part 1

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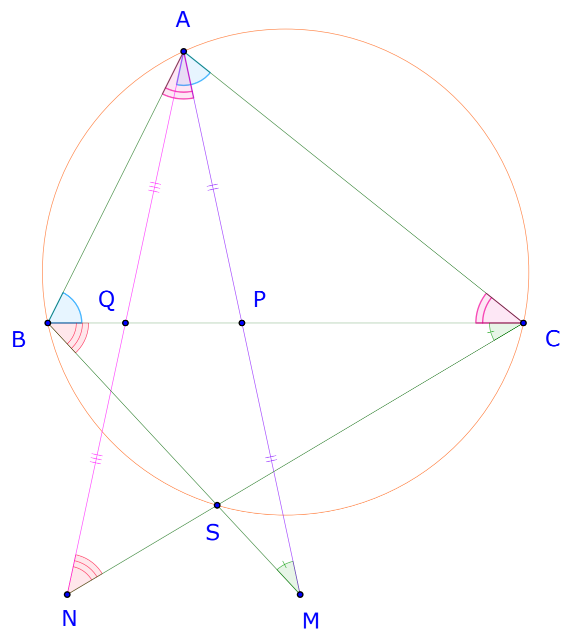
November 5, 2023

In this article, we explore different topics of geometry, each helps us to find a way for the same problem.

Example (IMO 2014, Problem 4)
Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA, \angle CAQ = \angle ABC$. Let M and N be points on lines AP and AQ, respectively, such that P and Q are midpoints of AM and AN, respectively. Prove that the intersection S of BM and CN is on the circumference of $\triangle ABC$.

1^{st} ***proof based on angle chasing and Cyclic Quadrilaterals.*** First, $\angle PAB = \angle BCA = \angle C$, and $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.



Thus, $\frac{PB}{PA} = \frac{QA}{QC}$, or $\frac{PB}{PM} = \frac{QN}{QC}$ (1).

Now, $\angle BPM = \angle PAB + \angle PAB = \angle B + \angle C = \angle QAC + \angle QCA = \angle NQC$ (2).

From (1) and (2) $\triangle BPM \sim \triangle NQC$, so $\angle SBC = \angle MBP,\ \angle SCB = \angle NCQ$.

Thus, $\triangle BSC \sim \triangle BPM$, or $\angle BSC = \angle BPM = \angle B + \angle C = 180° - \angle A$.

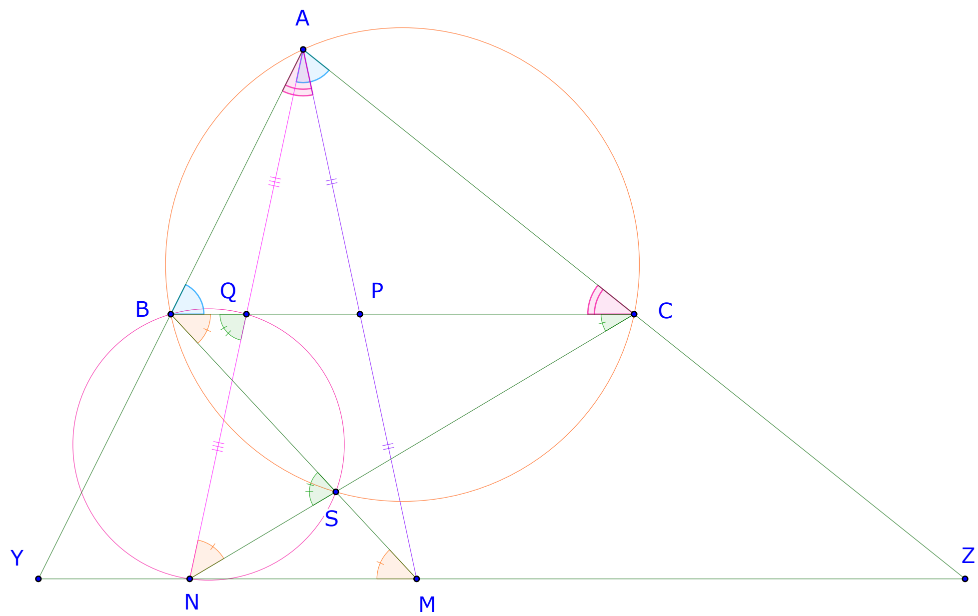
Therefore $ABSC$ is cyclic and S is on the circle (ABC). $\square$

2^{nd} ***proof based on similarity and Cyclic Quadrilaterals.*** First, $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.

Now, extend AB and AC to intersect line MN at Y and Z, respectively.

Since Q, P are median segment, thus $MN \parallel BC$. Therefore $\triangle AMY \sim \triangle ABP \sim \triangle CAQ \sim \triangle ZAN$.

B is the midpoint of AY in $\triangle AMY$, C is the midpoint of ZA in $\triangle ZAN$. By similarity $\angle BNY = \angle CNA$.

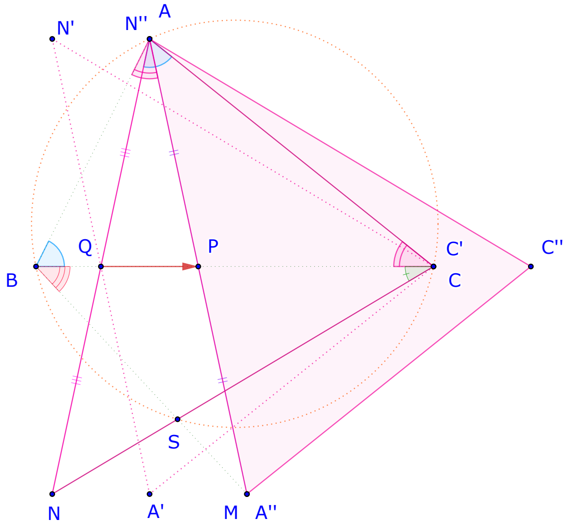


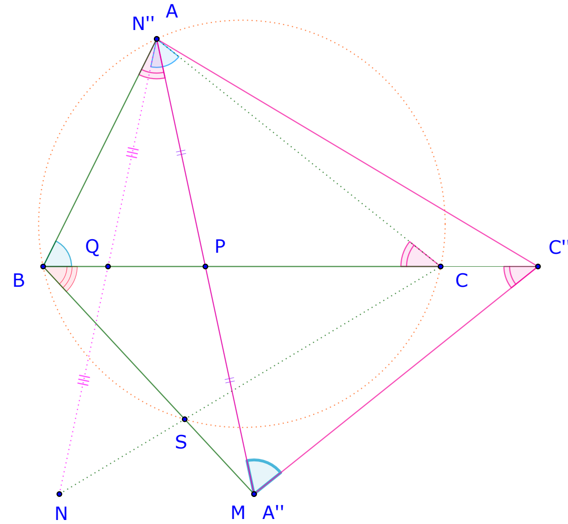
Therefore $\angle SBC = \angle BMN = \angle CNQ$, so $BQSN$ is cyclic, thus $\angle NSB = \angle NQB = \angle CQA = \angle A$.

Hence, $\angle BSC = 180^\circ - \angle A$, $ABSC$ is cyclic and S is on the circle (ABC). $\square$

3^{rd} ***proof based on rigid transformations.*** First, $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$,

$$\frac{AP}{AB} = \frac{AC}{BC}, \ \frac{AQ}{AC} = \frac{AB}{BC} \Rightarrow AP = AQ.$$





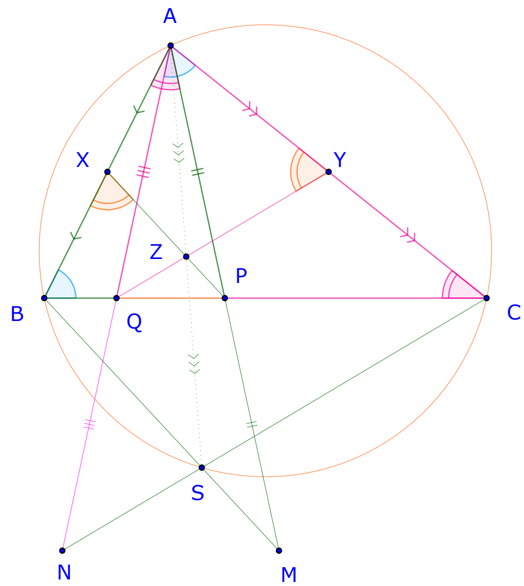
Thus, we can reflect the triangle ANC over the line BC to be $A'N'C'$, then translate it by the vector QP to be $A''N''C''$. Then segment $AM \equiv N''A''$.

P is still the midpoint of $AM \equiv N''A''$, and $\angle BPA = \angle CQA = 180° - \angle C''PA$. Thus C'', C, P, Q, B are collinear.

In quadrilateral $AC''A''B$, $\angle C''BA = \angle C''A''A$, thus it is cyclic, therefore $\angle A''BA + \angle A''C''A = 180°$.

Thus $\angle ABS + \angle ACS = 180°$. Therefore $ABSC$ is cyclic and S is on the circle (ABC). $\square$

4^{th} ***proof based on Homothety***. Let X and Y be the midpoint of AB and AC, respectively. Let Z be the intersection of PX and QY. $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC$, thus $\angle BXP = \angle AYQ$. Therefore $AXZY$ is cyclic.



The homothety $\mathcal{H}_{(A,2)}$ (center A and factor 2) sends X, Y, P, Q to B, C, M, N, respectively.

$S = MB \cap NC$ ($MB \cap NC$ denotes the intersection of MB and NC) thus it is the image of $PX \cap QY = Z$.

$AXZY$ is cyclic. $ABSC$ is similar to $AXZY$ and therefore cyclic, too. Hence, S is on circle (ABC). $\square$