# Counting in two ways

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In this article, we show a few example problems that can be solved when using the counting in two ways method.

Example (Sum of areas)

$ABCD$ is a parallelogram. E and F are points on AB and BC, respectively. Lines AF, CE, DE, and DF dissect the parallelogram into 8 regions, where some of them have known areas, as shown below in the diagram.

Find the sum of the areas $a + b + c$.
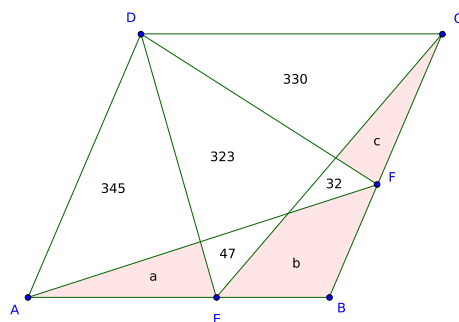
Solution. From the properties of the parallelogram $ABCD$, the following triangle pairs, with the same base and height, have the same area

$$\left.\begin{array}{l}[AFB] = [DFB]\\ [CEB] = [DEB]\end{array}\right\} \Rightarrow [AFB] + [CEB] = [DFB] + [DEB] = [DEBF]$$

Therefore

$$(a + 47 + b) + (c + 32 + b) = (323 + 47 + b + 32) \Rightarrow a + b + c = \boxed{323.}$$

$\square$

Solution. Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3$, and b_4 be the numbers of knights in each room as shown in the diagram. Let $S = a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4$, then

$$36 = (a_1 + b_1 + a_2) + (a_2 + b_2 + a_3) + (a_3 + b_4 + a_4) + (a_4 + b_4 + a_1) \Rightarrow \begin{cases} S = 36 - (a_1 + a_2 + a_3 + a_4) \\ S = \frac{1}{2}\left(36 + (b_1 + b_2 + b_3 + b_4)\right) \end{cases}$$

a_1	b_1	a_2
b_4	K	b_2
a_4	b_3	a_3

Figure 1: With variables

Thus, the least value of S is $\boxed{18,}$ if $b_1 = b_2 = b_3 = b_4 = 0$, and $a_1 = a_3 = 5, a_2 = a_4 = 4$. The most value of S is $\boxed{36,}$ if $a_1 = a_2 = a_3 = a_4 = 0$, and $b_1 = b_2 = b_3 = b_4 = 9$. $\qquad\square$

Solution. Let $x, 2y, 3z$, and $4w$ be the numbers of cookies that Danny Tran, Elvin Nguyen, Franklin Pham, and Williams Quach took. Their sisters took x, y, z, and w cookies respectively. Therefore,

$$x + 2y + 3z + 4w + x + y + z + w = 32 \Rightarrow 2x + 3y + 4z + 5w = 32.$$

Note that x, y, z, w are a permutation of $1, 2, 3, 4$, so $x + y + z + w = 10$. Thus

$$- x + z + 2w = 32 - 3(x + y + z + w) = 2 \Rightarrow x > z, x \text{ and } z \text{ have the same parity}$$
$$\Rightarrow -x + z = -2 \Rightarrow w = 2 \Rightarrow x = 3, z = 1, y = 4.$$

Now, Lan got one, Mai got two, Na got three, and Quynh got four cookies.

$x = 3$, so the girl who took three cookies, Na, is the sister of Danny Tran, so she is Na Tran.

$y = 4$, so the girl who took four cookies, Quynh, is the sister of Elvin Nguyen, so she is Quynh Nguyen.

$z = 1$, so the girl who took one cookie, Lan, is the sister of Franklin Pham, so she is Lan Pham.

$w = 2$, so the girl who took two cookies, Mai, is the sister of Williams Quach, so she is Mai Quach.

Hence, the girls's names are $\boxed{\text{Na Tran, Quynh Nguyen, Lan Pham, and Mai Quach.}}$ $\qquad\square$