

Nghia Doan

Victoria, British Columbia, Canada

(1) October 27: Geometric Transformations

(2) November 3: Carnot and Butterfly Theorems

(3) November 17: Combinatorial Geometry

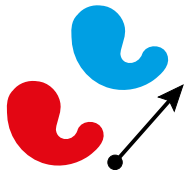
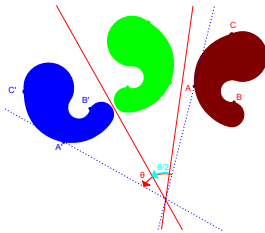
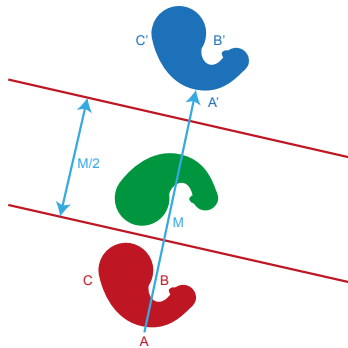
# Geometric Transformations

## Definitions

A transformation is an operation that moves, flips, or changes a figure to create a new figure.

A rigid transformation (also known as an isometry or congruence transformation) is a transformation that does not change the size or shape of a figure.

The rigid transformations are *reflections*, *rotations*, and *translations*.



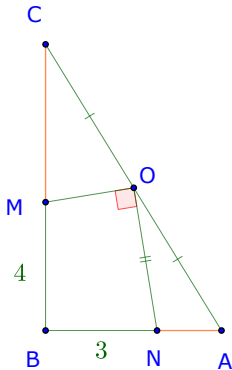
The new figure created by a transformation is called the image. The original is the preimage.

# Geometric Transformations

## Reflection - Example 1

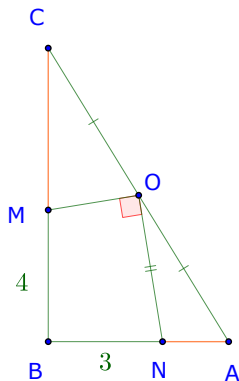
### Example

In the right triangle  $ABC$ ,  $O$  is the midpoint of the hypotenuse  $AC$ . Points  $M$  and  $N$  are chosen on sides  $BC$  and  $BA$  such that  $\angle MON = 90^\circ$ ,  $BM = 4$ , and  $BN = 3$ . Find  $AN^2 + CM^2$ .



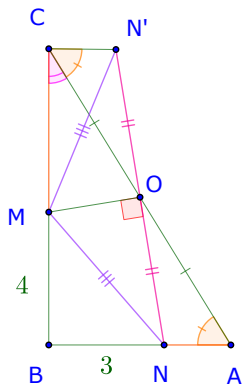
# Geometric Transformations

## Reflection - Example 1 - Solution



# Geometric Transformations

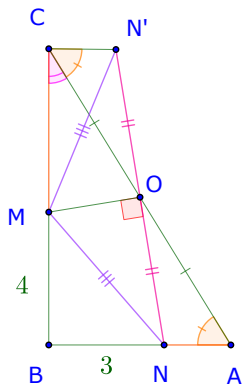
## Reflection - Example 1 - Solution



Let  $N'$  be the reflection of  $N$  over  $O$ .

# Geometric Transformations

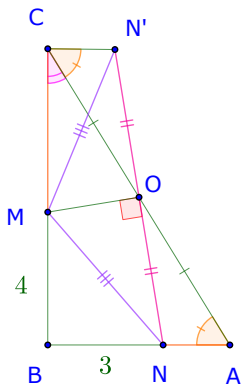
## Reflection - Example 1 - Solution



Let  $N'$  be the reflection of  $N$  over  $O$ .  $\triangle AON \cong \triangle CON'$  by SAS, so  $AN = CN'$ .

# Geometric Transformations

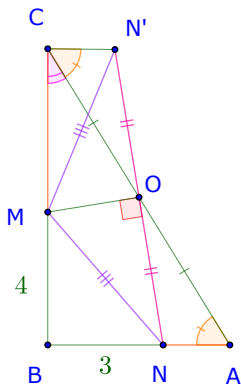
## Reflection - Example 1 - Solution



Let  $N'$  be the reflection of  $N$  over  $O$ .  $\triangle AON \cong \triangle CON'$  by SAS, so  $AN = CN'$ .  
 $\angle MCN' = \angle MCO + \angle OCN' = \angle BCA + \angle CAB = 90^\circ$ . Therefore,  $\triangle MCN'$  is a right triangle.

# Geometric Transformations

## Reflection - Example 1 - Solution



Let  $N'$  be the reflection of  $N$  over  $O$ .  $\triangle AON \cong \triangle CON'$  by SAS, so  $AN = CN'$ .  
 $\angle MCN' = \angle MCO + \angle OCN' = \angle BCA + \angle CAB = 90^\circ$ . Therefore,  $\triangle MCN'$  is a right triangle.  
 $AN^2 + CM^2 = CN'^2 + CM^2 = MN'^2 = MN^2 = BM^2 + BN^2 = \boxed{25}$ .

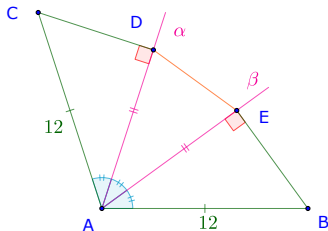


# Geometric Transformations

## Reflection - Example 2

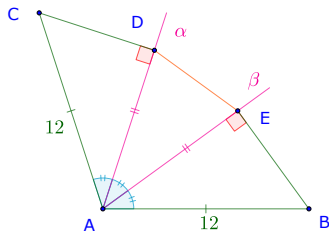
### Example

$\triangle ABC$  is an isosceles triangle,  $AB = AC = 12$ , and  $\angle BAC = 108^\circ$ . Two rays  $\vec{\alpha}$  and  $\vec{\beta}$  starting from  $A$ , trisect the  $\angle BAC$  into three equal angles. Points  $D$  and  $E$  are the feet of the perpendiculars from  $C$  and  $B$  to rays  $\alpha$  and  $\beta$ , respectively. Find  $DE$ .



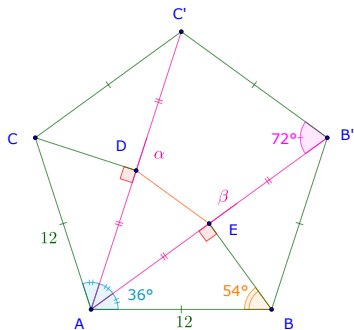
# Geometric Transformations

## Reflection - Example 2 - Solution



# Geometric Transformations

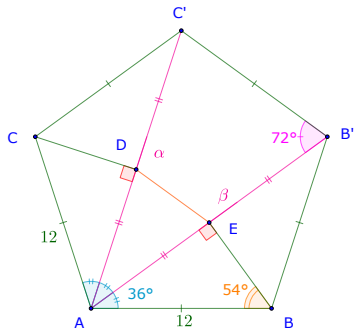
## Reflection - Example 2 - Solution



Let  $B'$  and  $C'$  be the reflections of  $A$  over lines  $BE$  and  $CD$ , respectively.

# Geometric Transformations

## Reflection - Example 2 - Solution

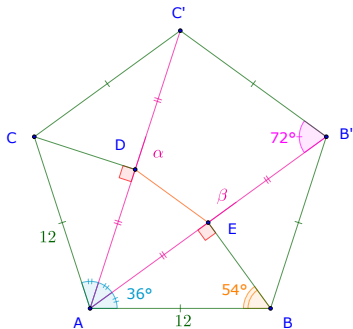


Let  $B'$  and  $C'$  be the reflections of  $A$  over lines  $BE$  and  $CD$ , respectively.

$$\angle B'AC + \angle C'CA = 2 \cdot 36^\circ + 2 \cdot 54^\circ = 180^\circ \Rightarrow CC' \parallel AB'.$$

# Geometric Transformations

## Reflection - Example 2 - Solution



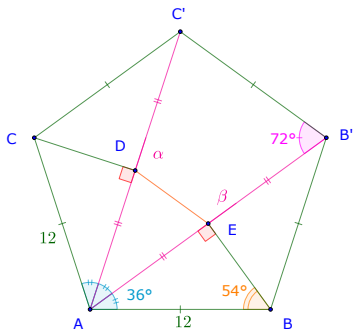
Let  $B'$  and  $C'$  be the reflections of  $A$  over lines  $BE$  and  $CD$ , respectively.

$$\angle B'AC + \angle C'CA = 2 \cdot 36^\circ + 2 \cdot 54^\circ = 180^\circ \Rightarrow CC' \parallel AB'.$$

Furthermore  $\triangle AB'C'$  is isosceles, so  $\angle B'AC = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$ .

# Geometric Transformations

## Reflection - Example 2 - Solution



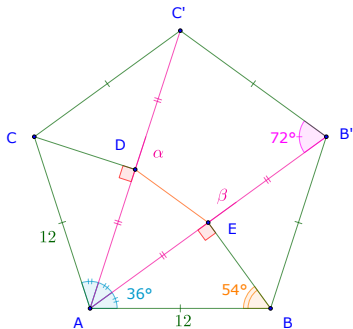
Let  $B'$  and  $C'$  be the reflections of  $A$  over lines  $BE$  and  $CD$ , respectively.

$$\angle B'AC + \angle C'CA = 2 \cdot 36^\circ + 2 \cdot 54^\circ = 180^\circ \Rightarrow CC' \parallel AB'.$$

Furthermore  $\triangle AB'C'$  is isosceles, so  $\angle B'AC = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$ .  
Thus,  $ACC'B'$  is isosceles trapezoid, so  $B'C' = AB = 12$ .

# Geometric Transformations

## Reflection - Example 2 - Solution



Let  $B'$  and  $C'$  be the reflections of  $A$  over lines  $BE$  and  $CD$ , respectively.

$$\angle B'AC + \angle C'CA = 2 \cdot 36^\circ + 2 \cdot 54^\circ = 180^\circ \Rightarrow CC' \parallel AB'.$$

Furthermore  $\triangle AB'C'$  is isosceles, so  $\angle B'AC = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$ .

Thus,  $ACC'B'$  is isosceles trapezoid, so  $B'C' = AB = 12$ .

$DE$  is a midsegment in  $\triangle AB'C'$ , hence  $DE = 6$ .

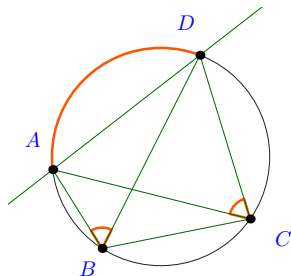
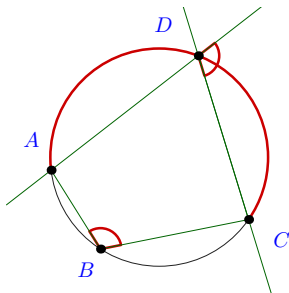
# Geometric Transformations

## Cyclic Quadrilaterals

### Theorem (Cyclic Quadrilaterals)

Let  $ABCD$  be a convex quadrilateral. Then the following are equivalent:

- $ABCD$  is cyclic.
- $\angle ABC + \angle CDA = 180^\circ$ .
- $\angle ABD = \angle ACD$ .



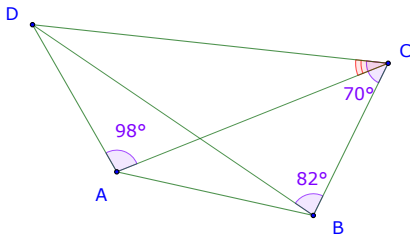


# Geometric Transformations

## Reflection - Example 3

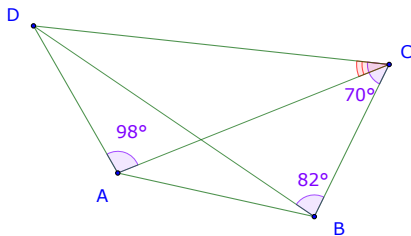
### Example

$ABCD$  is a convex quadrilateral.  $BC = AD$ ,  $\angle DAC = 98^\circ$ ,  $\angle DBC = 82^\circ$ ,  $\angle BCD = 70^\circ$ . Find  $\angle ACD$ .



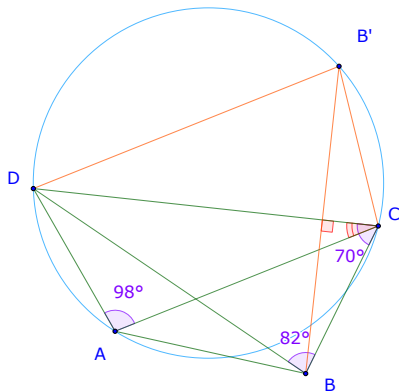
# Geometric Transformations

## Reflection - Example 3 - Solution



# Geometric Transformations

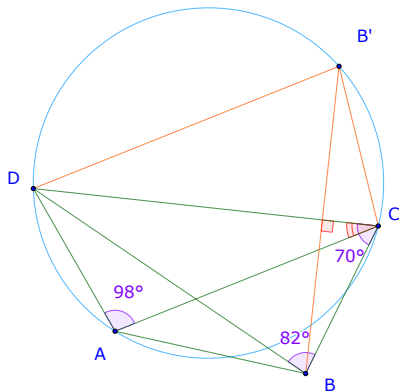
## Reflection - Example 3 - Solution



Let  $B'$  be the reflection of  $B$  across  $CD$ .

# Geometric Transformations

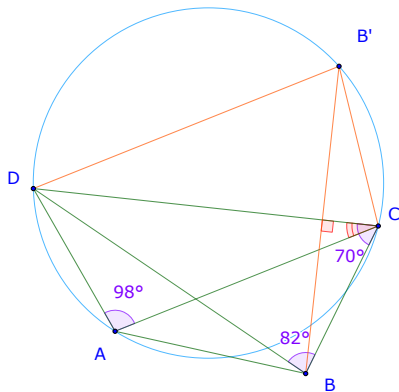
## Reflection - Example 3 - Solution



Let  $B'$  be the reflection of  $B$  across  $CD$ .  $\angle DAC + \angle CB'D = 180^\circ$ , so  $ACB'D$  is cyclic.  
 $AD = BC = CB'$ , thus  $\widehat{AD} = \widehat{B'C}$ .

# Geometric Transformations

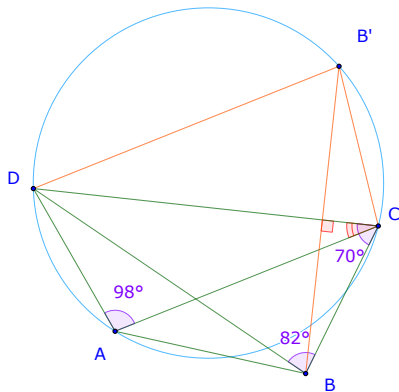
## Reflection - Example 3 - Solution



Let  $B'$  be the reflection of  $B$  across  $CD$ .  $\angle DAC + \angle CB'D = 180^\circ$ , so  $ACB'D$  is cyclic.  $AD = BC = CB'$ , thus  $\widehat{AD} = \widehat{B'C}$ . Therefore  $\angle ACB' = \angle DAC = 98^\circ$ .

# Geometric Transformations

## Reflection - Example 3 - Solution



Let  $B'$  be the reflection of  $B$  across  $CD$ .  $\angle DAC + \angle CB'D = 180^\circ$ , so  $ACB'D$  is cyclic.  $AD = BC = CB'$ , thus  $\widehat{AD} = \widehat{B'C}$ . Therefore  $\angle ACB' = \angle DAC = 98^\circ$ .

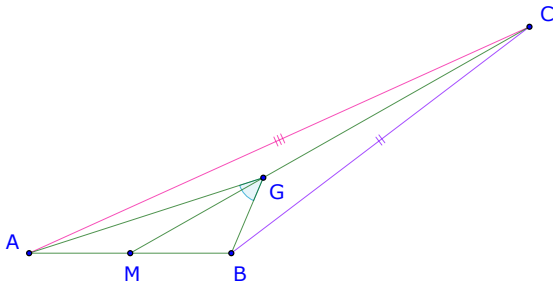
$$\angle DCB' = \angle BCD = 70^\circ \Rightarrow \angle ACD = \angle ACB' - \angle DCB' = 98^\circ - 70^\circ = \boxed{28^\circ}.$$

# Geometric Transformations

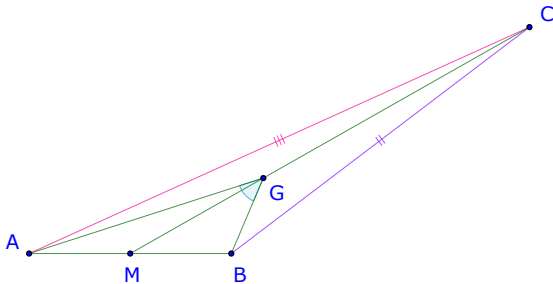
## Reflection - Example 4

### Example

Let  $G$  be the centroid of  $\triangle ABC$ . If  $\angle AGB \leq 90^\circ$ , find the largest possible value of  $n$  integer, such that  $AC + CB > nAB$ .



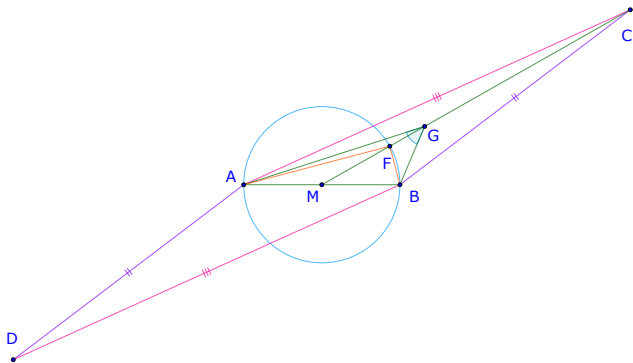
## Reflection - Example 4 - Solution





# Geometric Transformations

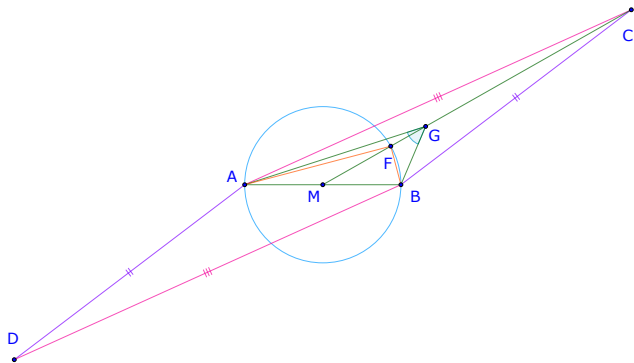
## Reflection - Example 4 - Solution



Let  $M$  be the midpoint of  $AB$ ,  $F$  be the intersection of the circle centred  $M$  diameter  $AB$ .  $\angle AGB \leq 90^\circ$  so  $G$  is outside the circle  $(M)$ , therefore  $GM \geq FM = \frac{1}{2}AB$ .

# Geometric Transformations

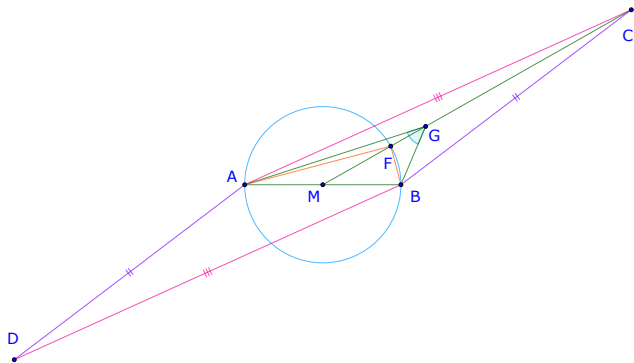
## Reflection - Example 4 - Solution



Let  $M$  be the midpoint of  $AB$ ,  $F$  be the intersection of the circle centred  $M$  diameter  $AB$ .  
 $\angle AGB \leq 90^\circ$  so  $G$  is outside the circle ( $M$ ), therefore  $GM \geq FM = \frac{1}{2}AB$ .  
Let  $D$  be the reflection of  $C$  over  $M$ . In triangle  $DAC$ ,  $DA + AC > DC \Rightarrow AC + CB > 2CM$ .

# Geometric Transformations

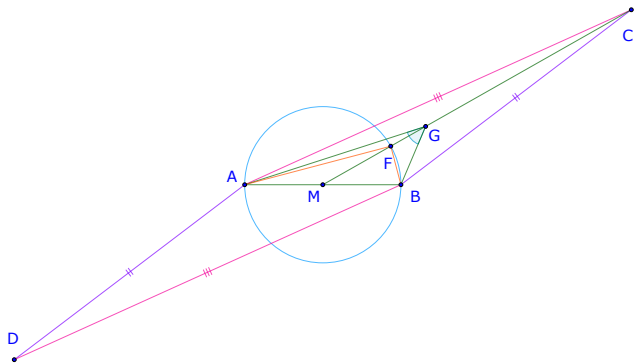
## Reflection - Example 4 - Solution



Let  $M$  be the midpoint of  $AB$ ,  $F$  be the intersection of the circle centred  $M$  diameter  $AB$ .  
 $\angle AGB \leq 90^\circ$  so  $G$  is outside the circle ( $M$ ), therefore  $GM \geq FM = \frac{1}{2}AB$ .  
Let  $D$  be the reflection of  $C$  over  $M$ . In triangle  $DAC$ ,  $DA + AC > DC \Rightarrow AC + CB > 2CM$ .  
Therefore  $AB \leq 2GM = \frac{2}{3}CM < \frac{1}{3}(AC + CB)$ , thus  $AC + CB > 3AB$ .

# Geometric Transformations

## Reflection - Example 4 - Solution



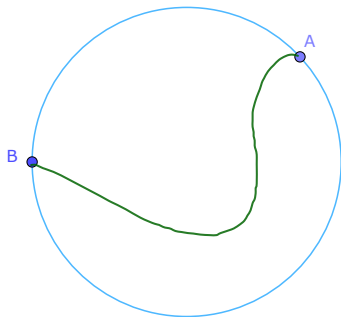
Let  $M$  be the midpoint of  $AB$ ,  $F$  be the intersection of the circle centred  $M$  diameter  $AB$ .  $\angle AGB \leq 90^\circ$  so  $G$  is outside the circle ( $M$ ), therefore  $GM \geq FM = \frac{1}{2}AB$ .  
 Let  $D$  be the reflection of  $C$  over  $M$ . In triangle  $DAC$ ,  $DA + AC > DC \Rightarrow AC + CB > 2CM$ .  
 Therefore  $AB \leq 2GM = \frac{2}{3}CM < \frac{1}{3}(AC + CB)$ , thus  $AC + CB > 3AB$ .  
 If  $AC = BC$ ,  $G$  on circle ( $M$ ), then  $AC + CB = \sqrt{10}AB < 4AB$ . Thus,  $\boxed{n = 3.}$

# Geometric Transformations

## Reflection - Example 5

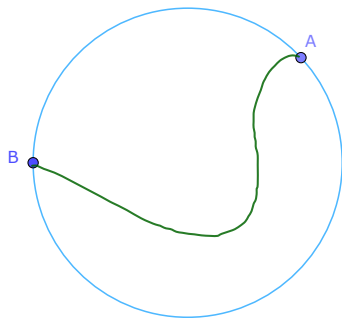
### Example

$A, B$  are two points on the circle. A curve through  $A, B$  bisects the area of the circle. Prove that the curve is at least as long as a diameter of the circle.



# Geometric Transformations

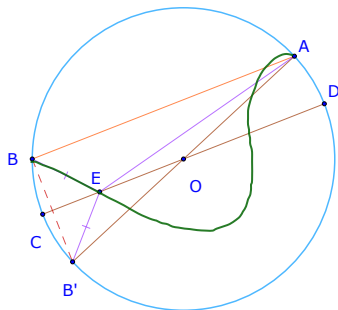
## Reflection - Example 5 - Solution



It is obvious that the curve is at least as long as a diameter of the circle if  $AB$  is a diameter. Thus, let's assume that  $AB$  is a chord that is shorter than the diameter of the circle.

# Geometric Transformations

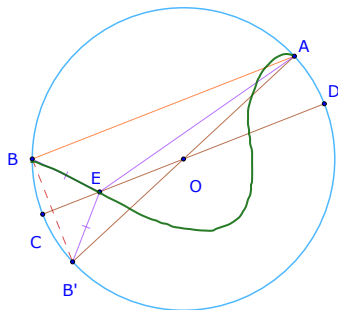
## Reflection - Example 5 - Solution



It is obvious that the curve is at least as long as a diameter of the circle if  $AB$  is a diameter. Thus, let's assume that  $AB$  is a chord that is shorter than the diameter of the circle. Let  $CD$  be the diameter of the circle that is parallel to  $AB$ . Let  $B'$  be the reflection of  $B$  over  $CD$ .

# Geometric Transformations

## Reflection - Example 5 - Solution

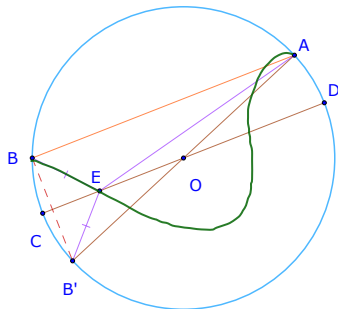


It is obvious that the curve is at least as long as a diameter of the circle if  $AB$  is a diameter. Thus, let's assume that  $AB$  is a chord that is shorter than the diameter of the circle. Let  $CD$  be the diameter of the circle that is parallel to  $AB$ . Let  $B'$  be the reflection of  $B$  over  $CD$ . If the curve does not intersect  $CD$ , then the curve cannot bisect the area of the circle, thus the curve must intersect  $CD$ .



# Geometric Transformations

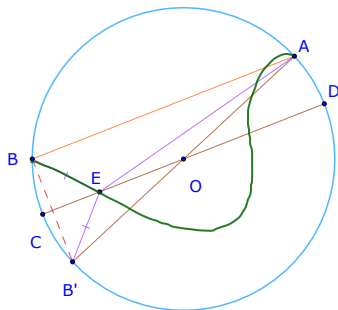
## Reflection - Example 5 - Solution



It is obvious that the curve is at least as long as a diameter of the circle if  $AB$  is a diameter. Thus, let's assume that  $AB$  is a chord that is shorter than the diameter of the circle. Let  $CD$  be the diameter of the circle that is parallel to  $AB$ . Let  $B'$  be the reflection of  $B$  over  $CD$ . If the curve does not intersect  $CD$ , then the curve cannot bisect the area of the circle, thus the curve must intersect  $CD$ . Let one of the intersections be  $E$ . The length of the curve must be at least as long as  $BE + EA$ . Now,  $BE + EA = B'E + EA \geq B'A$ , which is a diameter.

# Geometric Transformations

## Reflection - Example 5 - Solution



It is obvious that the curve is at least as long as a diameter of the circle if  $AB$  is a diameter. Thus, let's assume that  $AB$  is a chord that is shorter than the diameter of the circle. Let  $CD$  be the diameter of the circle that is parallel to  $AB$ . Let  $B'$  be the reflection of  $B$  over  $CD$ . If the curve does not intersect  $CD$ , then the curve cannot bisect the area of the circle, thus the curve must intersect  $CD$ .

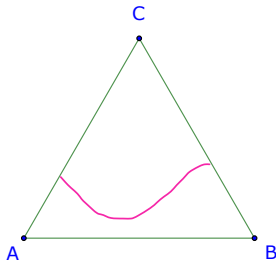
Let one of the intersections be  $E$ . The length of the curve must be at least as long as  $BE + EA$ . Now,  $BE + EA = B'E + EA \geq B'A$ , which is a diameter. Hence, the curve is at least as long as a diameter of the circle.

# Geometric Transformations

## Reflection - Example 6

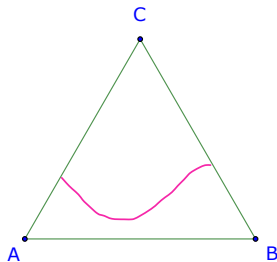
### Example

A continuous curve split an unit equilateral triangle  $ABC$  into two regions with equal area. What is the minimal length of the curve?



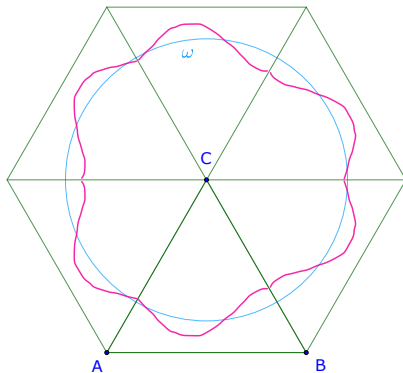
# Geometric Transformations

## Reflection - Example 6 - Solution



# Geometric Transformations

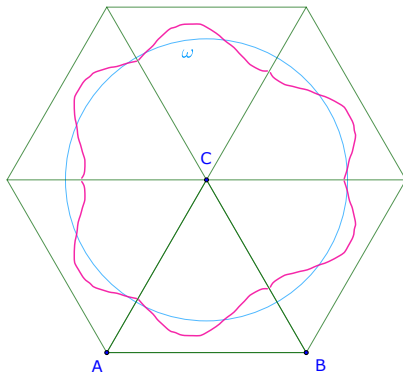
## Reflection - Example 6 - Solution



The continuous curve becomes a close curve encompassing an area half of a unit circle, or  $\pi$ .

# Geometric Transformations

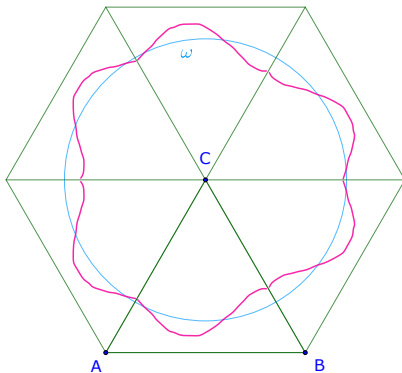
## Reflection - Example 6 - Solution



The continuous curve becomes a close curve encompassing an area half of a unit circle, or  $\pi$ .  
The close curve encompassing an area has minimal length if it forms the perimeter of a circle.

# Geometric Transformations

## Reflection - Example 6 - Solution



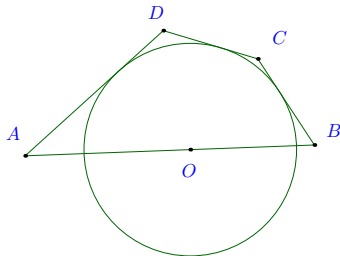
The continuous curve becomes a close curve encompassing an area half of a unit circle, or  $\pi$ . The close curve encompassing an area has minimal length if it forms the perimeter of a circle. Therefore the minimal length of the original curve is the perimeter of a circle area three times of an unit equilateral triangle.

# Geometric Transformations

## Reflection - Example 7

### Example

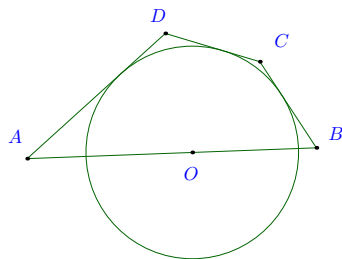
A circle has center on the side  $AB$  of the cyclic quadrilateral  $ABCD$ . The other three sides are tangent to the circle. Prove that  $AD + BC = AB$ .





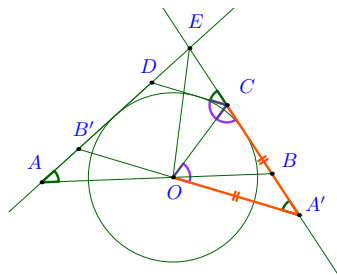
# Geometric Transformations

## Reflection - Example 7 - Solution



# Geometric Transformations

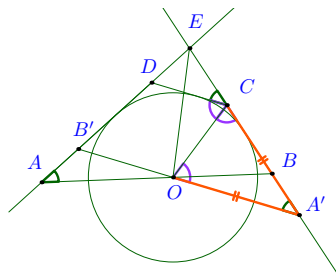
## Reflection - Example 7 - Solution



Let  $E$  be the intersection of  $AD$  and  $BC$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $EO$ .

# Geometric Transformations

## Reflection - Example 7 - Solution

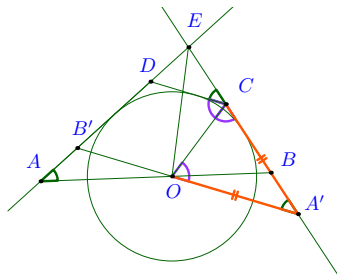


Let  $E$  be the intersection of  $AD$  and  $BC$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $EO$ .

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

# Geometric Transformations

## Reflection - Example 7 - Solution



Let  $E$  be the intersection of  $AD$  and  $BC$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $EO$ .

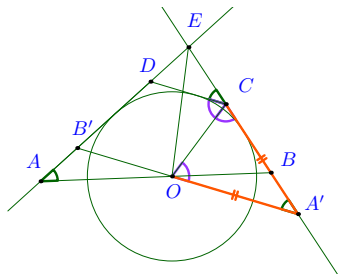
$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

$CB, CD$  are tangents, so  $CO$  is the bisector of  $\angle DCB$ , therefore

$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$

# Geometric Transformations

## Reflection - Example 7 - Solution



Let  $E$  be the intersection of  $AD$  and  $BC$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $EO$ .

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

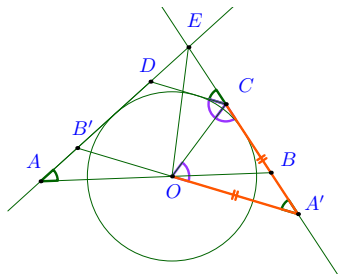
$CB$ ,  $CD$  are tangents, so  $CO$  is the bisector of  $\angle DCB$ , therefore

$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$

Thus,  $OA' = CA'$ ,  $OB = B'D$ .

# Geometric Transformations

## Reflection - Example 7 - Solution



Let  $E$  be the intersection of  $AD$  and  $BC$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $EO$ .

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

$CB$ ,  $CD$  are tangents, so  $CO$  is the bisector of  $\angle DCB$ , therefore

$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$

Thus,  $OA' = CA'$ ,  $OB = B'D$ . Hence,

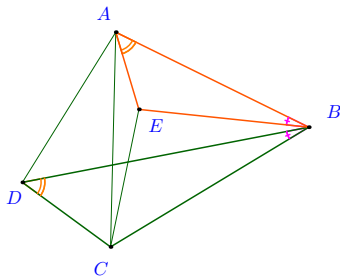
$$AB = A'B' = A'C + B'D = A'E - CE + B'E - DE = AE - ED + BE - EC = AD + BC.$$

### Theorem (Ptolemy Inequality)

The inequality states that in for four points  $A, B, C, D$  in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

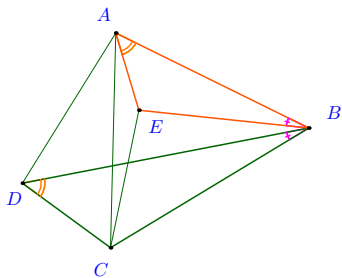
with equality for any cyclic quadrilateral  $ABCD$  with diagonals  $AC$  and  $BD$ .



Note: this also holds if  $A, B, C, D$  are four points not in the same plane, but the equality can't be achieved.

# Geometric Transformations

## Ptolemy Inequality - Proof



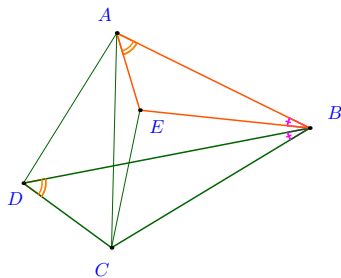
Proof.





# Geometric Transformations

## Ptolemy Inequality - Proof



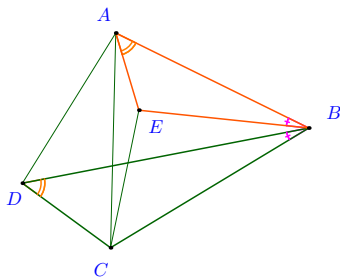
### Proof.

Let  $E$  be the point such that  $\angle EAB = \angle CDB$ ,  $\angle EBA = \angle CBD$ .



# Geometric Transformations

## Ptolemy Inequality - Proof



### Proof.

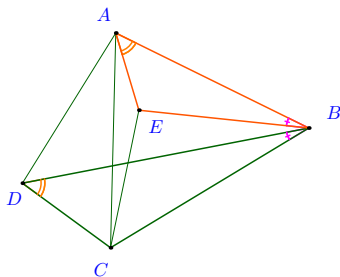
Let  $E$  be the point such that  $\angle EAB = \angle CDB$ ,  $\angle EBA = \angle CBD$ .

$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \quad \frac{CE}{AD} = \frac{BC}{BD}.$$



# Geometric Transformations

## Ptolemy Inequality - Proof



### Proof.

Let  $E$  be the point such that  $\angle EAB = \angle CDB$ ,  $\angle EBA = \angle CBD$ .

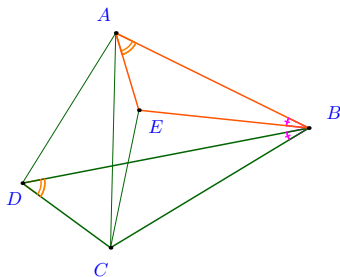
$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \quad \frac{CE}{AD} = \frac{BC}{BD}.$$

$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$



# Geometric Transformations

## Ptolemy Inequality - Proof



### Proof.

Let  $E$  be the point such that  $\angle EAB = \angle CDB$ ,  $\angle EBA = \angle CBD$ .

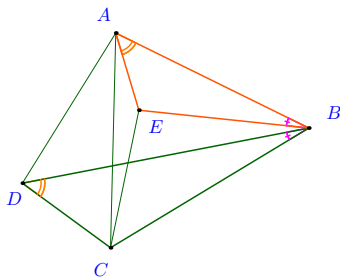
$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \quad \frac{CE}{AD} = \frac{BC}{BD}.$$

$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$



# Geometric Transformations

## Ptolemy Inequality - Proof



### Proof.

Let  $E$  be the point such that  $\angle EAB = \angle CDB$ ,  $\angle EBA = \angle CBD$ .

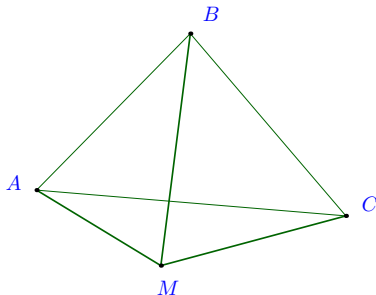
$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \quad \frac{CE}{AD} = \frac{BC}{BD}.$$

$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

The equality stands if and only if  $E \in AC$ , or  $\angle CAB = \angle EAB = \angle CDB$ , so  $ABCD$  is cyclic.  $\square$

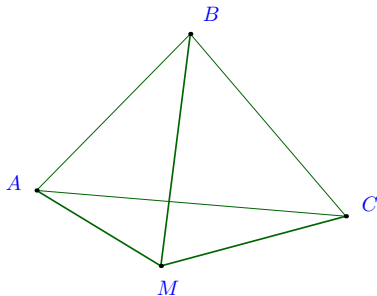
### Theorem (Pompeiu's Theorem)

*$\triangle ABC$  is equilateral. For any point  $M$ , the segments  $AM$ ,  $BM$  and  $CM$  form a triangle. This triangle degenerates if and only if  $M$  lies on the circumcircle of  $\triangle ABC$ .*



# Geometric Transformations

## Pompeiu's Theorem - Proof

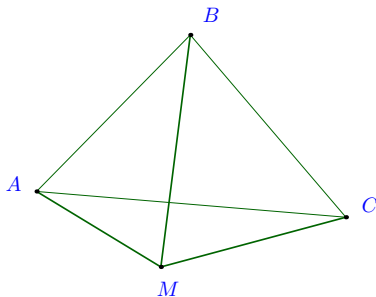


Proof.



# Geometric Transformations

## Pompeiu's Theorem - Proof



### Proof.

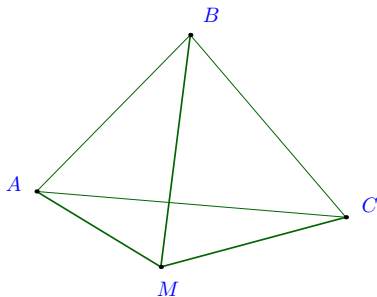
If  $M$  is inside of  $\triangle ABC$ , then  $AM + BM > AB > CM$ .





# Geometric Transformations

## Pompeiu's Theorem - Proof



### Proof.

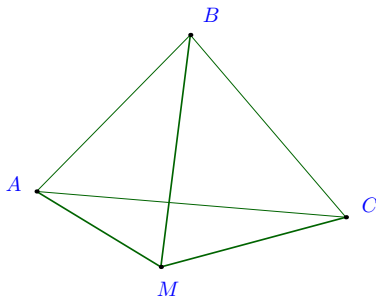
If  $M$  is inside of  $\triangle ABC$ , then  $AM + BM > AB > CM$ . If  $M$  is outside of  $\triangle ABC$ , by Ptolemy Inequality, for four points  $A, B, C, M$

$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$



# Geometric Transformations

## Pompeiu's Theorem - Proof



### Proof.

If  $M$  is inside of  $\triangle ABC$ , then  $AM + BM > AB > CM$ . If  $M$  is outside of  $\triangle ABC$ , by Ptolemy Inequality, for four points  $A, B, C, M$

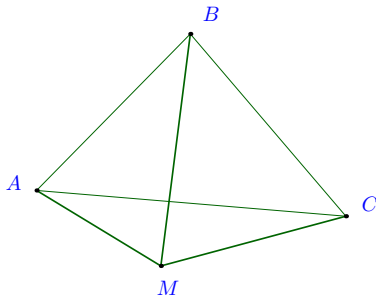
$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$

Similarly with other triangle inequalities. Hence,  $AM, BM$  and  $CM$  form a triangle.



# Geometric Transformations

## Pompeiu's Theorem - Proof



### Proof.

If  $M$  is inside of  $\triangle ABC$ , then  $AM + BM > AB > CM$ . If  $M$  is outside of  $\triangle ABC$ , by Ptolemy Inequality, for four points  $A, B, C, M$

$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$

Similarly with other triangle inequalities. Hence,  $AM, BM$  and  $CM$  form a triangle.

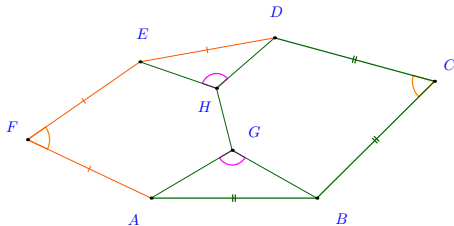
The equality stands if and only if  $ABCM$  is cyclic, or  $M$  is on the circumcircle of  $\triangle ABC$ . □

# Geometric Transformations

## Reflection - Example 8

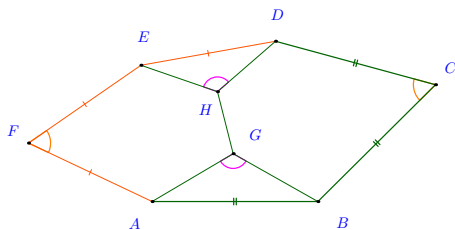
### Example

Let  $ABCDEF$  be a convex hexagon with  $AB = BC = CD$  and  $DE = EF = FA$ , such that  $\angle BCD = \angle EFA = \frac{\pi}{3}$ . Suppose  $G$  and  $H$  are points in the interior of the hexagon such that  $\angle AGB = \angle DHE = \frac{2\pi}{3}$ . Prove that  $AG + GB + GH + DH + HE \geq CF$ .

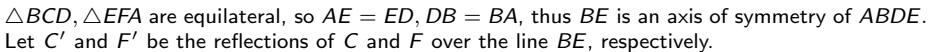


# Geometric Transformations

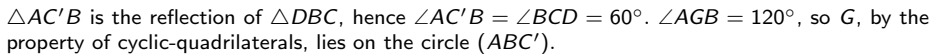
## Example 8 - Solution



### Example 8 - Solution

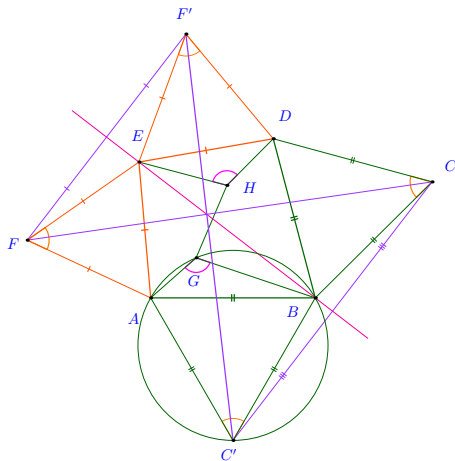


### Example 8 - Solution



# Geometric Transformations

## Example 8 - Solution

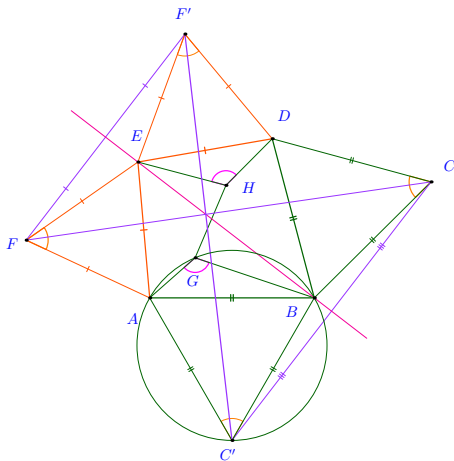


Similarly  $H$  lies on the circle  $(DEF')$ .



# Geometric Transformations

## Example 8 - Solution

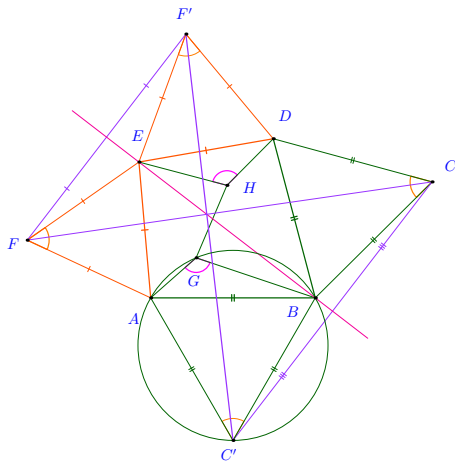


Thus, according to Pompeiu's Theorem,  $AG + GB = C'G$  and  $DH + HE = HF'$ , so

$$AG + GB + GH + DH + HE = C'G + GH + HF' \geq C'F' = CF$$

# Geometric Transformations

## Example 8 - Solution



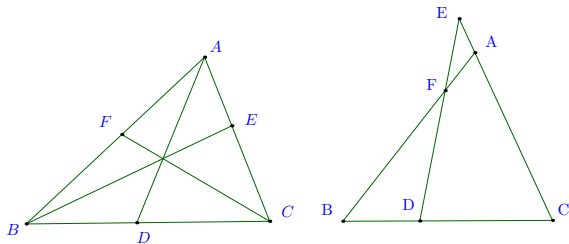
Hence,  $AG + GB + GH + DH + HE \geq CF.$

# Geometric Transformations

## Menelaus Theorem

### Theorem (Ceva Theorem)

Let  $ABC$  be a triangle, and let  $D, E, F$  be points on lines  $BC, CA, AB$ , respectively. Lines  $AD, BE, CF$  are **concurrent** if and only if:  $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$ .



### Theorem (Menelaus Theorem)

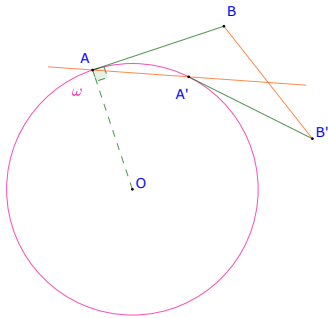
Let  $ABC$  be a triangle, and let  $D, F$  be points on lines  $BC, AB$ , respectively.  $E$  is on the extension of  $CA$ . Points  $D, E, F$  are **collinear** if and only if:  $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$ .

# Geometric Transformations

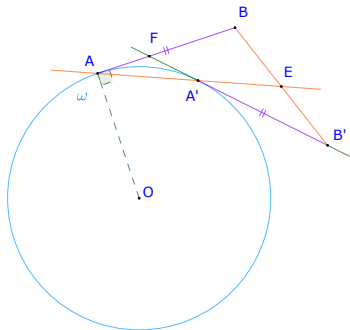
## Rotation - Example 1

### Example

Point  $B$  lies on a line which is tangent to circle  $\omega$  at point  $A$ . The line segment  $AB$  is rotated about the center of the circle by some angle to form segment  $A'B'$ . prove that the line  $AA'$  bisects the segment  $BB'$ .

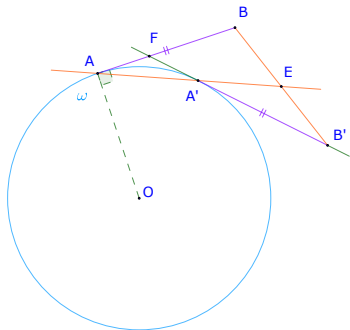


## Rotation - Example 1 - Solution



# Geometric Transformations

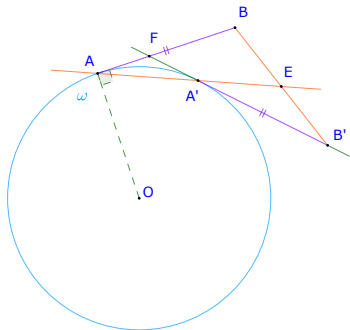
## Rotation - Example 1 - Solution



Let  $E$  be the intersections of lines through  $AA'$  and  $BB'$ . Let  $F$  be the intersections of lines through  $AB$  and  $A'B'$ .

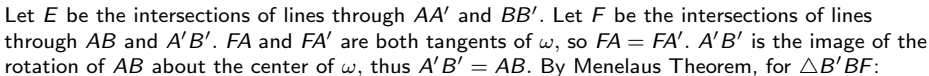
# Geometric Transformations

## Rotation - Example 1 - Solution



Let  $E$  be the intersections of lines through  $AA'$  and  $BB'$ . Let  $F$  be the intersections of lines through  $AB$  and  $A'B'$ .  $FA$  and  $FA'$  are both tangents of  $\omega$ , so  $FA = FA'$ .  $A'B'$  is the image of the rotation of  $AB$  about the center of  $\omega$ , thus  $A'B' = AB$ .

## Rotation - Example 1 - Solution



26 / 36

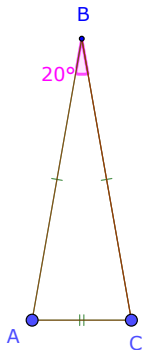


# Geometric Transformations

## Rotation - Example 2

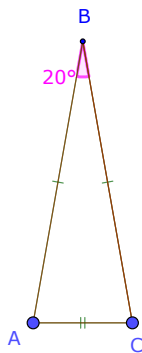
### Example

Given isosceles triangle  $ABC$ ,  $AB = BC$ , and  $\angle B = 20^\circ$ , prove that  $AB < 3AC$ .



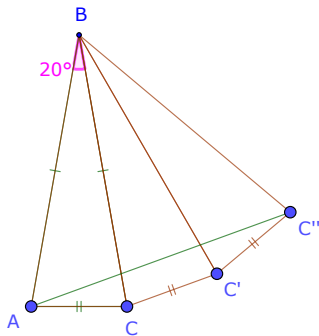
# Geometric Transformations

## Rotation - Example 2 - Solution



# Geometric Transformations

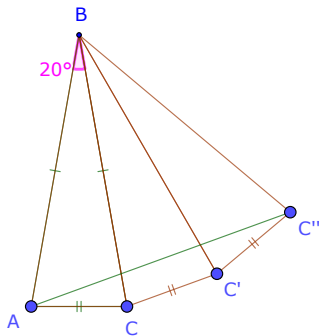
## Rotation - Example 2 - Solution



Let's rotate the triangle  $ABC$  around  $B$  twice  $20^\circ$  as shown in the diagram above.

# Geometric Transformations

## Rotation - Example 2 - Solution



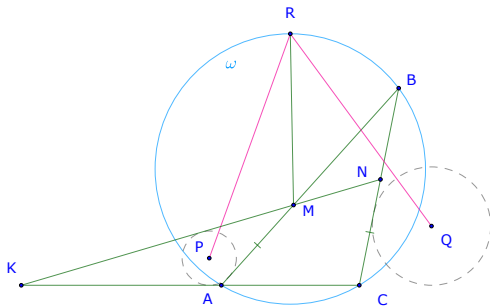
Let's rotate the triangle  $ABC$  around  $B$  twice  $20^\circ$  as shown in the diagram above. Then  $ABE$  is an equilateral triangle,  $AB = AC'' < AC + CC' + C'C'' = 3AC$ .

# Geometric Transformations

## Rotation - Example 3

### Example

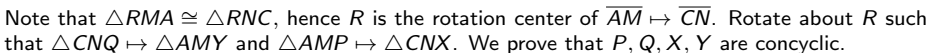
Given a triangle  $ABC$  with  $AB > BC$ , let  $\omega$  be the circumcircle. Let  $M, N$  lie on the sides  $AB, BC$  respectively, such that  $AM = CN$ . Let  $K$  be the intersection of  $MN$  and  $AC$ . Let  $P$  be the incentre of the triangle  $AMK$  and  $Q$  be the  $K$ -excentre of the triangle  $CNK$ . If  $R$  is midpoint of the arc  $ABC$  of  $\omega$  then prove that  $RP = RQ$ .



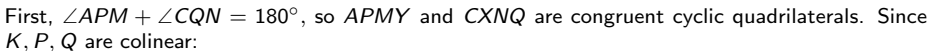
## Rotation - Example 3 - Solution



## Rotation - Example 3 - Solution



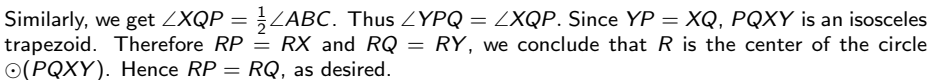
### Rotation - Example 3 - Solution



$$\angle YPQ = \angle APQ - \angle APY = (90^\circ - \frac{1}{2}\angle BMN) - \frac{1}{2}\angle BNM = \frac{1}{2}\angle ABC.$$



## Rotation - Example 3 - Solution

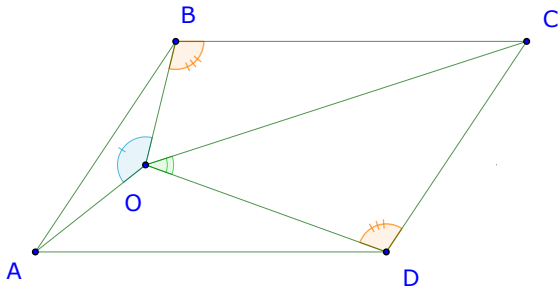


# Geometric Transformations

## Translation - Example 1

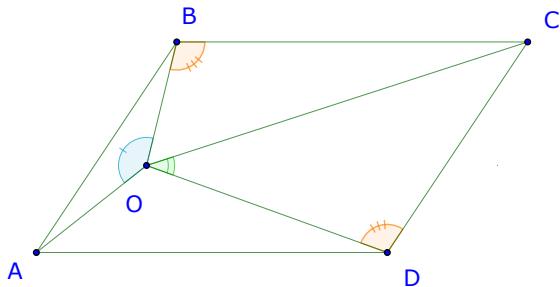
### Example

The point  $O$  is situated inside the parallelogram  $ABCD$  such that  $\angle AOB + \angle COD = 180^\circ$ . Prove that  $\angle OBC = \angle ODC$ .



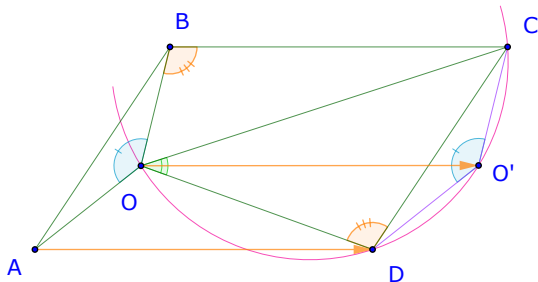
# Geometric Transformations

## Translation - Example 1 - Solution



# Geometric Transformations

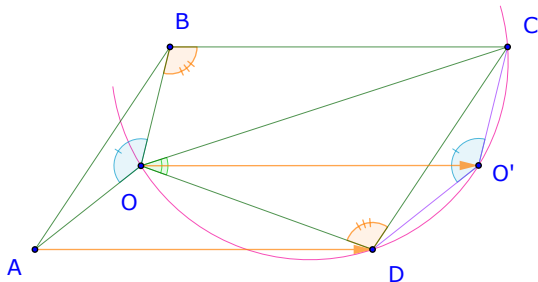
## Translation - Example 1 - Solution



The translation by  $\vec{AD}$  maps  $A$  to  $D$ ,  $B$  to  $C$ , and  $O$  to  $O'$ .

# Geometric Transformations

## Translation - Example 1 - Solution

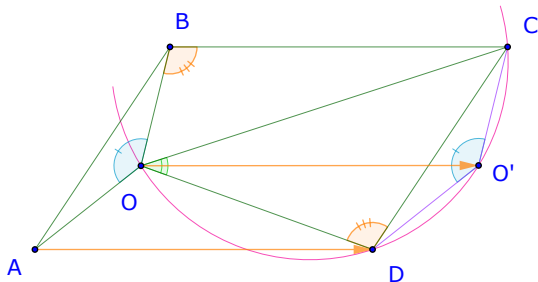


The translation by  $\overrightarrow{AD}$  maps  $A$  to  $D$ ,  $B$  to  $C$ , and  $O$  to  $O'$ .

$ABCD$  is a parallelogram,  $AD \parallel BC$ ,  $AD = BC$ . By the translation,  $OO' \parallel AD$ ,  $OO' = AD$ , thus  $OO' \parallel BC$ ,  $OO' = BC$ . Therefore  $OBCO'$  is a parallelogram. It implies that  $\angle OBC = \angle OO'C$ .

# Geometric Transformations

## Translation - Example 1 - Solution



The translation by  $\overrightarrow{AD}$  maps  $A$  to  $D$ ,  $B$  to  $C$ , and  $O$  to  $O'$ .

$ABCD$  is a parallelogram,  $AD \parallel BC$ ,  $AD = BC$ . By the translation,  $OO' \parallel AD$ ,  $OO' = AD$ , thus  $OO' \parallel BC$ ,  $OO' = BC$ . Therefore  $OBCO'$  is a parallelogram. It implies that  $\angle OBC = \angle OO'C$ .

Since  $\angle AOB + \angle COD = 180^\circ$ , so  $\angle DO'C + \angle COD = 180^\circ$ , or  $CODO'$  is cyclic. Therefore  $\angle ODC = \angle OO'C$ . Hence,  $\boxed{\angle OBC = \angle ODC}$ .

# Geometric Transformations

## Homothety

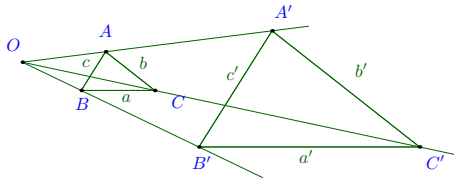
### Definition (Homothety)

A **homothety** (or homothecy) is a transformation of space which dilates distances *with respect to a fixed point*.

A homothety with center  $O$  and factor  $k$  sends point  $A$  to a point  $A'$ , and

$$OA' = k \cdot OA.$$

This is denoted by  $\mathcal{H}_{(O,k)}$ .



A homothety can be an *enlargement* (resulting figure is larger), *identity* transformation (resulting figure is congruent), or a *contraction* (resulting figure is smaller).

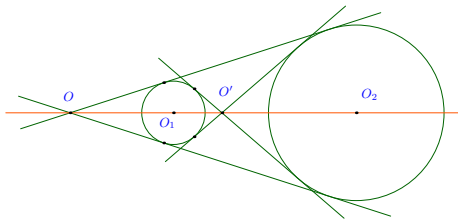
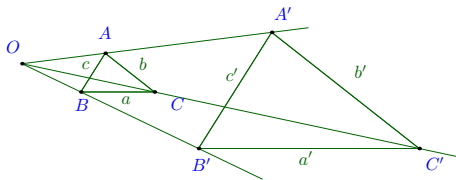
# Geometric Transformations

## Homothety Properties

### Theorem (Homothety Images)

Let  $\mathcal{H}_{(O,k)}$  be a homothety,

- 1 For point  $A$ ,  $\mathcal{H}_{(O,k)}(A) = A' \Rightarrow O, A, A'$  collinear. Thus, the lines connecting each point of a polygon to its corresponding point of a homothetic polygon are all concurrent.
- 2 For line  $\ell$ ,  $\mathcal{H}_{(O,k)}(\ell) = \ell' \Rightarrow \ell \parallel \ell'$ .
- 3 For polygon  $P$ ,  $\mathcal{H}_{(O,k)}(P) = P' \Rightarrow P \sim P'$ . Thus, the resulting image of a circle from a homothety is also a circle.



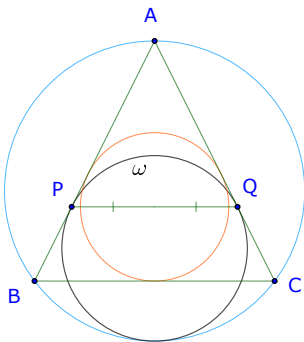


# Geometric Transformations

## Homothety - Example 1

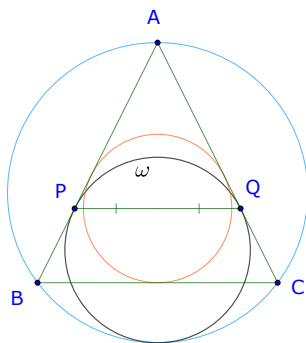
### Example

In a triangle  $ABC$  we have  $AB = AC$ . A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides  $AB, AC$  in the points  $P$ , respectively  $Q$ . Prove that the midpoint of  $PQ$  is the center of the inscribed circle of the triangle  $ABC$ .



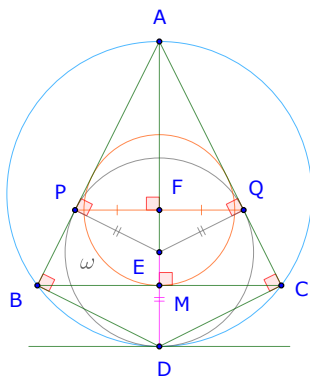
# Geometric Transformations

## Homothety - Example 1 - Solution



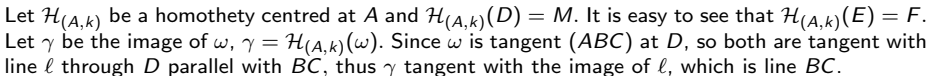
# Geometric Transformations

## Homothety - Example 1 - Solution



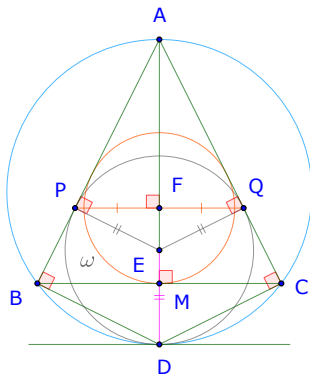
Let  $E$  be the center of the circle  $\omega$ , which is tangent with  $AB$ ,  $AC$ , and  $(ABC)$ . Let  $D$  be the tangent point of the two circles,  $M$  be the midpoint of  $BC$ , and  $F$  be the midpoint of  $PQ$ . It is easy to see that  $A, F, E, M, D$  are collinear.

## Homothety - Example 1 - Solution



# Geometric Transformations

## Homothety - Example 1 - Solution



Furthermore, the  $\mathcal{H}_{(A,k)}(D) = M$ , keeps  $B$  and  $C$  on rays  $AB$  and  $AC$ , and because  $\omega$  is tangent to  $AB$  and  $AC$ , so  $\gamma$  is also tangent to  $AB$  and  $AC$ . Therefore  $\gamma$  is tangent with all sides of  $\triangle ABC$ , so  $\gamma = I_{\triangle ABC}$ . Therefore, the image of  $\mathcal{H}_{(A,k)}(E)$  is  $I$ , the incentre of  $\triangle ABC$ , therefore  $F \equiv I$ .

Hence, the midpoint of  $PQ$  is the center of the inscribed circle of the triangle  $ABC$ .