Solving Forty Two Problems by the Induction Principle - Part II

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Problem 0.1 (Problem Six)**.** Let $f:\ \mathbb{Z}^+ \to \mathbb{Z}$ be a function such that:

$$f(f(n)) + f(n) = 2n.$$

Prove that $f(n) = n,\ \forall n \in \mathbb{Z}^+$.

Solution. For the inductive step, let $f(k) = k, \forall k < n$. Then

$$f(f(n)) + f(n) = 2n$$

Case 1: if $f(n) = k < n$, then

$$f(f(n)) = f(k) = k \Rightarrow f(f(n)) + f(n) = f(k) + k = 2k < 2n.$$

Case 2: if $f(n) = k > n$, then $f(f(n)) = 2n - k < n$. By the hypothesis

$$f(f(f(n))) + f(f(n)) = 2f(n).$$

However, $f(f(n)) < n$, so $f(f(f(n))) = f(n)$, thus $f(f(f(n))) + f(n) < 2f(n)$, a contradiction.

Therefore $f(n) = n$. The hypothesis follows. □

Problem 0.2 (Problem Seven)**.** Let $a_1, a_2, \ldots, a_n$ be any real numbers and $b_1, b_2, \ldots, b_n$ be any positive real numbers, prove that:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Let $a_1, a_2, \ldots, a_n$ be any positive real numbers, then:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Solution. The base case of $n = 1$ is clear. Let prove the case $n = 2$ in order to reuse it in the inductive step.

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} \geq \frac{(a_1 + a_2)^2}{b_1 + b_2} \Leftrightarrow (a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2) \geq (a_1 + a_2)^2 b_1 b_2 \Leftrightarrow a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2a_1 a_2 b_1 b_2.$$

It is easy to verify the last inequality. Now, for the inductive step,

$$\begin{aligned}
\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} + \frac{a_{n+1}^2}{b_{n+1}} &\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} + \frac{a_{n+1}^2}{b_{n+1}} \\
&\geq \frac{((a_1 + a_2 + \cdots + a_n) + a_{n+1})^2}{((b_1 + b_2 + \cdots + b_n) + b_{n+1})}
\end{aligned}$$

The last inequality is an application of the case $n = 2$ for two pairs of numbers $(a_1 + a_2 + \cdots + a_n), a_{n+1}$ and $(b_1 + b_2 + \cdots + b_n), b_{n+1}$. Hence the hypothesis follows. □

Problem 0.3 (Problem Eight)**.** Let $a_1, a_2, \ldots, a_n$ be any positive real numbers, then:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Solution. [Solution One] For the case $n \to 2n$,

$$\begin{aligned}\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{2+1} + \cdots + a_{2n}}{n}\right)\\ &\geq \sqrt{\sqrt[n]{a_1 a_2 \cdots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}} = \sqrt[2n]{a_1 a_2 \cdots a_{2n}}.\end{aligned}$$

The case $n \to n-1$ is as follow,

$$\begin{aligned}&\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}}{n}\\ &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1}\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)}\\ &\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n \geq a_1 a_2 \cdots a_{n-1}\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)\\ &\Rightarrow \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}\end{aligned}$$

Hence, by the base case and following $n \to 2n \to 2n-1 \to \cdots \to n+1$ chain of applications, the hypothesis follows. □

Solution. [Solution Two] We prove the following claim

> **Claim —** For $0 < x_1, x_2, \ldots, x_n$, where $x_1 x_2 \cdots x_n = 1$,
>
> $$x_1 + x_2 + \cdots + x_n \geq n \quad (*)$$

Proof. For $0 < x_1 \leq 1 \leq x_2$, then

$$x_2 - x_1 x_2 \geq 1 - x_1 \Rightarrow x_1 + x_2 \geq x_1 x_2 + 1.$$

Let's assume that (*) stand for $n-1$. Then $x_1 x_2 \cdots x_n = 1$. The trivial case when all x_i is 1 is obvious. WLOG, since the roles of x_i are interchangeable, let's assume that $x_{n-1} \leq 1 \leq x_n$.

From the case $n = 2$

$$x_1 + x_2 + \cdots + x_n = (x_1 + x_2 + \cdots + x_{n-2}) + (x_{n-1} + x_n) \geq x_1 + x_2 + \cdots + x_{n-2} + (x_{n-1} x_n) + 1$$

By the hypothesis,

$$x_1 + x_2 + \cdots + x_{n-2} + (x_{n-1} x_n) \geq n - 1 \Rightarrow x_1 + x_2 + \cdots + x_n \geq n.$$

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By applying $x_k = \frac{a_k}{\sqrt{\prod_{i=1}^n a_i}}$, $k = 1, 2, \ldots, n$, Then $x_1 x_2 \cdots x_n = 1$, thus

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{\prod_{i=1}^n a_i}} \geq n \Rightarrow \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

□

Problem 0.4 (Problem Nine)**.** Let $a_1, a_2, \dots, a_n$ be any real numbers satisfying $a_{i+j} \le a_i + a_j,\ \forall i, j \in \mathbb{Z}^+$. Prove that:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n,\ \forall n \ge 1.$$

Solution. The base case $n = 1$ is trivial. Let's now list all the inequalities for $1 \le k \le n$.

$$\begin{cases} a_1 & \ge a_1 \\ a_1 + \frac{a_2}{2} & \ge a_2 \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} & \ge a_3 \\ & \dots \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{n-1}}{n-1} & \ge a_{n-1} \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{n-1}}{n-1} + \frac{a_n}{n} & \ge a_n \end{cases}$$

By summing up

$$n \cdot a_1 + (n-1) \cdot \frac{a_2}{2} + \cdots + 1 \cdot \frac{a_n}{n} \ge a_1 + a_2 + \cdots + a_n.$$

By adding $a_1 + a_2 + \cdots + a_n$ to both sides, and note that $a_i + a_{n+1-i} \ge a_{n+1}$,

$$\begin{aligned} &(n+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right) \ge (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_n + a_1) \ge n a_{n+1} \\ &\Rightarrow a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \ge \frac{n}{n+1} a_{n+1} \Rightarrow a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \ge a_{n+1} \end{aligned}$$

The hypothesis follows. □

Problem 0.5 (Problem Ten)**.** Let $k \ge 3$ be an integer. Show that there exist odd integer x and y such that

$$2^k = 7x^2 + y^2.$$

Solution. For $k = 3$, $2^3 = 7 \cdot 1^2 + 1^2$. So $(1, 1)$ is a solution for $k = 3$.

Let assume that (x_k, y_k) is a solution for $2^k = 7x^2 + y^2$.

Now,

$$2^{k+1} = 2(7x_k^2 + y_k^2) = 7\left(\frac{x_k \pm y_k}{2}\right)^2 + \left(\frac{7x_k \mp y_k}{2}\right)^2.$$

It is easy to see that if x_k, y_k both odd, then $\dfrac{x_k + y_k}{2}$, or $\left|\frac{x_k - y_k}{2}\right|$ is odd (their sum is x_k or y_k).

If $\dfrac{x_k + y_k}{2}$ is odd, then we can choose $x_{k+1} = \dfrac{x_k + y_k}{2}$, $y_{k+1} = \left|\dfrac{7x_k - y_k}{2}\right|$, and both are odd.

If $\dfrac{x_k + y_k}{2}$ is even, then we can choose $x_{k+1} = \left|\dfrac{x_k - y_k}{2}\right|$, $y_{k+1} = \dfrac{7x_k + y_k}{2}$, and both are odd.

Thus, there exists a pair of odd integers (x_{k+1}, y_{k+1}), which is a solution for $2^{k+1} = 7x^2 + y^2$.

Therefore, the hypothesis is true for $k + 1$. Hence, it is true for all k. □