Pigeonhole Principle in Combinatorial Geometry

Nghia Doan

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In this article, we show two examples that can be solved with the Pigeonhole Principle.

Definition (Pigeonhole Principle)**.** The Pigeonhole Principle (also known as the Dirichlet box principle, Dirichlet principle or box principle) states that if $n+1$ or more holes are placed in n pigeons, then one pigeon must contain two or more holes. Another definition could be phrased as among any n integers, there are two with the same modulo-$n-1$ residue.

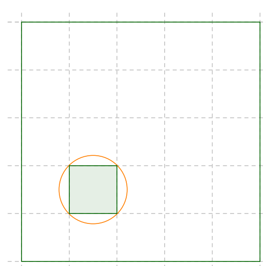
The extended version of the Pigeonhole Principle states that if k objects are placed in n boxes then at least one box must hold at least $\lceil \frac{k}{n} \rceil$ objects. Here $\lceil \cdot \rceil$ denotes the ceiling function.

Example (One)

There are 101 points in a unit square. Prove that five of them can be covered by a circle radius $\frac{1}{7}$.

Remark. *The idea: the idea comes from the given number of points* 101 *is just one more than* 4×25*, which means that if we divide the square into* 25 *holes, then among* 101 *points, or pigeons, there are* 5 *of them should be in a hole. The easiest way to divide a square into* 25 *pieces is to divide it into* 5×5 *grid.*

One. Lets divide the unit square into $5 \times 5 = 25$ squares as shown below.



According to the Piegonhole Principle, there exists a square where at least $\lceil \frac{101}{25} \rceil = 5$ points reside. The circle centered at the center of the square radius $\frac{1}{7}$ cover the whole square since,

$$\text{half of the diagonal of the square, } \frac{1}{2} \cdot \frac{\sqrt{2}}{5} < \boxed{\frac{1}{7}.}$$

□

Example (Two)

For $\mathbf{n} \geq 1$, on a $2n \times 2n$ board, $3n$ squares are marked. Prove that n rows and n columns can be selected so that they contain all marked squares.

Remark. First, we *get our hand dirty* by drawing some examples and see how it works. In the example below, where $n = 4$, in a 8×8 board we choose rows $2, 5, 7$ and 8; then columns d, f, g and h to cover all marked squares. Note that rows $2, 5, 7$, and 8 cover $8 = 2 \times 4$ marked squares.

Is it possible that n rows can be chosen such that they cover some $2n$ marked squares?

How about trying to select n rows so that they cover as many unmarked squares as possible?

This is called a *greedy* approach. A **greedy algorithm** is an algorithm that follows the problem-solving heuristic of making the locally optimal choice at some stage. A greedy strategy might not produce an optimal solution, but it can help to find a solution.

	a	b	c	d	e	f	g	h
8	x							
7		x						x
6							x	
5			x		x			
4						x		
3				x				
2			x	x		x		
1								x

Two. We first prove the claim we have noted in the remark.

Claim — Lets choose n rows such that **each of them cover as many squares as possible.** Then these rows cover at least $2n$ marked squares.

Proof. Assume that the number of marked squares covered by these square is less than $2n$, then there are more than $3n - 2n = n$ squares *not covered* by them, therefore the *non-selected* n rows cover at least $n + 1$ squares.

According to the Piegonhole Principle, there is one *non-selected* row that contains at least two marked squares. But by choice as above, **each of the selected rows should have at least as many markered squares as a non-selected rows,** thus the selected ones should cover at least $2n$ squares.

Then the number of marked squares covered by all square is at least $2n + (n + 1) = 3n + 1$. This exceeds the number of marked squares, which is $3n$, thus it is impossible. ■

Hence, at least $2n$ marked squares are covered by $\boxed{n \text{ selected rows.}}$ For at most n remaining uncovered squares, it is easy to choose $\boxed{n \text{ columns.}}$ □