Digits of a number

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In this article, we look at some problems with digits of number and the approaches to solve them using different methods.

1 Divisibility rules

In this section, we use the divisibility rules, i.e. what digits should a number have in order to be divisible by $3, 4$, or 8, and so on.

Example (One)

What is the greatest multiple of 8 whose digits are all different?

Solution. The *divisibility rule for* 8 states that the last three digits of a multiple of 8 must be divisible by 8. To create the largest $8-$digit number, the last three digits must be $0, 1$, and 2. Thus, the largest $3-$digit multiple of 8 with those digits is 120. Thus the desired number is $\boxed{9876543120.}$ □

Example (Two)

What is the least multiple of 36 that contains only digits 4 and 5.

Solution. Divisibility rule for 9 state that the sum of digits of the number must be $9, 18, \ldots$. Let examine the sum of the digits from the least possible value 9 and then going up. If the sum is 9, then 45 or 54 are not divisible by 4, so $9 = 4 + 5$ is not a possible sum. If the sum is 18, the $2-$digit multiple of 4 can be made from two pairs of 4 and 5 is 44. Thus the number is $\boxed{5544}$. □

Exercise (Three)**.** Find a $7-$digit number containing only digits 2 or digits 3 such that there are more of digits 2 than of digits 3 and the number is divisible by both 3 and 4.

2 Remainders of a perfect powers

In this section, we look at the remainders of a perfect power - a perfect square, a perfect cube, or a higher power of integer - when divided by an integer such as $3, 4, 8$, or 9, and so on.

Example (Four)

Is there a $5-$digit perfect square whose sum of digits is 29?

Solution. A perfect square is divisible by 3 or has a remainder of 1 when divided by 3 (why?). Since the remainder of a number when divided by 3 is the same as the remainder of its sum of digits when divided by 3, and 29 has a remainder of 2 when divided by 3 so there is no such number. $\square$

Example (Five)

Find the perfect cube **n** such that all digits of n are 9 except the unit digit, which is 5.

Solution. There is no such perfect cube since a perfect cube has a remainder $0, 1$, or 8 when divided by 9. $\square$

Exercise (Six)**.** Find $\mathbf{n} > 3$ such that the $(n+1)$-digit binary number $\overline{10\ldots01_2}$ is a perfect power of 3.

3 Digits as variables

In this section, we use some algebra tools to establish equations for the digits of a number, then solving those equations to obtain the value for them.

Example (Seven)

Digits **a**, **b**, and **c** are used to form $3-$digit numbers $\overline{abc}, \overline{bca}$, and $\overline{cab}$. The sum of these numbers is 1332, find $a + b + c$.

Solution. $\overline{abc} = 100a + 10b + c$, similarly with others. Their sum is $111(a + b + c) = 1332$, $a + b + c = 12$. $\square$

Example (Eight)

Find all $4-$digit number **n** whose sum of digits is $2010 - n$.

Solution. Let $n = \overline{abcd}$. Then $1001a + 101b + 11c + 2d = 2010$. If $a = 1$, then $b = 9$, so $11c + 2d = 100$, so $c = 8, d = 2$. If $a = 2$, then $b = c = 0, d = 4$. The solutions are $\boxed{1982,\ 2004}$. $\square$

Exercise (Nine)**.** Find a potitive integer **a** such that $(1 + 2 + \ldots + a) - 1000a$ is a $3-$digit number.

4 The last digits of a number

In this section, we show the use of so-called modular arithmetic in the easiest possible way to find the last digits of some numbers.

Example (Ten)

What is the last digit of $\left(\ldots\left((7)^7\right)^7\ldots\right)^7$? There are 1001 digits 7.

Solution. By testing $7 \equiv 7 \pmod{10}, 7^7 = (7)(7^2)^3 \equiv -7 \pmod{10}, (7^7)^7 \equiv (-7)^7 \equiv 7 \pmod{10}, \ldots$ By Induction Principle, it can be proved that the last digit of the generic expression is 7 if it has an odd amount of 7, otherwise it is 3. The given one has an odd number of 7, so its last digit is $\boxed{7.}$ □

Example (Eleven)

In how many zeros can the number $1^n + 2^n + 3^n + 4^n$ end for **n** positive integer?

Solution. For $n = 1$, and 2, the sum ends in one and two zeros. Now, for all $n \geq 3$, $2^n, 4^n$ are divisible by 8, and $1^n + 3^n$ congruent to 2 or 4 modulo 8. Thus, the sum cannot end in three or more zeros. □

Exercise (Twelf)**.** Find the last five digit of 5^{1981}.

5 Hints to the exercises

Hint 1 (Three)**.** First find the last two digits based on divisibility rule for 4. Then find the number of digits 2 in the first five digits.

Hint 2 (Six)**.** Let $\overline{10\ldots01}_2 = 2^n + 1 = 3^m$. Then casework based on the parity of m.

Hint 3 (Nine)**.** Investigate two cases, $a < 1999$ and $a \geq 2000$.

Hint 4 (Twelf)**.** Find the last 5 digits of $5^{1981} - 5^5 = 5^5(5^{1976} - 1)$.