Solving Forty Two Problems by the Induction Principle - Part I

Nghia Doan

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Problem 0.1 (Problem One)**.** $n \geq 3$ is a positive integer. Prove that $n!$ can be written as a sum of n distinct divisors of itself.

Solution. Our hypothesis is a *stronger* version of the problem's statement: for all $n \geq 3$, $n!$ can be written as a sum of n distinct divisors of itself with the smallest divisor is 1, in other words there exists positive integer k such that

$$n! = d_1 + d_2 + \cdots + d_k, \text{ where } d_i \mid n!, \ i = 1, 2, \ldots, k, \text{ and } d_1 = 1 < d_2 = n - 1 < \ldots < d_k \quad (*)$$

For the base case $n = 3$,

$$3! = 6 = 1 + 2 + 3, \ 1 \mid 3!, 2 \mid 3!, 3 \mid 3!.$$

Now, let's assume that the hypothesis is true for n, or (*) stands, then $\exists d_1 = 1 < d_2 = n - 1 < \ldots < d_k$.

$$(n+1)! = (n+1)(d_1 + d_2 + \cdots + d_k) = (n+1)(1 + d_2 + \cdots + d_k) = 1 + n + d_2 + \cdots + d_k$$

Thus there exists $\ell = k + 1$, and $e_1 = 1, e_2 = n, e_3 = (n+1)d_2, \ldots, e_{k+1} = (n+1)d_k$, such that

$$(n+1)! = e_1 + e_2 + \cdots + e_\ell, \text{ where } e_i \mid (n+1)!, \ i = 1, 2, \ldots, \ell, \text{ and } 1 = e_1 < n = e_2 < \ldots < e_\ell$$

Thus the hypothesis (*) is true for $n + 1$, therefore it is true for all $n \geq 3$. Hence, the weaker hypothesis of the problem follows. $\square$

Problem 0.2 (Problem Two)**.** $n \geq 1$ is a positive integer. Prove that:

$$x_n = \frac{1}{1} + \frac{1}{2^1} + \cdots + \frac{1}{2^n} < 2 \quad \text{and} \quad y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{7}{10}.$$

Solution. For the first question, by the formula for a geometric series

$$x_n = \frac{1}{1} + \frac{1}{2^1} + \cdots + \frac{1}{2^n} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n} < 2.$$

For the second question, we prove a *stronger* inequality, for $n \geq 4$,

$$y_n + \frac{1}{4n} < \frac{7}{10}$$

It is easy to verify that given inequality for $n \leq 3$, or $y_n < \frac{7}{10}$, $\forall n \leq 3$, and for $n = 4$,

$$y_4 + \frac{1}{4 \cdot 4} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{16} = \frac{1171}{1680} \approx 0.697 < 0.7 = \frac{7}{10}.$$

For the inductive step, if for $n \geq 4$,

$$y_n + \frac{1}{4n} < \frac{7}{10} \Rightarrow y_{n+1} + \frac{1}{4(n+1)} - y_n - \frac{1}{4n} = \left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right) + \left(\frac{1}{4(n+1)} - \frac{1}{4n}\right)$$
$$= \frac{1}{(2n+1)(2n+2)} - \frac{1}{4n(n+1)} = -\frac{1}{4n(n+1)(2n+1)} < 0 \Rightarrow y_{n+1} + \frac{1}{4(n+1)} < \frac{7}{10}$$

Thus the hypothesis (*) is true for $n+1$, therefore it is true for all $n \geq 1$. Hence, the weaker hypothesis of the problem follows. □

Problem 0.3 (Problem Three)**.** $n \geq 1$ is a positive integer. (f_n) is the Fibonacci sequence: $f_0 = f_1 = 1, f_{n+1} = f_n + f_{n-1}$. Prove that:

$$f_{2n+1} = f_{n+1}^2 + f_n^2 \quad (*)$$

Solution. [Solution One] We consider a *stronger* hypothesis, for m, n non-negative integers:

$$f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n \quad (**)$$

We prove it by induction based on $n \geq 0$.

The base case is trivial for $n = 0$, which means $f_{m+1} = f_{m+1}$, and for $n = 1$, which becomes $f_{m+2} = f_{m+1} + f_m$ that is the formula for the Fibonacci sequence.

For the inductive step, with the assumption that (**) is true for all $k = 0, 1, \ldots, n$,

$$f_{m+n+2} = f_{m+n+1} + f_{m+n} = (f_{m+1}f_{n+1} + f_m f_n) + (f_{m+1}f_n + f_m f_{n-1})$$
$$= f_{m+1}(f_{n+1} + f_n) + f_m(f_n + f_{n-1}) = f_{m+1}f_{n+2} + f_m f_{n+1}$$

Thus the hypothesis (*) is true for $n+1$, therefore it is true for all $n \geq 0$.

The desired identity is the case of (**) when $m = n$

$$f_{2n+1} = f_{(n)+(n)+1} = f_{(n)+1}f_{(n)+1} + f_{(n)}f_{(n)} = f_{n+1}^2 + f_n^2.$$

□

Solution. [Solution Two] Let $\alpha_1 = \frac{1+\sqrt{5}}{2}, \alpha_2 = \frac{1+\sqrt{5}}{2}$, the generic formula (closed-form) of a Fibonacci number is

$$f_n = \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n)$$

Note that

$$\alpha_1\alpha_2 = -1 \Rightarrow \alpha_1 + \frac{1}{\alpha_1} = \alpha_1 - \alpha_2 = \sqrt{5}, \alpha_2 + \frac{1}{\alpha_2} = \alpha_2 - \alpha_1 = -\sqrt{5}$$

Therefore

$$f_{n+1}^2 + f_n^2 = \frac{1}{5}\left[(\alpha_1^{2n+2} - 2(\alpha_1\alpha_2)^{n+1} + \alpha_2^{2n+2}) + (\alpha_1^{2n} - 2(\alpha_1\alpha_2)^n + \alpha_2^{2n})\right]$$
$$= \frac{1}{5}\left[\alpha_1^{2n+1}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \alpha_2^{2n+1}\left(\alpha_2 + \frac{1}{\alpha_2}\right)\right] = \frac{1}{5}\left(\alpha_1^{2n+1}\sqrt{5} + \alpha_2^{2n+1}(-\sqrt{5})\right)$$
$$= \frac{1}{\sqrt{5}}(\alpha_1^{2n+1} - \alpha_2^{2n+1}) = f_{2n+1}$$

□

Problem 0.4 (Problem Four)**.** $n \geq 1$ is a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right)$$

Solution. Let

$$a_n = \binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1}, \ b_n = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right)$$

We prove that for all $n \geq 1$,

$$\begin{cases} a_{n+1} = \frac{n+2}{2(n+1)} a_n + 1 & (*) \\ b_{n+1} = \frac{n+2}{2(n+1)} b_n + 1 & (**) \end{cases}$$

For the first identity (*), consider a term of a_n, for $i = 0, 1, \ldots n$,

$$\binom{n}{i}^{-1} = \frac{i!(n-i)!}{n!} = \frac{n+1}{n+2} \cdot \frac{(n+2)i!(n-i)!}{(n+1)!} = \frac{n+1}{n+2} \cdot \frac{((i+1)+(n+1-i))i!(n-i)!}{(n+1)!}$$
$$= \frac{n+1}{n+2} \cdot \left(\frac{(i+1)!(n-i)!}{(n+1)!} + \frac{i!(n+1-i)!}{(n+1)!} \right) = \frac{n+1}{n+2} \cdot \left(\binom{n+1}{i+1}^{-1} + \binom{n+1}{i}^{-1} \right)$$

By summing up

$$a_n = \frac{n+1}{n+2} \cdot \left(2a_{n+1} - \binom{n+1}{n+1}^{-1} - \binom{n+1}{0}^{-1} \right) = \frac{n+1}{n+2} \cdot (2a_{n+1} - 2) \Rightarrow a_{n+1} = \frac{n+2}{2(n+1)} a_n + 1.$$

For (**)

$$\frac{2^{n+2}}{n+2} b_{n+1} = \frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} + \frac{2^{n+2}}{n+2} = \frac{2^{n+1}}{n+1} b_n + \frac{2^{n+2}}{n+2} \Rightarrow b_{n+1} = \frac{n+2}{2(n+1)} b_n + 1.$$

Starting with $a_1 = b_1$, applying the Induction Principle on both sequences (a_n) and (b_n) at the same time, it is easy arrive at the conclusion that $a_n = b_n$ for all n. □

Problem 0.5 (Problem Five)**.** Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be a function with the following properties:

1. $f(2) = 2$;
2. $f(mn) = f(m)f(n), \ \forall m, n$;
3. $f(m) > f(n), \ \forall m > n$.

Prove that $f(n) = n, \ \forall n \in \mathbb{Z}^+$.

Solution. [Solution One] By (3):

$$f(2) = f(2 \cdot 1) = f(2)f(1) \Rightarrow f(1) = 1$$

Thus the base cases of $n = 1, 2$ are proven.

For the inductive step, let's assume that for all $k \leq 2n$, $f(k) = k$. We shall prove that

$$f(2n+1) = 2n+1, f(2n+2) = 2n+1.$$

Note that

$$f(2n+2) = f(2)f(n+1) = 2(n+1) \Rightarrow 2n = f(2n) < f(2n+1) < f(2n+2) = 2n+2 \Rightarrow f(2n+1) = 2n+1.$$

Thus for all n, $f(n) = n$. □

Solution. [Solution Two] By (3):

$$f(2) = f(2 \cdot 1) = f(2)f(1) \Rightarrow f(1) = 1$$

Thus the base cases of $n = 1, 2$ are proven.

For the inductive step, let's assume that for all $k < n$, $f(k) = k$.

Case 1: n is composite, then

$$\exists a, b \in \mathbb{Z}^+ : \ 1 < a, b < n, n = ab \Rightarrow f(n) = f(ab) = f(a)f(b) = ab = n$$

Case 2: n is prime, then $n - 1$ and $n + 1$ are composite. Following the same reasoning as in the previous case

$$f(n-1) = n-1, f(n+1) = n+1 \Rightarrow n-1 = f(n-1) < f(n) < f(n+1) = n+1 \Rightarrow f(n) = n.$$

The hypothesis follows. $\square$