

# Test Problems fore UMC K1

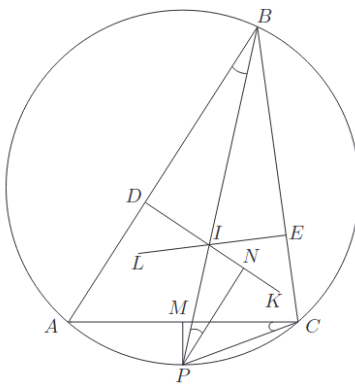
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**Problem 0.1** (Problem One). Given a triangle  $ABC$  satisfying  $AB + BC = 3 \cdot AC$ . The incircle of triangle  $ABC$  has center  $I$  and touches the sides  $AB$  and  $BC$  at the points  $D$  and  $E$ , respectively. Let  $K$  and  $L$  be the reflections of the points  $D$  and  $E$  with respect to  $I$ . Let  $P$  be the other intersection of  $BI$  with the circumcircle  $(ABC)$ . Prove that the points  $A, C, K, L$  lie on a circle centred at  $P$ .

*Proof.* Let  $M$  be the midpoint of  $AC$  and  $N$  the projection of  $P$  to  $IK$ . Since  $AB + BC = 3AC$ , we get  $BD = BE = AC$ , so  $BD = 2CM$ .

Furthermore,  $\angle ABP = \angle ACP$ , therefore  $\triangle DBI$  and  $\triangle MCP$  are similar in ratio 2.

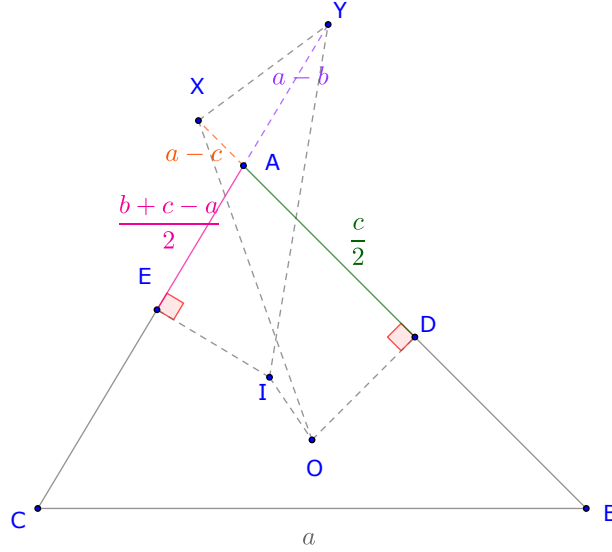


It is known that  $PA = PI = PC$ . Moreover,  $\angle NPI = \angle DBI$ , so that the triangles  $PNI$  and  $CMP$  are congruent. Hence  $ID = 2PM = 2IN$ ; i. e.  $N$  is the midpoint of  $IK$ .

This shows that  $PN$  is the perpendicular bisector of  $IK$ , so  $PK = PI$ . Analogously,  $PA = PL = PI$ . So  $P$  is the centre of the circle through  $A, K, I, L$  and  $C$ .  $\square$

**Problem 0.2** (Problem Two). In triangle  $ABC$ , let  $BC$  be the longest side. Point  $X$  is chosen on side  $AB$  such that  $BX = BC$ . Similarly, point  $Y$  is chosen on  $AC$  such that  $CY = BC$ . Prove that  $OI$  is perpendicular to  $XY$ , where  $O$  and  $I$  are the circumcenter and incenter, respectively, of triangle  $ABC$ .

*Proof.* Let  $AB = c, BC = a, CA = b$ . Then



$$XA = a - c, AD = \frac{c}{2}, DX = DA + XA = \frac{c}{2} + (a - c) = a - \frac{c}{2}$$

$$OX^2 = OD^2 + DX^2 = OA^2 - AD^2 + DX^2 = OA^2 - \left(\frac{c}{2}\right)^2 + \left(a - \frac{c}{2}\right)^2 = OA^2 + a(a - c)$$

$$\text{Similarly } OY^2 = OA^2 + a(a - b) \Rightarrow OY^2 - OX^2 = a(c - b).$$

Similarly for  $I$ ,

$$YA = a - b, EA = \frac{c + b - a}{2}, EY = EA + YA = \frac{c + a - b}{2}$$

$$IY^2 = IE^2 + EY^2 = IA^2 - EA^2 + EY^2 = IA^2 - \left(\frac{c + b - a}{2}\right)^2 + \left(\frac{c + a - b}{2}\right)^2 = IA^2 + c(a - b)$$

$$\text{Similarly } IX^2 = IA^2 + b(a - c) \Rightarrow IY^2 - IX^2 = a(c - b).$$

Hence, by the Perpendicularity Lemma  $OI$  is perpendicular to  $XY$ . □

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**Problem 0.3** (Problem Three). 25 flies are resting on the outdoor table in the garden, waiting for lunch to be served.

- It is known that for any 3 of them, 2 are at a distance less than 20 cm.
- There are at least a pair of flies that are further than 20 cm from each other.

Minh's mother gave him a fly swatter, shown below, with a hoop of radius 20 cm. With a single strike he can swat the flies where the hoop landed. In *at least* how many strikes can he swat all of them?

*Assume that Minh is so fast that the flies do not have time for reaction during and between his lightning strikes.*



*Solution.* If no 2 flies are further than 20 cm from each other, Minh can strike them all in 1 strike by aiming the center of the swatter at any fly. But this is not the case, so let's assume there are 2 flies,  $A$  and  $B$ , that are more than 20 cm apart.

Then, every other fly is either in a 20 cm radius of  $A$  or in a 20 cm radius of  $B$ . Out of the 23 remaining flies either at least 12 will be in the 20 cm radius of  $A$  or 12 will be in the 20 cm radius of  $B$ .

Swatting that the  $A$  or  $B$  fly with the center of the swatter kills at least 13. Thus, by  $\boxed{2}$  strikes, he can swat them all.  $\square$