A picture is worth a thousand words - Part 3

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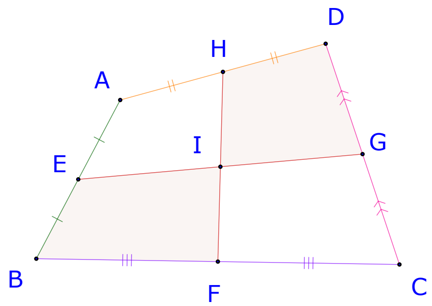
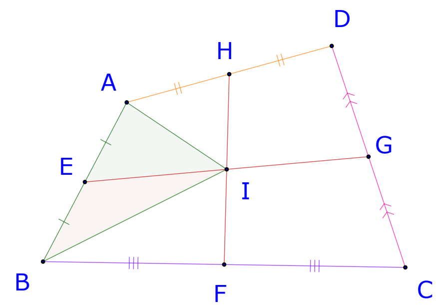
This article is the third part of the series on investigating a number of ways to *prove area equality without writing lengthy proof.*

Example (Example 11)

In the convex quadrilateral $ABCD$, E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. EG and FH intersect at I. Prove that

$$[EBFI] + [GDHI] = \frac{1}{2}[ABCD].$$



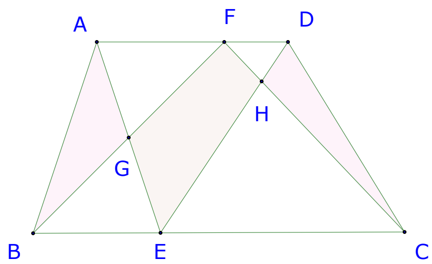


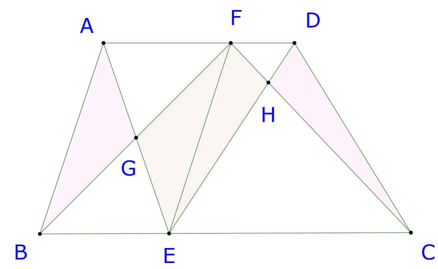
Proof. Connect AI. Since E is midpoint of AB, thus the triangles AEI and EIB have the same area. Similarly the area of BIF and CFI are the same. Thus $[EBIF] = \frac{1}{2}[ABCI]$. Similarly $[GDHI] = \frac{1}{2}[CDAI]$. Summing up, we have $[EBFI] + [GDHI] = \frac{1}{2}[ABCD]$. $\square$

Example (Example 12)

$ABCD$ is a trapezoid. E, F are arbitrary points on BC and AD, respectively. AE, BF intersect at G and CF, DE intersect at H. Prove that:

$$[ABG] + [CDH] = [EFGH].$$



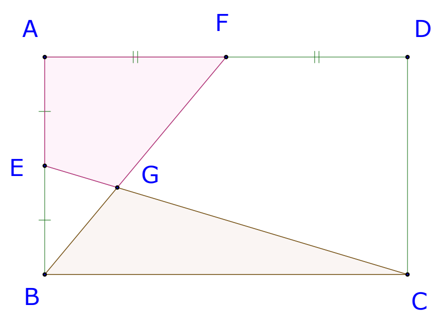


Proof. Connect EF. Since BE parallel with AF so $[ABF] = [AEF]$, thus $[ABG] = [FEG]$. Similarly $[CDH] = [FEH]$. Summing up, we have $[ABG] + [CDH] = [EFGH]$. □

Example (Example 13)

$ABCD$ is a rectangle. E, F are midpoints of AB, AD, respectively. CE, BF intersect at G. Prove that:

$$[AEGF] = [BGC].$$



Proof. It is easy to see that the triangles ABF and EBC have the same area

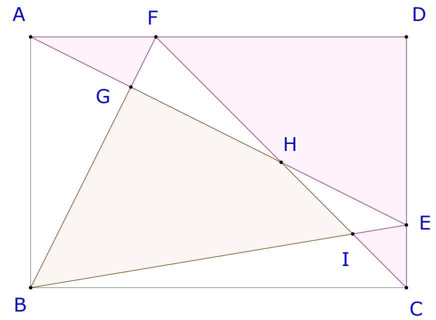
$$[ABF] = \frac{AB \cdot AF}{2} = \frac{AB \cdot AD}{4}, \ [EBC] = \frac{EB \cdot BC}{2} = \frac{AB \cdot AD}{4}$$
$$\Rightarrow [AEGF] = [ABF] - [EBG] = [EBC] - [EBG] = [BGC].$$

□

Example (Example 14)

$ABCD$ is a rectangle. E, F are arbitrary points on of CD, DA, respectively. AE intersects BF, BE at G, H, respectively. CF intersects BE at I. Prove that:

$$[AGF] + [FHED] + [CEI] = [BGHI].$$



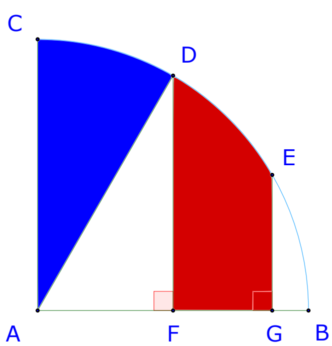
Proof. It is easy to see that the triangles BDF and CDF have the same area, similarly the triangles BDE and ADE have the same area, therefore: $[BFDE] = [CDF] + [ADE]$. But

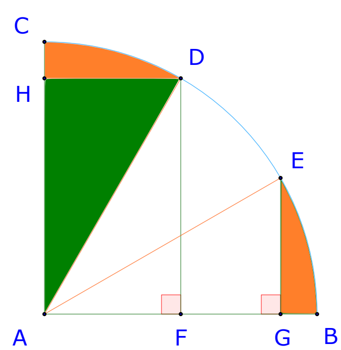
$$[BFDE] = [BGHI] + [GFH] + [FHED] + [HIE],$$
$$[CDF] + [ADE] = [AGF] + [GFH] + 2[FHED] + [HIE] + [CEI]$$

Hence, $[BGHI] = [AGF] + [FHED] + [CEI]$. □

Example (Example 15)

ABC is a quarter of circle. D, E trisect arc $\widehat{BC}$: $\widehat{CD} = \widehat{DE} = \widehat{EB}$. Prove that the blue circular sector ACD and the red region $DEGF$ have the same area.





Proof. Since D, E trisect arc $\widehat{BC}$, the circula sector ACD and AEB are congruent. Thus, if $DH \perp AC$, then the orange regions CDH and EGB are the same. It is easy to see that $\triangle AHD \cong \triangle ADF$, thus the sum of the area of $\triangle ADF$ and the region EGB is the same as the blue circular sector ACD. This means that the area $DEGF$ is one-third of the quarter circle ABC, thus it is the same as the area of the blue circular sector ACD □