# Solving Forty Two Problems by the Induction Principle - Part VII

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**Problem 0.1** (Problem Thirty Six)**.** On the cirle of radius 1 with the center $O$ there are given $2n+1$ points $P_1P_2\dots P_{2n+1}$, which lie on one side of a diameter. Prove that

$$|\overrightarrow{OP_1}+\overrightarrow{OP_2}+\ldots+\overrightarrow{OP_n}|\geq 1.$$

*Solution.* For $n=0$ we have $|\overrightarrow{OP_1}|=1$, thus the hypothesis stands.

Assume that it is true for $2n+1$ vectors. By the Extremal Principle, there exists the maximal angle any two vectors, WLOG, let there be $\overrightarrow{OP_1}$ and $\overrightarrow{OP_{2n+3}}$. Now, by induction hypothesis,

$$|\overrightarrow{OA}|=|\overrightarrow{OP_2}+\overrightarrow{OP_2}+\ldots+\overrightarrow{OP_{2n+2}}|\geq 1$$

Note that $\overrightarrow{OA}$ is inside the angle $\angle P_1OP_{2n+3}$, therefore it forms an acute angle with the vector

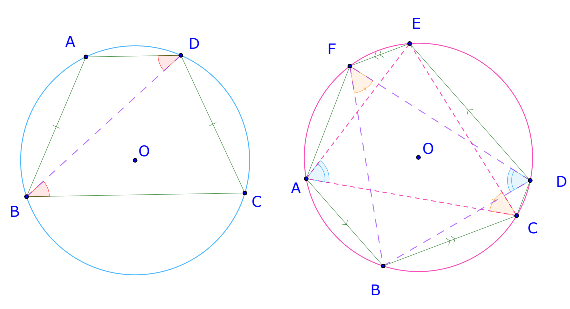
$$\overrightarrow{OB}=\overrightarrow{OP_1}+\overrightarrow{OP_{2n+3}}$$

which bisects the angle $\angle P_1OP_{2n+3}$. Thus

$$|\overrightarrow{OA}+\overrightarrow{OB}|\geq|\overrightarrow{OA}|=1.$$

□

**Problem 0.2** (Problem Thirty Seven)**.** $A_1A_2\dots A_{2n}$ is a polygon inscribed in a circle. It is known that all the pairs of its opposite sides except one are parallel. Prove that for any odd $n$, the remaining pair of sides is also parallel. Prove that for $n$ even, the length of the exceptional sides are equal.




*Solution.* We first prove case $n=2$, note that in the left diagram $AD\parallel BC$, then $\angle ADB=\angle CBD$, thus $AB=CD$.

For the case $n=3$, let $AB\parallel DE$, so $\angle ACE=\angle BFD$, and $BC\parallel EF$, so $\angle CAE=\angle BDF$, from here $\triangle CAE\sim\triangle FDB$, or $\angle AEC=\angle DBF$, thus $DC\parallel AF$.

Let assume that the statement is true for $(2n-2)-$gon $A_1A_2\ldots A_{2n}$, where $A_1A_2 \parallel A_{n+1}A_{n+2}, \ldots, A_{n-1}A_n \parallel A_{2n-1}A_{2n}$. Then considering the $(2n-2)-$gon $A_1A_2 \parallel A_{n-1}A_{n+1}\ldots A_{2n-1}$, by the hypothesis, for $n$ odd: $A_{n-1}A_{n+1} = A_{2n+1}A_1$ and for $n$ even $A_{n-1}A_{n+1} \parallel A_{2n-1}A_1$.

Consider now the triangles $A_{n-1}A_nA_{n+1}$ and $A_{2n-1}A_{2n}A_1$.

*Case 1:* if $n$ is even. then $\overrightarrow{A_{n-1}A_n}$ and $\overrightarrow{A_{2n-1}A_{2n}}$ as well as $\overrightarrow{A_{n-1}A_{n+1}}$ and $\overrightarrow{A_{2n-1}A_{2n}}$ are parallel and oppositely directed. Hence $\angle A_nA_{n-1}A_{n+1} = \angle A_1A_{2n-1}A_{2n}$ and $A_nA_{n+1} = A_{2n}A_1$ since they are chords that cut equal arcs.

*Case 2:* if $n$ is odd. then $A_{n-1}A_{n+1} = A_{2n-1}A_2$, i.e. $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$. In the hexagon $A_{n-1}A_nA_{n+1}A_{2n-1}A_{2n}A_1$ we have $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$, $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$, hence from the base case $n=3$, $A_nA_{n+1} \parallel A_{2n-1}A_1$. □

**Problem 0.3** (Problem Thirty Eight)**.** Let

$$a_1 = a_2 = 1,\ a_{n+2} = a_{n+1} + \frac{a_n}{3^n},\ \forall n \geq 1.$$

Prove that $a_n \leq 2,\ \forall n \geq 1$.

*Solution.* Let's prove that $a_n \leq 2 - \frac{1}{3^{n-2}},\ \forall n \geq 2 \quad (*)$

For $n=2$, $a_2 = 1 \leq 2 - \frac{1}{3^0} = 1$. For $n=3$, $a_3 = a_2 + \frac{a_1}{3^1} = 1 + \frac{1}{3} < 2 - \frac{1}{3}$.

Let's assume that (*) stand for all $k \leq n$, $n \geq 3$, then

$$a_{n+1} = a_n + \frac{a_{n-1}}{3^{n-1}} < 2 - \frac{1}{3^{n-2}} + \frac{2}{3^{n-1}} - \frac{1}{3^{2n-4}} = 2 - \frac{1}{3^{n-1}} - \frac{1}{3^{2n-4}} < 2 - \frac{1}{3^{n-1}}.$$

□

**Problem 0.4** (Problem Thirty Nine)**.** Let $x_0, x_1, \ldots, x_{1995}$ be positive real numbers,

$$x_0 = x_{1995} = 1,\ x_{n-1} + \frac{2}{x_{n-1}} = 2x_n + \frac{1}{x_n},\ \forall n = 1, 2, \ldots, 1995$$

Find the maximal value that $x_0$ can have.

*Solution.* First

$$x_{n-1} + \frac{2}{x_{n-1}} = 2x_n + \frac{1}{x_n} \Leftrightarrow (2x_n - x_{n-1})(x_nx_{n-1} - 1) = 0 \Rightarrow x_n = \frac{1}{2}x_{n-1},\ \text{or}\ x_n = \frac{1}{x_{n-1}}.$$

We prove by induction that

$$x_n = 2^{k_n}x_0^{e_n},\ k_n \in \mathbb{Z},\ |k_n| \leq n,\ e_n = (-1)^{n-k_n}.$$

This is true for $n=0$. Assume that it is true for some $n$, then

$$x_{n+1} = \frac{1}{2}x_n = 2^{k_n-1}x_0^{e_n} = 2^{k_{n+1}}x_0^{e_{n+1}},\ \text{where}\ k_{n+1} = k_n - 1, e_{n+1} = (-1)^{n-k_n} = (-1)^{(n+1)-(k_n-1)},$$
$$x_{n+1} = \frac{1}{x_n} = 2^{-k_n}x_0^{-e_n} = 2^{k_{n+1}}x_0^{e_{n+1}}\ \text{where}\ k_{n+1} = -k_n, e_{n+1} = (-1)^{n-k_n+1} = (-1)^{(n+1)-(k_n)}.$$

Now $x_0 = x_{1995} = 2^{k_{1995}}x_0^{e_{1995}}$, so, $e_{1995} = (-1)^{1995-k_{1995}}$. $e_{1995}$ cannot be 1 because then $k_{1995}$ would be odd, contradicting that $2^{k_{1995}} = 1$. So $e_{1995} = -1$, thus $x_0^2 = 2^{k_{1995}} \leq 2^{1994}$.

Thus the maximal value of $x_0$ is $\boxed{2^{997}.}$

When this can happen? $x_k = 2^{997-k}$ for $k = 0, 1, \ldots, 1994$, and $x_{1995} = (x_{1994})^{-1} = 2^{997}$. □

**Problem 0.5** (Problem Fourty)**.** Let $a_0 > 5$ be an odd integer,

$$a_{n+1} = \begin{cases} a_n^2 - 5 \text{ if } a_n \text{ is odd,} \\ \frac{1}{2}a_n \text{ otherwise} \end{cases} \quad \forall n \geq 0$$

Prove that this sequence is not bounded.

*Solution.* We prove that

**Claim** — $a_{3n}$ is odd, $a_{3n} > a_{3n-3} > \ldots > a_0 > 5$.

*Proof.* It is easy to verify the case $a_0 > 5$, odd integer. Let assume $a_{3n}$ is odd, so $a_{3n+1} = a_{3n}^2 - 5 \equiv 4 \pmod 8$. This means that $a_{3n+2} = \frac{1}{2}a_{3n+1}$, $a_{3n+3} = \frac{1}{2}a_{3n+2}$, and $a_{3n+3}$ is odd.

In addition $a_{3n+3}\frac{1}{4}(a_{3n}^2 - 5) > a_{3n}$, $(a_{3n} > 5)$ thus $a_{3n+3} > a_{3n}$. ■

□

**Problem 0.6** (Problem Fourty One)**.** Let $x$ be a real number and $n \geq 1$ positive integer. Prove that

$$|\sin(nx)| \leq n|\sin x|.$$

*Solution.* The case $n = 1$ is clear.

For the inductive step, consider

$$|\sin(n+1)x = |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \leq |\sin(nx)| + |\sin(x)|.$$

□

**Problem 0.7** (Problem Fourty Two)**.** Prove that for $a > 0$ and $n$ positive integer:

$$2(1 + a^{n+1})^3 \geq (1 + a^3)(1 + a^n)^3$$

Prove that for $a, b, c$ positive real numbers,

$$2(a^{2023} + 1)(b^{2023} + 1)(c^{2023} + 1) \geq (1 + abc)(a^{2022} + 1)(b^{2022} + 1)(c^{2022} + 1).$$

*Solution.* For $a > 0$ and $n$ positive integer, we prove by induction that:

**Claim** — $2(1 + a^{n+1})^3 \geq (1 + a^3)(1 + a^n)^3$.

*Proof.* For $n = 1$ then

$$2(1 + a^2)^3 \geq (1 + a^3)(1 + a)^3 \Leftrightarrow (a - 1)^4(a^2 + a + 1)$$

Assume that it is true for $n$, or

$$2(1 + a^{n+1})^3 \geq (1 + a^3)(1 + a^n)^3.$$

Since

$$(1 + a^{n+2})(1 + a^n) \geq (1 + a^{n+1})^2 \Rightarrow \frac{1 + a^{n+2}}{1 + a^{n+1}} \geq \frac{1 + a^{n+1}}{1 + a^n}$$

$$\Rightarrow 2(1 + a^{n+2})^3 = 2(1 + a^{n+1})^3 \left(\frac{1 + a^{n+2}}{1 + a^{n+1}}\right)^3$$

$$\geq (1 + a^3)(1 + a^n)^3 \left(\frac{1 + a^{n+1}}{1 + a^n}\right)^3 = (1 + a^3)(1 + a^{n+1})^3$$

■

Now it is easy to verify that

$$(1+a^3)(1+b^3(1+c^3) \geq (1+abc)^3.$$

Then

$$\begin{aligned}
&(2(a^{2023}+1)(b^{2023}+1)(c^{2023}+1))^3 = 2(a^{2023}+1)^3 \cdot 2(b^{2023}+1)^3 \cdot 2(c^{2023}+1)^3 \\
&\geq (1+a^3)(1+a^{2022}+1)^3 \cdot (1+b^3)(1+b^{2022}+1)^3 \cdot (1+c^3)(1+c^{2022}+1)^3 \\
&\geq (1+abc)^3((a^{2022}+1)(b^{2022}+1)(c^{2022}+1))^3
\end{aligned}$$

By taking cubic root, the result follows. $\square$