Solving Forty Two Problems by the Induction Principle - Part V

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October 11, 2024

Problem 0.1 (Problem Twenty Three)**.** The sequence (a_n) is defined by,

$$a_1 = \frac{1}{2}, a_{n+1} = \frac{2n-1}{2n+2}a_n$$

Prove that $a_1 + a_2 + \cdots + a_n < 1$ for $n \geq 1$.

Solution. Consider $s_n = a_1 + a_2 + \cdots + a_{n-1} + 2na_n,\ \forall n \geq 2$, then , so

$$a_1 = \frac{1}{2},\ a_2 = \frac{1}{4}\frac{1}{2},\ \Rightarrow s_2 = a_1 + 2(2)a_2 = \frac{1}{2} + \frac{1}{2} = 1.$$

We prove by induction $s_n = 1, \forall n \geq 2$, with the base case verified.

$$s_{n+1} = a_1 + a_2 + \cdots + a_n + 2(n+1)a_{n+1} = s_n + (1-2n)a_n + 2(n+1)a_{n+1} = s_n + (1-2n)a_n + (2n-1)a_n = s_n = 1.$$

Therefore the hypothesis is true, thus

$$a_1 + a_2 + \cdots + a_n = s_n - (2n-1)a_n < 1.$$

□

Problem 0.2 (Problem Twenty Four)**.** Show that for any positive integer n

$$\sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r} \quad (*)$$

Solution. For $n = 1$, both sides become 1, thus true.

Assuming that it is true for $n \geq 1$, then

$$\begin{aligned}
\sum_{r=1}^{n+1} \frac{1}{r}\binom{n+1}{r} - \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} &= \sum_{r=1}^{n} \frac{1}{r}\left(\binom{n+1}{r} - \binom{n}{r}\right) + \frac{1}{n+1}\binom{n+1}{n+1} \\
&= \sum_{r=1}^{n} \frac{1}{r}\left(\binom{n}{r-1}\right) + \frac{1}{n+1} = \frac{1}{n+1}\left(\sum_{r=1}^{n}\binom{n+1}{r}\right) + \frac{1}{n+1} \\
&= \frac{1}{n+1}\left(\sum_{r=0}^{n}\binom{n+1}{r} - 2 + 1\right) = \frac{2^{n+1} - 1}{n+1}
\end{aligned}$$

Thus

$$\sum_{r=1}^{n+1} \frac{1}{r}\binom{n+1}{r} = \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} + \frac{2^{n+1} - 1}{n+1} = \sum_{r=1}^{n+1} \frac{2^r - 1}{r}$$

□

Problem 0.3 (Problem Twenty Five)**.** Find all functions $f:\ \mathbb{Z} \to \mathbb{R}$ such that:

$$f(1) = \frac{5}{2}, f(m)f(n) = f(m+n) + f(m-n),\ \forall m, n \in \mathbb{Z}.$$

Solution. We prove that $f(n) = 2^n + 2^{-n},\ \forall n \geq 0 \quad (*)$ First

$$f(0)f(1) = 2f(1) \Rightarrow f(0) = 2$$

Assuming that (*) is true for all $k \leq n$, then

$$f(n)f(1) = f(n+1) + f(n-1) \Rightarrow f(n) = f(n)f(1) - f(n-1) = 2^{n+1} + 2^{-(n+1)}$$

For n negative, note that $f(0)f(n) = f(n) + f(-n) \Rightarrow f(-n) = f(n)$, in other words f is an even function, thus $f(-n) = f(n) = 2^n + 2^{-n},\ \forall n \in \mathbb{Z}$. □

Problem 0.4 (Problem Twenty Six)**.** Let $n \geq 1$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \leq x_n \leq x_{n-1} \leq x_2 \leq x_1$. Let

$$\begin{aligned} s_n &= x_1 - x_2 + \cdots + (-1)^n x_{n-1} + (-1)^{n+1} x_n \\ S_n &= x_1^2 - x_2^2 + \cdots + (-1)^n x_{n-1}^2 + (-1)^{n+1} x_n^2 \end{aligned}$$

Prove that $s_n^2 \leq S_n$.

Solution. First we prove that

Claim — $s_n \geq 0$, for all $n \geq 1$.

Proof. For the base case $s_1 = x_1 \geq 0$. Assume that $s_n \geq 0$.

Case 1: n is even

$$s_n \geq 0, x_{n+1} \geq 0 \Rightarrow s_{n+1} = s_n + (-1)^{n+2} x_{n+1} = s_n + x_{n+1} \geq 0.$$

Case 2: n is odd

$$s_n \geq 0, x_n \geq x_{n+1} \Rightarrow s_{n+1} = s_{n-1} + (-1)^{n+1} x_{n-1} + (-1)^{n+2} x_{n+1} = s_n + x_n - x_{n+1} \geq 0.$$

Thus $s_n \geq 0,\ \forall n \geq 1$. ■

Now we proof the problem statement by induction.

For the base case $n = 1$, $S_1 = s_1^2 \geq s_1^2$. For $n = 2$

$$S_2 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) \geq (x_1 - x_2)^2 = s_2^2.$$

Let assume that the hypothesis is true for n, then

Case 1: $n = 2k$

$$\begin{aligned} & s_{2k+2}^2 = (s_{2k+1} - x_{2k+2})^2 = s_{2k+1}^2 - 2s_{2k+1}x_{2k+2} + x_{2k+2}^2 \\ & \leq S_{2k+1} - x_{2k+2}^2 + 2x_{2k+2}(s_{2k+1} - x_{2k+2}) = S_{2k+2} - 2s_{2k+2}x_{2k+2} \\ & \leq S_{2k+2}. \end{aligned}$$

Case 2: $n = 2k+1$

$$\begin{aligned} s_{2k+3}^2 &= (s_{2k+1} - x_{2k+2} + x_{2k+3})^2 = s_{2k+1}^2 - 2s_{2k+1}(x_{2k+2} - x2k+3) + (x_{2k+2} - x_{2k+3})^2 \\ &\le S_{2k+1} - x_{2k+2}^2 + x_{2k+3}^2 - 2s_{2k+1}(x_{2k+2} - x2k+3) + 2x_{2k+2}^2 - 2x_{2k+2}x_{2k+3} \\ &= S_{2k+1} - x_{2k+2}^2 + x_{2k+3}^2 - 2(s_{2k+1} - x_{2k+2})(x_{2k+2} - x2k+3) \\ &= S_{2k+3} - 2s_{2k+2}(x_{2k+2} - x2k+3) \le S_{2k+3} \end{aligned}$$

□

Problem 0.5 (Problem Twenty Seven)**.** Let $n \ge 1$ be a non-negative integer. Prove that for all real number x,

$$\sum_{k=0}^{n} |\sin(2^k x)| \le 1 + \frac{\sqrt{3}}{2} n.$$

Solution. We prove that

Claim —

$$2|\sin x| + |\sin 2x| \le \frac{3\sqrt{3}}{2}.$$

Proof.

$$\begin{aligned} 2|\sin x| + |\sin 2x| &= 2|\sin x|(1 + |\cos x|) \\ &= 2\sqrt{(1 - (|\cos x|)^2)\,(1 + |\cos x|)^2} = 2\sqrt{(1 - |\cos x|)\,(1 + |\cos x|)^3} \\ &= \frac{2}{\sqrt{3}}\sqrt{3(1 - |\cos x|)(1 + |\cos x|)(1 + |\cos x|)(1 + |\cos x|)} \\ &\le \frac{2}{\sqrt{3}}\sqrt{\left(\frac{3(1 - |\cos x|) + (1 + |\cos x|) + (1 + |\cos x|) + (1 + |\cos x|)}{4}\right)^4} \\ &= \frac{2}{\sqrt{3}}\left(\frac{3}{2}\right)^2 = \frac{3\sqrt{3}}{2} \end{aligned}$$

■

For the base case for $n = 1$,

$$\sum_{k=0}^{n} |\sin(2^k x)| = \left(\frac{2}{3}|\sin x| + \frac{1}{3}|\sin 2x|\right) + \left(\frac{1}{3}|\sin x| + \frac{2}{3}|\sin 2x|\right) \le \frac{\sqrt{3}}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) = 1 + \frac{\sqrt{3}}{2}n.$$

Furthermore, from the base case, by replacing x with $2^n x$,

$$\frac{2}{3}|\sin 2^n x| + \frac{1}{3}|\sin 2^{n+1} x| \le \frac{\sqrt{3}}{2}.$$

Now, let's assume that the inequality stands for $n-1$,

$$\begin{aligned} &\sum_{k=0}^{n-1} |\sin(2^k x)| + \frac{2}{3}|\sin 2^n x| + \frac{1}{3}|\sin 2^{n+1} x| \le \frac{\sqrt{3}}{2}(n-1) + \frac{\sqrt{3}}{2} \\ \Rightarrow &\sum_{k=0}^{n+1} |\sin(2^k x)| \le \frac{\sqrt{3}}{2}(n) + \frac{1}{3}|\sin 2^n x| + \frac{2}{3}|\sin 2^{n+1} x| \le \frac{\sqrt{3}}{2}(n+1) \end{aligned}$$

□