# A picture is worth a thousand words - Part 2

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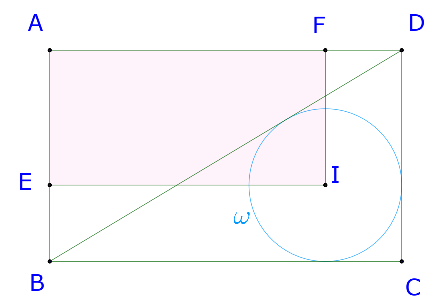
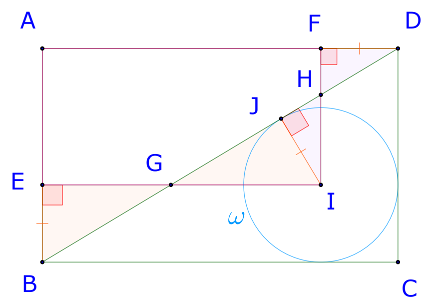
This article is the second part of the series on investigating a number of ways to *prove area equality without writing lengthy proof.*

**Example** (Example 6)

$ABCD$ is a rectangle. $E, F$ are arbitrary points on of $CD, DA$, respectively. Circle $\omega$ center at $I$ is tangent to sides $BC, CD$, and diagonal $BD$. Lines through $I$ parallel to the sides of the rectangle intersect $AB$ at $E$ and $DA$ at $F$. Prove that

$$[AEIF] = \frac{1}{2}[ABCD].$$




*Proof.* Let $I$ be the tangent point of circle $\omega$ with $BD$. $\triangle BEG$ and $\triangle CJG$ are both right triangles and have the short legs with the same length $EB = IJ$ (= the radius of $\omega$), thus they are congruent and have the same area. Similarly $\triangle IJH$ and $\triangle FDH$ have the same area.

$$[AEIF] = [AEGHF] + [GIH] = [AEGHF] + [GIJ] + [IJH] = [AEGHF] + [EGB] + [HFD] = \frac{1}{2}[ABCD].$$

□

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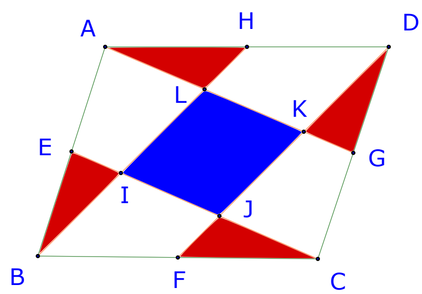
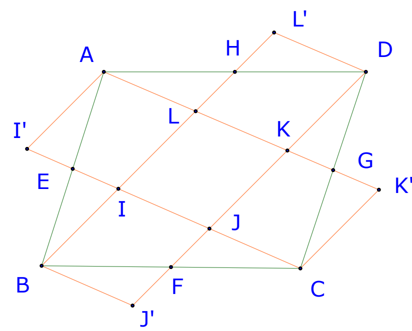
**Example** (Example 7)

$ABCD$ is a parallelogram. $E, F, G$, and $H$ are midpoints of $AB, BC, CD$, and $DA$, respectively. Segments $AG, BH, CE, EF$ intersect at $I, J, K$, and $L$, as shown below. Prove that the area of the blue region is the same as the sum of the areas of the red regions.

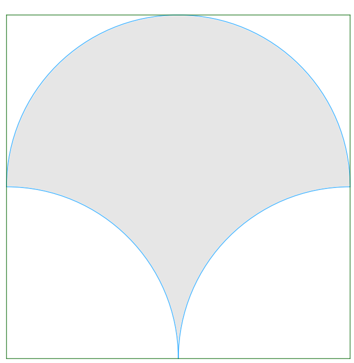


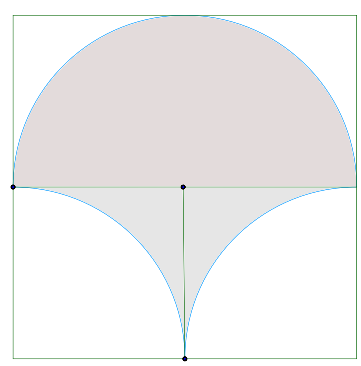




*Proof.* Let $I', J', K', L'$ be the reflections of $I, J, K, L$ over $E, F, G, H$, respectively. It is easy to prove that $ALII'$, $BIJJ'$, $CJKK'$, and $DKLL'$ as well as $IJKL$ are congruent parallelograms. Thus, the sum of the area of the red region is equivalent to the area of one of these parallelograms. □

**Example** (Example 8)

A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.

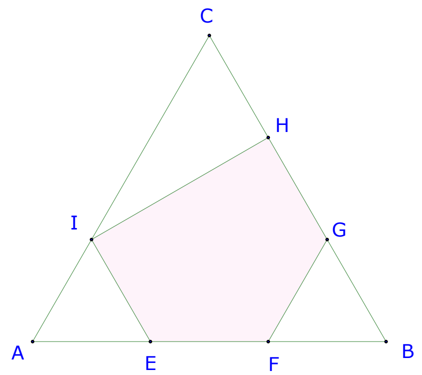




*Proof.* The area of the shaded region contains a half circle plus twice of the difference of a quarter of the square with the area of a quarter circle. The half circle and two quarter circles cancel out, leaving twice quarter of the square, thus the ratio is $\frac{1}{2}$. □

**Example** (Example 9)

$\triangle ABC$ is equilateral. Points $E, F$ trisect the side $AB$, point $GF, H$ trisect the side $BC$. $IE$ is parallel to $BC$. Find the ratio of the shaded area to the area of $\triangle ABC$.




*Proof.* Since $AE = \frac{1}{3}AB$, thus $[AIE] = \left(\frac{1}{3}\right)^2[ABC] = \frac{1}{9}[ABC]$. $[BGF] = [AIE]$. $\triangle CHI$ is right triangle at $H$, $CH = \frac{1}{3}BC$, $IH$ is two-third of the altitude from $A$ to $BC$, thus $[CIH] = \frac{1}{3} \cdot \frac{2}{3}[ABC]$. Hence

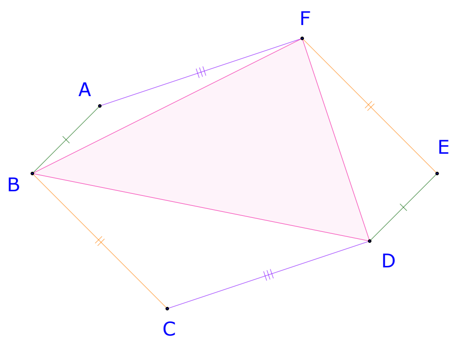
$$\frac{[EFGHI]}{[ABC]} = 1 - \frac{1}{9} - \frac{1}{9} - \frac{2}{9} = \frac{5}{9}.$$

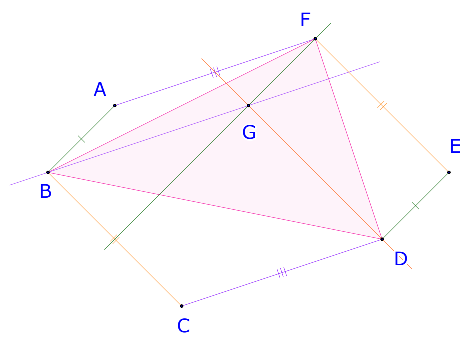
□

**Example** (Example 10)

In the hexagon $ABCDEF$, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$, $AB = DE$, $BC = EF$, and $CD = FA$. Prove that

$$[BDF] = \frac{1}{2}[ABCDEF].$$




*Proof.* Draw lines through $B$ parallel with $AD$ and $CD$, through $D$ parallel with $CB$ and $EF$, through $F$ parallel with $AB$ and $ED$. They are concurrent at $G$ (all meet at $G$, can you prove it?) and divide the hexagon into three parallelograms $ABGF$, $CDGB$, and $DEFG$. They also divide the shaded triangle $BDF$ into three triangles, each's area is half of the parallelogram whose contains it. Therefore the area of the shaded triangle is half of the hexagon. □