Perfect squares are everywhere - Part 3

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This article is the third part of the series on expedition to find *Perfect Squares*.

Example (Example 11)

Prove that, for all distinct values of a, b, and c, such that

$$\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}$$

is an integer (not necessarily positive), then the expression

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$$

is a perfect square.

Solution. [Solution 1] Let $x=\dfrac{1}{b-c},\ y=\dfrac{1}{c-a},\ z=\dfrac{1}{a-b}$, then

$$xy+yz+zx=\frac{(a-b)+(b-c)+(c-a)}{(a-b)(b-c)(c-a)}=0 \Rightarrow x^2+y^2+z^2=(x+y+z)^2.$$

$$\Rightarrow \frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}=\boxed{\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2.}$$

□

Solution. [Solution 2] Let $A=b-c, B=c-a, C=a-b$, then $A+B+C=0$, therefore:

$$\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)^2=\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}+\frac{2ABC(A+B+C)}{(ABC)^2}$$
$$=\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}.$$

The conclusion follows. □

Example (Example 12)

Find all (including negative) integer solutions of

$$n^2(n-1)^2 = 4(m^2-1).$$

Solution. [Solution 1] The left hand side of the equation is a perfect square, thus the right hand side must be a perfect square, too. It means that m^2-1 is a perfect square. Since m^2 is a perfect square and the only pair of perfect square with difference 1 is $(0,1)$. Therefore $m^2=1$, or $m=\pm 1$. Now, $n^2(n-1)^2=0$, so $n^2=0$, or $(n-1)^2=0$, thus $n=1$.

Hence, the solutions are $\boxed{\{(m,n)\} = \{(0,-1),(0,1),(1,-1),(1,1).\}.}$ □

Solution. [Solution 2]

$$n^2(n^2-1) = 4(m^2-1) \Rightarrow (n(n-1))^2 - (2m)^2 = 4 \Rightarrow (n(n-1)-2m)(n(n-1)+2m) = 4.$$

Note that both $n(n-1)$ and $2m$ are even, thus their sum and difference are even, too. Therefore

$$\begin{cases} n(n-1)-2m = 2, n(n-1)+2m = -2, & \text{or} \\ n(n-1)-2m = -2, n(n-1)+2m = 2, & \text{or} \end{cases} \Rightarrow \begin{cases} n(n-1) = 0, m = -1 \text{ or} \\ n(n-1) = 0, m = 1 \end{cases}$$

Hence, the solutions are $\boxed{\{(m,n)\} = \{(0,-1),(0,1),(1,-1),(1,1).\}.}$ □

Example (Example 13)

Prove that for n positive integer the following number is a perfect square:

$$m = 1\underbrace{77\ldots7}_{n}92\underbrace{88\ldots8}_{n-1}921.$$

Solution. Note that,

$$\underbrace{77\ldots7}_{n} = \frac{7}{9}\underbrace{99\ldots9}_{n} = \frac{7}{9}(10^n-1), \ \underbrace{88\ldots8}_{n-1} = \frac{8}{9}(10^{n-1}-1)$$

$$m = 1\cdot 10^{2n+4} + \frac{7}{9}(10^n-1)\cdot 10^{n+4} + 92\cdot 10^{n+2} + \frac{8}{9}(10^{n-1}-1)\cdot 10^3 + 921$$

$$= \frac{1}{9}(16\cdot 10^{2n+4} + 136\cdot 10^{n+2} + 289) = \frac{1}{9}\left(4\cdot 10^{n+2}+17\right)^2.$$

$4\cdot 10^{n+2}+17$ is divisible by 3, $(4\cdot 10^{n+2}+17)^2$ is divisible by 9, thus $\boxed{\frac{(4\cdot 10^{n+2}+17)^2}{9} \text{ is a perfect square.}}$ □

Example (Example 14)

N is 4−digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find N.

Solution. [Solution 1] Let $n^2 = N = \overline{abcd} = 10^3a + 10^2b + 10c + d$, where a, b, c, d are integers and $1 \le a \le 6$, $0 \le b, c, d\ le 6$, then there exists integer m such that:

$$m^2 = 10^3(a+3) + 10^2(b+3) + 10(c+3) + (d+3) = n^2 + 3333 \Rightarrow (m-n)(m+n) = 3 \cdot 11 \cdot 101.$$

Since $m < 99$, $n^2 \le 6666 \Rightarrow n \le 61$, thus $m + n = 101, m - n = 33$, or $n = 34$, hence $\boxed{N = 1156.}$ □

Solution. [Solution 2] Note that the unit digit of the perfect square N must be one of $0, 1, 4, 6, 9$. Since $N + 3333$ is also a perfect square thus the unit digit of N must be 1 or 6. By testing all 4-digit perfect numbers between 1000 and 9999 where its unit digit is one of 1, 6 and none of its other digits can be larger than 6, we find $\boxed{N = 1156.}$ □

Example (Example 15)

n is a non-negative integer, prove that

$$3^n + 2 \cdot 17^n$$

is never a perfect square.

Solution. [Solution 1] First $3^n + 2 \cdot 17^n = 3$ if $n = 0$, so it is not a perfect square.

For $n \ge 1$, $17^n \equiv 1 \pmod 8$, $3^{2k} \equiv 1 \pmod 8$, $3^{2k+1} \equiv 3 \pmod 8$, thus

$$3^n + 2 \cdot 17^n \equiv \begin{cases} 3 \pmod 8, & \text{if } n \text{ is even} \\ 5 \pmod 8, & \text{if } n \text{ is odd.} \end{cases}$$

However, for m integer,

$$m^2 \equiv \begin{cases} 0 \pmod 8, & \text{if } 4 \mid k \\ 4 \pmod 8, & \text{if } 2 \mid k,\ 4 \nmid k \\ 1 \pmod 8, & \text{if } 2 \nmid k \end{cases}$$

Hence, $\boxed{3^n + 2 \cdot 17^n \text{ is never a perfect square.}}$ □

Solution. [Solution 2] First $3^n + 2 \cdot 17^n = 3$ if $n = 0$, so it is not a perfect square.

For $n \ge 1$, $17^n \equiv 2^n \pmod 5$, thus the sequence of remainders of $2 \cdot 17^n$ when divided by 5 is periodic:

$$\underbrace{4, 3, 1, 2}_{\text{period}}, 4, 3, 1, 2, 4$$

The sequence of remainders of 3^n when divided by 5 is also periodic with period $3, 4, 2, 1$.

Therefore the sequence of remainders of $3^n + 2 \cdot 17^n$ when divided by 5 is periodic with period $2, 2, 3, 3$.

However any perfect square when divided by 5 leaves a remainder of $0, 1$, or 4.

Hence, $\boxed{3^n + 2 \cdot 17^n \text{ is never a perfect square.}}$ □