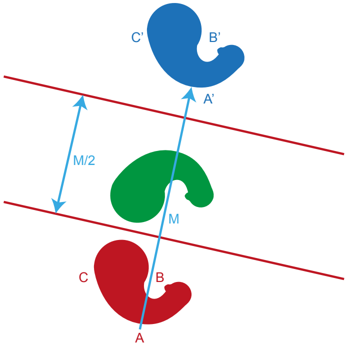
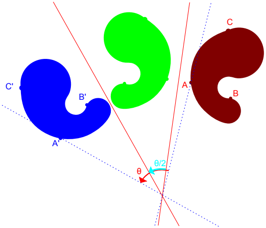
Geometric Transformations
Definitions

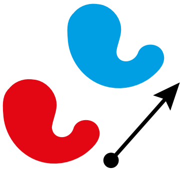
A transformation is an operation that moves, flips, or changes a figure to create a new figure.

A rigid transformation (also known as an isometry or congruence transformation) is a transformation that does not change the size or shape of a figure.

The rigid transformations are *reflections*, *rotations*, and *translations*.



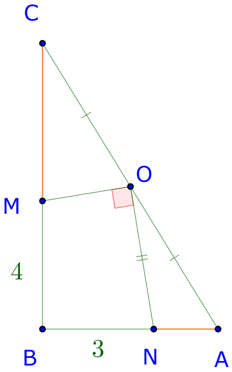


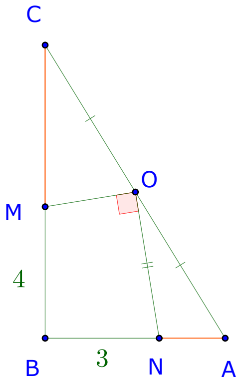


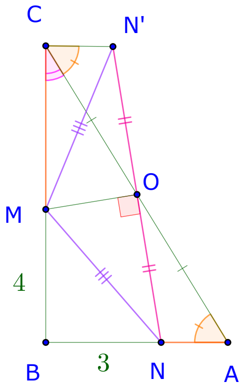
The new figure created by a transformation is called the image. The original is the preimage.

Example

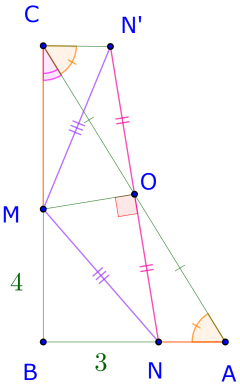
In the right triangle ABC, O is the midpoint of the hypotenuse AC. Points M and N are chosen on sides BC and BA such that $\angle MON = 90^\circ$, $BM = 4$, and $BN = 3$. Find $AN^2 + CM^2$.



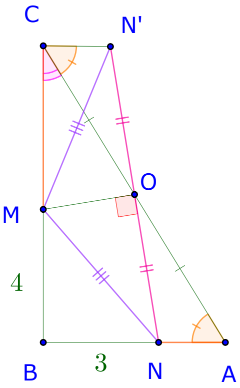

C
O
M
4
B
3
N
A



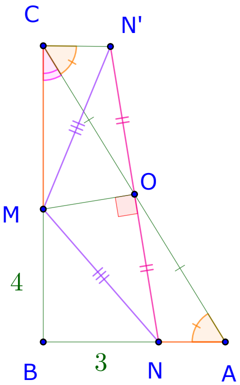
Let N' be the reflection of N over O.



Let N' be the reflection of N over O. $\triangle AON \cong \triangle CON'$ by ASA, so $AN = CN'$.



Let N' be the reflection of N over O. $\triangle AON \cong \triangle CON'$ by ASA, so $AN = CN'$.
$\angle MCN' = \angle MCO + \angle OCN' = \angle BCA + \angle CAB = 90^\circ$. Therefore, $\triangle ANC'$ is a right triangle.



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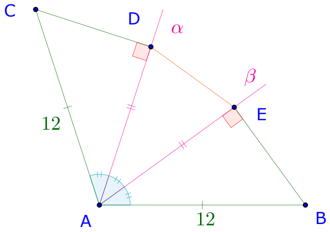
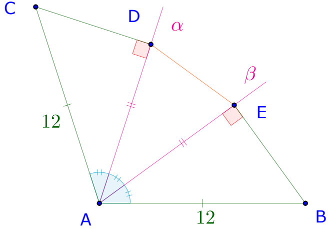
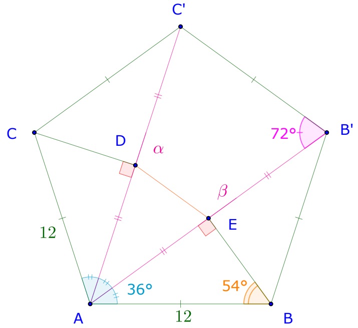
$AN^2 + CM^2 = CN'^2 + CN^2 = MN'^2 = MN^2 = BM^2 + BN^2 = \boxed{25.}$

Example

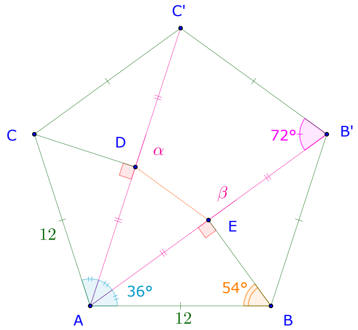
$\triangle ABC$ is an isosceles triangle, $AB = AC = 12$, and $\angle BAC = 108°$. Two rays $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ starting from A, trisect the $\angle BAC$ into three equal angles. Points D and E are the feet of the perpendiculars from C and B to rays α and β, respectively. Find DE.




C
D
α
β
E
12
12
A
B

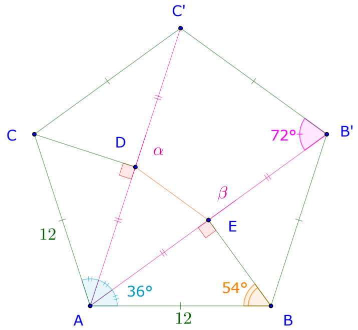


Let B' and C' be the reflections of A over lines BE and CD, respectively.



Let B' and C' be the reflections of A over lines BE and CD, respectively.

$$\angle B'AC + \angle CAC' = 2 \cdot 36^\circ + 2 \cdot 54^\circ = 180^\circ \Rightarrow CC' \parallel AB'.$$



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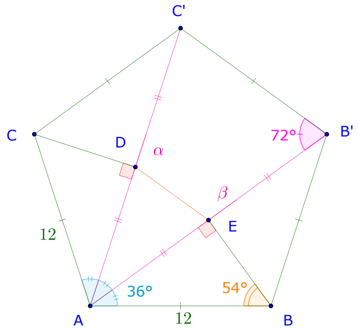
Furthermore $\triangle AB'C'$ is isosceles, so $\angle B'AC = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$.



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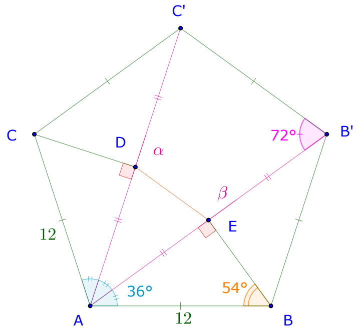
Thus, $ACC'B'$ is isosceles trapezoid, so $B'C' = AB = 12$.



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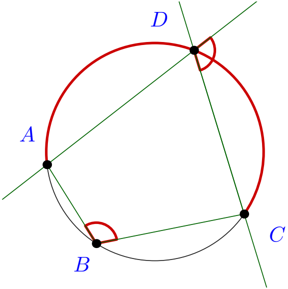
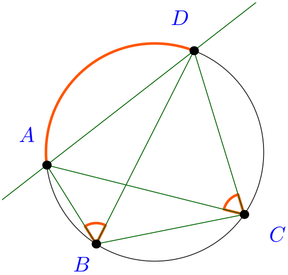
Thus, $ACC'B'$ is isosceles trapezoid, so $B'C' = AB = 12$.
DE is a midsegment in $\triangle AB'C'$, hence $\boxed{DE = 6.}$

Theorem (Cyclic Quadrilaterals)

Let ABCD be a convex quadrilateral. Then the following are equivalent:

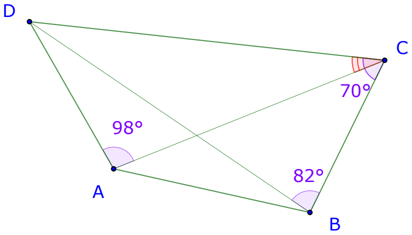
- *ABCD is cyclic.*
- $\angle ABC + \angle CDA = 180^\circ$.
- $\angle ABD = \angle ACD$.

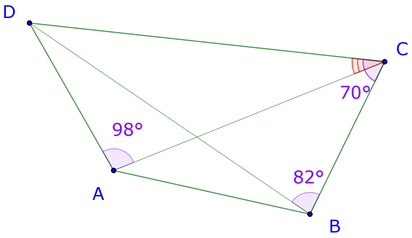


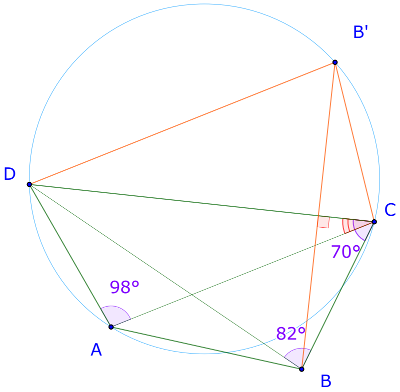


Example

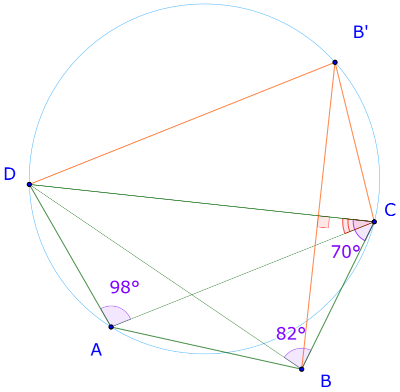
$ABCD$ is a convex quadrilateral. $BC = AD$, $\angle DAC = 98°$, $\angle DBC = 82°$, $\angle BCD = 70°$. Find $\angle ACD$.



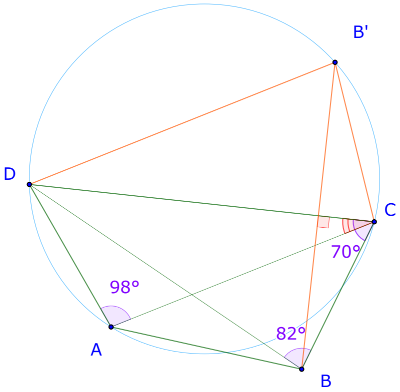

D
C
70°
98°
82°
A
B



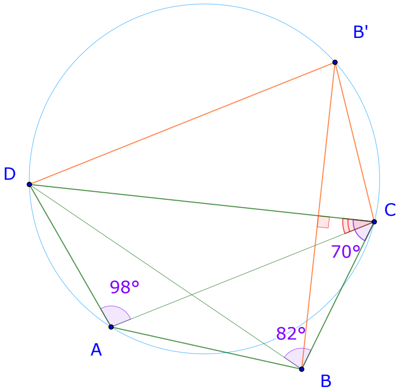
Let B' be the reflection of B across CD.



Let B' be the reflection of B across CD. $\angle DAC + \angle CB'D = 180^\circ$, so $ACB'D$ is cyclic.
$AD = BC$, thus $\overset{\frown}{AD} = \overset{\frown}{BC}$.



Let B' be the reflection of B across CD. $\angle DAC + \angle CB'D = 180^\circ$, so $ACB'D$ is cyclic.
$AD = BC$, thus $\widehat{AD} = \widehat{BC}$. Therefore $\angle ACB' = \angle DAC = 98^\circ$.

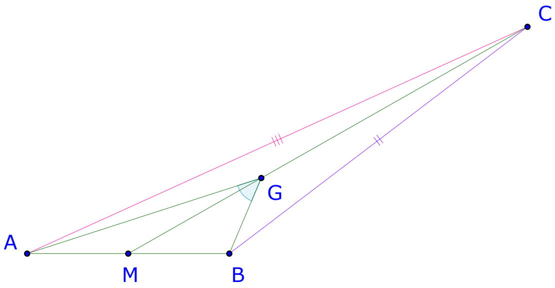


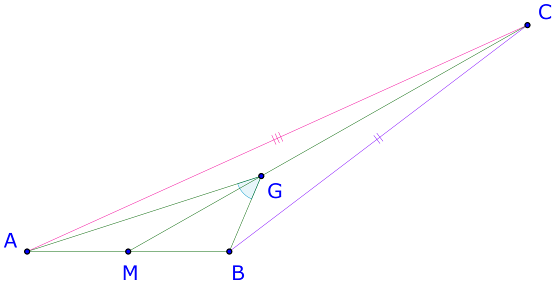
Let B' be the reflection of B across CD. $\angle DAC + \angle CB'D = 180^\circ$, so $ACB'D$ is cyclic. $AD = BC$, thus $\widehat{AD} = \widehat{BC}$. Therefore $\angle ACB' = \angle DAC = 98^\circ$.

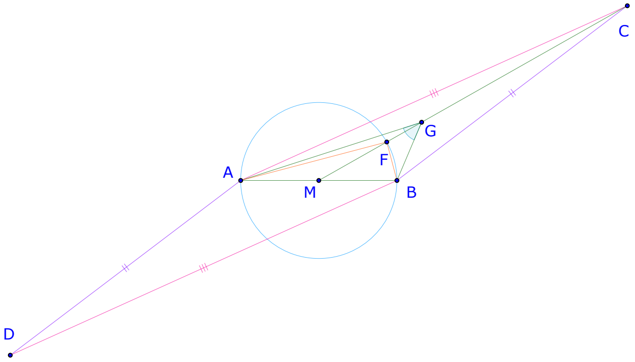
$$\angle DCB' = \angle BCD = 70^\circ \Rightarrow \angle ACD = \angle ACB' - \angle DCB' = 98^\circ - 70^\circ = \boxed{28^\circ.}$$

Example

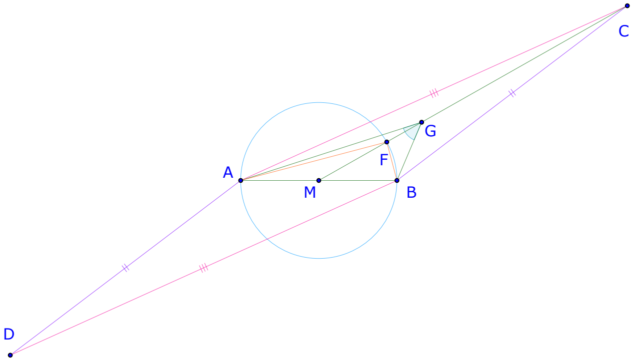
Let G be the centroid of $\triangle ABC$. If $\angle AGB \leq 90^\circ$, find the largest possible value of n integer, such that $AC + CB > nAB$.




C
G
A
M
B



Let M be the midpoint of AB, F be the intersection of the circle centred M diameter AB. $AGB \leq 90^\circ$ so G is outside the circle (M), therefore $GM \leq FM = \frac{1}{2}AB$.



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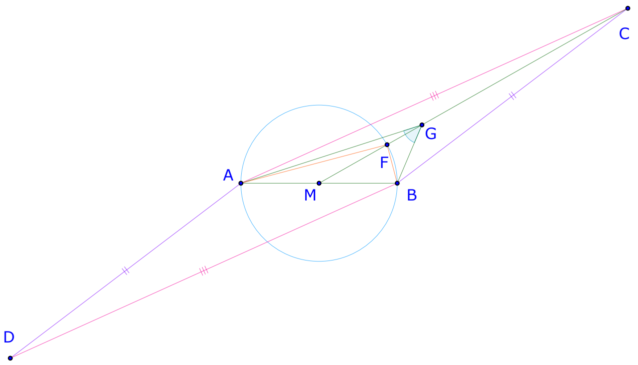
$AGB \leq 90^\circ$ so G is outside the circle (M), therefore $GM \leq FM = \frac{1}{2}AB$.
Let D be the reflection of C over M. In triangle DAC, $DA + AC > DC \Rightarrow AB + AC > 2CM$.



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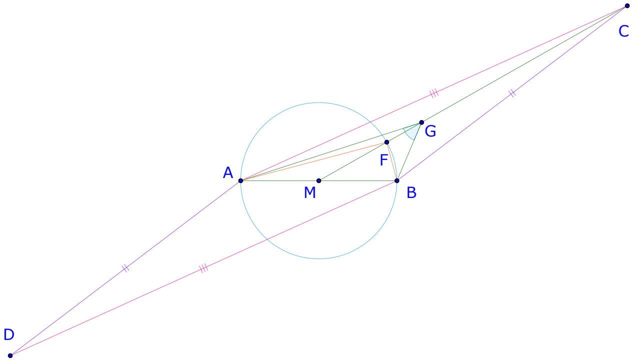
Therefore $AB \leq 2GM = \frac{2}{3}CM < \frac{1}{3}(AB + AC)$, thus $AB + AC > 3AB$.



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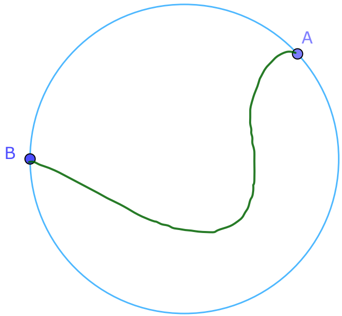
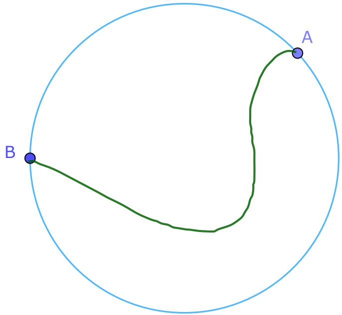
Therefore $AB \leq 2GM = \frac{2}{3}CM < \frac{1}{3}(AB + AC)$, thus $AB + AC > 3AB$.
If $AC = BC$, G on circle (M), then $AB + AC = \sqrt{10}AB < 4AB$. Thus, $\boxed{n = 3.}$

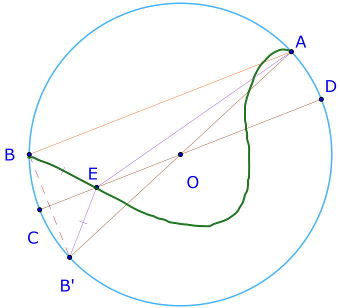
Example

A, B are two points on the circle. A curve through A, B bisects the area of the circle. Prove that the curve is at least as long as a diameter of the circle.

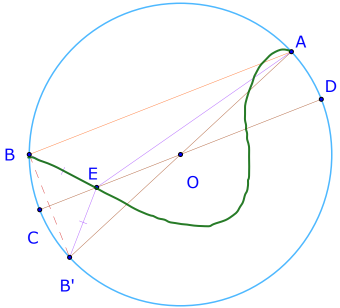




It is obvious that the curve is at least as long as a diameter of the circle if AB is a diameter. Thus, let's assume that AB is a chord that is shorter than the diameter of the circle.



It is obvious that the curve is at least as long as a diameter of the circle if AB is a diameter.
Thus, let's assume that AB is a chord that is shorter than the diameter of the circle.
Let CD be the diameter of the circle that is parallel to AB. Let B' be the reflection of B over CD.



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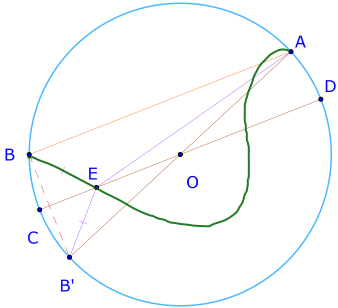
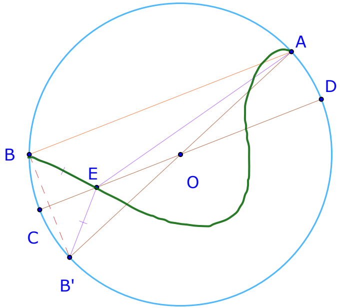
Let CD be the diameter of the circle that is parallel to AB. Let B' be the reflection of B over CD.
If the curve does not intersect CD, then the curve cannot bisect the area of the circle, thus the curve must intersect CD.



It is obvious that the curve is at least as long as a diameter of the circle if AB is a diameter. Thus, let's assume that AB is a chord that is shorter than the diameter of the circle.
Let CD be the diameter of the circle that is parallel to AB. Let B' be the reflection of B over CD. If the curve does not intersect CD, then the curve cannot bisect the area of the circle, thus the curve must intersect CD.
Let one of the intersections be E. The length of the curve must be at least as long as $BE + EA$. Now, $BE + EA = B'E + EA \geq B'A$, which is a diameter.



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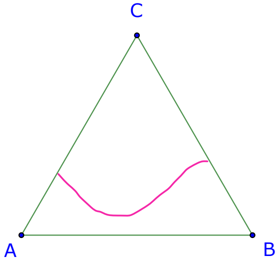
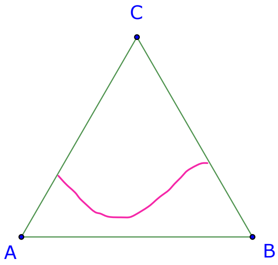
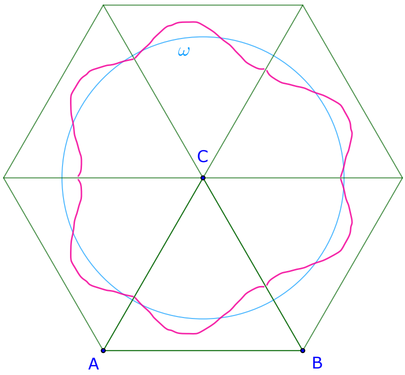
Thus, let's assume that AB is a chord that is shorter than the diameter of the circle.
Let CD be the diameter of the circle that is parallel to AB. Let B' be the reflection of B over CD.
If the curve does not intersect CD, then the curve cannot bisect the area of the circle, thus the curve must intersect CD.
Let one of the intersections be E. The length of the curve must be at least as long as $BE + EA$.
Now, $BE + EA = B'E + EA \geq B'A$, which is a diameter.
Hence, the curve is at least as long as a diameter of the circle.

Example

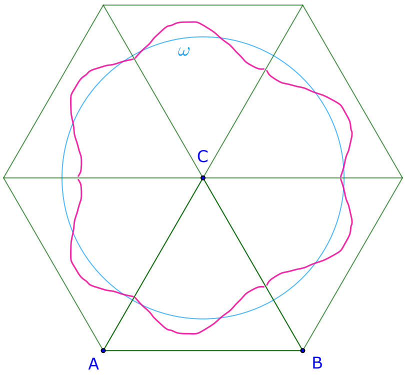
A continuous curve split an unit equilateral triangle ABC into two regions with equal area. What is the minimal length of the curve?



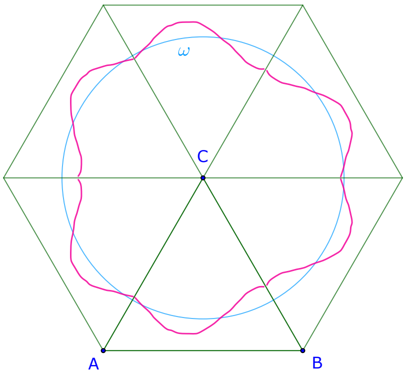

C
A
B



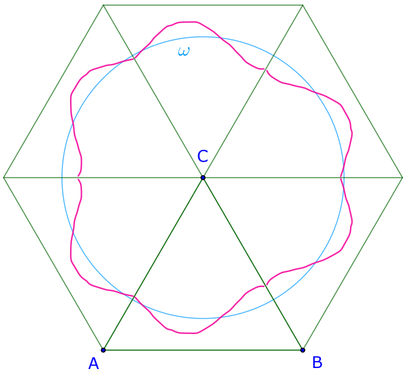
The continuous curve becomes a close curve encompassing an area half of a unit circle, or π.



The continuous curve becomes a close curve encompassing an area half of a unit circle, or π.
The close curve encompassing an area has minimal length if it forms the perimeter of a circle.



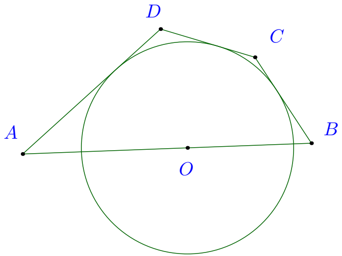
The continuous curve becomes a close curve encompassing an area half of a unit circle, or π. The close curve encompassing an area has minimal length if it forms the perimeter of a circle. This circle has an area of $\frac{1}{2}\pi$, thus its radius of $\frac{1}{\sqrt{2}}$ and a perimeter of $2\pi\left(\frac{1}{\sqrt{2}}\right) = \pi\sqrt{2}$.

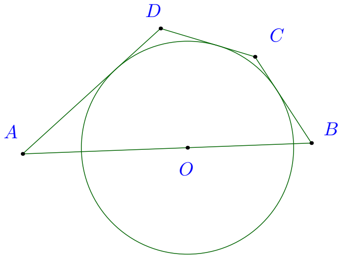


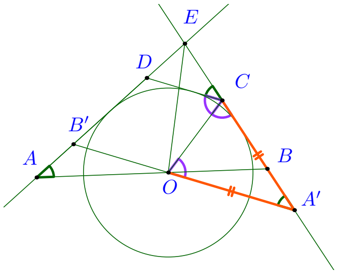
The continuous curve becomes a close curve encompassing an area half of a unit circle, or π.
The close curve encompassing an area has minimal length if it forms the perimeter of a circle.
This circle has an area of $\frac{1}{2}\pi$, thus its radius of $\frac{1}{\sqrt{2}}$ and a perimeter of $2\pi\left(\frac{1}{\sqrt{2}}\right) = \pi\sqrt{2}$.
Therefore the minimal length of the original curve is one-sixth of $\pi\sqrt{2}$, which is $\frac{\pi\sqrt{2}}{6}$.

Example

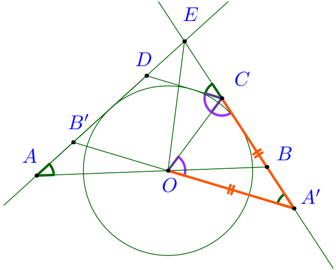
A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.




D
C
A
B
O

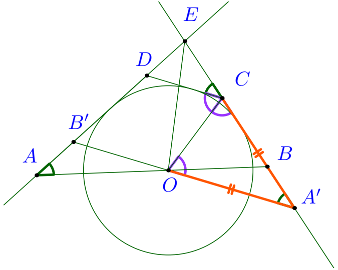


Let E be the intersection of AD and BC. Let A' and B' be the reflections of A and B over EO.



Let E be the intersection of AD and BC. Let A' and B' be the reflections of A and B over EO.

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

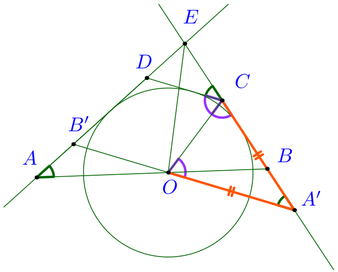


Let E be the intersection of AD and BC. Let A' and B' be the reflections of A and B over EO.

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

CB, CD are tangents, so CO is the bisector of $\angle DCB$, therefore

$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$



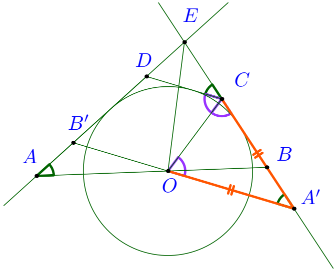
Let E be the intersection of AD and BC. Let A' and B' be the reflections of A and B over EO.

$$\angle EA'B' = \angle EAB = \angle ECD \Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'.$$

CB, CD are tangents, so CO is the bisector of $\angle DCB$, therefore

$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$

Thus, $OA' = CA'$, so $OA = OA' = CA' = DA$, similarly $OB = CB$.



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$$\angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA'.$$

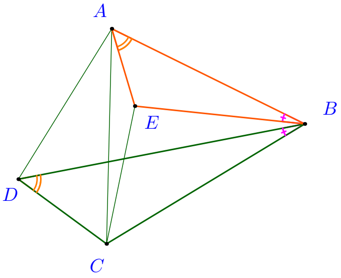
Thus, $OA' = CA'$, so $OA = OA' = CA' = DA$, similarly $OB = CB$. Hence, $\boxed{AD + BC = AB.}$

Theorem (Ptolemy Inequality)

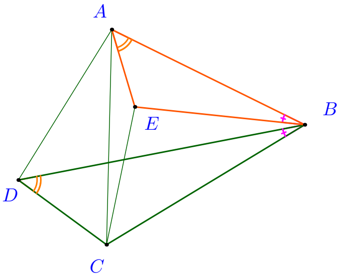
The inequality states that in for four points A, B, C, D in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality for any cyclic quadrilateral $ABCD$ with diagonals AC and BD.

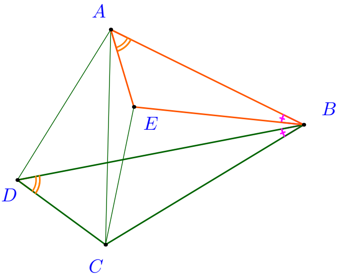


Note: this also holds if A, B, C, D are four points not in the same plane, but the equality can't be achieved.



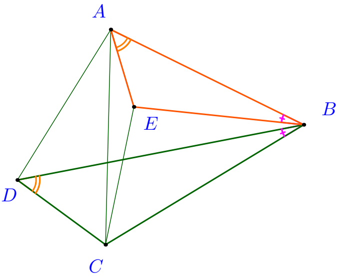
Proof.

□



Proof.

Let E be the point such that $\angle EAB = \angle CDB$, $\angle EBA = \angle CBD$.

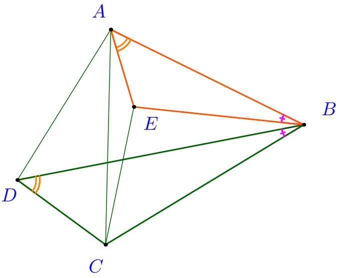


Proof.

Let E be the point such that $\angle EAB = \angle CDB$, $\angle EBA = \angle CBD$.

$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \ \frac{CE}{AD} = \frac{BC}{BD}.$$

$\square$



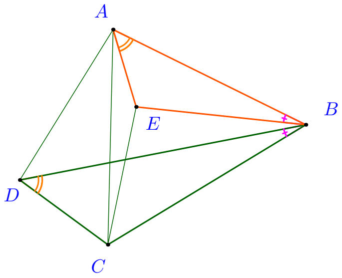
Proof.

Let E be the point such that $\angle EAB = \angle CDB,\ \angle EBA = \angle CBD$.

$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD},\ \frac{CE}{AD} = \frac{BC}{BD}.$$

$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

□



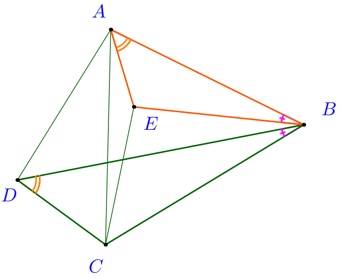
Proof.

Let E be the point such that $\angle EAB = \angle CDB$, $\angle EBA = \angle CBD$.

$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \ \frac{CE}{AD} = \frac{BC}{BD}.$$

$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

$\square$



Proof.

Let E be the point such that $\angle EAB = \angle CDB$, $\angle EBA = \angle CBD$.

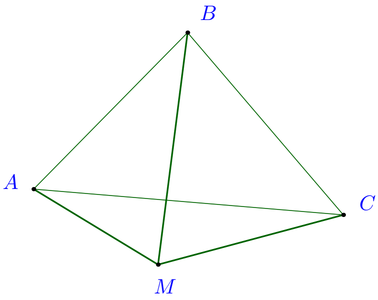
$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \ \frac{CE}{AD} = \frac{BC}{BD}.$$

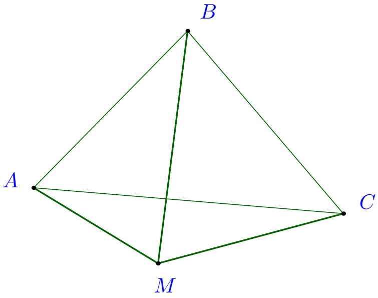
$$AE + CE \geq AC \Rightarrow AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \geq AC \Rightarrow AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

The equality stands if and only if $E \in AC$, or $\angle CAB = \angle EAB = \angle CDB$, so $ABCD$ is cyclic. □

Theorem (Pompeiu's Theorem)

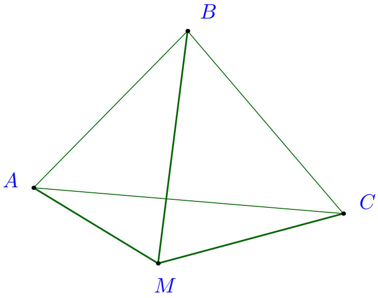
$\triangle ABC$ is equilateral. For any point M, the segments AM, BM and CM form a triangle. This triangle degenerates if and only if M lies on the circumcircle of $\triangle ABC$.





Proof.

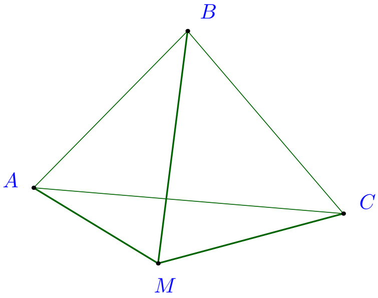
□



Proof.

If M is inside of $\triangle ABC$, then $AM + BM > AB > CM$.

□

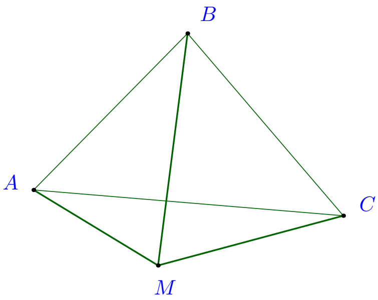


Proof.

If M is inside of $\triangle ABC$, then $AM + BM > AB > CM$. If M is outside of $\triangle ABC$, by Ptolemy Inequality, for four points A, B, C, M

$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$

$\square$



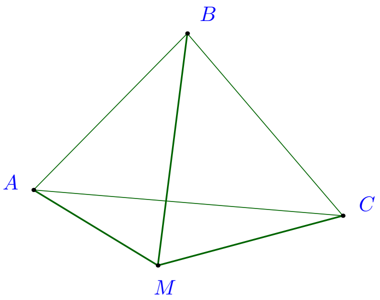
Proof.

If M is inside of $\triangle ABC$, then $AM + BM > AB > CM$. If M is outside of $\triangle ABC$, by Ptolemy Inequality, for four points A, B, C, M

$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$

Similarly with other triangle inequalities. Hence, $\boxed{AM,\ BM \text{ and } CM \text{ form a triangle.}}$

□



Proof.

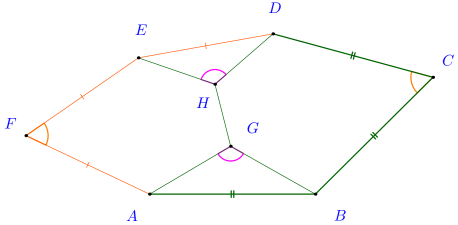
If M is inside of $\triangle ABC$, then $AM + BM > AB > CM$. If M is outside of $\triangle ABC$, by Ptolemy Inequality, for four points A, B, C, M

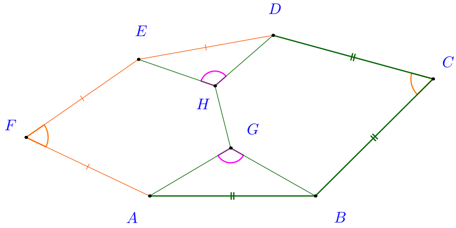
$$AM \cdot BC + CM \cdot AB \geq BC \cdot BM \Rightarrow AM + CM \geq BM.$$

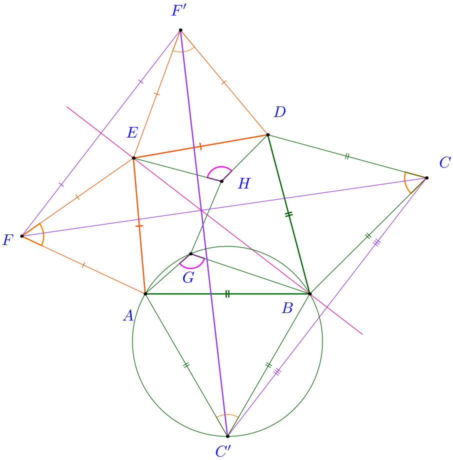
Similarly with other triangle inequalities. Hence, $\boxed{AM,\ BM \text{ and } CM \text{ form a triangle.}}$
The equality stands if and only if $ABCM$ is cyclic, or M is on the circumcircle of $\triangle ABC$. $\square$

Example

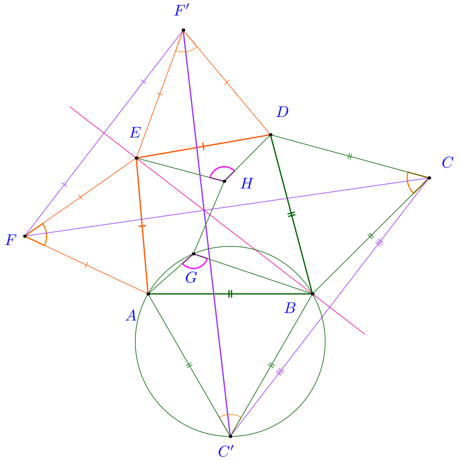
Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.



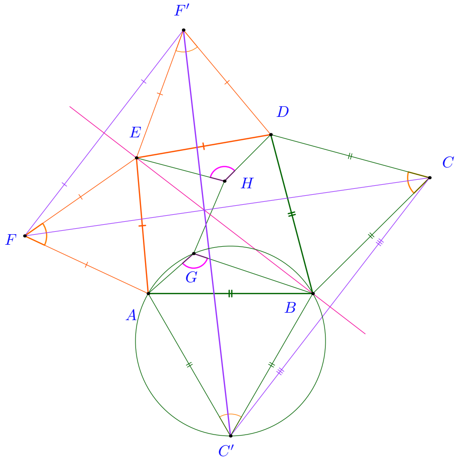

D
E
C
H
F
G
A
B



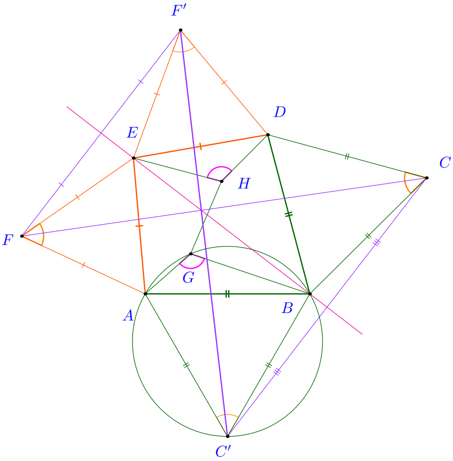
$\triangle BCD, \triangle EFA$ are equilateral, so $AE = ED, DB = BA$, thus BE is an axis of symmetry of $ABDE$. Let C' and F' be the reflections of C and F over the line BE, respectively.



$\triangle AC'B$ is the reflection of $\triangle DBC$, hence $\angle AC'B = \angle BCD = 60^\circ$. $\angle AGB = 120^\circ$, so G, by the property of cyclic-quadrilaterals, lies on the circle (ABC').

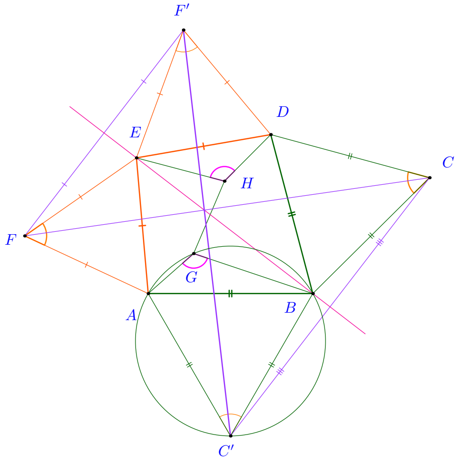


Similarly H lies on the circle (DEF').



Thus, according to by Pompeiu's Theorem, $AG + GB = C'G$ and $DH + HE = HF'$, so

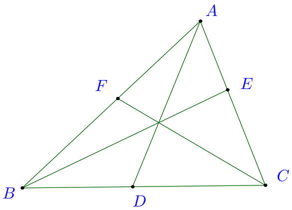
$$AG + GB + GH + DH + HE = C'G + GH + HF' \geq C'F' = CF$$

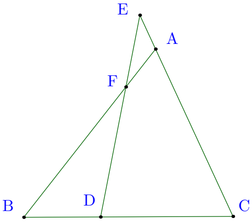


Hence, $\boxed{AG + GB + GH + DH + HE \geq CF.}$

Theorem (Ceva Theorem)

Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent if and only if: $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.



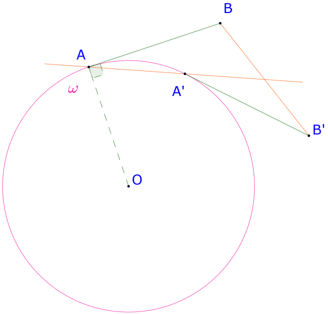


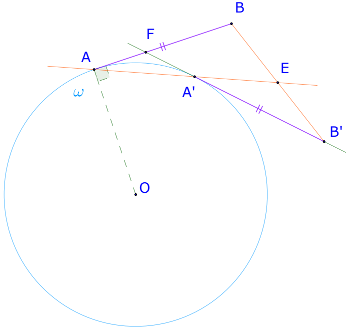
Theorem (Menelaus Theorem)

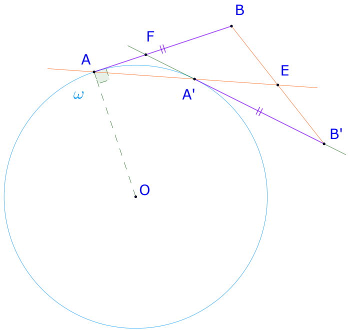
Let ABC be a triangle, and let D, F be points on lines BC, AB, respectively. E is on the extension of CA. Points D, E, F are collinear if and only if: $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

Example

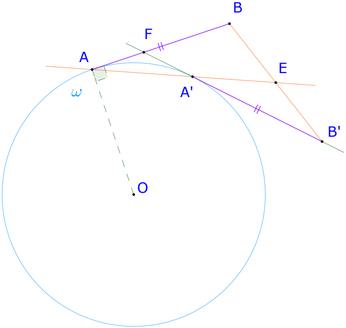
Point B lies on a line which is tangent to circle ω at point A. The line segment AB is rotated about the center of the circle by some angle to form segment $A'B'$. prove that the line AA' bisects the segment BB'.



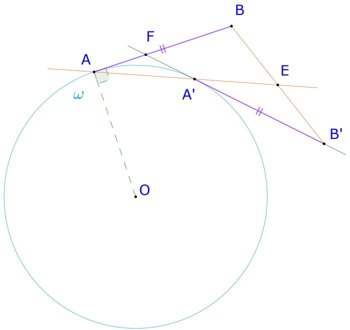

B
F
A
E
ω
A'
B'
O



Let E be the intersections of lines through AA' and BB'. Let F be the intersections of lines through AB and $A'B'$.



Let E be the intersections of lines through AA' and BB'. Let F be the intersections of lines through AB and $A'B'$. FA and FA' are both tangents of ω, so $FA = FA'$. $A'B'$ is the image of the rotation of AB about the center of ω, thus $A'B' = AB$.

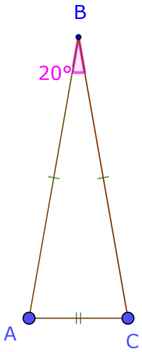


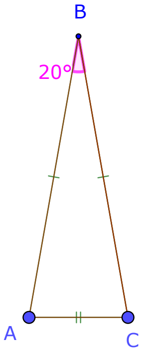
Let E be the intersections of lines through AA' and BB'. Let F be the intersections of lines through AB and $A'B'$. FA and FA' are both tangents of ω, so $FA = FA'$. $A'B'$ is the image of the rotation of AB about the center of ω, thus $A'B' = AB$. By Menelaus Theorem, for $\triangle B'BF$:

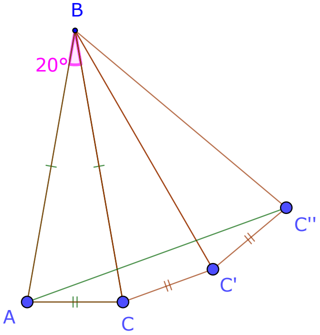
$$\frac{B'E}{EB} \cdot \frac{BA}{AF} \cdot \frac{FA'}{A'B'} = 1 \Rightarrow \frac{B'E}{EB} = 1 \Rightarrow \boxed{AA' \text{ bisects } BB'}.$$

Example

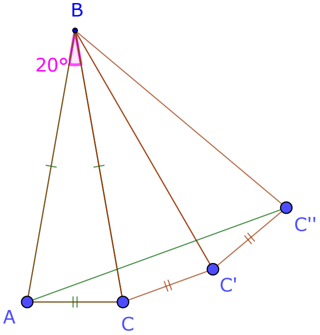
Given isosceles triangle ABC, $AB = BC$, and $\angle B = 20^\circ$, prove that $AB < 3AC$.




B
20°
A
C



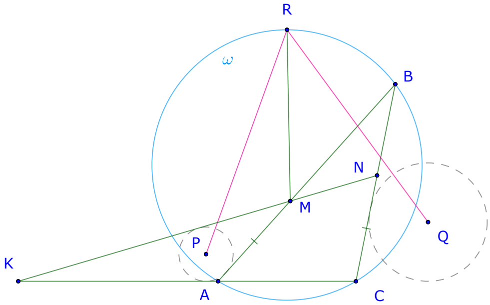
Let's rotate the triangle ABC around B twice 20° as shown in the diagram above.

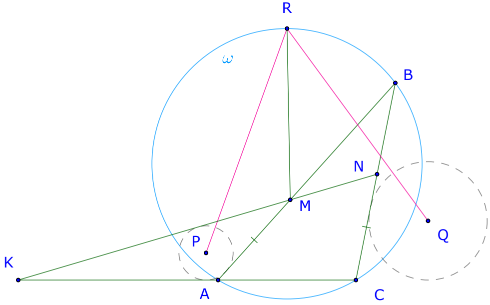


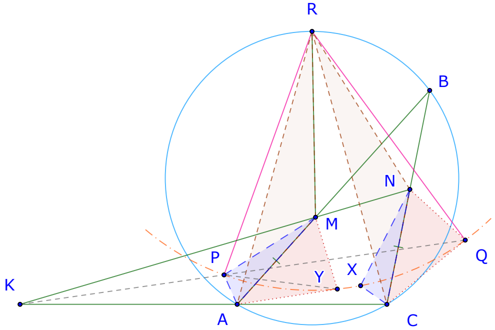
Let's rotate the triangle ABC around B twice 20° as shown in the diagram above.
Then ABE is a equilateral triangle, $AB = AC'' < AC + CC' + C'C'' = 3AC$.

Example

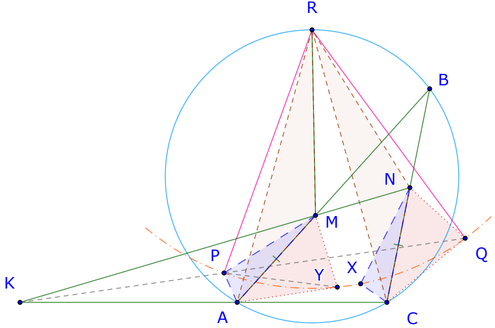
Given a triangle ABC with $AB > BC$, let ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of ω then prove that $RP = RQ$.




R
ω
B
N
M
Q
P
K
A
C



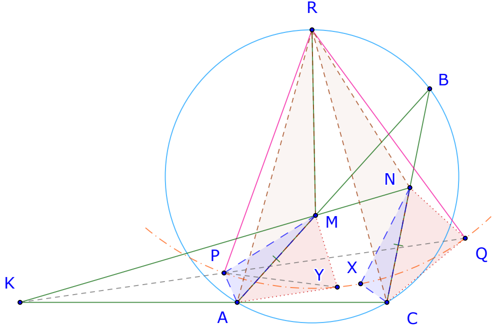
Note that $\triangle RMA \cong \triangle RNC$, hence R is the rotation center of $\overline{AM} \mapsto \overline{CN}$. Rotate about R such that $\triangle CNQ \mapsto \triangle AMY$ and $\triangle AMP \mapsto \triangle CNX$. We prove that P, Q, X, Y are concyclic.



First, $\angle APM + \angle CQN = 180^\circ$, so $APMY$ and $CXNQ$ are congruent cyclic quadrilaterals. Since K, P, Q are colinear:

$$\angle APQ = 180^\circ - \angle KPA = \angle PKA + \angle PAK = \frac{1}{2}\left(\angle KAM + \angle MKA\right) = 90^\circ - \frac{1}{2}\angle BMN$$

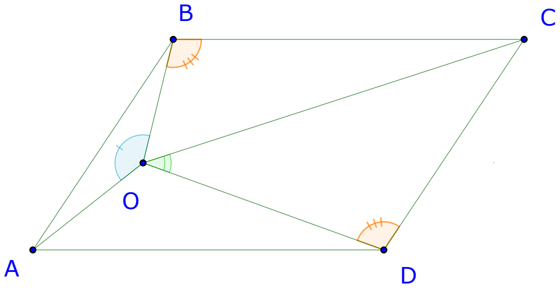
$$\angle YPQ = \angle APQ - \angle APY = (90^\circ - \frac{1}{2}\angle BMN) - \frac{1}{2}\angle BNM = \frac{1}{2}\angle ABC.$$

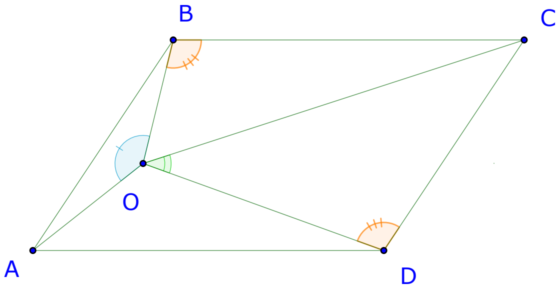


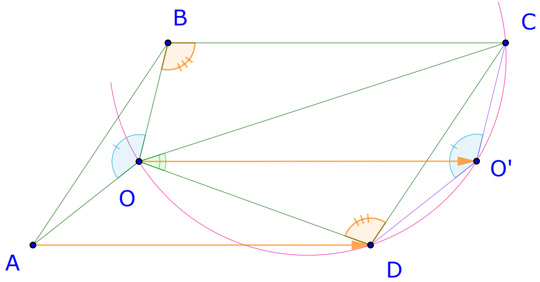
Similarly, we get $\angle XQP = \frac{1}{2}\angle ABC$. Thus $\angle YPQ = \angle XQP$. Since $YP = XQ$, $PQXY$ is an isosceles trapezoid. Therefore $RP = RX$ and $RQ = RY$, we conclude that R is the center of the circle $\odot(PQXY)$. Hence $RP = RQ$, as desired.

Example

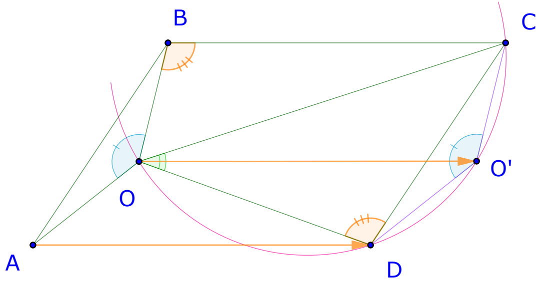
The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.




B
C
O
A
D

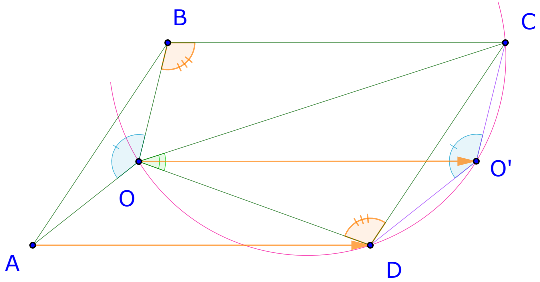


The translation by $\overrightarrow{AD}$ maps A to D, B to C, and O to O'.



The translation by $\overrightarrow{AD}$ maps A to D, B to C, and O to O'.

$ABCD$ is a parallelogram, $AD \parallel BC, AD = BC$. By the translation, $OO' \parallel AD, OO' = AD$, thus $OO' \parallel BC, OO' = BC$. Therefore $OBCO'$ is a parallelogram. It implies that $\angle OBC = \angle OO'C$.



The translation by $\overrightarrow{AD}$ maps A to D, B to C, and O to O'.

$ABCD$ is a parallelogram, $AD \parallel BC, AD = BC$. By the translation, $OO' \parallel AD, OO' = AD$, thus $OO' \parallel BC, OO' = BC$. Therefore $OBCO'$ is a parallelogram. It implies that $\angle OBC = \angle OO'C$.

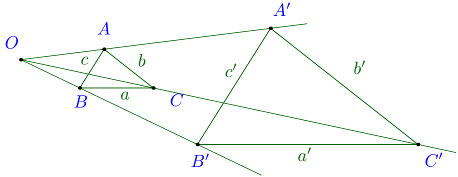
Since $\angle AOB + \angle COD = 180^\circ$, so $\angle DO'C + \angle COD = 180^\circ$, or $CODO'$ is cyclic. Therefore $\angle ODC = \angle OO'C$. Hence, $\boxed{\angle OBC = \angle ODC.}$

Definition (Homothety)

A **homothety** (or homothecy) is a transformation of space which dilates distances *with respect to a fixed point.*
A homothety with center O and factor k sends point A to a point A', and

$$OA' = k \cdot OA.$$

This is denoted by $\mathcal{H}_{(O,k)}$.

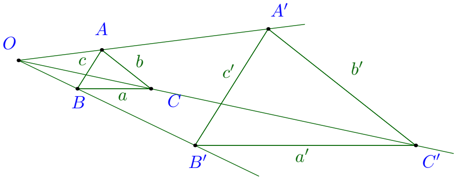


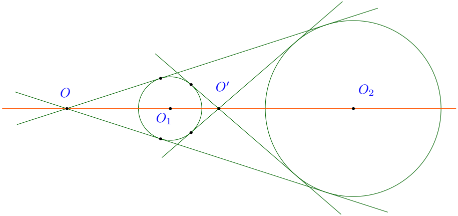
A homothety can be an *enlargement* (resulting figure is larger), *identity* transformation (resulting figure is congruent), or a *contraction* (resulting figure is smaller).

Theorem (Homothety Images)

Let $\mathcal{H}_{(O,k)}$ be a homothety,

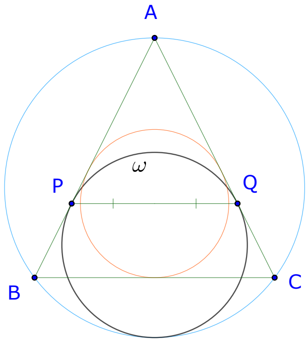
1. *For point A, $\mathcal{H}_{(O,k)}(A) = A' \Rightarrow O, A, A'$ collinear. Thus, the lines connecting each point of a polygon to its corresponding point of a homothetic polygon are all concurrent.*
2. *For line ℓ, $\mathcal{H}_{(O,k)}(\ell) = \ell' \Rightarrow \ell \parallel \ell'$.*
3. *For polygon P, $\mathcal{H}_{(O,k)}(P) = P' \Rightarrow P \sim P'$. Thus, the resulting image of a circle from a homothety is also a circle.*

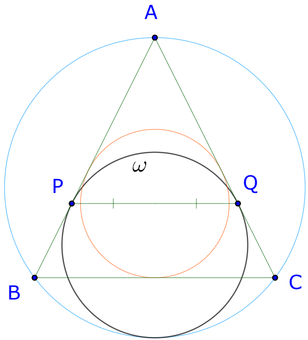


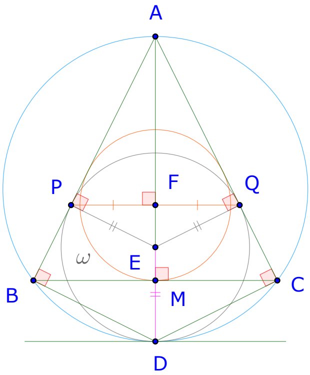


Example

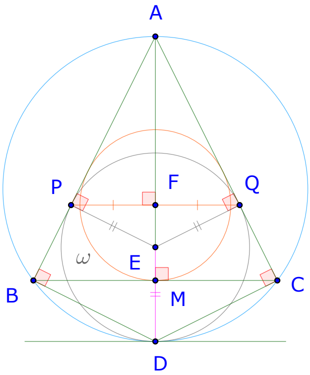
In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.



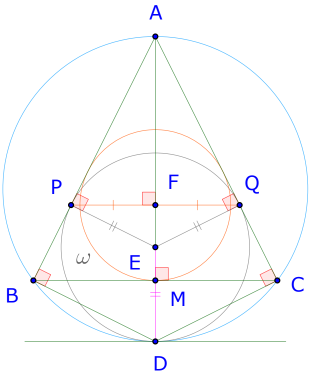

A
ω
P
Q
B
C



Let E be the center of the circle ω, which is tangent with AB, AC, and (ABC). Let D be the tangent point of the two circles, M be the midpoint of BC, and F be the midpoint of PQ. It is easy to see that A, F, E, M, D are collinear.



Let $\mathcal{H}_{(A,k)}$ be a homothety centred at A and $\mathcal{H}_{(A,k)}(D) = M$. It is easy to see that $\mathcal{H}_{(A,k)}(E) = F$. Let γ be the image of ω, $\gamma = \mathcal{H}_{(A,k)}(\omega)$. Since ω is tangent (ABC) at D, so both are tangent with line ℓ through D parallel with BC, thus γ tangent with the image of ℓ, which is line BC.



Furthermore, the $\mathcal{H}_{(A,k)}(D) = M$, keeps B and C on rays AB and AC, and because ω is tangent to AB and AC, so γ is also tangent to AB and AC. Therefore γ is tangent with all sides of $\triangle ABC$, so $\gamma = I_{\triangle ABC}$. Therefore, the image of $\mathcal{H}_{(A,k)}(E)$ is I, the incentre of $\triangle ABC$, therefore $F \equiv I$.

Hence, $\boxed{\text{the midpoint of } PQ \text{ is the center of the inscribed circle of the triangle } ABC.}$