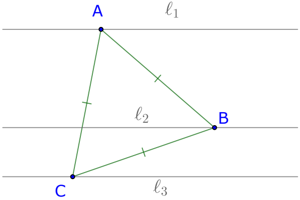
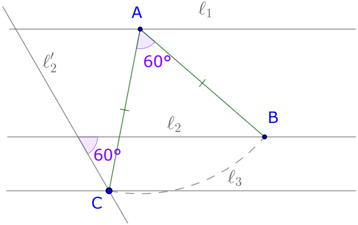
Example

Three parallel lines ℓ_1, ℓ_2, and ℓ_3 are given. A is a point on the line ℓ_1.

How can we determine points B and C on ℓ_2 and ℓ_3, respectively, such that ABC is an equilateral triangle.

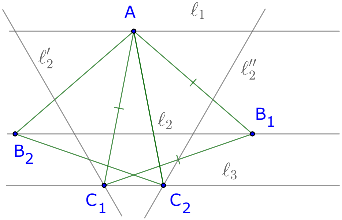


Assume that $\triangle ABC$ is equilateral, then **a rotation by** 60° **about** A **will carry** B **to** C.



That rotation also carries ℓ_2 (containing B) to ℓ_2'. The intersection of ℓ_2' and ℓ_3 is C.

Now we know how to do it. Rotate ℓ_2 by 60° about A to obtain ℓ_2'. The intersection of ℓ_2' with ℓ_3 is the position for C. B can be constructed easily as the intersection of circle centred at A radius AC.

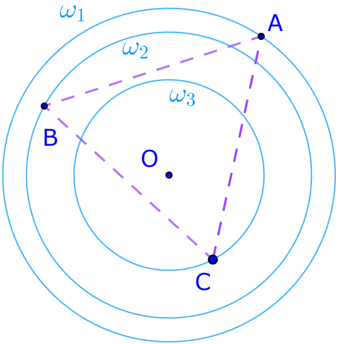


Note that there are two different solutions (why?)

Example

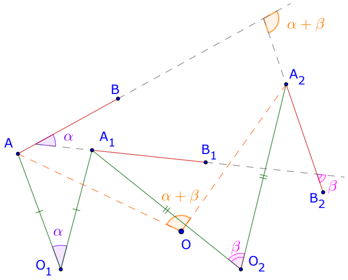
Three concentric circles ω_1, ω_2, and ω_3 are given. A is a point on ω_1.

How can we determine points B and C on ω_2 and ω_3, respectively, such that ABC is an equilateral triangle.



Let's take a look at a **sum of two rotations**:

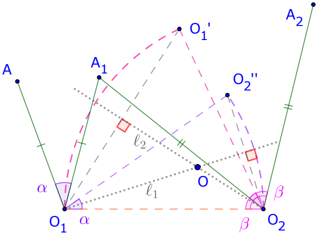
$$AB \overset{\text{rotate}(O_1,\alpha)}{\longrightarrow} A_1B_1 \overset{\text{rotate}(O_2,\beta)}{\longrightarrow} A_2B_2.$$



It is easy to see that the angle between A_2B_2 and AB is $\alpha+\beta$, thus it is a rotation by the angle $\alpha+\beta$, We need to determine the position of the center of rotation O.

Now, what happen with the centers O_1 and O_2:

$$O_1 \overset{\text{rotate}(O_1,\alpha)}{\to} O_1 \overset{\text{rotate}(O_2,\beta)}{\to} O_1' \quad \text{and} \quad O_2'' \overset{\text{rotate}(O_1,\alpha)}{\to} O_2 \overset{\text{rotate}(O_2,\beta)}{\to} O_2.$$

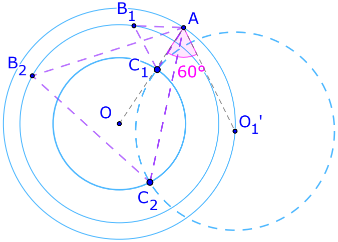


Therefore O is on both perpendicular bisectors of O_1O_1' and $O_2''O_2$.

Hence, $\angle OO_1O_2 = \frac{1}{2}\alpha$, $\angle OO_2O_1 = \frac{1}{2}\beta$.

Pretty much the same as in the solution for the previous example.

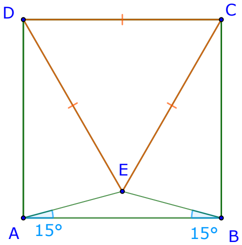
Rotate ω_2 by 60° about A to obtain ω_2'. The intersection of ω_2' with ω_3 is the position for C. B can be constructed easily as the intersection of circle centred at A radius AC.



Note that there are **at most four** different solutions (why?).

Example

Let E be a point in the square $ABCD$ such that $\angle EAB = \angle EBA = 15^\circ$. Prove that $\triangle CDE$ is equilateral.



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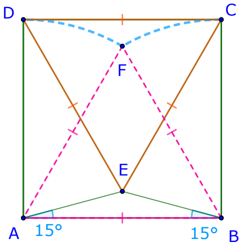

Let F be a point inside $ABCD$ such that $\triangle ABF$ is equilateral.

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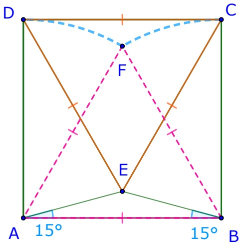

Let F be a point inside $ABCD$ such that $\triangle ABF$ is equilateral.

The rotation about A by 30° clockwise sends D to F, and the rotation about B by 30° clockwise sends F to C.

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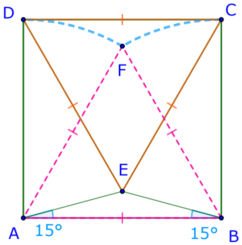

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Thus the sum of rotation is a rotation about O by 60° sends D to C, where O is the point such that $OC = OD$, $\angle DOC = 60^\circ$, and $\angle OAB = OBA = 15^\circ$.

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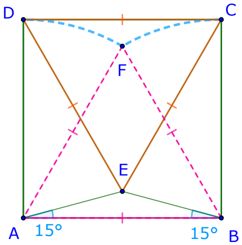

Let F be a point inside $ABCD$ such that $\triangle ABF$ is equilateral.

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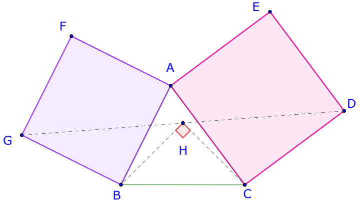
Thus the sum of rotation is a rotation about O by 60° sends D to C, where O is the point such that $OC = OD$, $\angle DOC = 60^\circ$, and $\angle OAB = OBA = 15^\circ$. Hence, $O \equiv E$.

Example

Let $ABGF$ and $ACDE$ be squares outside $\triangle ABC$. Let H be the midpoint of DG. Show that $HB = HC$ and $HB \perp HC$.



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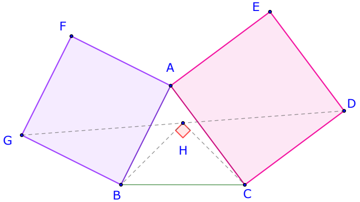

Since $CD = CA$ and $\angle DCA = 90^\circ$, so the rotation about C by 90° anti-clockwise sends D to A. Also, the rotation about B by 90° anti-clockwise sends A to G.

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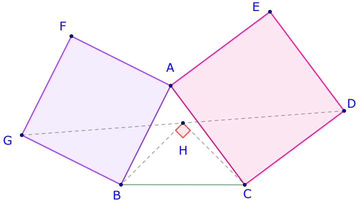

Since $CD = CA$ and $\angle DCA = 90^\circ$, so the rotation about C by 90° anti-clockwise sends D to A. Also, the rotation about B by 90° anti-clockwise sends A to G.

Thus the sum of rotation is a rotation about O by 180° sends D to G, where O is the point such that $\angle OCB = 45^\circ$ and $\angle OBC = 45^\circ$.

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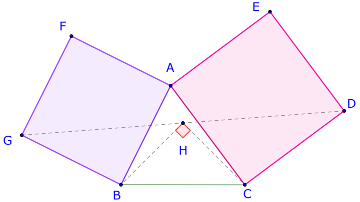

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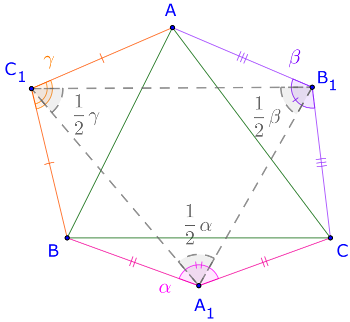
Thus the sum of rotation is a rotation about O by 180° sends D to G, where O is the point such that $\angle OCB = 45^\circ$ and $\angle OBC = 45^\circ$.

A rotation of about O by 180° sends D to G means that $OD = OG$, thus $O \equiv H$.

Example

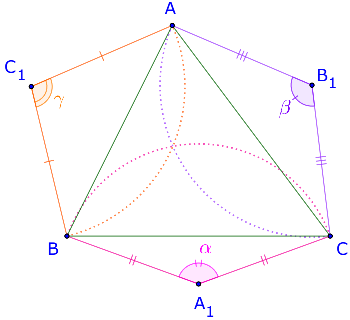
On the sides of an arbitrary triangle ABC, exterior to it, construct isosceles triangles BCA_1 ACB_1, CAB_1 with angles at the vertices A_1, B_1, and C_1, respectively equal to α, β and γ.

Prove that if $\alpha + \beta + \gamma = 360°$, then the angles of the triangle $A_1B_1C_1$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\gamma$, that is, they do not depend on the shape of the triangle ABC.



First, point A is taken into itself by the sum of three rotations through the angles β, α, and γ ($\alpha + \beta + \gamma = 360^\circ$) about the centers B_1, A_1, C_1:

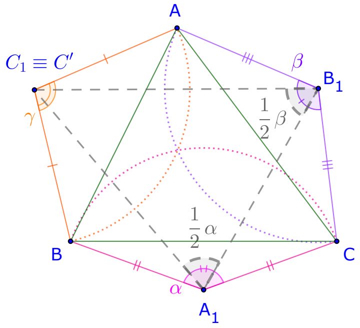
$$A \overset{\text{rotate}(B_1,\beta)}{\rightarrow} C \overset{\text{rotate}(A_1,\alpha)}{\rightarrow} B \overset{\text{rotate}(C_1,\gamma)}{\rightarrow} A.$$



Thus, the **sum of the these rotations** is the **identity transformation**.

Let C' be the center of the rotation equivalent to the sum of the rotations about B_1 and A_1. Then it is the rotation through $\alpha + \beta = 360^\circ - \gamma$ brings A to B.

However, the rotation about C_1 through γ brings A to B in opposite direction. Since a rotation through an angle θ is the same as the rotation through an angle $360^\circ - \theta$ about the same center in the opposite direction, thus $C_1 \equiv C'$.

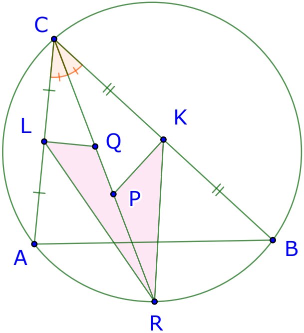


Therefore $\angle C_1A_1B_1 = \frac{1}{2}\alpha, \angle C_1B_1A_1 = \frac{1}{2}\beta$, and similarly $\angle B_1C_1A_1 = \frac{1}{2}\gamma$.

Example

In triangle ABC, the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R, P, and Q, respectively. The midpoints of BC and CA are K and L, respectively.

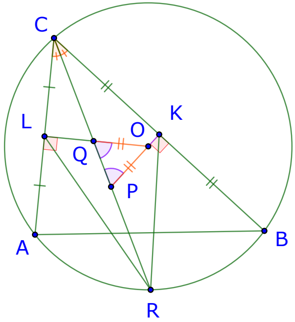
Prove that triangles RPK and RQL have the same area.



WLOG, assume $AC < BC$. Let $ACB = \gamma$. From the right triangles CLQ and CKP,

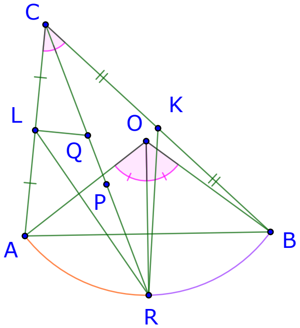
$$\angle OPQ = \angle OQP = 90^\circ - \frac{\gamma}{2}.$$

Thus, OPQ is isosceles, $OP = OQ$, and $\angle POQ = 180^\circ - 2\left(90^\circ - \frac{\gamma}{2}\right) = \gamma$.



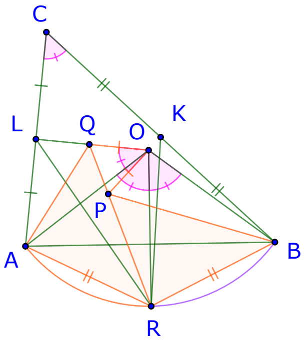
CR is the angle bisector, so R is midpoint of $\widehat{AB}$, and

$$\angle ROA = \angle ROB = \gamma, \; AR = RB.$$



Consider the **rotation around O by angle γ**, see the figure below. This transformation moves:

$$Q \to P,\ A \to R,\ R \to B \Rightarrow \triangle QAR \cong \triangle PRB \Rightarrow [QAR] = [PRB].$$

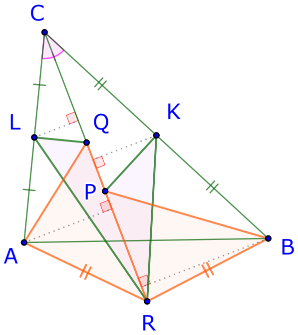


Finally:

$$\frac{[RQL]}{[RQA]} = \frac{\text{distance}(L, CR)}{\text{distance}(A, CR)} = \frac{CL}{CA} = \frac{1}{2}, \text{ and similarly } \frac{[RPK]}{[BPR]} = \frac{1}{2}$$

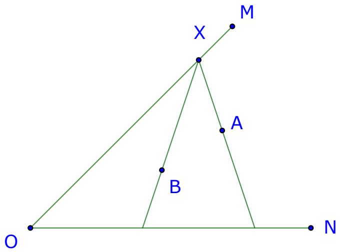
Therefore,

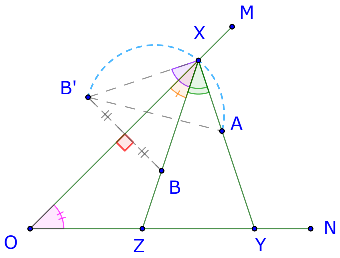
$$[RQL] = \frac{1}{2}[RQA] = \frac{1}{2}[BPR] = [RPK]$$



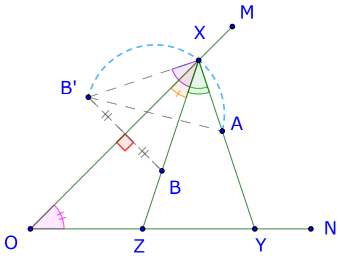
Example

$\angle MON$ is given, together with two points A and B. Find a point X on the side OM such that the triangle XYZ is isosceles: $XY = XZ$, where Y and Z are on the points of intersection of XA and XB with ON.



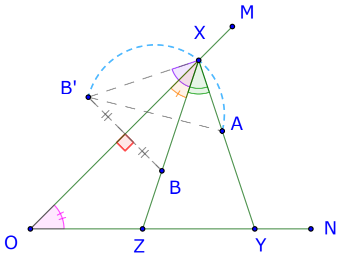


Let B' be the image of B over OM,



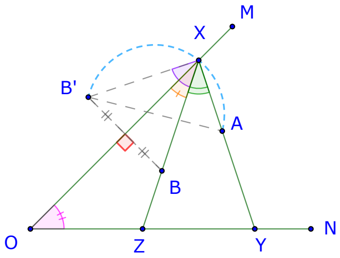
Let B' be the image of B over OM, then:

$\angle B'XA = \angle B'XB + \angle YXZ$,



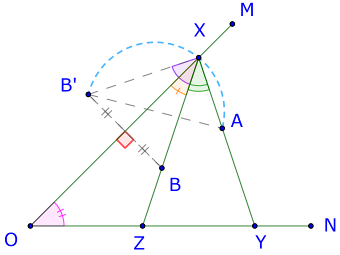
Let B' be the image of B over OM, then:

$\angle B'XA = \angle B'XB + \angle YXZ$, $\angle B'XB = 2\angle OXZ = 2(\angle XZY - \angle MON)$



Let B' be the image of B over OM, then:

$\angle B'XA = \angle B'XB + \angle YXZ$, $\angle B'XB = 2\angle OXZ = 2(\angle XZY - \angle MON) \Rightarrow \angle B'XA = 180^\circ - 2\angle MON$.



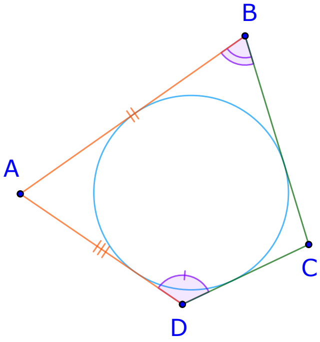
Let B' be the image of B over OM, then:

$\angle B'XA = \angle B'XB + \angle YXZ$, $\angle B'XB = 2\angle OXZ = 2(\angle XZY - \angle MON) \Rightarrow \angle B'XA = 180^\circ - 2\angle MON$.

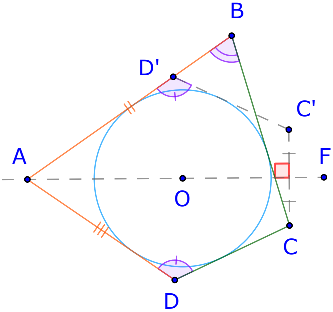
Thus, X is the intersection of OM with the arc constructed on the chord AB', that subtends an angle equal to $180^\circ - 2\angle MON$.

Example

Construct a quadrilateral $ABCD$ in which a circle can be inscribed, given the lengths of two adjacent sides AB and AD and the angles at the vertices B and D.



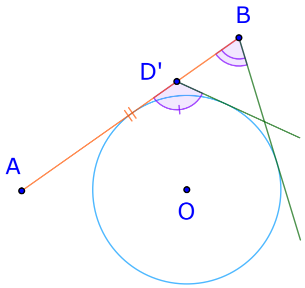
The key idea here is that the reflection of CD over the line through A and the center of the circle is a tangent to the circle!



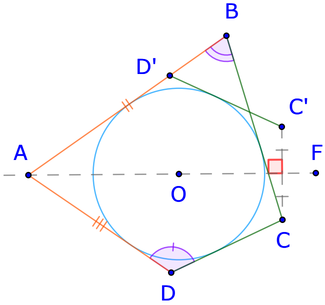
First, we start the construction by point A then segment AB, then segment $AD_1 = AD$ where D is on the line AB, same side as B in respect to A.

Second, because $\angle B$ and $\angle D_1 = \angle D$ are known, thus we can construct rays going from B and D_1.

Finally, we construct a circle tangents to all three lines.



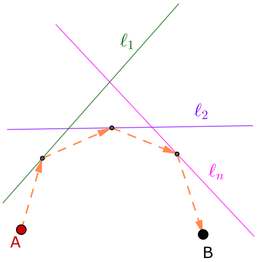
The rest is simple, we reflect D' and its ray over the line AO where O is the center of the circle. The reflected ray will intersect the ray from B at C. We are done.



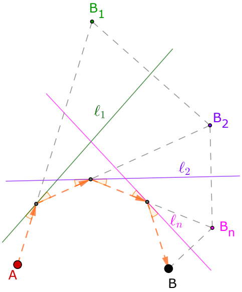
Example

A billiard ball bounces off a side of a billiard table in such a manner that the two lines along which it moves before and after hitting the sides are equally inclined to the side.

Suppose a billiard table were bordered by n lines $\ell_1, \ell_2, \ldots, \ell_n$. Let A and B be two given points on the billiard table. In what direction should one hit a ball placed at A so that it will bounce consecutively off the lines $\ell_1, \ell_2, \ldots, \ell_n$, and then pass through the point B (see the diagram below, where $n = 3$)?



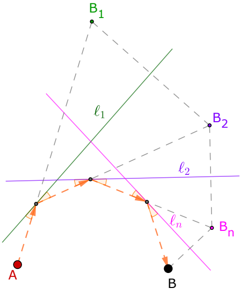
Assume that the problem has been solved, that is, that points $X_1, X_2, \ldots X_n$ have been found on the lines $\ell_1, \ell_2, \ldots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



It is easy to see that the point X_n, is the point of intersection of the line ℓ_n with the line $X_{n-1}B_n$, where B_n, is the image of B in ℓ_n, that is, the points B_n,X_n, X_{n-1} lie on a line.

But then the point X_{n-1} is the point of intersection of the line ℓ_{n-1} with the $X_{n-2}B_{n-1}$, where B_{n-1}, is the image of B_n in ℓ_{n-1} and so on.

Assume that the problem has been solved, that is, that points $X_1, X_2, \dots X_n$ have been found on the lines $\ell_1, \ell_2, \dots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



Here's the construction: Reflect the point B in l_n, obtaining the point B_n; next reflect B_n in l_{n-1} to obtain B_{n-1}, and so forth, until the image B_1 of the point B_2, in line ℓ_1 is obtained.

The point X_1, that determines the direction in which the billiard ball at A must be hit, is obtained as the point of intersection of the line ℓ_1 with the line AB_1. It is then easy to find the points $X_2, \dots X_n$ with the aid of the points $B_2, \dots B_n$ and X_1.

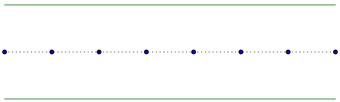
Example

A *center of symmetry* of a set S means a point O, not necessarily in S, such that for every point $A \in S$, there is another point $B \in S$ such that O is the midpoint of AB. We say that B is symmetric to A with respect to O.

Prove that a set S containing a *finite* number of points cannot have more than one center of symmetry.

Example from Geometric Transformations II session:

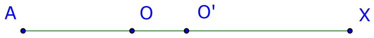
The strip formed by two parallel lines clearly has infinitely many centers of symmetry. Can a figure have more than one, but only a finite number of centers of symmetry (for example, can it have two and only two centers of symmetry)?

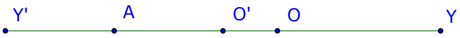


Let O be the center of symmetry. Suppose to the contrary that $\mathcal{F}$ has another center of symmetry $O' \not\equiv O$. Since $\mathcal{F}$ contains a finite number of points, by the Extremal Principle, **there exists** $A \in \mathcal{F}$ **with the greatest distance from** O. We consider two cases A, O, O' are collinear and A, O, O' are not collinear.

Case 1: A, O, O' are collinear. There are three sub-cases.

Case 1.1: O is between A and O'. Let X be the point symmetric to A with respect to O' (see the diagram below on the left). Then $XO > XO' = AO' > AO$, which is a contradiction to the maximality of AO.

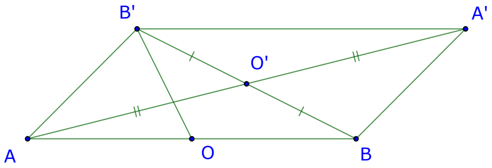




Case 1.2: O' is between A and O. Let Y be the point symmetric to A with respect to O, and let Y' be the point symmetric to Y with respect to O' (see the diagram above on the right). Then $Y'O > Y'O' = YO' > YO = AO$, which is a contradiction to the maximality of AO.

Case 1.3: A is between O and O'. Let Z be the point symmetric to A with respect to O' (see the diagram below) Then $ZO > AO$, which is a contradiction to the maximality of AO.

Case 2: A, O, O' are not collinear. Let B be the point symmetric to A with respect to O, let A' the point symmetric to A with respect to O', and let B' the point symmetric to B with respect to O' (see the diagram below).



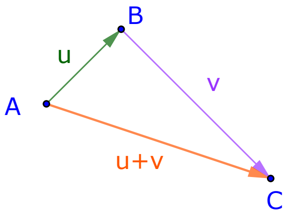
Because the quadrilateral $ABA'B'$ has 2 diagonals that bisect each other ($AO' = O'A', BO' = O'B'$), thus $ABA'B'$ is a parallelogram. Therefore $\angle B'AB + \angle A'BA = 180°$, so one of $\angle B'AB$ and $\angle A'BA$ is greater than or equal to $90°$.
Case 2.1: $\angle A'BA \geq 90°$, then $A'O > BO = AO$ (because $\angle A'BO$ is the largest angle in $\triangle A'BO$), which is a contradiction to the maximality of AO.
Case 2.2: $\angle B'AB \geq 90°$, then $B'O > AO$ (because $\angle B'AO$ is the largest angle in $\triangle B'AO$), which is a contradiction to the maximality of AO.

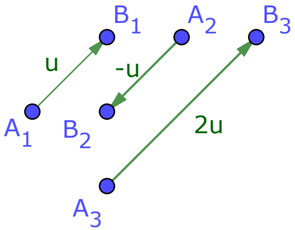
Hence, $\mathcal{F}$ cannot have more than one center of symmetry.

Definition (Vector)

Vector is a *directed line segment*, or as an arrow connecting an initial point A with a terminal point B, and denoted by $\overrightarrow{AB}$. Vectors are usually denoted in lowercase boldface, as in $\mathbf{u}, \mathbf{v}$, or $\mathbf{w}$.

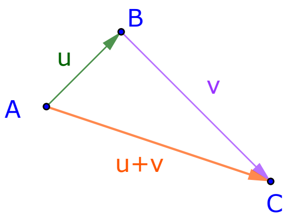
A vector is called *zero vector* if the initial point and the terminal point are the same, in other words its magnitude is 0. A zero vector has an arbitrary or indeterminate direction. A *unit vector* is any vector with a length of one.

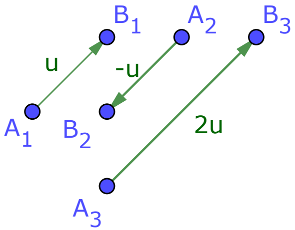




Definition (Equality)

Two vectors are said to be *equal* if they have the same magnitude and direction. The equality of $\mathbf{u}, \mathbf{v}$ is denoted as $\mathbf{u} = \mathbf{v}$.



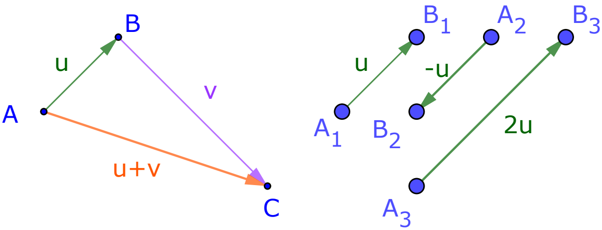


Definition (Parallel, Opposite, and Antiparallel)

Two vectors are *parallel* if they have the same direction but not necessarily the same magnitude: $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ ($\overrightarrow{AB} \parallel \overrightarrow{A_1B_1}$) or $\overrightarrow{AB}$ and $\overrightarrow{A_3B_3}$ ($\overrightarrow{AB} \parallel \overrightarrow{A_3B_3}$.)

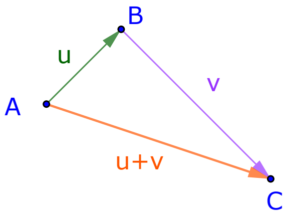
Two vectors are *opposite* if they have the same magnitude but opposite direction: $\overrightarrow{AB}$ and $\overrightarrow{A_2B_2}$

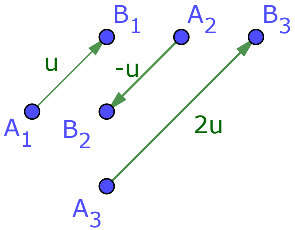
Two vectors are *antiparallel* if they have opposite direction: $\overrightarrow{A_2B_2}$ and $\overrightarrow{A_3B_3}$.



Definition (Addition and Subtraction)

The *sum* of $\mathbf{u}$ and $\mathbf{v}$ of two vectors may be defined as $\mathbf{u}+\mathbf{v}$. The resulting vector is sometimes called the resultant vector of $\mathbf{u}$ and $\mathbf{v}$.
The addition may be represented graphically by placing the tail of the arrow $\mathbf{v}$ at the head of the arrow $\mathbf{u}$, and then drawing an arrow from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$. The new arrow drawn represents the vector $\mathbf{u}+\mathbf{v}$, shown as below.
The *difference* of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}-\mathbf{v}$, or $\mathbf{u}+(-\mathbf{v})$, which is the addition of $\mathbf{u}$ to the opposite of $\mathbf{v}$.





Suppose that there were two centers of symmetry, P and Q. Let S be a set of pairwise distinct points in $\mathcal{F}$:

$$S = \{A_1, A_2, \ldots, A_n\}.$$

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By the definition of symmetry, we have $\overrightarrow{PA_k} + \overrightarrow{PB_k} = \mathbf{0}$.

Note that for each $k \in \{1, 2, \ldots, n\}$, there exists $j \in \{1, 2, \ldots, n\}$ $B_k = A_j$. So $(B_1, B_2, \ldots, B_n)$ is a permutation of $(A_1, A_2, \ldots, A_n)$.

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Summing this from $k = 1$ to $k = n$ implies that:

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By the same reasoning, we also have $\overrightarrow{QA_1} + \overrightarrow{QA_2} + \ldots + \overrightarrow{QA_n} = \mathbf{0}$. Subtracting,

$$\mathbf{0} = \sum_{k=1}^{n} \overrightarrow{PA_k} - \sum_{k=1}^{n} \overrightarrow{QA_k} = n\overrightarrow{QP} \Rightarrow \overrightarrow{QP} = 0 \Rightarrow \boxed{Q \equiv P.}$$