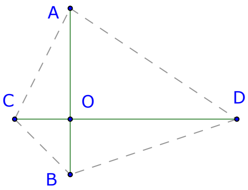
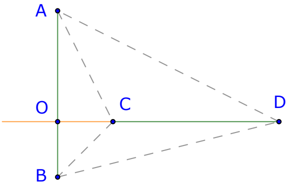
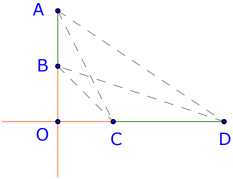
Theorem (Perpendicularity Lemma)

Let AB and CD be two intersecting lines. Then, $AB \perp CD \iff CA^2 - CB^2 = DA^2 - DB^2$.







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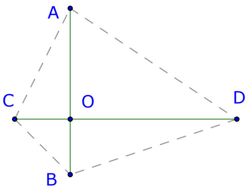
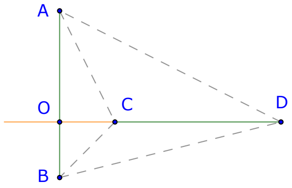
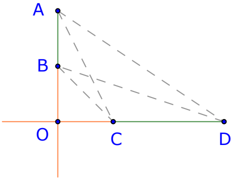


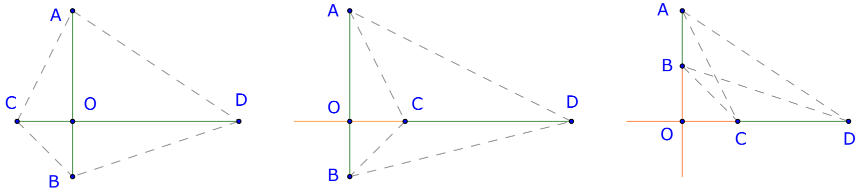




Let $AB \perp CD$. Let $AB \cap CD = O$.

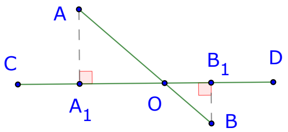
Theorem (Perpendicularity Lemma)

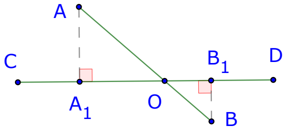
Let AB and CD be two intersecting lines. Then, $AB \perp CD \iff CA^2 - CB^2 = DA^2 - DB^2$.



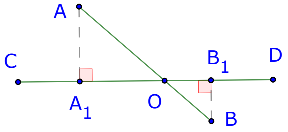
Let $AB \perp CD$. Let $AB \cap CD = O$.
$\triangle ACO$, $\triangle BCO$, $\triangle ADO$, and $\triangle BDO$ are right triangles, by the Pythagorean Theorem:

$$CA^2 - CB^2 = (OC^2 + OA^2) - (OC^2 + OB^2) = (OD^2 + OA^2) - (OD^2 + OB^2) = DA^2 - DB^2.$$


A
C
B_1
D
A_1
O
B

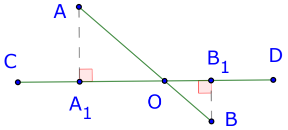


Let $CA^2 - CB^2 = DA^2 - DB^2$. We discuss the case where $O \in AB$ and $O \in CD$. Let $AA_1 \perp CD$, and $BB_1 \perp CD$.



Let $CA^2 - CB^2 = DA^2 - DB^2$. We discuss the case where $O \in AB$ and $O \in CD$. Let $AA_1 \perp CD$, and $BB_1 \perp CD$. $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$, and $\triangle DBB_1$ are right, by the Pythagorean Theorem:

$$CA^2 - CB^2 = DA^2 - DB^2 \Leftrightarrow (CA_1^2 + AA_1^2) - (CB_1^2 + BB_1^2) = (DA_1^2 + AA_1^2) - (DB_1^2 + BB_1^2)$$
$$\Rightarrow CA_1^2 - CB_1^2 = DA_1^2 - DB_1^2 \Rightarrow CA_1^2 - DA_1^2 = CB_1^2 - DB_1^2 \Rightarrow CA_1 - DA_1 = CB_1 - DB_1$$
$$\Rightarrow CA_1 - CB_1 = DA_1 - DB_1 \Rightarrow -A_1B_1 = A_1B_1 \Rightarrow A_1B_1 = 0 \Rightarrow A_1 \equiv B_1.$$



Let $CA^2 - CB^2 = DA^2 - DB^2$. We discuss the case where $O \in AB$ and $O \in CD$. Let $AA_1 \perp CD$, and $BB_1 \perp CD$. $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$, and $\triangle DBB_1$ are right, by the Pythagorean Theorem:

$$CA^2 - CB^2 = DA^2 - DB^2 \Leftrightarrow (CA_1^2 + AA_1^2) - (CB_1^2 + BB_1^2) = (DA_1^2 + AA_1^2) - (DB_1^2 + BB_1^2)$$
$$\Rightarrow CA_1^2 - CB_1^2 = DA_1^2 - DB_1^2 \Rightarrow CA_1^2 - DA_1^2 = CB_1^2 - DB_1^2 \Rightarrow CA_1 - DA_1 = CB_1 - DB_1$$
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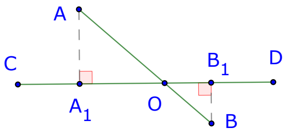
Therefore, the perpendiculars to CD from A and B pass through a common point on CD, so they must be the same line, i.e. $AB \perp CD$.



Let $CA^2 - CB^2 = DA^2 - DB^2$. We discuss the case where $O \in AB$ and $O \in CD$. Let $AA_1 \perp CD$, and $BB_1 \perp CD$. $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$, and $\triangle DBB_1$ are right, by the Pythagorean Theorem:

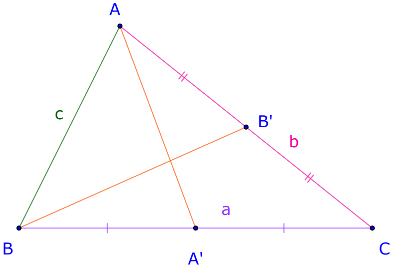
$$CA^2 - CB^2 = DA^2 - DB^2 \Leftrightarrow (CA_1^2 + AA_1^2) - (CB_1^2 + BB_1^2) = (DA_1^2 + AA_1^2) - (DB_1^2 + BB_1^2)$$
$$\Rightarrow CA_1^2 - CB_1^2 = DA_1^2 - DB_1^2 \Rightarrow CA_1^2 - DA_1^2 = CB_1^2 - DB_1^2 \Rightarrow CA_1 - DA_1 = CB_1 - DB_1$$
$$\Rightarrow CA_1 - CB_1 = DA_1 - DB_1 \Rightarrow -A_1B_1 = A_1B_1 \Rightarrow A_1B_1 = 0 \Rightarrow A_1 \equiv B_1.$$

Therefore, the perpendiculars to CD from A and B pass through a common point on CD, so they must be the same line, i.e. $AB \perp CD$.

In the cases where O is not between A and B or between C and D, the proof follows exactly the same steps. There might be a different operation when dealing with the line segments (addition or subtraction) depending on the configuration, but the result will always be the same.

Example

Prove that the medians AA', BB' of $\triangle ABC$ are perpendicular if and only if $a^2 + b^2 = 5c^2$, where $AB = c, BC = a$, and $CA = b$.



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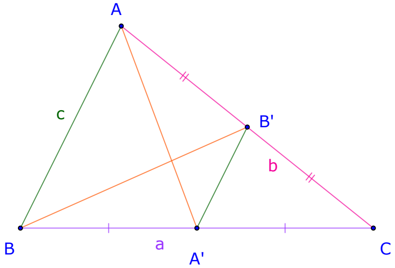
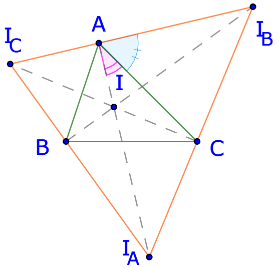


Note that $A'B' = \frac{c}{2}$, by the Perpendicularity Lemma AA', BB' are perpendicular if and only if:

$$AB^2 - AB'^2 = A'B^2 - A'B'^2 \Leftrightarrow c^2 - \frac{b^2}{4} = \frac{a^2}{4} - \frac{c^2}{4} \Leftrightarrow a^2 + b^2 = 5c^2.$$

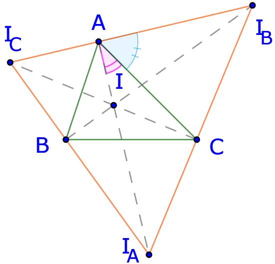
Example

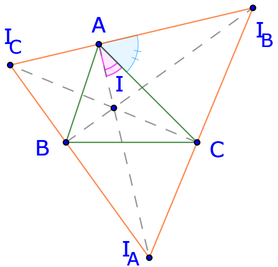
Let I_A, I_B, and I_C be the excenters opposite of A, B, and C in $\triangle ABC$, respectively. Prove that the incenter of $\triangle ABC$ is the orthocenter of $\triangle I_A I_B I_C$.



Carnot and Butterfly Theorems

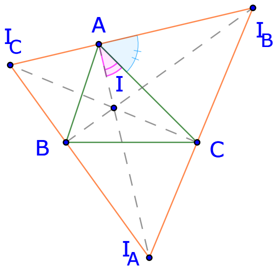
Orthocenter of excentres is the incenter





Let I be the incenter of $\triangle ABC$. AI and AI_B are internal and external angle bisectors.

$$\angle IAI_B = \angle IAC + \angle CAI_B = \frac{\angle A}{2} + \frac{180^\circ - \angle A}{2} = 90^\circ.$$



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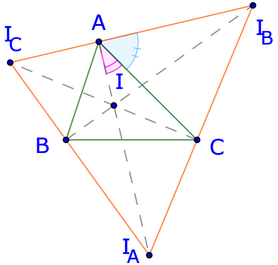
$$\angle IAI_B = \angle IAC + \angle CAI_B = \frac{\angle A}{2} + \frac{180^\circ - \angle A}{2} = 90^\circ.$$

Similarly, $\angle IAI_C = 90^\circ$. Therefore $\angle IAI_B + \angle IAI_C = 180^\circ$, thus $A \in I_B I_C$, and lines I_A and IA are the same, both are perpendicular to $I_B I_C$, so $I_A A$ is an altitude in $\triangle I_A I_B I_C$.



Let I be the incenter of $\triangle ABC$. AI and AI_B are internal and external angle bisectors.

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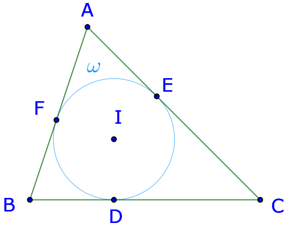
Similarly, $\angle IAI_C = 90^\circ$. Therefore $\angle IAI_B + \angle IAI_C = 180^\circ$, thus $A \in I_B I_C$, and lines I_A and IA are the same, both are perpendicular to $I_B I_C$, so $I_A A$ is an altitude in $\triangle I_A I_B I_C$.
Similar for $I_B B, I_C C$. Hence, $\boxed{I \text{ is the orthocenter of } \triangle I_A I_B I_C.}$

Theorem (Tangent Segments of the Incircle)

Let ω be the incircle in $\triangle ABC$. Let D be the tangent point of ω to the side BC. Prove that $AB + CD = AC + BD$.



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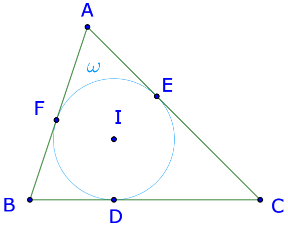

Let E and F be the tangent points of ω with the sides CA and AB, respectively.

Theorem (Tangent Segments of the Incircle)

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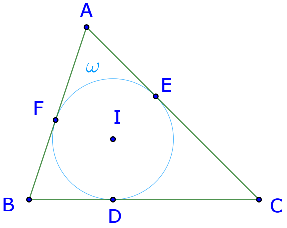


Let E and F be the tangent points of ω with the sides CA and AB, respectively.

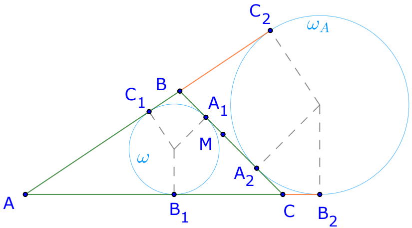
(AE, AF), (BF, BD), and (CD, CE) are pairs of tangent segments from A, B, and C to ω, thus:

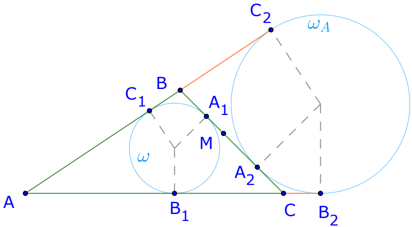
$$AF = AE, BF = BD, CD = CE \Rightarrow AB + CD = AF + FB + CD = AE + EC + BD = AC + BD.$$

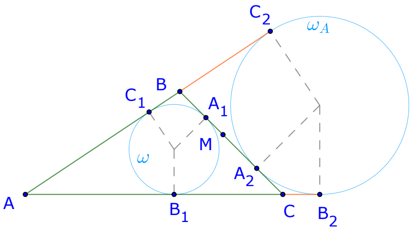
Theorem (Tangent Segments of the Excircles)

Let ω and ω_A be the incircle and the $A-$excircle in $\triangle ABC$. Let A_1, B_1, and C_1 be the tangent points of ω with the sides BC, CA, and AB, respectively. Let A_2, B_2, and C_2 be the tangent points of ω_A with the lines BC, CA, and AB. Prove that:

1. $AB + BA_2 = AC + CA_2$.
2. $BA_2 = CA_1$, *i.e.* $A_1M = MA_2$, *where M is the midpoint of BC.*

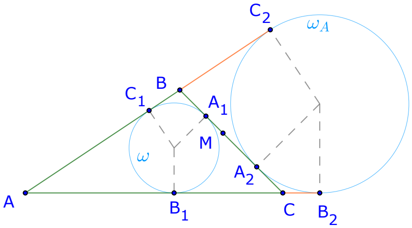



C_2
ω_A
B
C_1
A_1
M
ω
A_2
A
B_1
C
B_2



By tangent segments from A, B, and C to ω_A, $AB_2 = AC_2, BA_2 = BC_2, CA_2 = CB_2$. Thus:

$$AB + AB_2 = AB + BC_2 = AC_2 = AB_2 = AC + CB_2 = AC + CA_2.$$

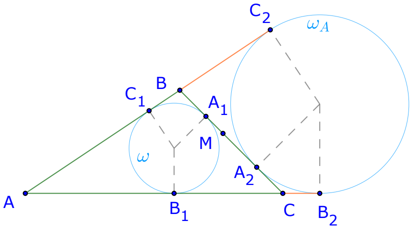


By tangent segments from A, B, and C to ω_A, $AB_2 = AC_2, BA_2 = BC_2, CA_2 = CB_2$. Thus:

$$AB + AB_2 = AB + BC_2 = AC_2 = AB_2 = AC + CB_2 = AC + CA_2.$$

The sum of both sides equals the perimeter of $\triangle ABC$, so if s denotes the semi-perimeter:

$$BA_2 = s - AB.$$



By tangent segments from A, B, and C to ω_A, $AB_2 = AC_2, BA_2 = BC_2, CA_2 = CB_2$. Thus:

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The sum of both sides equals the perimeter of $\triangle ABC$, so if s denotes the semi-perimeter:

$$BA_2 = s - AB.$$

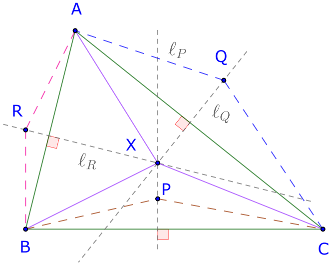
By the theorem Tangent Segments of the Incircle: $AC + BA_1 = AB + CA_1$. Similarly:

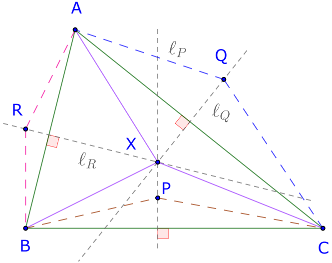
$$CA_1 = s - AB \Rightarrow BA_2 = CA_1 \Rightarrow A_1M = MA_2.$$

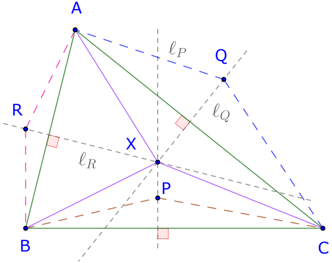
Theorem (Carnot's Extended Theorem)

Let P, Q, and R be points in the plane of triangle ABC. Then, the lines ℓ_P, ℓ_Q, and ℓ_R, which are the perpendiculars from P, Q, and R to BC, CA, and AB, respectively, are concurrent if and only if:

$$PB^2 - PC^2 + QC^2 - QA^2 + RA^2 - RB^2 = 0.$$

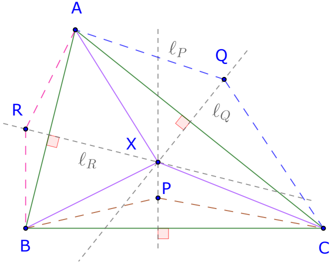



A
ℓ_P
Q
R
ℓ_Q
X
ℓ_R
P
B
C



For ($\Rightarrow$), let ℓ_P, ℓ_Q, and ℓ_R, be concurrent and let the point of concurrence be X. By the Perpendicularity Lemma, $XP \perp BC$, so $PB^2 - PC^2 = XB^2 - XC^2$, and similarly for others, then

$$XB^2 - XC^2 + XC^2 - XA^2 + XA^2 - XB^2 = 0.$$



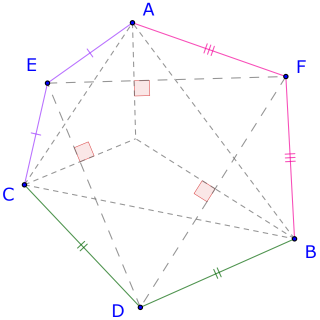
For $(\Rightarrow)$, let ℓ_P, ℓ_Q, and ℓ_R, be concurrent and let the point of concurrence be X. By the Perpendicularity Lemma, $XP \perp BC$, so $PB^2 - PC^2 = XB^2 - XC^2$, and similarly for others, then

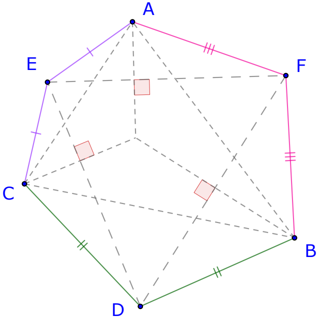
$$XB^2 - XC^2 + XC^2 - XA^2 + XA^2 - XB^2 = 0.$$

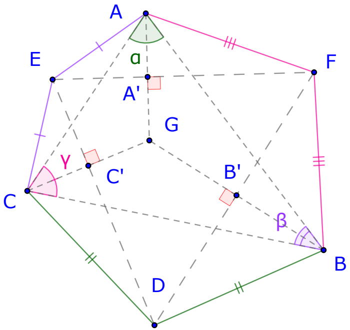
Now for $(\Leftarrow)$, let $PB^2 - PC^2 + QC^2 - QA^2 + RA^2 - RB^2 = 0$. Let $X = \ell_P \cap \ell_Q$. Then by $(\Rightarrow)$ $XB^2 - XC^2 + XC^2 - XA^2 + RA^2 - RB^2 = 0$, or $XB^2 - XA^2 = RA^2 - RB^2$, so by the Perpendicularity Lemma $XR \perp AB$, hence $\boxed{X \in \ell_R.}$

Example

Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines through EF, FE, and DE, respectively, are concurrent.




A
E
F
C
B
D



Since $AE = EC$, $CD = DB$, and $FB = FA$, therefore

$$AE^2 - AF^2 + BF^2 - BD^2 + CD^2 - CE^2 = 0.$$

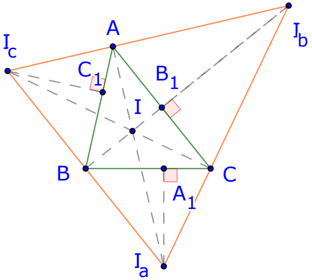
Since $AE = EC$, $CD = DB$, and $FB = FA$, therefore

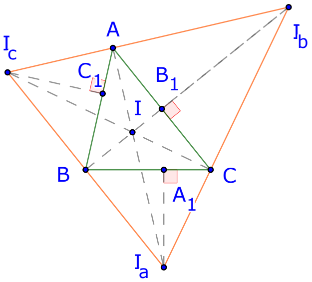
$$AE^2 - AF^2 + BF^2 - BD^2 + CD^2 - CE^2 = 0.$$

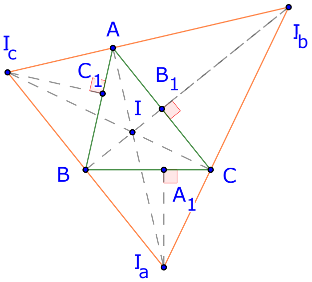
Thus by the Carnot's Extended Theorem, the lines through A, B, C perpendicular to the lines through EF, FE, and DE, respectively, are concurrent.

Example

Let I_a, I_b, and I_c be the excenters of triangle ABC opposite the vertices A, B and C, respectively. Let A_1, B_1, and C_1 be the tangent points of the $A-$, $B-$, and $C-$excircle with the sides BC, CA, and AB, respectively. Prove that the lines I_aA_1, I_bB_1, and I_cC_1 are concurrent.

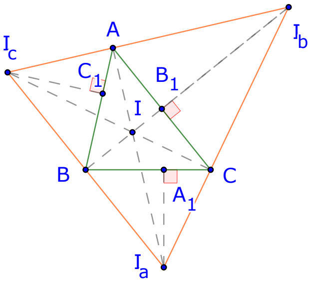



A
I_b
I_c
C_1
B_1
I
B
C
A_1
I_a



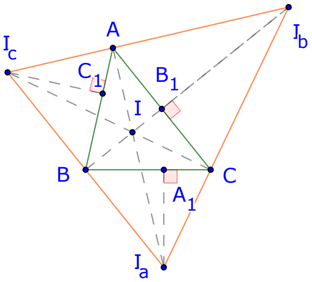
The three perpendiculars are concurrent if and only if, by Carnot's Extended Theorem,

$$I_aB^2 - I_aC =^2 + I_bC^2 - I_bA^2 + I_cA^2 - I_cB^2 = 0.$$



By the theorem Tangent Segments of The Excircles, $BA_1 = s - c = AB_1$, where s is the semi-perimeter of $\triangle ABC$. Similarly with other sides. Let $x = s - c, y = s - b$, and $z = s - a$. Let r_a, r_b, and r_c be the radii of the $A-$, $B-$, and $C-$excircle, respectively. Then

$$I_aB^2 = r_a^2 + x^2,\ I_aC^2 = r_a^2 + y^2$$
$$I_bC^2 = r_b^2 + z^2,\ I_bA^2 = r_b^2 + x^2$$
$$I_cA^2 = r_c^2 + y^2,\ I_cB^2 = r_c^2 + z^2$$



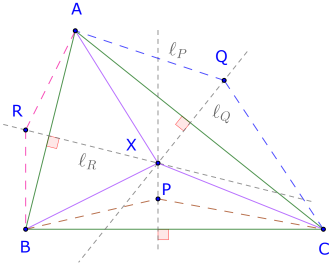
By applying Pythagorean Theorem six times:

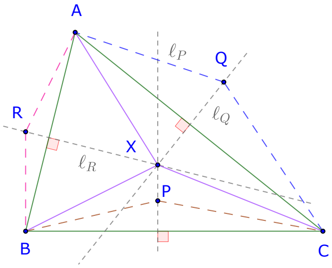
$$\underbrace{(r_a^2+x^2)}_{I_aB^2}-\underbrace{(r_a^2+y^2)}_{I_aC^2}+\underbrace{(r_b^2+z^2)}_{I_bC^2}-\underbrace{(r_b^2+x^2)}_{I_bA^2}+\underbrace{(r_c^2+y^2)}_{I_cA^2}-\underbrace{(r_c^2+z^2)}_{I_cB^2}=0.$$

Hence, $\boxed{I_aA_1, I_bB_1, \text{ and } I_cC_1 \text{ are concurrent.}}$

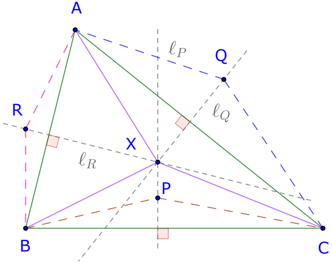
Example

Let P, Q, and R be points in the plane of triangle ABC. Then, the perpendiculars from P, Q, and R to BC, CA, AB, respectively, are concurrent if and only if the perpendiculars from C, A, and B to PQ, QR, and RP, respectively, are concurrent.



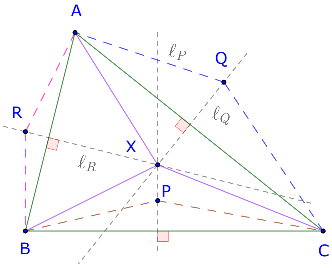


A
ℓ_P
Q
R
ℓ_Q
X
ℓ_R
P
B
C



Let ℓ_P, ℓ_Q, and ℓ_R, be the perpendiculars from P, Q, and R to BC, CA, and AB, respectively. By the Carnot's Extended Theorem, they are concurrent if and only if

$$PB^2 - PC^2 + QC^2 - QA^2 + RA^2 - RB^2 = 0.$$



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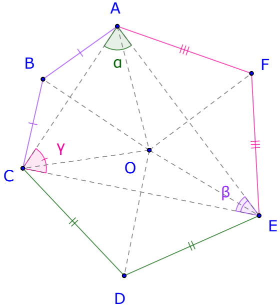
Now by rearranging the terms, we have $CP^2 - CQ^2 + AQ^2 - AR^2 + BR^2 - BP^2 = 0$, which stands if and only if the perpendiculars from C, A, and B to PQ, QR, and RP, respectively, are concurrent.

Example

$ABCDEF$ is a convex hexagon such that $AB = BC, CD = DE$ and $EF = FA$. Prove that the angle bisectors of $\angle ABC, \angle CDE$, and $\angle EFA$ are concurrent.



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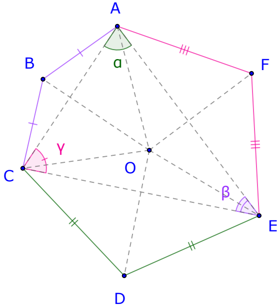

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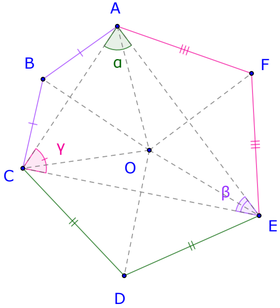

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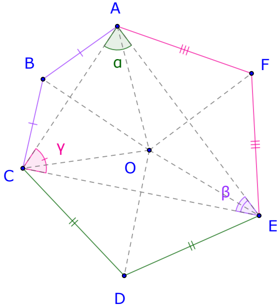

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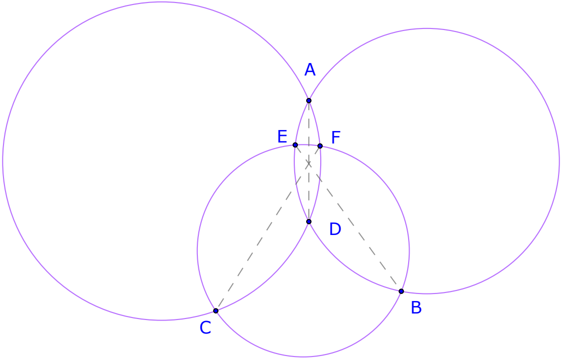
Example

Three circles intersect pairwise as shown. Prove that AD, BE, and CF are concurrent.



Example

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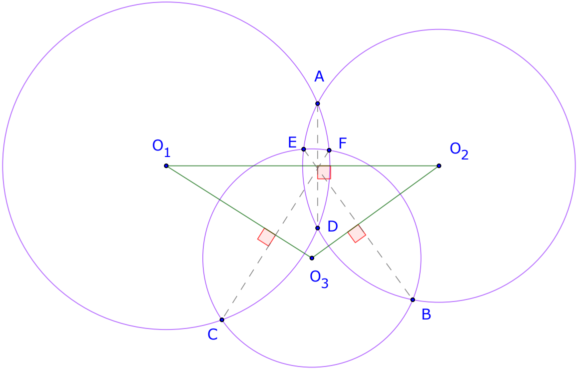
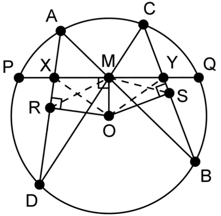


Since $O_1O_2 \perp AD$, $O_2O_3 \perp BE$, $O_3O_1 \perp CF$, and $AO_1^2 - AO_2^2 + BO_2^2 - BO_3^2 + CO_3^2 - CO_1^2 = 0$. By the Carnot's Extended Theorem, AD, BE, and CF are concurrent.

Theorem (Butterfly Theorem)

Let M be the midpoint of a chord PQ of a circle ω, through which two other chords AB and CD are drawn. Let $AD \cap PQ = X$ and $BC \cap PQ = Y$. Prove that M is also the midpoint of XY.



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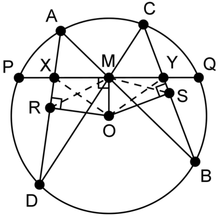

Let ω's center O. $MP = MQ$, so $OM \perp PQ$. To prove $XM = MY$, we need $\angle MOX = \angle MOY$.

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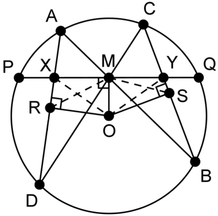

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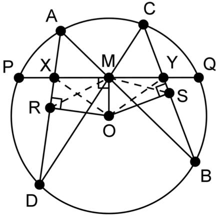

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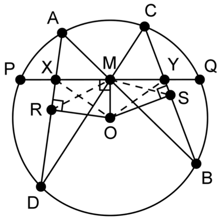
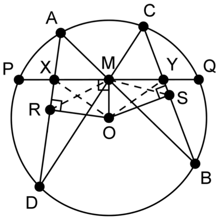
$$\angle DAM \equiv \angle DAB \overset{\omega}{=} \angle DCB \equiv \angle MCB \text{ and } \angle AMD = \angle CMB$$

$$\Rightarrow \triangle AMD \sim \triangle CMB \Rightarrow \frac{AD}{AM} = \frac{CB}{CM}$$



$$\frac{AD}{AM} = \frac{CB}{CM} \Rightarrow \frac{2AR}{AM} = \frac{2CS}{CM} \Rightarrow \frac{AR}{AM} = \frac{CS}{CM} \Longrightarrow \triangle AMR \sim \triangle CMS \Rightarrow \angle MRA = \angle MSC \quad (*)$$



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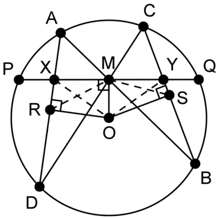
Since $OM \perp PQ$, $OR \perp AD$, and $\angle ORX + \angle OMX = 180^\circ$, so $OMXR$ is a cyclic quadrilateral. Similarly $OMYS$ is also a cyclic quadrilateral.



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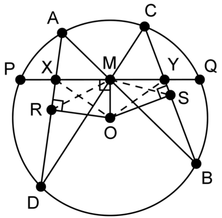

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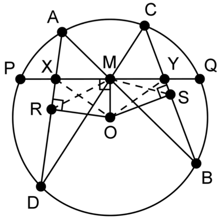
Thus $\triangle MXO \cong \triangle MYO$, or $MX = MY$, thus $\boxed{M \text{ is also the midpoint of } XY.}$

Theorem (The Butterfly Theorem in converse)

Denote by M the point of intersection of the chords AB and CD of a circle ω.

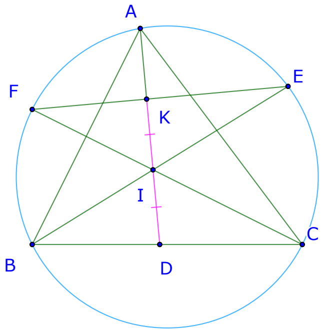
ℓ is a line passing through M such that $X = AD \cap \ell$ and $Y = BC \cap \ell$, and $MX = MY$.

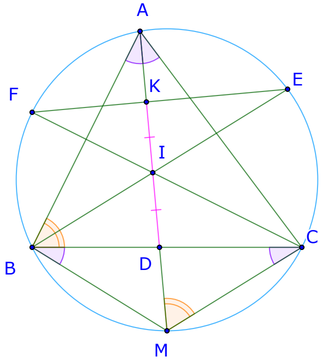
Then $OM \perp \ell$.

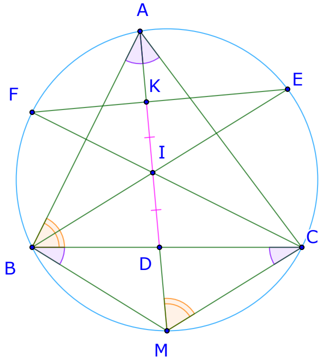


Example

(O) and (I) are circumcircle and incircle, respectively, of $\triangle ABC$. Lines through BI and CI intersect (O) at E and F,respectively. Let K and D be the intersections of AI with EF and BC. If $AB + AC = 2BC$, prove that $IK = ID$.

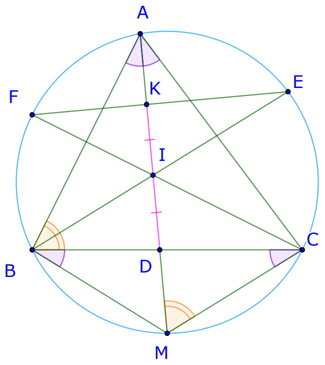



A
E
F
K
I
C
B
D
M



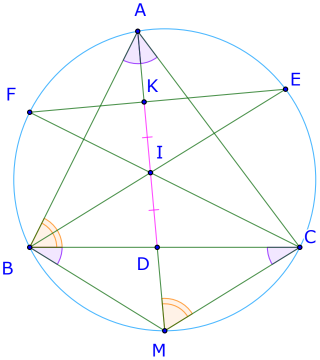
Let M be the intersection of AI and the circle (O), $M \equiv A$. See above on the right.

$$\angle AMC = \angle ABD, \angle BAD = \angle CAM \Rightarrow \triangle BAD \sim \triangle MAC \Rightarrow \frac{MC}{MA} = \frac{BD}{BA}.$$



Note that BI is the angle bisector in $\triangle DBA$, and CI is the angle bisector in $\triangle DCA$, so

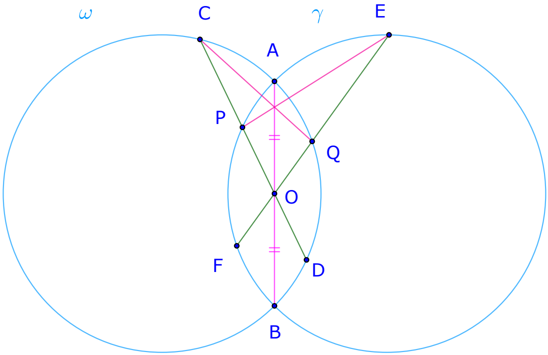
$$\frac{BD}{BA} = \frac{ID}{IA} = \frac{CD}{CA} = \frac{BD + CD}{BA + CA} = \frac{BC}{2BC} = \frac{1}{2} \Rightarrow MA = 2MC.$$

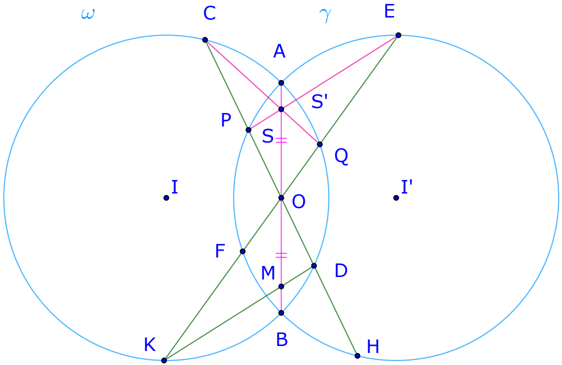


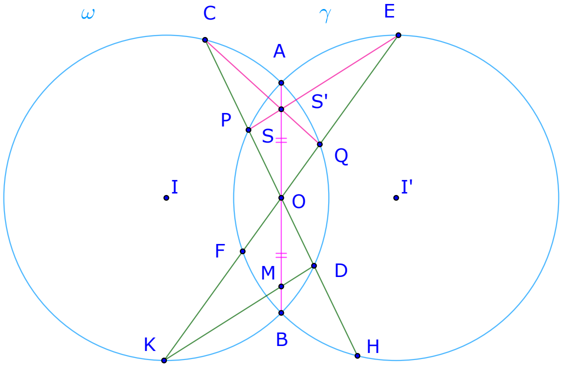
Furthermore $\angle MIC = \frac{\angle A + \angle C}{2} = \angle ICM$, thus $\triangle MIC$ is isosceles at M, so $MI = MC$. therefore $MI = \frac{1}{2}MA$, so $MI = IA$.
Consider the circle (O). Chords BE and CF intersecting at I. A line through I intersects BC and EF at D and K, respectively. Since $IM = IA$, by the *Butterfly Theorem* $\boxed{IK = ID.}$

Example

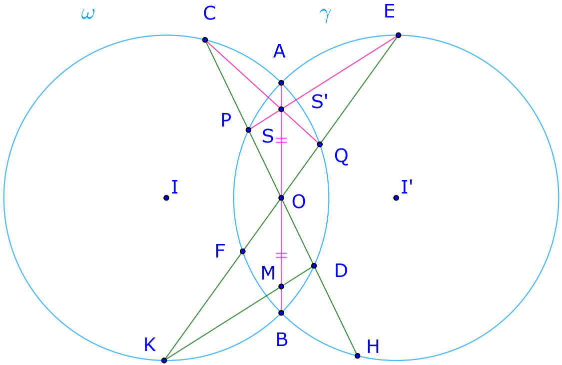
The radii of the circles ω and γ have the same length. The circles intersect each other at A and B. Let O be the midpoint of AB. Chord CD of ω through O intersects γ at P. Chord EF of γ through O intersects ω at Q. Prove that AB, CQ, and EP are concurrent.



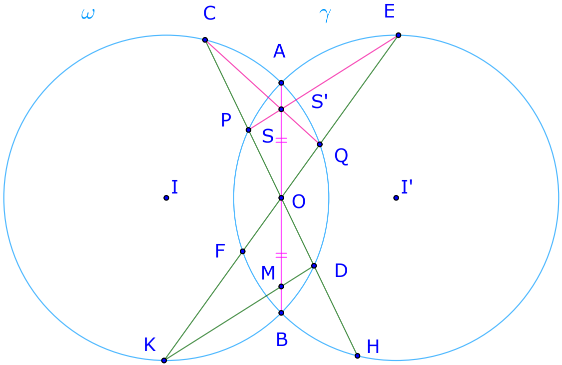

ω
C
γ
E
A
S'
P
S
Q
I
I'
O
F
D
M
K
B
H



Let H and K be the second intersections of CD and EF with γ and *omega*, respectively. Let $S = CQ \cap AB$, $S' = EF \cap AB$, and $M = DK \cap AB$.



Consider the circle ω, chords CD and KQ intersecting at O. A line through O intersects CQ and KD at S and M, respectively. Since $OA = OA$, by the *Butterfly Theorem* $OS = OM$.



Now, the radii of the circles ω and γ have the same length, thus O is midpoint of PD ($\triangle IOP \cong \triangle I'OD$) and KE. Therefore $PKDE$ is a parallelogram, so $OS' = OM$.
Hence, AB, CQ, and EP are concurrent.