

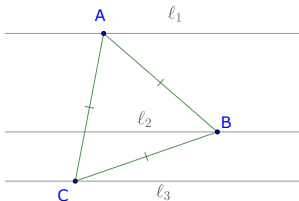
Geometric Transformations III

Rotations by an Angle - Example 1

Example

Three parallel lines ℓ_1 , ℓ_2 , and ℓ_3 are given. A is a point on the line ℓ_1 .

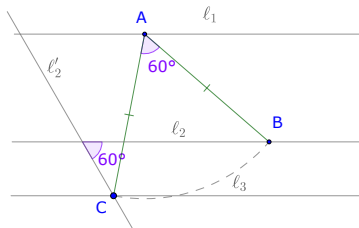
How can we determine points B and C on ℓ_2 and ℓ_3 , respectively, such that ABC is an equilateral triangle.



Geometric Transformations III

Rotations by an Angle - Example 1

Assume that $\triangle ABC$ is equilateral, then a **rotation by 60° about A will carry B to C** .

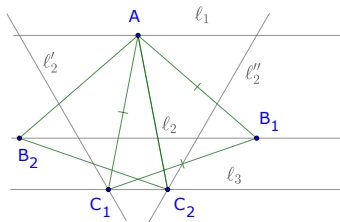


That rotation also carries ℓ_2 (containing B) to ℓ'_2 . The intersection of ℓ'_2 and ℓ_3 is C .

Geometric Transformations III

Rotations by an Angle - Example 1

Now we know how to do it. Rotate ℓ_2 by 60° about A to obtain ℓ'_2 . The intersection of ℓ'_2 with ℓ_3 is the position for C . B can be constructed easily as the intersection of circle centred at A radius AC .



Note that there are two different solutions (why?)

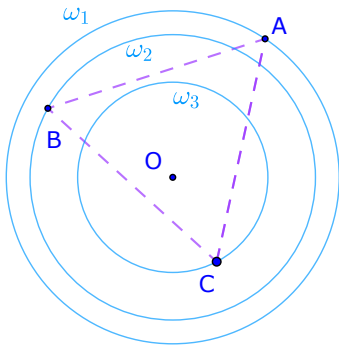
Geometric Transformations III

Rotations by an Angle - Example 2

Example

Three concentric circles ω_1 , ω_2 , and ω_3 are given. A is a point on the line ω_1 .

How can we determine points B and C on ω_2 and ω_3 , respectively, such that ABC is an equilateral triangle.

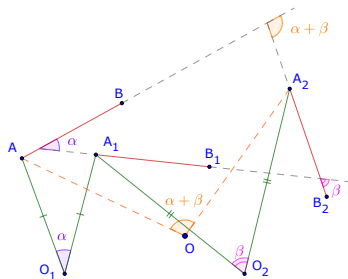


Geometric Transformations III

Sum of Rotations by an Angle

Let's take a look at a **sum of two rotations**:

$$AB \xrightarrow{\text{rotate}(O_1, \alpha)} A_1B_1 \xrightarrow{\text{rotate}(O_2, \beta)} A_2B_2.$$



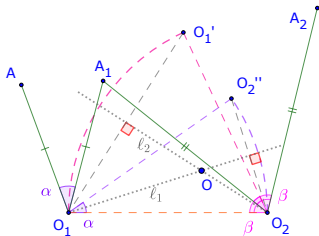
It is easy to see that the angle between A_2B_2 and AB is $\alpha + \beta$, thus it is a rotation by the angle $\alpha + \beta$. We need to determine the position of the center of rotation O .

Geometric Transformations III

Sum of Rotations by an Angle

Now, what happen with the centers O_1 and O_2 :

$$O_1 \xrightarrow{\text{rotate}(O_1, \alpha)} O_1 \xrightarrow{\text{rotate}(O_2, \beta)} O_1' \quad \text{and} \quad O_2'' \xrightarrow{\text{rotate}(O_1, \alpha)} O_2 \xrightarrow{\text{rotate}(O_2, \beta)} O_2.$$

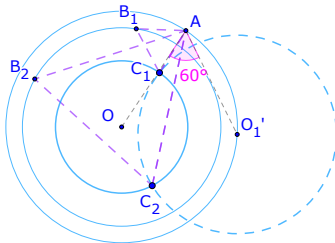


Therefore O is on both perpendicular bisectors of O_1O_1' and $O_2''O_2$.

Hence, $\angle OO_1O_2 = \frac{1}{2}\alpha$, $\angle OO_2O_1 = \frac{1}{2}\beta$.

Rotations by an Angle - Example 3

Rotate ω_2 by 60° about A to obtain ω'_2 . The intersection of ω'_2 with ω_3 is the position for C . B can be constructed easily as the intersection of circle centred at A radius AC .





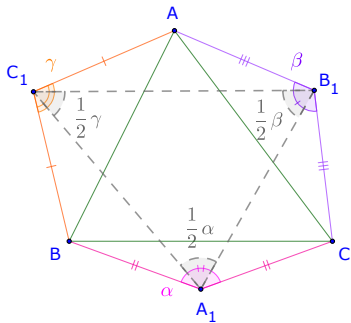
Geometric Transformations III

Rotations by an Angle - Example 3

Example

On the sides of an arbitrary triangle ABC , exterior to it, construct isosceles triangles BCA_1 , ACB_1 , CAB_1 with angles at the vertices A_1 , B_1 , and C_1 , respectively equal to α , β and γ .

Prove that if $\alpha + \beta + \gamma = 360^\circ$, then the angles of the triangle $A_1B_1C_1$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\gamma$, that is, they do not depend on the shape of the triangle ABC .

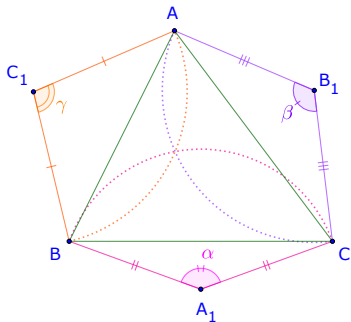


Geometric Transformations III

Rotations by an Angle - Example 3

First, point A is taken into itself by the sum of three rotations through the angles β , α , and γ ($\alpha + \beta + \gamma = 360^\circ$) about the centers B_1 , A_1 , C_1 :

$$A \xrightarrow{\text{rotate}(B_1, \beta)} C \xrightarrow{\text{rotate}(A_1, \alpha)} B \xrightarrow{\text{rotate}(C_1, \gamma)} A.$$



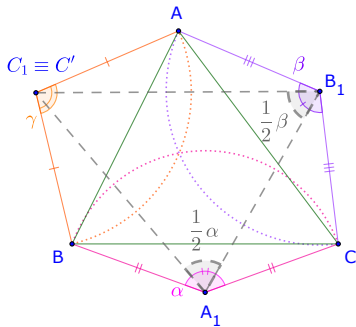
Thus, the **sum of the these rotations** is the **identity transformation**.

Geometric Transformations III

Rotations by an Angle - Example 3

Let C' be the center of the rotation equivalent to the sum of the rotations about B_1 and A_1 . Then it is the rotation through $\alpha + \beta = 360^\circ - \gamma$ brings A to B .

However, the rotation about C_1 through γ brings A to B in opposite direction. Since a rotation through an angle θ is the same as the rotation through an angle $360^\circ - \theta$ about the same center in the opposite direction, thus $C_1 \equiv C'$.



Therefore $\angle C_1 A_1 B_1 = \frac{1}{2}\alpha$, $\angle C_1 B_1 A_1 = \frac{1}{2}\beta$, and similarly $\angle B_1 C_1 A_1 = \frac{1}{2}\gamma$.

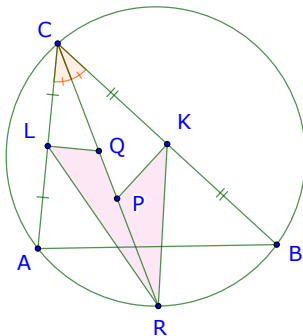
Geometric Transformations III

Rotations by an Angle - Example 4

Example

In triangle ABC , the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R , P , and Q , respectively. The midpoints of BC and CA are K and L , respectively.

Prove that triangles RPK and RQL have the same area.



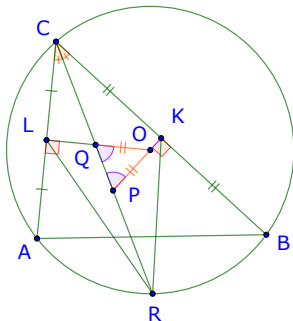
Geometric Transformations III

Rotations by an Angle - Example 4

WLOG, assume $AC < BC$. Let $ACB = \gamma$. From the right triangles CLQ and CKP ,

$$\angle OPQ = \angle OQP = 90^\circ - \frac{\gamma}{2}.$$

Thus, OPQ is isosceles, $OP = OQ$, and $\angle POQ = 2(90^\circ - \frac{\gamma}{2}) = \gamma$.

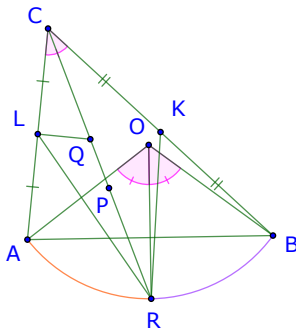


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Rotations by an Angle - Example 4

CR is the angle bisector, so R is midpoint of \widehat{AB} , and

$$\angle ROA = \angle ROB = \gamma, \quad AR = RB.$$

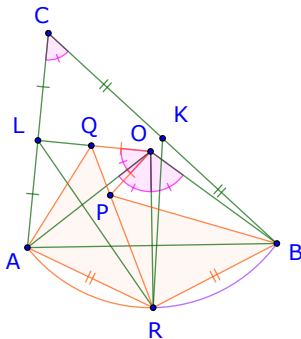


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Rotations by an Angle - Example 4

Consider the **rotation around O by angle γ** , see the figure below. This transformation moves:

$$Q \rightarrow P, A \rightarrow R, R \rightarrow B \Rightarrow \triangle QAR \cong \triangle PRB \Rightarrow [QAR] = [PRB].$$



Geometric Transformations III

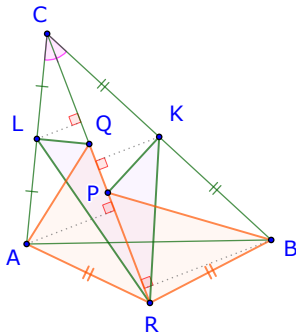
Rotations by an Angle - Example 4

Finally:

$$\frac{[RQL]}{[RQA]} = \frac{\text{distance}(L, CR)}{\text{distance}(A, CR)} = \frac{CL}{CA} = \frac{1}{2}, \text{ and similarly } \frac{[RPK]}{[BPR]} = \frac{1}{2}$$

Therefore,

$$[RQL] = \frac{1}{2}[RQA] = \frac{1}{2}[BPR] = [RPK]$$

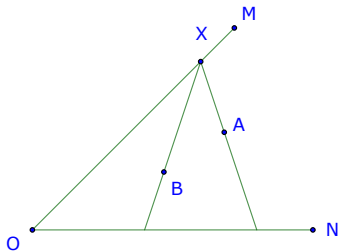


Geometric Transformations III

Symmetry - Example 1

Example

$\angle MON$ is given, together with two points A and B . Find a point X on the side OM such that the triangle XYZ is isosceles: $XY = XZ$, where Y and Z are on the points of intersection of XA and XB with ON .

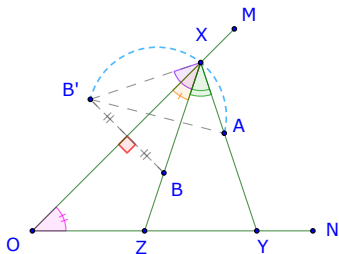


Geometric Transformations III

Symmetry - Example 1

Let B' be the image of B over OM , then:

$$\angle B'XA = \angle B'XB + \angle YXZ, \angle B'XB = 2\angle OXZ = 2(\angle XZY - \angle MON) \Rightarrow \angle B'XA = 180^\circ - \angle MON.$$



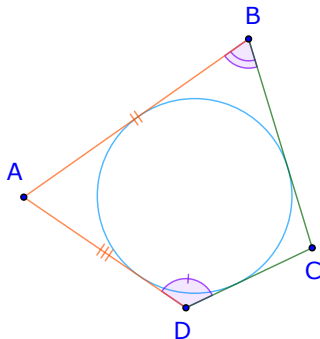
Thus, X is the intersection of OM with the arc constructed on the chord AB' , that subtends an angle equal to $180^\circ - \angle MON$.

Geometric Transformations III

Symmetry - Example 2

Example

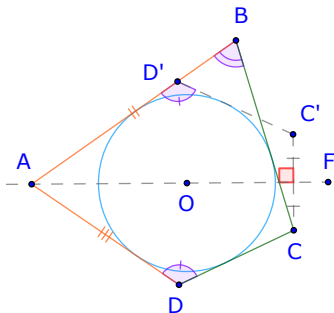
Construct a quadrilateral $ABCD$ in which a circle can be inscribed, given the lengths of two adjacent sides AB and AD and the angles at the vertices B and D .



Geometric Transformations III

Symmetry - Example 2

The key idea here is that the reflection of CD over the line through A and the center of the circle is a tangent to the circle!



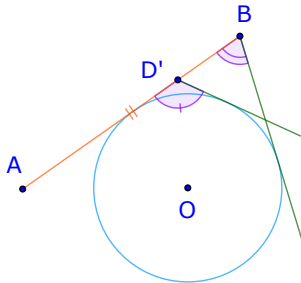
Geometric Transformations III

Symmetry - Example 2

First, we start the construction by point A then segment AB , then segment $AD_1 = AD$ where D is on the line AB , same side as B in respect to A .

Second, because $\angle B$ and $\angle D_1 = \angle D$ are known, thus we can construct rays going from B and D_1 .

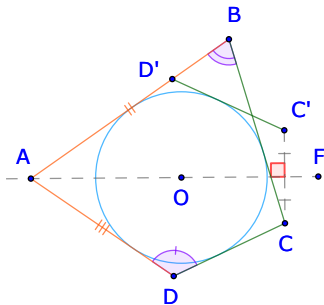
Finally, we construct a circle tangents to all three lines.



Geometric Transformations III

Symmetry - Example 2

The rest is simple, we reflect D' and its ray over the line AO where O is the center of the circle. The reflected ray will intersect the ray from B at C . We are done.



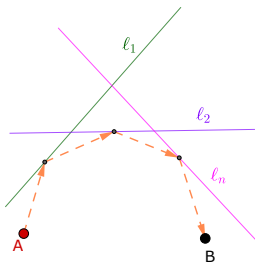
Geometric Transformations III

Symmetry - Example 3

Example

A billiard ball bounces off a side of a billiard table in such a manner that the two lines along which it moves before and after hitting the sides are equally inclined to the side.

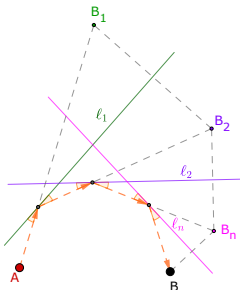
Suppose a billiard table were bordered by n lines $\ell_1, \ell_2, \dots, \ell_n$. Let A and B be two given points on the billiard table. In what direction should one hit a ball placed at A so that it will bounce consecutively off the lines $\ell_1, \ell_2, \dots, \ell_n$, and then pass through the point B (see the diagram below, where $n = 3$)?



Geometric Transformations III

Symmetry - Example 3

Assume that the problem has been solved, that is, that points X_1, X_2, \dots, X_n have been found on the lines $\ell_1, \ell_2, \dots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



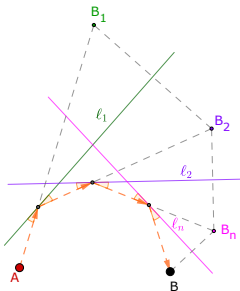
It is easy to see that the point X_n , is the point of intersection of the line ℓ_n with the line $X_{n-1}B_n$, where B_n , is the image of B in ℓ_n , that is, the points B_n, X_n, X_{n-1} lie on a line.

But then the point X_{n-1} is the point of intersection of the line ℓ_{n-1} with the $X_{n-2}B_{n-1}$, where B_{n-1} , is the image of B_n in ℓ_{n-1} and so on.

Geometric Transformations III

Symmetry - Example 3

Assume that the problem has been solved, that is, that points X_1, X_2, \dots, X_n have been found on the lines $\ell_1, \ell_2, \dots, \ell_n$ such that $AX_1X_2 \cdots X_nB$ is the path of a billiard ball (the case $n = 3$).



Here's the construction: Reflect the point B in ℓ_n , obtaining the point B_n ; next reflect B_n in ℓ_{n-1} , and so forth, until the image B_1 of the point B_2 , in line ℓ_1 is obtained.

The point X_1 , that determines the direction in which the billiard ball at A must be hit, is obtained as the point of intersection of the line ℓ_1 with the line AB_1 . It is then easy to find the points X_2, \dots, X_n with the aid of the points B_2, \dots, B_n and X_1 .

Geometric Transformations III

Symmetry - Example 4

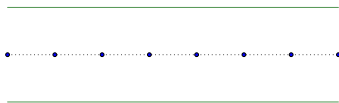
Example

A *center of symmetry* of a set S means a point O , not necessarily in S , such that for every point $A \in S$, there is another point $B \in S$ such that O is the midpoint of AB . We say that B is symmetric to A with respect to O .

Prove that a set S containing a *finite* number of points cannot have more than one center of symmetry.

Example from Geometric Transformations II session:

The strip formed by two parallel lines clearly has infinitely many centers of symmetry. Can a figure have more than one, but only a finite number of centers of symmetry (for example, can it have two and only two centers of symmetry)?



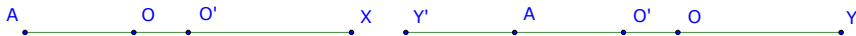
Geometric Transformations III

Symmetry - Example 4 - Solution by the Extremal Principle

Let O be the center of symmetry. Suppose to the contrary that \mathcal{F} has another center of symmetry $O' \neq O$. Since \mathcal{F} contains a finite number of points, by the Extremal Principle, **there exists** $A \in \mathcal{F}$ **with the greatest distance from** O . We consider two cases A, O, O' are collinear and A, O, O' are not collinear.

Case 1: A, O, O' are collinear. There are three sub-cases.

Case 1.1: O is between A and O' . Let X be the point symmetric to A with respect to O' (see the diagram below on the left). Then $XO > XO' = AO' > AO$, which is a contradiction to the maximality of AO .



Case 1.2: O' is between A and O . Let Y be the point symmetric to A with respect to O , and let Y' be the point symmetric to Y with respect to O' (see the diagram above on the right). Then $Y'O > Y'O' = YO' > YO = AO$, which is a contradiction to the maximality of AO .

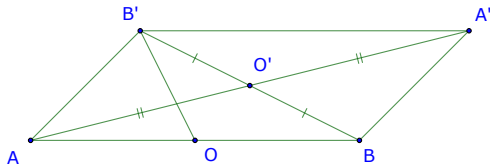
Case 1.3: A is between O and O' . Let Z be the point symmetric to A with respect to O' (see the diagram below) Then $ZO > AO$, which is a contradiction to the maximality of AO .



Geometric Transformations III

Symmetry - Example 4 - Solution by the Extremal Principle

Case 2: A, O, O' are not collinear. Let B be the point symmetric to A with respect to O , let A' be the point symmetric to A with respect to O' , and let B'' the point symmetric to B with respect to O (see the diagram below).



Because the quadrilateral $ABA'B'$ has 2 diagonals that bisect each other ($AO' = O'A', BO' = O'B'$), thus $ABA'B'$ is a parallelogram. Therefore $\angle B'AB + \angle A'BA = 180^\circ$, so one of $\angle B'AB$ and $\angle A'BA$ is greater than or equal to 90° .

Case 2.1: $\angle A'BA \geq 90^\circ$, then $A'O > BO = AO$ (because $\angle A'BO$ is the largest angle in $\triangle A'BO$), which is a contradiction to the maximality of AO .

Case 2.2: $\angle B'AB \geq 90^\circ$, then $B'O > AO$ (because $\angle B'AO$ is the largest angle in $\triangle B'AO$), which is a contradiction to the maximality of AO .

Hence, \mathcal{F} cannot have more than one center of symmetry.

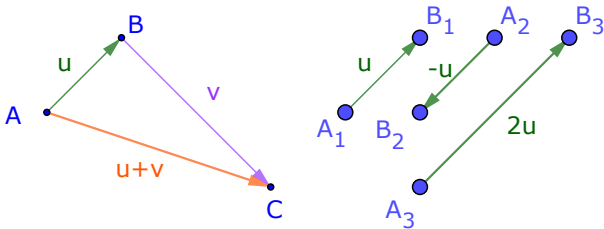
Geometric Transformations III

Vector Introduction

Definition (Vector)

Vector is a *directed line segment*, or as an arrow connecting an initial point A with a terminal point B, and denoted by \overrightarrow{AB} . Vectors are usually denoted in lowercase boldface, as in \mathbf{u} , \mathbf{v} , or \mathbf{w} .

A vector is called *zero vector* if the initial point and the terminal point are the same, in other words its magnitude is 0. A zero vector has an arbitrary or indeterminate direction. A *unit vector* is any vector with a length of one.

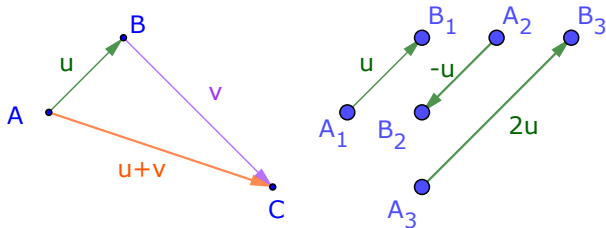


Geometric Transformations III

Vector Introduction

Definition (Equality)

Two vectors are said to be *equal* if they have the same magnitude and direction. The equality of \mathbf{u}, \mathbf{v} is denoted as $\mathbf{u} = \mathbf{v}$.



Geometric Transformations III

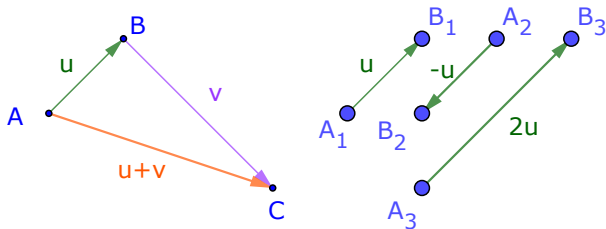
Vector Introduction

Definition (Parallel, Opposite, and Antiparallel)

Two vectors are *parallel* if they have the same direction but not necessarily the same magnitude: \overrightarrow{AB} and $\overrightarrow{A_1B_1}$ ($\overrightarrow{AB} \parallel \overrightarrow{A_1B_1}$) or \overrightarrow{AB} and $\overrightarrow{A_3B_3}$ ($\overrightarrow{AB} \parallel \overrightarrow{A_3B_3}$.)

Two vectors are *opposite* if they have the same magnitude but opposite direction: \overrightarrow{AB} and $\overrightarrow{A_2B_2}$.

Two vectors are *antiparallel* if they have opposite direction: $\overrightarrow{A_2B_2}$ and $\overrightarrow{A_3B_3}$.



Geometric Transformations III

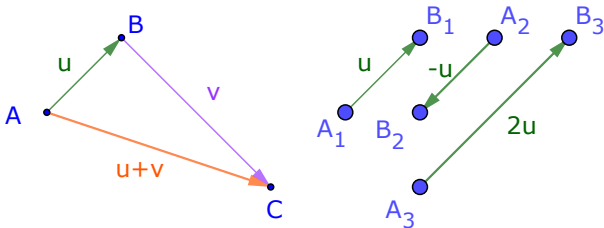
Vector Introduction

Definition (Addition and Subtraction)

The *sum* of \mathbf{u} and \mathbf{v} of two vectors may be defined as $\mathbf{u} + \mathbf{v}$. The resulting vector is sometimes called the *resultant* vector of \mathbf{u} and \mathbf{v} .

The addition may be represented graphically by placing the tail of the arrow \mathbf{v} at the head of the arrow \mathbf{u} , and then drawing an arrow from the tail of \mathbf{u} to the head of \mathbf{v} . The new arrow drawn represents the vector $\mathbf{u} + \mathbf{v}$, shown as below.

The *difference* of \mathbf{u} and \mathbf{v} is $\mathbf{u} - \mathbf{v}$, or $\mathbf{u} + (-\mathbf{v})$, which is the addition of \mathbf{u} to the opposite of \mathbf{v} .



Geometric Transformations III

Symmetry - Example 4 - Solution by Vectors

Suppose that there were two centers of symmetry, P and Q . Let S be a set of pairwise distinct points in \mathcal{F} :

$$S = \{A_1, A_2, \dots, A_n\}.$$

Geometric Transformations III

Symmetry - Example 4 - Solution by Vectors

Suppose that there were two centers of symmetry, P and Q . Let S be a set of pairwise distinct points in \mathcal{F} :

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For each $k = 1, 2, \dots, n$, let B_k denotes the point in S that is symmetric to A_k with respect to P .

Geometric Transformations III

Symmetry - Example 4 - Solution by Vectors

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By the definition of symmetry, we have $\overrightarrow{PA_k} + \overrightarrow{PB_k} = \mathbf{0}$.

Note that for each $k \in \{1, 2, \dots, n\}$, there exists $j \in \{1, 2, \dots, n\}$ $B_k = A_j$. So (B_1, B_2, \dots, B_n) is a permutation of (A_1, A_2, \dots, A_n) .

Geometric Transformations III

Symmetry - Example 4 - Solution by Vectors

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Summing this from $k = 1$ to $k = n$ implies that:

$$\sum_{k=1}^n \overrightarrow{PA_k} + \overrightarrow{PB_k} = \mathbf{0}, \quad \sum_{k=1}^n \overrightarrow{PA_k} = \sum_{k=1}^n \overrightarrow{PB_k} \Rightarrow \sum_{k=1}^n \overrightarrow{PA_1} = \mathbf{0}.$$

Geometric Transformations III

Symmetry - Example 4 - Solution by Vectors

Suppose that there were two centers of symmetry, P and Q . Let S be a set of pairwise distinct points in \mathcal{F} :

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Summing this from $k = 1$ to $k = n$ implies that:

$$\sum_{k=1}^n \overrightarrow{PA_k} + \overrightarrow{PB_k} = \mathbf{0}, \quad \sum_{k=1}^n \overrightarrow{PA_k} = \sum_{k=1}^n \overrightarrow{PB_k} \Rightarrow \sum_{k=1}^n \overrightarrow{PA_1} = \mathbf{0}.$$

By the same reasoning, we also have $\overrightarrow{QA_1} + \overrightarrow{QA_2} + \dots + \overrightarrow{QA_n} = \mathbf{0}$. Subtracting,

$$\mathbf{0} = \sum_{k=1}^n \overrightarrow{PA_k} - \sum_{k=1}^n \overrightarrow{QA_k} = n\overrightarrow{QP} \Rightarrow \overrightarrow{QP} = \mathbf{0} \Rightarrow \boxed{Q \equiv P}.$$