Example (HC-2021-SM2-R3-P11)

On the blackboard six segments are drew. Every two segments intersect one another at a point. The 1st segment contains three of the intersections, the 2nd segment contain 4 of the intersections, the 3rd, 4th, and 5th segments each contains five intersections. What segments does the 6th segment intersect with?

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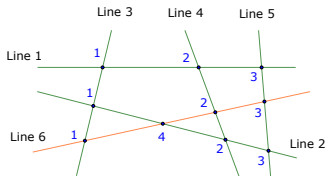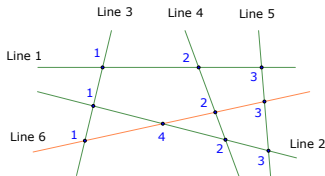

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Thus, the 2nd segment intersect with the 3rd, 4th, 5th, and 6th segments.

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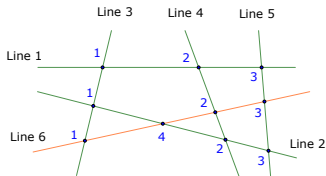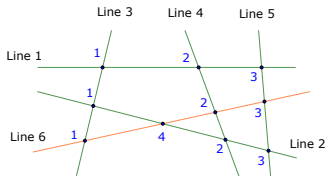

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Thus, the 2nd segment intersect with the 3rd, 4th, 5th, and 6th segments.

Hence, the 6th intersects with the $\boxed{\text{2nd, 3rd, 4th, and 5th}}$ segments.

Example (PCT 2021/Mar/MT/3)

There are 9 distinct lines on the plane, no two of them are parallel, and no three of them meet at a single point. How many non-overlapping regions they divide the plane into?

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These newly added parts divide n existing regions, adding n new regions. Therefore, the number of regions is

$$2 + 2 + 3 + 4 + \cdots + n = 1 + \frac{1}{2}n(n+1)2 = \frac{n^2 + n + 2}{2}.$$

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For $n = 9$,

$$\frac{n^2 + n + 2}{2} = \frac{81 + 9 + 2}{2} = \boxed{46.}$$

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The problem is equivalent to find the possible numbers of intersection of 4 distinct lines.

The maximum number of intersections of 4 lines is the number of pairs of lines can be chosen from 4 lines, or $\binom{4}{2} = 6$. The minimum number of intersections of 4 lines, obviously, is 0.

The diagram shows examples for any number between 0 and 6, except 2.

Example (AMC 12 2019/A/8)

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?
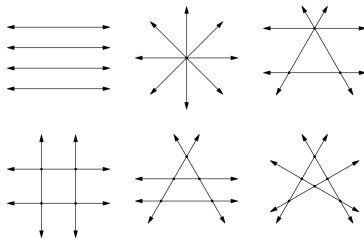
Now, assume that there are two intersections A and B so that all four lines go through them.

Case 1: if there is no line through both A and B. The two lines that intersecting at A must be pairwise parallel with the two lines that intersecting at B. In that case there will be more intersections among the non-parallel pairs.

Case 2: if there is a line through both A and B. The other line passing through A must be parallel with the other lines passing through B. The fourth line shall pass through only A or B and cannot be parallel with those parallel lines. In that case more intersection points shall exist.
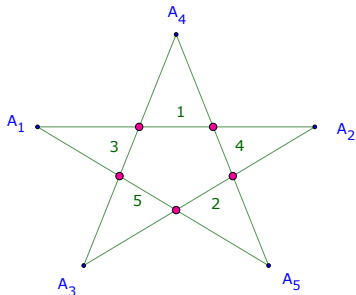
Hence, the sum of possible numbers of intersections is $\boxed{0 + 1 + 3 + 4 + 5 + 6 = 19.}$

Example (HC-2022-SM1-R10-P6)

A *close path* $A_1A_2 \ldots A_5$ contains 5 segments $A_1A_2, A_2A_3, \ldots, A_5A_1$ that has 5 *self-intersection* points . This close path has the *maximum* number of self-intersection points.
Find the *maximum* number of self-intersection points for a $9-$segment close path $A_1A_2 \ldots A_9$.

For two consecutive points, says A_i and A_{i+1}, except $A_{i-1}A_i$, A_iA_{i+1}, and $A_{i+1}A_{i+2}$ (where $A_0 \equiv A_n$, and $A_{n+1} \equiv A_1$) the remaining $n-3$ segments of the path $A_1A_2 \ldots A_nA_1$ can intersect with A_iA_{i+1} at at most $n-3$ points.
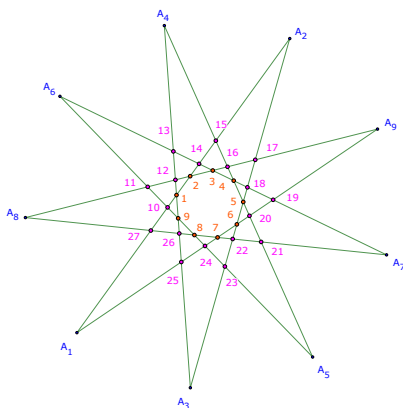
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For a $n = 9$, the maximum number of intersections is $\boxed{27.}$

Example (MIC-2022-SM2-R1-S9)

On a straight line there are 10 distinct points. For every pair of points, Anna *marks* the midpoint of the segment connected them. What is the minimal number of distinct *marked* points?

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Let A and B be the points with the greatest distance between them. Connect point A by segments with all other points except B. The midpoints of the obtained $10 - 2 = 8$ segments do not coincide, otherwise the second endpoints of the segments would coincide too, and are situated inside a circle centred at A with radius $\frac{AB}{2}$. *In the diagram below an example is shown with the given points colored red and the midpoints slightly moved out of their positions for better visualization.*

Example (MIC-2022-SM2-R1-S9)

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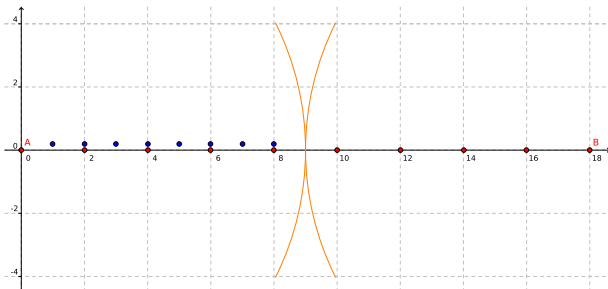
Similarly there are 8 distinct midpoints situated in a circle centred at B of radius with radius $\frac{AB}{2}$. The two circles have exactly one common point, which is the midpoint of the segment AB. Thus, together with the midpoint of AB, we have implicitly constructed $8 + 8 + 1 = \boxed{17}$ midpoints.
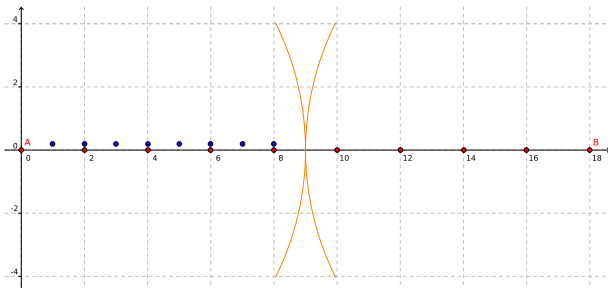
All given points lie on the same straight line at $0, 2, 4, \ldots, 18$, *so with a constant step between them. It is easy to see that there are exactly* 17 *distinct midpoints at* $1, 2, 3, \ldots, 17$.

Example (PCT 2021/Mar/MT/12)

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For an $m \times n$ rectangle, if the diagonal passes through no corner, then it passes through the top left square, then $m - 1$ squares (one for each remaining row), and then $n - 1$ squares (one for each remaining column), in total

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With $(m, n) = (42, 30)$, the answer is $\boxed{42 + 30 - \gcd(42, 30) = 72 - 6 = 66.}$

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Twelve points are equally spaced on the circumference of a circle. How many chords can be drawn that connect pairs of these points and which are longer than the radius of the circle but shorter than its diameter?

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We must first determine which diagonals are greater than the radius of the circle and less than the diameter. Diagonals such as AC are equal in length to the radius. This can be seen by noting that $ACEGIK$ is a regular hexagon, and the sides of a regular hexagon are equal in length to the radius of the circumscribed circle. Diagonals such as AG are diameters of the the circle.
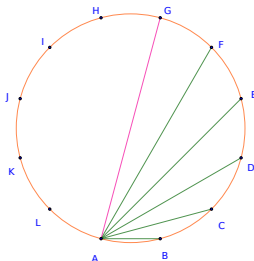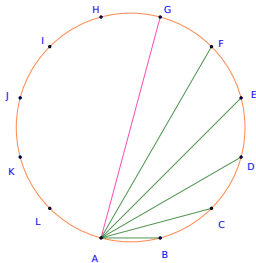
Example (MIC-2022-SM2-R3-J2)

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This leaves diagonals like AD, AE, and AF, which are longer than AC and shorter than AG.

From each vertex there are 6 such diagonals, for a total of $6(12) = 72$. Hence, there are $\frac{72}{2} = \boxed{36}$ diagonals longer than the radius of the circle and less than the diameter.

Example (HC-2022-SM2-R2-P9)

In a ceremony of the guild, the master asked the members to form 10 lines, each line consists of 9 persons, so that he can stand in *a place that has the same distance to every row*. The guild has 81 members., including the master. How can they do that?

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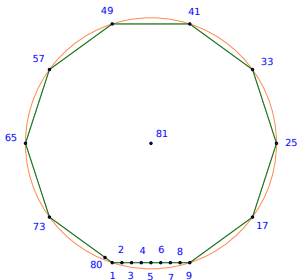Thus, there should be 10 rows, all are at the same distant from a point. Thus these are 10 chords of a circle with the same length. *To arrange* 80 *persons into* 10 *chords, each has* 9 *persons, we create a regular polygon with* 9 *persons on each side.*

Example (MIC-2021-SM2-R1-P7)

10 vertices are placed on a circle. What is the maximum number of line segments, connecting two of the given vertices, that can be drawn such that no two intersect each other, except at the vertices?

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If we count the segments by the triangles, each triangles has three segments as sides, each segment is counted once if it is a side of the $10-$gon, and twice if it is a diagonal (shared by two triangles), so

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Therefore, from (1) and (2), $d = 7$, $k = 8$.

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Thus, the number of segment is $\boxed{d + 10 = 17.}$

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We prove that n ($n \geq 2$) distinct points can be placed on a circle of radius 1, such that the distance between any two points is a rational number.

For 2 points, it is simple to see that the two vertices of diameter is such a pair of points.

For 3 points, for a Pythagorean triple (a_1, b_1, c_1) of integers: $c_1^2 = a_1^2 + b_1^2$, a point P_1 can be chosen on the semicircle $\widehat{AB}$ diameter 2, such that both P_1A and P_1B are rational:

$$AB = 2, \ P_1A = \frac{2a}{c}, \ P_1B = \frac{2b}{c} \Rightarrow P_1A^2 + P_1B^2 = \frac{4(a^2 + b^2)}{c^2} = AB^2 \Rightarrow P_1 \in \widehat{AB}.$$

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Therefore, by using $n \geq 1$ distinct Pythagorean triples (a_i, b_i, c_i), where $\forall i = 1, \ldots, n$, $c_i^2 = a_i^2 + b_i^2$, n distinct points $P_1, P_2, \ldots, P_n$ can be chosen on the semicircle $\overset{\frown}{AB}$, so that all P_iA, P_iB are rationals.
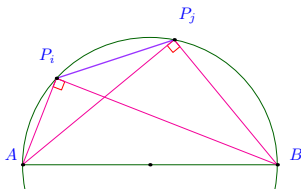
For any $1 \leq i \neq j \leq n$, ABP_jP_i is a cyclic quadrilateral, by the Ptolemy Theorem,

$$AB, \; P_iA, \; P_iB, \; P_iA, \; P_jB \text{ are rational} \; \Rightarrow P_iP_j = \frac{P_iB \cdot P_jA - P_iA \cdot P_jB}{AB} \text{are rational}$$
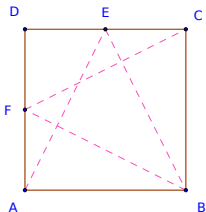
Example (HC-2022-SM2-R3-P11)

Find the minimal value *m* so that no matter how you partition a unit square into two distinct sets of points, the diameter of one of the sets is at least *m*. *Diameter of a set of points is the largest distance between two points of the set*.

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Let $ABCD$ be the unit square, E and F be the midpoints of CD and DA. We prove that m is equal to the length of AE. Let assume that $ABCD$ can be divided into two sets $\mathcal{G}$ and $\mathcal{R}$ such that *none of the sets has a diameter larger than or equal to AE.*

Example (HC-2022-SM2-R3-P11)

Find the minimal value m so that no matter how you partition a unit square into two distinct sets of points, the diameter of one of the sets is at least m. *Diameter of a set of points is the largest distance between two points of the set.*

WLOG, let $A \in \mathcal{G}$, then $E \notin \mathcal{G}$, thus $E \in \mathcal{R}$, therefore $B \in \mathcal{G}$, so $F \in \mathcal{R}$, hence $C \in \mathcal{G}$, but then the diameter of $\mathcal{G}$ is at least $AC > AE$. Contradiction.
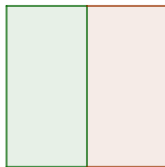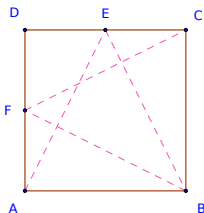
The diagram on the right of the figure above shows the case when such diameter is equal to AE
Therefore one of the sets has a diameter larger than or equal to AE.
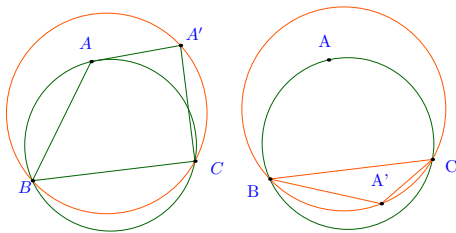
Example (LPS V2B/2.44)

Prove that there exist three consecutive vertices A, B, C in every convex $n-$gon ($n \geq 3$), such that the circumcircle of $\triangle ABC$ covers the whole $n-$gon.

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Consider all circles through three vertices of the $n-$gon. There is a finite number of such circles. Let Ω be the maximal such circle. We will prove the following statements: (1) Ω covers the whole $n-$gon; (2) Ω passes through three consecutive vertices.
For the first statement, suppose that vertex A' lies outside Ω.

Because Ω is the circumcircle of $\triangle ABC$, where A, B, C are denoted such that A, B, C, A' form convex quadrilateral. Easy to see that the circumcircle of $\triangle A'BC$ is larger than Ω, see the left diagram. This is contradiction.

Example (LPS V2B/2.44)

Prove that there exist three consecutive vertices A, B, C in every convex $n-$gon $(n \geq 3)$, such that the circumcircle of $\triangle ABC$ covers the whole $n-$gon.

For the second statement, let A, B, C be vertices on Ω. Suppose there exists vertex A' between B, C and A' not on Ω.
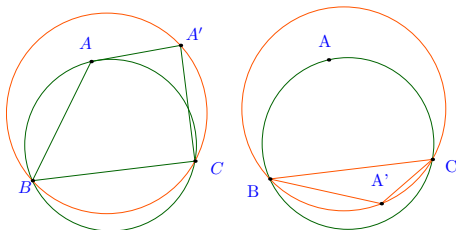
Because of the first statement, Ω covers the whole $n-$gon, so A' is inside Ω. Hence the circumcircle of $\triangle A'BC$ is larger than Ω, see the right diagram. This is a contradiction.