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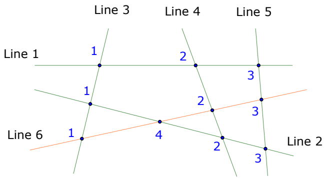
On the blackboard six segments are drew. Every two segments intersect one another at a point. The 1st segment contains three of the intersections, the 2nd segment contain 4 of the intersections, the 3rd, 4th, and 5th segments each contains five intersections. What segments does the 6th segment intersect with?



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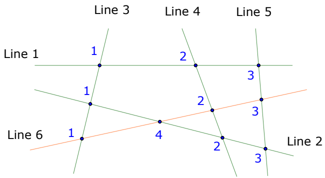

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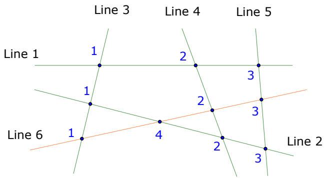

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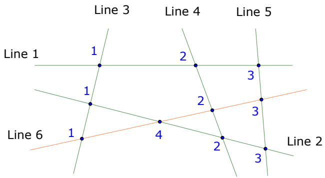

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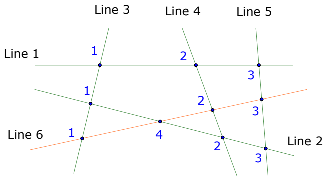

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Thus, the 2$^{\text{nd}}$ segment intersect with the 3$^{\text{rd}}$, 4$^{\text{th}}$, 5$^{\text{th}}$, and 6$^{\text{th}}$ segments.

Hence, the 6$^{\text{th}}$ intersects with the $\boxed{\text{2}^{\text{nd}}\text{, 3}^{\text{rd}}\text{, 4}^{\text{th}}\text{, and 5}^{\text{th}}}$ segments.

Example (PCT 2021/Mar/MT/3)

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These newly added parts divide n existing regions, adding n new regions. Therefore, the number of regions is

$$2+2+3+4+\cdots+n = 1+\frac{1}{2}n(n+1)2 = \frac{n^2+n+2}{2}.$$

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For $n=9$,

$$\frac{n^2+n+2}{2} = \frac{81+9+2}{2} = \boxed{46.}$$

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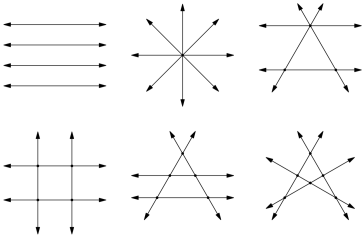
The problem is equivalent to find the possible numbers of intersection of 4 distinct lines.

The maximum number of intersections of 4 lines is the number of pairs of lines can be chosen from 4 lines, or $\binom{4}{2} = 6$. The minimum number of intersections of 4 lines, obviously, is 0.



The diagram shows examples for any number between 0 and 6, except 2.

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For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

Now, assume that there are two intersections A and B so that all four lines go through them.

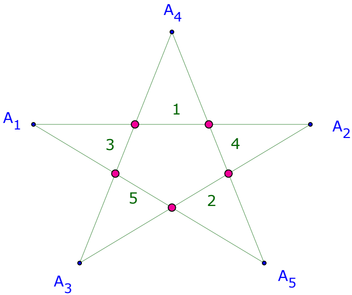
Case 1: if there is no line through both A and B. The two lines that intersecting at A must be pairwise parallel with the two lines that intersecting at B. In that case there will be more intersections among the non-parallel pairs.

Case 2: if there is a line through both A and B. The other line passing through A must be parallel with the other lines passing through B. The fourth line shall pass through only A or B and cannot be parallel with those parallel lines. In that case more intersection points shall exist.

Hence, the sum of possible numbers of intersections is $\boxed{0+1+3+4+5+6=19.}$

Example (HC-2022-SM1-R10-P6)

A *close path* $A_1A_2\dots A_5$ contains 5 segments $A_1A_2, A_2A_3, \dots, A_5A_1$ that has 5 *self-intersection* points . This close path has the *maximum* number of self-intersection points.
Find the *maximum* number of self-intersection points for a 9−segment close path $A_1A_2\dots A_9$.



For two consecutive points, says A_i and A_{i+1}, except $A_{i-1}A_i, A_iA_{i+1}$, and $A_{i+1}A_{i+2}$ (where $A_0 \equiv A_n$, and $A_{n+1} \equiv A_1$) the remaining $n-3$ segments of the path $A_1A_2 \dots A_nA_1$ can intersect with A_iA_{i+1} at at most $n-3$ points.

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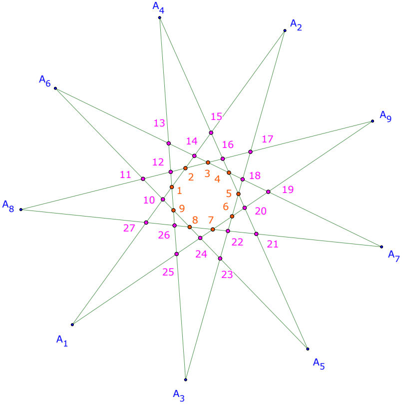
For a $n = 9$, the maximum number of intersections is $\boxed{27.}$



Example (MIC-2022-SM2-R1-S9)

On a straight line there are 10 distinct points. For every pair of points, Anna *marks* the midpoint of the segment connected them. What is the minimal number of distinct *marked* points?

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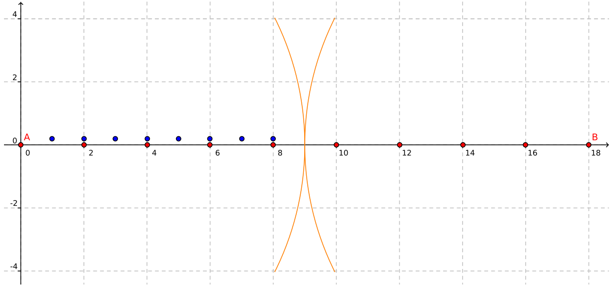
Let A and B be the points with the greatest distance between them. Connect point A by segments with all other points except B. The midpoints of the obtained $10-2=8$ segments do not coincide, otherwise the second endpoints of the segments would coincide too, and are situated inside a circle centred at A with radius $\frac{AB}{2}$. *In the diagram below an example is shown with the given points colored red and the midpoints slightly moved out of their positions for better visualization.*



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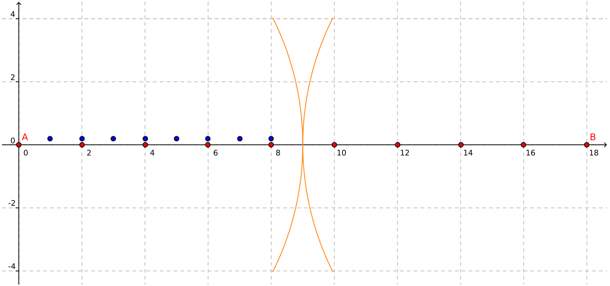
Similarly there are 8 distinct midpoints situated in a circle centred at B of radius with radius $\frac{AB}{2}$. The two circles have exactly one common point, which is the midpoint of the segment AB. Thus, together with the midpoint of AB, we have implicitly constructed $8+8+1=\boxed{17}$ midpoints.



All given points lie on the same straight line at $0, 2, 4, \ldots, 18$, *so with a constant step between them. It is easy to see that there are exactly* 17 *distinct midpoints at* $1, 2, 3, \ldots, 17$.

Example (PCT 2021/Mar/MT/12)

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For an $m \times n$ rectangle, if the diagonal passes through no corner, then it passes through the top left square, then $m - 1$ squares (one for each remaining row), and then $n - 1$ squares (one for each remaining column), in total

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With $(m, n) = (42, 30)$, the answer is $\boxed{42 + 30 - \gcd(42,\ 30) = 72 - 6 = 66.}$

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Twelve points are equally spaced on the circumference of a circle. How many chords can be drawn that connect pairs of these points and which are longer than the radius of the circle but shorter than its diameter?

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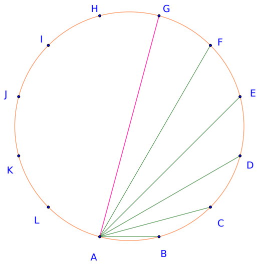
We must first determine which diagonals are greater than the radius of the circle and less than the diameter. Diagonals such as AC are equal in length to the radius. This can be seen by noting that $ACEGIK$ is a regular hexagon, and the sides of a regular hexagon are equal in length to the radius of the circumscribed circle. Diagonals such as AG are diameters of the the circle.



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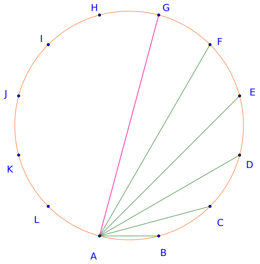
This leaves diagonals like AD, AE, and AF, which are longer than AC and shorter than AG.



From each vertex there are 6 such diagonals, for a total of $6(12) = 72$. Hence, there are $\frac{72}{2} = \boxed{36}$ diagonals longer than the radius of the circle and less than the diameter.

Example (HC-2022-SM2-R2-P9)

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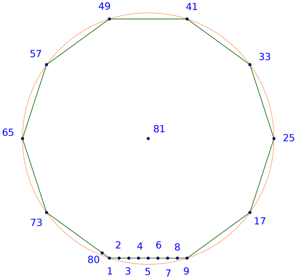
Thus, there should be 10 rows, all are at the same distant from a point. Thus these are 10 chords of a circle with the same length. *To arrange* 80 *persons into* 10 *chords, each has* 9 *persons, we create a regular polygon with* 9 *persons on each side.*



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10 vertices are placed on a circle. What is the maximum number of line segments, connecting two of the given vertices, that can be drawn such that no two intersect each other, except at the vertices?

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On the other hand, by connecting the segments, we create a so-called *planar graph*, by Euler's graph formula $V - E + F = 2$ for 10 vertices, $d + 10$ edges, and $k + 1$ faces:

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Therefore, from (1) and (2), $d = 7, \; k = 8$.

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Thus, the number of segment is $\boxed{d + 10 = 17.}$

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We prove that n ($n \geq 2$) distinct points can be placed on a circle of radius 1, such that the distance between any two points is a rational number.

For 2 points, it is simple to see that the two vertices of diameter is such a pair of points.

For 3 points, for a Pythagorean triple (a, b, c) of integers: $c^2 = a^2 + b^2$, a point P_1 can be chosen on the semicircle $\widehat{AB}$ diameter 2, such that both P_1A and P_1B are rational:

$$AB = 2,\ P_1A = \frac{2a}{c},\ P_1B = \frac{2b}{c} \Rightarrow P_1A^2 + P_1B^2 = \frac{4(a^2+b^2)}{c^2} = AB^2 \Rightarrow P_1 \in \widehat{AB}.$$

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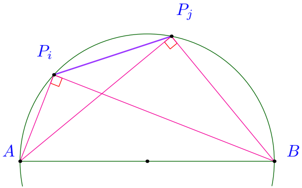
Therefore, by using $n \geq 1$ distinct Pythagorean triples (a_i, b_i, c_i), where $\forall i = 1, \ldots, n,\ c_i^2 = a_i^2 + b_i^2$, n distinct points $P_1, P_2, \ldots, P_n$ can be chosen on the semicircle $\widehat{AB}$, so that all P_iA, P_iB are rationals.



For any $1 \leq i \neq j \leq n$, ABP_jP_i is a cyclic quadrilateral, by the Ptolemy Theorem,

$$AB,\ P_iA,\ P_iB,\ P_iA,\ P_jB \text{ are rational } \Rightarrow P_iP_j = \frac{P_iB \cdot P_jA - P_iA \cdot P_jB}{AB} \text{are rational}$$

Example (HC-2022-SM2-R3-P11)

Find the maximal value m so that no matter how you partition a unit square into two distinct sets of points, the length of the diameter of one of the sets is at least m. *Diameter of a set of points is the largest distance between two points of the set.*

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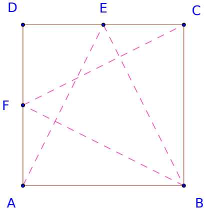
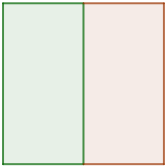
Let $ABCD$ be the unit square, E and F be the midpoints of CD and DA. We prove that m is equal to the length of AE. Let assume that $ABCD$ can be divided into two sets $\mathcal{G}$ and $\mathcal{R}$ such that *none of the sets has a diameter larger than or equal to AE*.

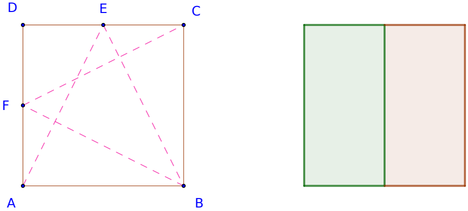




Example (HC-2022-SM2-R3-P11)

Find the maximal value m so that no matter how you partition a unit square into two distinct sets of points, the length of the diameter of one of the sets is at least m. *Diameter of a set of points is the largest distance between two points of the set.*

WLOG, let $A \in \mathcal{G}$, then $E \notin \mathcal{G}$, thus $E \in \mathcal{R}$, therefore $B \in \mathcal{G}$, so $F \in \mathcal{R}$, hence $C \in \mathcal{G}$, but then the diameter of $\mathcal{G}$ is at least $AC > AE$. Contradiction.



The diagram on the right of the figure above shows the case when such diameter is equal to AE Therefore one of the sets has a diameter larger than or equal to AE.

Example (LPS V2B/2.44)

Prove that there exist three consecutive vertices A, B, C in every convex $n-$gon ($n \geq 3$), such that the circumcircle of $\triangle ABC$ covers the whole $n-$gon.

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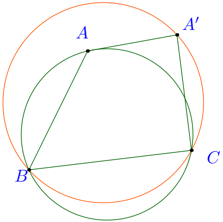
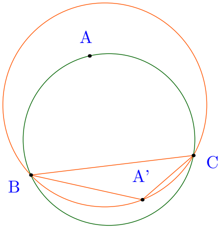
Consider all circles through three vertices of the $n-$gon. There is a finite number of such circles. Let Ω be the maximal such circle. We will prove the following statements: (1) Ω covers the whole $n-$gon; (2) Ω passes through three consecutive vertices.
For the first statement, suppose that vertex A' lies outside Ω.



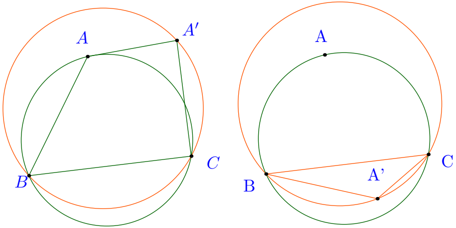


Because Ω is the circumcircle of $\triangle ABC$, where A, B, C are denoted such that A, B, C, A' form convex quadrilateral. Easy to see that the circumcircle of $\triangle A'BC$ is larger than Ω, see the left diagram. This is contradiction.

Example (LPS V2B/2.44)

Prove that there exist three consecutive vertices A, B, C in every convex $n-$gon ($n \geq 3$), such that the circumcircle of $\triangle ABC$ covers the whole $n-$gon.

For the second statement, let A, B, C be vertices on Ω. Suppose there exists vertex A' between B, C and A' not on Ω.



Because of the first statement, Ω covers the whole $n-$gon, so A' is inside Ω. Hence the circumcircle of $\triangle A'BC$ is larger than Ω, see the right diagram. This is a contradiction.

Example (IMO 1972/2)

Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

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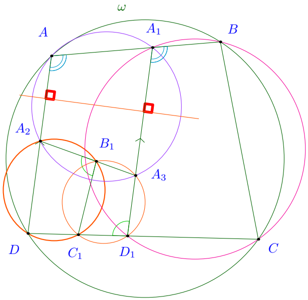
First, we prove a minor claim: any cyclic quadrilateral can be divided into four cyclic quadrilaterals, and at least one of them is an isosceles trapezoid.

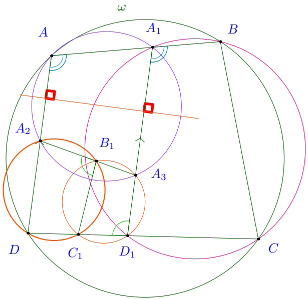


The diagram above shows how to divide a cyclic quadrilateral into four cyclic quadrilaterals: $A_1D_1 \parallel AD$, $AA_1A_3A_2$ is isosceles trapezoid, and B_1C_1 chosen such that $A_2B_1C_1D$ is cyclic.

Example (IMO 1972/2)

Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

By the claim, first the given cyclic quadrilateral can be divided into four cyclic quadrilaterals.



Among the cyclic quadrilateral there is an isosceles trapezoid, so by drawing a segment parallel to its bases, an additional isosceles trapezoid can be gained, which is also a cyclic quadrilateral. By repeated this operation $n-5$ times more, there are n cyclic quadrilaterals.

Example (IMO 1969/5)

Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals with vertices amongst the n points.

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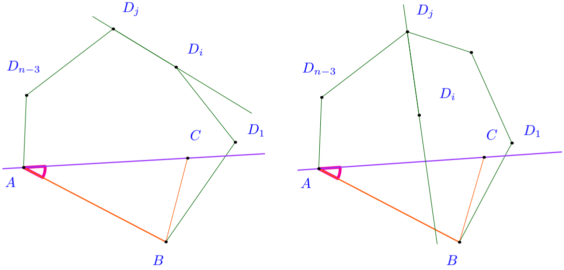
WLOG, *assume that* $AB\ldots$ *is the convex hull* of all n given points. Now, since no three points collinear, there exists a point C, such that for all other points $D_1, \ldots, D_{n-3}$,



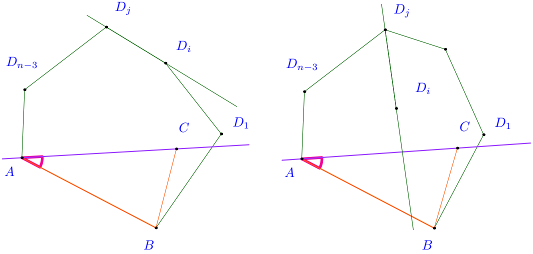
$$\angle BAC < \min\{\angle BAD_k \mid \forall k \in \{1, \ldots, n-3\}\}.$$

Thus, *line through* AC *divides the plane into two separate half-planes*, one of which contains only B, the other contains $n-3$ remaining points $D_1, \ldots, D_{n-3}$.

Example (IMO 1969/5)

Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals with vertices amongst the n points.

Consider any two distinct points P_i, P_j, $i, j \in \{1, \ldots, n-3\}$.



Case 1: Line D_iD_j does not intersect any sides of $\triangle ABC$ (it may intersects the extensions of those segments), then ACD_iD_j is convex, see the left diagram.
Case 2: Line D_iD_j intersects two sides AC, BC of $\triangle ABC$ and divides the plane into two half-planes, WLOG, separating A, B and C, then BCD_jD_i is convex, see the right diagram.
Hence, there are at least $\binom{n-3}{2}$ convex quadrilaterals with vertices amongst the n points.

Example (IMO SL 1972/2)

We are given $3n$ points $A_1, A_2, \ldots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct n disjoint triangles.

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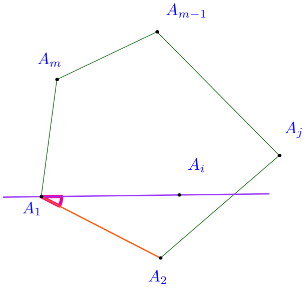
We are given $3n$ points $A_1, A_2, \ldots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct n disjoint triangles.

Lets prove by induction. For $n = 1$ it is obvious. Assume that the hypothesis is true for n. WLOG, *assume that* $A_1A_2 \ldots A_m$ *is the convex hull* of $A_1, A_2, \ldots, A_{3n+3}$ given points.

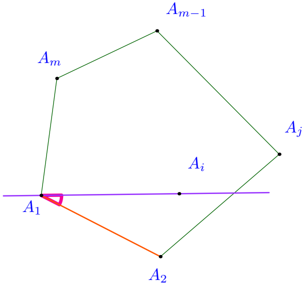


Since no three points collinear, $\exists A_i,\ \angle A_iA_1A_2 = \min\{\angle A_kA_1A_2 \mid \forall k \in \{3, \ldots, 3n+3\}\}$.

Example (IMO SL 1972/2)

We are given $3n$ points $A_1, A_2, \ldots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct n disjoint triangles.

The line through A_1A_2 divides the plane into two separate half-planes.



One half-plane contains only A_2, the other contains $3n$ remaining points, which form n disjoint triangles. These and $\triangle A_1A_2A_i$ form $n+1$ disjoint triangles. Thus, the hypothesis is proved.