The Induction Principle for Beginners - Part II

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Example 0.1 (Example Six)
Show that for all $n \geq 1$,
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

Solution. Our hypothesis is that for all $n \geq 1$,
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

For the base case $n = 2$, it is easy to verify that
$$\frac{1}{2} < \frac{1}{\sqrt{3}}.$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}. \quad (*)$$

We shall prove that
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \leq \frac{1}{\sqrt{2n+3}}. \quad (**)$$

By the assumption (*),
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{2n+1}} \cdot \frac{2n+1}{2n+2} = \frac{\sqrt{2n+1}}{2n+2}$$

Since
$$\frac{\sqrt{2n+1}}{2n+2} < \frac{1}{\sqrt{2n+3}} \Leftrightarrow (2n+1)(2n+3) < (2n+2)^2 \Leftrightarrow 4n^2 + 8n + 3 < 4n^2 + 8n + 4, \text{ which is true.}$$

Thus the hypothesis is true for $n + 1$, therefore it is true for all $n \geq 2$. $\square$

Example 0.2 (Example Seven)

Prove that every positive integer can be represented as a sum of several distinct powers of 2.

Solution. Our hypothesis is that every positive integer can be represented as a sum of several distinct powers of 2.

It is easy to verify the base cases $n = 1$ and $n = 2$.

Now, for the Inductive step, let's assume that the hypothesis is true for n or let's assume that every positive integer less than or equal to n can be represented as a sum of several distinct powers of 2.

We shall prove that $n + 1$ can be represented as a sum of several distinct powers of 2.

Now, for $n + 1 \geq 4$, there exists an positive integer m such that

$$2^m \leq (n + 1) < 2^{m+1}.$$

If $n + 1 = 2^m$, then we are done, if not then $n + 1 = 2^m + (n + 1 - 2^m)$, where $n + 1 - 2^m < n$ and can be represented as a sum of several distinct powers of 2. It is easy to see that any power of 2 in the sum representing $n + 1 - 2^m$ is less than 2^m, otherwise $n + 1 > 2^{m+1}$.

Thus the hypothesis is true for $n + 1$, therefore it is true for all $n \geq 2$. □

Example 0.3 (Example Eight)

There are $n \geq 1$ real numbers with non-negative sum written on a circle. Prove that one can enumerate them $a_1, a_2, \ldots, a_n$ such that they are consecutive on the circle and

$$a_1 \geq 0, a_1 + a_2 \geq 0, \ldots, a_1 + a_2 + \cdots + a_{n-1} \geq 0, a_1 + a_2 + \cdots + a_{n-1} \geq 0.$$

Solution. Our hypothesis is based on n.

It is easy to verify the base cases when $n = 1$ or we have only one number.

For the Inductive step, let's assume that the hypothesis is true for $n - 1$. We shall prove for n.

As the sum of these numbers are non-negative, there are non-negative numbers. If all of them are non-negative, we can chose any number to be a_1 and then enumerate the rest clockwise, and we have the desired inequalities.

Now, let's assume that there exists $a_n < 0$, then by applying the hypothesis for

$$a_1, a_2, \ldots, a_{n-2}, a_{n-1} + a_n \text{ (note that the last number is a sum)}$$

we can find a_j such that

$$a_j, a_j + a_{j+1}, \ldots, a_j + \ldots a_{n-2}, a_j + \ldots a_{n-1} + a_n, \ldots \text{ are all non-negative.}$$

Since $a_n < 0$, thus $a_j + \ldots a_{n-1} > 0$, therefore this sum plus $n - 1$ of the above sums are the n desired sums with a_j as the first number in the re-enumeration. □

Example 0.4 (Example Nine)

The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined as follow,

$$a_1 = 3, a_2 = 5, a_{n+1} = 3a_n - 2a_{n-1}, \text{ for } n \geq 2.$$

Prove that $a_n = 2^n + 1$, for all n positive integer.

Solution. Our hypothesis is that $a_n = 2^n + 1$, for all n positive integer

It is easy to verify the base cases when $n = 1$ and $n = 2$.

For the Inductive step, let's assume that the hypothesis is true for all positive integers less than or equal to n. We shall prove for $n + 1$.

It is easy to verify that $a_{n+1} = 3a_n - 2a_{n-1} = 3(2^{n-1} + 1) - 2(2^{n-2} + 1) = 2^n + 1$. Thus the hypothesis is true for $n + 1$, therefore it is true for all $n \geq 1$. □

Example 0.5 (Example Ten)

A bank has an unlimited supply of 3-peso and 5-peso notes. Prove that it can pay any number of pesos greater than 7.

Solution. Our hypothesis is that any positive integer larger than 7 can be expressed a sum of 3s and 5s.

It is easy to verify the base cases of 8, 9, and 10:

$$8 = 3 + 5, 9 = 3 + 3 + 3, 10 = 5 + 5.$$

For the Inductive step, let's assume that the hypothesis is true for $k, k + 1, k + 2$. We can easily add 3 to any of the number to prove for $k + 3, k + 4, k + 5$.

This means that this induction proof with a compound base may be split into three standard inductions using the following schemes:

$$8 \to 11 \to 14 \to \cdots, 9 \to 12 \to 15 \to \cdots, 10 \to 13 \to 16 \to \cdots$$

□