A straight line is the shortest distance between two points

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In this article, we explore some basic properties of broken lines.

Fact (Triangle Inequality)**.** For any three points A, B, C,

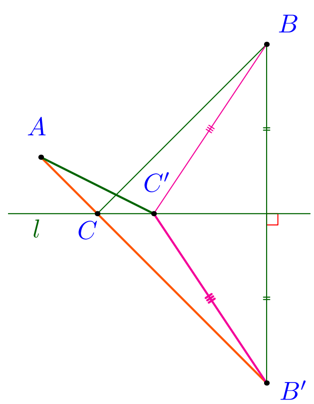
$$AB + BC \geq AC,$$

The equality holds if and only if A, B, and C are collinear.

Fact (Broken Line Inequality)**.** For any points $A_1, A_2, \ldots, A_n$, $A_1A_2 + A_2A_3 + \ldots + A_{n-1}A_n \geq A_1A_n$. The equality holds if and only if $A_1, A_2, \ldots, A_{n-1}$, and A_n are collinear.

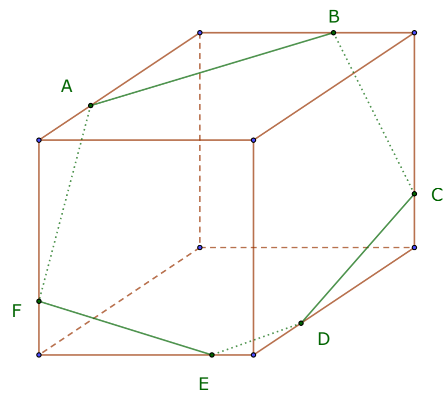
Lemma (Heron's Problem)

Two points A and B lie on one side of a straight line l. C is a point on on l. The sum $CA + CB$ is minimal if and only if $C = BC' \cap \ell$, where B' is the reflection of B over l.

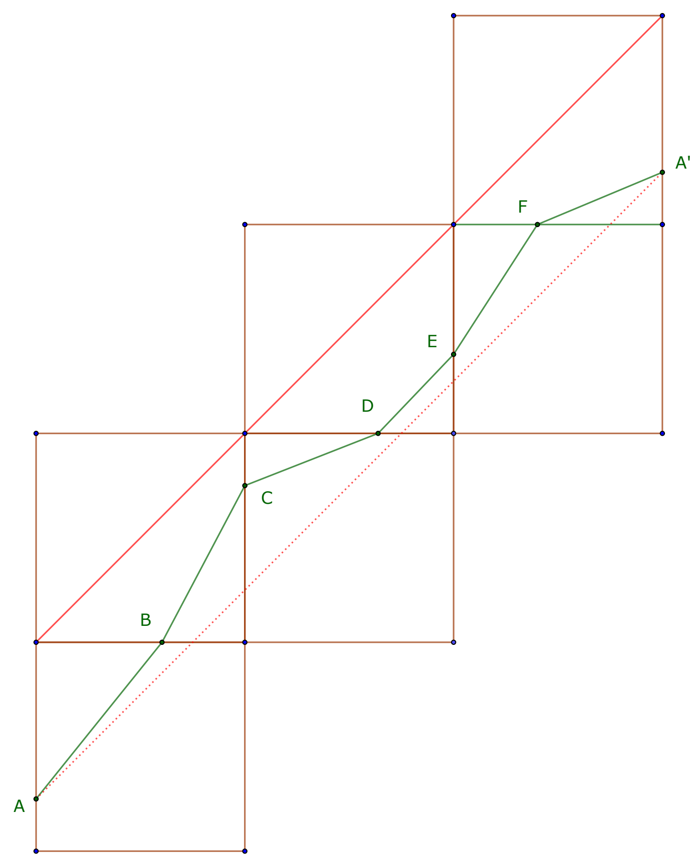


Example (Cross-section of a cube)

Lilian cuts a cube with side length 1. She got a with a hexagon cross-section as shown below. What is the minimal value of the hexagon perimeter $AB + BC + CD + DE + EF + FA$?



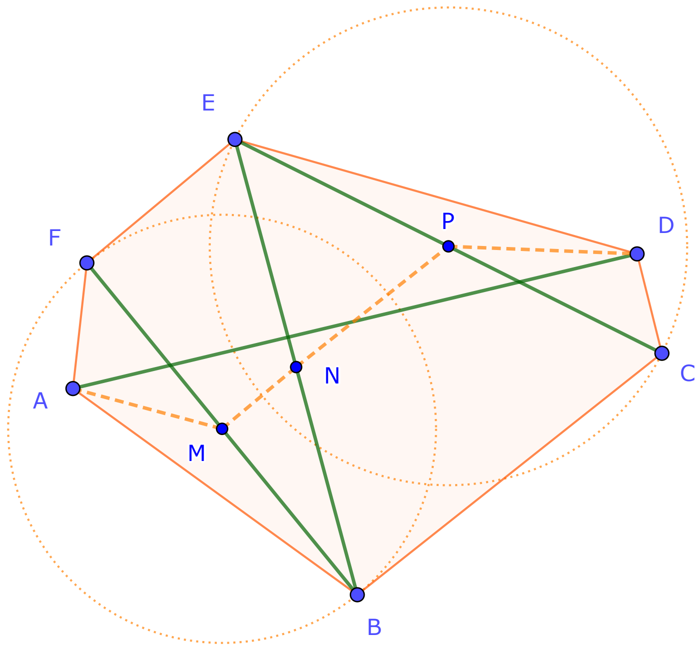
Solution. The diagram below is obtained by unfolding the cube into a net. The hexagon perimeter forms a broken line $ABCDEFA'$.



This is always larger or equal the distance AA', which is same as three times the diagonal of the unit square. Hence the perimeter is always at least $\boxed{3\sqrt{2}}$. □

Example (Diagonal of a hexagon)

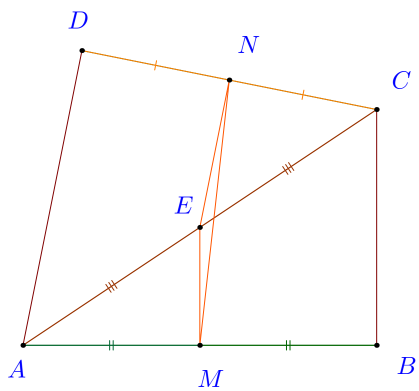
$ABCDEF$ is a convex hexagon, where $\angle A \geq 90°$ and $\angle D \geq 90°$. Prove that $BC+CE+EF+FB \geq 2AD$.



Proof. Let M, N, and P be the midpoints of BF, BE, and CE, respectively. Since any broken line is longer or equal the distance between two endpoints, so $AD \leq AM + MN + NP + PD$. MN is the median segment in $\triangle BEF$, thus $FE = 2MN$. Similarly $BC = 2NP$. In $\triangle ABF$, $\angle A \geq 90°$, thus $BF \geq 2AM$. Similarly $CE \geq 2DP$. Therefore $BC + CE + EF + FB \geq 2(AM + MN + NP + PD) = 2AD$. □

Example (Romanian Math Olympiad)

Let $ABCD$ be a convex quadrilateral. It is known that the circles with diameter AB and CD are externally tangent, and so are the circles with diameters AD and BC. Prove that $ABCD$ is a rhombus.



Proof. We first prove a claim.

Claim — Let M and N be the midpoint of AB and CD, respectively, then $AD + BC \geq 2MN$.

Proof. Let E be the midpoint of AC. It is easy to see that

$$MN \leq ME + EN = \frac{BC}{2} + \frac{AD}{2} = \frac{AD + BC}{2}$$

The equality can happen if and only if MN intersect AC at the midpoint of AC, so $MN \parallel AD \parallel BC$. ■

By the claim $AD + BC \geq 2MN,$ similarly $AB + CD \geq 2PQ$, thus

$$AB + BC + CD + DA \geq 2(MN + PQ) \qquad (*)$$

Now, let P and Q be the midpoints of BC and AD, respectively. Since the circles of diameters AB and CD are externally tangent so $AB + CD = 2MN$, similarly $AD + BC = 2$. Thus

$$AB + BC + CD + DA = 2(MN + PQ) \qquad (**)$$

(**) implies the existence of equality in (*), so $MN \parallel AD \parallel BC$ and $PQ \parallel AB \parallel CD$. Thus $ABCD$ is a parallelogram, and $MN = AD = BC$. Similarly $AB = CD$. Since $AB + CD = 2MN$ (see above), therefore

$$AD = BC = MN = AB = CD.$$

Hence, $\boxed{ABCD \text{ is a rhombus.}}$ □