On the Decoding of Polar Codes on Permuted Factor Graphs

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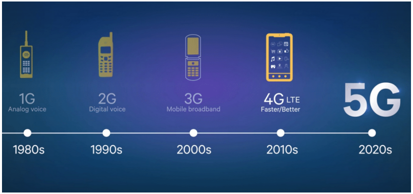
Problem Statement



- Polar codes: selected for the eMBB control channel in 5G
- Successive Cancellation (SC) List (SCL): good error-correction performance, serial nature
- Belief Propagation (BP): reasonable error-correction performance, **highly parallel** $\rightarrow$ **enable high decoding throughput !**

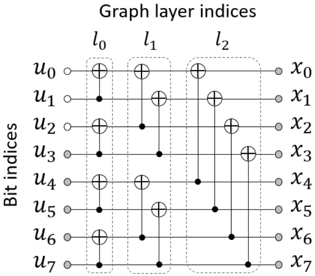
Problem Statement

This paper

- Improve the error-correction performance of polar-code decoding by exploiting factor-graph permutations
- Provide a hardware friendly representation of the factor-graph permutations

Polar codes

- Introduced by Arıkan[1] in 2009
- $\mathcal{P}(N, K)$, N: code length, K: message length
- Code construction: based on polarization phenomenon
  - K most reliable channels: information bits
  - $(N - K)$ least reliable channels: frozen bits

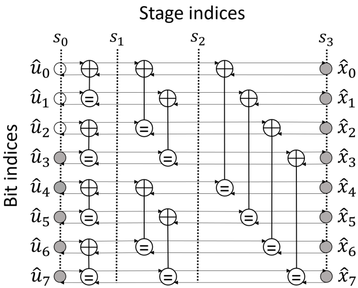


$\mathcal{P}(8, 5)$ with u_0, u_1, and u_2 are frozen bits

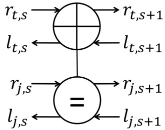
[1] E. Arıkan. "Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels". In: 55.7 (2009), pp. 3051–3073. ISSN: 0018-9448. DOI: 10.1109/TIT.2009.2021379.

Belief Propagation (BP) Decoding

- An iterative message passing algorithm
- Belief messages are updated through Processing Elements (PEs)
- Termination conditions: CRC-based condition[2], maximum number of iterations



BP decoding for $\mathcal{P}(8, 5)$

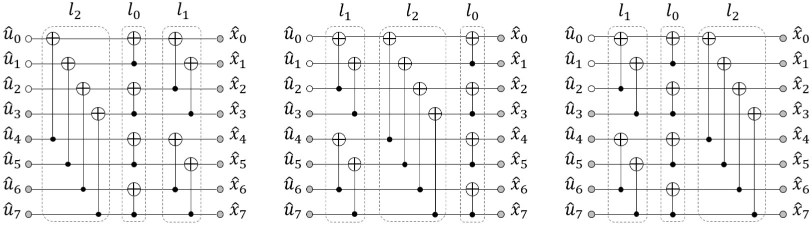


A polar PE

[2] Y. Ren et al. "Efficient early termination schemes for belief-propagation decoding of polar codes". In: *IEEE 11th Int. Conf. on ASIC*. 2015, pp. 1–4. DOI: 10.1109/ASICON.2015.7517046.

BP Decoding on Permuted Factor Graphs

- Permuting the factor-graph layers preserves the code[3,4]
- Running BP decoding on multiple factor-graph permutations improves the error-correction performance[5,6]



Various factor-graph permutations of $\mathcal{P}(8,5)$

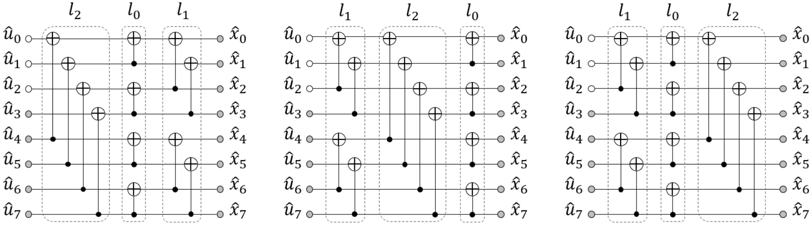
[3] S. B. Korada. "Polar Codes for Channel and Source Coding". PhD thesis. Lausanne, Switzerland: EPFL, 2009.

[4] Nadine Hussami, Satish Babu Korada, and Rudiger Urbanke. "Performance of polar codes for channel and source coding". In: *IEEE Int. Symp. on Inf. Theory*. 2009, pp. 1488–1492.

[5] A. Elkelesh et al. "Belief propagation decoding of polar codes on permuted factor graphs". In: *IEEE Wireless Commun. and Net. Conf.* 2018, pp. 1–6. DOI: 10.1109/WCNC.2018.8377158.

[6] Ahmed Elkelesh et al. "Belief Propagation List Decoding of Polar Codes". In: *IEEE Communications Letters* 22 (2018), pp. 1536–1539.

BP Decoding on Permuted Factor Graphs



Permuted factor graph representations for $\mathcal{P}(8, 5)$

- Problem: each factor-graph permutation requires a different decoding schedule $\rightarrow$ **require different decoders in hardware**
- Solution: transform factor-graph layer permutation to codeword-position permutation

Factor-graph Permutation to Codeword-position Permutation

- Definitions:
  - Permutation $\pi : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ where $n = \log_2(N)$
  - Apply a permutation π to the original graph layer $L = \{l_{n-1}, \ldots, l_1, l_0\}$:

$$L = \{l_{n-1}, \ldots, l_1, l_0\} \xrightarrow{\pi} L_\pi = \{l_{\pi(n-1)}, \ldots, l_{\pi(1)}, l_{\pi(0)}\}$$

  - Binary expansion of the integer i

$$\{b_{n-1}^{(i)}, \ldots, b_1^{(i)}, b_0^{(i)}\}$$

where $b_j^{(i)} \in \{0, 1\}$ $(0 \leq i \leq N-1; 0 \leq j \leq n-1)$

Factor-graph Permutation to Codeword-position Permutation

Theorem 1

Apply permutation π to the original factor graph of a polar code

$$L = \{l_{n-1}, \ldots, l_1, l_0\} \xrightarrow{\pi} L_\pi = \{l_{\pi(n-1)}, \ldots, l_{\pi(1)}, l_{\pi(0)}\}.$$

Then, the synthetic channel associated with the binary expansion

$$\{b_{n-1}^{(i)}, \ldots, b_1^{(i)}, b_0^{(i)}\}$$

*of the **original** factor graph **L** is the same as the synthetic channel associated with the binary expansion*

$$\{b_{\pi(n-1)}^{(i)}, \ldots, b_{\pi(1)}^{(i)}, b_{\pi(0)}^{(i)}\}$$

*of the **permuted** factor graph $\boldsymbol{L_\pi}$.*

Factor-graph Permutation to Codeword-position Permutation

Proof.

- On L, the synthetic channel associated with the binary expansion $\{b_{n-1}^{(i)}, \ldots, b_1^{(i)}, b_0^{(i)}\}$ is

$$W_L^{(i)} = ((((W^{b_{n-1}^{(i)}})^{\cdots})^{b_1^{(i)}})^{b_0^{(i)}})$$

- On L_π, the synthetic channel associated with the binary expansion $\{b_{\pi(n-1)}^{(i)}, \ldots, b_{\pi(1)}^{(i)}, b_{\pi(0)}^{(i)}\}$ is

$$\begin{aligned} W_{L_\pi}^{(i)} &= ((((W^{b_{\pi(\pi(n-1))}^{(i)}})^{\cdots})^{b_{\pi(\pi(1))}^{(i)}})^{b_{\pi(\pi(0))}^{(i)}}) \\ &= ((((W^{b_{n-1}^{(i)}})^{\cdots})^{b_1^{(i)}})^{b_0^{(i)}}) = W_L^{(i)} \end{aligned}$$

$\square$

Factor-graph Permutation to Codeword-position Permutation

Apply Theorem 1 to a **permuted** factor graph $\boldsymbol{L_\pi}$:

$$L_\pi = \{l_{\pi(n-1)}, \ldots, l_{\pi(1)}, l_{\pi(0)}\} \xrightarrow{\pi} L = \{l_{n-1}, \ldots, l_1, l_0\}.$$

Then, the synthetic channel associated with the binary expansion

$$\{b^{(i)}_{n-1}, \ldots, b^{(i)}_1, b^{(i)}_0\}$$

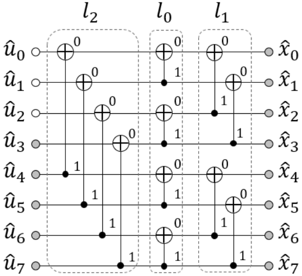
of the **permuted** factor graph $\boldsymbol{L_\pi}$ is the same as the synthetic channel associated with the binary expansion

$$\{b^{(i)}_{\pi(n-1)}, \ldots, b^{(i)}_{\pi(1)}, b^{(i)}_{\pi(0)}\}$$

of the **original** factor graph $\boldsymbol{L}$.

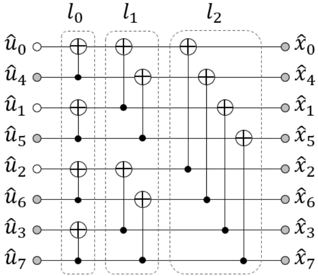
Factor-graph Permutation to Codeword-position Permutation

An example:



Original codeword positions
Permuted graph layers

$$\pi(0) = 000_2 = 0$$
$$\pi(1) = 010_2 = 2$$
$$\pi(2) = 100_2 = 4$$
$$\pi(3) = 110_2 = 6$$
$$\pi(4) = 001_2 = 1$$
$$\pi(5) = 011_2 = 3$$
$$\pi(6) = 101_2 = 5$$
$$\pi(7) = 111_2 = 7$$

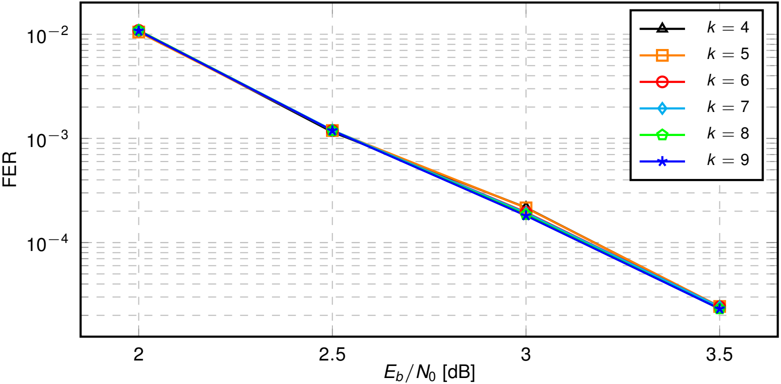


Permuted codeword positions
Original graph layers

Selection of Good Permutations

- Construct a set of permutations $\mathbb{P}$
  - Fix $(n-k)$ left-most layers of the original graph
  - Construct $k!$ permutations of the k right-most layers
  - $|\mathbb{P}| = k!$
- If the decoding algorithm fails on the original graph
  - Run the decoding algorithm on each permutation p of $\mathbb{P}$
  - Numerically evaluate the probability of successful decoding for each permutation p
  - Select the M best permutations of $\mathbb{P}$

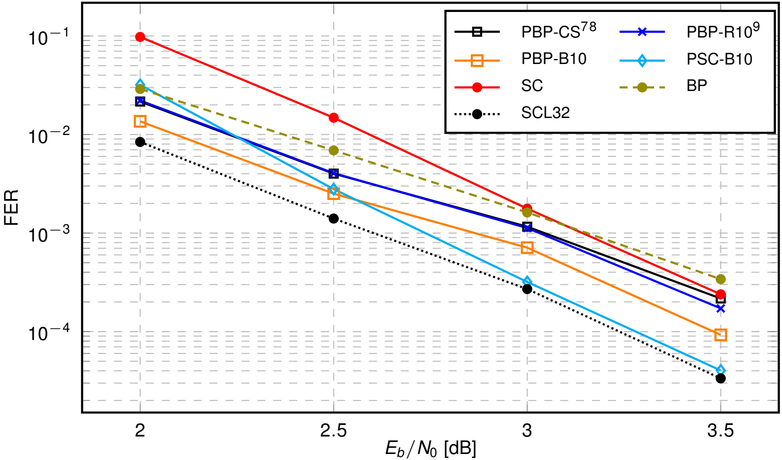
Selection of Good Permutations



FER performance of BP decoding on 16 best permuted factor graphs of $\mathcal{P}(1024, 512)$ with 24-bit CRC, and $|\mathbb{P}| = k!$.

- Good permutations: found by permuting the layers on the right-most side of the original factor graph

5G $\mathcal{P}(1024, 512)$, no CRC, AWGN channel

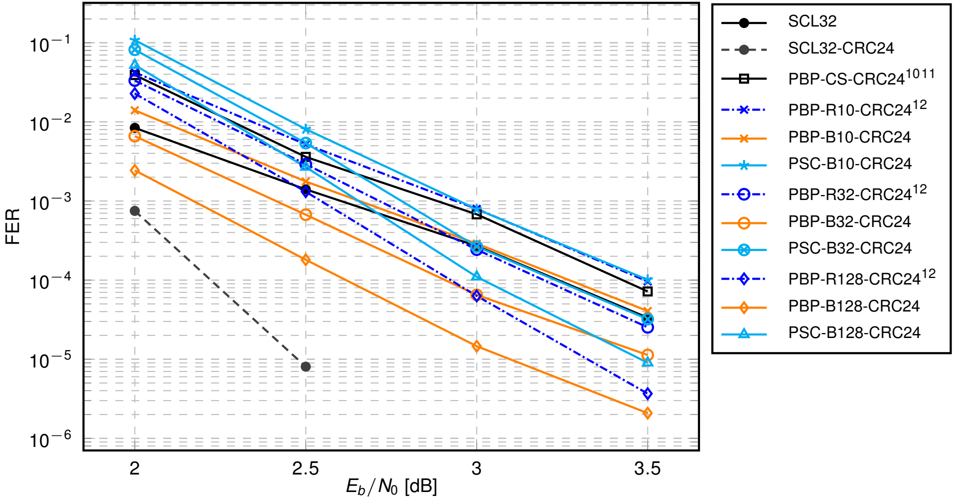


[7] Korada, "Polar Codes for Channel and Source Coding".

[8] Hussami, Korada, and Urbanke, "Performance of polar codes for channel and source coding".

[9] Elkelesh et al., "Belief propagation decoding of polar codes on permuted factor graphs".

5G $\mathcal{P}(1024, 512)$, 24-bit CRC, AWGN channel



[10] Korada, "Polar Codes for Channel and Source Coding".

[11] Hussami, Korada, and Urbanke, "Performance of polar codes for channel and source coding".

[12] Elkelesh et al., "Belief propagation decoding of polar codes on permuted factor graphs".

Conclusion

- Factor-graph permutations can be mapped to codeword-position permutations
  $\rightarrow$ require a single decoder for hardware implementation
- Propose a method to construct good permutations for different polar-code decoders
  - 5G $\mathcal{P}(1024, 512)$, no CRC, at $FER = 10^{-4}$: PSC-B10 is 0.4 dB better than PSC-R10, and almost equivalent to SCL32
  - 5G $\mathcal{P}(1024, 512)$, 24-bit CRC, at $FER = 10^{-4}$: PBP-B128 is 0.25 dB better than PBP-R128, but 0.3 dB worse than SCL32

Thank you !