

# ***Sinusoidal Velocity Profiles for Motion Control***

*A review on traditional solutions and a proposal for a solution with reduced piece-wise constructors and reduced harmonic content of the kinetic perturbations*

Victor M. Arevalo

Augen Opticos, S.A., Ensenada, B.C., Mexico 22830

## **Background**

When facing the simplest and first question to be made with a motorized motion: how to perform the change of position from one point to another, the present technical literature provides similar answers to those given 30 years ago.

The available solutions can be summarized as follows:

- **Constant Velocity Window.**- For mass-reduced, robust mechanisms, with enough power to tolerate the acceleration impulses at the beginning and the end of this type of motion. This method is not regarded as a "solution" herein.
- **Trapezoidal Velocity Profile.**- Produces rectangular acceleration windows. The jerk obtained is in the form of impulses (4 in total), aligned with the edges of the acceleration windows.
- **S-Curve Velocity Profile.**- Part of the work reported in this paper is the development of a procedure that generalizes the treatment of this family of profiles. The generalization includes the special cases of "short travel" and allows to "select" the particular profile by means of a single parameter (denoted as  $\gamma$  in the example figures). By varying continuously  $\gamma$  from 0.0 to 1.0, the profile respectively evolves from "impulsive jerks" (i.e. the Trapezoidal Velocity Profile, appearing as a particular and extreme case of the S-Curve **family** of functions), to its maximum "smoothness", namely, triangular acceleration profile and minimization of both, jerk and peak-power values.

The analytical flexibility offered by these solutions is a tradeoff with their features. As shown in the final summary table, the benefit of reduced peak values of power and jerk has a cost of additional components in the piece-wise construct of the functions. Even after the identification and utilization of symmetry points for mirror simplification, the constructors count is still high.

As a final word, the traditional velocity profiles are solutions of such numerical simplicity that can be used when computing power is expensive, limited or absent. However, the use of computerized control increases to the point that in some cases it is mandatory.

As long as assertions like: "*Over the past 30 years the price of computer processing power has declined by an annual average of 30% in real terms*", and "*... the cost of a unit of computer processing power has fallen by 99% over the decade*"[1] can be made, it is clear that computer utilization in motion control is not an affected trend, but a strong foundation.

## **Proposal**

This work proposes a motion profile named Sinusoidal Velocity Profile. It is a piece-wise construct with 3 components: An acceleration zone, a constant speed zone and the corresponding decelerating zone, similar to the construction used in trapezoidal profiles.

## **Procedures**

The calculation procedures will be developed in a dual fashion: As a procedural review of the traditional S-Curve, and as a presentation of the proposed Sinusoidal profile.

The following parameters are required:

- $V_m$   $\equiv$  Maximum Velocity allowed (absolute value).
- $A_m$   $\equiv$  Maximum Acceleration peak value allowed (absolute value).
- $Throw$   $\equiv$  The travel distance (positive or negative).

Also, when dealing with the S-Curve profile, the parameters set must include  $\gamma$ , dimensionless quantity varying from 0.0 to 1.0 as explained above. The effects of some typical values are shown in the final summary table.

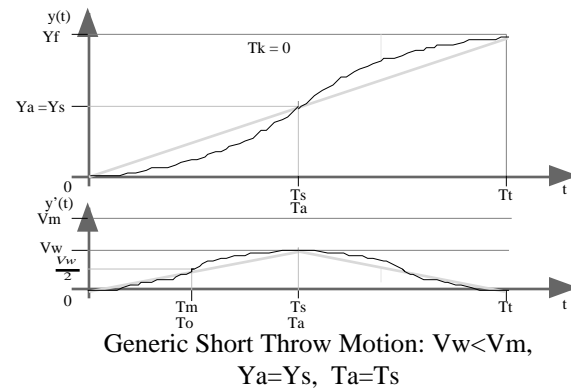
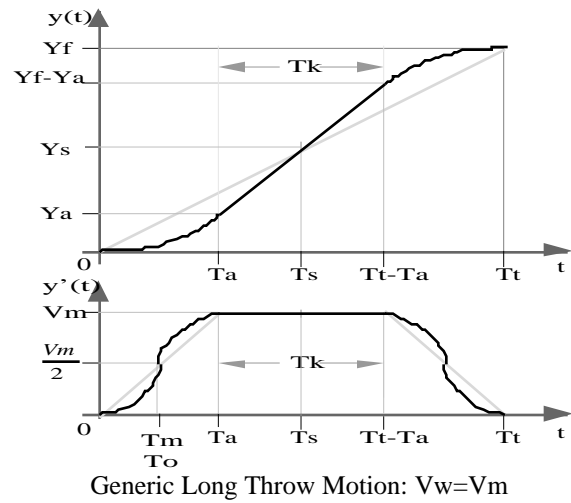
The numerical format for these and the rest of the variables employed herein is not restricted to be integer and the units adopted for these variables can be freely selected according to the natural specification of the motion environment (resolver counts, micrometers, millimeters, meters, inches, etc.). The general idea with this approach is that, for the sake of self-documentation and clarity, the calculation part of the control system contains the user's abstraction of the problem, and the required conversions (resolver integer counts, stepper pulses, etc.) can be made at the final "delivery" stage of the process. Conversely, when "reading" from the physical devices, the conversion of units made at the "source" of the information favors maintenance and inspection versatility.

With the “user” set of parameters in hand, another set of 12 (with a temporary  $Y_{aux}$  added for a total of 13) “calculation” parameters is obtained prior to the utilization of the procedural formulas. In terms of software implementation it is convenient that this initialization procedure be confined in a function (or subroutine). This will be referred to, as the Initialization Sequence:

| No. | Profile Type | Initialization  |
|-----|--------------|---|
| 1   | Both         | $Sgn = \text{sgn}(\text{Throw})$  |
| 2   | Both         | $Y_f =  \text{Throw} $  |
| 3   | Both         | $Y_s = \frac{Y_f}{2}$   |
| 4   | S-Curve      | $Y_{aux} = \frac{1}{2} \cdot (1 + \gamma) \cdot \left( \frac{V_m^2}{A_m} \right)$   |
|     | Sinusoidal   | $Y_{aux} = \frac{V_m^2}{A_m}$   |
| 5   | Both         | $Y_a = \begin{cases} Y_s & \forall Y_s \leq Y_{aux} \\ Y_{aux} & \forall Y_s > Y_{aux} \end{cases}$   |
| 6   | S-Curve      | $V_w = \begin{cases} \sqrt{\frac{Y_s \cdot A_m}{\frac{1}{2} \cdot (1 + \gamma)}} & \forall Y_s \leq Y_{aux} \\ V_m & \forall Y_s > Y_{aux} \end{cases}$ |
|     | Sinusoidal   | $V_w = \begin{cases} \sqrt{Y_s \cdot A_m} & \forall Y_s \leq Y_{aux} \\ V_m & \forall Y_s > Y_{aux} \end{cases}$  |
| 7   | Both         | $T_o = \frac{V_w}{A_m}$   |
| 8   | S-Curve      | $T_a = T_o \cdot (1 + \gamma)$  |
|     | Sinusoidal   | $T_a = 2 \cdot T_o$   |
| 9   | S-Curve      | $\tau = \gamma \cdot T_o$   |
|     | Sinusoidal   | $\omega = \frac{2 \cdot \pi}{T_a}$  |
| 10  | S-Curve      | $T_m = \frac{T_a}{2}$   |
|     | Sinusoidal   | $K_s = \frac{T_a \cdot V_w}{4 \cdot \pi^2}$   |

| No. | Profile Type | Initialization (continued...)         |
|-----|--------------|---------------------------------------|
| 11  | Both         | $T_k = 2 \cdot \frac{Y_s - Y_a}{V_m}$ |
| 12  | Both         | $T_s = T_a + \frac{T_k}{2}$           |
| 13  | Both         | $T_t = 2 \cdot T_s$                   |

The following figures are generic outlines for arbitrary (S-Curve or Sinusoidal) profiles, intended as a graphical description of some of the the variables employed in this section.



The procedural equations can be stated in several ways and the adoption of a particular style is a matter of taste, and available space. That is the reason for the adoption of the “recursive” style. Although the software implementation was experimented following both, the “classical” style (using auxiliary functions) and the “recursive” style, apart from the characteristic compactness of the later, there are not improvements in performance in following a particular style.

|  |  |
|--|--|
| $y(t) = Sgn \cdot \hat{y}(t)$ $\hat{y}(t) = \begin{cases} 0 & \forall t \leq 0 \\ \left( \frac{1}{2} \cdot \frac{V_w}{T_o} \right) \cdot \left( \frac{t^3}{3 \cdot \tau} \right) & \forall 0 < t < \tau \\ \left( \frac{1}{2} \cdot \frac{V_w}{T_o} \right) \cdot \left[ \left( t - \frac{\tau}{2} \right)^2 + \frac{\tau^2}{12} \right] & \forall \tau < t \leq T_o \\ Y_s + V_w \cdot (t - T_s) + \hat{y}(T_a - t) & \forall T_o < t \leq T_s \\ Y_f - \hat{y}(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 1-a: S-Curve Profile Position</i></p> | $y(t) = Sgn \cdot \hat{y}(t)$ $\hat{y}(t) = \begin{cases} 0 & \forall t \leq 0 \\ \frac{Am}{4} \cdot t^2 + K_s \cdot (\cos(\omega \cdot t) - 1) & \forall 0 < t \leq T_a \\ Y_s + V_w \cdot (t - T_s) & \forall T_a < t \leq T_s \\ Y_f - \hat{y}(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 2-a: Sinusoidal Profile Position</i></p> |
| $y'(t) = Sgn \cdot \hat{y}'(t)$ $\hat{y}'(t) = \begin{cases} 0 & \forall t \leq 0 \\ \left( \frac{Am}{2} \right) \cdot \left( \frac{t^2}{\tau} \right) & \forall 0 < t < \tau \\ \left( \frac{Am}{2} \right) \cdot (2 \cdot t - \tau) & \forall \tau < t \leq T_o \\ V_w - \hat{y}'(T_a - t) & \forall T_o < t \leq T_s \\ \hat{y}'(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 1-b: S-Curve Profile Velocity</i></p>  | $y'(t) = Sgn \cdot \hat{y}'(t)$ $\hat{y}'(t) = \begin{cases} 0 & \forall t \leq 0 \\ K_s \cdot \omega \cdot (\omega \cdot t - \sin(\omega \cdot t)) & \forall 0 < t \leq T_a \\ V_w & \forall T_a < t \leq T_s \\ \hat{y}'(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 2-b: Sinusoidal Profile Velocity</i></p>                        |
| $y''(t) = Sgn \cdot \hat{y}''(t)$ $\hat{y}''(t) = \begin{cases} 0 & \forall t \leq 0 \\ \left( \frac{Am}{\tau} \right) \cdot t & \forall 0 < t < \tau \\ Am & \forall \tau < t \leq T_o \\ \hat{y}''(T_a - t) & \forall T_o < t \leq T_s \\ -\hat{y}''(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 1-c: S-Curve Profile Acceleration</i></p>   | $y''(t) = Sgn \cdot \hat{y}''(t)$ $\hat{y}''(t) = \begin{cases} 0 & \forall t \leq 0 \\ \frac{Am}{2} \cdot (1 - \cos(\omega \cdot t)) & \forall 0 < t \leq T_a \\ 0 & \forall T_a < t \leq T_s \\ -\hat{y}''(T_t - t) & \forall T_s < t \end{cases}$ <p><i>Equations Set 2-c: Sinusoidal Profile Acceleration</i></p>                                  |

The presented sets of equations are for  $y(t)$  –position–,  $y'(t)$  –velocity–, and  $y''(t)$  –acceleration– (sets 1-a, 1-b and 1-c respectively for S-Curve and sets 2-a, 2-b and 2-c respectively for Sinusoidal). The independent variable  $t$  is the local time in terms of a particular profile motion which starts at  $t = 0$ , and finishes with  $t = T_t$ . Any subsequent motion restates its own  $t$ .

Note that the following validation checks are performed:  $y(t) = 0 \quad \forall t \leq 0$  and  $y(t) = Throw \quad \forall t > T_t$  for position boundaries.

Also, all the velocity and acceleration boundaries (when  $t \leq 0$  and  $t \geq T_t$ ) are validated to ZERO values.

## Results

Higher order derivatives, jerk (acceleration derivative) and the so called “jounce” (jerk derivative) can be similarly obtained. Note that the jounce for the S-Curve profile

is a set of impulses even for the smooth case. Conversely, the jounce for the Sinusoidal profile is a pair of windowed cosine functions. This result is associated to the fact that the harmonic content of the sinusoidal curves (acceleration and jerk), by definition is the minimum attainable, when compared with the triangular and rectangular counterparts.

The experimental work with these procedures in a target servomechanism confirmed the expectations made with the Sinusoidal profile. Although the observations were made while comparing the different profiles, the test results are qualitative rather than quantitative. The bottom line is that the difference between the canonical S-Curve ( $\gamma = 1$ ) and the sinusoidal profile is perceptible but far from dramatic. Probably the major advantage, apart from marginal reductions in servo positioning errors, lies in the procedural simplicity of the implementation.

A word about the test implementation: In a i486-DX4 type processor at 66 Mhz (with FPU integrated), running an ISR (interrupt service routine) that feeds a servo controller at a rate of 250 Hz, under a multitasking OS (Linux), there is not an appreciable difference for the computational load of calculating any of the profiles reviewed.

Also available as a complement of these procedures is the solution to the opposite problem: For a given travel time, What is the required parameter change (velocity in this case) that satisfies the motion conditions?.

This can be useful when synchronizing motion in several axes, or in the case when the calculated  $Tt$  in Initialization has to be adjusted upwards to the nearest multiple of the sampling period. In this case the parameters to be given are:

- $A_m$   $\equiv$  Maximum Acceleration peak value allowed.
- $Tt$   $\equiv$  Elapsed time required for the motion.
- $Throw$   $\equiv$  The travel distance (positive or negative).
- $V_m$   $\equiv$  The original Maximum Velocity allowed (absolute value) for validation of the  $Tt$  provided.

The procedure verifies the validity of the parameters and finds in closed form (a quadratic) the result: Two values of velocity that, when adopted as  $V_m$ , will work as a solution to the problem of traveling the **Throw** in  **$Tt$**  seconds.

The  $Tt$  equation for the S-Curve is

$$Tt = \frac{Yf}{V_w} + \frac{V_w}{A_m} \cdot (1 + \gamma),$$

and for the Sinusoidal profile

$$Tt = \frac{Yf}{V_w} + 2 \cdot \frac{V_w}{A_m}.$$

Both formulas are similar and, for the purpose of this analysis, a convenient simplification consists in using only the S-Curve formula and regarding the Sinusoidal profile case as an S-Curve with  $\gamma = 1$ .

The pertinent solutions are [2]

$$V_1 = \left( \frac{A_m}{1 + \gamma} \right) \cdot \frac{1}{2} \cdot \left( Tt + \sqrt{Tt^2 - 4 \cdot \left( \frac{1 + \gamma}{A_m} \right) \cdot Yf} \right),$$

$$V_2 = \frac{2 \cdot Yf}{Tt + \sqrt{Tt^2 - 4 \cdot \left( \frac{1 + \gamma}{A_m} \right) \cdot Yf}}.$$

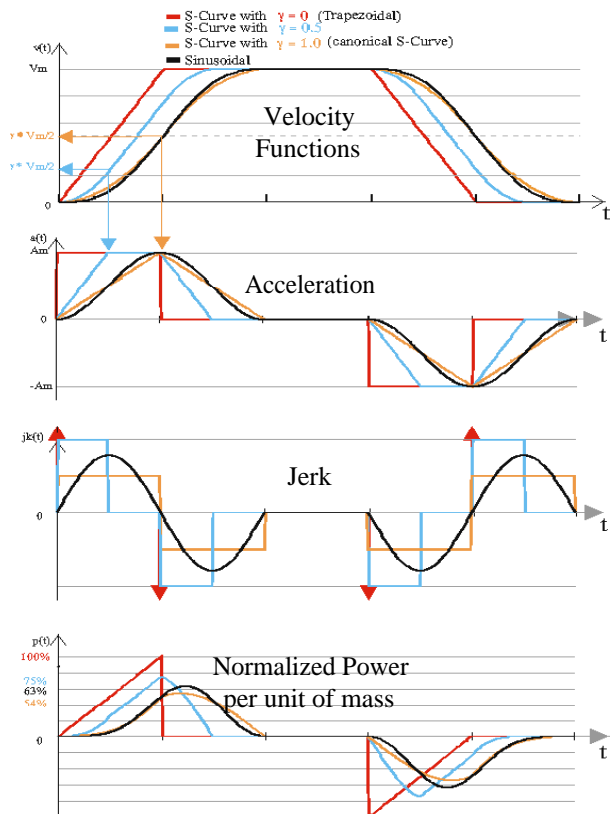
$V_1 \equiv$  The solution that minimizes (or cancels) the constant velocity zone.

$V_2 \equiv$  The solution that maximizes the constant velocity zone.

Both solutions are valid, but only  $V_2$  guarantees the security condition  $V_2 \leq V_m$ .

## Summary Table of Velocity Profiles

| Velocity Profile Type       | Normalized Transition Time ( $Tt$ ) | Total Piece Wise Constructors | Normalized Jerk Value          | Jerk Shape                | Normalized Peak Power per Unity of Mass |
|-----------------------------|-------------------------------------|-------------------------------|--------------------------------|---------------------------|---|
| Trapezoidal<br>$\gamma = 0$ | 1.00<br>(Reference and lowest)      | 3                             | $\infty$                       | 4 Impulses                | 1.00<br>(Reference)                     |
| S-Curve<br>$\gamma = 0.25$  | 1.25                                | 7                             | 4.00                           | 4 windows                 | 0.88                                    |
| S-Curve<br>$\gamma = 0.50$  | 1.50                                | 7                             | 2.00                           | 4 windows                 | 0.75                                    |
| S-Curve<br>$\gamma = 0.75$  | 1.75                                | 7                             | 1.33                           | 4 windows                 | 0.63                                    |
| S-Curve<br>$\gamma = 1.0$   | 2.00                                | 7                             | 1.00<br>(Reference and lowest) | 2 pairs of linked windows | 0.54<br>(Lowest)                        |
| Sinusoidal                  | 2.00                                | 3                             | 1.57                           | 2 windowed sinusoids      | 0.63                                    |



## Final Comment

Contrary to the traditional methods, the Sinusoidal Profile requires computing power for the trigonometric evaluation (FPU, series expansion or lookup tables), but provides a simplified algorithmic implementation.

- 1.- "The price of age," The Economist, Dec 23rd 2000, pp 91-94.
- 2.- Press, W.H. et al, "Numerical Recipes", 2nd ed., Cambridge UP, New York, 1995.