3.2: Determinants and Matrix inverses.

① $A, B: n \times n \Rightarrow \det(A.B) = \det A . \det B$

② $\det(I) = 1$

③ $\det(A^{-1}) = \frac{1}{\det A}$

④ $\det A^T = \det A$

⑤ $\det(cA) = c^n \det A$

⑥ $\det A^n = (\det A)^n$

EX: Let $\det A = 2$, $\det B = 5$. Calculate $\det(A^3 . B^{-1} . A^T . B^2)$

$= \det(A^3) . \det B^{-1} \det A^T \det(B^2)$

$= (\det A)^3 . \frac{1}{\det B} \det A \, (\det B)^2$

$= (\det A)^4 . \det B = 2^4 . 5 = 80$

EX2: Let $\det A = 4$, $\det B = 3$. $A, B: 2 \times 2$

a) find $\det(A^5 B^{-1} A^T B^T B^2) = ?$

b) $\det(3AB) = ?$

a) $= \det(A^5) . \det B^{-1} \det A^T \det B^T \det(B^2)$

$= (\det A)^5 . \frac{1}{\det B} . \det A . \det B . (\det B)^2$

$= (\det A)^6 (\det B)^2 = 4^6 . 3^2 = 36864.$

b) $\det(3AB) = 3^2 \det A . \det B = 9.4.3 = 108$

$$\# \quad A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A \qquad (\det A \neq 0)$$

$\downarrow$

(adjugate: MT phụ hợp)

$A: 2\times 2 \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \operatorname{adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$A: n\times n \ (n \geq 3)$

$$\operatorname{adj} A = [C_{ij}(A)]^T \qquad \left(\begin{array}{l} C_{ij}(A) = (-1)^{i+j} \cdot \det A_{ij} \\ A_{ij} = A - \begin{cases} \text{row } i \\ \text{column } j \end{cases} \end{array} \right)$$

EX: Compute the adjugate of $A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{pmatrix}$ and calculate $A.(\operatorname{adj} A)$

$$C_{ij}(A) = \begin{pmatrix} C_{11}(A) & C_{12}(A) & C_{13}(A) \\ C_{21}(A) & C_{22}(A) & C_{23}(A) \\ C_{31}(A) & C_{32}(A) & C_{33}(A) \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{1+1} \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \\ - \begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & - \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{pmatrix}$$

$$\rightarrow \quad \operatorname{adj} A = \begin{pmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{pmatrix}^T = \begin{pmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & \underline{1} \end{pmatrix}$$

b) $B = \begin{pmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$C_{ij}(B) = \begin{pmatrix} 4 & 11 & -6 \\ 4 & 37 & -19 \\ -3 & -18 & 11 \end{pmatrix} \Rightarrow \operatorname{adj} B = \begin{pmatrix} 4 & 4 & -3 \\ 11 & 37 & -18 \\ -6 & -19 & 11 \end{pmatrix}$$

* Theorem

① $A : n \times n \Rightarrow A \cdot \operatorname{adj} A = \operatorname{adj}(A) \cdot A = (\det A) \cdot I$

② $\det A \neq 0 \Rightarrow A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$

$$\det(\operatorname{adj} A) = (\det A)^{n-1}$$

EX: Find the (2,3)-entry of A^{-1} if $A = \begin{pmatrix} 2 & 1 & 3 \\ \cancel{5} & \cancel{-7} & \cancel{1} \\ 3 & 0 & -6 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A = \frac{1}{180} \left[C_{ij}(A) \right]^{T}$$

$$= \frac{1}{180} C_{ji}(A) = \frac{1}{180} C_{32}(A) = \frac{1}{180} (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 5 & \underline{1} \end{vmatrix}$$

$$= \frac{13}{180}$$

b) find the (1, 2)-entry of A^{-1}.

$$A^{-1} = \frac{1}{180} \cdot \operatorname{adj} A = \frac{1}{180} \cdot C_{ji}(A) = \frac{1}{180} C_{21}(A)$$

$$= \frac{1}{180} (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 0 & -6 \end{vmatrix} = \frac{6}{180} = \frac{1}{30}$$

Exercise 3.1.9 In each case either prove the statement or give an example showing that it is false:

a. $\det(A+B) = \det A + \det B$. F

$A \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det A = 18 \rightarrow \det 1$

b. If $\det A = 0$, then A has two equal rows. F

c. If A is 2×2, then $\det(A^T) = \det A$. T

d. If R is the reduced row-echelon form of A, then $\det A \cancel{=} \det R$. F

→ chẻm chú.

e. If A is 2×2, then $\det(7A) = 49 \det A$. T

$= 7^2 \cdot \det A$

$\det(cA) = c^n \det A$

f. $\det(A^T) = -\det A$. F

g. $\det(-A) = -\det A$. F

$\det(-A) = \det A$

$\det(7A) = 49 \det A$

h. If $\det A = \det B$ where A and B are the same size, then $A = B$. F

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det = 1$

$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \det = 1$