Chapter 3

Determinants and Diagonalization (Định thức)

3.1. The cofactor expansion

+) Determinant of A. $|A|$, $\det A$: $n \times n$

+) $A = (a) \rightarrow \det A = a$

+) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \times 3 - 2 \times 2 = -1$$

+) Theorem : $A_{ij} = A$ - row i, column j

$$C_{ij}(A) = (-1)^{i+j} . \det(A_{ij})$$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$, find the cofactor of position $(1,2)$, $(3,1)$ $(2,3)$ $(3,2)$

$$C_{12}(A) = (-1)^{1+2} . \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -1 . (0 \times 0 - 1 \times 2) = 2$$

$$C_{31}(A) = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 . (2 \times 2 - 1 \times 3) = 1$$

$$C_{23}(A) = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 . (1 \times 1 - 1 \times 2) = 1$$

$$C_{32}(A) = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = -1 \cdot (1 \cdot 2 - 0 \cdot 3) = -2.$$

$A : n \times n \quad (n \geqslant 3)$

$$A = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\det A = a_{11} \cdot C_{11}(A) + a_{12}\, C_{12}(A) + \cdots + a_{1n}\, C_{1n}(A)$$

EX: $A = \begin{pmatrix} 3 & \underline{0} & \underline{0} \\ 5 & 1 & 2 \\ 2 & 6 & 1 \end{pmatrix}$ $\det A = ?$

$$|A| = \underbrace{3 \cdot C_{11}(A)} + 0\, C_{12}(A) + 0\, C_{13}(A)$$

$$= 3 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} = 3 \cdot (1 \cdot 1 - 6 \cdot 2) = 3 \cdot (-11) = -33$$

EX: $B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\det B = 1\, C_{11}(B) + 2\, C_{12}(B) + 0\, C_{13}(B)$$

$$= (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 2\cdot(-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1 \cdot 3 + (-2)\cdot 5 = -7.$$

$$C = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ 6 & 3 & 1 & 0 \end{pmatrix}$$

$$= 3 \cdot C_{11}(C) = 3 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= 3 \cdot \boxed{-1 \cdot C_{23}} = -3 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 3 \cdot -5 = -15$$

$$A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 0 \\ 0 & 7 & -1 & 0 \\ 0 & 1 & 8 & 2 \end{pmatrix}$$

$$|A| = 1(-1)^{1+1} \begin{vmatrix} 6 & 4 & 0 \\ 7 & -1 & 0 \\ 1 & 8 & 2 \end{vmatrix}$$

$$= 2(-1)^{3+3} \begin{vmatrix} 6 & 4 \\ 7 & -1 \end{vmatrix} = 2 \cdot (-34) = -68$$

Ex: find the values of x for which $\det A = 0$

Where $A = \begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}$$

$$= 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} + x \cdot (-1)^{1+2} \begin{vmatrix} x & x \\ x & 1 \end{vmatrix} + x \cdot (-1)^{1+3} \begin{vmatrix} x & 1 \\ x & x \end{vmatrix}$$

$$= 1 - x^2 - x(x - x^2) + x(x^2 - x) = 0$$

$$= 2x^3 - 3x^2 + 1 = 0 \Rightarrow \begin{cases} x = -\frac{1}{2} \\ x = 1 \end{cases}$$

I. Properties

1/ A has one [row / column] of zeros $\Rightarrow |A| = 0$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \Rightarrow |A| = 0$$

2/ A : Triangular matrix $\Rightarrow \det A = \prod_{r=1}^{n} a_{ii}$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \det A = 1 \times 2 \times 4 = 8$$

$$B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix} \Rightarrow |A| = 3 \times 2 \times 3 \times 1 = 18$$

3/ B = A interchange two [rows / columns] $\Rightarrow \det B = -\det A$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -\left(- \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 5 \end{vmatrix} \right)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 5 \end{vmatrix}$$

4/ A: two $\left[\begin{array}{l}\text{rows}\\ \text{columns}\end{array}\right.$ is the same $\Rightarrow \det A = 0$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix} \Rightarrow \quad |A| = 0$$

5/ A. one $\left[\begin{array}{l}\text{row}\\ \text{column}\end{array}\right. = k \left[\begin{array}{l}\text{row}\\ \text{column}\end{array}\right. \Rightarrow A = k|B|$

$$|A| = \begin{vmatrix} 3 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= \underline{3.2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

6/ $\left[\begin{array}{l}\text{row}\\ \text{column}\end{array}\right. \Rightarrow |A|$ constant

Ex: $A = \begin{pmatrix} 1 & a & a+1 \\ 2 & b & b+2 \\ 3 & c & c+3 \end{pmatrix}$ $|A| = ?$

$$|A| = \begin{vmatrix} 1 & a & a \\ 2 & b & b \\ 3 & c & c \end{vmatrix} = 0$$

EX2: let $\det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = 6$

a) Evaluate $\det A = \begin{vmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{vmatrix}$

b) $B = \begin{vmatrix} p+x & q+y & r+z \\ 2a-x & 2b-y & 2c-z \\ x & y & z \end{vmatrix}$

$$A = -3 \begin{vmatrix} a+x & b+y & c+z \\ x & y & z \\ p & q & r \end{vmatrix} = -3 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$= 3 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 3 . 6 = 18$$

$$B = \begin{vmatrix} p & q & r \\ 2a & 2b & 2c \\ x & y & z \end{vmatrix} = 2 \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix}$$

$$= -2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -2 \cdot 6 = -12$$