Chapter 5

The Vector Space $\mathbb{R}^n$

Def: $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, \forall i = \overline{1,n}\}$

① Linear Combination (tổ hợp tuyến tính)

$(a) = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$

x is linear combination of (a)

$\Longleftrightarrow$ $\exists\, r_1, r_2, \ldots, r_m \in \mathbb{R}$, such that

$\boxed{x = r_1 a_1 + r_2 a_2 + \ldots + r_m a_m}$ $\Rightarrow$ has solution

Ex1. Let $x = (-1, -2, -2)$, $u = (0, 1, 4)$, $v = (-1, 1, 2)$ and $w = (3, 1, 2)$ in $\mathbb{R}^3$

find scalar a, b, and c such that $x = au + bv + cw$

$$a\begin{pmatrix}0\\1\\4\end{pmatrix} + b\begin{pmatrix}-1\\1\\2\end{pmatrix} + c\begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

$$\Rightarrow \begin{cases} 0a - b + 3c = -1 \\ a + b + c = -2 \\ 4a + 2b + 2c = -2 \end{cases} \rightarrow \begin{cases} a = 1 \\ b = -2 \\ c = -1 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 0 & -1 & 3 & -1 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 2 & -2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 4 & 2 & 2 & -2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & -2 & -2 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -8 & 8 \end{array}\right)$$

$\Rightarrow x = 4 - 2v - w$

$\Rightarrow$ x is linear combination of $\{u, v, w\}$

$\left[\begin{array}{l} \text{Rank}(A) = 3 = \text{variable} \Rightarrow \text{unique solution} \\ \text{Rank}(A) < \text{variable} \Rightarrow \text{infinity many solution} \end{array}\right.$

2. Write v as a linear combination of u and w, if possible, where $u = (1,2), w = (1,-1)$

a. $v = (0,1)$ b. $v = (2,3)$ c. $v = (1,4)$ d. $(-5,1)$

$V = au + bw$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} a + b = 0 \\ 2a - b = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a = 1/3 \\ b = -1/3 \end{cases} \Rightarrow \text{unique solution}$$

$$\boxed{V = \frac{1}{3}u - \frac{1}{3}w}$$

② Span (Hệ sinh)

$(a) = \{a_1, a_2, \dots, a_m\} \subset R^n$, $x \in R^n$

If $\forall x \in R^n$ is linear combination of system (a)

$\exists x_1, x_2, \dots, x_m$, such that

$\forall x = x_1 a_1 + x_2 a_2 + \dots + x_m a_m$

$\Rightarrow \{a_1, a_2, \dots, a_m\}$ is Span of R^n

EX2: Determine whether the set S is span of R^2

$$S = \{(-1,2), (3,1), (2,1)\}$$

$$\boxed{x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -\alpha_1 + 3\alpha_2 + 2\alpha_3 = x_1 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = x_2 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} -1 & 3 & 2 & x_1 \\ 2 & 1 & 1 & x_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -3 & -2 & -x_1 \\ 2 & 1 & 1 & x_2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & -2 & -x_1 \\ 0 & 7 & 5 & x_2 + 2x_1 \end{array}\right)$$

Rank(A) = 2 < variable => infinity many solution

=> S is span of R^2.

③ Linearly independent and linearly dependent

(độc lập tt & phụ thuộc tt)

$(a) = \{a_1, a_2, \dots, a_m\} \in R^n$

If $\boxed{\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m = 0} \to$ (vector 0)

$\Leftrightarrow$ $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0 \to$ (number 0)

=> (a) : Linearly independent

If : $\exists \lambda_i \neq 0$, such that

$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_m a_m = 0$

$\Rightarrow$ (a) : Linearly dependent.

3. Determine whether the set S is linearly independent or linearly dependent in corresponding vector spaces.

a. $S = \{(-1,2),(3,1),(2,1)\}$

b. $S = \{\overset{a_1}{(-1,2,3)},\overset{a_2}{(1,3,5)}\}$

c. $S = \{(1,-2,2),(2,3,5),(3,1,7)\}$

d. $S = \{(-1,2,1),(2,4,0),(3,1,1)\}$

e. $S = \{(1,-2,2,1),(1,2,3,5),(-1,3,1,7)\}$

b) $\lambda_1 a_1 + \lambda_2 a_2 = 0$

$\Leftrightarrow \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} -\lambda_1 + \lambda_2 = 0 \\ 2\lambda_1 + 3\lambda_2 = 0 \\ 3\lambda_1 + 5\lambda_2 = 0 \end{cases}$

$A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 5 \\ 0 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow S$: Linearly independent

Rank(A) = 2 = variable

(a) $= \{a_1, a_2, \ldots, a_m\} \in R^n$

$$A = \begin{pmatrix} a_1 & a_2 & & a_m \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} \longrightarrow \text{Rank}(A)$$

(+) Rank (A) = variable $\Rightarrow$ linearly independent

(+) Rank (A) < variable $\Rightarrow$ linearly dependent.

Ex/ $S = \{(-1,-2,2), (2,3,5), (3,1,7)\}$

$$A = \begin{pmatrix} -1 & 2 & 3 \\ -2 & 3 & 1 \\ 2 & 5 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & -3 \\ -2 & 3 & 1 \\ 2 & 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -3 \\ 0 & 8 & 8 \\ 0 & 9 & 13 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$\Rightarrow$ Ran(A) = 3 = variable $\Rightarrow$ S: linearly independent

4. For which values of k is each set linearly independent in corresponding vector spaces.

a. $S = \{(-1,2,1),(k,4,0),(3,1,1)\}$

b. $S = \{(-1,k,1),(1,1,0),(2,-1,1)\}$

c. $S = \{(k,1,1),(1,k,1),(1,1,k)\}$

d. $S = \{(1,2,1,0),(-2,1,1,-1),(-1,3,2,k)\}$

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix}_{n \times n}$$

(+) $\det A \neq 0 \Rightarrow$ linearly independent

(+) $\det A = 0 \Rightarrow$ linearly dependent a/ $A = \begin{pmatrix} -1 & k & 3 \\ 2 & 4 & 1 \\ 1 & 0 & 1 \end{pmatrix}_{3\times 3}$

$$\det A = \begin{vmatrix} -1 & k & 3 \\ 2 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} \xrightarrow{C_1 - C_3} \begin{vmatrix} -4 & k & 3 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\,(-1)^{3+3} \begin{vmatrix} -4 & k \\ 1 & 4 \end{vmatrix} = -16 - k$$

$$\Rightarrow \quad \det A \neq 0 \iff -16 - k \neq 0$$

$$\Rightarrow \boxed{k = -16}$$

d/ $A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & -1 & k \end{pmatrix}_{4\times 3} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & 3 & 3 \\ 0 & -1 & k \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & k+1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & k+1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\iff \operatorname{rank}(A) = 3 \Rightarrow k+1 \neq 0 \Rightarrow k \neq -1$$