Ex: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}, \quad T\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} \quad \text{find } T\begin{pmatrix}5\\2\end{pmatrix} = ?$$

$$\begin{pmatrix}5\\2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-3\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 5 = a + b \\ 2 = a - 3b \end{cases} \Rightarrow \begin{cases} a = \frac{17}{4} \\ b = 3/4 \end{cases}$$

$$T\begin{pmatrix}5\\2\end{pmatrix} = T\left(a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-3\end{pmatrix}\right) \qquad T(au) = aT(u)$$

$$= \frac{17}{4}T\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{4}T\begin{pmatrix}1\\-3\end{pmatrix}$$

$$= \frac{17}{4}\begin{pmatrix}2\\3\end{pmatrix} + \frac{3}{4}\begin{pmatrix}4\\1\end{pmatrix} =$$

## Matrix Transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \Rightarrow A_{m\times n}$$

- $T_A(x) = Ax$ $\qquad x$: vector
- $T(x) = Ax$

·) Standard basis of $\mathbb{R}^n$ : $\{e_1, e_2 \dots e_n\}$

$\downarrow$ column

$$I_2 = \begin{array}{c} \begin{matrix} e_1 & e_2 \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \to e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , e_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

·) $A = [T(e_1)\ T(e_2)\ T(e_3) \dots T(e_n)]$

$\mathbb{R}^2$ : $A = [T(e_1) . T(e_2)]$

$\mathbb{R}^3$ : $A = [T(e_1) . T(e_2) . T(e_3)]$

EX: Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ , $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + 3y \end{pmatrix}$

a) Find the Matrix T ?

b) Find $T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = ?$

$A = [T(e_1)\ T(e_2)]$

$$T(e_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 - 0\\1 + 3.0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 - 1\\0 + 3.1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$$

$$\Rightarrow A = [T(e_1)\ T(e_2)] = \begin{pmatrix}1 & -1\\1 & 3\end{pmatrix}$$

Ex. $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 + 2x_2\\x_2 - x_3\\x_3\end{pmatrix}$

find $A$ = ?

$$A = [T(e_1)\ T(e_2)\ T(e_3)]$$

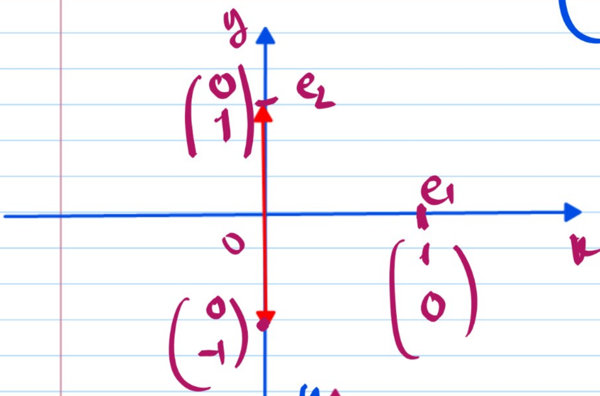
$$\left.\begin{aligned} T(e_1) &= T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\\ T(e_2) &= T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}\\ T(e_3) &= T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}\end{aligned}\right\} \Rightarrow A = \begin{pmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix}$$

$A: \mathbb{R}^2 \to \mathbb{R}^2$ : find $A$, reflection in the $x$ axis

(đối xứng qua $Ox$)

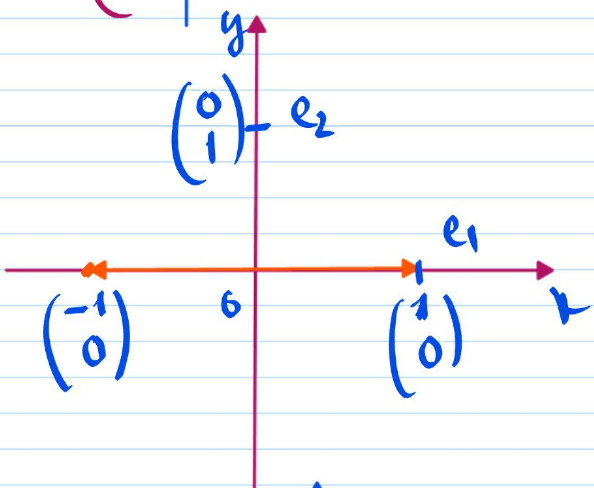
rotation through $\frac{\pi}{2}$

(phép quay)



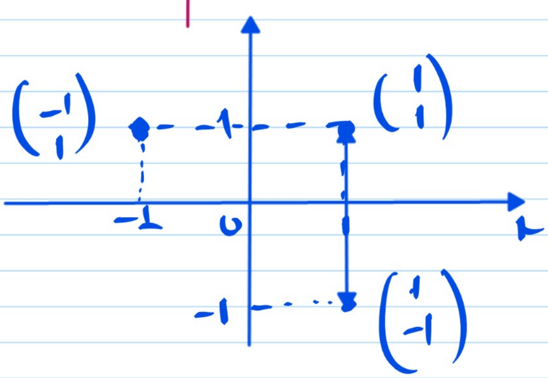

$R_{ox}$

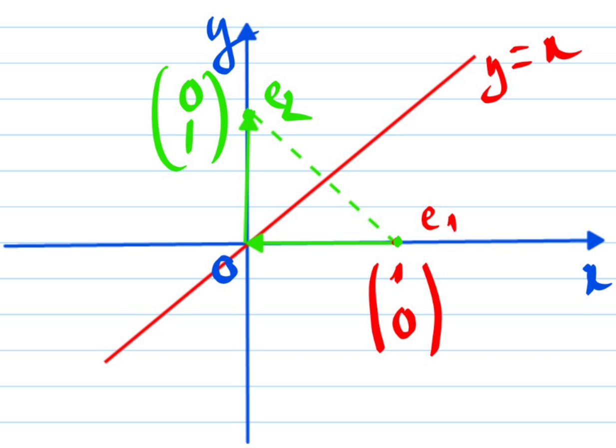
$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$




$R_{oy}$

$e_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

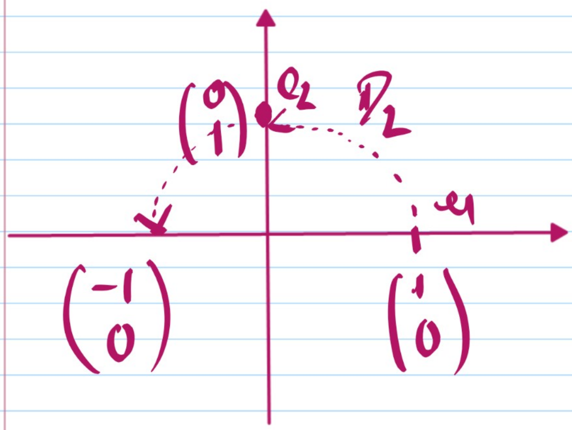



Reflection in the line $y = x$

$R_{y=x}$

$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

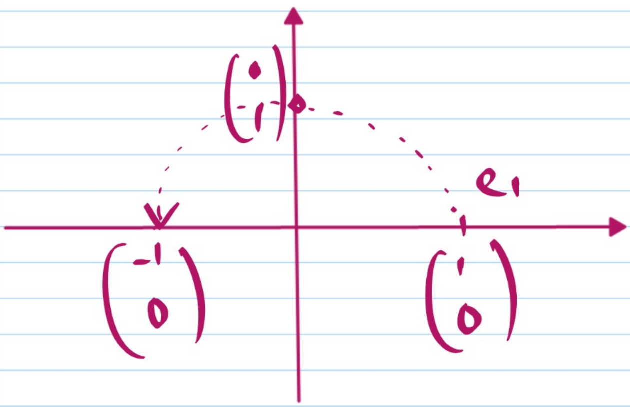



rotation through $\pi/2$

$Q_{\pi/2}$

$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

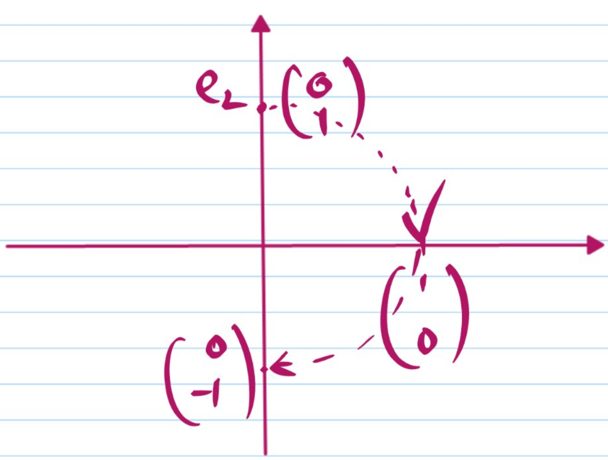
$e_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$




$Q_{\pi}$

$e_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$e_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$




$Q_{-\frac{\pi}{2}}$

$e_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Ex: Find the matrix of a Reflection in the x axis followed by rotation through $\frac{\pi}{2}$

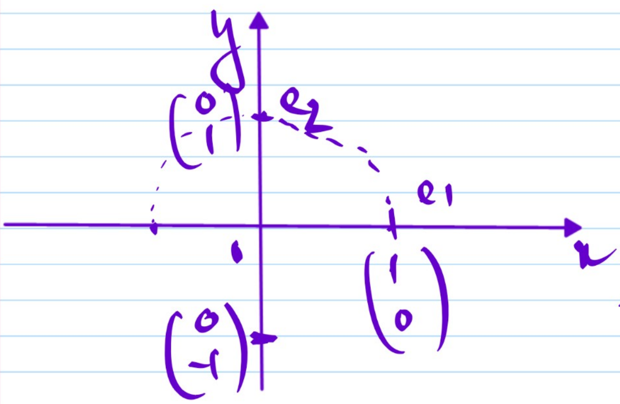
dưới lên trên

| | | | |
|---|---|---|---|
| transformation | $S$ | $T$ | $S \circ T$ |
| Matrix | $A$ | $\leftarrow B$ | $B.A$ |

$R_{Ox}$ $\qquad\qquad\qquad$ $Q_{\frac{\pi}{2}}$

$A \qquad\qquad\qquad B$

$$A = \left[T(e_1)\ T(e_2)\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$




$$\Longrightarrow \quad B.A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b) Reflection in the y axis, followed by rotation through $\frac{\pi}{2}$.

$$R_{oy} \qquad\qquad Q_{\pi/2}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow B.A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

c) Rotation through π followed by reflection in the *x* axis.

$$Q_{\pi} \qquad\qquad R_{ox}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow B.A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

d) Rotation through π/2 followed by reflection in the line *y* = *x*.

$$Q_{\pi/2} \qquad\qquad R_{y=x}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow B.A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\#\ P_m : \mathbb{R}^2 \to \mathbb{R}^2$

be projection on the line $y = mx$. Then $Pm$ is a linear transformation with matrix

$$\boxed{\frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}}$$

EX In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle

a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{1+2^2}\begin{pmatrix} 1 & 2 \\ 2 & 2^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & (m)^2 \end{pmatrix}$$

$\Rightarrow m = 2 \longrightarrow \boxed{y = 2x}$

b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x - y \\ y - x \end{pmatrix}$

$$= \frac{1}{1+1^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{1+(-1)^2}\begin{pmatrix} 1 & -1 \\ -1 & (-1)^2 \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

$$\Rightarrow m = -1 \rightarrow y = -x$$

c) $$T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3x + 4y \\ 4x + 3y \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{1+2^2}\begin{pmatrix} 1-2^2 & 2.2 \\ 2.2 & 2^2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

$$\Rightarrow m = 2 \rightarrow \boxed{y = 2x}$$