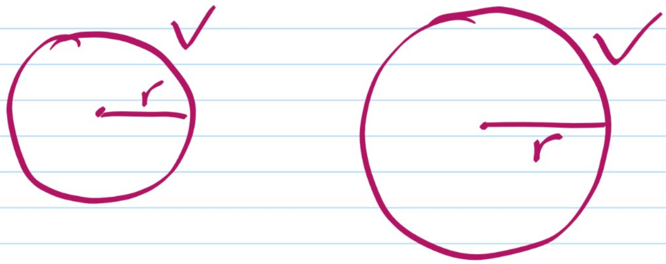
Related Rates



V, r : related

The rate of change in the volume V : $\frac{dV}{dt}$

" " r : $\frac{dr}{dt}$

$\Rightarrow \frac{dV}{dt}, \frac{dr}{dt} : \rightarrow$ Related rates.

Derivative $\frac{d?}{d?}$

EX: A spherical ballon is being filled with air at the constant rate of $2\ cm^3/sec$. How fast is the radius increasing when the radius is $3cm$?

Solution: $\frac{dV}{dt} = 2\ cm^3/sec$ $\qquad \frac{dr}{dt} = ?$

$\Rightarrow \frac{dr}{dt} = ?$ when $r = 3$ cm.

Formula $V_{cầu} = \frac{4}{3}\pi r^3$

$V(t) = \frac{4}{3}\pi r^3(t)$

$\Rightarrow \frac{d}{dt}(V(t)) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3(t)\right)$

$\frac{dV}{dt} = \frac{4}{3}\pi r^2 . 3 \frac{dr}{dt}$

$$\Rightarrow 2 = \frac{4}{3}\pi \cdot 3^2 \cdot 3 \, \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{18\pi} \text{ (cm/sec)}$$

b) What is the instantaneous rate of change of the radius when r = 6m?

Solution: $\frac{dv}{dt} = \frac{4}{3}\pi r^2 \cdot 3 \cdot \frac{dr}{dt}$

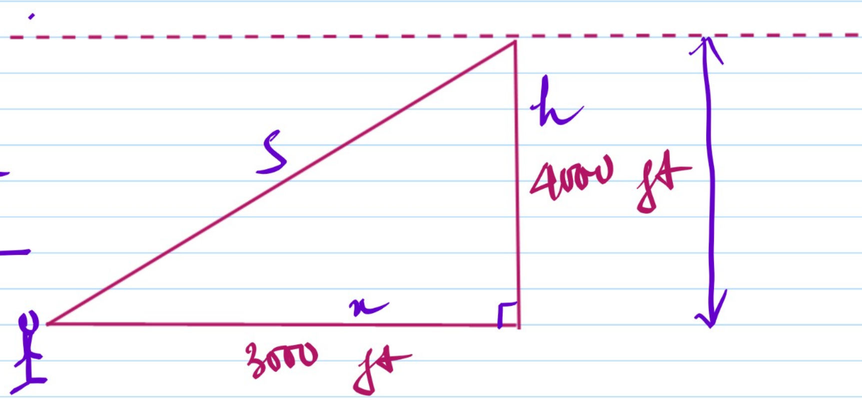
$$\Rightarrow \frac{dr}{dt} = 2 : \left(\frac{4}{3}\pi \cdot 6^2 \cdot 3\right) = \dots$$

EX:

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

horizontal

$\frac{dx}{dt} = 600$ ft/sec



Find $\frac{ds}{dt} = ?$

when x = 3000 ft.

Solution: $x^2 + \underset{\text{constant}}{h^2} = s^2 \Leftrightarrow x^2(t) + 4000^2 = s^2(t)$

$$\Leftrightarrow \frac{d}{dt}\left(x^2(t)\right) + 4000^2 = \frac{d}{dt}\left(s^2(t)\right)$$

$$\Leftrightarrow 2x \frac{dx}{dt} = 2s \frac{ds}{dt}$$

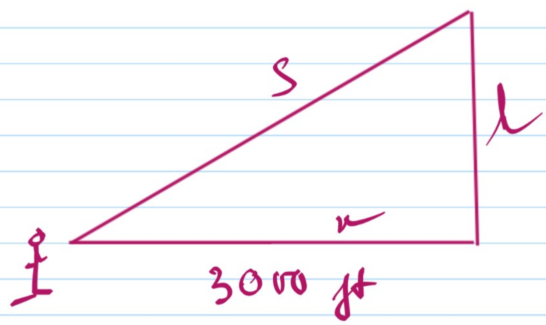
$$\Leftrightarrow 3000 \cdot 600 = \sqrt{3000^2 + 4000^2} \cdot \frac{ds}{dt}$$

$$\Rightarrow \frac{ds}{dt} = 360 \text{ ft/sec}$$

b>

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

$$\frac{ds}{dt} = 300 \text{ ft/sec}$$



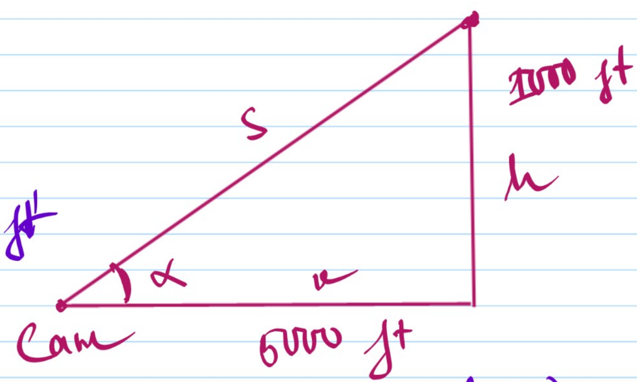
$$x^2(t) + 4000^2 = s^2(t)$$

$$\Leftrightarrow 2x \frac{dx}{dt} = 2s \frac{ds}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{5000 \cdot 300}{3000} = 500 \text{ ft/sec.}$$

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

$$\frac{dh}{dt} = 600 \text{ ft/sec.}$$



$$\frac{d\alpha}{dt} = ? \quad \text{when } h = 1000 \text{ ft}, \quad x = 5000 \text{ ft}$$

Solution: $\tan\alpha = \frac{h}{x} \Rightarrow \tan\alpha(t) = \frac{h(t)}{5000}$

$$\frac{d}{dt}\tan\alpha(t) = \frac{d}{dt}\left(\frac{h(t)}{5000}\right)$$

$$\Leftrightarrow \frac{1}{\cos^2\alpha(t)} \cdot \frac{d\alpha}{dt} = \frac{1}{5000} \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{d\alpha}{dt} = \frac{1}{5000} \cdot 600 \cdot \left(\frac{5000^2}{1000^2 + 5000^2}\right)$$

$$\Rightarrow \frac{d\alpha}{dt} = \frac{3}{26} \text{ rad/s}$$

b)

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Find $\frac{d\alpha}{dt}$ when $h = 2000$ ft

At that time, $\frac{dh}{dt} = 500$ ft/sec

$$\frac{1}{\cos^2\alpha(t)} \cdot \frac{d\alpha}{dt} = \frac{1}{5000} \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{d\alpha}{dt} = \left(\frac{1}{4000} \cdot 500\right) \cdot \left(\frac{4000^2}{2000^2 + 4000^2}\right)$$

$$= \frac{1}{10} \text{ rad/sec}$$

EX

For the following exercises, find the quantities for the given equation.

1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$

$$\Leftrightarrow \frac{d}{dt}\left(y(t)\right) = \frac{d}{dt}\left(x^2+3\right)(t)$$

$$\Leftrightarrow \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\Leftrightarrow \frac{dy}{dt} = 2 \cdot 1 \cdot 4 = 8$$

2) Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$

$$\frac{d}{dt}(y(t)) = \frac{d}{dt}((2x^2+1)(t))$$

$$\Leftrightarrow \frac{dy}{dt} = 4x\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-1}{4.-2} = \frac{1}{8}$$

5) Find $\frac{dz}{dt}$ at $(x;y)=(1;3)$

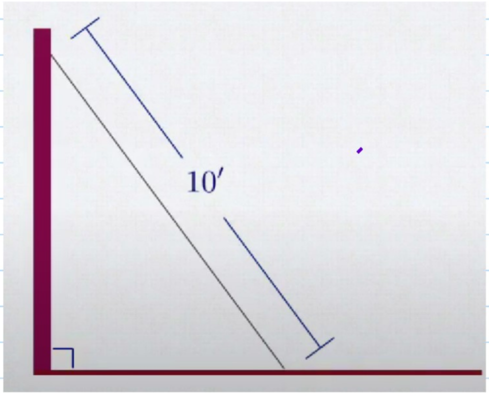
and $z^2 = x^2+y^2$ if $\frac{dx}{dt}=4$ and $\frac{dy}{dt}=3$

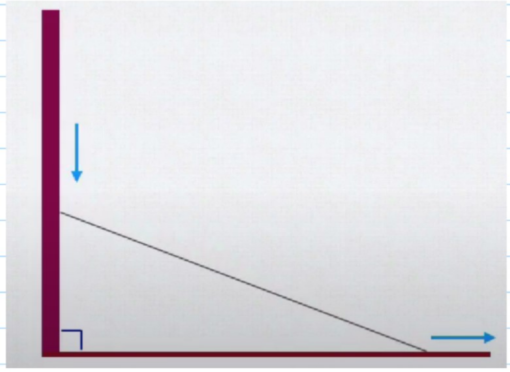
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

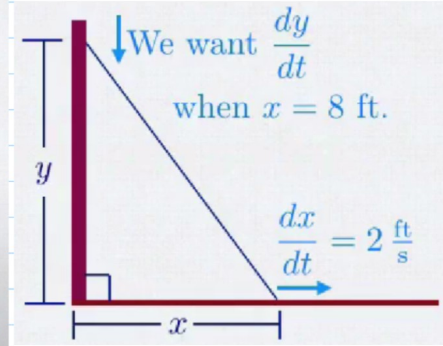
$$\Leftrightarrow \frac{dz}{dt} = \frac{2.1.4+2.3.3}{2.\sqrt{10}} = \frac{8+18}{2\sqrt{10}}$$

$$= \frac{13}{\sqrt{10}}$$

A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?







$$x^2 + y^2 = 10^2 = 100$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100)$$

$$\Leftrightarrow \quad 2x\frac{dx}{dt} + \frac{2y\,dy}{dt} = 0$$

$$\Leftrightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

$$= \frac{-8 \cdot 2}{y} = \frac{-16}{6} = \frac{-8}{3} \text{ ft/s}$$

$$y = \sqrt{10^2 - 8^2} = 6$$

Linear Approximations
(Sấp xỉ tuyến tính)

If x near a: $f(x) \approx f(a) + f'(a)(x-a)$

$\underline{L}(x) = f(a) + f'(a)(x-a)$

linear approximation

or

tangent line approximation

$(f(x) \approx L(x))$

EX: Find the linear approximation of

$f(x) = \sqrt{x}$ at $x = 9$

Solution: $L(x) = f(9) + f'(9)(x-9)$

$= 3 + \frac{1}{6}(x-9)$

$= \frac{1}{6}x + \frac{3}{2}$

b) Use the approximation to the estimate $\sqrt{9.1}$

$L(9.1) = f(9) + f'(9)(9.1-9)$

≈ 3.0167

EX: Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x=8$ and use it to approximation $\sqrt[3]{8.1}$.

$$L(x) = f(8) + f'(8)(x-8)$$

$$= 2 + \frac{1}{12}(x-8)$$

$$L(\sqrt[3]{8.1}) = 2 + \frac{1}{12}(\sqrt[3]{8.1} - 8) \approx$$

EX3: a) Find the linear approximation of $f(x) = \sin x$ at $x = \pi/3$ and use it to approximation $\sin(62^\circ)$

$$\Rightarrow L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

b) $f(x) = \cos x$ at $x = \frac{\pi}{2}$

c) $f(x) = (1+x)^n$ at $x = 0$

$$\Rightarrow L(x) = 1 + n(x-0) = 1 + nx$$

d) $f(x) = (1+x)^4$ at $x = 0$

II. Differentials (vi phân)

dy, dx : differentials

$y = f(x) \Rightarrow \dfrac{dy}{dx} = f'(x)$

$\Rightarrow dy = f'(x)\, \underline{dx}$

EX: For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$

a) $y = x^2 + 2x$

Solution: $f'(x) = (2x + 2)\, dx$

When $x = 3$, $dx = 0,1 \Rightarrow f'(x) = 0,8$

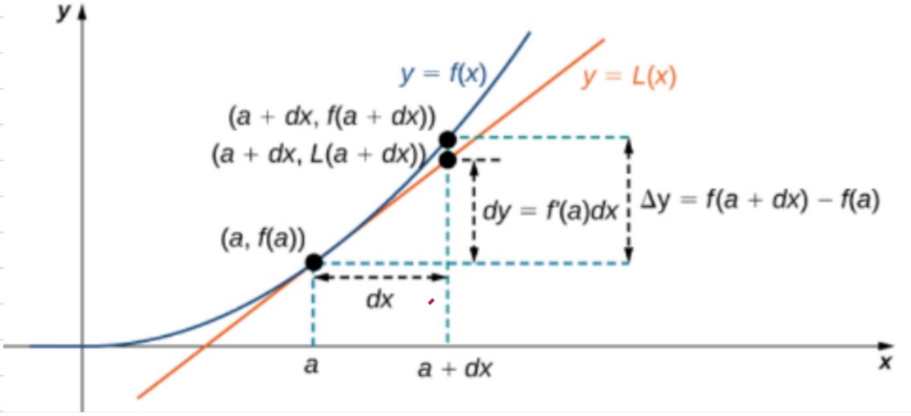
b) $y = \cos x$

$y'(x) = -\sin x\, dx$

When $x = 3$, $dx = 0,1 \Rightarrow f'(x) = -0,1 \sin(3)$

EX2: for $y = e^{x^2}$, find dy

$f'(x) = 2x \cdot e^{x^2} dx$



The differential $dy = f'(a)dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

$$\Delta y = f(a+dx) - f(a)$$

EX: Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0{,}1$

Solution: $\Delta y = f(3 + 0{,}1) - f(3)$

$= [3{,}1]^2 + 2 \cdot 3{,}1 - [(3)^2 + 2 \cdot 3]$

$= 0{,}81$

$\therefore\ dy = f'(3)\, dx$

$= (2 \cdot 3 + 2)\, 0{,}1 = 0{,}8$

Ex 2: For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0{,}2$?