

# Linear Algebra

## Chapter 1: Systems of linear equations.

### 1/ Solutions and elementary operations.

(Ngo v沦为.php bñi đor & cip')

\* Linear equations:  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

The diagram shows the equation  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ . Red arrows point from the coefficients  $a_1, a_2, \dots, a_n$  to the label "Coefficients". Blue arrows point from the variables  $x_1, x_2, \dots, x_n$  to the label "Variables (unknown)" with a brace underneath.

Solutions:  $s_1, s_2, \dots, s_n \Rightarrow a_1s_1 + a_2s_2 + \dots + a_ns_n = b$

- No Solution (VS)
- Unique solution (No dung what')
- infinite family of solutions (USN)

Ex:

$$\begin{cases} x+2y = 3 \\ x+2y = -1 \end{cases}$$

[No Solution] [inconsistent (10° fusing thick)]

Ex:

$$\begin{cases} x+y-2=t \\ x+y+t=3 \end{cases} \rightarrow \begin{cases} x=1-y+t \\ 1-y+t+y+t=3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=2-y \\ t=z \end{cases} \Leftrightarrow \begin{cases} x=2-t \\ y=t \\ z=t \end{cases} \quad (t \in \mathbb{R})$$

$\Rightarrow$  Solution  $(2-t, t, 1) \in$  general solution

$t$  is parameter ( $t$  is arbitrary)

$t$ : then  $\vec{x}^*$  (bat b)

Matrix:

$$\left\{ \begin{array}{l} 3x_1 + 2x_2 - x_3 + x_4 = -1 \\ 2x_1 - x_3 + 2x_4 = 0 \quad (3 \text{ equations}, \\ 3x_1 + x_2 + 2x_3 + 5x_4 = 2 \quad 4 \text{ variables}) \end{array} \right.$$

$$\left( \begin{array}{cccc|c} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array} \right)$$

Coefficient Matrix

Constant Matrix

augmented Matrix

# Elementary Operations (phép biến đổi sơ cấp)

- ① Interchange two equation
- ② Multiply one equation by a nonzero number
- ③ Add a multiple of one equation to a different equation.

Ex: 
$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow{\text{Interchange}} \begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases}$$

$$\left( \begin{array}{cc|c} 2 & 3 & 5 \\ 1 & -2 & -1 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_2} \left( \begin{array}{cc|c} 1 & -2 & -1 \\ 2 & 3 & 5 \end{array} \right)$$

Ex 
$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow{\text{Multiplied by } (-5)} \begin{cases} 2x + 3y = 5 \\ -5x + 10y = 5 \end{cases}$$

$$\begin{matrix} r_1 \rightarrow (2 & 3 & 5) \\ r_2 \rightarrow (1 & -2 & -1) \end{matrix} \xrightarrow{\begin{matrix} r_1 = 3r_1 \\ r_2 = -5r_2 \end{matrix}} \left( \begin{array}{cc|c} 6 & 9 & 15 \\ -5 & 10 & 5 \end{array} \right)$$

Ex3: 
$$\begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases} \xrightarrow{\begin{matrix} \text{add a multiple} \\ r_2 = r_2 - 2r_1 \end{matrix}} \begin{cases} x - 2y = -1 \\ 0x + 7y = 7 \end{cases}$$

$\Leftrightarrow \begin{cases} x - 2y = -1 \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$

$$\left( \begin{array}{cc|c} 4 & -2 & -1 \\ 2 & 3 & 5 \end{array} \right) \xrightarrow{r_2=r_2-2r_1} \left( \begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 7 & 7 \end{array} \right)$$

$$\xrightarrow{r_2=\frac{1}{7}r_2} \left( \begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

## Gaussian Algorithm

$$\left( \begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right) \longrightarrow \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & \cdot \\ 0 & 1 & 2 & 2 & \cdot \\ 0 & 0 & 1 & 1 & \cdot \end{array} \right)$$

$\longrightarrow \left\{ \begin{array}{l} x + y + z = c \\ y + 2z = b \\ z = a \end{array} \right.$

Ex: find all solution

$$\begin{cases} 3x + 4y + z = 1 \\ 2x + 3y = 0 \\ 4x + 3y - z = -1 \end{cases}$$

$$\left( \begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -1 \end{array} \right) \xrightarrow{r_1=r_1-r_2} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -1 \end{array} \right)$$

$$\begin{array}{l}
 r_2 = r_2 - 2r_1 \\
 r_3 = r_3 - 4r_1
 \end{array}
 \rightarrow
 \left( \begin{array}{ccc|c}
 1 & 1 & 1 & 1 \\
 0 & 1 & -2 & -2 \\
 0 & -1 & -5 & -5
 \end{array} \right)
 \rightarrow
 \left( \begin{array}{ccc|c}
 1 & 1 & 1 & 1 \\
 0 & 1 & -2 & -2 \\
 0 & 0 & -7 & +7
 \end{array} \right)$$

$$\rightarrow
 \left( \begin{array}{ccc|c}
 1 & 1 & 1 & 1 \\
 0 & 1 & -2 & -2 \\
 0 & 0 & 1 & 1
 \end{array} \right)$$

$$r_3 = -\frac{1}{7}r_3$$

$$\rightarrow \begin{cases} x + y + z = 1 \\ y - 2z = -2 \\ z = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases}$$

→ Solution  $(0, 0, 1)$

EX:

$$\begin{cases} 3x + y - 4z = 1 \\ x + 10z = 5 \\ 4x + y + 6z = 1 \end{cases}$$

$$\left( \begin{array}{ccc|c}
 3 & 1 & -4 & 1 \\
 1 & 0 & 10 & 5 \\
 4 & 1 & 6 & 1
 \end{array} \right)
 \xrightarrow{r_1 \leftrightarrow r_2}
 \left( \begin{array}{ccc|c}
 1 & 0 & 10 & 5 \\
 3 & 1 & -4 & 1 \\
 4 & 1 & 6 & 1
 \end{array} \right)$$

$$\begin{array}{l}
 r_2 = r_2 - 3r_1 \\
 r_3 = r_3 - 4r_1
 \end{array}
 \rightarrow
 \left( \begin{array}{ccc|c}
 1 & 0 & 10 & 5 \\
 0 & 1 & -34 & -14 \\
 0 & 1 & -34 & -19
 \end{array} \right)
 \rightarrow
 \left( \begin{array}{ccc|c}
 1 & 0 & 10 & 5 \\
 0 & 1 & -34 & -14 \\
 0 & 0 & 0 & -5
 \end{array} \right)$$

$$\Rightarrow \left\{ \begin{array}{l} x + 10z = 5 \\ y - 34z = -14 \\ \underline{0 = -5} \end{array} \right.$$

$\rightarrow$  No Solution.