Rank of Matrix.

* Def: Column and Row space of a Matrix.

+ The column space, Col A, of A is the subspace of $\mathbb{R}^m$ spanned by the column of A.

+ The Row space, Row A, of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

* Lemma 5.4.1: Let A and B denote $m \times n$ Matrix

+ If $A \to B$ by elementary row operations, then row A = row B.

+ If $A \to B$ by elementary column operations, then Col A = Col B.

* Lemma 5.4.2: If A is a Row-echelon Matrix, then

1. The nonzero rows of A are a basis of row A.
2. The column of A containing leading ones are a basis of Col A. chứa
3. $A: m \times n \Rightarrow \text{Rank } A \leq \text{Min}\{m, n\}$

EX: $A: 4 \times 8 \Rightarrow \text{Rank } A \leq \text{Min}\{4, 8\}$

$\Rightarrow \text{Rank } A \leq 4.$

* Theorem : $A: m \times n$, Rank $A = r$

(+) Basis of null A : $\{x_1, x_2, \dots, x_{n-r}\}$

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- Def : null $A = \{x \in R^n \mid Ax = 0\}$

Solution

(+) Im A = Col A

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- Def : Im $A = \{Ax \mid x \in R^n\}$

EX 1: Find null $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 - x_2 = 0 \\ 5x_2 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = t \\ x_3 = -5t \end{cases}$$

$$\Rightarrow \text{null } A = \begin{pmatrix} t \\ t \\ -5t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

Find Basis of null $A = \left\{ \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} \right\}$

EX2:

$$A = \begin{array}{cccc} c_1 & & c_3 & \\ \end{array}$$
$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix}_{3\times 4}$$ find basis of null A, Im A and dimension ?

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{pmatrix} \to \begin{array}{cccc} c_1 & & c_3 & \\ \end{array} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to r(A) = 2$$

leading one

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2s + t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \end{cases}$$

$$= \begin{pmatrix} 2s \\ s \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ -2t \\ t \end{pmatrix} = s\underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{X_1} + t\underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}}_{X_2}$$

$\Rightarrow$ Basis of null $A = \{(2,1,0,0), (1,0,-2,1)\}$

$$\boxed{\text{Basis of Im } A = \text{Basis of Col } A}$$

Basis of Im $A = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

* $A : \overset{3\times 4}{m \times n}$, Rank $A = \overset{2}{r}$

$\Rightarrow$ $\boxed{\dim \text{null } A = n - r}$

$\Rightarrow$ dim null $A = 4 - 2 = 2$

EX: Find the dimension of null A, B.

$$A = \begin{pmatrix} 1 & 2 & -9 \\ 2 & 8 & -38 \\ 5 & 14 & -65 \end{pmatrix}_{\substack{m\times n \\ 3\times 3}}, \quad B = \begin{pmatrix} 1 & -2 & 3 & -3 & -1 \\ -2 & 5 & -5 & 4 & -4 \\ -1 & 3 & -2 & 1 & -5 \end{pmatrix}_{3\times 5}$$

$$A = \begin{pmatrix} 1 & 2 & -9 \\ 0 & 4 & -20 \\ 0 & 4 & -20 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -9 \\ 0 & 4 & -20 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow r(A) = 2$

$\Rightarrow$ dim null $A = 3 - 2 = 1$

$$B = \begin{pmatrix} 1 & -2 & 3 & -3 & 1 \\ 0 & 1 & 1 & -2 & -6 \\ 0 & 1 & 1 & -2 & -6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 3 & -3 & 1 \\ 0 & 1 & 1 & -2 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow r(B) = 2$

$\rightarrow$ dim null $B = 5 - 2 = 3$

EX: Compute the rank of $A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$

and find bases for row A and col A.

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Rank $A = 2$

Basis of row $A = \{(1,2,2,-1),(0,0,1,-3)\}$

Basis of col $A = \left\{\begin{pmatrix}1\\3\\1\end{pmatrix},\begin{pmatrix}2\\5\\1\end{pmatrix}\right\}$

EX: Find bases of col A, row A and $\dim(\text{row} A)$, $\dim(\text{col} A)$?

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & 0 & 5 \\ -2 & -3 & 3 & -4 \\ 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

$$\boxed{\dim(\text{row} A) = \dim(\text{col} A) = \text{rank} A}$$

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Rank} A = 4$$

Basis of row $A = \{r_1, r_2, r_3, r_4\}$

" " col $A = \{c_1, c_2, c_3, c_4\}$

$\Rightarrow$ $\dim(\text{row } A) = \dim(\text{col } A) = 4$