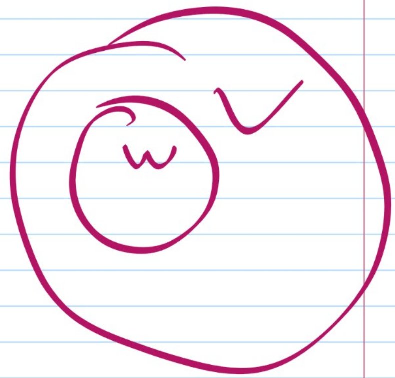
Subspace of $\mathbb{R}^n$

W is call a Subspace of $\mathbb{R}^n$ if:



$$\iff \begin{cases} \emptyset \neq W \subset V \; ✓ \\ u+v \in W, \; \forall u, v \in W \; ✓ \\ k.u \in W, \; \forall u \in W, \; \forall k \in V \end{cases}$$

EX1: $U = \{(a, a, 0) \mid a \in \mathbb{R}\}$

① the zero vector of $\mathbb{R}^3$ $(0,0,0) \in U \neq \emptyset$

② $(a,a,0) \in U$, $(b,b,0) \in U$

$(a,a,0) + (b,b,0) = (a+b, a+b, 0) \in U$

③ If $(a,a,0) \in U$ and $k \in \mathbb{R}$, then $k(a,a,0)$
$= (ka, ka, 0) \in U$

U is a Subspace $\mathbb{R}^3$.

EX2: $U = \{(a, |a|, 0) \mid a \in \mathbb{R}\}$

② $(-1, |-1|, 0)$, $(1, |1|, 0) \in U$

$(-1, |-1|, 0) + (1, |1|, 0) = (0, 2, 0) \notin U$

$\Rightarrow U$ is not a Subspace

EX3: which of the following are subspace of $\mathbb{R}^2$?

$W_1 = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$

① $W_1 \neq \emptyset$

$u = (a, a),\ v = (b, b) \in W_1,\ \forall k \in R$

② $u + v = (a+b, a+b) \in W_1$

③ $ku = (ka, ka) \in W_1$

W_1 is a Subspace R^2.

EX4: $W_2 = \{(x, y) \in R^2 \mid 3x - y = 5\}$

$W_2 \neq \emptyset$

$u = (x_1, y_1),\ v = (x_2, y_2) \in W_2$

$\Rightarrow 3x_1 - y_1 = 5 \quad \Big| \quad \Rightarrow 3x_2 - y_2 = 5$

$u + v = (3x_1 + 3x_2 - y_1 - y_2 = 10)$

$= [3(x_1 + x_2) - (y_1 + y_2) = 10]$

$\Rightarrow$ $u = (0, -5) \in W_2$

$v = (1, -2) \in W_2$

$u + v = (1, -7) \notin W_2$

$\Rightarrow W_2$ is not a subspace of R^2.

* Basis and dimension

① Def of basis: Suppose U is a Subspace of R^n

a set $\{x_1, x_2, \dots, x_k\}$ is basis of U if

$$\begin{cases} U = \text{Span}\{x_1, x_2, \dots, x_k\} \\ \{x_1, x_2, \dots x_k\} \text{ is Linear independent} \end{cases}$$

(+) Let U be a Subspace of R^n and $B = \{X_1, X_2, \dots, X_m\} \subset U$, where $\underbrace{\dim U = m}$

•) $B = \{X_1, X_2, \dots X_m\} \subset R^n$

$$B \text{ is Basis} \Leftrightarrow \begin{cases} B \text{ is linear independent} \\ m = n \end{cases}$$

EX: find all values of m such that the set S is basis of R^3.

a. $S = \{(1,2,1), (m,1,0), (-2,1,1)\}$

$$|A| = \begin{vmatrix} 1 & m & -2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \longrightarrow \begin{vmatrix} 3 & m & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot (-1)^{3+3} \begin{vmatrix} 3 & m \\ 1 & 1 \end{vmatrix} = 3 - m$$

$$|A| \neq 0 \Leftrightarrow \boxed{m \neq 3}$$

b) $S = \{(-1, m, 1), (1, 1, 0), (m, -1, -1)\}$

$$A = \begin{vmatrix} -1 & 1 & m \\ m & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} m-1 & 1 & m \\ m-1 & 1 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= -1(-1)^{3+3} \begin{vmatrix} m-1 & 1 \\ m-1 & 1 \end{vmatrix} = -(m-1-m+1) = 0$$

$$m = \emptyset.$$

$$= 1(-1)^{3+1}\begin{vmatrix} 1 & m \\ 1 & -1 \end{vmatrix} + (-1)(-1)^{3+3}\begin{vmatrix} -1 & 1 \\ m & 1 \end{vmatrix}$$

$$= -1 - m \quad -1(-1-m)$$

$$= -1 - m + 1 + m = 0$$

6. Find a basis for and the dimension of the subspace U

a. $U = \{(2s-t, s, s+t) \mid s,t \in R\}$

b. $U = \{(s-t, s, t, s+t) \mid s,t \in R\}$

c. $U = \{(0, t, -t) \mid t \in R\}$

d. $U = \{(x,y,z) \mid x+y+z=0\}$

e. $U = \{(x,y,z) \mid x+y+z=0, x-y=0\}$

f. $U = span\{(1,2,3),(2,3,4),(3,5,7)\}$

A: $\{(2,1,1),(-1,0,1)\}$

a. $$U = \begin{pmatrix} 2s-t \\ s \\ s+t \end{pmatrix} = \begin{pmatrix} 2s \\ s \\ s \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$= s\underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{a_1} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{a_2}$$

$\forall u = s\underline{a_1} + t\underline{a_2} \rightarrow$ linear combination

$\Rightarrow \{a_1, a_2\}$ is span (1)

$\cdot)$ $$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & -3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$ $\Rightarrow$ rank $= 2 =$ variable

$\Rightarrow$ linear independent (2)

(1), (2) $\Rightarrow \{a_1, a_2\}$ is a basis

$\dim U = 2$

d. $U = \{(x,y,z) \mid x+y+z = 0\} \rightarrow$

$x = -a-b$
$y = b$
$z = a$

$$\Leftrightarrow u = \begin{pmatrix} -a-b \\ b \\ a \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} -b \\ b \\ 0 \end{pmatrix}$$

$$= a\underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{a_1} + b\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{a_2}$$

$\forall u = a a_1 + b a_2 \rightarrow \{a_1, a_2\}$ is span

· $\begin{pmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow r(A) = 2$

$\Rightarrow \{a_1, a_2\}$ is a basis

$\dim U = 2$.

9. Find the dimension of the subspace

$U = span\{\underset{a_1}{(-2,\ 0,\ 3)}, \underset{a_2}{(1,\ 2,\ -1)}, \boxed{\cancel{(-2,\ 8,\ 5), (-1,\ 2,\ 2)}}\}$

$$A = \begin{pmatrix} -2 & 1 & -2 & -1 \\ 0 & 2 & 8 & 2 \\ 3 & -1 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & -2 & -1 \\ 0 & 2 & 8 & 2 \\ 0 & 1 & 4 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -2 & 1 & -2 & -1 \\ 0 & 2 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rank } A = 2.$$

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Some important theorems

- ***Theorem 3***. Let U≠0 be a subspace of R^n. Then:
  ❶ U has a basis and dimU⍰n.
  ❷ Any independent set of U can be enlarged (by adding vectors) to a basis of U.
  ❸ If B spans U, then B can be cut down (by deleting vectors) to a basis of U.

  Ex1. Let U=span{(1,1,1), (1,0,1), (1,-2,1)} be a subspace of R^3. This means, B={(1,1,1), (1,0,1), (1,-2,1)} spans U.
  ❶ ⍰U has a basis and dimU⍰3,
  a_1 a_2
  ❸ ⍰B can be cut down to a basis of U: {(1,0,1), (1,1,1)} is a basis of U, dimU=2
  ❷ ⍰construct a basis for U: ⍰⍰{(1,0,1)} ⍰{(1,0,1), (1,1,1)}.

1	1	1
1	0	-2
1	1	1

1	1	1
0	-1	-3
0	0	0

1	1	1
0	1	3
0	0	0

- ***Theorem 4***. Let U be a subspace of R^n and B={X_1,X_2,…,X_m}⍰U, where dimU=m. Then B is independent if and only if B spans U.
- ***Theorem 5***. Let U⍰V be subspaces of R^n. Then:
  ❶ dimU ⍰ dimV.
  ❷ If dimU=dimV, then U=V.

Choose $(a) = (a_1, a_2)$

is linear independent

$\Rightarrow \{a_1, a_2\}$ is a basis

$\dim U = 2$