4.3 More on the Cross Product

(T) Theorem

If $u = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, $v = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $w = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ then

$$u \cdot (v \times w) = \det \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}$$

(T) Theorem : Lagrange Identity

If u and v are two vectors in $\mathbb{R}^3$, then

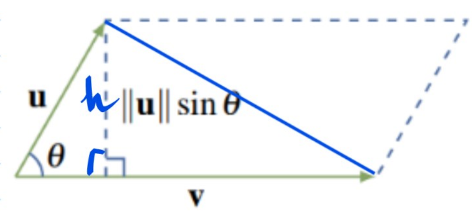
$$\|u + v\|^2 = \|u\|^2 . \|v\|^2 - (u \cdot v)^2$$

(T) Theorem

If u and v are two nonzero vectors and θ is the angle between u and v, then

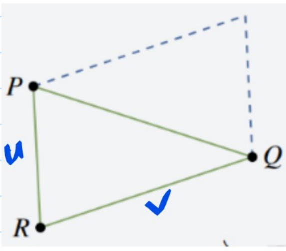
$\|u \times v\| = \|u\| . \|v\| \sin\theta$ = the area of the parallelogram

$$S = \|v\| . \|h\| = \|u\| . \|v\| . \sin\theta$$



(T) Area of triangle $= \frac{1}{2} \|u \times v\|$

EX: Find the area of the triangle with vertices $P(2,1,0)$, $Q(3,-1,1)$ and $R(1,0,1)$



$$\vec{RP} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{RQ} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

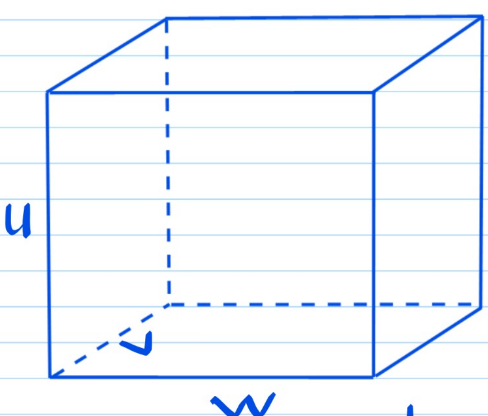
$$A = \frac{1}{2} \| \underbrace{\vec{RP} \times \vec{RQ}}_{= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \| = \frac{1}{2}\sqrt{1^2+2^2+3^2} = \frac{1}{2}\sqrt{14}$$

Area $= 0$

① Theorem

The volume of the parallelepiped determined by three vectors w, u, and v is given by

$$\boxed{|w \cdot (u \times v)|}$$

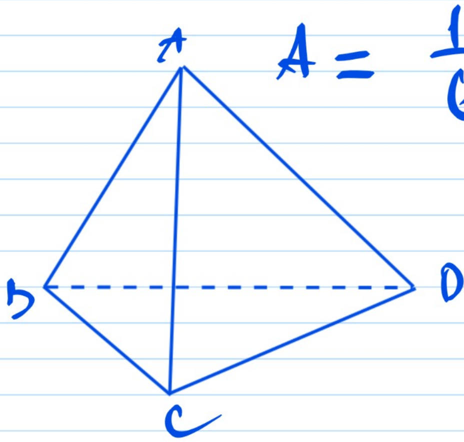


EX: Find the volume of the parallelepiped determined by the vectors

$$w = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$|w \cdot (u \times v)| = \left| \det \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \right| = 3$$

(+) Area of the [tetrahedron ABCD / pyramid A.BCD]



$$A = \frac{1}{6} \left| u . (v \times w) \right|$$

$$A = \frac{1}{6} \left| \vec{AB} . (\vec{AC} \times \vec{AD}) \right|$$

Exercise 4.3.5 Find the volume of the parallelepiped determined by **w**, **u**, and **v** when:

a. $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

b. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

EX: Let area of the parallelogram by u, v is 10

Find the area the parallelogram by 2u, 3v is ?

$$\|u \times v\| = 10$$

$$\|2u \times 3v\| = 6 \|u \times v\| = 6 . 10 = 60$$

4.4 Linear Operators on $\mathbb{R}^3$

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

is called a ***linear operator*** *on* $\mathbb{R}^n$.

(+) Distance preserving: $\|T(v) - T(w)\| = \|v - w\|$ $\forall u, w \in \mathbb{R}^3$

(bảo toàn k/cách)

Rotations   Reflections

(f) No distance preserving: Projections

Reflections and Projections

(*) Let L be the line in $\mathbb{R}^2$ passing through the origin and has slope m.

(1) The Matrix of Rotation through θ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(2) The Matrix of projection is

$$\frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

(3) The Matrix of reflection in L is

$$\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

(*) $\mathbb{R}^3$

(1) The Matrix of projection on L is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

② The Matrix of reflection in L is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix}$$

* Let M be the plane in $\mathbb{R}^3$

① The Matrix of projection on M is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

② The Matrix of reflection in M is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}$$

Exercise 4.4.4

a. Find the rotation of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{4}$.

b. Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

c) $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ about the x-axis through $\theta = \pi/4$

$$\begin{array}{cccc} & i & j & k \\ A = & \multicolumn{3}{c}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \end{array}$$

$$A = \begin{pmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

b) $B = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise 4.4.3 In each case solve the problem by finding the matrix of the operator.

a. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x - 5y + 2z = 0$.

b. Find the projection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ on the plane with equation $2x - y + 4z = 0$.

c. Find the reflection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$.

d. Find the reflection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ in the plane with equation $2x + y - 5z = 0$.

e. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

f. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

g. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.

h. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

(1) The Matrix of projection on M is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

$$B = \frac{1}{2^2+(-1)^2+4^2}\begin{pmatrix} 17 & 2 & -8 \\ 2 & 20 & 4 \\ -8 & 4 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$