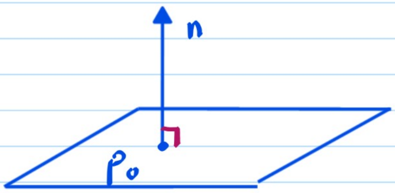
Planes



The plane through $P_0(x_0, y_0, z_0)$ with normal $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$

$$\left[\begin{array}{l} a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\ ax + by + cz + d = 0 \end{array}\right.$$

EX: Find an equation of the plane through $\underline{P_0}(1, -1, 3)$ with $n = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ as normal.

•) $3(x-1) - (y+1) + 2(z-3) = 0$

$\Leftrightarrow$ $3x - y + 2z - 10 = 0$

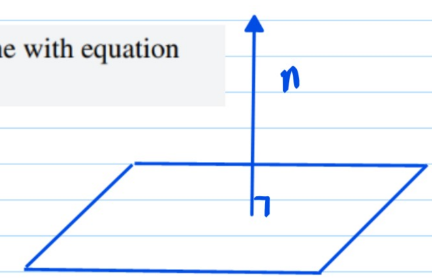
•) $3x - y + 2z - 10 = 0$

EX2: Find an equation of the plane through $\underline{P_0}(3, -1, 2)$ that is parallel to the plane with equation $2x - 3y = 6$.

$n = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

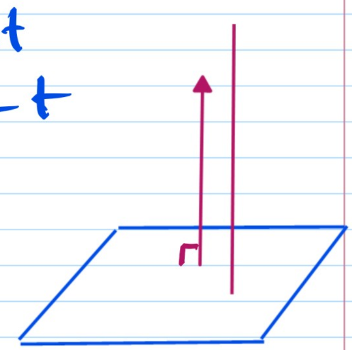
$2(x-3) - 3(y+1) = 0$

$\Leftrightarrow$ $2x - 3y - 9 = 0$



EX3: ... Through $P_0(1, -1, 3)$ perpendicular to the line

$$\begin{cases} x = 2 + t \\ y = 1 + 3t \\ z = -2 - t \end{cases}$$



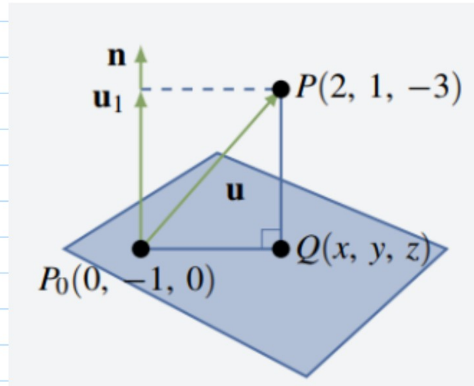
$d = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \Rightarrow n = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$1(x-1) + 3(y+1) - (z-3) = 0$

$$\Leftrightarrow \quad x + 3y - z + 5 = 0$$

Find the shortest distance from the point $P(2,\ 1,\ -3)$ to the plane with equation $3x - y + 4z = 1$. Also find the point Q on this plane closest to P.

$$n = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad u = \overrightarrow{P_0P} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$



$$\Rightarrow u_1 = \operatorname{proj}_n u = \frac{u \cdot n}{\|n\|^2} \cdot n$$

$$= \frac{6 - 2 - 12}{3^2 + (-1)^2 + 4^2} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \frac{-4}{13} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$\Rightarrow \|PQ\| = \|u_1\| = \sqrt{\left(\frac{-12}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{-16}{13}\right)^2} = \frac{4\sqrt{26}}{13} \rightarrow \text{Shortest distance}$$

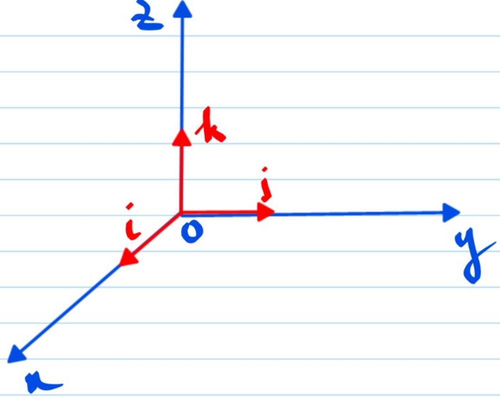
$$\overrightarrow{QP} = u_1 \Leftrightarrow \begin{cases} 2 - x = \frac{-12}{13} \\ 1 - y = 4/13 \\ -3 - z = -16/13 \end{cases} \Rightarrow \begin{cases} x = 38/13 \\ y = 9/13 \\ z = -23/13 \end{cases}$$

$$\Rightarrow Q = \left(38/13,\ 9/13,\ -23/13\right)$$

⊕ The cross product. (Tích có hướng).

If $V_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ are two vectors, then

$$V_1 \times V_2 = \det \begin{vmatrix} i & x_1 & x_2 \\ j & y_1 & y_2 \\ k & z_1 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} i - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$



$\|i\| = \|j\| = \|k\| = 1$

$i = (1, 0, 0)^T$

$j = (0, 1, 0)^T$

$k = (0, 0, 1)^T$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xi + yi + zk$$

$$\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 2i - 3j + 5k$$

Given vector $v_1 = \begin{pmatrix} \cancel{x_1} \\ y_1 \\ z_1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \cancel{x_2} \\ y_2 \\ z_2 \end{pmatrix}$

$$\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}$$

Cross product : $v_1 \times v_2 = \begin{pmatrix} y_1 z_1 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$

EX : $v = \begin{pmatrix} \cancel{2} \\ -1 \\ 4 \end{pmatrix}$ $w = \begin{pmatrix} \cancel{1} \\ 3 \\ 7 \end{pmatrix}$

$$\begin{array}{cc} 2 & 1 \\ -1 & 3 \end{array}$$

$$v \times w = \begin{pmatrix} -19 \\ -10 \\ 7 \end{pmatrix}$$

b) $u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$1 \quad\quad 3$

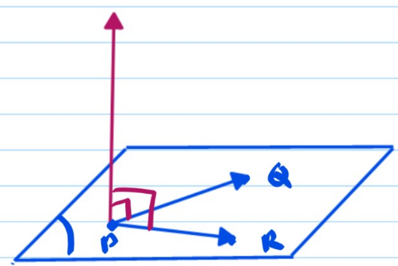
$-1 \quad\quad 1$

$$u \times v = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$$

EX: Find the equation of the plane through $P(1, 3, -2)$, $Q(1, 1, 5)$, and $R(2, -2, 3)$.

$$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \overrightarrow{QP} \times \overrightarrow{QR}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$$



$$n = \begin{pmatrix} 25 \\ 7 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad 25x + 7y + 2z - 42 = 0$$

Exercise 4.2.23 Find the shortest distance between the following pairs of parallel lines.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

$M_1 M_2 \quad , \quad u$

$$\boxed{\Rightarrow \text{Shortest distance} = \frac{\| \overrightarrow{M_1 M_2} \times u \|}{\|u\|}}$$

a) $M_1 = (2, -1, 3)$, $M_2 = (1, 0, 1)$

$$u = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \overrightarrow{M_1 M_2} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$\overrightarrow{M_1M_2} \times u = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \frac{\|\overrightarrow{M_1M_2} \times u\|}{\|u\|} = \frac{\sqrt{2^2+2^2+0^2}}{\sqrt{1^2+(-1)^2+4^2}} = \frac{2}{3}$$

$$\overrightarrow{M_1M_2} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{M_1M_2} \times u = \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix}$$

$$\Rightarrow \quad = \frac{\sqrt{10^2}}{\sqrt{10}} = \sqrt{10}$$

Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then find the points A and B on the lines that are closest together.

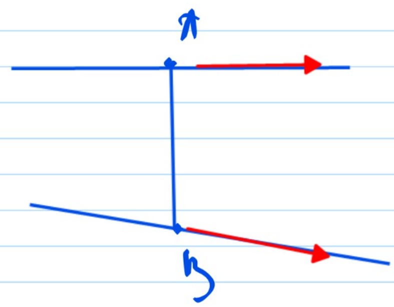
$$\boxed{\text{Shortest distan} = \frac{|(u_1 \times u_2) \cdot \overrightarrow{M_1M_2}|}{\|u_1 \times u_2\|}}$$

$$u_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \overrightarrow{M_1M_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$u_1 \times u_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \Rightarrow (u_1 \times u_2) \cdot \overrightarrow{M_1M_2} = -2+3+2 = 3$$

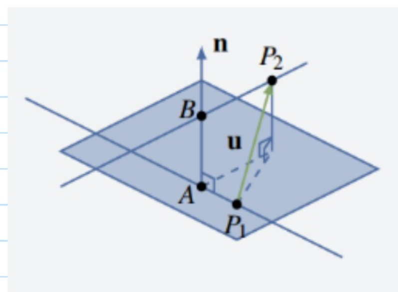
$$\Rightarrow \quad = \frac{|3|}{\sqrt{(-1)^2+3^2+2^2}} = \frac{3}{\sqrt{14}}$$

$A = (1+2t,\ 0,\ -1+t)$, $B = (3+s,\ 1+s,\ -s)$



$\vec{AB} \perp u_1, u_2$

$\vec{AB} = \begin{pmatrix} \ \end{pmatrix}$



$$\begin{cases} \vec{AB} \cdot u_1 = 0 \\ AB \cdot u_2 = 0 \end{cases} \Rightarrow A, B$$