Linear Approximation

(Sấp xỉ tuyến tính)

If x near a : $f(x) \approx f(a) + f'(a)(x-a)$

$$L(x) = f(a) + f'(a)(x-a)$$

- linear approximation
- tangent line approximation

EX: Find the linear approximation

$$f(x) = \sqrt{x} \quad \text{at } x = 9$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$L(x) = f(9) + f'(9)(x-9)$$

$$= 3 + \frac{1}{6}(x-9)$$

$$\boxed{L(x) = \frac{1}{6}x + \frac{3}{2}}$$

b) Use the approximation to the estimate $\sqrt{9.1}$

$$f(x) = \sqrt{x} = \sqrt{9.1} \Rightarrow x = 9.1$$

$$L(9.1) = f(9) + f'(9)(9.1-9)$$

$$= 3 + \frac{1}{6}(0.1)$$

$$= \frac{181}{60} \approx 3.0167.$$

*) $L(x) = \frac{1}{6}x + \frac{3}{2} \Rightarrow L(9.1) = \frac{1}{6} \times 9.1 + \frac{3}{2}$

$$\approx 3.0167.$$

EX2: find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$ and use it to approximation $\sqrt[3]{8.1}$

$$L(x) = f(8) + f'(8)(x-8)$$
$$= 2 + \frac{1}{12}(x-8)$$
$$= \frac{1}{12}x + \frac{4}{3}$$

$$L(8.1) = \frac{1}{12}(8.1) + \frac{4}{3} \simeq$$

EX3: Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximation $\sin(62°)$.

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$
$$L(62°) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(62° - \frac{\pi}{3}\right) \simeq 0,8834.$$

EX: find the approximation of $f(x) = (1+x)^n$ at $x = 0$

$$L(x) = f(0) + f'(0)(x-0)$$
$$= 1 + n(x-0)$$
$$= nx + 1$$

Differentials (Vi phân)

$dy, dx \ldots$: differentials

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$$

$$\Rightarrow dy = f'(x)\,\underline{dx}$$

EX: for each of the following function, find dy and evaluate when $\underline{x = 3}$ and $dx \underline{= 0.1}$

a) $f(x) = x^2 + 2x$

$$dy = (2x + 2)\,dx$$

$$dy = (2.3 + 2).0{,}1 = 0.8$$

b) $y = \cos x$

$$dy = -\sin x\,dx$$

$$dy = -\sin 3 . 0{,}1 =$$

EX: for $y = e^{x^2}$, find dy.

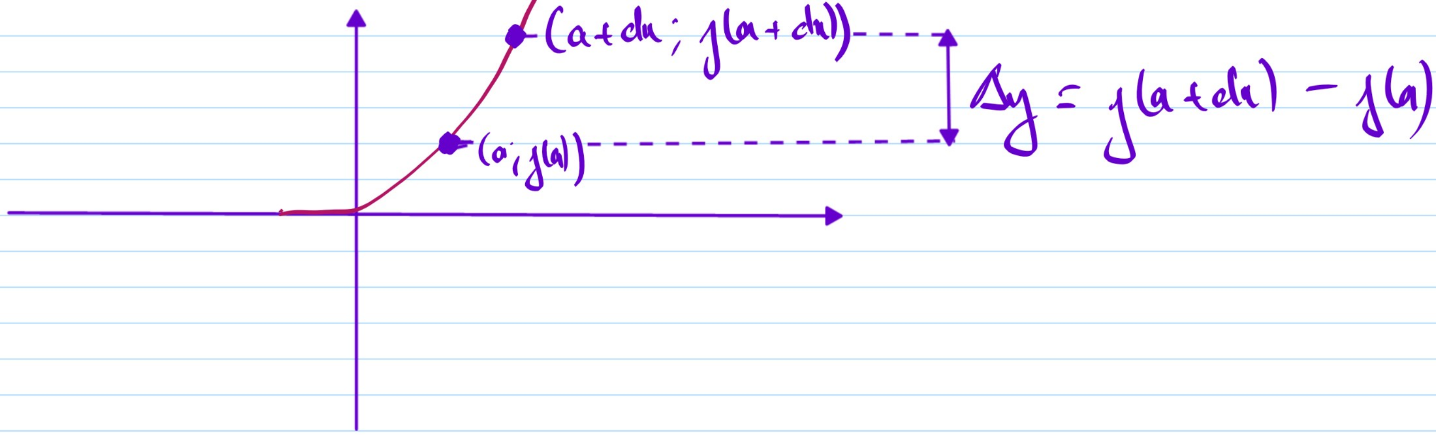
$$dy = 2x . e^{x^2}\,dx$$

$$\Rightarrow dy = 2x . e^{x^2}\,dx$$

$$\left((e^u)' = u' . e^u\right)$$

Find Δy.

$$\Delta y = f(a + dx) - f(a)$$



EX: Let $y = x^2 + 2x$ compute Δy and dy at $x = 3$ & $dx = 0.1$

$$\Delta y = f(3 + 0.1) - f(3)$$
$$= f(3.1) - f(3)$$
$$= (3.1)^2 + 2 \times 3.1 - [(3)^2 + 2.3]$$
$$= 0.81$$

$$dy = f'(x)\,dx$$
$$= f'(3)\,dx = (2.3 + 2)\,0.1 = 0.8$$

The Mean Value Theorem
(ĐL giá trị trung bình)

Suppose.

1. f is continuous on $[a; b]$
2. f is differentiable on $(a; b)$

$\exists c \in (a; b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}}$$

EX: for each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all value c in the given interval

a) $f(x) = x^2 + 2x$ over $[-2; 0]$

$$f'(c) = \frac{f(0) - f(-2)}{0 - (-2)} = \frac{0 - 0}{2} = 0$$

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2 = 0$$

$$\Rightarrow \boxed{c = -1}$$

b) $f(x) = x^3 - 4x$ over $[-2; 2]$

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{0 - 0}{4} = 0$$

$$f'(x) = 3x^2 - 4 \Rightarrow f'(c) = 3c^2 - 4 = 0$$

$$\Rightarrow c = \pm \frac{2}{\sqrt{3}}.$$

EX 2: $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1; 3]$. find all points c guaranteed by Rolle's theorem.

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{0 - 0}{2} = 0$$

$$f'(x) = 4x - 8 \Rightarrow f'(c) = 4c - 8 = 0$$
$$\Rightarrow c = 2.$$

EX:

For $f(x) = \sqrt{x}$ over the interval [0, 9], show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting ((0, f(0)) and ((9, f(9)). Find these values c guaranteed by the Mean Value Theorem.

$$f'(c) = \frac{f(9) - f(0)}{9 - 0} = \frac{3 - 0}{9} = \frac{1}{3}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{3}$$
$$\Rightarrow c = 9/4$$

EX:

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t2 + 100$. $(-16t^2 + 100)$
a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity vavg of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is ~~v vg.~~ Vavg

a) $$s(t) = 0 \Leftrightarrow -16t^2 + 100 = 0$$
$$\Rightarrow t = \pm \frac{5}{2} \ (s)$$
$$(t \geq 0) \Rightarrow t = 5/2 \ (s)$$

b) $$V_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{1 - 100}{5/2} = -40 \text{ ft/s}$$

c) $$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40$$

$$s'(t) = -32t \Rightarrow s'(c) = -32c = -40$$
$$\Rightarrow c = 5/4.$$