3.3 Diagonalization and $\left[\begin{array}{l}\text{Eigenvalues} \to \text{(giá trị riêng)} \\ \text{Eigenvectors} \to \text{(vectơ riêng)}\end{array}\right.$

$\downarrow$
chéo hóa

(+) Theorem: The following statements are equivalent

1) A: invertible
2) $\det A \neq 0$
3) $Ax = B$ has unique solution
4) $Ax = 0$ has only trivial solution

(+) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow A^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}$

(+) Diagonal Matrix : $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$

(+) Eigenvalues and eigenvectors

Def: $A: n \times n$, λ eigenvalues of A if $\boxed{Ax = \lambda x}$, $x \neq 0$: column

$\left[\begin{array}{l}x\text{: eigenvector of } A \text{ corresponding to the eigenvalues } \lambda \\ x\text{: } \lambda\text{-eigenvector}\end{array}\right.$

(+) Characteristic polynomial of $A: n \times n$: $C_A(x) = \det(xI - A)$
(đa thức đặc trưng)

Ex: $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} \Rightarrow C_A(x) = \det(xI - A) = \begin{vmatrix} x-3 & -5 \\ -1 & x+1 \end{vmatrix}$

$xI = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

$= (x-3)(x+1) - 5 = x^2 - 2x - 8 = \boxed{(x-4)(x+2)}$

(f) Find eigenvalues and eigenvectors

+) $C_A(x) = \det(xI - A) = 0 \Rightarrow$ eigenvalues λ.

+) $(\lambda I - A)x = 0 \xrightarrow{\text{gaussian}}$ find basis solution X

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow X: \lambda\text{- eigenvector.}$$

Ex: $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$ find eigenvalues and eigenvectors of A.

$$C_A(x) = \begin{vmatrix} x-3 & -5 \\ -1 & x+1 \end{vmatrix} = (x-3)(x+1) - 5 = (x-4)(x+2) = 0$$

$$\Rightarrow \left[\begin{array}{l} x-4=0 \\ x+2=0 \end{array}\right. \Rightarrow \left[\begin{array}{l} x=4 \\ x=-2 \end{array}\right.$$

$\Rightarrow$ eigenvalues: $\lambda_1 = 4$, $\lambda_2 = -2$

Case 1: $\lambda_1 = 4$

$$(\lambda_1 I - A)x = 0 \Leftrightarrow \begin{pmatrix} 1 & -5 \\ -1 & 5 \end{pmatrix} x = 0 \longrightarrow x_1 - 5x_2 = 0 \Rightarrow \begin{cases} x_1 = 5t \\ x_2 = t \end{cases}$$

$$\lambda_1 I - A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -1 & 5 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} 5t \\ t \end{pmatrix} = t\underbrace{\begin{pmatrix} 5 \\ 1 \end{pmatrix}}_{\text{basis solution}}$$

$X_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$: λ_1- eigenvector.

Case 2: $\lambda_2 = -2$

$$(\lambda_2 I - A)x = 0 \Leftrightarrow \begin{pmatrix} -5 & -5 \\ -1 & -1 \end{pmatrix} x = 0$$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = 0 \Leftrightarrow x_1 + x_2 = 0$$

$$\Rightarrow \begin{cases} x_1 = -t \\ x_2 = t \end{cases} \Rightarrow X = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{l} X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} : \lambda_2 \text{ - eigenvector} \\ X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}\right.$$

EX2: find the characteristic polynomial, eigenvalues and basic eigenvectors for

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$$

$$C_A(x) = \det \begin{vmatrix} x-2 & 0 & 0 \\ -1 & x-2 & 1 \\ -1 & -3 & x+2 \end{vmatrix} = x-2 \begin{vmatrix} x-2 & 1 \\ -3 & x+2 \end{vmatrix}$$

$$= (x-2)\left[(x-2)(x+2) + 3\right]$$

$$= (x-2)(x^2-1) = (x-2)(x-1)(x+1) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} x = 2 \\ x = 1 \\ x = -1 \end{array}\right.$$

$\Rightarrow$ eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$

Case 1: $\lambda_1 = 2$.

$$(\lambda_1 I - A)x = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{pmatrix} x = 0$$

$$\Rightarrow \begin{cases} -x_1 + x_3 = 0 \\ -x_1 - 3x_2 + 4x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = t \\ x_3 = t \end{cases}$$

$$\Rightarrow X = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \lambda_1\text{- eigenvector}$$

Case 2: $\lambda_2 = 1$

$$(\lambda_2 I - A)x = 0 \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{pmatrix} x = 0$$

$$\xrightarrow[r_3 - r_1]{r_2 - r_1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{pmatrix} x = 0 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

$$\rightarrow \begin{cases} x_1 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = t \end{cases}$$

$$X = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : \lambda_2\text{- eigenvector.}$$

Case 3: $\lambda_3 = -1$

$$(\lambda_3 I - A)x = 0 \rightarrow \begin{pmatrix} -3 & 0 & 0 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{pmatrix} x = 0$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

$$\rightarrow \begin{cases} x_1 = 0 \\ x_1 + 3x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = t/3 \\ x_3 = t \end{cases}$$

$$\Rightarrow X = t \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow X_3 = \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix} : \lambda_3 \text{- eigenvector.}$$