4.2 projections and planes.

- The dot product.

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$u.v = u_1.v_1 + u_2.v_2 + u_3.v_3$$

EX: If $v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ find do product.

$$v.w = 2.1 + (-1).4 + 3.(-1) = -5$$

Theorem

1. $v.w$ is a real number
2. $v.w = w.v$
3. $v.0 = 0.v$
4. $v.v = \|v\|^2$
5. $(kv).w = k(v.w) = v(k.w)$
6. $u.(v \pm w) = u.v \pm u.w$

EX: Verify that $\|v-3w\|^2 = 1$ when $\|v\| = 2$, $\|w\| = 1$ and $v.w = 2$

$$VT = (v-3w)(v-3w) = v^2 - 6v.w + 9w^2$$

$$= \|v\|^2 - 6v.w + 9\|w\|^2$$

$$= 4 - 12 + 9 = 1 = VP$$

(+) Theorem.

Let v and w be nonzero vectors. If θ is the angle between v and w. Then

$$v.w = \|v\| . \|w\| . \cos \theta$$ theta

obtuse

$$\Rightarrow \cos\theta = \frac{v.w}{\|v\| . \|w\|}$$

EX: Compute the angle between $u = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$$\cos\theta = \frac{-1.2 + 1.1 + 2.(-1)}{\sqrt{(-1)^2+1^2+2^2} . \sqrt{2^2+1^2+(-1)^2}} = -\frac{1}{2}$$

$$\Rightarrow \quad \theta = 2\pi/3$$

b) $u = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{2+3}{\sqrt{10} . \sqrt{5}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

- $u.v > 0$ if and only if θ is acute $(0 < \theta < \frac{\pi}{2})$
- $u.v < 0$ if and only if θ is obtuse $(\pi/2 < \theta < \pi)$

$\cdot \rightarrow$ $u.v = 0$ if and only if $\theta = \frac{\pi}{2}$

(F) Theorem.

Two vectors v and w are orthogonal ($\perp$) if and only if $v.w = 0$

EX: Find all real number x such that.

a/ $u = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} x \\ -2 \\ 1 \end{pmatrix}$ are orthogonal.

b/ $u = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ x \\ 2 \end{pmatrix}$ are at an angle of $\frac{\pi}{3}$

a/ $u.v = 0 \Leftrightarrow 2x + (-1)(-2) + 3.1 = 0$

$\Leftrightarrow x = -\frac{5}{2}$

b/ $\cos\frac{\pi}{3} = \dfrac{2 - x + 2}{\sqrt{6}.\sqrt{x^2+5}}$

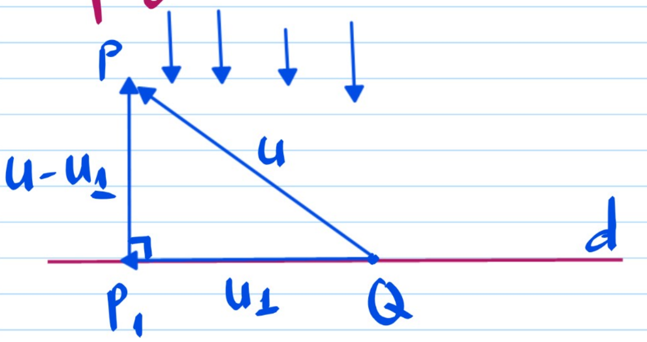
$\Leftrightarrow \sqrt{6}.\sqrt{x^2+5} = 8 - 2x$

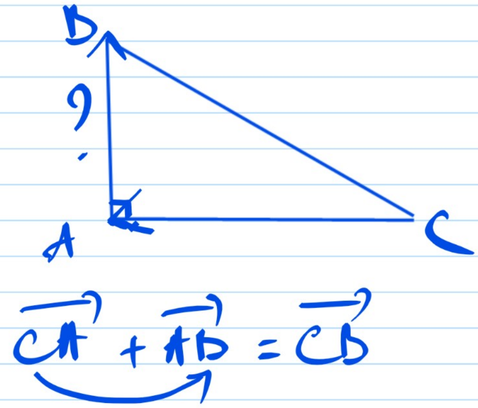
$\Leftrightarrow 6x^2 + 30 = 64 - 32x + 4x^2$

$\Leftrightarrow 2x^2 + 32x - 34 = 0$

$\Leftrightarrow \left[\begin{array}{l} x = 1 \\ x = -17 \end{array}\right.$

(F) projections.





$\overrightarrow{QP_1} = u_1$, $\overrightarrow{QP} = u$, $\overrightarrow{P_1P} = u - u_1$

$\overrightarrow{QP_1} = u_1$ is call projection of u on d.

$$\boxed{u_1 = \text{proj}_d u = \frac{u \cdot d}{\|d\|^2} \cdot d}$$

EX: a/ Find the projection of $u = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ on $d = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

b/ $u = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}$ on $v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

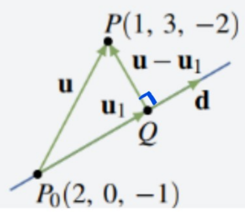
c/ $u = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ on $w = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

a) $$u_1 = \text{proj}_d u = \frac{u \cdot d}{\|d\|^2} \cdot d$$

$$= \frac{2 + 3 + 3}{\left(\sqrt{1^2 + (-1)^2 + 3^2}\right)^2} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \frac{8}{11} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

b) $$\text{proj}_{v} u = \frac{10-7+3}{14}\begin{pmatrix}2\\-1\\3\end{pmatrix} = \frac{3}{7}\begin{pmatrix}2\\-1\\3\end{pmatrix}$$

c/ $$\text{proj}_{w} u = \frac{11}{18}\begin{pmatrix}4\\1\\1\end{pmatrix}$$



Find the shortest distance (see diagram) from the point $P(1, 3, -2)$ to the line through $P_0(2, 0, -1)$ with direction vector $\underline{\mathbf{d}} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

$$\underline{\|\overrightarrow{PQ}\|}, \quad \underline{Q\,?}$$

$$\underline{u} = \overrightarrow{P_0P}, \quad \underline{u_1} = P_0Q, \quad \underline{\overrightarrow{QP}} = u - u_1$$

$$u = \begin{pmatrix}-1\\3\\-1\end{pmatrix}, \quad d = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$u_1 = \text{proj}_{d} u = \frac{-1-3}{1^2+(-1)^2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-2\\2\\0\end{pmatrix}$$

$$\overrightarrow{QP} = u - u_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix} \Rightarrow \|\overrightarrow{QP}\| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$$Q(x,y,z) \Rightarrow \overrightarrow{QP} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\Leftrightarrow \begin{cases}1-x = 1\\ 3-y = 1\\ -2-z = -1\end{cases} \Rightarrow \begin{cases}x = 0\\ y = 2\\ z = -1\end{cases}$$

$\Rightarrow \quad G = (0, 2, -1)$