Exercises

9. Find all values of m such that the system $\begin{cases} x+2y+z=1 \\ 2x+5y+3z=5 \\ 3x+7y+m^2z=6 \end{cases}$

has infinitely many solution.

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 5 \\ 3 & 7 & m^2 & 6 \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & m^2-3 & 3 \end{array}\right)$$

$$\xrightarrow{r_3-r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & m^2-4 & 0 \end{array}\right) \Rightarrow \text{infinitely} : m^2-4=0 \Leftrightarrow \boxed{m=\pm 2}$$

$$\rightarrow \begin{cases} x+2y+z=1 \\ y+z=3 \\ (m^2-4)z=0 \end{cases} \Leftrightarrow \begin{cases} x+2y+z=1 \\ y+z=3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=\dots \\ y=3-t \\ z=t \end{cases} \quad (t\in\mathbb{R})$$

2. Find all values of m such that the system has **no solution**, (another word: inconsistent)

$$\begin{cases} x-2y+z=3 \\ 3x-5y+z=m \\ -x+y+z=-1 \end{cases}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 3 & -5 & 1 & m \\ -1 & 1 & 1 & -1 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-3r_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & m-9 \\ 0 & -1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow[r_3+r_2]{} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & m-9 \\ 0 & 0 & 0 & m-7 \end{array}\right)$$

$$0 \neq a$$

$\Rightarrow$ no Solution: $m-7 \neq 0$

$\Leftrightarrow \boxed{m \neq 7}$.

Row-echelon matrix

(MT bậc thang theo dòng)

$$\begin{pmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{dài}$$

leading one

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading ones

Which is a row-echelon matrix?

a) $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & * & * \\ 0 & \cancel{2} & * \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ no

c) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{bmatrix}$ no $\leftarrow$; $1 \rightarrow 0$

d) $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ no

* Reduced row - echelon form
(MT Bậc thang thu gọn)

- It is a row-echelon matrix
- Each leading 1 is the **only nonzero** entry in its column

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 1.2.1 Which of the following matrices are in reduced row-echelon form? Which are in row-echelon form?

a. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → no

b. $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → no $\neq 1$

c. $\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\neq 0$ row-echelon

d. $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ → row

e. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Find Rank A ....

(hang)

$\xrightarrow{\text{gauss}}$ rows $= n$

$= \text{Rank} = 2$

$\begin{pmatrix} 1 & 2 & 8 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$

EX: Compute the Rank of $A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{pmatrix}$

$\xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \text{Rank} = 2$

4) $\begin{pmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ Rank $= 2$

Exercise 1.2.11 Find the rank of each of the following matrices.

a. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ b. $\begin{bmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}$ d. $\begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 2 & a^2 \\ 1 & 1-a & 2 & 0 \\ 2 & 2-a & 6-a & 4 \end{bmatrix}$

Exercise 1.2.12 Consider a system of linear equations with augmented matrix A and coefficient matrix C. In each case either prove the statement or give an example showing that it is false.

F a. If there is more than one solution, A has a row of zeros.

F b. If A has a row of zeros, there is more than one solution.

T c. If there is no solution, the reduced row-echelon form of C has a row of zeros.

F d. If the row-echelon form of C has a row of zeros, there is no solution.

a) $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$

$\begin{cases} x + z = 0 \\ y + z = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = \dots \\ y = -t \\ z = t \end{cases}$ (VSNo)

$(t \in \mathbb{R})$

b) $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$\Leftrightarrow \begin{cases} x + y = 1 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 2 \end{cases}$

$\longrightarrow$ (Unique Solution)

c) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 1 & x \\ 0 & 0 & 0 & 3 \end{array}\right)$

$0 = 3 \longrightarrow$ No Solution

d) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ (USN.)

1.3.Homogeneous Equations (phương trình thuần nhất)

- The system is called **homogeneous** (thuần nhất) if the constant matrix has all the entry are zeros
- Note that every homogeneous system **has at least one solution (0,0,…,0),** called **trivial solution** (nghiệm tầm thường)
- If a homogeneous system of linear equations has **nontrivial solution** (nghiệm không tầm thường) then it has infinite family of solutions (vô số nghiệm)

.) Linear equation : $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

.) Homogeneous equation : $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$

.) $x_1 = x_2 = \dots = x_n = 0 \Rightarrow$ [trivial Solution / unique Solution

.) $\exists x_1 = a \neq 0 \rightarrow$ [nontrivial Solution / infinite family of Solution

Example 1: Show that the following homogeneous system has nontrivial solutions.

$$x_1 - x_2 + 2x_3 + x_4 = 0$$
$$2x_1 + 2x_2 - x_4 = 0$$
$$3x_1 + x_2 + 2x_3 + x_4 = 0$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 4 & -4 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[r_2 - r_3]{} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[r_2 + 4r_3]{} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \begin{cases} x_1 = -t \\ x_2 = t \\ x_3 = t \\ x_4 = 0 \end{cases} \quad (t \in \mathbb{R}) \longrightarrow \text{nontrivial Solution}$$

$$x = (-t, t, t, 0) = t\underbrace{(-1, 1, 1, 0)}_{\text{Basic Solution}}$$