SWING-UP OF A TRIPLE ARM INVERTED PENDULUM
Master Semester Project

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Abstract

The swing-up of a triple pendulum. In this report, the subject of the swing-up of a triple arm inverted pendulum is approached. The swing-up strategy is based on recording the motions of the triple pendulum and then replay the same movements but in the backward direction. The main focus of the report is on the modeling of the triple pendulum and the simulation of the swing-up phase by using the described strategy on MATLAB.

LQR controller

1 Introduction

The swing-up of a triple arm inverted pendulum is presented. The main focus of this semester project is to simulate a strategy for the swing-up of the pendulum. It is done to study the feasability of the strategy. The goal of this project is to provide a solution for the swing-up of a triple arm inverted pendulum.

2 Principle of the swing-up strategy

The main focus of this project is the swing-up phase of the pendulum. This phase consists in leading the cart of the triple pendulum from the stable equilibrium to an unstable equilibrium where all the arms are in the upright position.

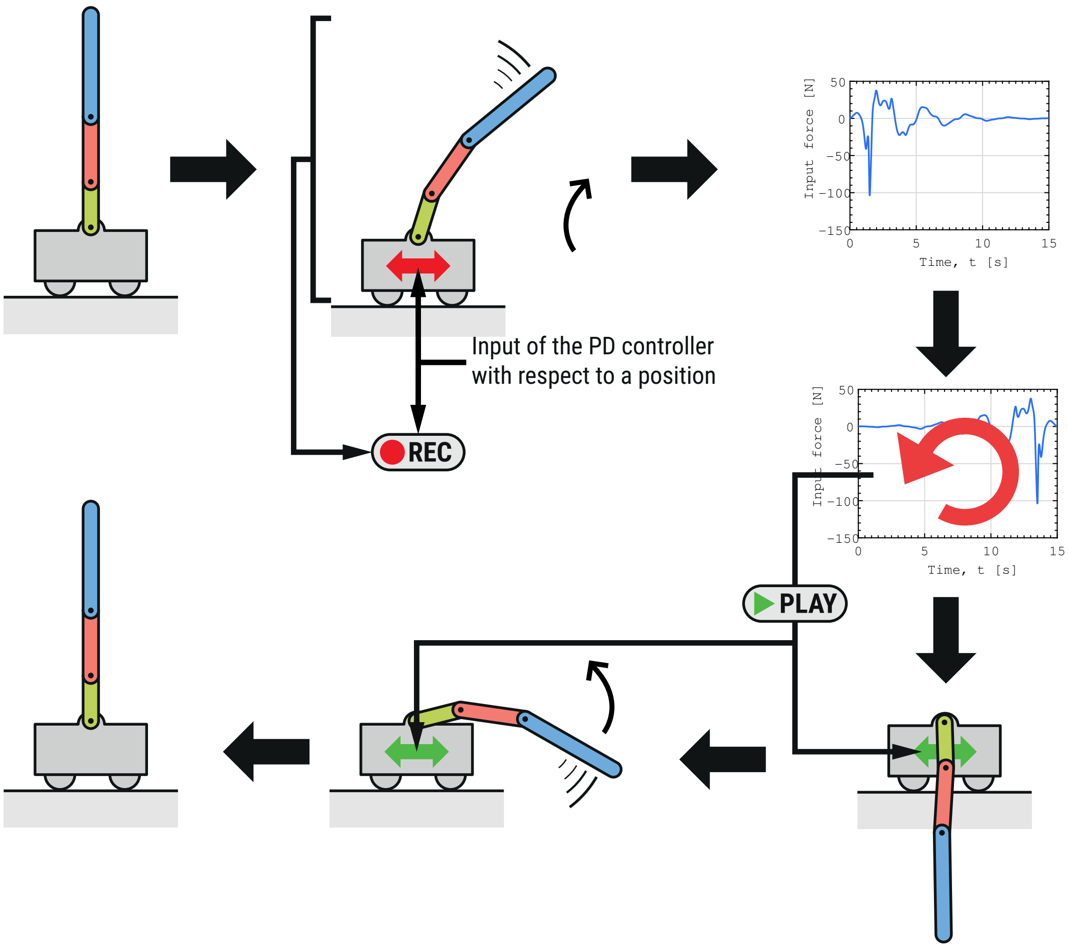

Figure 1 – Principle of the swing-up strategy

3 Modeling of the triple pendulum

For the purpose of this project, it is required to develop the equations of motion of the triple pendulum system. A *Lagrangian approach* is used for the modeling of the triple pendulum in the following pages.

3.1 Description of the triple pendulum

The pendulum consists in three links which are linked together to a cart. The cart is the only actuated part of the system. The other parts are free to move in their degree of freedom. For each link, a length and a mass is defined. The links are supposed to be made of homogeneous material so the gravity center for each one are on the geometrical middle of each bars. The viscous friction in the bears of the cart and the links is not considered for the modeling of the triple pendulum.

A schematic representation of the system is showned in Figure 2. The dispositions of the positions and angles, and the coordinate system are showned in Figure 2 and will be used for the modeling of the pendulum.

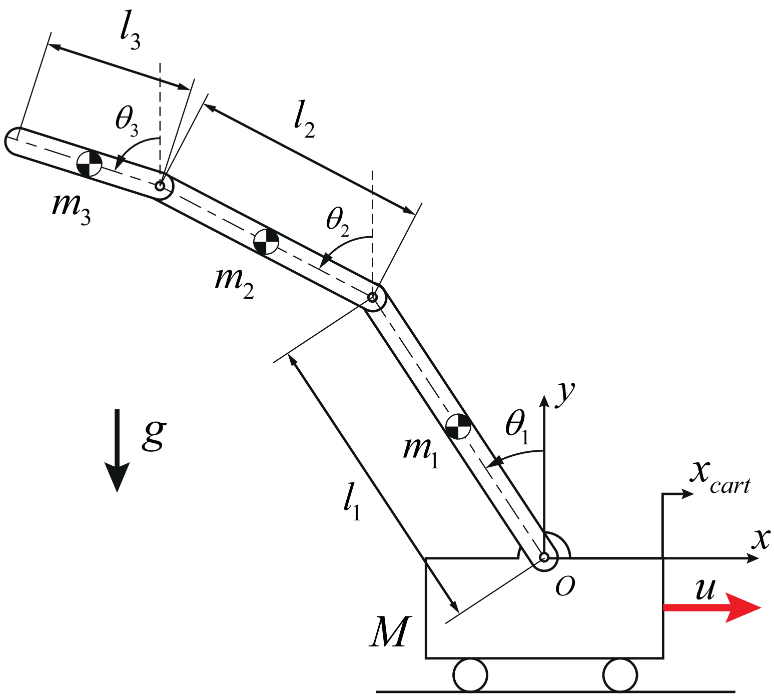

Figure 2 – Triple arm inverted pendulum with assumed coordinate systems, dimensions and angles

Since there are four degrees of freedom for the system, there are also four generalized coordinates which are defined for this pendulum. The first one is the position of the cart x_{cart} with respect to a reference. The last three generalized coordinates are the angles (θ_1, θ_2, θ_3) of links at the joints. Their zero value are when the bars are in down position and they take their positives values when turning counterclockwise. The assignement of each degree of freedom to a generalized coordinate is according to the following:

$$x_{cart} \rightarrow q_1, \qquad \theta_1 \rightarrow q_2, \qquad \theta_2 \rightarrow q_3, \qquad \theta_3 \rightarrow q_4$$

The parameters of the system are given values in Table 1 and will be used for every simulation of the triple pendulum.

Table 1 – Parameters for the following simulations

Parameter	Symbol	Value	Unit
Mass of the cart	m_{cart}	1	kg
Mass of first arm	m_1	0.5	kg
Mass of second arm	m_2	0.5	kg
Mass of third arm	m_3	0.5	kg
Length of first arm	l_1	1	m
Length of second arm	l_2	1	m
Length of third arm	l_3	1	m
Inertia of first arm	$J_1 = \frac{m_1 l_1^2}{12}$	0.0416	$\mathrm{kg\,m^2}$
Inertia of second arm	$J_2 = \frac{m_2 l_2^2}{12}$	0.0416	$\mathrm{kg\,m^2}$
Inertia of third arm	$J_3 = \frac{m_3 l_3^2}{12}$	0.0416	$\mathrm{kg\,m^2}$
Constant of gravity	g	9.81	$\mathrm{m/s^2}$

3.2 Mathematical model

After defining the coordinate system and the generalized coordinates, the core of the modeling takes place here. The equations of motion for the system are derived from Lagrange's equations with a generalized force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = u_i \quad \text{with } i = 1,2,3,4 \tag{1}$$

Knowing that there is only one force input which exert to the cart, the different u_i are defined such as $u_1 = u$ and $u_i = 0$ for $i = 2,3,4$. To derive these equations of motion, the Lagrangian of the triple pendulum needs to be founded:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{2}$$

where $\mathcal{T}$ is the kinetic energy and $\mathcal{V}$ the potential energy. Furthermore, the vectors from the origin of the coordinate system to the center of mass of each arms are defined by:

$$\mathbf{p}_1 = \begin{bmatrix} x_{cart} - l_1 \sin(\theta_1)/2 \\ l_1 \cos(\theta_1)/2 \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} x_{cart} - l_1 \sin(\theta_1) - l_2 \sin(\theta_2)/2 \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_2)/2 \end{bmatrix}$$

$$\mathbf{p}_3 = \begin{bmatrix} x_{cart} - l_1 \sin(\theta_1) - l_2 \sin(\theta_2) - l_3 \sin(\theta_3)/2 \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_2) + l_3 \cos(\theta_3)/2 \end{bmatrix}$$

In a derivation parallel to that of reference [], the energies The reasoning for deriving kinetic and potential energies is borrowed from []. The kinetic energy of the whole system is given by:

$$\mathcal{T} = \frac{1}{2} m_{cart} \dot{x}_{cart}^2 + \frac{1}{2} \sum_{i=1}^{3} m_i \dot{\mathbf{p}}_i^T \dot{\mathbf{p}}_i + \frac{1}{2} \sum_{i=1}^{3} J_i \dot{\theta}_i^2 \tag{3}$$

The potential energy is computed with the y components of each vector $\mathbf{p}_i$. Hence it is given by:

$$\mathcal{V} = g \cdot \sum_{i=1}^{3} m_i \mathbf{p}_{i,y} \tag{4}$$

The Lagrange's equations obtained with (1) are:

$$
\begin{gathered}
-\ddot{\theta}_1\big((l_1 m_1 \cos(\theta_1))/2 + l_1 m_2 \cos(\theta_1) + l_1 m_3 \cos(\theta_1)\big) - \ddot{\theta}_2\big(l_2 m_2 \cos(\theta_2)/2 + l_2 m_3 \cos(\theta_2)\big) \\
- \ddot{\theta}_3 l_3 m_3 \cos(\theta_3)/2 + \ddot{x}\big(M + m_1 + m_2 + m_3\big) + \dot{\theta}_1^2 l_1\big(m_1/2 + m_2 + m_3\big)\sin(\theta_1) \\
+ \dot{\theta}_2^2 l_2\big(m_2/2 + m_3\big)\sin(\theta_2) + \dot{\theta}_3^2 l_3 m_3 \sin(\theta_3)/2 = u_1
\end{gathered}
\tag{5}
$$

$$
\begin{gathered}
\big[\ddot{x}\,(-3m_1\cos(\theta_1) - 6m_2\cos(\theta_1) - 6m_3\cos(\theta_1)) + \ddot{\theta}_1 l_1\,(2m_1 + 6m_2 + 6m_3) \\
+ \ddot{\theta}_2 l_2\,(3m_2 + 6m_3)\cos(\theta_1 - \theta_2) + \ddot{\theta}_3 3 l_3 m_3 \cos(\theta_1 - \theta_3) \\
+ \dot{\theta}_2^2 l_2 (3m_2 + 6m_3)\sin(\theta_1 - \theta_2) + \dot{\theta}_3^2 3 l_3 m_3 \sin(\theta_1 - \theta_3) \\
- 6 g m_2 \sin(\theta_1) - 6 g m_3 \sin(\theta_1) - 3 g m_1 \sin(\theta_1) \\
- g(3m_1 + 6m_2 + 6m_3)\sin(\theta_1)\big]\, l_1/6 = 0
\end{gathered}
\tag{6}
$$

3.3 Resolving the differential equations

3.4 Verification of the Lagrange model

4 Swing-up of the triple pendulum

4.1 Generation of a downward trajectory

4.2 Simulation swing-up of the triple pendulum

4.3 Analysis

5 Implementation of the LQR stabilization

5.1 Theory of the LQR regulator

5.2 Linearization of the equations of motion

6 Conclusion

References

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