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APPENDIX A

Probability Theory

This appendix briefly reviews some elements of probability theory. The review is short because there is an abundance of excellent references on the subject [1]. The topics included here either are not widely known outside the teletraffic community or are of fundamental importance for the traffic analysis presented in this book. This second category is limited to the fundamental results of birth-and-death equations and of the $GI/M/N/\infty/N$ queue. Applications of the general theory to the particular queues of interest in traffic theory are found in the main text.

A.1 Some Functions Used in Probability Models

The following definitions and simple properties are either used extensively in the text or are not well known outside the teletraffic community.

Binomial Coefficients and Related Polynomials

The binomial coefficients are defined for positive integer n as the coefficients of x in the expansion of

$$(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$$

and are given by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

which can be written

$$= \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

This definition can be extended to nonnegative real a by defining the binomial coefficient as the coefficient of x in the expansion of

$$(1+x)^a = \sum_{k=0}^{\infty} x^k \frac{a(a-1)\ldots(a-k+1)}{k!},$$

from which we get

$$\binom{a}{k} \triangleq \frac{a(a-1)\dots(a-k+1)}{k!} \quad \forall a \in \Re, \quad \binom{a}{0} \triangleq 1 \tag{A.1}$$

From this, we define the *factorial polynomials*:

$$(a)_k \triangleq a(a-1)\dots(a-k+1). \tag{A.2}$$

The binomial coefficients can be extended to arbitrary real a by looking at the power series of

$$(1+x)^{-a} = \sum_{k=0}^{\infty} x^k (-1)^k \frac{a(a+1)\dots(a+k-1)}{k!}, \quad a \geq 0$$

By defining the extended binomial moment as the coefficient of $-x^k$, we get

$$\binom{-a}{k} = \frac{a(a+1)\dots(a+k-1)}{k!}.$$

A useful relation is obtained if we note that the polynomial $a(a+1)\dots(a+k-1)$ can also be viewed as a factorial polynomial starting at $a+k-1$. Rewriting the identity for the polynomial, we get

$$\binom{-x}{k} = \binom{x+k-1}{k}, \quad x > 0$$

The binomial coefficients also have the following useful inversion property:

$$a_n = \sum_{j=n}^{N} \binom{j}{n} b_j$$

$$b_j = \sum_{n=j}^{N} (-1)^{n-j} \binom{n}{j} a_j$$

The factorial polynomials are related to the $a(a+1)\dots(a+k-1)$ polynomials by the functions

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \frac{a(a+1)\dots(a+k-1)x^k}{k!} \tag{A.3}$$

$$(1+x)^{a} = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{k!} \tag{A.4}$$

Factorial polynomials have a double representation that is related by the so-called *Stirling numbers* [2] $s_{q,i}$ and $S_{q,i}$ of the first and second kind, respectively. We have

$$(x)_q = \sum_{i=0}^{q} s_{q,i} x^i$$

$$x(x+1)\dots(x+q-1) = \sum_{i=0}^{q} S_{q,i}(x)_i$$

with the Stirling numbers defined recursively by

$$\begin{aligned} s_{q+1,i} &= s_{q,i-1} - q s_{q,i} \\ S_{q,i+1} &= S_{q,i-1} + i S_{q,i} \\ s_{0,0} &= S_{0,0} = 1 \\ s_{q,i} &= S_{q,i} = 0 \quad i > q \text{ and } i < -1 \end{aligned}$$

The Kosten and Brockmeyer Polynomials

The Brockmeyer polynomials [3] are defined as

$$S(l,k) = \sum_{i=0}^{l} \frac{A^{l-i}}{(l-i)!}\binom{k-1+i}{i}, \quad S(l,k) = 0 \text{ if } l < 0 \text{ or } m < 0 \quad \text{(A.5)}$$

and the closely related Kosten polynomials [4] by

$$R(k,l) = e^{-A} S(k,l). \quad \text{(A.6)}$$

These functions have some useful properties, such as

$$\begin{aligned} S(k,l) &= S(k-1,l) + S(k,l-1) \\ kS(k,l) &= (A+k-1+l)S(k-1,l) - AS(k-2,l) \\ kS(k,l) &= AS(k-1,l) + lS(k-1,l+1) \\ S(k,l) &= \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} S(k-i+j, l+i) \\ S(k,l) &= \sum_{i=0}^{k} S(i,l-1) \end{aligned}$$

These polynomials are frequently useful in analyzing queuing systems with loss.

Moments of a Distribution

The various moments of a distribution ranging from 0 to $N \leq \infty$ are defined as

$$\alpha_q = \sum_{j=0}^{N} j^q p_j \quad \text{(A.7)}$$

for the ordinary moments;

$$\mu_q = \sum_{j=0}^{N} (j-\alpha_1)^q p_j \quad \text{(A.8)}$$

for the central moments;

$$M_q = \sum_{j=q}^{N} (j)_q p_j \tag{A.9}$$

for the factorial moments; and

$$\beta_q = \sum_{j=q}^{N} \binom{j}{q} p_j = \frac{M_q}{q!} \tag{A.10}$$

for the binomial moments.

The various types of moments are interrelated in such a way that, given one set, the other sets can be computed directly. In particular, we have the mean and variance of a given distribution directly from the moments by

$$M \triangleq \alpha_1 = M_1 = \beta_1 \tag{A.11}$$

$$V \triangleq \mu_2 = \alpha_2 - \alpha_1^2 = M_2 - \alpha_1(\alpha_1 - 1) = 2\beta_2 + \beta_1(\beta_1 - 1) \tag{A.12}$$

Generating Functions and Transforms

The generating function, or the z-transform, of a function defined over the nonnegative integers is given by

$$G(z) = \sum_{k=0}^{\infty} p_k z^k, \quad |z| \leq 1$$

This sum converges if p_k does not grow faster than geometrically; the generating function is analytic in the disk $|z| \leq 1$. For instance, the binomial coefficients have the generating function

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

The generating function contains all the information about a distribution. From this function, we can recover either the probabilities or the moments, as needed. We have

$$G(0) = \sum_{n=0}^{N} p_n$$

$$p_n = \frac{1}{n!} \left(\frac{d^n G(z)}{dz^n} \right)_{z=0}$$

$$\beta_n = \frac{1}{n!} \left(\frac{d^n G(z)}{d^n z} \right)_{z=1}$$

and, in particular, the mean and variance of a distribution:

$$M = G'(1)$$
$$V = G''(1) + G'(1) - (G'(1))^2$$

For example, we have the expression of the distribution in terms of the factorial moments as

$$p_j = \sum_{k=0}^{\infty} (-1)^{k-j} \binom{k}{j} \frac{M_k}{k!}. \tag{A.13}$$

For continuous distributions $F(t)$, we define the Laplace-Stieltjes transform $\Phi(s)$ as follows:

$$\Phi(s) = \int_0^{\infty} e^{-st} dF(t)$$

and, if $F(t)$ is differentiable everywhere, the ordinary Laplace transform

$$\phi(s) = \int_0^{\infty} e^{-st} f(t) dt$$
$$\Phi(s) = s\phi(s)$$

Here too, we can get the moments of the distribution $F(t)$ directly from the Laplace-Stieltjes transform:

$$\int_0^{\infty} t^n dF(t) = (-1)^n \left(\frac{d^n \Phi(s)}{ds^n} \right)_{s=0}$$

Also, if we define the generating function for the binomial moment,

$$F(z) = \sum_{k=0}^{N} \beta_k z^k,$$

it is not hard to show that this function is related to the probability generating function by

$$F(z) = G(z+1).$$

A.2 Birth-and-Death Processes

Birth-and-death (BD) processes are the simplest type of processes that can be used to analyze queuing systems [5]. BD processes are characterized uniquely by their state-dependent arrival and departure rates. For one-dimensional processes, the state is represented by k, the number of customers present in the

system. The arrival rate when the system is in state k is λ_k; the departure rate is given by μ_k. The time-dependent state equation is then given by

$$\frac{dp_k}{dt} = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}, \tag{A.14}$$

and the stationary form

$$0 = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1}p + \mu_{k+1}p_{k+1}. \tag{A.15}$$

The general solution for the stationary state probabilities is given by

$$p_k = p_0 \prod_{j=0}^{k-1} \frac{\lambda_j}{\mu_{j+1}}, \tag{A.16}$$

which satisfies the useful recurrence

$$p_{k+1} = \frac{\lambda_k}{\mu_{k+1}} p_k$$

and the normalization condition

$$\sum_{k=0}^{\infty} p_k = 1.$$

As we saw in Chapter 3, a large number of queuing systems of interest can be modeled by a suitable choice of λ_k and μ_k.

A.3 Renewal Processes and the $GI/M/N/\infty/N$ Queue

A general analysis of overflow processes is based on the $GI/M/N/\infty/N$ queue. In such a queue, the arrival process is of the renewal type, and the service periods have a negative exponential distribution. A complete description of the results available on the $GI/M/N$ system can be found in [6,7]. Here we briefly summarize the derivation of the general equations that can be used to describe this queue.

Distribution of Arrivals in a Fixed Interval

When considering the sequence of requests for service on a particular link, we are naturally led to consider the random variable that represents the length of time that separates two successive arrivals. Following Syski [7], we denote this variable by L_r, calling it the *life* of the interval between the $r-1^{\text{th}}$ and r^{th} calls. The arrival process is a *renewal process* if the L_r variables are independent, nonnegative, identically distributed variables. Although the theory is explained in terms of call arrivals, it applies equally well to the call-departure process, or to any other type of event that is of interest.

We denote the distribution of L_r by $F(t)$, and its density by $f(t)$. For an arbitrary interval $(0, t]$ two quantities are of interest: (1) $N(t)$, the number

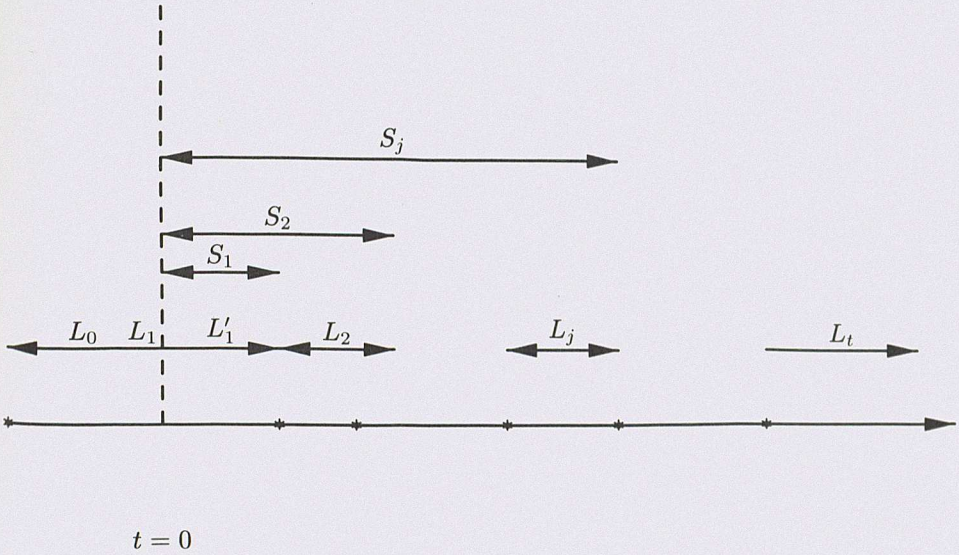

Figure A.1 Definition of Variables Related to a Renewal Process

of arrivals during this interval, and (2) $L(t)$, the time since the last arrival before time t, called the *age* of the interval, with the distribution $Y(s,t) \triangleq Pr\{L(t) \leq s\}$. Note that Y is not simply the value of t because we can select the origin of time at an arbitrary instant, which need not be the time of an arrival.

Calculating the distribution of these random variables require some auxiliary variables. Let S_j be the total lifetime of j calls that arrive in this interval. The distribution functions of the S_js are denoted $F_j(t) = Pr\{S_j \leq t\}$. This is the total duration of the j intervals, and it is obviously given by $S_j = L_1' + L_2 + \ldots + L_j$. Note the special notation for the first interval. Because we wish to be able to select the origin at an arbitrary point, this origin generally does not coincide with an arrival. In this case, the time origin is inside an interarrival period, and the age of the first interval as measured from the origin is different from the ages of the other intervals. This first partial interval is denoted $L_1' = L_1 - L_0$; this term appears in the definition of S_j. In other words, L_1' is the residual life of the first interval after we start looking at the system. Note also that we have

$$L_t = \begin{cases} t - S_{N(t)} & \text{if } N(t) \geq 1 \\ t + L_0 & \text{if } N(t) = 0 \end{cases}$$

The meaning of these variables is shown in Fig A.1.

APPENDIX C

Notation and List of Symbols

In this appendix we collect the main conventions for notation used in this book. We also include an alphabetic list of symbols used most frequently.

C.1 Notation

Vectors are denoted in boldface $\mathbf{x}$, and matrices in uppercase italic A. Unless absolutely necessary, we do not explicitly distinguish between row and column vectors, and do not use the transpose notation to simplify notation. Thus the expression $A\mathbf{x}$ denotes the product of the vector $\mathbf{x}$ by the matrix A; we assume that the matrix and vectors have the adequate dimensions. In this case, the assumption is that $\mathbf{x}$ is a column vector.

It is sometimes necessary to use row vectors, especially for matrix multiplications. Most such cases are recognizable because the vector appears on the left of the matrix, as in $\mathbf{x}A$, which represents the left multiplication of the row vector $\mathbf{x}$ by the matrix A. This is equivalent to $A^T\mathbf{x}$, where now $\mathbf{x}$ is a column vector, but avoids the need for yet another superscript in situations where the vector may already have many others.

The ith component of $\mathbf{x}$ is denoted by x_i, the jth column of matrix A by $\mathbf{a}_j$, and the ith row by $\mathbf{a}^i$. Recall that the product $A\mathbf{x}$ can be read $\sum_i x_i\mathbf{a}_i$ — that is, as the linear combination of the columns of A, with the components of $\mathbf{x}$ as the coefficients. When it is necessary to identify a particular component of the vector $A\mathbf{x}$, we denote this by $(A\mathbf{x})_i$. The inner product of two vectors is denoted $<\mathbf{x}, \mathbf{y}>$. Vector inequalities or equalities hold component-wise.

Many variables appear in a network context, where it is often necessary to indicate to which arc the variable belongs, which commodity it represents, or both. This is simplified as much as possible by the following convention for indices. Many variables in networks are double-index variables, referring either to arcs in a graph or to some matrix element. The double-index notation is sometimes essential, and sometimes too cumbersome. Something described by two indices may be described equally well with a single index if we assume that there is a mapping from one representation to the other. For instance, we may

denote the flow on the arc (i, j) of a graph by $f_{i,j}$, or, if we know that arc (i, j) is the kth in some given list of arcs, by f_k. In the text, we assume that such a unique transformation always exists, and we use the single and double indices freely according to the needs of the exposition.

In general, superscripts refer to a commodity, and subscripts to a link. Thus $f_{i,j}^{k,l}$ denotes the flow of commodity having its source at k and its sink at l on the link (i, j). This same quantity could as well have been denoted f_s^m, or $f_s^{k,l}$ if it is important for the discussion to know the origin and destination of the flow, but not the origin and extremity of the arc in question.

A similar simplification is made for denoting paths in networks. A path from i to j can be specified by giving the list of nodes encountered on the path, as in $(i, k_1, k_2, \ldots k_l, j)$, or the list of arcs $((i, k_1), (k_1, k_2), \ldots (k_l, j))$. Using the simplified notation for arcs, we could denote it as well by $(s_1, s_2, \ldots s_l)$. In many cases, it is not useful to know the precise composition of the path. In this case, we assume that all paths have been placed in a list, referring to the kth path between i and j as (i, j, k). Even more generally, if the particular origin and destination nodes are not needed, we may assume that there is a global list of all paths in the network, (or of all paths of interest in the particular situation) and refer to the kth path.

There is another convention regarding traffic and the relation between offered, carried, and overflow traffic. If x is a symbol representing traffic, then the variable x by itself denotes offered traffic, $\overline{x}$ denotes carried traffic, and $\hat{x}$ denotes blocked, or overflow, traffic. This convention applies to link, path, or route traffic, with the appropriate notation in each case.

C.2 List of Symbols

Here we list the symbols that occur regularly throughout the text. A particular symbol may be used with a different meaning in some short section, in which case this is indicated in the text. The reason is the convenience of conforming with specialized notations that are customary in certain areas.

$A^{i,j}$ = The value of the external traffic from i to j. This is the intensity of the Poisson process that describes the arrival of new calls into the network.

$A_k^{i,j}$ = The value of the external traffic from i to j that is offered to the kth path available to this stream.

$a_{s,t}$ = The total traffic offered to link s, t. This is generally made up of two components, direct trafic $A^{s,t}$ and internal traffic, that is, calls that either originate at a node other than s or terminate at a node other than t.

$a_{s,t}^{i,j}$ = The traffic offered to link (s,t) that belongs to the stream (i,j). This is often called a traffic *parcel*.

B_s = The blocking probability of link s. This is the probability that all circuits on link s are occupied.

B'_s = The probability of reaching or exceeding the protection threshold on link s. This is the probability that m or more circuits on link s are occupied.

C = A cost (most often). It can be a function of the group size, or a constant when the cost model is linear. It is also used occasionally in the discussion of flow models to denote the arc capacity.

$d(k)$ = The destination (sink) node of commodity k.

$E(A,N)$= The Erlang B function.

f = A flow variable or a generic function.

$F(t)$ = A distribution of interarrival times. Its density is represented by $f(t)$.

H = The marginal occupancy on a link. This is the incremental amount of traffic that can be carried on a link for a small increment of capacity, given a constant offered traffic.

L^k = The end-to-end blocking probability for stream k. This is the probability that a k-call cannot be connected.

M = The mean of a traffic. In some cases, this can also be represented by the lowercase m.

m = The protection level of a link. This is the number of busy circuits at which some calls begin to be blocked.

N_s= The number of circuits on link s.

$o(k)$ = The origin (source) of commodity k.

$P_k^{i,j}$ = The probability of overflow to the kth path in the route tree of stream (i,j).

Q = The connection probability; $1-P$.

p_k = The probability of being in state k for a stochastic process.

r_s = The reservation level for link s. We always have $r = N - m$.

$r_{i,j}$ = The reward for the transition from state i to state j in a Markov model.

R_s = The residual capacity of link s. This can also be used for paths, with the appropriate change in indices.

V = The variance of a traffic. In some cases, this is represented by the lowercase variable v.

$\mathbf{x}$ = A path-flow vector. This is the flow carried on all paths in a network.

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ = Vectors of Kuhn-Tucker multipliers.

$\mathcal{I}$ = The incidence matrix of a network.

$\mathcal{L}$ = The Lagrange function, also called the Lagrangian.

$\mathcal{P}_k^{i,j}$ = The probability that the kth path available to stream (i, j) is blocked.

$\alpha_k^{i,j}$ = The load-sharing coefficient for the kth path available to stream (i, j). This is also the tandem selection probability in residual capacity adaptive routing.

β = The marginal capacity of a link. This is the increment of traffic that can be offered to a link for a small increment in the link size such that the blocking probability remains constant.

γ = The marginal overflow of a link. This is the increment of overflow traffic corresponding to a small increase in the offered traffic, for a given constant link size.

$\delta_{i,j}$ = The Kroenecker symbol. $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise.

λ_k = The arrival rate when the system is in state k for an arrival process.

μ_k = The average service rate when the system is in state k. In other words, $1/\mu$ is the average service time.

$\Phi(s)$ = The Laplace-Stieltjes transform of the distribution $F(t)$. The corresponding transform for the density is denoted by $\phi(s)$.

APPENDIX *D*

Numerical Data

In Figs. D.1–D.7, this appendix provides the information necessary to do those exercises that require numerical calculations. The information is by no means intended for real design problems. The reason is that the data is applicable only to the case of Poisson traffic — not a realistic approximation for all hierarchical networks and some nonhierarchical networks. Also, as cheap microcomputers are being introduced, manual network design is rapidly becoming obsolete; such design should be done only for educational purpose, as in the present case.

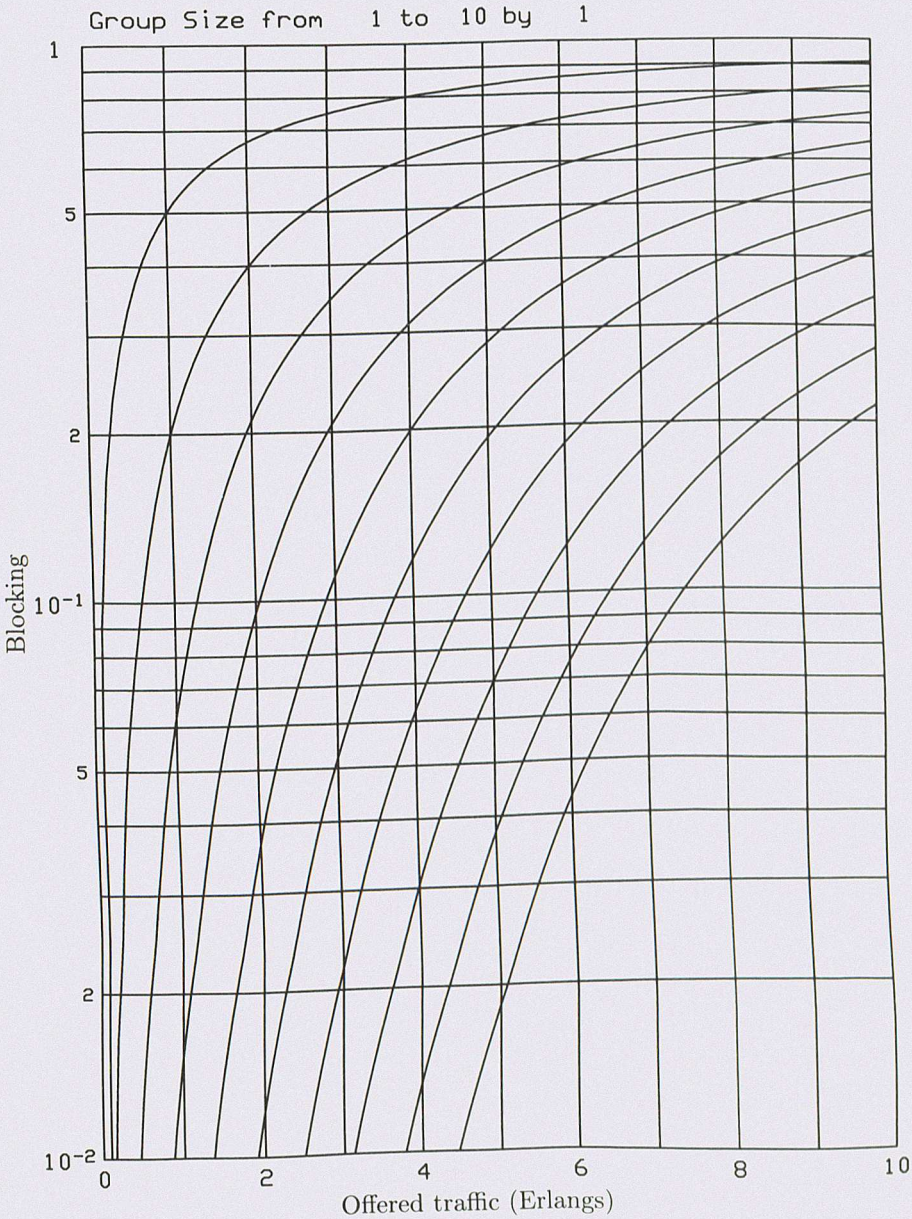

Figure D.1 Erlang B Function

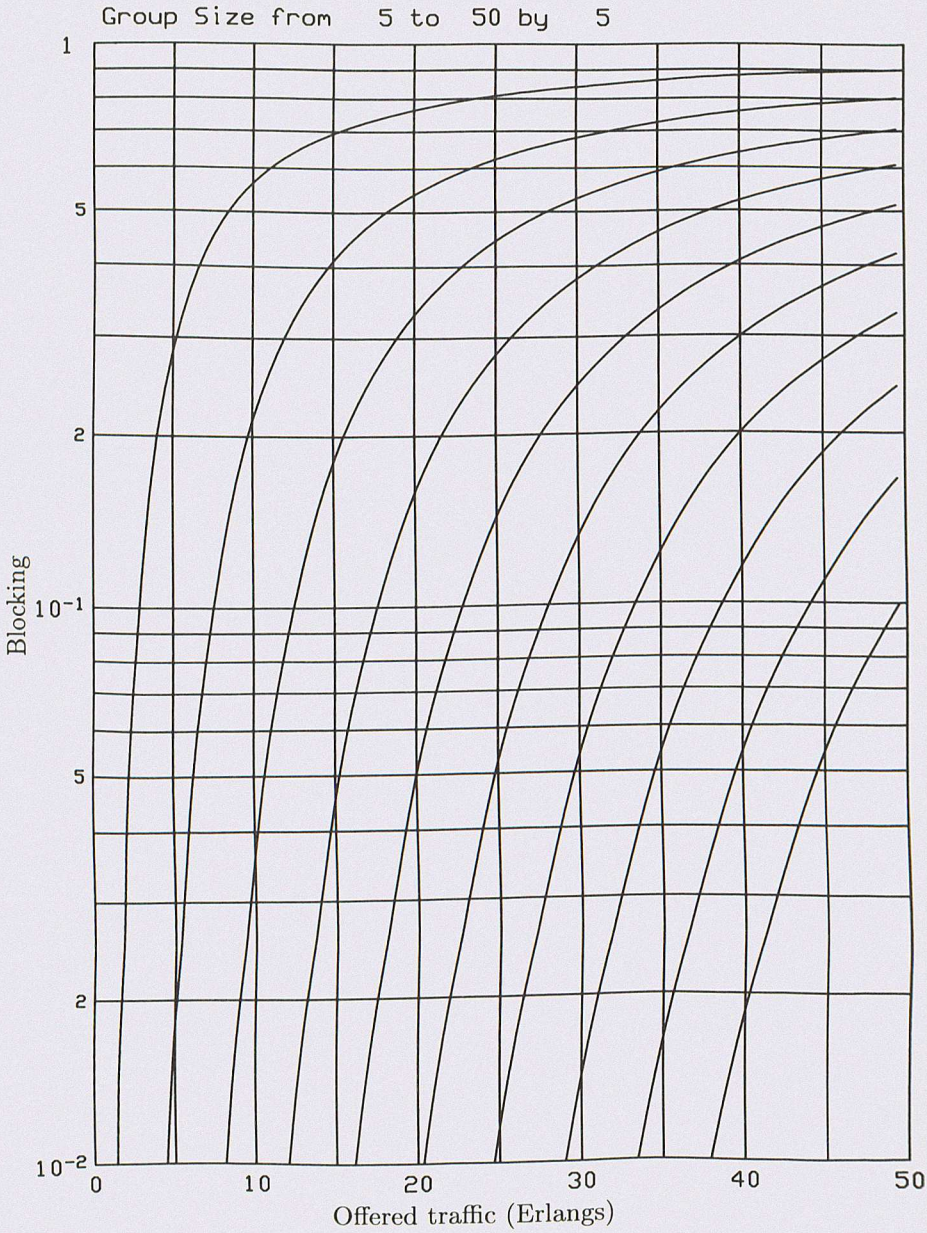

Figure D.2 Erlang B Function

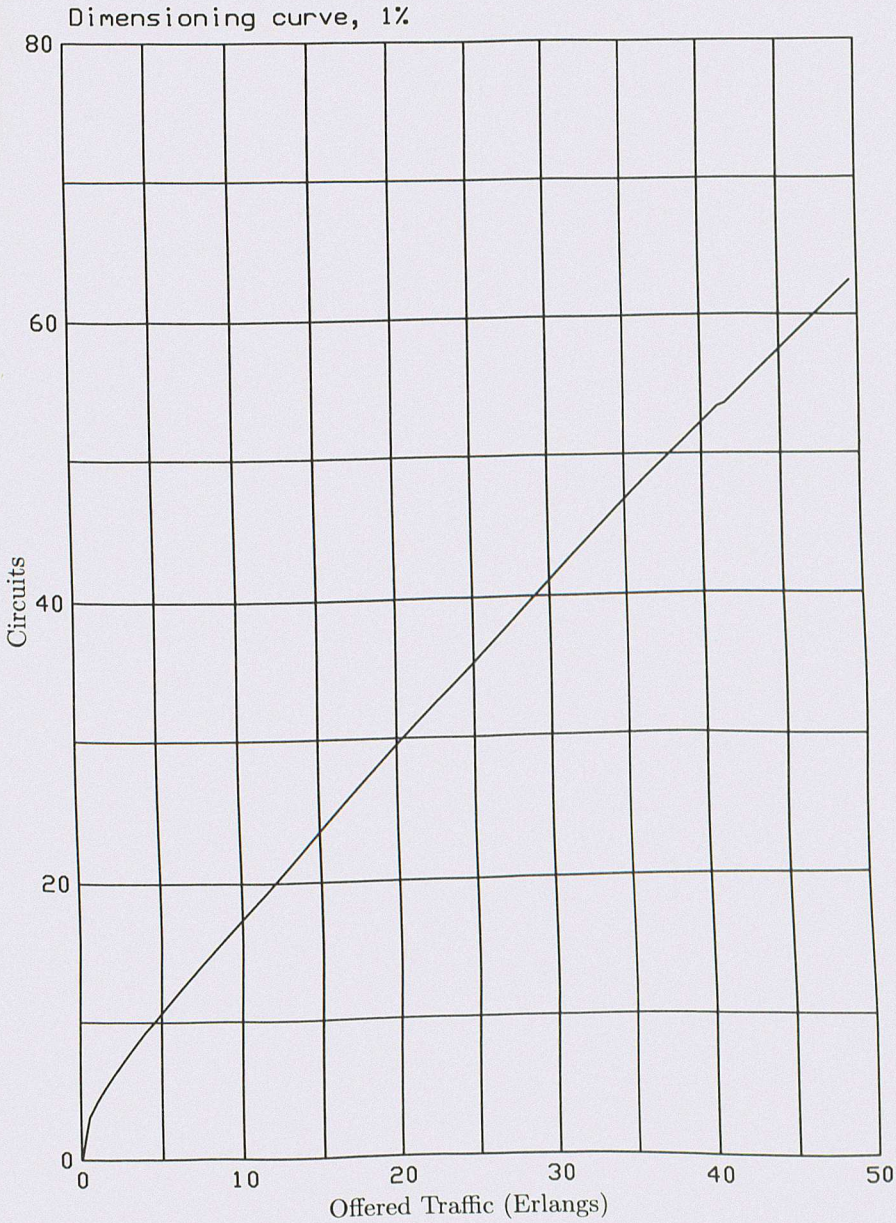

Figure D.3 Dimensioning Curve

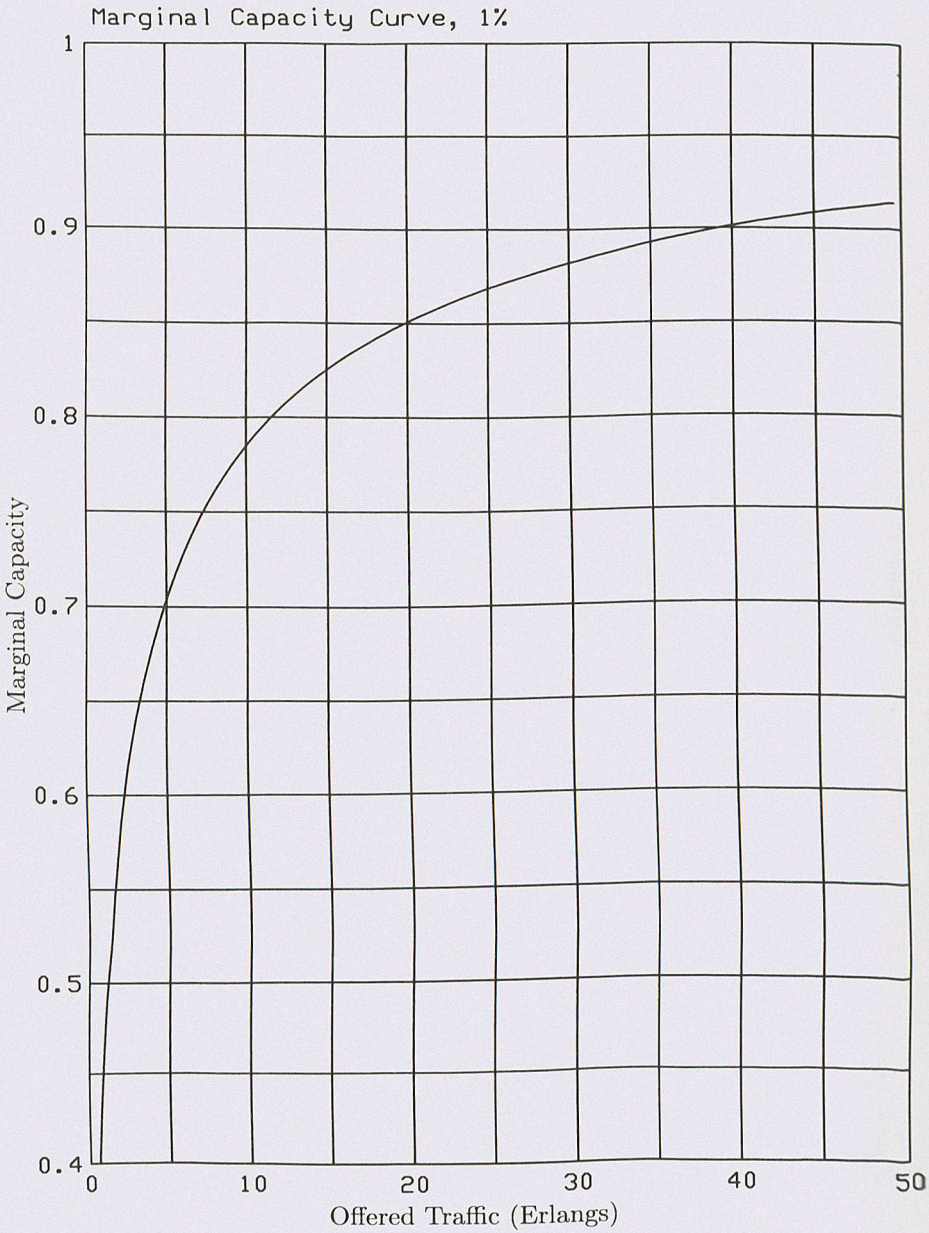

Figure D.4 Marginal Capacity Curve

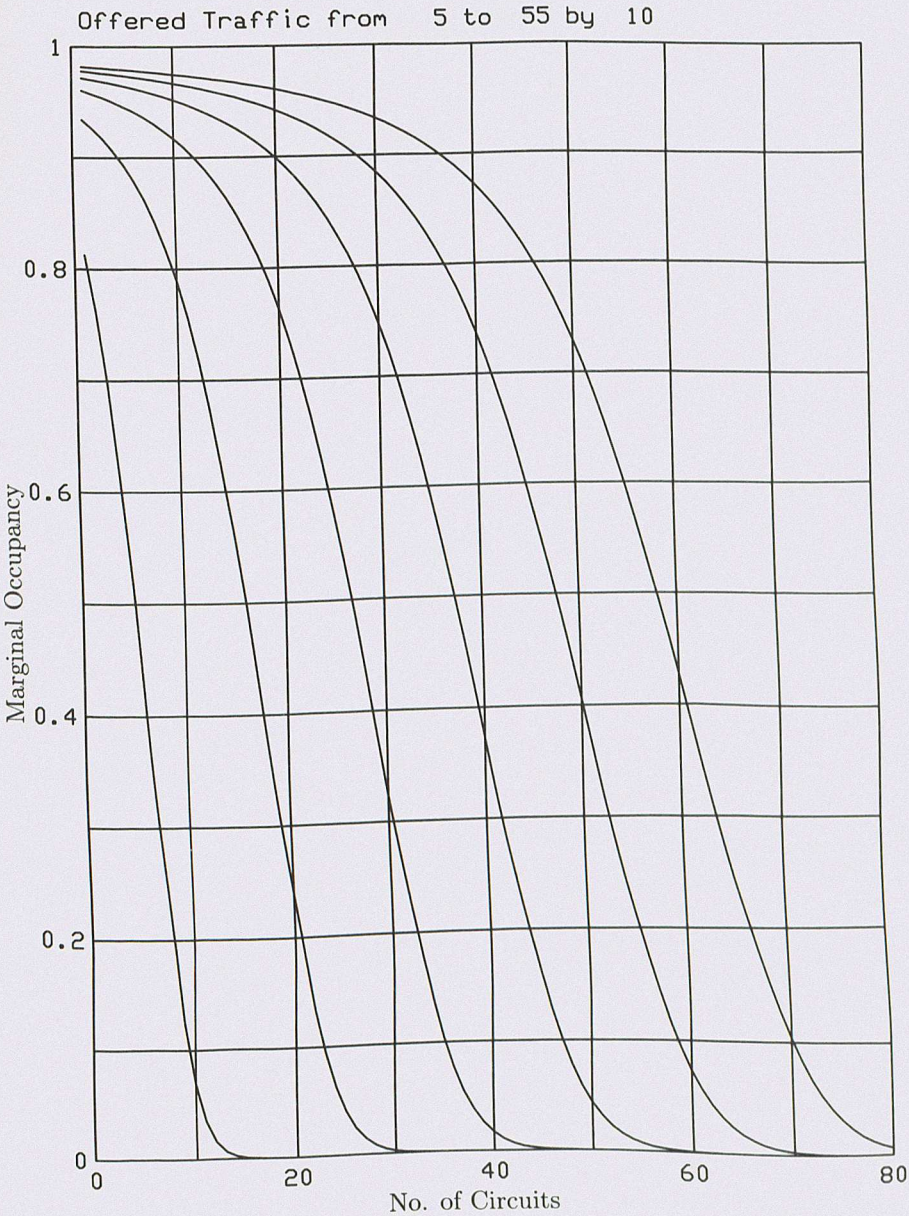

Figure D.5 Marginal Occupancy

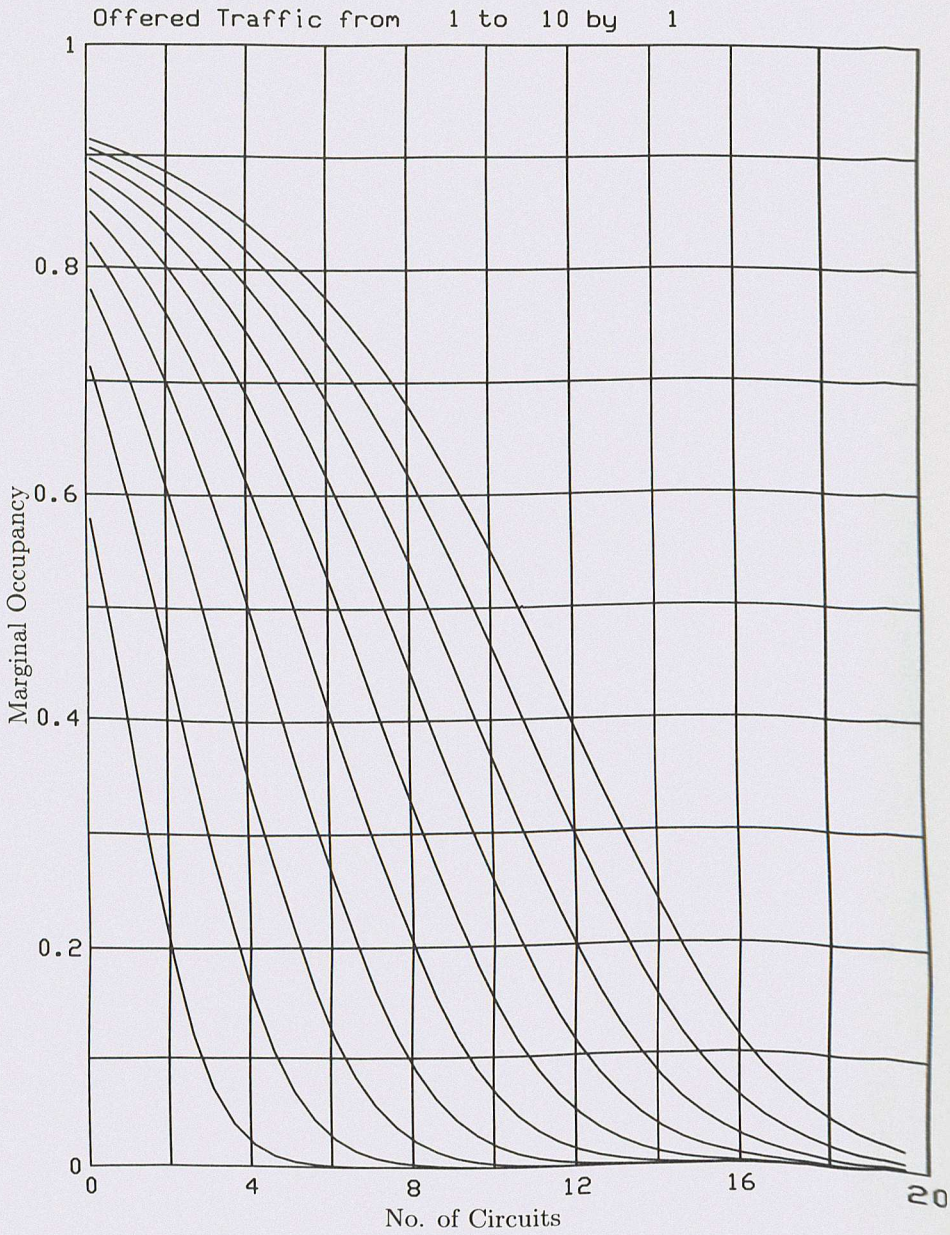

Figure D.6 Marginal Occupancy

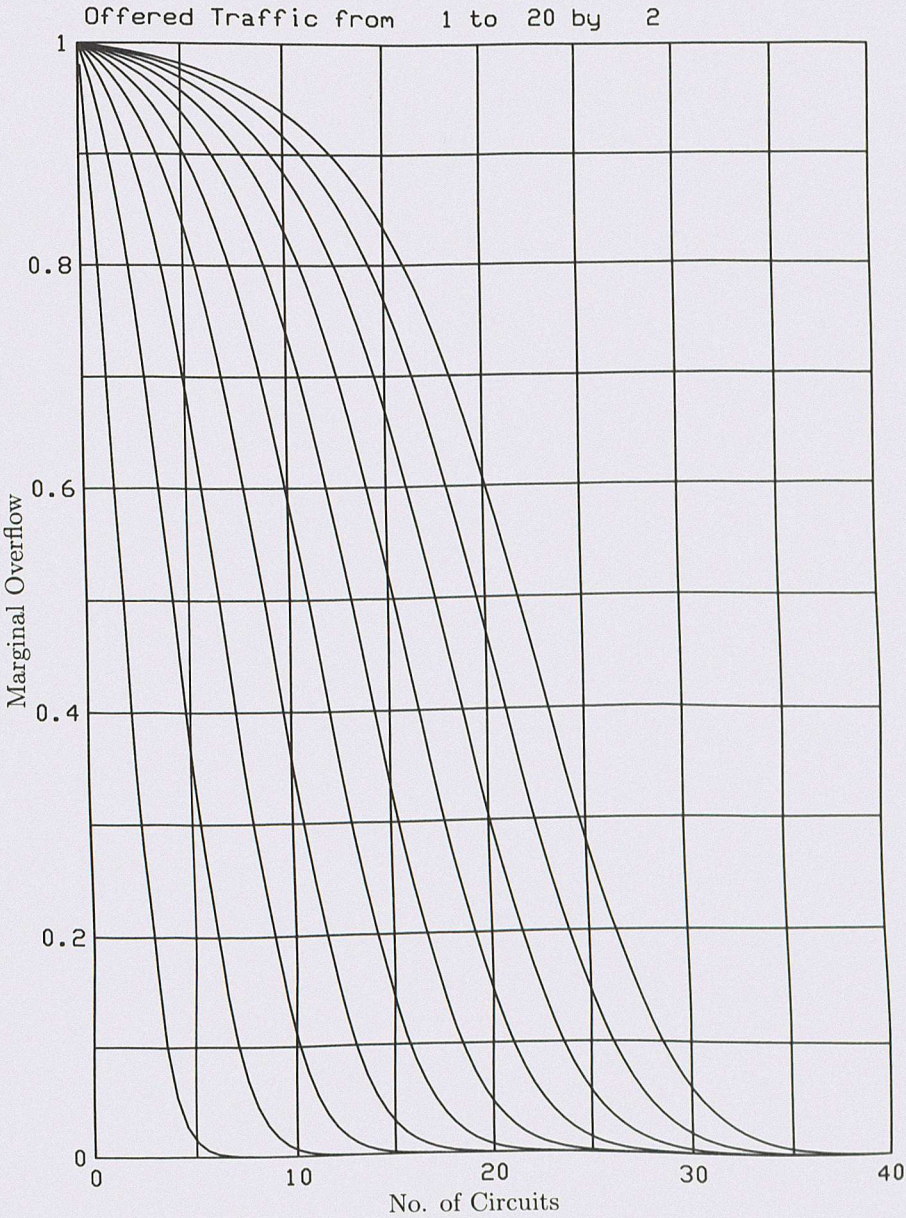

Figure D.7 Marginal Overflow