A COMPUTATIONAL APPROACH TO ODD CYCLE GRAPH ENTROPY

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INTRODUCTION

A cycle graph consists of a series of vertices connected in a loop. Such graphs can model a noisy communication channel, where each vertex represents a transmitted symbol and edges indicate confusability between symbols. What is the most efficient communication schema to transmit data with no errors and maximize precious bandwidth? This information density is encapsulated by the quantity known as **graph entropy**.

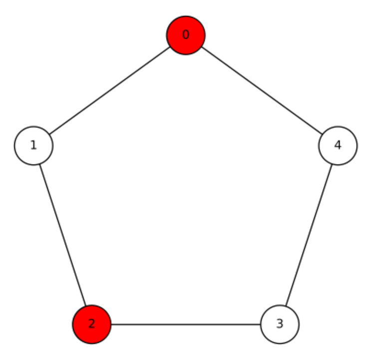


Figure 1: An independent set in the graph C_5.

DEFINITIONS

- **Coding theory** concerns itself with communication in noisy environments.
- A **cycle graph** is a graph consisting of vertices connected in a loop.
- **Graph entropy** (**Shannon capacity**) measures the amount of information transmitted by the code modelled by a graph.
- An **independent set** is a set of vertices of a graph which are mutually non-adjacent.
- The **strong product** $G \boxtimes H$ of graphs G and H has vertex set $V = V(G) \times V(H)$ and edges between points (u, v) and $(u', v') \iff$ at least one of the following holds: (1) u is adjacent u' or (2) v is adjacent to v'.
- An **affine subspace** is a vector subspace with a constant shift from the origin.

RIGIDITY & BOUNDS

Theorem 1. *Let the vertices of C_7 be the elements of $\mathbb{F}_7$ and define the power C_7^5 (assuming the strong product) to have an edge between vertices x and y when $x - y \in \mathcal{B} = \{-1, 0, 1\}^5$. Any maximal affine subspace of $C_7^5 = \mathbb{F}_7^5$ which is an independent set is rigid.*

Corollary 1. *This characterization can in fact also be applied to the affine subspaces constructed in [1] for the lower $\alpha(G)$ bounds for $G = C_{11}^4$, C_{13}^4, and C_{15}^3.*

Corollary 2. *Furthermore, a similar argument can be utilized to show that constructions utilizing affine subspaces for C_5^5, C_7^4, and C_9^4 are also rigid. Note that this relies only upon the ring structure of these spaces, and so is also applicable for structures like C_9^2.*

p\d	**1**	**2**	**3**	**4**	**5**
5	✓[1]	✓[1]	✓[1]	✓[1]	✓[3]
7	✓[1]	✓[1]	✓[1]	✓[4]	✓[2,4]
9	✓[1]	✓[1]	✓[1]	✓[3]	✓[3]
11	✓[1]	✓[1]	✓[1]	✓[2,4]	
13	✓[1]	✓[1]	✓[1]	✓[2,4]	
15	✓[1]	✓[1]	✓[2,4]		

Table 1: Rigidity of lower bound independent set constructions for C_p^d.

Key:
[1] $\alpha(G)$ known.
[2] Corollary 1 (Theorem 1).
[3] Corollary 2.
[4] Computer proof.

BACKGROUND

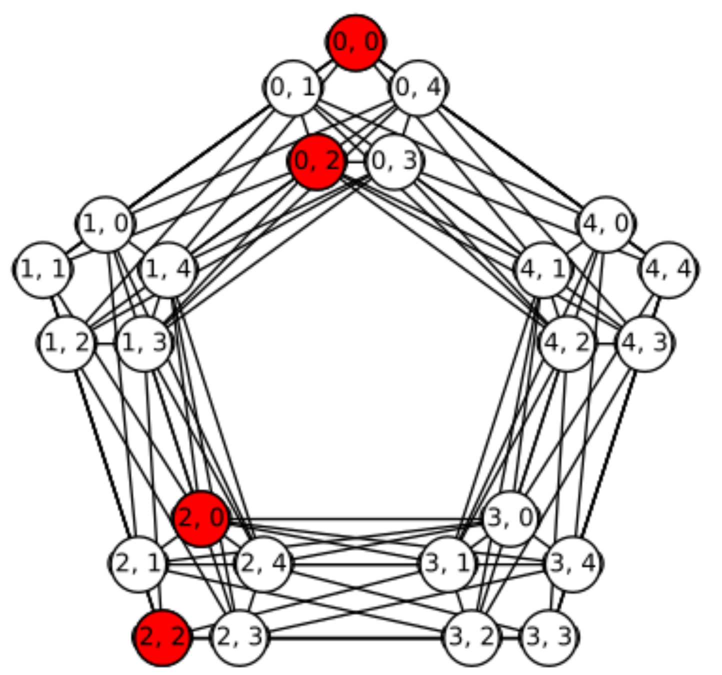


Figure 2: A naïve independent set in C_5^2, obtained by taking the Cartesian product of the independent set of C_5 with itself.

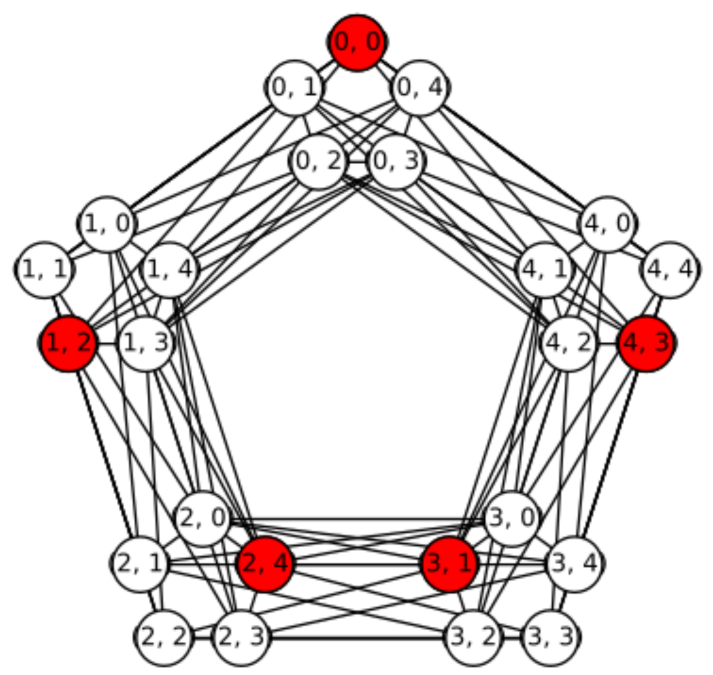
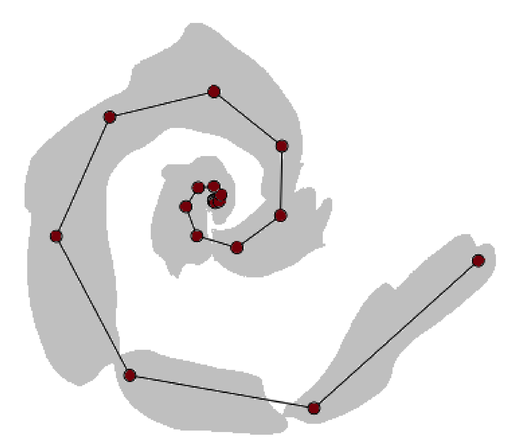


Figure 3: The maximal independent set of C_5^2, which increases the lower bound of the Shannon capacity to $\sqrt{5}$.

The size of the independent set of a graph G to the power k, $\alpha(G^{\boxtimes k})$, is bounded according to Equation (1).
The entropy, or Shannon capacity, is obtained by taking the kth root of the central quantity.

$$\alpha(G)^k \leq \alpha(G^{\boxtimes k}) \leq \chi(\bar{G})^k \tag{1}$$

GRAPH CYCLONE



The **Graph Cyclone** package is available via PyPI at `https://pypi.org/project/graph-cyclone/`. It can also be installed with `pip install graph-cyclone`. Features include:

- Create cycle graphs and their powers
- Add/remove points
- Check if a point is present
- Calculate size
- Count neighbors of a point

FUTURE RESEARCH

This research can be extended by implementing some of the following approaches:

- Improve bounds with new methods (local search, annealing, "energy" characterization)
- More information about structure of sets in [1] (discrete Fourier transform)
- Smarter parallelization (e.g. genetic algorithm)
- Improvements to the current local improvement algorithm (bookkeeping and design)
- Optimize point removal

REFERENCES

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