CS 450: Homework 3

University of Illinois at Urbana-Champaign

Author: Nathan Glaser

Net-ID: nglaser3

Professor: Andreas Kloeckner

September 25, 2024

Contents

Question 1

To begin, some prerequisite definitions:

$$\mathbf{A}^{-1}\mathbf{A}\vec{x} = \vec{x} \tag{1}$$

$$\|\mathbf{A}\vec{x}\|_2 \leq \|\mathbf{A}\|_2 \cdot \|\vec{x}\|_2 \tag{2}$$

To begin, we take the 2-norm of Eq. 1:

$$\|\mathbf{A}^{-1}\mathbf{A}\vec{x}\|_2 = \|\vec{x}\|_2 \tag{3}$$

substituting in $\mathbf{A}^{-1}$ in for $\mathbf{A}$ and $\mathbf{A}\vec{x}$ for $\vec{x}$ in Eq. 2, we obtain the following inequality:

$$\|\vec{x}\|_2 \leq \|\mathbf{A}^{-1}\|_2\|\mathbf{A}\vec{x}\|_2 \tag{4}$$

and simply rearranging:

$$\frac{\|\vec{x}\|_2}{\|\mathbf{A}^{-1}\|_2} \leq \|\mathbf{A}\vec{x}\|_2 \tag{5}$$

Question 2

To begin, add and subtract $\|\mathbf{B}\vec{x}\|_2$ from some arbitrary norm $\|\mathbf{A}\vec{x}\|_2$ and apply a total absolute value to the equation:

$$\big|\|\mathbf{A}\vec{x}\|_2\big| = \big|\|\mathbf{A}\vec{x}\|_2 + \|\mathbf{B}\vec{x}\|_2 - \|\mathbf{B}\vec{x}\|_2\big| \tag{6}$$

Then applying the triangle-rule, pulling out $\|\mathbf{A}\vec{x}\|_2$ and $-\|\mathbf{B}\vec{x}\|_2$:

$$\|\mathbf{A}\vec{x}\|_2 \leq \big|\|\mathbf{A}\vec{x}\|_2 - \|\mathbf{B}\vec{x}\|_2\big| + \|\mathbf{B}\vec{x}\|_2 \tag{7}$$

note the absolute value bars dropped from the two standalone norms, this is because a norm is always positive. Next, the 'inverse' triangle rule (Page 53 in textbook) is:

$$\big|\|\vec{x}\|_2 - \|\vec{y}\|_2\big| \leq \|\vec{x} - \vec{y}\|_2 \tag{8}$$

applying this to Eq. 7, we obtain:

$$\|\mathbf{A}\vec{x}\|_2 \leq \big|\|\mathbf{A}\vec{x} - \mathbf{B}\vec{x}\|_2\big| + \|\mathbf{B}\vec{x}\|_2 \tag{9}$$

And then pulling out $\vec{x}$ from the large chunk, and then because A-B linearly scales vector $\vec{x}$, we can pull that out of the norm (maintaining a norm on it):

$$\|\mathbf{A}\vec{x}\|_2 \leq \|\mathbf{A} - \mathbf{B}\|_2 \cdot \|\vec{x}\|_2 + \|\mathbf{B}\vec{x}\|_2 \tag{10}$$

Question 3

To begin, we assume the following condition to be true:

$$\frac{\|\mathbf{A}-\mathbf{B}\|_2}{\|\mathbf{A}\|_2} < \frac{1}{\kappa_2(\mathbf{A})} \tag{11}$$

If we multiply the L.H.S. by $\frac{\|\vec{x}\|_2}{\|\vec{x}\|_2}$ (1):

$$\frac{\|\mathbf{A}-\mathbf{B}\|_2}{\|\mathbf{A}\|_2} \cdot \frac{\|\vec{x}\|_2}{\|\vec{x}\|_2} < \frac{1}{\kappa_2(\mathbf{A})} \tag{12}$$

And then applying a similair method used at the end of the previous question:

$$\frac{\|\mathbf{A}\vec{x}\|_2 - \|\mathbf{B}\vec{x}\|_2}{\|\mathbf{A}\|_2\|\vec{x}\|_2} < \frac{1}{\kappa_2(\mathbf{A})} \tag{13}$$

And because we know A is invertible / non-singular, we can utilize Eq. 5:

$$\frac{\frac{\|\vec{x}\|_2}{\|\mathbf{A}^{-1}\|_2} - \|\mathbf{B}\vec{x}\|_2}{\|\mathbf{A}\|_2\|\vec{x}\|_2} < \frac{1}{\kappa_2(\mathbf{A})} \tag{14}$$

substituting in the definiton for a matrix conditioning number, $\kappa = \|\mathbf{A}^{-1}\|_2\|\mathbf{A}\|_2$, and rearranging:

$$\frac{\|\vec{x}\|_2}{\|\mathbf{A}^{-1}\|_2} - \|\mathbf{B}\vec{x}\|_2 < \frac{\|\vec{x}\|_2}{\|\mathbf{A}^{-1}\|_2} \tag{15}$$

and further, subtracting out the fraction:

$$\|\mathbf{B}\vec{x}\|_2 > 0 \tag{16}$$

Because the norm of $\mathbf{B}\vec{x}$ is non-zero, we know it is non-zero. Assuming that $\mathbf{B} \in \mathbb{R}^{n\times n}$ and that $\vec{x}$ is a non-zero vector, then we know that $\mathbf{B}$'s Null space must only contain the null vector, and is therefor not-singular.