CS 450: Homework 2

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Contents

Question 1

$$\tilde{p}(r) = p(r) + \delta(r) \tag{1}$$

$$\tilde{r} = r + \Delta r \tag{2}$$

$$\epsilon_{b,rel} = \frac{|\tilde{p} - p|}{|p|} = \frac{|\delta|}{|p|} \tag{3}$$

$$f(a) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!}(x-a)^n \tag{4}$$

$$|\delta'(r)| \leq 2\,|\Delta r| \tag{5}$$

Expanding $\tilde{p}(\tilde{r})$ about r we obtain a definition for $\delta(r)$.

First the preliminary expansion, ignoring second+ order terms:

$$\tilde{p}(\tilde{r}) = \tilde{p}(r) - \tilde{p}'(r)\Delta r \tag{6}$$

Substituting definitions of $\tilde{p}$, the L.H.S. is 0 and every instance of $\tilde{p}$ is substituted with Eq. 1:

$$0 = p(r) + \delta(r) - (p'(r) + \delta'(r))\Delta r \tag{7}$$

Next, Eq. 5:

$$0 \leq p(r) + \delta(r) - p'(r)\Delta(r) + 2(\Delta r)^2 \tag{8}$$

Finally, cancelling the second order term and rearranging:

$$\delta(r) \leq p'(r)\Delta r \tag{9}$$

and then substituting into Eq. 3:

$$\epsilon_{b,rel} = \frac{|\delta|}{|p|} \leq \frac{p'(r)\Delta r}{p(r)} \cdot \frac{r}{r} \tag{10}$$

Question 2

Inserting the polynomials into Eq. 3, we obtain:

$$\frac{|c^n|}{|(x-r)^n|} \tag{11}$$

recognizing we are computing the p norm where p=1, and so the norm is simply the sum of coefficients. Further, This denominator is a binomial series of input x and r. So, the denominator becomes:

$$|(x-r)^n| = \left| \sum_{i=0}^{n} \binom{r}{x} r^{n-i} x^i \right| = \sum_{i=0}^{n} \binom{r}{x} r^{n-i} \geq r^n \tag{12}$$

Thus, because r^n is less than the full series, and because the series is in the denominator of Eq. 11, the relative backward error simplifies to:

$$\leq \frac{|c^n|}{|(r)^n|} \tag{13}$$

The root of $p(x)$ is simply r. The root of $\tilde{p}(x)$ is found through inverting the function:

$$root = r \pm c \tag{14}$$

Where $\tilde{p}$ has a root only at $r + c$ when n is odd, and $r \pm c$ when n is even.

The relative forward error is simply Eq. a15, and then from Eq. 14:

$$\epsilon_{f,rel} = \frac{|\tilde{r} - r|}{|r|} \tag{15}$$

$$\epsilon_{f,rel} = \frac{|c|}{|r|} \tag{16}$$

Finally, as n increases the problem becomes worse and worse conditioned. This can be seen blatantly from the definition of the conditioning number:

$$\left|\frac{c}{r}\right| \leq \kappa \left|\frac{c}{r}\right|^n \tag{17}$$

and solving for kappa, we see that, because $r\ c$, as n increases so does kappa:

$$\left|\frac{r}{c}\right|^{n-1} \leq \kappa \tag{18}$$

And because kappa is increasing, the problem is worse and worse conditioned.

Question 3

To begin, because f is diffeomorphic (f is differentiable, f has an inverse, f's inverse is differentiable), the following theorem applies:

$$\left[f^{-1}\right]'(x) = \frac{1}{f'(f^{-1}(x))} \tag{19}$$

utilizing this definition, as well as the definition of κ_{abs} below:

$$\kappa_{abs} = |f'(x)| \tag{20}$$

we determine that:

$$\kappa_{inv} = \left|\frac{1}{p'(p^{-1}(x))}\right| \tag{21}$$

recognizing that both $p^{-1}(x)$ and $p'(x)$ are defined such that their domains are x $\in \Re$, then the above denominator is identical to saying $p'(x)$, and again using the definition for κ_{abs} from above:

$$\kappa_{inv} = \frac{1}{p'(x)} = \frac{1}{\kappa_{abs}} \tag{22}$$

Now for the next part. To recap:

$$\epsilon_{b,rel} \geq \left|\frac{\Delta r}{r}\right| \left|\frac{rp'(r)}{p(r)}\right| \tag{23}$$

$$\epsilon_{f,rel} = \left|\frac{\Delta r}{r}\right| \tag{24}$$

$$\epsilon_{b,rel} = \left|\frac{\tilde{p} - p}{p}\right| \tag{25}$$

$$\kappa_{rel} = \left|\frac{rp'(r)}{p(r)}\right| \tag{26}$$

$$\epsilon_{f,rel} \leq \kappa_{rel}\epsilon_{b,rel} \tag{27}$$

$$\kappa_{inv} = \frac{1}{\kappa_{abs}} = \frac{r}{\kappa_{rel}p} \tag{28}$$

To begin, solving Eq. 27 for κ:

$$\kappa_{rel} \geq \frac{\epsilon_{f,rel}}{\epsilon_{b,rel}} \tag{29}$$

And then inserting Eqs. 24 and 26 into Eq. 23:

$$\epsilon_{b,rel} \geq \epsilon_{f,rel}\kappa_{rel} \tag{30}$$

solving for κ:

$$\kappa_{rel} \leq \frac{\epsilon_{b,rel}}{\epsilon_{f,rel}} \tag{31}$$

Solving Eq. 28 for κ_{rel}:

$$\kappa_{rel} = \frac{r}{\kappa_{inv}p} \tag{32}$$

Inserting this into the previous equation, multiplying both sides by p and r (to convert the relative forward and backward errors to absolute):

$$\frac{1}{\kappa_{inv}} \leq \frac{\epsilon_{b,abs}}{\epsilon_{f,abs}} \tag{33}$$

Finally, recognizing the R.H.S is simply $\frac{1}{\kappa_{abs}}$, and taking the inverse of the equation:

$$\kappa_{inv} \geq \kappa_{abs} \tag{34}$$