CS 450: Homework 1

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August 31, 2024

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1 Question 1

To begin, the following equation must be satisfied (problem statement):

$$f^{(n)}(x) = \frac{f(x) - 2 \cdot f(x-h) + f(x-2h)}{h^2} \tag{1}$$

The Taylor series expansion definition:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2}$$

Expanding $f(x-h)$ using Eq. 2, setting $x - a = h$:

$$f(x-h) = f(x) - h \cdot f'(x) + h^2 \cdot \frac{f''(x)}{2}.... \tag{3}$$

Similarly, expanding $f(x-2h)$:

$$f(x-2h) = f(x) - 2h \cdot f'(x) + 2h^2 \cdot f''(x).... \tag{4}$$

plugging in the expanded forms of $f(x-h)$ and $f(x-2h)$ into Eq. 1, we see which terms cancel out.

$f(x)$ cancels out as $1 - 2 + 1 = 0$.

$f'(x)$ also cancels out, as $(-2 \cdot -1) + (1 \cdot -2) = 0$.

For $f''(x)$, $(-2 \cdot \frac{1}{2}) + (1 \cdot 2) = 1$.

Thus, this approximation is for the second derivative of $f(x)$.

2 Question 2

The truncation error is $\mathcal{O}(h^3)$, or third order accurate, as the lowest order term 'truncated' from the approximation is $\alpha h^3 f'''(x)$. This approximation is exact when the function being approximated is a 2nd order or lower polynomial.

3 Question 3

To begin, simply substituting in the finite difference scheme:

$$|g(x,h) - \hat{g}(x,h)| = \frac{|f(x-2h) - f(x\Theta 2h) - 2f(x-h) + 2f(x\Theta h)|}{h^2} \tag{5}$$

Then, by the triangle in-equality:

$$R.H.S. \leq \frac{|f(x-2h) - f(x\Theta 2h)|}{h^2} + 2\frac{|-f(x-h) + f(x\Theta h)|}{h^2} \tag{6}$$

Then:

$$|f(x) - f(\hat{x})| = |f(x)| \frac{|x - \hat{x}|}{|x|} \tag{7}$$

And:

$$\frac{|(x\Theta\alpha h) - (x - \alpha h)|}{(x - \alpha h)} \leq \epsilon \tag{8}$$

where this epsilon equation replaces much of the R.H.S. of Eq. 7:

$$|f(x)| \frac{|x - \hat{x}|}{|x|} = \epsilon \, |f(x)| \tag{9}$$

substituting this in, where 'x' can be any $x - \alpha h$:

$$\frac{\epsilon(|f(x-2h)| + 2\,|f(x-h)|)}{h^2} \tag{10}$$

And finally, the upper bound of $f(x - \alpha h) = 2f(x)$, thus the upper bound is:

$$|g(x,h) - \hat{g}(x,h)| \leq \frac{6\epsilon f(x)}{h^2} \tag{11}$$