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# NPRE 449: HOMEWORK 3

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SEPTEMBER 18, 2024

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## Question 1

To begin, the Navier-Stokes equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \quad (1a)$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v} \vec{v} = -\nabla P + \nabla \cdot \vec{\tau} + \rho \vec{g} \quad (1b)$$

$$\frac{\partial \rho u}{\partial t} + \nabla \cdot \rho \vec{v} u = -\nabla \cdot q'' - P \nabla \cdot \vec{v} + \vec{\tau} : \nabla \vec{v} + q''' \quad (1c)$$

First, we expand the mass equation into Cartesian coordinates. This is rather rudimentary as the mass equation is a scalar equation, and there is only the divergence of a vector to be done:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} &= 0 \\ \frac{\partial \rho}{\partial t} + \left[ \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} \right] &= 0 \end{aligned} \quad (2)$$

And so, for the mass equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \left[ \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} \right] = 0} \quad (3)$$

Next, the momentum equation. To do this we will break into x, y, and z equations, as the momentum equation is a vector equation. The equations x, y, and z will all have the same form just with different subscripts. The expansion will be done with x. So as to not have a multi-line equation, each term will be defined for specifically x. To do this various steps are undertaken. Firstly, the divergence of  $\vec{v} \vec{v}$  can be seen as, in the x direction,  $\nabla(v_x \vec{v})$ . Next, the pressure is simply just the derivative of pressure in the x direction. Then, the divergence of the stress tensor is functionally similar to the divergence of the vector-vector multiplication,  $\nabla \vec{\tau}_x$ . Finally, the gravity form is simply the x-component of g.

$$\begin{aligned} \nabla \cdot \rho \vec{v} \vec{v} &= \frac{\partial \rho v_x v_x}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} + \frac{\partial \rho v_x v_z}{\partial z} \\ \nabla P &= \frac{\partial P_x}{\partial x} \\ \nabla \cdot \vec{\tau} &= \frac{\partial \vec{\tau}_{xx}}{\partial x} + \frac{\partial \vec{\tau}_{xy}}{\partial y} + \frac{\partial \vec{\tau}_{xz}}{\partial z} \\ \rho \vec{g} &= \rho g_x \end{aligned} \quad (4)$$

Substituting these into our separated momentum equations, we get the momentum equation expanded into x, y, and z:

$$\boxed{\frac{\partial \rho v_x}{\partial t} + \left[ \frac{\partial \rho v_x v_x}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} + \frac{\partial \rho v_x v_z}{\partial z} \right] = -\frac{\partial P_x}{\partial x} + \left[ \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{xy}}{\partial y} + \frac{\partial \bar{\tau}_{xz}}{\partial z} \right] + \rho g_x} \quad (5a)$$

$$\boxed{\frac{\partial \rho v_y}{\partial t} + \left[ \frac{\partial \rho v_y v_x}{\partial x} + \frac{\partial \rho v_y v_y}{\partial y} + \frac{\partial \rho v_y v_z}{\partial z} \right] = -\frac{\partial P_y}{\partial y} + \left[ \frac{\partial \bar{\tau}_{yx}}{\partial x} + \frac{\partial \bar{\tau}_{yy}}{\partial y} + \frac{\partial \bar{\tau}_{yz}}{\partial z} \right] + \rho g_y} \quad (5b)$$

$$\boxed{\frac{\partial \rho v_z}{\partial t} + \left[ \frac{\partial \rho v_z v_x}{\partial x} + \frac{\partial \rho v_z v_y}{\partial y} + \frac{\partial \rho v_z v_z}{\partial z} \right] = -\frac{\partial P_z}{\partial z} + \left[ \frac{\partial \bar{\tau}_{zx}}{\partial x} + \frac{\partial \bar{\tau}_{zy}}{\partial y} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right] + \rho g_z} \quad (5c)$$

Finally, for the energy equation. This is fortunately a scalar equation, and is thus not needed to be separated into different equations. First,  $\nabla \cdot \rho \vec{v} u$  is simply the divergence of a vector, in which each component is multiplied by  $\rho$  and  $u$ . Next,  $\nabla \cdot q''$  is again just the divergence of the  $q''$  vector. Further,  $P \nabla \cdot \vec{v}$  is again just the divergence of a vector multiplied by the pressure.

$$\begin{aligned} \nabla \cdot \rho \vec{v} u &= \frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \\ \nabla \cdot q'' &= \frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y} + \frac{\partial q''_z}{\partial z} \\ P \nabla \cdot \vec{v} &= P \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] \end{aligned} \quad (6)$$

Then to explain the  $\bar{\tau} : \nabla \vec{v}$ . First, the  $\nabla \vec{v}$ . This is actually written as  $\nabla^T \vec{v}$ :

$$\nabla^T \vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [v_x \ v_y \ v_z] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (7)$$

Thus the double dot product has the true form of:

$$\bar{\tau} : \nabla \vec{v} = \begin{bmatrix} \tau_{ii} & \tau_{ij} & \tau_{ik} \\ \tau_{ji} & \tau_{jj} & \tau_{jk} \\ \tau_{ki} & \tau_{kj} & \tau_{kk} \end{bmatrix} : \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (8)$$

Next, the double dot product of two tensors is very similar to a vector dot product. Each component in the tensor will be multiplied by its corresponding component in the other tensor, then all of the products will be summed.

$$\bar{\tau} : \nabla \vec{v} = \tau_{ii} \frac{\partial v_x}{\partial x} + \tau_{ij} \frac{\partial v_x}{\partial y} + \tau_{ik} \frac{\partial v_x}{\partial z} + \tau_{ji} \frac{\partial v_y}{\partial x} + \tau_{jj} \frac{\partial v_y}{\partial y} + \tau_{jk} \frac{\partial v_y}{\partial z} + \tau_{ki} \frac{\partial v_z}{\partial x} + \tau_{kj} \frac{\partial v_z}{\partial y} + \tau_{kk} \frac{\partial v_z}{\partial z} \quad (9)$$

Or, in shorter notation (where i, j, k are x, y, z):

$$\vec{\tau} : \nabla \vec{v} = \sum_n \sum_m^{i,j,k} \tau_{nm} \frac{\partial v_n}{\partial m} \quad (10)$$

Finally, we can plug these all back into our energy equation, yielding the energy equation expanded into Cartesian. Both the short-hand and long form are presented. Short form:

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \left[ \frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \right] = & - \left[ \frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} + \frac{\partial q_z''}{\partial z} \right] \\ & - P \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] + \sum_n \sum_m^{i,j,k} \tau_{nm} \frac{\partial v_n}{\partial m} + q''' \quad (11) \end{aligned}$$

Long form:

$$\boxed{\begin{aligned} \frac{\partial \rho u}{\partial t} + \left[ \frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \right] = & - \left[ \frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} + \frac{\partial q_z''}{\partial z} \right] \\ & - P \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] + q''' + \left[ \tau_{ii} \frac{\partial v_x}{\partial x} + \tau_{ij} \frac{\partial v_x}{\partial y} + \tau_{ik} \frac{\partial v_x}{\partial z} \right. \\ & \left. + \tau_{ji} \frac{\partial v_y}{\partial x} + \tau_{jj} \frac{\partial v_y}{\partial y} + \tau_{jk} \frac{\partial v_y}{\partial z} + \tau_{ki} \frac{\partial v_z}{\partial x} + \tau_{kj} \frac{\partial v_z}{\partial y} + \tau_{kk} \frac{\partial v_z}{\partial z} \right] \quad (12) \end{aligned}}$$

## Question 2

From our assumptions, the mass and momentum equations simplify, in cylindrical coordinates, to:

$$\nabla \cdot \vec{v} = 0 \quad (13a)$$

$$0 = -\frac{\partial P}{\partial r} + \rho g_r \quad (13b)$$

$$0 = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \rho g_\theta \quad (13c)$$

$$0 = -\frac{\partial P}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \right] \quad (13d)$$

Finally the energy equation simplifies as follows, because specific internal energy  $u$  is simply  $c_p T$ :

$$\nabla \cdot \rho \vec{v} u = -\nabla \cdot \vec{q}'' \quad (14a)$$

$$\rho c_p \left[ \frac{1}{r} \frac{\partial}{\partial r} (r v_r T) + \frac{1}{r} \frac{\partial v_\theta T}{\partial \theta} + \frac{\partial v_z T}{\partial z} \right] = k \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \quad (14b)$$

$$v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_p} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \quad (14c)$$

Finally we can solve these equations for axial velocity and temperature.

First, we will solve for the pressure profile. We know that  $\vec{g}$  is equal to  $\langle -g \sin(\theta), -g \cos(\theta), 0 \rangle$ .

Thus:

$$P = -\rho g r \sin \theta + C(\theta, z) \quad (15a)$$

$$P = -\rho g r \sin \theta + C(r, z) \quad (15b)$$

And because the result from both radial and azimuthal solving yields the same functional form, we know that  $C$  must only be a function of  $z$ . Further, we know that pressure must be a linear function of  $z$ , as its derivative with respect to  $z$  must not be a function of  $z$ . Thus, we can solve for the velocity profile.

$$\frac{P_0 r}{\mu} = \frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \quad (16a)$$

$$\frac{P_0 r}{2\mu} + \frac{C_1}{r} = \frac{\partial v_z}{\partial r} \quad (16b)$$

$$v_z(r) = \frac{P_0 r^2}{4\mu} + C_1 \ln r + C_2 \quad (16c)$$

Next for our boundary conditions:

- ① No-Slip Condition:  $v_z(R_o) = 0$
- ② Symmetry:  $\frac{\partial v_z(0)}{\partial r} = 0$ , or Finiteness at the centerline

Thus,  $C_1$  must be 0 by ②, and then investigating ①,  $C_2$  must be:

$$v_z(R_0) = \frac{P_0 R_0^2}{4\mu} + C = 0 \quad (17a)$$

$$C = -\frac{P_0 R_0^2}{4\mu} \quad (17b)$$

Thus our axial velocity, recognizing that  $P_0 = \frac{-\Delta P}{l}$ , is:

$$v_z(r) = \frac{\Delta P R^2}{4\mu l} - \frac{\Delta P r^2}{4\mu l} = \frac{\Delta P}{4\mu l} (R^2 - r^2) = \langle v_z \rangle \left(1 - \frac{r^2}{R^2}\right) \quad (18)$$

Such that  $\langle v_z \rangle = \frac{\Delta P R^2}{4\mu l}$ .

Now, for the temperature. Importantly,  $\frac{\partial T}{\partial z}$  is a constant because of the definition of fully developed temperature flow. .

$$v_z(r) \frac{\partial T}{\partial z} = \frac{k}{\rho c_p r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \quad (19a)$$

$$\frac{\langle v_z \rangle}{k} \frac{\rho c_p}{\partial z} \frac{\partial T}{\partial z} \left(r - \frac{r^3}{R^2}\right) = \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \quad (19b)$$

$$\frac{\langle v_z \rangle}{k} \frac{\rho c_p}{\partial z} \frac{\partial T}{\partial z} \left(\frac{r}{2} - \frac{r^3}{4R^2}\right) + \frac{C_1}{r} = \frac{\partial T}{\partial r} \quad (19c)$$

$$T(r, z) = \frac{\langle v_z \rangle}{k} \frac{\rho c_p}{\partial z} \frac{\partial T}{\partial z} \left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln(r) + C_2 \quad (19d)$$

And then again for simplicity, I define  $\frac{\langle v_z \rangle}{k} \frac{\rho c_p}{\partial z} \frac{\partial T}{\partial z}$  as equal to  $\xi$ :

$$T(r, z) = \xi \left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln(r) + C_2 \quad (20)$$

Now for our boundary conditions:

① Symmetry:  $\frac{\partial T(0)}{\partial r}$ , or Finiteness at the center line

②  $T(R, z) = T_s(z)$

Investigating ①,  $C_1$  must be 0. Then from ②:

$$T(R, z) = \xi \left[ \frac{R^2}{4} - \frac{R^2}{16} \right] + C_2 \quad (21a)$$

$$T_s(z) - \xi \left[ \frac{3R^2}{16} \right] = C_2 \quad (21b)$$

Thus, our temperature distribution is:

$$T(r, z) = \xi \left[ \frac{r^2}{4} - \frac{r^4}{16R^2} \right] + T_s(z) - \xi \frac{3R^2}{16} \quad (22)$$

Then to find the normalized temperature,  $\frac{T_s(z) - T(r, z)}{T_s(z) - T_{cl}(z)}$ :

$$T_s(z) - T(r, z) = \xi \left[ \frac{r^4}{16R^2} - \frac{r^2}{4} + \frac{3R^2}{16} \right] \quad (23a)$$

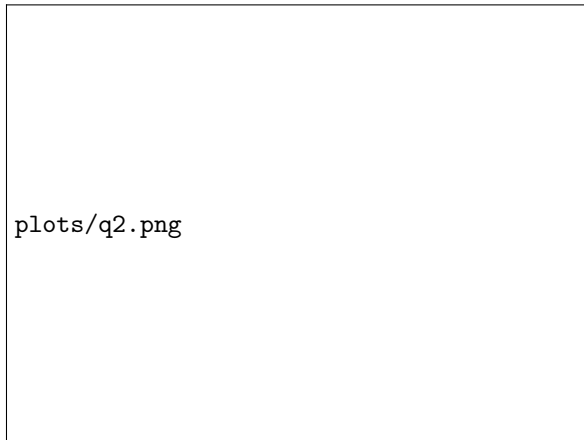
$$T_s(z) - T_{cl}(z) = \xi \left[ \frac{3R^2}{16} \right] \quad (23b)$$

Thus:

$$\frac{T_s(z) - T(r, z)}{T_s(z) - T_{cl}(z)} = \frac{\frac{r^4}{16R^2} - \frac{r^2}{4} + \frac{3R^2}{16}}{\frac{3R^2}{16}} \quad (24a)$$

$$\frac{T_s(z) - T(r, z)}{T_s(z) - T_{cl}(z)} = \frac{r^4}{3R^4} - \frac{4r^2}{3R^2} + 1 \quad (24b)$$

Finally, the normalized temperature and velocity distributions are:





### Question 3

To begin, we can skip all of the beginning cruft. We already know our velocity profile:

$$v_z(r) = \frac{\Delta P}{4\mu l}(r^2 - R^2) \quad (25)$$

And taking the area average of this:

$$\langle v_z \rangle = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R r v_z(r) dr d\theta \quad (26a)$$

$$\langle v_z \rangle = \frac{1}{\pi R^2} \frac{\pi \Delta P}{2\mu l} \left( \frac{r^4}{4} - \frac{R^2 r^2}{2} \right) \Big|_0^R \quad (26b)$$

$$\langle v_z \rangle = \frac{-R^2 \Delta P}{8\mu l} \quad (26c)$$

Thus, the velocity profile is equal to  $2\langle v_z \rangle(1 - \frac{r^2}{R^2})$ .

Next, our energy equation is:

$$\rho c_p v_z(r) \frac{\partial T}{\partial z} = \frac{k}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial z} + \mu \left( \frac{\partial v_z}{\partial r} \right)^2 \quad (27a)$$

$$2\rho c_p \langle v_z \rangle \left(1 - \frac{r^2}{R^2}\right) \frac{\partial T}{\partial z} = 16\mu \langle v_z \rangle^2 \frac{r^2}{R^4} \quad (27b)$$

Next, to integrate both sides through the volume:

$$\iiint_V 2r\rho c_p \langle v_z \rangle \left(1 - \frac{r^2}{R^2}\right) \frac{\partial T}{\partial z} dV = \iiint_V 16r\mu \langle v_z \rangle^2 \frac{r^2}{R^4} dV \quad (28a)$$

$$2\rho c_p \int_0^{2\pi} \int_0^R \int_0^l \left(r - \frac{r^3}{R^2}\right) \frac{\partial T}{\partial z} dz d\theta dr = 16\mu \langle v_z \rangle \int_0^{2\pi} \int_0^R \int_0^l \frac{r^3}{R^4} dz d\theta dr \quad (28b)$$

$$4\pi\rho c_p \Delta T \left( \frac{r^2}{2} - \frac{r^4}{4R^2} \right) \Big|_0^R = 32\pi\mu \langle v_z \rangle \frac{r^4}{4R^4} \Big|_0^R \Big|_0^l \quad (28c)$$

$$\rho c_p \Delta T R^2 = 8\mu \langle v_z \rangle l \quad (28d)$$

$$l = \frac{\rho c_p \Delta T R^2}{8\mu \langle v_z \rangle} \quad (28e)$$

Finally, we can find the average velocity,  $\langle v_z \rangle$ , with the reynolds number of laminar flow.

$$Re = \frac{2\rho \langle v_z \rangle R}{\mu} \langle v_z \rangle = \frac{Re\mu}{2R\rho} \quad (29)$$

Finally, using a reynolds number of 2300, we can solve for the two lengths:

$$\boxed{l = 56473.337m, D = 0.1m} \tag{30}$$

$$\boxed{l = 56.473m, D = 0.01m} \tag{31}$$