NPRE 449: Homework 3

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Question 1

To begin, the Navier-Stokes equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \tag{1a}$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v}\,\vec{v} = -\nabla P + \nabla \cdot \overleftrightarrow{\tau} + \rho \vec{g} \tag{1b}$$

$$\frac{\partial \rho u}{\partial t} + \nabla \cdot \rho \vec{v} u = -\nabla \cdot q'' - P\nabla \cdot \vec{v} + \overleftrightarrow{\tau}{:}\nabla \vec{v} + q''' \tag{1c}$$

First, we expand the mass equation into Cartesian coordinates. This is rather rudimentary as the mass equation is a scalar equation, and there is only the divergence of a vector to be done:

$$\begin{gathered} \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \\ \frac{\partial \rho}{\partial t} + \left[\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right] = 0 \end{gathered} \tag{2}$$

And so, for the mass equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \left[\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right] = 0} \tag{3}$$

Next, the momentum equation. To do this we will break into x, y, and z equations, as the momentum equation is a vector equation. The equations x, y, and z will all have the same form just with different subscripts. The expansion will be done with x. So as to not have a multi-line equation, each term will be defined for specifically x. To do this various steps are undertaken. Firstly, the divergence of $\vec{v}\,\vec{v}$ can be seen as, in the x direction, $\nabla(v_x \vec{v})$. Next, the pressure is simply just the derivative of pressure in the x direction. Then, the divergence of the stress tensor is functionally similair to the divergence of the vector-vector multiplication, $\nabla \overleftrightarrow{\tau}_x$. Finally, the gravity form is simply the x-component of g.

$$\begin{gathered} \nabla \cdot \rho \vec{v}\,\vec{v} = \frac{\partial \rho v_x v_x}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} + \frac{\partial \rho v_x v_z}{\partial z} \\ \nabla P = \frac{\partial P_x}{\partial x} \\ \nabla \cdot \overleftrightarrow{\tau} = \frac{\partial \overleftrightarrow{\tau}_{xx}}{\partial x} + \frac{\partial \overleftrightarrow{\tau}_{xy}}{\partial y} + \frac{\partial \overleftrightarrow{\tau}_{xz}}{\partial z} \\ \rho \vec{g} = \rho g_x \end{gathered} \tag{4}$$

Substituting these into our seperated momentum equations, we get the momentum equation expanded into x, y, and z:

$$\boxed{\frac{\partial \rho v_x}{\partial t} + \left[\frac{\partial \rho v_x v_x}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} + \frac{\partial \rho v_x v_z}{\partial z}\right] = -\frac{\partial P_x}{\partial x} + \left[\frac{\partial \overleftrightarrow{\tau}_{xx}}{\partial x} + \frac{\partial \overleftrightarrow{\tau}_{xy}}{\partial y} + \frac{\partial \overleftrightarrow{\tau}_{xz}}{\partial z}\right] + \rho g_x} \tag{5a}$$

$$\boxed{\frac{\partial \rho v_y}{\partial t} + \left[\frac{\partial \rho v_y v_x}{\partial x} + \frac{\partial \rho v_y v_y}{\partial y} + \frac{\partial \rho v_y v_z}{\partial z}\right] = -\frac{\partial P_y}{\partial y} + \left[\frac{\partial \overleftrightarrow{\tau}_{yx}}{\partial x} + \frac{\partial \overleftrightarrow{\tau}_{yy}}{\partial y} + \frac{\partial \overleftrightarrow{\tau}_{yz}}{\partial z}\right] + \rho g_y} \tag{5b}$$

$$\boxed{\frac{\partial \rho v_z}{\partial t} + \left[\frac{\partial \rho v_z v_x}{\partial x} + \frac{\partial \rho v_z v_y}{\partial y} + \frac{\partial \rho v_z v_z}{\partial z}\right] = -\frac{\partial P_z}{\partial z} + \left[\frac{\partial \overleftrightarrow{\tau}_{zx}}{\partial x} + \frac{\partial \overleftrightarrow{\tau}_{zy}}{\partial y} + \frac{\partial \overleftrightarrow{\tau}_{zz}}{\partial z}\right] + \rho g_z} \tag{5c}$$

Finally, for the energy equation. This is fortunately a scalar equation, and is thus not needed to be separated into different equations. First, $\nabla \cdot \rho \vec{v} u$ is simply the divergence of a vector, in which each component is multiplied by ρ and u. Next, $\nabla \cdot q''$ is again just the divergence of the q'' vector. Further, $P\nabla \cdot \vec{v}$ is again just the divergence of a vector multiplied by the pressure.

$$\begin{aligned}
\nabla \cdot \rho \vec{v} u &= \frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \\
\nabla \cdot q'' &= \frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y} + \frac{\partial q''_z}{\partial z} \\
P\nabla \cdot \vec{v} &= P\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]
\end{aligned} \tag{6}$$

Then to explain the $\overleftrightarrow{\tau} : \nabla \vec{v}$. First, the $\nabla \vec{v}$. This is actually written as $\nabla^T \vec{v}$:

$$\nabla^T \vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [v_x \;\; v_y \;\; v_z] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{7}$$

Thus the double dot product has the true form of:

$$\overleftrightarrow{\tau} : \nabla \vec{v} = \begin{bmatrix} \tau_{ii} & \tau_{ij} & \tau_{ik} \\ \tau_{ji} & \tau_{jj} & \tau_{kk} \\ \tau_{ki} & \tau_{kj} & \tau_{kk} \end{bmatrix} : \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{8}$$

Next, the double dot product of two tensors is very similair to a vector dot product. Each component in the tensor will be multiplied by its corresponding component in the other tensor, then all of the products will be summed.

$$\overleftrightarrow{\tau} : \nabla \vec{v} = \tau_{ii}\frac{\partial v_x}{\partial x} + \tau_{ij}\frac{\partial v_x}{\partial y} + \tau_{ik}\frac{\partial v_x}{\partial z} + \tau_{ji}\frac{\partial v_y}{\partial x} + \tau_{jj}\frac{\partial v_y}{\partial y} + \tau_{jk}\frac{\partial v_y}{\partial z} + \tau_{ki}\frac{\partial v_z}{\partial x} + \tau_{kj}\frac{\partial v_z}{\partial y} + \tau_{kk}\frac{\partial v_z}{\partial z} \tag{9}$$

Or, in shorter notation (where i, j, k are x, y, z):

$$\overleftrightarrow{\tau} : \nabla \vec{v} = \sum_{n}^{i,j,k} \sum_{m}^{i,j,k} \tau_{nm} \frac{\partial v_n}{\partial m} \tag{10}$$

Finally, we can plug these all back into our energy equation, yielding the energy equation expanded into Cartesian. Both the short-hand and long form are presented. Short form:

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \left[\frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \right] = &- \left[\frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y} + \frac{\partial q''_z}{\partial z} \right] \\ &- P \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] + \sum_{n}^{i,j,k} \sum_{m}^{i,j,k} \tau_{nm} \frac{\partial v_n}{\partial m} + q''' \end{aligned} \tag{11}$$

Long form:

$$\boxed{\begin{aligned} \frac{\partial \rho u}{\partial t} + \left[\frac{\partial \rho v_x u}{\partial x} + \frac{\partial \rho v_y u}{\partial y} + \frac{\partial \rho v_z u}{\partial z} \right] = &- \left[\frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y} + \frac{\partial q''_z}{\partial z} \right] \\ &- P \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] + q''' + \Bigg[\tau_{ii} \frac{\partial v_x}{\partial x} + \tau_{ij} \frac{\partial v_x}{\partial y} + \tau_{ik} \frac{\partial v_x}{\partial z} \\ &+ \tau_{ji} \frac{\partial v_y}{\partial x} + \tau_{jj} \frac{\partial v_y}{\partial y} + \tau_{jk} \frac{\partial v_y}{\partial z} + \tau_{ki} \frac{\partial v_z}{\partial x} + \tau_{kj} \frac{\partial v_z}{\partial y} + \tau_{kk} \frac{\partial v_z}{\partial z} \Bigg] \end{aligned}} \tag{12}$$

Question 2

From our assumptions, the mass and momentum equations simplify, in cylindrical coordinates, to:

$$\nabla \cdot \vec{v} = 0 \tag{13a}$$

$$0 = -\frac{\partial P}{\partial r} + \rho g_r \tag{13b}$$

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta \tag{13c}$$

$$0 = -\frac{\partial P}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r}\right] \tag{13d}$$

Finally the energy equation simplifies as follows, because specific internal energy u is simply $c_p T$:

$$\nabla \cdot \rho \vec{v} u = -\nabla \cdot \vec{q}'' \tag{14a}$$

$$\rho c_p \left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r T) + \frac{1}{r}\frac{\partial v_\theta T}{\partial \theta} + \frac{\partial v_z T}{\partial z}\right] = k\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] \tag{14b}$$

$$v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{14c}$$

Finally we can solve these equations for axial velocity and temperature.

First, we will solve for the pressure profile. We know that $\vec{g}$ is equal to $\langle -g\sin(\theta),\, -g\cos(\theta),\, 0\rangle$. Thus:

$$P = -\rho g r \sin\theta + C(\theta, z) \tag{15a}$$

$$P = -\rho g r \sin\theta + C(r, z) \tag{15b}$$

And because the result from both radial and azimuthal solving yields the same functional form, we know that C must only be a function of z. Further, we know that pressure must be a linear function of z, as its derivative with respect to z must not be a function of z. Thus, we can solve for the velocity profile.

$$\frac{P_0 r}{\mu} = \frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \tag{16a}$$

$$\frac{P_0 r}{2\mu} + \frac{C_1}{r} = \frac{\partial v_z}{\partial r} \tag{16b}$$

$$v_z(r) = \frac{P_0 r^2}{4\mu} + C_1 \ln r + C_2 \tag{16c}$$

Next for our boundary conditions:

1. No-Slip Condition: $v_z(R_o) = 0$

2. Symmetry: $\frac{\partial v_z(0)}{\partial r} = 0$, or Finiteness at the centerline

Thus, C_1 must be 0 by 2, and then investigating 1, C_2 must be:

$$v_z(R_0) = \frac{P_0 R_0^2}{4\mu} + C = 0 \tag{17a}$$

$$C = -\frac{P_0 R_0^2}{4\mu} \tag{17b}$$

Thus our axial velocity, recognizing that $P_0 = \frac{-\Delta P}{l}$, is:

$$v_z(r) = \frac{\Delta P R^2}{4\mu l} - \frac{\Delta P r^2}{4\mu l} = \frac{\Delta P}{4\mu l}(R^2 - r^2) = \langle v_z \rangle (1 - \frac{r^2}{R^2}) \tag{18}$$

Such that $\langle v_z \rangle = \frac{\Delta P R^2}{4\mu l}$.

Now, for the temperature. Importantly, $\frac{\partial T}{\partial z}$ is a constant because of the definition of fully developed temperature flow. .

$$v_z(r)\frac{\partial T}{\partial z} = \frac{k}{\rho c_p r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{19a}$$

$$\frac{\langle v_z \rangle \ \rho \ c_p}{k}\frac{\partial T}{\partial z}(r - \frac{r^3}{R^2}) = \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{19b}$$

$$\frac{\langle v_z \rangle \ \rho \ c_p}{k}\frac{\partial T}{\partial z}(\frac{r}{2} - \frac{r^3}{4R^2}) + \frac{C_1}{r} = \frac{\partial T}{\partial r} \tag{19c}$$

$$T(r,z) = \frac{\langle v_z \rangle \ \rho \ c_p}{k}\frac{\partial T}{\partial z}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln(r) + C_2 \tag{19d}$$

And then again for simplicity, I define $\frac{\langle v_z \rangle \ \rho \ c_p}{k}\frac{\partial T}{\partial z}$ as equal to ξ:

$$T(r,z) = \xi\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln(r) + C_2 \tag{20}$$

Now for our boundary conditions:

(1) Symmetry: $\frac{\partial T(0)}{\partial r}$, or Finiteness at the center line

(2) $T(R,z) = T_s(z)$

Investigating (1), C_1 must be 0. Then from (2):

$$T(R,z) = \xi\left[\frac{R^2}{4} - \frac{R^2}{16}\right] + C_2 \tag{21a}$$

$$T_s(z) - \xi\left[\frac{3R^2}{16}\right] = C_2 \tag{21b}$$

Thus, our temperature distribution is:

$$T(r,z) = \xi\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + T_s(z) - \xi\frac{3R^2}{16} \tag{22}$$

Then to find the normalized temperature, $\frac{T_s(z)-T(r,z)}{T_s(z)-T_{cl}(z)}$:

$$T_s(z) - T(r,z) = \xi\left[\frac{r^4}{16R^2} - \frac{r^2}{4} + \frac{3R^2}{16}\right] \tag{23a}$$

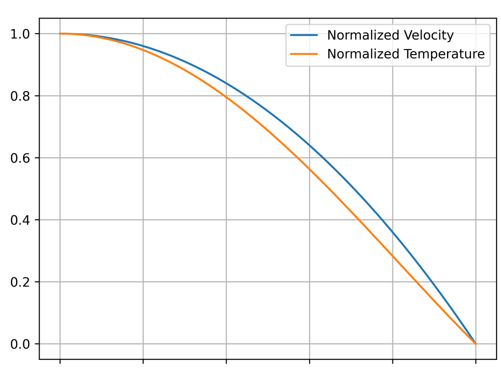
$$T_s(z) - T_{cl}(z) = \xi[\frac{3R^2}{16}] \tag{23b}$$

Thus:

$$\frac{T_s(z) - T(r,z)}{T_s(z) - T_{cl}(z)} = \frac{\frac{r^4}{16R^2} - \frac{r^2}{4} + \frac{3R^2}{16}}{\frac{3R^2}{16}} \tag{24a}$$

$$\frac{T_s(z) - T(r,z)}{T_s(z) - T_{cl}(z)} = \frac{r^4}{3R^4} - \frac{4r^2}{3R^2} + 1 \tag{24b}$$

Finally, the normalized temperature and velocity distributions are:



Question 3

To begin, we can skip all of the beginning cruft. We already know our velocity profile:

$$v_z(r) = \frac{\Delta P}{4\mu\ l}(r^2 - R^2) \tag{25}$$

And taking the area average of this:

$$\langle v_z \rangle = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R r v_z(r) dr d\theta \tag{26a}$$

$$\langle v_z \rangle = \frac{1}{\pi R^2} \frac{\pi \Delta P}{2\mu l} (\frac{r^4}{4} - \frac{R^2 r^2}{2}) \Big|_0^R \tag{26b}$$

$$\langle v_z \rangle = \frac{-R^2 \Delta P}{8\mu l} \tag{26c}$$

Thus, the velocity profile is equal to $2\langle v_z \rangle (1 - \frac{r^2}{R^2})$.

Next, our energy equation is:

$$\rho c_p v_z(r) \frac{\partial T}{\partial z} = \frac{k}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial z} + \mu \left(\frac{\partial v_z}{\partial r} \right)^2 \tag{27a}$$

$$2\rho c_p \langle v_z \rangle (1 - \frac{r^2}{R^2}) \frac{\partial T}{\partial z} = 16\mu \langle v_z \rangle^2 \frac{r^2}{R^4} \tag{27b}$$

Next, to integrate both sides through the volume:

$$\iiint_V 2r\rho c_p \langle v_z \rangle (1 - \frac{r^2}{R^2}) \frac{\partial T}{\partial z} dV = \iiint_V 16 r\mu \langle v_z \rangle^2 \frac{r^2}{R^4} dV \tag{28a}$$

$$2\rho c_p \int_0^{2\pi} \int_0^R \int_0^l (r - \frac{r^3}{R^2}) \frac{\partial T}{\partial z} dz d\theta dr = 16\mu \langle v_z \rangle \int_0^{2\pi} \int_0^R \int_0^l \frac{r^3}{R^4} dz d\theta dr \tag{28b}$$

$$4\pi \rho c_p \Delta T \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right) \Big|_0^R = 32\pi\mu \langle v_z \rangle \frac{r^4}{4R^4} \Big|_0^R (z) \Big|_0^l \tag{28c}$$

$$\rho c_p \Delta T R^2 = 8\mu \langle v_z \rangle l \tag{28d}$$

$$l = \frac{\rho c_p \Delta T R^2}{8\mu \langle v_z \rangle} \tag{28e}$$

Finally, we can find the average velocity, $\langle v_z \rangle$, with the reynolds number of laminar flow.

$$Re = \frac{2\rho \langle v_z \rangle R}{\mu} \langle v_z \rangle = \frac{Re\mu}{2R\rho} \tag{29}$$

Finally, using a reynolds number of 2300, we can solve for the two lengths:

$$\boxed{l = 56473.337m, D = 0.1m} \tag{30}$$

$$\boxed{l = 56.473m, D = 0.01m} \tag{31}$$